Stein domains with exotic contact boundaries

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Mu Zhao

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

Mark McLean - Dissertation Advisor
Associate Professor, Department of Mathematics

Aleksey Zinger - Chairperson of Defense
Professor, Department of Mathematics

Kenji Fukaya
Professor, Department of Mathematics

Mohammed Abouzaid
Professor, Department of Mathematics, Columbia University

This dissertation is accepted by the Graduate School

Eric Wertheimer
Dean of the Graduate School

Abstract of the Dissertation

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We introduce a new invariant, the ***positive idempotent group***, for strongly asymptotically dynamically convex contact manifolds. This invariant can be used to distinguish different contact structures. As an application, for any complex dimension $n > 8$ and any positive integer k, we can construct $n-$dimensional Stein manifolds $V_0, V_1, \cdots, V_k$ such that $\tilde{H}_j(V_i) = 0, j \neq n-1, n$. $V_i's$ are almost symplectomorphic, their boundaries are in the same almost contact class but not contactomorphic.

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1 Introduction

In this paper, we will introduce a new invariant $I_+(\Sigma)$, the ***positive idempotent group***, for strongly asymptotically dynamically convex contact manifolds (Σ, ξ, Φ)(see definition in Section 3.1). The definition of positive idempotent group $I_+(W)$ depends on the filling W: it is well defined when $SH_*(W) \neq 0$ for some Liouville filling W, and it is independent of filling when (Σ, ξ, Φ) is a strongly ADC contact manifold.

The main purpose of this paper is to prove the following theorem:

Theorem 1.1. *If (Σ, ξ, Φ) is a strongly asymptotically dynamically convex contact structure with a Liouville filling W such that $SH_*(W) \neq 0$, then all connected Liouville fillings of (Σ, ξ, Φ) with nonzero symplectic homology have isomorphic positive idempotent group I_+.*

Remark 1.2. Here a Liouville filling W of (Σ, ξ, Φ) means that W is a filling of (Σ, ξ) and the trivialization Φ of the canonical bundle extends over W. Now that all these Liouville fillings have isomorphic positive idempotent group, we can regard I_+ as an invariant for strongly ADC contact manifold. We will prove the result in section 4.

As an application, we will use the ***positive idempotent group*** to distinguish contact boundaries of Stein manifolds, which has a long history. Y.Eliashberg [E+91] constructed an exotic contact structure representing the standard almost contact structure on S^{4k+1}, and I.Ustilovsky [Ust99] proved that every almost contact class on S^{4k+1} has infinitely many different contact structures. M.McLean [McL07] has shown that there are infinitely many exotic Stein structures $\mathbb{C}^n_k$ on $\mathbb{C}^n, n \geq 4$. Using flexible Weinstein structures, O.Lazarev [Laz16] proved that any contact manifold admitting an almost Weinstein filling admits infinitely many exotic contact structures with flexible fillings. We have the following theorem:

Theorem 1.3. *For any complex dimension $n > 8$ and any positive integer k, there are Stein domains $V_0, V_1, \cdots, V_k$ such that:*

- *$V_i's$ are almost symplectomorphic,*
- *the contact boundaries ∂V_i of V_i are in the same almost contact class,*
- *∂V_i are mutually non-contactomorphic.*
- *$\tilde{H}_j(V_i) = 0$ for $j \neq n, n-1$.*

Remark 1.4. In Theorem 1.14 [Laz16], O.Lazarev proved that if V is almost symplectomorphic to a domain containing a closed (regular) Lagrangian, then there are infinitely symplectic structures V_k almost symplectomorphic to V that are not symplectomorphic and their contact boundaries are not contactomorphic either. The Stein domains constructed in this paper are different from Lazarev's examples.

1.1 Sketch of the proof

The contact structure on ∂V_i in Theorem 1.3 is asymptotically dynamically convex. In the case when $I_+(\Sigma)$ is finite, we can define the ***positive idempotent index*** $i(\Sigma) := |I_+(\Sigma)|$ (see Section 3.2). The theorem 1.3 is based on the following theorem, which will be proved in Section 7:

Theorem 1.5. *There exists connected Weinstein domains* (W^{2n}, λ, ψ), *for any* $n > 8$ *such that*

- $(\partial W, \lambda)$ *is asymptotically dynamically convex,*
- $SH_*(W, \mathbb{Z}/2\mathbb{Z}) \neq 0$*,*
- $|I(W, \mathbb{Z}/2\mathbb{Z})| < \infty$.
- $\tilde{H}_i(W, \mathbb{Z}/2\mathbb{Z}) = 0$*, for* $i \neq n, n-1$.

Remark 1.6. The definition of I is in equation 3.3.

The basic idea to construct the Weinstein domain is to use Brieskorn variety. First we take the complement of a specific Brieskorn variety and then attach a Weinstein 2-handle to kill the fundamental group. With the help of a covering trick we can show that the resultant manifold has asymptotically dynamically convex boundary. The full proof is at the end of this paper, see Section 7.

We will need the fact that any almost Weinstein domain admits a flexible Weinstein structure in the same almost symplectic class (See Section 2.8). Moreover, if a contact manifold admits a flexible filling, then it is asymptotically dynamically convex, as stated in the following lemma:

Lemma 1.7 (Corollary 4.1 [Laz16]). *If* $(Y^{2n-1}, \xi), n \geq 3$*, has a flexible filling, then* (Y, ξ) *is asymptotically dynamically convex.*

Proof of Theorem 1.3. Let (W, λ, ψ) be in Theorem 1.5. There is a flexible Weinstein domain (W_1, λ_1, ψ_1) that is almost symplectomorphic to W. Let

$$(W_i, \lambda_i, \psi_i) := \underbrace{(W, \lambda, \psi)\natural(W, \lambda, \psi)\natural \cdots \natural(W, \lambda, \psi)}_{i} \natural \underbrace{(W_1, \lambda_1, \psi_1)\natural(W_1, \lambda_1, \psi_1)\natural \cdots \natural(W_1, \lambda_1, \psi_1)}_{k-i}.$$

That is, W_i is the boundary connect sum of i copies of W and $k-i$ copies of the flexibilization of W_1. The boundary connect sum is equivalent to attaching a Weinstein 1-handle, so W_i is a Weinstein domain. By construction, they are all almost symplectomorphic, see subsection 2.8.3, and their boundaries are in the same almost contact class by lemma 2.35. Theorem 2.7 allows us to deform a Weinstein structure into a Stein structure, which is denoted by V_i. The last condition is obvious. There's only the third condition left to be verified. Indeed, we have $(\partial V_i, \lambda_i)$ is asymptotically dynamically convex. Furthermore, we have:

Proposition 1.8. $|I_+(\partial W_i)| \neq |I_+(\partial W_j)|, i \neq j$.

The proof of Proposition 1.8 will be defer to Subsection 3.3.

□

2 Background

2.1 Conventions and notation

(See Section 2 for detailed definitions.)

Let λ be a Liouville 1-form on a Liouville manifold W.

$$
\begin{aligned}
&d\lambda(\cdot, J\cdot) = g_J \qquad \text{(Riemannian metric)},\\
&d\lambda(X_H, \cdot) = -dH, \qquad X_H = J\nabla H \qquad \text{(Hamiltonian vector field)},\\
&\mathcal{L}\widehat{W} := C^\infty(S^1, \widehat{W}), \qquad S^1 = \mathbb{R}/\mathbb{Z} \qquad \text{(loop space)},\\
&A_H : \mathcal{L}\widehat{W} \to \mathbb{R}, \qquad A_H(x) := \int_{S^1} x^*\lambda - \int_{S^1} H(t, x(t))\, dt \qquad \text{(action)},\\
&\nabla A_H(x) = -J(x)(\dot{x} - X_H(t, x)) \qquad (L^2\text{-gradient}),\\
&u : \mathbb{R} \to \mathcal{L}W, \qquad \partial_s u = \nabla A_H(u(s, \cdot)) \qquad \text{(gradient line)}\\
&\iff \partial_s u + J(u)(\partial_t u - X_H(t, u)) = 0 \qquad \text{(Floer equation)}, \qquad (2.1)\\
&\mathcal{P}(H) := \text{Crit}(A_H) = \{\text{1-periodic orbits of the Hamiltonian vector field } X_H\},
\end{aligned}
$$

$$
\text{For each } h \in H_1(W), \mathcal{P}^h(H) := \text{Crit}_h(A_H) = \{x \in \mathcal{P}(H) \big| [x] = h \in [S^1 \to W]\}
$$

$$
\mathcal{M}(x_-, x_+; H, J) = \{u : \mathbb{R} \times S^1 \to W \mid \partial_s u = \nabla A_H(u(s, \cdot)),\ u(\pm\infty, \cdot) = x_\pm\}/\mathbb{R}
$$

$$
\text{(moduli space of Floer trajectories connecting } x_\pm \in \mathcal{P}(H)),
$$

$$
\mathcal{M}_h(x_-, x_+; H, J) = \{u : \mathbb{R} \times S^1 \to W \mid \partial_s u = \nabla A_H(u(s, \cdot)),\ u(\pm\infty, \cdot) = x_\pm \in \mathcal{P}^h(H)\}/\mathbb{R}
$$

$$
\text{(moduli space of Floer trajectories connecting } x_\pm \in \mathcal{P}^h(H)),
$$

$$
\begin{aligned}
\dim \mathcal{M}(x_-, x_+; H, J) &= \mu_{CZ}(x_+) - \mu_{CZ}(x_-) - 1,\\
A_H(x_+) - A_H(x_-) &= \int_{\mathbb{R}\times S^1} |\partial_s u|^2 ds\, dt = \int_{\mathbb{R}\times S^1} u^*(d\lambda - dH \wedge dt).
\end{aligned}
$$

Here the formula expressing the dimension of the moduli space in terms of Conley-Zehnder indices is to be understood with respect to a symplectic trivialization of u^*TW.

Let $\mathbb{K}$ be a field and $a < b$ with $a, b \notin \text{Spec}(\partial W, \alpha)$. We define the filtered Floer chain groups with coefficients in $\mathbb{K}$ by

$$
SC_*^{<b}(H) := \bigoplus_{\substack{x \in \mathcal{P}(H)\\ A_H(x) < b}} \mathbb{K} \cdot x, \qquad SC_*^{(a,b)}(H) = SC_*^{<b}(H)/SC_*^{<a}(H),
$$

with the differential $d : SC_*^{(a,b)}(H) \to SC_{*-1}^{(a,b)}(H)$ given by

$$
dx_+ = \sum_{\mu_{CZ}(x_-) = \mu_{CZ}(x_+) - 1} \#\mathcal{M}(x_-, x_+; H, J) \cdot x_-.
$$

Here # denotes the signed count of points with respect to suitable orientations. We think of the cylinder $\mathbb{R} \times S^1$ as the twice punctured Riemann sphere, with the positive puncture at $+\infty$ as incoming, and the negative puncture at $-\infty$ as outgoing. This terminology makes reference to the corresponding asymptote being an input, respectively an output for the Floer differential. Note that the differential decreases both the action A_H and the Conley-Zehnder index. The filtered Floer homology is now defined as

$$SH_*^{(a,b)}(H) = \ker d/\mathbf{im}\, d.$$

Note that for $a < b < c$ the short exact sequence

$$0 \to SC_*^{(a,b)}(H) \to SC_*^{(a,c)}(H) \to SC_*^{(b,c)}(H) \to 0$$

induces a *tautological exact triangle*

$$SH_*^{(a,b)}(H) \to SH_*^{(a,c)}(H) \to SH_*^{(b,c)}(H) \to SH_*^{(a,b)}(H)[-1]. \tag{2.2}$$

Remark. We will suppress the field $\mathbb{K}$ from the notation. As noted in the Introduction, the definition can also be given with coefficients in a commutative ring. In this paper, $\mathbb{K} = \mathbb{Z}_2$.

Notation. Let $\mathbf{a} = (a_0, a_1, \cdots, a_n)$ be an $(n+1)$-tuple of integers $a_i > 1, \mathbf{z} := (z_0, z_1, \cdots, z_n) \in \mathbb{C}^{n+1}$, and set $f(\mathbf{z}) := z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n}$, and let $B(s)$ to be the closed ball of radius s.

$$V_{\mathbf{a}}(t) := \{(z_0, z_1, \cdots, z_n) \in \mathbb{C}^{n+1} | f(\mathbf{z}) = t\}.$$

We will often suppress $\mathbf{a}$ from the notation. Let

$$X_t^s = V(t) \cap B(s).$$

and let $\beta \in C^\infty(\mathbb{R})$ be a smooth monotone decreasing cut-off function with $\beta(x) = 1, x \leq \frac{1}{4}$ and $\beta(x) = 0, x \geq \frac{3}{4}$,

$$U_{\mathbf{a}}(\epsilon) := \{\mathbf{z} \in \mathbb{C}^{n+1} | z_0^{a_0} + \cdots + z_n^{a_n} = \epsilon \cdot \beta(||\mathbf{z}||^2)\}.$$

Likewise $\mathbf{a}$ will often be suppressed. Moreover let

$$W_\epsilon^s = U(\epsilon) \cap B(s).$$

2.2 Symplectic and contact structures

A *symplectic manifold* (M, ω) is a smooth $2n$-dimensional manifold M together with a nondegenerate, closed 2-form ω. A function $H \in C^\infty(M)$ on a symplectic manifold (M, ω) is called *Hamiltonian.* We define its *Hamiltonian vector field* X_H via

$$dH = -\iota_{X_H}\omega = -\omega(X_H, \cdot) = \omega(\cdot, X_H).$$

A *contact manifold* Σ is a smooth $(2n-1)$-dimensional manifold together with a completely non-integrable smooth hyperplane distribution $\xi \in T\Sigma$. The distribution is called a *contact structure.* It can be locally defined as $\xi = \ker\alpha$ for some local 1-form α such that $\alpha \wedge (d\alpha)^{n-1} \neq 0$

pointwise. If α is globally defined, then α is called a ***contact form***. We will always assume α is globally defined. Under this assumption $\alpha \wedge (d\alpha)^{n-1}$ gives rise to a volume form and hence Σ is orientable. Once an orientation is chosen we require that $\alpha \wedge (d\alpha)^{n-1} > 0$. Associated with a contact form α one has a ***Reeb vector field*** R, uniquely defined by the equations

$$\iota_R(d\alpha) = 0,$$
$$\iota_R\alpha = 1.$$

Clearly R is transverse to ξ. If we have two different forms $\alpha, \alpha^{'}$ which define the same contact structure, then we can find a nowhere vanishing function f such that $\alpha^{'} = f \cdot \alpha$. Indeed, $f = \alpha^{'}(R)$. The flow of a Reeb vector field is called Reeb flow, and closed trajectories of Reeb flow are called the ***Reeb orbits***. The ***action*** of a Reeb orbit γ is defined as

$$A(\gamma) := \int_{S^1} \gamma^*\alpha$$

Note that $A(\gamma)$ is always positive and equals the period of γ. The ***spectrum*** $spec(\Sigma, \alpha)$ is the set of actions of all Reeb orbits of α. We will need the following definition for Reeb trajectories which is part of a closed Reeb orbit.

Definition 2.1. $\gamma : [0, T] \to X$ is called a ***fractional Reeb orbit*** for contact manifold (X, ξ) if there is a closed Reeb orbit γ_0 of (X, ξ) such that $\gamma(t) = \gamma_0(t), t \in [0, T]$.

We say that a Reeb orbit γ of α is ***non-degenerate*** if the linearized Reeb flow along γ from ξ_p to itself for some $p \in \gamma$ has no eigenvalue 1. Moreover we say that a contact form is ***non-degenerate*** if all Reeb orbits of α are non-degenerate. We can always assume a contact form is non-degenerate after a $\mathcal{C}^0$-small perturbation, since a generic contact form is non-degenerate. Notice that when α is non-degenerate, $spec(\Sigma, \alpha)$ is a discrete subspace of $\mathbb{R}^+$.

2.3 Liouville and Weinstein domains

A ***Liouville domain*** is a pair (W^{2n}, λ) such that

- W^{2n} is a compact manifold with boundary,
- $d\lambda$ is a symplectic form on W ,
- the Liouville field X_λ, defined by $i_X d\lambda = \lambda$, is outward transverse along ∂W.

Let $\alpha := \lambda|_{\partial W}$ be a contact one-form on ∂W. The negative flow of X gives rise to a collar:

$$\phi : (1 - \epsilon, 1] \times \partial W \to W,$$
$$\phi^*\lambda = r\alpha, \quad \phi^*X = r\partial_r.$$

We can attach an infinite cone to it, which is called the ***completion*** of (W, λ):

$$\widehat{W} = W \cup_{\partial W} ([1, \infty) \times \partial W), \qquad \hat{\lambda}|_W = \lambda$$
$$\hat{\lambda}|([1, \infty) \times \partial W) = r\alpha, \quad \hat{X}|([0, \infty) \times \partial W) = r\partial_r, \quad \hat{\omega} = d\hat{\lambda}.$$

A ***Liouville isomorphism*** between domains W_0, W_1 is a diffeomorphism $\psi : \widehat{W_0} \to \widehat{W_1}$ satisfying $\psi^*\hat{\lambda}_1 = \hat{\lambda}_0 + df$, for some f compactly supported. We also say that $\widehat{W_0}$ and $\widehat{W_1}$ are Liouville isomorphic. Clearly ψ is compatible with the Liouville flow at infinity.

Definition 2.2. A Liouville domain (W, λ) is called G-equivariant if a group G acts on W and λ is G-invariant, i.e, $g^*\lambda = \lambda, \forall g \in G$. A diffeomorphism f between two G-equivariant Liouville domains is called G-equivariant if the following diagram commutes, for all $g \in G$:

$$\begin{array}{ccc} (W_1, \lambda_1) & \xrightarrow{f} & (W_0, \lambda_0) \\ \downarrow{\scriptstyle g*} & & \downarrow{\scriptstyle g*} \\ (W_1, \lambda_1) & \xrightarrow{f} & (W_0, \lambda_0) \end{array}$$

Remark 2.3. A manifold M is called G-equivariant if G acts on it.

Proposition 2.4 (Proposition 11.8 [CE12]). *Let W be a compact symplectic manifold with contact type boundary and $\lambda_t, t \in [0,1]$ be a homotopy of Liouville forms on W. Then there exits a diffeomorphism of the completions $f : \widehat{W_0} \to \widehat{W_1}$ such that $f^*\hat{\lambda_1} - \hat{\lambda_0} = dg$ where g is a compactly supported function.*

We have an immediate corollary for Proposition 2.4:

Corollary 2.5. *Let $(\lambda_t)_{0 \le t \le 1}$ be a family of ($G-$equivariant) Liouville structures on W. Then all the (W, λ_t) (($\widehat{W}, \hat{\lambda_t}$)) are mutually ($G-$equivariantly) Liouville isomorphic.*

A *Weinstein domain* is a triple (W^{2n}, λ, ϕ) such that

- (W, λ) is a Liouville domain,
- $\phi : W \to \mathbb{R}$ is an exhausting Morse function with ∂W being a regular level set,
- X_λ is a gradient-like vector field for ϕ.

Since W is compact and ϕ is an exhausting Morse function with ∂W as a regular level set, ϕ has finitely many critical points. Liouville and Weinstein *cobordisms* are defined similarly. If a contact manifold (Y, ξ) is contactomorphic to $\partial(W, \lambda)$, then we say that (W, λ) is a Liouville or Weinstein *filling* of (Y, ξ).

Definition 2.6. A *Stein manifold* (M, J, ϕ) is a complex manifold (M, J) with an exhausting plurisubharmonic function $\phi : M \to \mathbb{R}$. A manifold of the form $\phi^{-1}((-\infty, c])$ is called a *Stein domain*, where c is a regular value of ϕ.

We also have the following famous theorem by Eliashberg:

Theorem 2.7 (Theorem 1.1 [CE12]). *Given a Weinstein structure $\mathfrak{M} = (\omega, X, \phi)$ on V, there exists a Stein structure (J, ϕ) on V such that $\mathfrak{M}(J, \phi)$ is Weinstein homotopic to $\mathfrak{M}$ with fixed ϕ.*

2.4 Symplectic homology

This section is mainly taken out from [Laz16]. The convention used here agrees with [CO18] .

2.4.1 Admissible Hamiltonians and almost complex structures

Let $\mathcal{H}_{std}(W)$ denote the class of ***admissible Hamiltonians***, which are functions on $\widehat{W}$ defined up to smooth approximation as follows:

- $H^s \equiv 0$ in W,
- H^s is linear in r with slope $s \notin Spec(Y, \alpha)$ in $\widehat{W} \backslash W = Y \times [1, \infty)$. We will often suppress s.

To be precise, H is a $\mathcal{C}^2$-small Morse function in W and $H = h(r)$ in $\widehat{W} \setminus W$ for some function h that is increasing convex in a small collar $(Y \times [1, 1+\epsilon], r\alpha)$ of Y and linear with slope s outside this collar.

For $H \in \mathcal{H}_{std}(W)$, recall the Hamiltonian vector field X_H is defined by the condition $d\hat{\lambda}(\cdot, X_H) = dH$. The time-1 orbits of X_H are called the Hamiltonian orbits of H and fall into two categories depending on their location in $\widehat{W}$:

- In W, the only Hamiltonian orbits are constants corresponding to critical points of $H|_W$
- In $\widehat{W} \setminus W$, we have $X_H = h'(r)R_\alpha$, where R_α is the Reeb vector field of (Y, α). So all Hamiltonian orbits lie on level sets of r and come in S^1-families corresponding to reparametrizations of some Reeb orbit of α with period $h'(r)$.

Since the slope s of H at infinity is not in $Spec(Y, \alpha)$, all non-constant Hamiltonian orbits lie in a small neighborhood of Y in $\widehat{W}$. After a $\mathcal{C}^2$-small time-dependent perturbation of H, the orbits become ***non-degenerate***, i.e. the linearized Hamiltonian flow from T_pW to T_pW, for some p in the Hamiltonian orbit, does not have 1 as an eigenvalue. These non-degenerate orbits also lie in a neighborhood of W and so their number is finite. In fact, under this perturbation, each S^1-family of Hamiltonian orbits degenerates into two Hamiltonian orbits.

We say that an almost complex structure J is ***cylindrical*** on the symplectization $(Y \times (0, \infty), r\alpha)$ if it preserves $\xi = \ker \alpha$, $J|_\xi$ is independent of r, is compatible with $d(r\alpha)|_\xi$, and $J(r\partial_r) = R_\alpha$. Let $\mathcal{J}_{std}(W)$ denote the class of ***admissible*** almost complex structures J on $\widehat{W}$ which satisfy

- J is compatible with ω on $\widehat{W}$
- J is cylindrical on $\widehat{W} \backslash W = (Y \times [1, \infty), r\alpha)$.

2.4.2 Floer complex

For $H \in \mathcal{H}_{std}(W), J \in \mathcal{J}_{std}(W)$, the Floer complex $SC(W, \lambda, H, J)$ is generated as a free abelian group by Hamiltonian orbits of H. In this paper we need to consider all Hamiltonian orbits, as opposed to only the contractible ones, see [Wen].

First, let's fix a reference loop

$$l_h : S^1 \to W$$

with $[l_h] = h \in H_1(W, \mathbb{Z})$. Denote by $\mathcal{P}^h(H)$ the set of all $1-$periodic orbits of X_{H_t} in the homology class h.

For a fixed reference class h, we will often write the chain complex generated as a free abelian group by orbits in $\mathcal{P}^h(H)$ as $SC^h(H,J)$ when we do not need to specify (W,λ). We will suppress h when it causes no confusion.

The differential is given by counts of Floer trajectories. In particular, for two Hamiltonian orbits x_-, x_+ of H, let $\widehat{\mathcal{M}}(x_-, x_+; H, J)$ be the moduli space of smooth maps $u : \mathbb{R} \times S^1 \to \widehat{W}$ such that $\lim_{s\to\pm\infty} u(s,\cdot) = x_\pm$ and u satisfies Floer's equation

$$\partial_s u + J(\partial_t u - X_H) = 0.$$

Here s, t denotes the $\mathbb{R}, S^1$ coordinates on $\mathbb{R} \times S^1$ respectively. Since the Floer equation is $\mathbb{R}$-invariant, there is a free $\mathbb{R}$-action on $\widehat{\mathcal{M}}(x_-, x_+; H, J)$ for $x_- \neq x_+$. Let $\mathcal{M}(x_-, x_+ : H, J)$ be the quotient by this $\mathbb{R}$-action, i.e. $\widehat{\mathcal{M}}(x_-, x_+; H, J)/\mathbb{R}$. After a small time-dependent perturbation of (H, J), $\mathcal{M}(x_-, x_+, H, J)$ is a smooth finite-dimensional manifold.

A maximal principle ensures that Floer trajectories do not escape to infinity in $\widehat{W}$. For the following, let $V \subset (W, \lambda_W)$ be a ***Liouville subdomain***, i.e. $(V, \lambda_W|_V)$ is a Liouville domain. Then $(Z, \alpha_Z) = \partial(V, \lambda)$ is a contact manifold. Since V is a Liouville subdomain, there is a collar of Z in W that is symplectomorphic to $(Z \times [1, 1+\delta], d(t\alpha_Z))$ for some small δ.

Lemma 2.8. *[AS10] Consider $H : \widehat{W} \to \mathbb{R}$ such that $H = h(r)$ is increasing near Z, where r is the cylindrical coordinate and $J \in \mathcal{J}_{std}(W)$ is cylindrical near Z. If both asymptotic orbits of a (H, J)-Floer trajectory $u : \mathbb{R} \times S^1 \to \widehat{W}$ are contained in V, then u is contained in V.*

Applying this result to $V = W$, we can proceed as if W were closed and conclude by the Gromov-Floer compactness theorem that $\mathcal{M}(x_-, x_+; H, J)$ has a codimension one compactification. This implies that $\mathcal{M}_h(x_-, x_+; H, J)$, the zero-dimensional component of $\mathcal{M}(x_-, x_+; H, J)$, has finitely many elements and the map $d : SC(H, J) \to SC(H, J)$, defined by

$$dx_+ = \sum_{x_-} \#\mathcal{M}_h(x_-, x_+; H, J)x_-$$

on generators and extended to $SC(H, J)$ by linearity, is a differential. Here $\#\mathcal{M}_h(x_-, x_+; H, J)$ denotes the $\bmod 2$ count of elements of $\mathcal{M}_h(x_-, x_+; H, J)$. We have that $(SC(H, J), d)$ is a chain complex. Note that the underlying vector space $SC(H, J)$ depends only on H while the differential d depends on both H and J. The resulting homology $HF(H, J)$ is independent of J and compactly supported deformations of H.

If $c_1(W, \omega) = 0$, as will always be the case in this paper, $HF(H, J)$ has a $\mathbb{Z}$-grading. More precisely, if $c_1(W, \omega) = 0$, the canonical line bundle of (W, ω) is trivial. After choosing a global trivialization of this bundle, we can assign an integer called the Conley-Zehnder index $\mu_{CZ}(x)$ to each Hamiltonian orbit x; see Subsection 2.7. For a general orbit x, $\mu_{CZ}(x)$ depends on the choice of trivialization of the canonical bundle. For a Hamiltonian orbit corresponding to a critical point p of the Morse function $H|_W$, the Conley-Zehnder index coincides with $n - Ind(p)$, where $Ind(p)$ is the Morse index of $H|_W$ at p.

Furthermore, $\dim \mathcal{M}(x, y; H, J) = \mu_{CZ}(y) - \mu_{CZ}(x) - 1$ so the differential, which counts the zero-dimensional components of $\mathcal{M}(x, y; H, J)$, decreases the degree by one.

2.4.3 Continuation map

Although $HF(H,J)$ is independent of J and compactly supported deformations of H, $HF(H,J)$ does depend on the slope of H at infinity and so is not an invariant of W. In particular, $HF(H,J)$ only sees Reeb orbits of period less than the slope of H at infinity. To incorporate all Reeb orbits, we need to consider Hamiltonians with arbitrarily large slope. More formally, this can be done by considering continuation maps between $SC(H,J)$ for different H. Given $H_-, H_+ \in \mathcal{H}_{std}(W)$, let $H_s \in \mathcal{H}_{std}(W), s \in \mathbb{R}$, be a family of Hamiltonians such that $H_s = H_-$ for $s \ll 0$ and $H_s = H_+$ for $s \gg 0$. Similarly, let $J_s \in \mathcal{J}_{std}(W)$ interpolate between J_-, J_+. For Hamiltonian orbits x_-, x_+ of H_-, H_+ respectively, let $\mathcal{M}(x_-, x_+; H_s, J_s)$ be the moduli space of parametrized Floer trajectories, i.e. maps $u : \mathbb{R} \times S^1 \to \widehat{W}$

$$\partial_s u + J_s(\partial_t u - X_{H_s}) = 0$$

To ensure that parametrized Floer trajectories do not escape to infinity, we again need to use a maximal principle. For this principle to hold, it is crucial that the homotopy of Hamiltonian functions is decreasing, i.e. $\partial H_s/\partial s \leq 0$. If J_s is s-independent, we use the following parametrized version of 'no escape' Lemma 2.8, which is proven in Proposition 3.1.10 of [Gut15]. If J_s does depend on s and $V = W$, then we use the maximal principle from [Sei06]. We state both in the following lemma.

Lemma 2.9. *[Gut15], [Sei06] Consider a decreasing homotopy $H_s : \widehat{W} \to \mathbb{R}$ such that $H_s = h_s(t)$ is increasing in t near $Z = \partial V$ and $H_s|_Z$ is s-independent; let $J \in \mathcal{J}_{std}(W)$ be cylindrical near Z. If $u : \mathbb{R} \times S^1 \to \widehat{W}$ is a (H_s, J)-Floer trajectory with both asymptotes in V, then u is contained in V. If $V = W$, the same claim also holds for a homotopy $J_s \in J_{std}(W)$ that is cylindrical near Z.*

By applying the second part of Lemma 2.9, we can proceed as if W were closed and conclude that $\mathcal{M}(x_-, x_+; H_s, J_s)$ has a codimension one compactification. Then the continuation map $\phi_{H_s,J_s} : SC(H_+, J_+) \to SC(H_-, J_-)$ defined by

$$\phi_{H_s,J_s}(x_+) = \sum_{x_-} \#\mathcal{M}_h(x_-, x_+; H_s, J_s) x_-$$

on generators and extended to $SC(H_+, J_+)$ by linearity is a chain map. Up to chain homotopy, this map is independent of J_s and H_s. Note that there is no $\mathbb{R}$-action since the parametrized Floer equation is not $\mathbb{R}$-invariant. As a result, ϕ_{H_s,J_s} is degree-preserving. Finally, we define symplectic homology as the direct limit

$$SH(W, \lambda) := \lim_{\rightarrow} HF(H, J).$$

The direct limit is taken over continuation maps $\phi_{H_s,J_s} : HF(H_+, J_+) \to HF(H_-, J_-)$ on homology. One key property is that $SH(W, \lambda)$ depends only on the symplectomorphism type of $(\widehat{W}, d\hat{\lambda})$ [Sei06].

2.5 Positive symplectic homology

Positive symplectic homology $SH^+(W)$ is defined using the action functional. For a small time-dependent perturbation of $H \in \mathcal{H}_{std}(W)$, the action functional $A_H : C^\infty(S^1, \widehat{W}) \to \mathbb{R}$ is

$$A_H(x) := \int_{S^1} x^*\lambda - \int_{S^1} H(x(t))dt.$$

Under our conventions, the Floer equation is the ***positive*** gradient flow of the action functional and so action increases along Floer trajectories, i.e. if $u \in \mathcal{M}(x_-, x_+)$ is a non-constant Floer trajectory, then $A_H(x_+) > A_H(x_-)$. Let $SC^{<a}(H, J)$ be generated by orbits of action less than a. Since action increases along Floer trajectories, the differential decreases action and hence $SC^{<a}(H, J)$ is a subcomplex of $SC(H, J)$; we define $SC^{>a}(H, J)$ to be the quotient complex $SC(H, J)/SC^{<a}(H, J)$.

For $H \in \mathcal{H}_{std}(W)$, the constant orbits corresponding to Morse critical points $x \in W$ have action $-H(x)$. The non-constant orbits that correspond to Reeb orbits have action close to the action of the corresponding Reeb orbit, which is positive. In fact, for sufficiently small ϵ, $SC^{<\epsilon}(H, J)$ corresponds to the Morse complex of $-H|_W$ with a grading shift. More precisely, $H_k(SC^{<\epsilon}(H, J)) \cong H^{n-k}(W; \mathbb{Z})$. Define $SC^+(H, J)$ to be the quotient complex $SC(H, J)/SC^{<\epsilon}(H, J)$ and let $HF^+(H, J)$ be the resulting homology. Using a direct limit construction, we can also define $HF^+(W)$. More precisely, suppose H_s is a decreasing homotopy such that $H_s = H_-$ for $s \ll 0$ and $H_s = H_+$ for $s \gg 0$. Then the continuation Floer trajectories are also action increasing and hence there is an induced chain map $\phi^+_{H_s,J_s} : SC^+(H_+, J_+) \to SC^+(H_-, J_-)$. As with $SH(W)$, we define $SH^+(W)$ by

$$SH^+(W, \lambda) := \lim_{\rightarrow} HF^+(H, J).$$

The direct limit is taken over the continuation maps $\phi^+_{H_s,J_s} : HF^+(H_+, J_+) \to HF^+(H_-, J_-)$ on homology.

Like $SH(W)$, $SH^+(W)$ depends only on the symplectomorphism type of $(\widehat{W}, d\hat{\lambda})$. Note that as a vector space, $SC^+(H, J)$ essentially depends only on (Y, α) and not on the interior (W, λ). This is because $SC^+(H, J)$ is generated by non-constant Hamiltonian orbits, which live in the cylindrical end of W and correspond to Reeb orbits of (Y, α). On the other hand, the differential for $SC^+(H, J)$ may depend on the filling W of (Y, α) since Floer trajectories between non-constant orbits may go into the filling, so different Liouville fillings of (Y, ξ) might have different SH^+.

The chain-level short exact sequence

$$0 \to SC^{<\epsilon}(H, J) \to SC(H, J) \to SC^+(H, J) \to 0$$

induces the 'tautological' long exact sequence in homology

$$\cdots \to H^{n-k}(W; \mathbb{Z}) \to SH_k(W, \lambda) \to SH^+_k(W, \lambda) \to H^{n-k+1}(W; \mathbb{Z}) \to \cdots .$$

2.6 Summary of the TQFT structure on $SH_*(W)$

This is taken out of chapter 6 in [Rit13]. For a detailed construction, see chapter 16 of [Rit13]. Note that both the grading and action functional differ from ours by a negative sign, and our homology $SH_*(W^{2n})$ is cohomology $SH^*(W^{2n})$ in [Rit13]. We summarize here the TQFT structure. Suppose we are given:

1. a Riemann surface (S, j) with $p + q$ punctures, with fixed complex structure j;

2. ***ends***: a cylindrical parametrization $s + it$ near each puncture, with $j\partial_s = \partial_t$;

3. $p \geq 1$ of the punctures are ***negative*** (i.e, we converge to the puncture as $s \to -\infty$), they are indexed by $a = 1, \ldots, p$;

4. $q \geq 0$ of the punctures are ***positive*** (i.e, we converge to the puncture as $s \to +\infty$), they are indexed by $b = 1, \ldots, q$;

5. ***weights***: constants $A_a, B_b > 0$ satisfying $\sum A_a - \sum B_b \geq 0$;

6. a 1-form β on S with $d\beta \leq 0$, and on the ends $\beta = A_a\, dt$, $\beta = B_b\, dt$ for large $|s|$.

Remark 2.10. Negative/positive parametrizations are modelled on $(-\infty, 0] \times S^1$ and $[0, \infty) \times S^1$, respectively. In (6), $d\beta \leq 0$ means $d\beta(v, jv) \leq 0$ for all $v \in TS$. By Stokes' theorem, $\sum A_a - \sum B_b = -\int_S d\beta \geq 0$. This forces $p \geq 1$ and (5). Subject to this inequality, such β exists. See Lemma 16.1 [Rit13].

Fix a Hamiltonian $H : \widehat{W} \to \mathbb{R}$ linear at infinity with $H \geq 0$ (required in Section 16.3 [Rit13]), this defines $X = X_H$. Fix an almost complex structure J on W of contact type at infinity.

The moduli space $\mathcal{M}(x_a; y_b; S, \beta)$ of ***Floer solutions*** consists of smooth maps $u : S \to \widehat{W}$ such that $du - X \otimes \beta$ is (j, J)-holomorphic, and u converges on the ends to 1-orbits x_a, y_b of $A_a H$, $B_b H$ which we call the *asymptotics*.

After a small generic S-dependent perturbation J_z of J, $\mathcal{M}(x_a; y_b; S, \beta)$ is a smooth manifold. One can ensure that on the ends J_z does not depend on $z = s + it \in S$ for $|s| \gg 0$. Just as for Floer continuations maps (2.4.3), a maximum principle and an a priori energy estimate $E(u) = \sum \mathcal{A}_{B_b H}(y_b) - \sum \mathcal{A}_{A_a H}(x_a)$ holds, so the $\mathcal{M}(x_a; y_b; S, \beta)$ have compactifications by broken Floer solutions: Floer trajectories for $A_a H, B_b H$ can break off at the respective ends. When gradings are defined (2.7),

$$\dim \mathcal{M}(x_a; y_b; S, \beta) = -\sum \mu_{CZ}(x_a) + \sum \mu_{CZ}(y_b) + n\chi(S) \tag{2.3}$$

$$= \sum \mu_{CZ}(y_b) - \sum \mu_{CZ}(x_a) + n(2 - 2g - p - q). \tag{2.4}$$

Define $\psi_S : \otimes_{b=1}^{q} SC_*(B_b H) \to \otimes_{a=1}^{p} SC_*(A_a H)$ on generators by counting isolated Floer solutions

$$\psi_S(y_1 \otimes \cdots \otimes y_q) = \sum_{u \in \mathcal{M}_0(x_a; y_b; S, \beta)} \epsilon_u\, x_1 \otimes \cdots \otimes x_p,$$

where $\epsilon_u \in \{\pm 1\}$ are orientation signs (In this paper we use $\mathbb{Z}_2$ coefficients, so these signs don't matter. In general, see Section 17 of [Rit13]). Then extend ψ_S linearly.

The ψ_S are chain maps. On homology,

$\psi_S : \otimes_{b=1}^{q} SH_*(B_b H) \to \otimes_{a=1}^{p} SH_*(A_a H)$

is independent of the choices (β, j, J) relative to the ends. Taking direct limits, we get induced maps:

$$\psi_S : SH_*(W)^{\otimes q} \to SH_*(W)^{\otimes p} \qquad (p \geq 1, q \geq 0).$$

So $SH_*(W)$ has a unit $\psi_C(1)$.

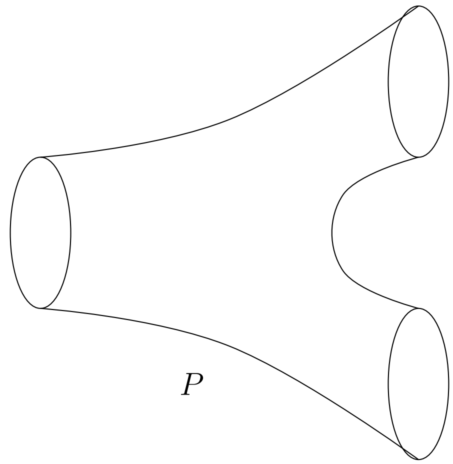

Figure 1: Pair of pants product: the operation ψ_P receives inputs at positive punctures of P and emits output at the negative puncture. So it goes "from right to left".

2.6.1 The product

The pair of pants surface P (Figure 1) defines the product

$$\psi_P : SH_i(W) \otimes SH_j(W) \to SH_{i+j}(W),\; x \cdot y = \psi_P(x, y),$$

which is graded-commutative and associative.

Remark 2.11. The pair of pants product also respects the action filtration. As mentioned in [Ueb15] and in Section 16.3 of [Rit13], we have

$$\mathcal{A}_{2H}(x_3) \leq \mathcal{A}_H(x_1) + \mathcal{A}_H(x_2).$$

Hence the product restricts to a map

$$SH_*^{[a,b)}(W) \times SH_*^{[a',b')}(W) \to SH_*^{[\max\{a+b',a'+b\},b+b')}(W),$$

where on the right hand side it is necessary to divide out all generators with action less than $\max\{a + b', a' + b\}$ to make the map well defined. So one does not get a product on the whole positive symplectic homology, but we can define maps:

$$SH_*^{[\delta,b)}(W) \times SH_*^{[\delta,b)}(W) \to SH_*^{[b+\delta,2b)}(W)$$

2.6.2 The unit

Let $C = \mathbb{C}$ with $p = 1$, $q = 0$. The end is parametrized by $(-\infty, 0] \times S^1$ via $s + it \mapsto e^{-2\pi(s+it)}$. On this end, $\beta = f(s)dt$ with $f'(s) \leq 0$, $f(s) = 1$ for $s \leq -2$ and $f(s) = 0$ for $s \geq -1$. Extend by $\beta = 0$ away from the end (See Figure 2). Thus we get a map $\psi_C : \mathbb{K} \to SH_*(H)$.

Definition 2.12. Let $e_H = \psi_C(1) \in SH_n(H)$. We can define $e = \varinjlim e_H \in SH_n(W)$.

Theorem 2.13 (Theorem 6.1 [Rit13]). *e is the unit for the production on $SH_*(W)$.*

Proof. By the gluing illustrated in the Figure 3, $\psi_P(e, \cdot) = \psi_{P\#C}(\cdot) = \psi_Z(\cdot) = \text{id}$. $\square$

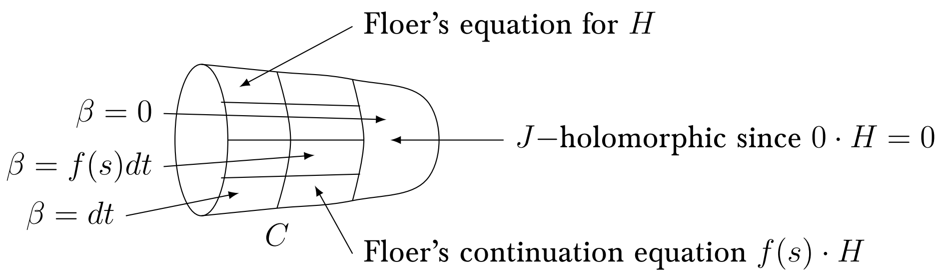

Figure 2: A cap C, and its interpretation as a continuation cylinder.

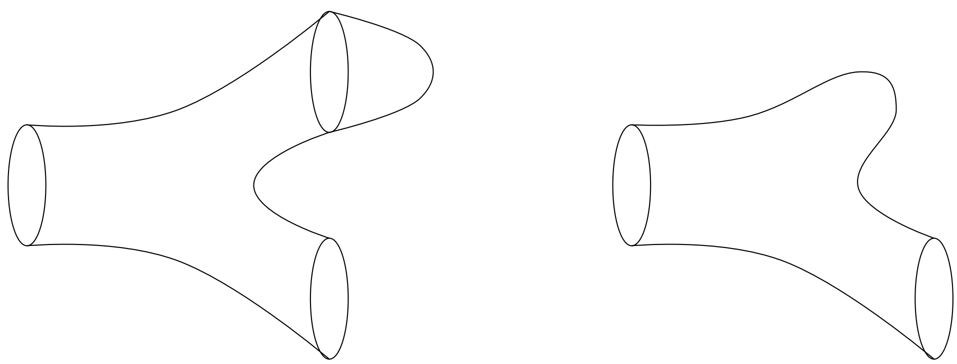

Figure 3: Unit for pair of pants product

Remark 2.14. For "gluing = compositions" results, see Theorems 16.10, 16.12, 16.14 in [Rit13]. Before taking direct limits, the above is the continuation map

$$SH_*(H) \xrightarrow{e_H \otimes \cdot} SH_*(H)^{\otimes 2} \xrightarrow{\psi_P} SH_*(2H).$$

Lemma 2.15 (Lemma 6.2 [Rit13]). *e_H is a count of the isolated finite energy Floer continuation solutions $u : \mathbb{R} \times S^1 \to \widehat{W}$ for the homotopy $f(s)H$ from H to 0.*

Lemma 2.16 (Lemma 6.3 [Rit13]). *For H as in Section 2.4.1, e_H = sum of the local minima of H.*

Theorem 2.17 (Theorem 6.4 [Rit13]). *$e = \varinjlim e_H$ is the image of 1 under $c_* : H_*(W) \to SH_*(W)$, and $e_H = c_{*,H}(1)$ where $c_{*,H} : H_*(M) \cong SH_*^{<\delta}(H) \to SH_*(H)$ is the inclusion map.*

2.6.3 The TQFT structure on $SH_*(W)$ is compatible with the grading by $H_1(W)$

We can grade $SC_*(H) = \bigoplus_{h \in H_1(W)} SC_*^h(H)$ by the homology classes $h \in H_1(\widehat{W})$ of the generators. The Floer differential preserves the H_1 grading, and so do Floer operations on a cylinder and a cap. The pair of pants product respects this grading as follows: $\psi_S : SH_*^{h_1}(W) \otimes SH_*^{h_2}(W) \to SH_*^{h_1+h_2}(W)$. We can also grade $SH_*(W) = \bigoplus_h SH_*^h(M)$ by the free homotopy classes $h \in [S^1, M]$ of the generators. The TQFT operations for genus zero surfaces are compatible with the grading (the equation above holds after replacing $\sum$ by concatenation of free loops).

Remark 2.18. Let $SH_*^0(W)$ denote the summand corresponding to the contractible loops. Considering only contractible loops determines a TQFT with operations $\psi_S : SH_*^0(W)^{\otimes q} \to SH_*^0(W)^{\otimes p}$ ($p \geq 1, q \geq 0$). Also $c_* : H_*(W) \to SH_*^0(W) \subset SH_*(W)$ naturally lands in $SH_*^0(W)$.

2.6.4 Viterbo Functoriality

For Liouville subdomains $W \subset \widehat{M}$, Viterbo [Vit99] constructed a restriction map $SH_*(M) \to SH_*(W)$ and McLean [McL07] proved that it is a ring homomorphism.

Theorem 2.19 ([McL07] [CO18]). *Let W and V be compact symplectic manifolds with contact type boundary and assume that the Conley-Zehnder index is well-defined on W. If V is obtained from W by attaching to $\partial W \times [0,1]$ a subcritical symplectic handle H_k^{2n}, $k < n$, then it holds that*

$$SH_*(V, \mathbb{Z}_2) \cong SH_*(W, \mathbb{Z}_2)$$

as rings.

Remark 2.20. A.Ritter proved a stronger statement in Theorem 9.5 of [Rit13].

2.7 Conley-Zehnder index

In this section we discuss Conley-Zehnder index as in Fauck [Fau16]. To define μ_{CZ}, let $Sp(2n)$ denote the group of $2n \times 2n$ symplectic matrices. We will discuss a generalization, called the Robbin-Salamon index as follows: any smooth path $\Psi : [a,b] \to Sp(2n)$ satisfies an ordinary differential equation

$$\Psi'(t) = J_0 S(t)\Psi(t), \qquad \Psi(a) \in Sp(2n),$$

Where $t \to S(t) = S(t)^T$ is a smooth path of symmetric matrices and J_0 is the standard almost complex structure. We say $t \in [a,b]$ is called a crossing if $\det(id - \Psi(t)) = 0$. The crossing form at time t is a quadratic form $\Gamma(\Psi, t)$ defined for $v \in \ker(id - \Psi(t))$ by

$$\Gamma(\Psi, t)v =< v, S(t)v >$$

A crossing t is called regular if $\Gamma(\Psi, t)$ is non-degenerate. For a path with only regular crossings, the Robbin-Salamon index is defined by

$$\mu_{CZ}(\Psi, a, b) := \frac{1}{2} sign\Gamma(\Psi, a) + \sum_{a<t<b} sign\Gamma(\Psi, t) + \frac{1}{2} sign\Gamma(\Psi, b)$$

where the sum runs all over crossings $t \in (a,b)$, and sign(M) denotes the signature of the matrix M, which equals the number of positive eigenvalues minus the number of negative eigenvalues. Here we use μ_{CZ} to denote the Robbin-Salamon index.The fact that the Robbin-Salamon index coincides with Conley-Zehnder index when $\det(id - \Psi(b)) \neq 0$ sort of justifies this abuse of notation.

We have the following properties for μ_{CZ}:

- (*Naturality*) For any path $\Phi : [a,b] \to Sp(2n)$, $\mu_{CZ}(\Phi\Psi\Phi^{-1}) = \mu_{CZ}(\Psi)$

- (*Homotopy*) $\mu_{CZ}(\Psi_s)$ is constant for any homotopy Ψ_s with fixed endpoints.
- (*Product*) If $Sp(2n) \oplus Sp(2n')$ is identified with a subgroup of $Sp(2(n+n'))$ in the natural way, then $\mu_{CZ}(\Psi \oplus \Psi') = \mu_{CZ}(\Psi) + \mu_{CZ}(\Psi')$.

The homotopy property allows us to define $\mu_{CZ}(\Psi, a, b)$ also for paths with non-regular crossings, given that having regular crossings is a $\mathcal{C}^\infty$ generic property among paths with fixed endpoints.

Remark 2.21 (Lemma 59 [Fau16]). Let $\Psi_1, \Psi_2, \Psi_3 : [0, T] \to Sp(2)$ be the following paths:

$$\Psi_1(t) = e^{it}, \quad , \Psi_2(t) = e^{-it}, \quad \Psi_3(t) = \text{diag}\big(e^{f(t)}, e^{-f(t)}\big), f \in C^1(\mathbb{R}).$$

Then, their Conley-Zehnder indices are given as follows:

$$\begin{aligned}
\mu_{CZ}(\Psi_1) &= \left\lfloor \frac{T}{2\pi} \right\rfloor + \left\lceil \frac{T}{2\pi} \right\rceil, \\
\mu_{CZ}(\Psi_2) &= \left\lfloor \frac{-T}{2\pi} \right\rfloor + \left\lceil \frac{-T}{2\pi} \right\rceil = -\mu_{CZ}(\Psi_1), \\
\mu_{CZ}(\Psi_3) &= 0.
\end{aligned}$$

Trivialization

Suppose we have a symplectic manifold (M, ω) with $c_1(M) = 0$ and J is an $\omega-$compatible almost complex structure. Then the anti-canonical bundle of M is the highest exterior power of (TM, J), i.e, $\kappa_J^* = \wedge^n(TM, J)$. The canonical bundle κ_J is the dual of κ_J^*. In the same manner, we can define the canonical bundle of a contact manifold (C, ξ) with a choice of one form α and $d\alpha$-compatible almost complex structure on ξ.

A ***trivialization of the canonical bundle*** is a bundle isomorphism $\Phi : \kappa_J \to M \times \mathbb{C}$. A ***trivialization*** of (γ^*TM, J)(where γ is a loop in M) is a bundle isomorphism $\Psi : \gamma^*TM \to S^1 \times \mathbb{C}^n$. Such a trivialization has a one-to-one correspondence (up to homotopy) with the trivialization of $\gamma^*\kappa_J^*$ and hence the trivialization of the canonical bundle via:

$$\det{}_{\mathbb{C}}(\Psi) : \wedge^n(\gamma^*TM) = \gamma^*\kappa_J^* \to S^1 \times \mathbb{C}.$$

For a 1-periodic Hamiltonian orbit x, we fix a trivialization of (x^*TM, J) along x as:

$$\Psi : x^*TM \to S^1 \times \mathbb{C}^n$$

Suppose ψ is the Hamiltonian flow and $d\psi_t : TM|_{x(0)} \to TM|_{x(t)}$ is its linearization, then define

$$M_t(x) := \Psi_t \circ d\psi_t \circ \Psi_0^{-1}$$

The Conley-Zehnder index of x is defined as $\mu_{CZ}(x) := \mu_{CZ}(M_t(x))$. Similarly, we can define the Conley-Zehnder index of a Reeb orbit. In particular, let (W, λ) be a Liouville domain and $(C := \partial W, \xi := \ker \lambda|_C)$ its boundary. We have $TM|_C = \xi \oplus < X_{Reeb} > \oplus < X >$, where X_{Reeb}, X are a Reeb vector field and a Liouville vector field, respectively. Since $< X_{Reeb} >=$

$J < X >$, we can identify $< X_{Reeb} > \oplus < X >$ with $\mathbb{C}$, i.e. $\gamma^*TM = \gamma^*\xi \oplus \mathbb{C}$, where γ is a Reeb orbit. Due to the fact that Reeb flow preserves X_{Reeb} and extends to the symplectization, we have

$$M_{t,M}(\gamma) = M_{t,\xi}(\gamma) \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $M_{t,M}(\gamma)$ is the symplectic matrix associated to the linearization of the Reeb flow with respect to a trivialization of $TM|_{\gamma}$, i.e, $M_{t,\xi}(\gamma)$ is defined in the same manner. The product property of Conley-Zehnder index implies that $\mu_{CZ}(M_{t,M}(\gamma)) = \mu_{CZ}(M_{t,\xi}(\gamma))$. Hence we will not specify which index we are referring to in the rest of this paper.

Now consider a $G-$equivariant Liouville domain (W, λ). Suppose the group action is free and $|G| < \infty$. Then we have that the quotient map

$$\pi_G : (W, \lambda) \to (W/G, \lambda)$$

is a finite covering map. Each Reeb orbit γ in $\partial(W/G)$ then lifts to a fractional orbit $\widetilde{\gamma}$ in ∂W. That is, $\widetilde{\gamma}(t) = \gamma_0(t), t \in [0, T]$ for some closed Reeb orbit γ_0 in $\partial(W)$. In particular, we can choose γ_0 with period of $|G| \cdot T$. If we choose a $G-$equivariant trivialization for the canonical bundle κ_W, then such trivialization descends down to $\kappa_{W/G}$. Equivalently, if we choose G-equivariant trivialization of $\xi|_{\gamma_0}$, and $M_{t,\xi}(\gamma_0)$ is the matrix of the linearized map, then we have for some $M_G \in Sp(2n, \mathbb{R})$,

$$M_G \cdot M_{t,\xi}(\gamma_0) = M_{t+T,\xi}(\gamma_0).$$

where M_G satisfies $M_G^{|G|} = M_{|G|\cdot T,\xi}(\gamma_0)$ is a constant matrix, which only depends on the homotopy class of our G-equivariant trivialization. In particular, $M_{T,\xi}(\gamma_0) = M_G$, so $\mu_{CZ}(M_{t,\xi}(\gamma_0)), t \in [0, T]$ is well defined since the Conley-Zehnder index is constant for any homotopy with fixed endpoints. We can therefore define the Conley-Zehnder index of such a fractional Reeb orbit of γ to be the Conley-Zehnder index of $M_{t,\xi}(\gamma_0), t \in [0, T]$.

As a consequence, we have

$$\mu_{CZ}(\gamma) = \mu_{CZ}(\widetilde{\gamma}).$$

Lemma 2.22. *Let $(\mathbb{R} \times S^1, d(rd\theta))$ be the symplectization of (S^1, θ). Choose the canonical trivialization of $T(\mathbb{R} \times S^1) = T\mathbb{R} \times TS^1$, then all fractional Reeb orbits of (S^1, θ) have Conley-Zehnder index (Robbin-Salamon index) zero, with respect to any cyclic group action rotating the cylinder.*

Proof. Since Reeb flow preserves $(\partial_r, \partial_\theta)$, so the matrix for linearized return map is

$$M(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the Conley-Zehnder index is zero. □

The following lemma gives a formula for the Reeb vector field in terms of the Hamiltonian and Liouville vector field.

Lemma 2.23. *Let (W, λ) be a Liouville manifold. Suppose H is a function on W with 0 as its regular value and the Liouville vector field X is transverse to the 0-level set. Then $(\Sigma := H^{-1}(0), \lambda)$ is a contact manifold whose Reeb vector field is given by $X_{Reeb} = \frac{X_H}{X(H)}$, where X_H is the Hamiltonian vector field of H.*

Proof. (Σ, λ) is well known to be contact. We only need to prove the latter part of the lemma. Since

$$\iota_{X_H} d\lambda|_\Sigma = -dH|_\Sigma = 0$$

and

$$\iota_{X_H}\lambda = \iota_{X_H}\iota_X d\lambda = d\lambda(X, X_H) = dH(X) = X(H),$$

it follows $X_{Reeb} = \frac{X_H}{X(H)}$. □

Lemma 2.24 (Lemma 5.20 [McL16]). *Let (C, ξ) be a contact manifold with associated contact form α and let $h : \mathbb{R} \to \mathbb{R}$ be a function with $h' > 0, h'' > 0$ and $h'(0) = 1$. Let $\widehat{C} := C \times \mathbb{R}$ be the symplectization of C with symplectic form $d(e^r \alpha)$ where r parameterizes $\mathbb{R}$. Let $\gamma(t)$ be a Reeb orbit of α of period L with a choice of trivialization of the symplectic vector bundle $\oplus_{j=1}^N TM$ along this orbit. This choice of trivialization induces a choice of trivialization of $\gamma^* \oplus_{j=1}^N \xi$ in a natural way. Then the Hamiltonian $Lh(e^r)$ has a 1 periodic orbit x equal to $\gamma(Lt)$ inside $C \times \{0\} = C$ and its Conley-Zehnder index is equal to $\mu_{CZ}(\gamma) + \frac{1}{2}$.*

Remark 2.25. Notice that the Hamiltonian vector field in [McL16] differs from ours by a minus sign. We have

$$M_{t,M}(x) = M_{t,\xi}(\gamma) \oplus \begin{bmatrix} 1 & 0 \\ ah''t & 1 \end{bmatrix},$$

for some constant $a > 0$.

If instead, $h'' < 0$, then the index equals $\mu_{CZ}(\gamma) - \frac{1}{2}$. And if $h' < 0$, then the Hamiltonian orbit goes in the opposite direction of the Reeb orbit, and the index differs by a minus sign.

We will conclude this subsection with a lemma relating Morse index of critical point with Conley-Zehnder index of the corresponding constant Hamiltonian orbit.

Lemma 2.26. *If S is an invertible symmetric matrix with $||S|| < 2\pi$ and $\Psi(t) = \exp(tJ_0S)$, then*

$$\mu_{CZ}(\Psi) = n - Ind(S)$$

where $Ind(S)$ is the number of negative eigenvalues of S.

Corollary 2.27 (Corollary 7.2.2 **[AD14]**). *Let W be a symplectic manifold of dimension $2n$, let*

$$H : W \to \mathbb{R}$$

be a Hamiltonian and x be a critical point of H. We assume that H is $\mathcal{C}^2$-small (in this case, we can choose a Darboux chart centered at x such that the usual norm $||Hess_x(H)|| < 2\pi$). Then the Conley-Zehnder index $\mu_{CZ}(x)$ of x as a periodic solution of the Hamiltonian system and its Morse index Ind(x) as a critical point of the function H are connected by

$$\mu_{CZ}(x) = n - Ind(x).$$

2.8 Weinstein handle attachment and contact surgery

2.8.1 Contact surgery

This section is already included in Chapter 6 of [Gei08]. We will highlight the parts which should be paid attention to in this paper, namely, the trivialization of the conformal symplectic normal bundle.

Definition 2.28. Let (M, ξ) be a contact manifold. A submanifold L of (M, ξ) is called an *isotropic submanifold* if $T_pL \subset \xi_p$ for all point $p \in L$.

Let $L \subset (M, \xi = \ker \alpha)$ be an isotropic submanifold in a contact manifold with cooriented contact structure. Let $(TL)^\perp \subset \xi_L$ be the subbundle of ξ_L that is symplectically orthogonal to TL with respect to the symplectic bundle structure $d\alpha|_\xi$.The conformal structure of this bundle does not depend on the choice of contact form and therefore $(TL)^\perp$ is determined by ξ. The fact L is isotropic implies that $TL \subset (TL)^\perp$. So we have the following definition,

Definition 2.29. The quotient bundle

$$CSN_M(L) := (TL)^\perp/TL$$

with the conformal symplectic structure induced by $d\alpha$ is called the *conformal symplectic normal bundle* of L in M.

So we have

$$\xi|_L = \xi|_L/(TL)^\perp \oplus (TL)^\perp/TL \oplus TL = TL \oplus \xi|_L/(TL)^\perp \oplus CSN_M(L).$$

Let $J : \xi \to \xi$ be a complex bundle structure on ξ compatible with the symplectic structure given by $d\alpha$. Then the bundle $\xi|_L/(TL)^\perp$ is isomorphic to $J(TL)$. So the contact structure has the following natural splitting on the isotropic submanifold:

Lemma 2.30.

$$\xi|_L = TL \oplus J(TL) \oplus CSN_M(L)$$

Therefore, if we fix a trivialization of $TL \oplus J(TL)$, then the trivialization of $CSN_M(L)$ is determined by the trivialization of $\xi|_L$. Now we can state the contact surgery theorem:

Theorem 2.31 (Theorem 6.2.5 [Gei08]). *Let Λ^{k-1} be an isotropic sphere in a contact manifold $(M, \xi = ker\alpha)$ with a trivialization of the conformal symplectic normal bundle $CSN_M(\Lambda^{k-1})$. Then there is a symplectic cobordism from (M, ξ) to the manifold M' obtained from M by surgery along Λ^{k-1} with the natural framing. In particular, the surgered manifold M' carries a contact structure that coincides with the one on M away form the surgery region.*

Remark 2.32. The resulting contact structure on M' is uniquely determined up to isotopy by the isotopic isotropy class of Λ^{k-1} and the homotopy class of the trivialization of $CSN_M(\Lambda^{k-1})$.

2.8.2 Weinstein handlebodies

For the purposes of this paper, we need to attach a handle to a Weinstein domain. We will follow Section 13 in [CE06]. The standard handle of index k will be the bidisk in $\mathbb{C}^n$:

$$\Big\{\sum_{j=1}^{k} x_j^2 \leq (1+\epsilon)^2, \sum_{j=1}^{k} y_j^2 + \sum_{j=k+1}^{n} |z_j|^2 \leq \epsilon^2\Big\},$$

where $z_j = x_j + iy_j, j = 1, 2, \cdots, n$, are the complex coordinates in $\mathbb{C}^n$. In particular, the handle H carries the standard complex structure i, along with the standard symplectic structure ω_{std}. The symplectic form ω_{std} on H admits a hyperbolic Liouville field

$$X_{std} = \sum_{j=1}^{k} \big(- x_j \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial y_j}\big) + \frac{1}{2} \sum_{l=k+1}^{n} \big(x_l \frac{\partial}{\partial x_l} + y_l \frac{\partial}{\partial y_l}\big).$$

Let us denote by ξ^- the contact structure $\alpha_{st}|_{\partial^- H} = 0$ defined on $\partial^- H$ by the Liouville form $\alpha_{st} = \iota_{X_{std}}\omega_{std}$, where $\partial^- H := \partial D_1^k \times D_\epsilon^{2n-k}$ is the *lower boundary*. Notice that the bundle $\xi^-|_{\Lambda^{k-1}}$ canonically splits as $T\Lambda^{k-1} \oplus J(T\Lambda^{k-1}) \oplus \epsilon^{n-k}$, where ϵ^{n-k} is a trivial $(n-k)-$dimensional complex bundle. We will denote by σ_Λ the isomorphism

$$T\Lambda^{k-1} \oplus J(T\Lambda^{k-1}) \oplus \epsilon^{n-k} \to \xi^-|_\Lambda.$$

Suppose we are given a real $k-$dimensional bundle E, a complex $n-$dimensional bundle $\tau, n \geq k$, and an injective totally real homomorphism $\phi : E \to \tau$. Then ϕ canonically extends to a complex homomorphism $\phi \otimes \mathbb{C} : E \otimes \mathbb{C} \to \tau$. If $\phi \otimes \mathbb{C}$ extends to a fiberwise complex isomorphism $\Phi : E \otimes \mathbb{C} \oplus \epsilon^{n-k}$ then Φ is called a *saturation* of E covering ϕ. When $n = k$ the saturation is unique.

Let (V, ω, X, ϕ) be a Weinstein manifold, p a critical point of index k of the function ϕ, $a < b = \phi(p)$ a regular value of ϕ. Denote $W := \{\phi \leq a\}$. Suppose that the stable manifold of p intersects $V \setminus IntW$ along a disc D^k, and let $\Lambda^{k-1} = \partial D^k$ be the attaching sphere. The inclusion $T\Lambda^{k-1} \hookrightarrow \xi$ extends canonically to an injective complex homomorphism $T\Lambda^{k-1} \oplus J(T\Lambda^{k-1}) \hookrightarrow \xi$, while the inclusion $TD^k \hookrightarrow TV$ extends to an injective complex homomorphism $TD^k \oplus J(TD^k) \hookrightarrow TV$. There exists a homotopically unique complex trivialization of the conformal symplectic normal bundle $CSN_{\partial W}(\Lambda^{k-1})$ in ξ which extends to D^k as a trivialization of the conformal symplectic normal bundle to D^k in TV. This trivialization provides a canonical isomorphism $\Phi_{D^k} : T\Lambda \oplus J(T\Lambda) \oplus \epsilon^{n-k} \to \xi|_{\Lambda^{k-1}}$, and we will call this the canonical saturation of the inclusion $\Lambda^{k-1} \hookrightarrow \partial W$.

We have the following theorem on attaching a handle to a Weinstein domain:

Theorem 2.33 (Prop13.11 [CE06], [W+91]). *Let (W, ω, X, ϕ) be a $2n-$dimensional Weinstein domain with boundary ∂W and ξ the induced contact structure $\{\alpha|_{\partial W} = 0\}$ on W defined by the Liouville form $\alpha = \iota_X \omega$. Let $h : \Lambda \to \partial W$ be an isotropic embedding of the $(k-1)-$sphere Λ. Let $\Phi : T\Lambda \oplus J(T\Lambda) \oplus \epsilon^{n-k} \to \xi$ be a saturation covering the differential $dh : T\Lambda \to \xi$. Then there exists a Weinstein domain $(\tilde{W}, \tilde{\omega}, \tilde{X}, \tilde{\phi})$ such that $W \subset Int\tilde{W}$, and*

(i) $(\tilde{\omega}, \tilde{X}, \tilde{\phi})|_W = (\omega, X, \phi)$;

(ii) the function $\tilde{\phi}|_{\tilde{M}\setminus IntW}$ has a unique critical point p of index k.

(iii) the stable disc D of the critical point p is attached to ∂W along the sphere $h(\Lambda)$, and the canonical saturation Φ_D coincides with Φ.

Given any two Weinstein extensions $(W_0, \omega_0, X_0, \phi_0)$ and $(W_1, \omega_1, X_1, \phi_1)$ of (W, ω, X, ϕ) which satisfy properties (i)-(iii), there exists a diffeomorphism g fixed on W such that $g : W_0 \to W_1$ satisfying (ω_0, X_0, ϕ_0) and $(g^*\omega_1, g^*X_1, g^*\phi_1)$ are homotopic in the class of Weinstein structures which satisfy (i)-(iii). In particular, the completion of these two Weinstein domains are symplectomorphic via a symplectomorphism fixed on W.

We say that the Weinstein domain $(\tilde{W}, \tilde{\omega}, \tilde{X}, \tilde{\phi})$ is obtained from (W, ω, X, ϕ) by attaching a handle of index k along an isotropic sphere $h : \Lambda \to \partial W$ with the given trivialization Φ.

Definition 2.34. A Weinstein domain (W^{2n}, λ, ϕ) is *flexible* if there exist regular values $c_1, \cdots, c_k$ of ϕ such that $c_1 < \min \phi < c_2 < \cdots < c_{k-1} < \max \phi < c_k$ and for all $i = 1, \cdots, k-1$, $\{c_i \leq \phi \leq c_{i+1}\}$ is a Weinstein cobordism with a single critical point p whose attaching sphere Λ_p is either subcritical or a loose Legendrian in $(Y^{c_i}, \lambda|_{Y^{c_i}})$.

Flexible Weinstein *cobordisms* are defined similarly. Also, a Weinstein handle attachment or contact surgery is called flexible if the attaching Legendrian is loose. So any flexible Weinstein domain can be constructed by iteratively attaching subcritical or flexible handles to (B^{2n}, ω_{std}). A Weinstein domain that is Weinstein homotopic to a Weinstein domain satisfying Definition 2.34 will also be called flexible. Loose Legendrians have dimension at least 2 so if (Y_+, ξ_+) is the result of flexible contact surgery on (Y_-, ξ_-), then by Proposition 2.36 $c_1(Y_+)$ vanishes if and only if $c_1(Y_-)$ does. Finally, we note that subcritical domains are automatically flexible.

Since they are built using loose Legendrians and subcritical spheres, which satisfy an h-principle, flexible Weinstein domains also satisfy an h-principle [CE12]. Again, the h-principle has an existence and uniqueness part:

- any almost Weinstein domain admits a flexible Weinstein structure in the same almost symplectic class
- any two flexible Weinstein domains that are almost symplectomorphic are Weinstein homotopic (and hence have exact symplectomorphic completions and contactomorphic boundaries).

2.8.3 Formal structures

There are also formal versions of symplectic, Weinstein, and contact structures that depend on just the underlying algebraic topological data. For example, an *almost symplectic structure* (W, J) on W is an almost complex structure J on W; this is equivalent to having a non-degenerate (but not necessarily closed) 2-form on W. An almost symplectomorphism between two almost symplectic manifolds $(W_1, J_1), (W_2, J_2)$ is a diffeomorphism $\phi : W_1 \to W_2$ such that $\phi^* J_2$ can be deformed to J_1 through almost complex structures on W_1. Equivalently, it also means that there is a family of non-degenerate 2-forms ω_t interpolating between ω_1 and ω_2.

An *almost Weinstein domain* is a triple (W, J, ϕ), where (W, J) is a compact almost symplectic manifold with boundary and ϕ is a Morse function on W with no critical points of index greater than n and maximal level set ∂W. An ***almost contact structure*** (Y, J) on Y is an almost complex structure J on the stabilized tangent bundle $TY \oplus \epsilon^1$ of Y. Therefore an almost symplectic domain (W, J) has almost contact boundary $(\partial W, J|_{\partial W})$; it is an almost symplectic filling of this almost contact manifold. Therefore a family of almost symplectic structures give rise to a family of almost contact structures on the boundary:

Lemma 2.35. *Almost symplectomorphic Liouville domains have almost contactomorphic boundaries.*

Note that any symplectic, Weinstein, or contact structure can also be viewed as an almost symplectic, Weinstein, or contact structure by considering just the underlying algebraic topological data.

Note that the first Chern class $c_1(J)$ is an invariant of almost symplectic, almost Weinstein, or almost contact structures. In this paper, we will often need to assume that $c_1(J)$ vanishes. The following proposition, which will be used several times in this paper, shows that the vanishing of $c_1(Y, J)$ is often preserved under contact surgery and furthermore implies the vanishing of $c_1(W, J)$.

Proposition 2.36 (Proposition 2.1 [Laz16]). *Let $(W^{2n}, J), n \geq 3$, be an almost Weinstein cobordism between $\partial_- W = (Y_-, J_-)$ and $\partial_+ W = (Y_+, J_+)$. If $H^2(W, Y_-) = 0$, the following are equivalent:*

- $c_1(J_-) = 0, c_1(J_+) = 0$
- $c_1(J) = 0$.

If $\partial_- W = \emptyset$, the vanishing of $c_1(J_+)$ and $c_1(J)$ are equivalent.

Proof. Let $i_\pm : Y_\pm \hookrightarrow W$ be inclusions. Then $i_\pm^* c_1(J) = c_1(J_\pm)$ so the vanishing of $c_1(J)$ implies the vanishing of $c_1(J_-)$ and $c_1(J_+)$. To prove the converse, consider the cohomology long exact sequences of the pairs (W, Y_-) and (W, Y_+):

$$H^2(W, Y_\pm; \mathbb{Z}) \to H^2(W; \mathbb{Z}) \xrightarrow{i_\pm^*} H^2(Y_\pm; \mathbb{Z}).$$

By assumption, $H^2(W, Y_-; \mathbb{Z})$ vanishes and hence i_-^* is injective. By Poincaré-Lefschetz duality, $H^2(W, Y_+; \mathbb{Z}) \cong H_{2n-2}(W, Y_-; \mathbb{Z})$. Since $2n - 2 \geq n + 1$ for $n \geq 3$ and W is a Weinstein cobordism, $H_{2n-2}(W, Y_-; \mathbb{Z})$ vanishes and hence i_+^* is also injective. Then if either $c_1(J_-) = i_-^* c_1(J)$ or $c_1(J_+) = i_+^* c_1(J)$ vanish, so does $c_1(J)$.

If $\partial_- W = \emptyset$, we just need the vanishing of $H^2(W, Y_+; \mathbb{Z})$, which holds for $n \geq 3$. □

2.9 Morse-Bott case

The results of this section largely come from [McL16].

Definition 2.37. A ***Morse-Bott family of Reeb orbits of*** (C, α) ***of period*** T is a closed path connected submanifold $B \subset C$ where B is contained in the image of the union of closed Reeb orbits of period T, satisfying $\ker(D\psi_T)|_B = TB$, where if $\psi_t : B \to B$ is the Reeb flow of α.

We are interested in indices of Reeb orbits and so from now on we assume that we work with a fixed trivialization of a fixed power of the canonical bundle of (C, α).

Note that the Conley-Zehnder index of the period T orbits starting in B are all the same because B is path connected. Hence we define the ***Conley-Zehnder index of*** B, $\mu_{CZ}(B)$, to be the Conley-Zehnder index of one of its period T Reeb orbits.

We can define an index closely related to the Conley-Zehnder index, called ***lower SFT index***, $lSFT(\gamma)$, as follows:

$$lSFT(\gamma) := \mu_{CZ}(\gamma) - \frac{1}{2}\dim\ker(D_{\gamma(0)}\psi_T|_\xi - \mathbf{id}) + (n-3).$$

Similarly, we have the following definition:

Definition 2.38. Let K be a Hamiltonian on a symplectic manifold (X, ω_X) and B is a set of fixed points of its time T flow. We say that B is ***isolated*** if any such fixed point near B is contained in B. suppose B is a path connected topological space and we have fixed a symplectic trivialization of the canonical bundle of TX. Then every such Hamiltonian orbit has the same Conley-Zehnder index and we will write $\mu_{CZ}(B, K)$ for the Conley-Zehnder index. The set B is said to be ***Morse-Bott*** if B is a submanifold and $\ker(D\psi_K^T - id) = TB$ along B where $\psi_K^T : X \to X$ is the time T Hamiltonian flow of K.

The following lemma is a technical lemma which relates the index of Reeb orbits in a contact hypersurface (which is a regular level set of a Hamiltonian) and the index of the corresponding Hamiltonian orbits.

Lemma 2.39 (Lemma 5.22 [McL16]). *Let (W, ω_W) be a symplectic manifold with a choice of symplectic trivialization of the canonical bundle of TW. Let θ_W be a 1-form satisfying $d\theta_W = \omega_W$, and K be an Hamiltonian with the property that $b := \iota_{X_{\theta_W}} dK > 0$. This means $C_r := K^{-1}(r)$ is a contact manifold with contact form $\alpha_r := \theta_W|_{C_r}$. Let $B \subset W$ be a connected submanifold transverse to C_r for each r so that $B_r := C_r \cap B$ is a Morse-Bott submanifold of the contact manifold (C_r, α_r) of period L_r, where L_r smoothly depends on r. Suppose that $b = L_0$ along B_0 and that $db(V) > \frac{d(L_r)}{dr}|_{r=0}$ along B_0, where V is a vector field tangent to B satisfying $dK(V) = 1$. Then B_0 is Morse-Bott for K and $\mu_{CZ}(B_0, K) = \mu_{CZ}(B_0, \alpha_0) + \frac{1}{2}$.*

Remark 2.40. Our sign convention is different from McLean's in [McL16] since we use $\omega(\cdot, X_H) = dH$. So the condition on $b := \iota_{X_{\theta_W}} dK > 0$ differs by a minus sign. If $b \neq L_0$ along B_0, we have to either rescale b or L_t.

Remark 2.41. In light of lemma 2.22, we have $\mu_{CZ}(\gamma, r^2) = \frac{1}{2}$, where γ is any Morse-Bott manifold of Hamiltonian orbits.

3 ADC structures and positive idempotent group

We will define (strongly) asymptotically dynamically convex contact structure first. Then we introduce a new invariant called the ***positive idempotent group*** base on $SH_*(W)$. It does not depend on the filling for ADC contact structures and therefore can be seen as a contact invariant. The proof will be deferred to section 4. In subsection 3.3, we show that the (strongly) ADC property is preserved under subcritical contact surgery.

3.1 Asymptotically dynamically convex contact structures

Let's take a moment to look at the degree of Reeb orbits, which is essential for the definition of ADC contact structures. For any contact manifold (Σ, α) with $c_1(\Sigma, \xi)$, the canonical line bundle of ξ is trivial, as will always be the case in this paper. After choosing a global trivialization of this bundle, we can assign an integer to each Reeb orbit γ of (Σ, α)-the reduced Conley-Zehnder index:

$$|\gamma| := \mu_{CZ}(\gamma) + n - 3.$$

For a general Reeb orbits, $|\gamma|$ depends on the choice of trivialization of the canonical bundle. However, if the Reeb orbit $|\gamma|$ is contractible in Σ, then the grading does not depend on the trivialization. We will consider both the contractible and non-contractible Reeb orbits in this paper.

Let $\mathcal{P}^{<D}_{\Phi}(\Sigma, \alpha)$ be the set of Reeb orbits γ of (Σ, α) satisfying $A(\gamma) < D$, where Φ is a specific trivialization of the canonical bundle. In a similar manner , we can define $\mathcal{P}^{<D}_{0}(\Sigma, \alpha)$ to be the set of contractible Reeb orbits γ of (Σ, α) satisfying $A(\gamma) < D$. Here we dropped the subscript Φ since the degree of contractible Reeb orbits does not depends on the choice of trivialization.

Lemma 3.1 (Proposition 3.1 [Laz16]). *For any $D, s > 0$, there is a grading preserving bijection between $\mathcal{P}^{<D}_{\Phi}(Y, s\alpha)$ and $\mathcal{P}^{<D/s}_{\Phi}(Y, \alpha)$.*

Proof. Note that $R_{s\alpha} = \frac{1}{s}R_\alpha$. So if $\gamma_\alpha : [0, T] \to Y$ is a Reeb trajectory of α with action T, then $\gamma_{s\alpha} = \gamma_\alpha \circ m_{\frac{1}{s}} : [0, sT] \to Y$ is a Reeb trajectory of $s\alpha$ with action sT; here $m_{\frac{1}{s}} : [0, sT] \to [0, T]$ is multiplication by $\frac{1}{s}$. The map $\gamma_\alpha \to \gamma_{s\alpha}$ is a bijection between the set of Reeb orbits. If $T < D/s$, then $sT < D$ and so it is a bijection between $\mathcal{P}^{<D/s}_{\Phi}(Y, \alpha)$ and $\mathcal{P}^{<D}_{\Phi}(Y, s\alpha)$. This bijection is grading-preserving since the Conley-Zehnder index of a Reeb orbit is determined by the linearized Reeb flow on the trivialized contact planes ξ but does not depend on the speed of the flow. □

We will also need the following notation. If α_1, α_2 are contact forms for ξ, then there exists a unique $f : Y \to \mathbb{R}^+$ such that $\alpha_2 = f\alpha_1$. We write $\alpha_2 > \alpha_1, \alpha_2 \geq \alpha_1$ if $f > 1, f \geq 1$ respectively. Note that if $\alpha_2 > \alpha_1, \alpha_2 \geq \alpha_1$, then for any diffeomorphism $\Psi : Y' \to Y$, we have $\Psi^*\alpha_2 > \Psi^*\alpha_1, \Psi^*\alpha_2 \geq \Psi^*\alpha_1$, respectively.

Definition 3.2. A contact manifold (Σ, ξ) is *asymptotically dynamically convex* (*strongly asymptotically dynamically convex with respect to* Φ) if there exists a sequence of non-increasing contact forms $\alpha_1 \geq \alpha_2 \geq \alpha_3 \cdots$ for ξ and increasing positive numbers $D_1 < D_2 < D_3 \cdots$ going to infinity such that all elements of $\mathcal{P}^{<D_k}_{0}(\Sigma, \alpha_k)$ ($\mathcal{P}^{<D_k}_{\Phi}(\Sigma, \alpha_k)$) have positive lower SFT index.

Remark 3.3. The ADC property defined in definition 3.6 [Laz16] requires the non-degeneracy of α_i. Here we define the strongly ADC property (with respect to Φ) using lower SFT index. Therefore the contact form α_i in the definition doesn't have to be non-degenerate. It is an immediate corollary of lemma 3.4, also see remark 3.7 (2) of [Laz16].

Lemma 3.4 (Lemma 4.10 [McL16]). *Let γ be any Reeb orbit of α of period T and define $K := \dim\ker(D\psi_T|\xi(\gamma(0)) - id)$. Fix some Riemannian metric on C. There is a constant $\delta > 0$ and a neighborhood N of $\gamma(0)$ so that for any contact form α_1 with $||\alpha - \alpha_1||_{\mathcal{C}^2} < \delta$ and any Reeb orbit γ_1 of α_1 starting in N of period in $[T - \delta, T + \delta]$ we have $\mu_{CZ}(\gamma_1) \in [\mu_{CZ}(\gamma) - \frac{K}{2}, \mu_{CZ}(\gamma) + \frac{K}{2}]$.*

3.2 Positive idempotent group I_+

Now we consider a strongly asymptotically dynamically convex contact manifold (Σ, ξ, Φ) with Liouville filling (W, λ). We have the following result due to Lazarev:

Theorem 3.5 (Proposition 3.8 [Laz16]). *If (Σ, ξ, Φ) is a strongly asymptotically dynamically convex contact structure, then all Liouville fillings of (Σ, ξ, Φ) have isomorphic SH^+.*

3.2.1 Definition of positive idempotent group I_+

We also want to define the ring structure. However as in Remark 2.11, we can not define a product on SH^+. Having said that, we can use the pair-of-pants product on $SH_*(W)$ to define an invariant for SH_*^+ which is independent of the Liouville filling.

First, let's recall the tautological short exact sequence:

$$0 \to SC_*^{<\epsilon}(W) \to SC_*(W) \to SC_*(W)/SC_*^{<\epsilon}(W) \to 0.$$

We have long exact sequence:

$$\cdots \to SH_*^{<\epsilon}(W) \to SH_*(W) \to SH_*^+(W) \to SH_{*-1}^{<\epsilon}(W) \to \cdots \tag{3.1}$$

We also have $H^{n-*}(W,H) \cong SH_*^{<\epsilon}(W,H)$ since the admissible Hamiltonian H is $\mathcal{C}^2$ small in W. Therefore we can replace $SH_*^{<\epsilon}$ terms in equation 3.1 by H^{n-*}, in particular, we have a long exact sequence

$$\cdots \to H^0(W) \to SH_n(W) \to SH_n^+(W) \to H^1(W) \to \cdots \tag{3.2}$$

In fact, the map $H^0(W) \to SH_n(W)$ in equation 3.2 is a ring homomorphism, see Appendix. A of [CO18]. Suppose $SH_*(W) \neq 0$, then 1_W does ***not*** maps to the unit in $SH_*(W)$, where 1_W is the unit of $H^0(W)$, by Theorem 2.17 (also see Lemma A.3 of [CO18]). Therefore $H^0(W) \to SH_n(W)$ is injective, and we will regard it as a subring of $SH_n(W)$. We can thus identify elements in $SH_n(W)/H^0(W)$ with elements in $SH_n^+(W)$. In particular, $SH_n(W)/H^0(W) \cong SH_n^+(W)$ if $H^1(W) = 0$.

Now let's consider the subgroup of $SH_n(W)$ as follows:

$$I(W) := \{\, \alpha \in SH_n(W) \,|\, \alpha^2 - \alpha \in H^0(W)\}. \tag{3.3}$$

Notice the group action here is "addition".

Define the ***positive idempotent group*** $I_+(W)$ by

$$I_+(W) := I(W)/H^0(W).$$

By the previous analysis, we can regard $I_+(W)$ as a subgroup of $SH_n^+(W)$. In the case $I_+(W)$ is finite, we can further define ***positive idempotent index*** $i(W) := |I_+(W)|$.

3.2.2 Properties of I_+

Since $H^0(W,\mathbb{Z}_2) \cong \mathbb{Z}_2$, $I_+(W,\mathbb{Z}_2)$ is determined by $I(W,\mathbb{Z}_2)$. Recall that $SH_*(W,\mathbb{Z}_2)$ has a $H_1(W,\mathbb{Z})/Tors$ grading. The first observation is that elements in $H^0(W)$ have $H_1/Tors$ grading zero. Indeed, it's true for all elements in $I(W)$. Suppose R is an algebra over $\mathbb{Z}_2$ which is graded by a finitely generated torsion-free abelian group K. This means that as a vector space, $R = \bigoplus_{k\in K} R_k$ with the property that if $a \in R_{k_1}, b \in R_{k_2}$ then $ab \in R_{k_1\cdot k_2}$. Define $I_0(R) := \{0,1\}$ and $I(R) := \{x \in R | x^2 - x \in I_0\}$.

Lemma 3.6 (Lemma 7.6 [McL07]). *If $a \in I(R)$ then $a \in R_e$ where e is identity of group K.*

Proof. We argue by contradiction. Suppose we have $a = a_{k_1} + \cdots + a_{k_n}$ where $k_i \in K$ and $a_{k_i} \in R_{k_i}$, $k_1 \neq e$. Then $a^2 = a_{k_1^2} + \cdots + a_{k_n^2}$. Since K is torsion free, there is a group homomorphism $p : K \to \mathbb{Z}$ such that $p(k_1) \neq 0$. This map actually gives R a $\mathbb{Z}$ grading. Let b be an element in R, then it can be uniquely written as $b = b_1 + \cdots + b_k$ where b_k are non-zero elements of R with grading $d_i \in \mathbb{Z}$. We can define a function f as follows:

$$f(b) := \min\{|d_i| \neq 0\}$$

Note that f is well-defined only if at least one of the $d_i's$ is non-zero. And when it is well-defined, $f(b+1) = f(b)$ because $1 \in R_e$ and has grading 0. The assumption $p(k_1) \neq 0$ implies that $f(a)$ is well defined and positive. On the other hand, we have $a \in I(R)$, which means $a^2 = a$ or $a^2 = a + 1$. Either way, it implies $f(a^2) = f(a)$, which contradicts the fact that $f(a^2) \geq 2f(a)$. □

Corollary 3.7. *Any element in $I(W)$ is null-homologous in $H_1(W,\mathbb{Z})/Tors$.*

Therefore, we can refine our definition of $I(W)$ to be

$$I(W) := \{\, \alpha \in SH^0_n(W) \,|\, \alpha^2 - \alpha \in H^0(W)\}.$$

where $SH^0_*(W)$ is generated by all null-homologous Reeb orbits. In the case of strongly asymptotically dynamically convex contact manifolds (with respect to certain framing Φ), different Liouville fillings have isomorphic positive idempotent group, as stated in Theorem 1.1, the proof will be deferred to section 4.

3.3 Effect of contact surgery

Theorem 3.8 (Theorem 3.15 [Laz16], [Yau04]). *If $(Y_1^{2n-1}, \xi_1), n \geq 2$, is an asymptotically dynamically convex contact structure and (Y_2,ξ_2) is the result of index $k \neq 2$ subcritical contact surgery on (Y_1,ξ_1), then (Y_2,ξ_2) is also asymptotically dynamically convex.*

Now we are in the position to prove Proposition 1.8.

Proof of Proposition 1.8. Recall that W_1 is a flexible Weinstein domain, so $SH_*(W) = 0$ (see [BEE12]). By lemma 1.7, ∂W_1 is asymptotically dynamically convex and so is W by assumption. Moreover, ∂V_k is obtained by attaching a Weinstein $1-$handle to asymptotically dynamically convex contact manifold, therefore it is asymptotically dynamically convex by Theorem 3.8. A well known fact is that subcritical surgery does not change symplectic homology as a ring, see Theorem 2.19. We have

$$SH_n(W_i) \cong \underbrace{SH_n(W) \oplus \cdots \oplus SH_n(W)}_{i} \oplus \underbrace{SH_n(W_1) \oplus \cdots \oplus SH_n(W_1)}_{k-i} = \bigoplus_{j=1}^{i} SH_n(W)$$

so we have

$$I(W_i) \cong \underbrace{I(W) \oplus \cdots \oplus I(W)}_{i}.$$

Since $SH_*(W) \neq 0$ and is finite dimensional, $\{0, 1_W\} \subset I(W)$. We therefore have $2 \leq |I(W)| < \infty$, so $|I(W_i)| = |I(W)|^i$ are mutually distinct. Therefore, $|I_+(W_i)| \neq |I_+(W_j)|$ for $i \neq j$.

□

Theorem 3.9 ([Laz16], [Yau04]). *Let (Σ_1, ξ_1) be a strongly asymptotically dynamically convex contact structure with respect to Φ, and (α_k, D_k) as in Definition 3.2 and (Σ_2, ξ_2) be the result of index 2 contact surgery on $\Lambda^1 \subset \Sigma_1$ so that the trivialization Φ extends to the handle. Then (Σ_2, ξ_2) is also strongly asymptotically dynamically convex with respect to Φ.*

Remark 3.10. Since the trivialization Φ of the canonical bundle of (Σ_1, ξ_1) extends to the attaching handle, so by abuse of notation, the trivialization of the canonical bundle of (Σ_2, ξ_2) which is obtained by extending Φ to the attaching handle is still denoted by Φ.

Proposition 3.11 (Proposition 5.5 [Laz16], [Yau04]). *Let $\Lambda^{k-1} \subset (\Sigma_1^{2n-1}, \alpha_1), n > 1$, be an isotropic sphere with $k < n$. For any $D > 0$ and integer $i > 0$, there exists $\epsilon = \epsilon(D, i) > 0$ such that if (Σ_2, α_2) is the result of contact surgery on $U^\epsilon(\Lambda, \alpha)$ with respect to the trivialization Φ, then there is a grading preserving bijection between $\mathcal{P}_\Phi^{<D}(\Sigma_2, \alpha_2)$ and $\mathcal{P}_\Phi^{<D}(\Sigma_1, \alpha_1) \cup \{\gamma^1, \cdots, \gamma^l\}$ where $|\gamma^i| = 2n - k - 4 + 2i$.*

Remark 3.12. The proof largely follows [Laz16] proposition 5.5, with only minor changes regarding the non-contractible Reeb orbits. The difference in the Strongly ADC case is that we need to choose the trivialization to define the Conley-Zehnder index.

Proof of Proposition 3.11. As explained in [Yau04], the surgery belt sphere S^{2n-k-1} contains a contact sphere $(S^{2n-2k-1}, \xi_{std})$. After taking appropriate sequence of contact forms on (Σ_2, ξ_2), the Reeb orbits of (Σ_2, ξ_2) correspond to the old Reed orbits of (Σ_1, ξ_1), plus the new orbits of $(S^{2n-2k-1}, \xi_{std})$ inside the belt sphere of action less than D. The correspondence is natural since the trivialization of the canonical bundle extends over the surgery. These new orbits corresponds to the iterations $\gamma^1, \cdots, \gamma^l$ of a single Reeb orbit γ, see [Yau04]. Moreover, $\mu_{CZ}(\gamma^i) = n - k - 1 + 2i$ and therefore $|\gamma^i| = 2n - k - 4 + 2i$. Meanwhile, by shrinking the handle, the action can be made arbitrarily small and therefore we can ensure that arbitrarily large iterations of γ have action less than D. □

For $\Lambda \subset \Sigma$, Since $J^1(\Lambda) \simeq T^*\Lambda \times \mathbb{R}$, choose a Riemannian metric on Λ. Let $U^\epsilon(\Lambda) \subset (J^1(\Lambda), \alpha_{std})$ be $\{||y|| < \epsilon, |z| < \epsilon\}$, the metric on Λ to define $||y||$ on the fiber, z is the coordinate on $\mathbb{R}$. If $\Lambda \subset (Y, \alpha)$ is Legendrian, let $U^\epsilon(\Lambda, \alpha) \subset (Y, \alpha)$ be a neighborhood of Λ that is strictly contactomorphic to $U^\epsilon(\Lambda)$.

Proposition 3.13 (Proposition 6.7 [Laz16]). *Let $\alpha_1 > \alpha_2$ be contact forms for (Σ, ξ) and let $\Lambda \subset (\Sigma, \xi)$ be an isotropic submanifold with trivial symplectic conormal bundle. Then for any sufficiently small δ_1, δ_2, there exists a contactomorphism h of (Σ, ξ) such that*

- *h is supported in $U^\epsilon(\Lambda, \alpha_1)$, $h|_\Lambda = Id$, and $h^*\alpha_2 < 4\alpha_1$*
- *$h^*\alpha_2|_{U^{\delta_1}(\Lambda,\alpha_1)} = c\alpha_1|_{U^{\delta_1}(\Lambda,\alpha_1)}$ for some constant c (depending on δ_1, δ_2)*
- *$h(U^{\delta_1}(\Lambda, \alpha_1)) \subset U^{\delta_2}(\Lambda, \alpha_2)$.*

Proposition 3.14 (Remark 6.5 [Laz16]). *Let $\Lambda \subset (\Sigma_1^{2n-1}, \xi_1), n > 2$ be an isotropic sphere and (Σ_2, ξ_2) be the result of contact surgery on Λ which extends the chosen trivialization Φ of the canonical bundle. Suppose (Σ_1, ξ_1) is a strongly asymptotically dynamically convex contact structure with respect to the trivialization Φ and has (α_k, D_k) as in Definition 3.2. If $\alpha_k|_{U^\epsilon(\Lambda,\alpha_1)} = c_k\alpha_1|_{U^\epsilon(\Lambda,\alpha_1)}$ for some constants ϵ, c_k, then (Σ_2, ξ_2) is also strongly asymptotically dynamically convex with respect to the trivialization Φ.*

Proof of Theorem 3.9. Now we will proceed exactly as Lazarev did, keeping in mind that we are dealing with the strongly ADC property. We can apply Proposition 3.13 so that the conditions of Proposition 3.14 are satisfied.

□

4 I_+ is an invariant of ADC contact manifolds

We will follow Lazarev's approach. First we will introduce the procedure called stretching-the-neck. Here we use the notation in [Laz16].

Let $V \subset \widehat{W}$ be a Liouville subdomain with contact boundary (Z, α_Z). Consider a collar of Z in V symplectomorphic to $(Z \times [1-\delta, 1], d(t\alpha_Z))$ for small δ. Let $J \in \mathcal{J}_{std}(W)$ be cylindrical in $Z \times [1-\delta, 1]$ and set $J' := J|_{Z\times[1-\delta,1]}$. For $0 < R < 1-\delta$, we extend J' to a cylindrical almost complex structure on $Z \times [R, 1]$, which we also call J'. Now let f_R be any diffeomorphism $[R, 1] \to [1-\delta, 1]$ whose derivative equals 1 near the boundary. We can define $J_R \in \mathcal{J}_{std}(W)$ to be $(Id \times f_R)_*J'$ on $Z \times [1-\delta, 1]$ and J outside $Z \times [1-\delta, 1]$. It is smooth because of the derivative condition on f_R. If $J_s \in \mathcal{J}_{std}(W)$ is a homotopy that is cylindrical and s-independent in $Z \times [1-\delta, 1]$, then we can apply the same construction to obtain a homotopy $J_{R,s}$.

Let $(H_s, J_s), s \in \mathbb{R}$ be a homotopy with $(H_s, J_s) = (H_-, J_-)$ for $s \ll 0$ and $(H_s, J_s) = (H_+, J_+)$ for $s \gg 0$, and $H_s \equiv 0$ in $Z \times [1-\delta, 1] \subset V$. Furthermore, let x_+, x_- be Hamiltonian orbits of H_+, H_- respectively in the source and target of the maps induced by (H_s, J_s).

Proposition 4.1 (Proposition 3.10 [Laz16]). *Suppose that (Z, α) is strongly ADC with respect to the trivialization Φ and all elements of $\mathcal{P}_\Phi^{<D}(Z, \alpha)$ have positive degree. If $A_{H_+}(x_+) - A_{H_-}(x_-) < D$, then there exists $R_0 \in (0, 1-\delta)$ such that for any $R \le R_0$, all rigid $(H_s, J_{R,s})$-Floer trajectories are contained in $\widehat{W} \setminus V$.*

Remark 4.2. If H_s is independent of s, then the Floer trajectories define the differential; if H_s is an decreasing homotopy, then $(H_s, J_{R,s})$-Floer trajectories define the continuation map.

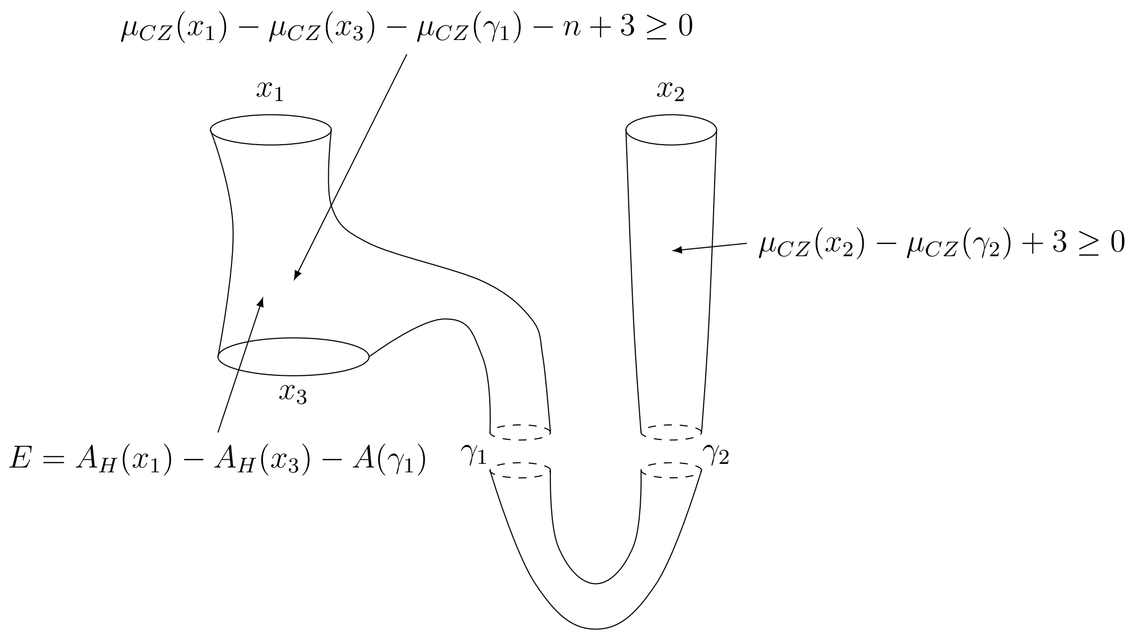

Figure 4: Top of Floer building is connected (such breaking does *not* occur). Hamiltonian orbits are represented by continuous lines, Reeb orbits by dashed lines.

In P.Uebele's paper [Ueb15], the pair-of-pants product is defined for "index-positive" contact manifold, where the symplectic homology used is actually Rabinowitz-Floer homology. Though as the paper points out, the ring structure is not well defined on SH^+. However, at the chain level, if the pair-of-pants product is asymptotic to Hamiltonian orbits of positive action, then by the stretching-the-neck technique, we can prove the pair-of-pants does not enter the interior of the Liouville filling. In [Ueb15], this is proved for index-positive contact manifolds. However, it is not true for Strongly ADC contact manifolds in general. That being said, the pair-of-pants does *not* enter the interior of the filling when the indices of the Hamiltonian orbits of the asymptotes are high enough. To be precise, we have the following:

Proposition 4.3. *Suppose that (Z, α) is strongly ADC with respect to the trivialization Φ and all elements of $\mathcal{P}^{<D}_{\Phi}(Z, \alpha)$ have positive reduced Conley-Zehnder index. Furthermore, let $A_H(x_i) < D/2 (i = 1, 2, 3)$ be non-constant Hamiltonian orbits such that $\mu_{CZ}(x_1) + \mu_{CZ}(x_2) - \mu_{CZ}(x_3) = n$ and $\mu_{CZ}(x_i) \geq n, i = 1, 2$, then there exists $R_0 \in (0, 1 - \delta)$ such that for any $R \leq R_0$, all pair-of-pants products are contained in $\widehat{W} \setminus V$.*

Proof. The proof is a combination of proposition 3.10 of [Laz16] and lemma 3.12 of [Ueb15]. First of all, we have to rule out the breaking as in Figure 4 (Similarly with x_1 and x_2 exchanged).

Suppose we have the breaking as in Figure 4, then the top level has positive dimension, and we have (see lemma 3.10 of [Ueb15])

$$\mu_{CZ}(x_1) - \mu_{CZ}(x_3) - \mu_{CZ}(\gamma_1) - n + 3 \geq 0$$

and

$$\mu_{CZ}(x_2) - \mu_{CZ}(\gamma_2) + 3 \geq 0.$$

Then, since

$$\mu_{CZ}(x_1) + \mu_{CZ}(x_2) - \mu_{CZ}(x_3) = n,$$

these conditions are reduced to

$$\mu_{CZ}(\gamma_1) \leq 3 - \mu_{CZ}(x_2) \quad \text{and} \quad \mu_{CZ}(\gamma_2) \leq 3 + \mu_{CZ}(x_2).$$

In particular,

$$\mu_{CZ}(\gamma_1) \leq 3 - n.$$

Meanwhile, the Floer energy of the top level would be

$$0 \leq E = A_H(x_1) - A_H(x_3) - A(\gamma_1).$$

So

$$0 < A(\gamma_1) \leq A_H(x_1) - A_H(x_3) < D.$$

We have $\gamma_1 \in \mathcal{P}_\Phi^{<D}(Z, \alpha)$, which implies $\mu_{CZ}(\gamma_1) > 3 - n$, which is a contradiction. Now we know that the top of the Floer building is connected, so we can proceed as in the proof of proposition 3.10 of [Laz16]. We prove this by contradiction. Suppose the pair-of-pants product breaks after neck-stretching and γ_k are the Reeb orbits in the top of the Floer building as in [Laz16]. The virtual dimension of the moduli space of the top Floer building is

$$|x_1| + |x_2| - |x_3| - \sum |\gamma_k| < 0.$$

Contradiction. □

Now if we further require that the admissible Hamiltonian H has a unique minimum (which is always possible and compatible with our requirements on admissible Hamiltonians), then the Floer chain complex $SC_*(W,H,J) = O_*(W,H,J) \oplus C_*(W,H,J)$, where $O_*(W,H,J)$ is generated by all non-constant Hamiltonian orbits and $C_*(W,H,J)$ is generated by all constant Hamiltonian orbits (critical points of H). Since H is $\mathcal{C}^2$ small in W, the action of the critical points is small, and the Floer differential d coincides with the Morse boundary operator d_1. We therefore have $(SC_*^{<\delta}(W,H,J), d) = (C_*(W,H,J), d_1)$ and $SC_*^+(W,H,J) = O_*(W,H,J)$.

For degree reasons, $C_n(W,H,J) = \mathbb{Z}_2 < p >$, where p is the unique minimum of H. Note that $d(p) = d_1(p) = 0$ and we have the fact that $d(x+p) = 0$ implies $d(x) = d(p) = 0$.

Proof of Theorem 1.1. As shown in Figure 5, let W, V be two different Liouville fillings for a strongly ADC contact manifold (Σ, λ). Suppose H_W^D, H_V^D are Hamiltonians (as in Subsection 2.4.1) whose slopes at infinity are $D \notin Spec(\Sigma, \lambda)$. We can further assume that they have unique minima which are denoted by p, q respectively. Note that any element $x \in O_*(W, H_W, J_W)$ has action $\mathcal{A}_{H_W}(x) < D$. As shown above, $O_*(W, H_W, J_W) = SC_*^+(W, H_W, J_W)$.

After neck-stretching, we can assume that

$$(H_W, J_W)|_{\Sigma\times[R,\infty)} \equiv (H_V, J_V)|_{\Sigma\times[R,\infty)}$$

So we have $O_*(W, H_W, J_W) = O_*(W, H_V, J_V)$. Proposition 4.1 shows that Floer cylinders with asymptotes in $SC_*^+(W, H_W, J_W)$ are entirely contained in $\Sigma \times [R, \infty)$. Therefore Floer differentials of $O_*(W, H_W, J_W)$ and $O_*(W, H_V, J_V)$ coincide. We will suppress W and V in the

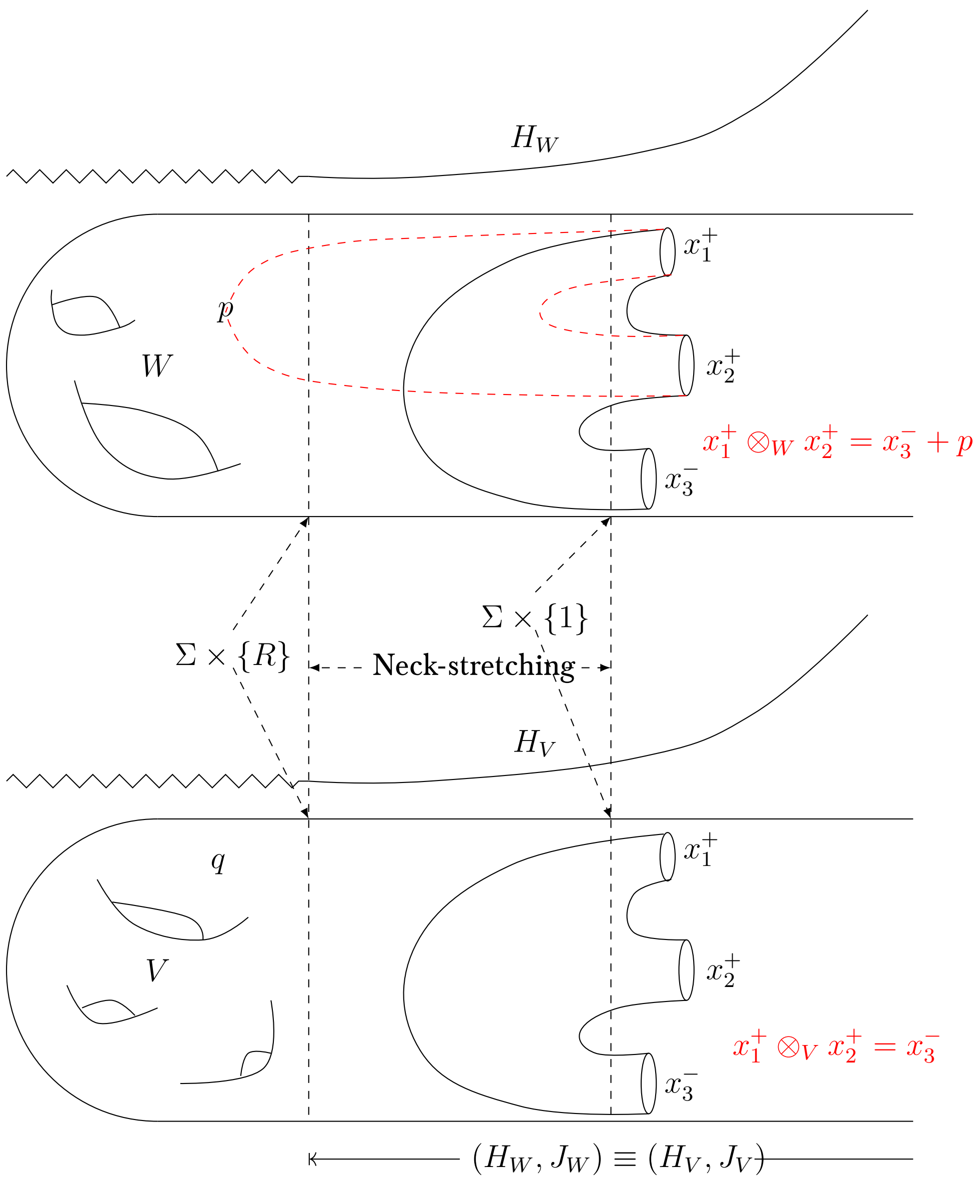

Figure 5: Pair-of-pants product for different fillings W and V and the natural identification of $C_n^{<D}(W, H_W, J_W)$ and $C_n^{<D}(V, H_V, J_V)$. On the chain level, pair-of-pants product are the same, up to a difference in $O_n^{<D}$. Morally, the difference vanishes when elements are quotiented by $O_n^{<D}$; $I_+(W)$ is therefore isomorphic to $I_+(V)$.

notation and denote them by $(O_*(H,J),\partial)((O_*^{<K}(H,J),\partial)$ if it is filtered above by action K). We have the pair-of-pants product $\otimes_W$ on

$$SC_n^{<D/2}(W,H^D,J) = O_n^{<D/2}(H^D,J) \oplus C_n^{<D/2} = O_n^{<D/2}(H^D,J) \oplus \mathbb{Z}_2 < p >$$

defined as

$$SC_n^{<D/2}(W,H^D,J) \otimes SC_n^{<D/2}(W,H^D,J) \to SC_n^{<D}(W,H^D,J) \tag{4.1}$$

$$(x,y) \mapsto x \otimes_W y. \tag{4.2}$$

By Proposition 4.3, $\otimes_W$ coincides with $\otimes_V$ on components in $O_n^{<D}(H^D,J)$, that is, for $x,y \in O_n^{<D/2}(H^D,J)$, $x \otimes_W y = z + \delta_W(x,y)$, where $z \in O_n^{<D}(H^D,J)$ and $\delta_W(x,y) \in \mathbb{Z}_2 < p >$. Note that $\delta_W(x,y)$ is closed in $SC_n^{<D}(H,J)$. Likewise, we have $x \otimes_V y = z + \delta_V(x,y)$, where $z \in O_n^{<D}(H^D,J)$ and $\delta_V(x,y) \in \mathbb{Z}_2 < q >$. Now for any $\alpha \in I^{<D/2}(W,H_W,J_W) \subset SH_n^{<D/2}(W,H_W,J_W)$, we have

$$\alpha = [x + \epsilon p]_W = [x]_W + \epsilon[p]_W = [x]_W + \epsilon e_{H_W}$$

where $x \in O_n^{D/2}(H,J), \epsilon = 0$ or 1. $x \otimes_W x = z + \delta_W(x,x)$ implies

$$\alpha^2 - \alpha = [x]_W^2 + \epsilon^2 e_{H_W}^2 - [x]_W - \epsilon e_{H_W} = [z - x + \delta(x,x)]_W = [z-x]_W + [\delta(x,x)]_W$$

So $\alpha \in I^{<D/2}(W,H_W,J_W)$ is equivalent to

$$[z-x]_W + [\delta(x,x)]_W \in H^0(W).$$

But since $[\delta(x,x)]_W \in H^0(W)$, $\alpha \in I^{<D/2}(W,H_W,J_W)$ is equivalent to $[z-x]_W \in H^0(W)$. Hence for $x \in O_n^{<D/2}(H,J), \partial(x) = 0$ (∂ is Floer differential on $O_n^{<D/2}(H,J)$),

$$[x]_W^+ \in I_+(W,H_W,J_W) \iff [z-x]_W^+ \in SH_n^{+,<D/2}(H,J)$$

where $[y]_W^+$ stands for the equivalence class of $y \in O_*(H,J)$ in $SH_n^+(W,H_W,J_W)$. We can prove the same results for V similarly. Therefore we have an isomorphism between $I_+^{<D/2}(W,H_W,J_W)$ and $I_+^{<D/2}(V,H_V,J_V)$:

$$[x]_W^+ \mapsto [x]_V^+.$$

Since $SH_n^{+,<D/2}(W,H_W,J_W), SH_n^{+,<D/2}(V,H_V,J_V)$ can be defined by $(\Sigma \times [R,\infty),H,J)$ as the Floer cylinder never enters the interior. Therefore we have the identity

$$SH_*^{+,<D/2}(W,H_W,J_W) \cong H_*(O_*^{<D/2}(H,J),\partial) \cong SH_n^{+,<D/2}(V,H_V,J_V),$$

the inclusion map $SH_n^{+,<D/2}(W,H_W,J_W) \to SH_n^+(W,H_W,J_W)$ commute with the above isomorphism,

$$\begin{array}{ccccc}
I_+^{<D/2}(W,H_W,J_W) & \xrightarrow{i} & SH_n^{+,<D/2}(W,H_W,J_V) & \xrightarrow{i} & SH_n^+(W,H_W,J_V) \\
\downarrow{\cong} & & \downarrow{\cong} & & \downarrow{\cong} \\
I_+^{<D/2}(V,H_V,J_V) & \xrightarrow{i} & SH_n^{+,<D/2}(V,H_V,J_V) & \xrightarrow{i} & SH_n^+(V,H_V,J_V)
\end{array}$$

and we can therefore take the direct limit with respect to H_W. Since we already know $SH_*^+(W)$ is isomorphic to $SH_*^+(V)$ by Theorem 3.5, it follows that $I_+(W) \cong I_+(V)$.

□

Remark 4.4. We can also proceed exactly as in proof of proposition 3.8 in [Laz16]. The key point is to use the *essential complex* as defined in that proof.

5 Brieskorn Manifolds

5.1 Definition of Brieskorn manifolds

Let $\mathbf{a} = (a_0, a_1, \cdots, a_n)$ be an $(n+1)$-tuple of integers $a_i > 1, \mathbf{z} := (z_0, z_1, \cdots, z_n) \in \mathbb{C}^{n+1}$, and set $f(\mathbf{z}) := z_0^{a_0} + z_1^{a_1} + \cdots + z_n^{a_n}$, we define *Brieskorn Variety* as

$$V_{\mathbf{a}}(t) := \{(z_0, z_1, \cdots, z_n) \in \mathbb{C}^{n+1} | f(\mathbf{z}) = t\} \quad \text{for each} \quad t \in \mathbb{C}. \tag{5.1}$$

We will often suppress $\mathbf{a}$ when it causes no confusion, and define $X_t^s = V(t) \cap B(s)$.

Further, with S^{2n+1} denoting the unit sphere in $\mathbb{C}^{n+1}$, we define the *Brieskorn Manifold* as the intersection of Brieskorn Variety $V_{\mathbf{a}}(0)$with the unit sphere:

$$\Sigma(\mathbf{a}) := V_{\mathbf{a}}(0) \cap S^{2n+1}.$$

Lemma 5.1 (Lemma 96 [Fau16], Lemma 7.1.1 [Gei08]). *$\Sigma(\mathbf{a})$ and $V_{\mathbf{a}}(t), t \neq 0$ are smooth manifolds.*

Proof. We set $\rho(z) := ||z||^2 = \sum z_k \bar{z_k}$ and consider the maps

$$f : \mathbb{C}^{n+1} \to \mathbb{C} \qquad \textit{and} \qquad (f, \rho) : \mathbb{C}^{n+1} \to \mathbb{C} \times \mathbb{R}$$

Since $V_{\mathbf{a}}(t) = f^{-1}(t)$ and $\Sigma(\mathbf{a}) = (f, \rho)^{-1}(0, 1)$, it suffices to show that t (respectively $(0, 1)$) are regular values. With a little Wirtinger calculus (and using the fact that f is holomorphic) we find the Jacobian matrix

$$D(f, \rho) = \begin{bmatrix} a_0 z_0^{a_0 - 1} & \cdots & a_n z_n^{a_n - 1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_0 \bar{z_0}^{a_0 - 1} & \cdots & a_n \bar{z_n}^{a_n - 1} \\ \bar{z_0} & \cdots & \bar{z_n} & z_0 & \cdots & z_n \end{bmatrix}$$

For $\mathbf{z} \neq 0$ the first two rows of $D(f, \rho)$ are linearly independent, which implies that $\epsilon \neq 0$ is a regular value of f. If $\mathbf{z}$ is a point where this matrix has rank smaller than 3, there exists a non-zero complex number λ such that $\bar{z_k} = \lambda a_k z_k^{a_k - 1}$ for all k and hence

$$\sum_{k=0}^{n} \frac{z_k \bar{z_k}}{a_k} = \lambda \sum_{k=0}^{n} z_k^{a_k} = \lambda \cdot f(\mathbf{z})$$

This equality is incompatible with the conditions $\rho(\mathbf{z}) = 1$ and $f(\mathbf{z}) = 0$ for a point $\mathbf{z} \in \Sigma(\mathbf{a})$. □

5.2 Topology of Brieskorn manifolds

Now, we give some topological facts about Brieskorn manifolds without proof.

Proposition 5.2 (Theorem 5.2 [Mil16]). *A Brieskorn manifold $\Sigma(\mathbf{a})^{2n-1}$ is $(n-2)$-connected.*

5.3 Trivialization and Conley-Zehnder index

Let us consider on $\mathbb{C}^{n+1}$ the following Hermitian form given by

$$<\xi,\zeta>_{\mathbf{a}}:=\frac{1}{2}\sum_{k=0}^{n}a_k\xi_k\bar{\zeta}_k.$$

It defines a symplectic 2-form

$$\omega_{\mathbf{a}}:=\frac{i}{4}\sum_{k=0}^{n}a_k dz_k\wedge d\bar{z}_k.$$

Notice that $Y_\lambda(\mathbf{z}):=\frac{\mathbf{z}}{2}$ is a Liouville vector field for $\omega_{\mathbf{a}}$, with the corresponding 1-form

$$\lambda_{\mathbf{a}}:=\omega_{\mathbf{a}}(Y_\lambda,\cdot)=\frac{i}{8}\sum_{k=0}^{n}a_k(z_k d\bar{z}_k-\bar{z}_k dz_k).$$

Proposition 5.3 (Proposition 97 [Fau16], [LM76]). *The restriction $\alpha_a:=\lambda_a|_\Sigma$ is a contact form on $\Sigma(\mathbf{a})$ with Reeb vector field $R_{\mathbf{a}}$ given by*

$$R_{\mathbf{a}}=4i(\frac{z_0}{a_0},\frac{z_1}{a_1},\cdots,\frac{z_n}{a_n}).$$

Proof. The gradient of f with respect to $<\cdot,\cdot>_{\mathbf{a}}$ is given by

$$\nabla_{\mathbf{a}}f:=2(\bar{z_0}^{a_0-1},\bar{z_1}^{a_1-1},\cdots,\bar{z_n}^{a_n-1}).$$

The Liouville vector field Y_V of the restricted 1-form $\lambda_{\mathbf{a}}|_{V_{\mathbf{a}}(0)}$ with respect to the restricted symplectic form $\omega_{\mathbf{a}}|_{V_{\mathbf{a}}(0)}$is given by

$$Y_V:=Y_\lambda-\frac{<\nabla_{\mathbf{a}}f,Y_\lambda>_{\mathbf{a}}}{||\nabla_{\mathbf{a}}f||^2_{\mathbf{a}}}\cdot\nabla_{\mathbf{a}}f.$$

Note that $TV_{\mathbf{a}}(t)=\ker df=\ker<\nabla_{\mathbf{a}}f,\cdot>_{\mathbf{a}}$, which shows that $Y_V\in TV_{\mathbf{a}}(0)$. Furthermore, we have for any $\xi\in TV_{\mathbf{a}}(0)$,

$$\omega_{\mathbf{a}}(Y_V,\xi)=\omega_{\mathbf{a}}(Y_\lambda,\xi)-\frac{<\nabla_{\mathbf{a}}f,Y_\lambda>_{\mathbf{a}}}{||\nabla_{\mathbf{a}}f||^2_{\mathbf{a}}}\cdot\omega_{\mathbf{a}}(\nabla_{\mathbf{a}}f,\xi)=\lambda_{\mathbf{a}}(\xi)+\frac{<\nabla_{\mathbf{a}}f,Y_\lambda>_{\mathbf{a}}}{||\nabla_{\mathbf{a}}f||^2_{\mathbf{a}}}\cdot\underbrace{Im<\nabla_{\mathbf{a}}f,\xi>)_{\mathbf{a}}}_{=0}=\lambda_{\mathbf{a}}(\xi)$$

So this indicates that Y_V is the Liouville vector field for the pair $(\omega_{\mathbf{a}}|_{V_{\mathbf{a}}(0)},\lambda_{\mathbf{a}}|_{V_{\mathbf{a}}(0)})$. Now notice that $d\rho=\sum_{k=0}^{n}\bar{z}_k dz_k+z_k d\bar{z}_k$ (ρ is defined in the proof of lemma 5.1) and we have

$$d\rho(Y_V)=\sum\frac{z_k\bar{z}_k}{2}-\frac{<\nabla_{\mathbf{a}}f,Y_\lambda>_{\mathbf{a}}}{||\nabla_{\mathbf{a}}f||^2_{\mathbf{a}}}\sum 2\bar{z}_k\cdot\bar{z_k}^{a_k-1}=\frac{\rho(\mathbf{z})}{2}-\frac{<\nabla_{\mathbf{a}}f,Y_\lambda>_{\mathbf{a}}}{||\nabla_{\mathbf{a}}f||^2_{\mathbf{a}}}\cdot 2\bar{f(\mathbf{z})}=\frac{1}{2}.$$

since $\rho(\mathbf{z})=1$ and $f(\mathbf{z})=0$. It follows that Y_V points out of the unit sphere and hence out of $\Sigma(\mathbf{a})$ in $V_{\mathbf{a}}(0)$. It follows that $\Sigma(\mathbf{a})$ is a contact hypersurface in $V_{\mathbf{a}}(0)$. Now we are going to check that $R_{\mathbf{a}}$ is the Reeb vector field of $\alpha_{\mathbf{a}}$. For any $\mathbf{z}\in\Sigma(\mathbf{a})$, we have

$$<R_{\mathbf{a}},\nabla_{\mathbf{a}}f>_{\mathbf{a}}=4i\sum_{k=0}^{n}z_k^{a_k}=4if(\mathbf{z})=0,$$

$$d\rho(R_{\mathbf{a}}) = \sum_{k=0}^{n} z_k(-4i)\bar{z_k} + \bar{z_k}4iz_k = 0$$

The two equations above shows that $R_{\mathbf{a}}$ is a tangent vector. We also have

$$\alpha_{\mathbf{a}}(R_{\mathbf{a}}) = \lambda_{\mathbf{a}}(R_{\mathbf{a}}) = \frac{i}{8}\sum_{k=0}^{n} a_k(\frac{4i}{a_k}\bar{z_k} - \bar{z_k}\frac{4i}{a_k}z_k) = \rho(\mathbf{z}) = 1,$$

$$\iota_{R_{\mathbf{a}}} d\alpha_{\mathbf{a}} = \frac{i}{4}\sum_{k=0}^{n}(4i\frac{z_k}{a_k}d\bar{z_k} - (-4i)\frac{\bar{z_k}}{a_k}) = -\sum_{k=0}^{n}(z_k d\bar{z_k} + \bar{z_k}dz_k) = -d\rho.$$

The latter form is zero for vectors in $T\Sigma(\mathbf{a})$, therefore, $R_{\mathbf{a}}$ is the Reeb vector field. □

Proposition 5.4 (Corollary 98 [Fau16]). *The symplectic complement $\xi_{\mathbf{a}}^{\perp}$ with respect to $\omega_{\mathbf{a}}$ of the contact structure $\xi_{\mathbf{a}} := \ker \alpha_{\mathbf{a}}$ inside $\mathbb{C}^{n+1}$ is symplectically trivialized by the following 4 vector fields:*

- $X_1 := \frac{\nabla_{\mathbf{a}} f}{||\nabla_{\mathbf{a}} f||_{\mathbf{a}}}$
- $Y_1 := i \cdot X_1$
- $X_2 := Y_V$
- $Y_2 := R_{\mathbf{a}}$.

Proof. X_1, Y_1 generate the complex complement of $TV_{\mathbf{a}}(0)$ while X_2, Y_2 generate the symplectic complement of $\xi_{\mathbf{a}}$ in $TV_{\mathbf{a}}(0)$, so we have

$$\omega_{\mathbf{a}}(X_1, X_2) = \omega_{\mathbf{a}}(X_1, Y_2) = \omega_{\mathbf{a}}(Y_1, X_2) = \omega_{\mathbf{a}}(Y_1, Y_2) = 0.$$

Meanwhile we have

$$\omega_{\mathbf{a}}(X_1, Y_1) = 1, \quad \omega_{\mathbf{a}}(X_2, Y_2) = \lambda_{\mathbf{a}}(R_{\mathbf{a}}) = 1.$$

The latter equation comes from the proof of proposition 5.3. □

The Reeb vector field $R_{\mathbf{a}} = 4i(\frac{z_0}{a_0}, \frac{z_1}{a_1}, \cdots, \frac{z_n}{a_n})$ generates the following flow:

$$\psi_{\mathbf{a}}^t(\mathbf{z}) = (e^{\frac{4it}{a_0}} \cdot z_0, \cdots, e^{\frac{4it}{a_n}} \cdot z_n)$$

The submanifolds Σ_T of period $T \in \pi\mathbb{Z}/2$ are given by

$$\Sigma_T = \left\{ \mathbf{z} \in \Sigma(\mathbf{a}) \,\middle|\, z_k = 0 \text{ if } \frac{T}{a_k} \in \pi\mathbb{Z}/2 \right\}.$$

Σ_T is not empty if and only if the relation $\frac{T}{a_k} \in \pi\mathbb{Z}/2$ is satisfied by at least 2 different k, as $\mathbf{z} \in \Sigma(\mathbf{a})$ has at least 2 non-zero entries. Note that Σ_T is the intersection $\Sigma(\mathbf{a}) \cap V(\mathbf{a}, T)$, where $V(\mathbf{a}, T)$ denotes the complex linear subspace

$$V(\mathbf{a}, T) := \left\{ \mathbf{z} \in \mathbb{C}^{n+1} \,\middle|\, z_k = 0 \text{ if } \frac{T}{a_k} \notin \frac{\pi}{2}\mathbb{Z} \right\}$$

whose complex dimension is given by

$$\dim_{\mathbb{C}} V(\mathbf{a},T) := \left| \left\{ k \,\middle|\, 0 \le k \le n,\ \frac{T}{a_k} \in \frac{\pi}{2}\mathbb{Z} \right\} \right|,$$

where $|S|$ denotes the cardinality of the set S. We notice that Σ_T is therefore isomorphic to the Brieskorn manifold $\Sigma(\mathbf{a}(T))$, where

$$\mathbf{a}(T) = (a_0, \cdots, \hat{a}_i, \cdots, a_n)$$

is a subset of $\mathbf{a}$. Here $\hat{a}_i$ means the term a_i is omitted, when $\frac{T}{a_i} \notin \frac{\pi}{2}\mathbb{Z}$. The differential of $\phi_{\mathbf{a}}$ at time t is given by

$$D\psi^t_{\mathbf{a}} = diag\big(e^{4it/a_0}, \cdots, e^{4it/a_n}\big)$$

It follows that

$$\ker(D_{\mathbf{z}}\psi^T_{\mathbf{a}}\big|_{T_{\mathbf{z}}\Sigma(\mathbf{a})} - id) = T_{\mathbf{z}}\Sigma(\mathbf{a}) \cap V(\mathbf{a},T) = T_{\mathbf{z}}\Sigma_T$$

Therefore Σ_T is Morse-Bott submanifold.

The calculation of the indices of all closed Reeb orbits can be found in various literature, see [KvK16], [Ust99]. We conclude this subsection with the following proposition:

Proposition 5.5 ([KvK16], [Fau16]). *Let $\gamma \in \Sigma(\mathbf{a})$ be a fractional Reeb of period t. We have*

$$\mu_{CZ}(\gamma) = \sum_{k=0}^{n} \left(\left\lfloor \frac{2t}{a_k\pi} \right\rfloor + \left\lceil \frac{2t}{a_k\pi} \right\rceil \right) - \left(\left\lfloor \frac{2t}{\pi} \right\rfloor + \left\lceil \frac{2t}{\pi} \right\rceil \right)$$

Proof. First we notice that the indices are canonically defined when $n \ge 4$, by Proposition 5.2. Recall the Reeb vector field in Proposition 5.3, $R_{\mathbf{a}} = 4i(\frac{z_0}{a_0}, \frac{z_1}{a_1}, \cdots, \frac{z_n}{a_n})$. The associated Reeb flow is

$$\psi^t_{\mathbf{a}}(\mathbf{z}) = (e^{\frac{4it}{a_0}} \cdot z_0, \cdots, e^{\frac{4it}{a_n}} \cdot z_n).$$

We regard this as a flow on $\mathbb{C}^{n+1}$ as opposed to $\Sigma(\mathbf{a})$. This perspective gives us the advantage of calculating the indices directly on $\mathbb{C}^{n+1}$. If we take the standard trivialization of $T\mathbb{C}^{n+1}$, then the linearized return map is

$$D\psi^t_{\mathbf{a}} = diag\big(e^{4it/a_0}, \cdots, e^{4it/a_n}\big) =: \Psi_t.$$

By Proposition 5.4, we have the trivialization of the symplectic complement $\xi^\perp_{\mathbf{a}}$. The linearized return map of the flow on $\xi^\perp_{\mathbf{a}}$ gives:

- $D\psi^t_{\mathbf{a}}(X_1(\mathbf{z})) = e^{4it} \cdot X_1(\psi^t_{\mathbf{a}}(\mathbf{z}))$,
- $D\psi^t_{\mathbf{a}}(Y_1(\mathbf{z})) = e^{4it} \cdot Y_1(\psi^t_{\mathbf{a}}(\mathbf{z}))$,
- $D\psi^t_{\mathbf{a}}(X_2(\mathbf{z})) = X_2(\psi^t_{\mathbf{a}}(\mathbf{z}))$,
- $D\psi^t_{\mathbf{a}}(Y_2(\mathbf{z})) = Y_2(\psi^t_{\mathbf{a}}(\mathbf{z}))$.

It follows that the linearized map of $D\psi^t_{\mathbf{a}}$ on $\xi^{\perp}_{\mathbf{a}}$ under the prescribed trivialization is the diagonal matrix:

$$\Psi^t_2 := \begin{bmatrix} e^{4it} & 0 \\ 0 & 1 \end{bmatrix}$$

A trivialization of $\xi_{\mathbf{a}}$ along the Reeb orbit gives us the linearization of Ψ^t_1 of $\psi^t_{\mathbf{a}}$ on $\xi_{\mathbf{a}}$. Any trivialization of $\xi_{\mathbf{a}}$ and $\xi^{\perp}_{\mathbf{a}}$ combined gives rise to a trivialization of $T\mathbb{C}^{n+1}$, which is homotopic to the standard one. Therefore by the product property of the Conley-Zehnder index and using remark 2.21, we find that

$$\begin{aligned}\mu_{CZ}(\gamma) = \mu_{CZ}(\Psi_1) =& \mu_{CZ}(\Psi) - \mu_{CZ}(\Psi_2) \\ =& \sum_{k=0}^{n}\left(\left\lfloor \frac{2t}{a_k\pi}\right\rfloor + \left\lceil \frac{2t}{a_k\pi}\right\rceil\right) - \left(\left\lfloor \frac{2t}{\pi}\right\rfloor + \left\lceil \frac{2t}{\pi}\right\rceil\right).\end{aligned}$$

□

Lemma 5.6. *Let* $\mathbf{a} = (a_0, a_1, a_2, \cdots, a_n)$*, where the* $a_i's$ *are positive integers, and* $\sum \frac{1}{a_k} \geq 1$*. Then the following function* $f_{\mathbf{a}} : \mathbb{R}_+ \to \mathbb{Z}$*,*

$$f_{\mathbf{a}}(x) = \sum_{k=0}^{n}\left(\left\lfloor \frac{x}{a_k}\right\rfloor + \left\lceil \frac{x}{a_k}\right\rceil\right) - \left(\lfloor x\rfloor + \lceil x\rceil\right)$$

has a minimum, denoted by $m(\mathbf{a})$*. In particular, if* $\mathbf{a} = (2, 2, 2, a_1, \cdots, a_n)$*, then* $m(\mathbf{a}) \geq 2$*, where* $a_k's$ *are positive integers,* $n \geq 2$*.*

Proof. We notice that $2x - 1 < \lfloor x\rfloor + \lceil x\rceil < 2x + 1$, we have

$$f_{\mathbf{a}}(x) > 2\big(\sum_{k=0}^{n}\frac{1}{a_k} - 1\big)x - n - 1 \geq -n - 1,$$

which proves the first part. For the second part, we have

$$f_{\mathbf{a}}(x) = 3\left(\left\lfloor \frac{x}{2}\right\rfloor + \left\lceil \frac{x}{2}\right\rceil\right) + \sum_{k=1}^{n}\left(\left\lfloor \frac{x}{p_k}\right\rfloor + \left\lceil \frac{x}{p_k}\right\rceil\right) - \left(\lfloor x\rfloor + \lceil x\rceil\right)$$

Note that $f_{\mathbf{a}}(x + 2) \geq f_{\mathbf{a}}(x) + 2$, so the minimum is obtained in $x \in (0, 2]$. On this interval, we have $f_{\mathbf{a}}(x) = 3\left(\left\lfloor \frac{x}{2}\right\rfloor + \left\lceil \frac{x}{2}\right\rceil\right) + n - \left(\lfloor x\rfloor + \lceil x\rceil\right)$, which is

$$f_{\mathbf{a}}(x) = \begin{cases} 2 + n & x \in (0, 1), \\ 1 + n & x = 1, \\ n \quad x \in (1, 2), \\ 4 + n & x = 2. \end{cases}$$

hence our conclusion. □

6 Exotic contact manifolds

6.1 Liouville domains admitting group actions

We need to find a Liouville domain (W, λ) with the contact manifold $\Sigma(\mathbf{a})$ as its boundary. While $V_{\mathbf{a}}(0)$ has a singularity at the origin, $V_{\mathbf{a}}(\epsilon)$ is smooth. Therefore we will follow Alexander Fauck's approach [Fau16] to overcome this by constructing an interpolation between $V_{\mathbf{a}}(0)$ and $V_{\mathbf{a}}(\epsilon)$. First, we choose a smooth monotone decreasing cut-off function $\beta \in C^{\infty}(\mathbb{R})$ with $\beta(x) = 1, x \leq \frac{1}{4}$ and $\beta(x) = 0, x \geq \frac{3}{4}$. Then we define (we will often omit $\mathbf{a}$)

$$U_{\mathbf{a}}(\epsilon) := \{\mathbf{z} \in \mathbb{C}^{n+1} | z_0^{a_0} + \cdots + z_n^{a_n} = \epsilon \cdot \beta(||\mathbf{z}||^2)\}.$$

Let

$$W_{\epsilon}^{s} := U_{\epsilon} \cap B(s)$$

we have

Proposition 6.1 (Proposition 99 [Fau16]). *For sufficiently small* ϵ,$(X_{\epsilon}^{1}, \lambda)$ *is a Liouville domain with boundary* $(\Sigma(\mathbf{a}), \alpha_{\mathbf{a}})$ *and vanishing first Chern class.*

Moreover, we have a cyclic group

$$C(L) := \{\, e^{\frac{2\pi k i}{L}} \in \mathbb{C} \,|\, k \in \mathbb{Z} \,\} =< \zeta >$$

acting on $(\mathbb{C}^{n+1})^{*}$, which is generated by :

$$\zeta_{*} : (\mathbb{C}^{n+1})^{*} \longrightarrow (\mathbb{C}^{n+1})^{*}$$
$$(z_0, z_1, \cdots, z_n) \mapsto (z_0\zeta^{b_0}, z_1\zeta^{b_1}, \cdots, z_n\zeta^{b_n})$$

where $L := \operatorname{lcm} a_j, b_j := L/a_j, \zeta := e^{\frac{2\pi i}{L}}$. We can easily see that the 1-form $\lambda_{\mathbf{a}}$ is $C(L)$-invariant.

We can restrict this group action to the subsets of $(\mathbb{C}^{n+1})^{*}$ mentioned above and obtain a $C(L)-$action on the manifolds X_{ϵ}^{s} and W_{ϵ}^{s}. By definition, $X_{\epsilon}^{1/2} = U(\epsilon) \cap B(1/2) = V(\epsilon) \cap B(1/2) = W_{\epsilon}^{1/2}$. We have the following proposition:

Proposition 6.2. *For sufficiently small* $\epsilon > 0$, *there is a* $C(L)$*-equivariant isotopy between the following pairs of Liouville domains:*

- X_{ϵ}^{1} *and* $X_{\epsilon}^{1/2}$,
- W_{ϵ}^{1} *and and* $W_{\epsilon}^{1/2}$.

Proof. We only give a proof for the existence of a $C(L)$-equivariant isotopy between W_{ϵ}^{1} and $W_{\epsilon}^{1/2}$. We can prove the same results for X_{ϵ}^{1} and $X_{\epsilon}^{1/2}$ verbatim. Consider the function $\rho(\mathbf{z}) = ||\mathbf{z}||^2$ on V_{ϵ}. If for sufficiently small ϵ, the critical values of ρ restricted to W_{ϵ}^{1} are less that $1/4$, then we are done, by lemma 6.3. Indeed, we have $f_{\epsilon}(\mathbf{z}) := f(\mathbf{z}) - \epsilon \cdot \beta(||\mathbf{z}||^2)$ on $\mathbb{C}^{n+1}$ and its differential is given by

$$Df_{\epsilon} = Df - \epsilon \cdot \beta^{'}(||\mathbf{z}||^2) \cdot D\rho$$

so the map

$$(f_\epsilon, \rho) : \mathbb{C}^{n+1} \to \mathbb{C} \times \mathbb{R}$$

has Jacobian matrix $(Df - \epsilon \cdot \beta'(||\mathbf{z}||^2) \cdot D\rho, D\rho)$, which has the same rank as $(Df, D\rho)$. So by the same argument in the proof of lemma 5.1, if $\mathbf{z}$ is a point where the Jacobian is not full rank, then we have for some complex number λ, $\bar{z_k} = \lambda a_k z_k^{a_k - 1}$ for all k. For $||\mathbf{z}|| \geq 1/2$, we have $|z_{k_0}| \geq \frac{1}{2\sqrt{n}}$ for some k_0, so

$$|z_{k_0}| = |\lambda| \cdot a_{k_0} \cdot |z_{k_0}|^{a_{k_0} - 1}$$

i.e,

$$|\lambda| = \frac{|z_{k_0}|^{2-a_{k_0}}}{a_{k_0}} \leq \frac{(2\sqrt{n})^{a_{k_0}-2}}{a_{k_0}} \leq C(\mathbf{a}) \tag{6.1}$$

where $C(\mathbf{a}) := \max\limits_{0 \leq k \leq n} \{\frac{(2\sqrt{n})^{a_k - 2}}{a_k}\}$ only depends on $\mathbf{a}$ and n. Meanwhile, we have

$$\sum_{k=0}^{n} \frac{z_k \bar{z}_k}{a_k} = \lambda \sum_{k=0}^{n} z_k^{a_k} = \lambda \cdot f(\mathbf{z}) = \lambda \cdot \epsilon\beta(||\mathbf{z}||^2) \tag{6.2}$$

Combining equations (6.2) and (6.1), we have

$$\sum_{k=0}^{n} \frac{z_k \bar{z}_k}{a_k} = \lambda \cdot \epsilon\beta(||\mathbf{z}||^2) \leq \epsilon \cdot C(\mathbf{a}) \tag{6.3}$$

On the other,

$$\sum_{k=0}^{n} \frac{z_k \bar{z}_k}{a_k} \geq \frac{1}{\max\{a_j\}} \sum_{k=0}^{n} z_k \bar{z}_k \geq \frac{1}{\max\{a_j\}} \cdot ||\mathbf{z}||^2 = \frac{1}{4\max\{a_j\}} \tag{6.4}$$

Equations (6.4) and (6.3) cannot hold for sufficiently small ϵ at the same time, and therefore the function ρ has no critical points in $||\mathbf{z}|| \geq 1/2$, hence all critical values are less than $1/4$. □

Lemma 6.3 (Theorem 2.2.2 [Nic11]). *Suppose finite group G acts on a manifold M and f is a G-invariant exhausting function on M. Moreover, assume that no critical value of f is contained in $[a, b] \subset \mathbb{R}$, then there is a G-equivariant isotopy ϕ_t between the sublevel sets $M^a := f^{-1}((-\infty, a])$ and $M^b := f^{-1}((-\infty, b])$, and ϕ_t coincides with Id outside a compact set.*

Proof. Since there are no critical values of f in [a, b] and the sublevel sets are compact, we deduce that there exists $\epsilon > 0$ such that

$$\{a - \epsilon < f < b + \epsilon\} \subset M \setminus Crit(f).$$

First we fix a gradient-like G-invariant vector field Y and construct a compactly supported G-equivariant smooth function

$$g : M \to [0, \infty)$$

such that

$$g(x) = \begin{cases} \frac{1}{|Yf|}, & a \leq f(x) \leq b, \\ 0, & f(x) \notin (a - \epsilon, b + \epsilon). \end{cases}$$

We can now construct a G-invariant vector field $X := gY$ on M and we denote by

$$\phi : \mathbb{R} \times M \to M, \quad (t, x) \to \phi_t(x)$$

the flow generated by X. Clearly the flow commutes with the group action, so ϕ_t is G-equivariant. If $u(t)$ is an integral curve of X, then differentiating f along $u(t)$ in the region $\{a \le f \le b\}$ and get

$$\frac{df}{dt} = Xf = \frac{1}{Yf} Yf = 1$$

This implies

$$\phi_{b-a}(M^a) = M^b$$

and ϕ_t is identity outside the region $\{a - \epsilon < f < b + \epsilon\}$. □

Remark 6.4. By proposition 6.1, $(X_\epsilon^{1/2}, \Phi_t^* \lambda)$ is a family of $C(L)-$equivariant Liouville structures. Then by corollary 2.5, we have $(X_\epsilon^{1/2}, \lambda)$ is $C(L)-$ equivariant Liouville isomorphic to (X_ϵ^1, λ). By the same token, (W_ϵ^1, λ) is $C(L)-$ equivariant Liouville isomorphic to $(W_\epsilon^{1/2}, \lambda)$ and therefore to (X_ϵ^1, λ).

Let $\phi_t(\mathbf{z}) := \frac{1}{8} \sum_{j=0}^{n} c_j(t)|z_j|^2$, where $c_j(t)$ is a linear interpolation such that $c_j(0) = 1, c_j(1) = a_j$. It's easy to check that ϕ_t is plurisubharmonic on $V_\mathbf{a}(\epsilon)$. Indeed, ϕ_t is i-convex on $\mathbb{C}^{n+1}$ since $\Delta\phi_t > 0$ and $V_\mathbf{a}(\epsilon)$ is a smooth complex submanifold. So $(V_\mathbf{a}(\epsilon), i, \phi_t)$ are $C(L)-$ equivariant Stein manifolds.

Since $X_\epsilon^1 = \phi_0^{-1}((-\infty, 1/8])$, $(X_\epsilon^1, i, \phi_0)$ is a $C(L)-$equivariant Stein domain. Seen as a Liouville domain, $(X_\epsilon^1, -d^{\mathbb{C}}\phi_0)$ is $C(L)-$equivariant Liouville isomorphic to (X_ϵ^1, λ) as follows:

Proposition 6.5. *There is a $C(L)$-equivariant Liouville homotopy between $(X_\epsilon^1, -d^{\mathbb{C}}\phi_0)$ and (X_ϵ^1, λ), for sufficiently small ϵ.*

Proof. Notice that for $\lambda = -d^{\mathbb{C}}\phi_1$, it suffices to prove the critical points of ϕ_t are contained in a compact set $\{||\mathbf{z}|| \le 1/3\}$, then $\nabla_{\phi_t}\phi_t$ will be transversal to the boundary, and $-d^{\mathbb{C}}\phi_t$ will be a family of $C(L)-$equivariant Liouville structures on X_ϵ^1, so we can conclude the result by corollary 2.5. In the following we are going to prove that all critical points satisfy $||\mathbf{z}|| \le 1/3$. Consider the map

$$(f, \phi_t) : \mathbb{C}^{n+1} \to \mathbb{C} \times \mathbb{R}$$

Its Jacobian matrix is

$$D(f, \rho) = \begin{bmatrix} a_0 z_0^{a_0-1} & \cdots & a_n z_n^{a_n-1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_0 \bar{z_0}^{a_0-1} & \cdots & a_n \bar{z_n}^{a_n-1} \\ \frac{1}{8}c_0(t)\bar{z_0} & \cdots & \frac{1}{8}c_n(t)\bar{z_n} & \frac{1}{8}c_0(t)z_0 & \cdots & \frac{1}{8}c_n(t)z_n \end{bmatrix}$$

If $\mathbf{z}$ is a point where this matrix has rank smaller than 3, there exists a non-zero complex number $\lambda \in \mathbb{C}$ such that $\frac{c_k(t)}{8}\bar{z_k} = \lambda a_k z_k^{a_k-1}$ for all k and

$$\sum_{k=0}^{n} \frac{c_k(t) z_k \bar{z_k}}{8 a_k} = \lambda \sum_{k=0}^{n} z_k^{a_k} = \lambda \cdot f(\mathbf{z}) = \lambda \cdot \epsilon \tag{6.5}$$

For $||\mathbf{z}|| > \frac{1}{3}$, we have $|z_r| > \frac{1}{3(n+1)}$ for some $0 \leq r \leq n$. So we have

$$\frac{c_r(t)}{8} \cdot |\bar{z}_r| = |\lambda| \cdot a_r \cdot |z_r^{a_r - 1}|$$

i.e,

$$|\lambda| = \frac{c_r(t)}{8a_r|z_r|^{a_r-2}} < \frac{(3(n+1))^{a_r-2}}{8} \leq C$$

where $C = \max\limits_{0\leq i\leq n} \frac{(3(n+1))^{a_i-2}}{8}$, only depends on $\mathbf{a}$. On one hand,we have

$$\sum_{k=0}^{n} \frac{c_k(t) z_k \bar{z}_k}{8a_k} = \lambda \sum_{k=0}^{n} z_k^{a_k} = \lambda \cdot f(\mathbf{z}) = \lambda \cdot \epsilon < C \cdot \epsilon. \tag{6.6}$$

On the other hand, we have

$$\sum_{k=0}^{n} \frac{c_k(t) z_k \bar{z}_k}{8a_k} \geq \sum_{k=0}^{n} \frac{|z_k|^2}{8a_k} \geq \frac{1}{72 \max\limits_{0\leq i\leq n}\{a_i\}}. \tag{6.7}$$

So for ϵ small enough, equations (6.6) and (6.7) cannot both hold, which implies the critical points of ϕ_t is contained in $\{||\mathbf{z}|| \leq 1/3\}$.

□

Remark 6.6. Since we have $\phi_0(\mathbf{z}) = \frac{||\mathbf{z}||^2}{8}$, $\nabla_{\phi_0}\phi_0 = \sum\limits_{i=0}^{n}(z_i\partial\bar{z}_i + \bar{z}_i\partial z_i)/2$ is complete in $\mathbb{C}^{n+1}$. Therefore ϕ_0 is a completely exhausting function on $V_{\mathbf{a}}(\epsilon)$. By the proof of Proposition 6.5, all critical points of ϕ_0 are in the interior of X_ϵ^1. It follows that $V_{\mathbf{a}}(\epsilon)$ is the completion of X_ϵ^1 by matching the corresponding trajectories of the Liouville fields.

6.2 Topology of manifolds M_0 and M_1

Now let's consider $C(L)$-equivariant Stein manifold $(\mathbb{C}^*, i, (\log|z|)^2/2)$ where the $C(L)-$action is multiplication given by

$$\mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z}) \longrightarrow \mathbb{C}^*, \quad (r, \theta) \mapsto e^{r+\theta i}.$$

The map gives rise to polar coordinates form of the same Stein manifold $(\mathbb{R} \times S^1, j, r^2/2)$ and the Liouville vector is $r\partial_r$, which is complete.

Now we consider the product of the Stein manifolds $(\mathbb{C}^*, i, (\log|z|)^2/2)$ and $(V_{\mathbf{a}}(\epsilon), i, \phi_0)$. It has a free $C(L)$ action as follows:

$$\begin{aligned} \zeta_* : V_{\mathbf{a}}(\epsilon) \times \mathbb{C}^* &\longrightarrow V_{\mathbf{a}}(\epsilon) \times \mathbb{C}^* \\ (z_0, z_1, \cdots, z_n, \eta) &\mapsto (z_0\zeta^{b_0}, z_1\zeta^{b_1}, \cdots, z_n\zeta^{b_n}, \eta\zeta), \end{aligned}$$

where $b_i = L/a_i, \zeta \in C(L)$. The product function $\phi := (\log|z|)^2/2) + \phi_0$ is a completely exhausting $J-$convex Morse function, and the product Stein manifold is of finite type. By abuse of the notation, we use ϕ to denote the function on the quotient manifold as well. Also,

$M_0 := \{\phi \leq C\}$ is a Stein domain, where C is greater than all critical values of ϕ. Hence the completion $\widehat{M_0(\mathbf{a})} = (V_{\mathbf{a}}(\epsilon) \times \mathbb{C}^*)/C(L)$ since ϕ is complete. Oftentimes we will suppress $\mathbf{a}$. If we consider the Weinstein structure instead, the Weinstein domain can be cut out in other ways, as stated in the following lemma:

Lemma 6.7. *Suppose (W, λ, ϕ) is a finite type Weinstein manifold. Let $\psi : W \to \mathbb{R}$ be an exhausting Morse function. Suppose X_λ is nondegenerate and gradient-like for ψ outside $\{\psi \leq 0\}$. Then $\{\psi \leq 0\}$ together with λ is Liouville homotopic to a Weinstein domain $W_1 := \{\phi \leq K\}$, for K sufficiently large.*

Proof. Let K satisfy

$$\{\psi \leq 0\} \subset W_1 \subset W_2 := \{\psi \leq C\}$$

for some large enough C (conditions will be evident along the line of proof). Notice that $\{\psi \leq 0\}$ is Liouville homotopic to W_2. Fix a smooth function ρ (it can be constructed on the level sets of ϕ) such that

- $\rho = 1$ in W_1, $\rho = 0$ outside W_2.
- $X_\lambda(\rho) \leq 0$.

Let $M := \max_{p \in W_2 \setminus W_1} (\phi - \psi)$. Now consider the function $f = \rho\phi + (1-\rho)(\psi + M)$. We will show that f is Morse and X_λ is gradient-like for f. We only need to verify X is gradient-like in $W_2 \setminus W_1$. We have

$$X_\lambda(f) = \rho X_\lambda(\phi) + (1-\rho)X_\lambda(\psi) + (\phi - \psi - M)(X_\lambda(\rho)) \geq \rho X_\lambda(\phi) + (1-\rho)X_\lambda(\psi) > 0$$

So X_λ is gradient-like for f and f doesn't have new critical points outside W_1. Because $f|_{W_1} = \phi|_{W_1}$, f is Morse. Hence (λ, f) is also a Weinstein structure on W_2, and a linear interpolation between f and ϕ gives rise to a family of Weinstein structures. In particular, it gives rise to a Liouville homotopy. □

In fact, we have an explicit form for the topology of M_0. The following quotient map

$$\pi : V_{\mathbf{a}}(\epsilon) \times \mathbb{C}^* \to \mathbb{C}^{n+1} \setminus V_{\mathbf{a}}(0)$$
$$(z_0, z_1, \cdots, z_n, t) \mapsto (z_0 t^{b_0}, z_1 t^{b_1}, \cdots, z_n t^{b_n})$$

coincides with the $C(L)-$ action quotient.

Therefore $\widehat{M_0(\mathbf{a})}$ (hence M_0) is diffeomorphic $\mathbb{C}^{n+1} \setminus V_{\mathbf{a}}(0)$.We have the following proposition about M_0:

Proposition 6.8. *Let $M_0(\mathbf{a})$, $n \geq 3$ be the manifold defined above. Then $\pi_1(M_0) = \mathbb{Z}, H_i(M_0) = 0, i \geq 2, i \neq n, n+1$.*

Proof. It suffices to prove the results for $\mathbb{C}^{n+1} \setminus V_{\mathbf{a}}(0)$. We have a deformation retraction

$$r : \mathbb{C}^{n+1} \setminus V_{\mathbf{a}}(0) \to S^{2n+1} \setminus \Sigma(\mathbf{a}),$$

and we have the Milnor fibration:

$$S^{2n+1} \setminus \Sigma(\mathbf{a}) \longrightarrow S^1$$
$$(z_0, z_1, \cdots, z_n) \mapsto \frac{f(\mathbf{z})}{||f(\mathbf{z})||}$$

The fibers are homotopic to a bouquet of $n-$spheres, which is simply connected since $n \geq 3$, the long exact sequence gives us $\pi_1(M_0) = \mathbb{Z}$. Meanwhile, $H_*(M_0) = H_*(S^{2n+1} \setminus \Sigma(\mathbf{a}))$, and for $1 < i < 2n$, by Alexander duality we have

$$\tilde{H}_i(S^{2n+1} \setminus \Sigma(\mathbf{a})) = \tilde{H}^{2n-i}(\Sigma(\mathbf{a})).$$

The conclusion follows Theorem 5.2. □

Proposition 6.9. *Let M_0 be a manifold with $\pi_1(M_0) = \mathbb{Z}, H_i(M_0) = 0, i \geq 2, i \neq n, n+1$. Suppose γ is a generator for $\pi_1(M_0)$ and M_1 is the result of attaching a 2-handle along γ. Then $\tilde{H}_i(M_1) = 0, i \neq n, n+1$.*

Proof. The attaching 2-handle kills the generator $[\gamma]$ so $\pi_1(M_1) = 0$. Meanwhile, $H_i(M_0) = 0, i \geq 2, i \neq n, n+1$ implies $H_k(M_1) = 0, k \geq 3, k \neq n, n+1$ since attaching a 2-handle does not change higher homology. Let's denote the 2-handle by H. We have the Mayer-Vietoris sequence:

$$\cdots \to H_2(H) \oplus H_2(M_0) \to H_2(M_1) \to H_1(M_0 \cap H) \xrightarrow{i_*} H_1(H) \oplus H_1(M_0) \to H_1(M_1) \to \cdots$$

Here $[\gamma]$ is the generator of both $H_1(M_0 \cap H)$ and $H_1(M_0)$, so i_* is isomorphism. Hence we have

$$\cdots \to 0 \to H_2(M_1) \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \to 0 \to \cdots$$

So $H_2(M_1) = 0$. The conclusion follows. □

6.2.1 Handle attachment and trivialization

Now we need to fix a trivialization of the canonical bundle $\kappa_{\widehat{M_0}}$ of $(T\widehat{M}_0, J)$. Since we have the $C(L)-$ equivariant quotient map $V_{\mathbf{a}}(\epsilon) \times \mathbb{C}^* \to \widehat{M}_0$, it suffices to fix $C(L)-$ trivializations on both $V_{\mathbf{a}}(\epsilon)$ and $\mathbb{C}^*$ since

$$T(V_{\mathbf{a}}(\epsilon) \times \mathbb{C}^*) = TV_{\mathbf{a}}(\epsilon) \times T\mathbb{C}^*$$

Notice that the trivialization of the symplectic complement in Proposition 5.4 is $C(L)-$equivariant, and the standard trivialization of $T\mathbb{C}^{n+1}$ is also $C(L)-$equivariant, as long as $\sum_{i=0}^{n} \frac{1}{a_i} \in \mathbb{Z}$. Indeed, if we take $\Omega = dz_0 \wedge dz_2 \wedge \cdots \wedge dz_n$, then the $C(L)-$actions on Ω is

$$\eta^*(\Omega) = e^{\frac{2\pi i}{a_0}} dz_0 \wedge \cdots \wedge e^{\frac{2\pi i}{a_n}} dz_n = e^{2\pi i \sum \frac{1}{a_i}} \Omega = \Omega.$$

Therefore a $C(L)-$equivariant trivialization of $TV_{\mathbf{a}}(\epsilon)$ exists. Since $V_{\mathbf{a}}(\epsilon)$ is simply connected, the trivialization of $TV_{\mathbf{a}}(\epsilon)$ is homotopically unique. We will take the natural trivialization of $T\mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}$, which determines the trivialization Φ of $T(V_{\mathbf{a}}(\epsilon) \times \mathbb{C}^*)$. We will also fix Φ for the rest of this paper, which will be crucial in two places:

- Determining the framing for the Weinstein 2-handle attachment in Proposition 6.10.
- Determining the trivialization for the calculation of Conley-Zehnder index in Proposition 6.21.

Proposition 6.10. *There is a contractible Weinstein domain* $(M_1, \omega_1, X_1, \psi_1)$ *obtained from the Weinstein domain* $(M_0, -dd^{\mathbb{C}}\phi, \nabla_\phi \phi, \phi)$ *by attaching a 2-handle such that the canonical saturation (see Subsection 2.8.2) coincides with the trivialization* Φ.

Proof. If we can find an isotropic circle in M_0 which generates the fundamental group, then by Theorem 2.33, we can attach a Weinstein handle in such a way that the trivialization of the contact structure extends to the Weinstein handle body. The existence of such an isotropic circle is guaranteed by the $h-$principle in lemma 6.11, which states a subcritical embedding can be perturbed into an isotropic embedding. □

Let M be a contact manifold of dimension $2n+1$ and V a smooth manifold of subcritical dimension, i.e. $\dim V \leq n$. Let $Mono^{emb}$ be the space of monomorphisms $TV \to TM$ which cover embeddings $V \to M$, and $Mono^{emb}_{isot}$ its subspace which consists of isotropic monomorphisms $F : TV \to TM$. Let $Mono^{emb}_{isot}$ be the space of homotopies

$$Mono^{emb}_{isot} = \{F_t, t \in [0,1] | F_t \in Mono^{emb}, F_0 = df_0, F_1 \in Mono^{emb}_{isot}\}.$$

The space Emb_{isot} of isotropic embeddings $V \to M$ can be viewed as a subspace of $Mono^{emb}_{isot}$. Indeed, we can associate to $f \in Emb_{isot}$ the homotopy $F_t \equiv df, t \in [0,1]$, in $Mono^{emb}_{isot}$.

Lemma 6.11 (Proposition 12.4.1 [EM02]). *The inclusion*

$$Emb_{isot} \hookrightarrow Mono^{emb}_{isot}$$

is a homotopy equivalence.

The above $h-$principle also holds in the relative and $\mathcal{C}^0-$dense forms.

Remark 6.12. By Theorem 2.7, $(M_1, \omega_1, X_1, \psi_1)$ is homotopic to a Stein domain through Weinstein structures. We denote the Stein structure by the same notation (M_1, J_1, ϕ_1).

6.3 The Weinstein domain M_0

We notice $(V_{\mathbf{a}}(\epsilon), -d^{\mathbb{C}}\phi_0) = (\widehat{X^1_\epsilon}, -d^{\mathbb{C}}\phi_0)$ while $(X^1_\epsilon, -d^{\mathbb{C}}\phi_0)$ is $C(L)-$equivariant Liouville homotopic to (W^1_ϵ, λ), and in light of lemma 6.7, we can define different Weinstein domains in $(\widehat{W^1_\epsilon} \times \mathbb{C}^*, \lambda_0 := \lambda + rd\theta)$ by different functions.

First of all, we need the following technical proposition.

Proposition 6.13. *Let* (M, λ) *be a* G*-equivariant Liouville domain,and* R *be the coordinate for its cylindrical end. Assume* ϕ *is a* G*-equivariant Morse function on* M *such that* $X(\phi) < 0$ *near the boundary of* M. *Then for any* $\epsilon > 0$, *there exists* $\delta_1 \gg \delta_2 > 0$ *and a* G*-equivariant Morse function* f *(see Figure 6) such that:*

- $||1 - f||_{\mathcal{C}^2} < \epsilon$ *in the region* $M \setminus \{R > 1 - \delta_1 + 2\delta_2\}$.

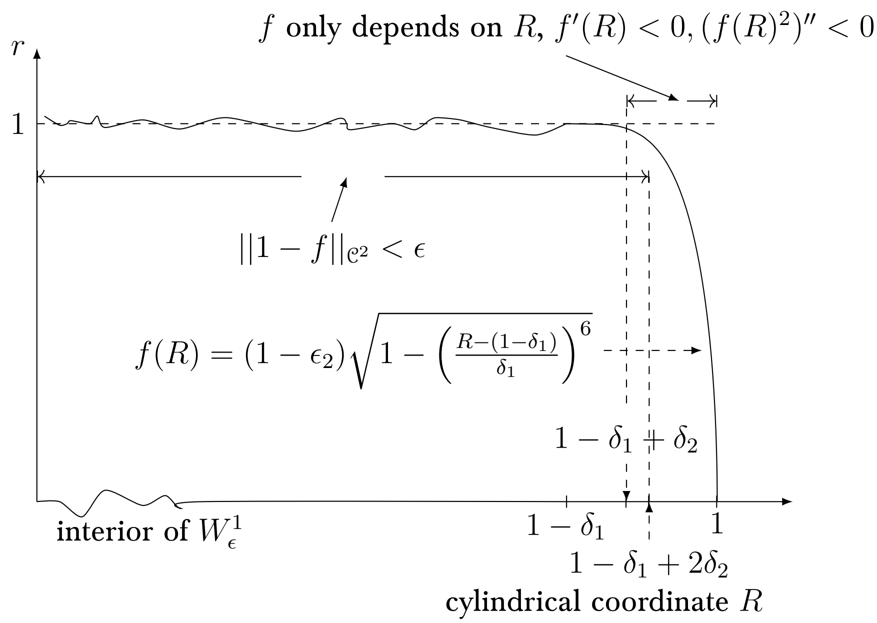

Figure 6: $G-$equivariant Morse function f.

- *f and ϕ have same set of critical points, and the Morse indices are the same.*
- *f satisfies the equation*

$$\left(\frac{f}{a}\right)^2 + \left(\frac{R-(1-\delta_1)}{\delta_1}\right)^6 = 1 \tag{6.8}$$

on the region $1-\delta_1+\delta_2 < R \leq 1$, *for some* $0 < a < 1$.

Proof. First, we can fix the canonical collar of the boundary using the negative Liouville flow

$$\iota : (1-\epsilon_1, 1] \times \partial M \longrightarrow M$$
$$\iota^*\lambda = R\lambda, \quad \iota^* X = R\partial_R$$

where $\epsilon_1 > 0$ is sufficiently small, so that $X(\phi) < 0$ in the canonical collar and R is the cylindrical coordinate. Notice that R is G-equivariant and so is any function in R.

Now let's fix a sufficiently small $\epsilon_1 > \delta_1 \gg \delta_2 \gg \epsilon_2 > 0$ (the exact constraints on $\delta_1, \delta_2, \epsilon_2$ will be clear along the proof), and an increasing bump function ρ such that $\rho(R) = 1$ for $R \geq 1$ and $\rho(R) = 0$ for $R \leq 0$. Let $\hat{\rho}(R) := \rho(\frac{R-(1-\delta_1)}{\delta_2})$, then we have

$$||\hat{\rho}||_{\mathcal{C}^2} \leq \frac{1}{\delta_2^2}||\rho||_{\mathcal{C}^2} \tag{6.9}$$

Define a bump function $\hat{\rho}$ on M to be $\hat{\rho(R)}$ on its canonical collar and extended by 0. Apparently $\hat{\rho}$ is G-equivariant. Let $h > 0$ be a function of radial coordinate on $[1-\delta_1, 1] \times \partial M$ satisfying the conditions:

$$\left(\frac{h}{1-\epsilon_2}\right)^2 + \left(\frac{R-(1-\delta_1)}{\delta_1}\right)^6 = 1.$$

Then h can be extended to a smooth function on M. Without loss of generality, we can assume $||1-\phi||_{\mathcal{C}^2} < \epsilon_2$. Otherwise we can simply replace ϕ by $1+c\phi$ for $c>0$ sufficiently small. We claim the function

$$f = \phi \cdot (1-\hat{\rho}) + h\hat{\rho}$$

satisfies all conditions in this proposition. Firstly, h is well-defined and G-equivariant, and since f coincides with h on the region $\{R \geq 1-\delta_1+\delta_2\}$, equation 6.8 is satisfied.

Secondly, we only need to show that f has no critical points in the region $\{1-\delta_1 \leq R \leq 1-\delta_1+\delta_2\}$, for which we have

$$\partial_R(f) = h'\hat{\rho} + h\hat{\rho}' + (1-\hat{\rho})\partial_R(\phi) - \phi\hat{\rho}' = (h-\phi)\hat{\rho} + h'\hat{\rho} + (1-\hat{\rho})\partial_R(\phi) < 0$$

since $R\partial_R(\phi) = X(\phi) < 0$ and $h \leq 1-\epsilon_2 \leq \phi$. Therefore f has no critical point in the canonical collar. Since outside the canonical collar $f \equiv \phi$, the second condition follows.

Now we show that f also satisfies the first condition. In the region $M \setminus \{R > 1-\delta_1\}$, we have $f \equiv \phi$, so we only need to check the region $\{1-\delta_1 \leq R \leq 1-\delta_1+2\delta_2\}$, where

$$\begin{aligned}
||f-1||_{\mathcal{C}^2} &= ||(\phi-1) + (h-\phi)\hat{\rho}||_{\mathcal{C}^2} \\
&\leq ||\phi-1||_{\mathcal{C}^2} + ||(\phi-h)\hat{\rho}||_{\mathcal{C}^2} \\
&\leq \epsilon_2 + 2||(\phi-1)+(1-h)||_{\mathcal{C}^2} \cdot ||\hat{\rho}||_{\mathcal{C}^2} \\
&\leq \epsilon_2 + \frac{1}{\delta_2^2}(||\phi-1||_{\mathcal{C}^2} + ||1-h||_{\mathcal{C}^2})||\rho||_{\mathcal{C}^2} \\
&\leq \epsilon_2 + \frac{1}{\delta_2^2}(\epsilon_2 + ||1-h||_{\mathcal{C}^2})||\rho||_{\mathcal{C}^2}
\end{aligned}$$

The Taylor expansion of $1-h$ at $R = 1-\delta_1$ is:

$$1 - h((1-\delta_1)+t) = \epsilon_2 + Ct^6 + o(t^{11}), \quad C = \frac{1-\epsilon_2}{2\delta_1^6}$$

Therefore $||1-h||_{\mathcal{C}^2} \leq \epsilon_2 + C_1\delta_2^6$, for $t < 2\delta_2 \ll \delta_1$, where $C_1 = C_1(\delta_1)$. Thus we have

$$||1-f||_{\mathcal{C}^2} \leq \epsilon_2 + \frac{2\epsilon_2 + C_1\delta_2^4}{\delta_2^2} < \epsilon$$

The last inequality holds as long as $\epsilon_2 \leq \delta_2^4$ and $\delta_2 \ll \delta_1$. □

Lemma 6.14. *Let f, ρ be defined as above, $g := \hat{\rho}(\delta_2 - R)$. Then $g \cdot X_f$ is $\mathcal{C}^1$ small, where X_f is the Hamiltonian vector field of f with respect to $d\lambda_0$.*

Proof. We only need to prove this in $\{1-\delta_1+\delta_2 < R < 1-\delta_1+2\delta_2\}$. Notice that

$$||X_f||_{\mathcal{C}^1} \leq ||1-f||_{\mathcal{C}^2} < K\delta_2^2,$$

where K is independent of δ_2. Meanwhile, we have

$$\begin{aligned}||g\cdot X_f||_{\mathcal{C}^1} &\le |g|\cdot||X_f|| + ||dg||\cdot||X_f|| + |g|\cdot||dX_f|| \\ &\le (|g| + ||dg||)\cdot(||X_f|| + ||dX_f||) \\ &\le ||g||_{\mathcal{C}^1}\cdot||X_f||_{\mathcal{C}^1} \\ &\le \frac{||\rho||_{\mathcal{C}^1}}{\delta_2}\cdot K\delta_2^2 \\ &\le K'\delta_2\end{aligned}$$

□

Suppose G is a finite group and M is a $G-$manifold. Let $\mathcal{M}^G(M,\mathbb{R})$ denote the set of $G-$equivariant Morse functions on M and $C(M,\mathbb{R})$ the set of smooth functions.

Lemma 6.15 (Density Lemma 4.8 [Was69]). *$\mathcal{M}^G(M,\mathbb{R})$ is dense in $C(M,\mathbb{R})$ with respect to the C^k topology.*

Remark 6.16. Note that (W^1_ϵ, λ) is $G-$equivariantly Liouville isomorphic to $(X^1_\epsilon, -d^{\mathbb{C}}\phi_0)$. Since ϕ_0 is $i-$convex on X^1_ϵ (and we can perturb it into a $G-$equivariant Morse function if necessary), the index of each critical point of $-\phi_0$ is at least n (half of the dimension of a Stein Manifold). Therefore we can find such function ϕ' on W^1_ϵ as well.

Apply proposition 6.13 to (W^1_ϵ, λ), with ϕ' as in remark 6.16. Then consider the function F on the product Liouville manifold $(\widehat{W^1_\epsilon}\times\mathbb{R}\times S^1, \lambda_0 := \lambda + rd\theta)$ defined as:

$$F : \widehat{W^1_\epsilon}\times\mathbb{R}\times S^1 \to \mathbb{R}, \tag{6.10}$$

$$(p,(r,\theta)) \mapsto r^2 - f(p)^2 \quad \text{for} \quad p\in W^1_\epsilon \tag{6.11}$$

$$((q,R),(r,\theta) \mapsto r^2 - a^2\left(1-\left(\frac{R-(1-\delta_1)}{\delta_1}\right)^6\right) \quad \text{for} \quad (q,R)\in\partial W^1_\epsilon\times(1-\delta_1+2\delta_2,\infty). \tag{6.12}$$

where $a = 1-\epsilon_2$. It is easy to check that F is a smooth $C(L)$-equivariant function on $\widehat{W^1_\epsilon}\times\mathbb{R}\times S^1$, and 0 is a regular value. Furthermore, the following lemma shows that the Liouville vector filed $Y := Y_\lambda + r\partial_r$(where Y_λ is the Liouville field on (W^1_ϵ,λ)) is gradient-like for F on $\{F\ge 0\}$.

Lemma 6.17. *The Liouville vector field Y of $(\widehat{W^1_\epsilon}\times\mathbb{R}\times S^1, \lambda + rd\theta)$ is gradient-like for F outside $W_0 := \{F\le 0\}$.*

Proof. We will verify the statement on the regions $\{R > 1-\delta_1+\delta_2\}$ and $\{R > 1-\delta_1+2\delta_2\}^c$ separately. In the region $\{R > 1-\delta_1+\delta_2\}$, $Y = r\partial_r + R\partial_R$ with $F = r^2 - a^2\left(1-\left(\frac{R-(1-\delta_1)}{\delta_1}\right)^6\right)$, the claim is trivial. In the region $\{R > 1-\delta_1+2\delta_2\}^c$, we have $Y = r\partial_r + Y_\lambda$. Notice that this region is a product $W' \times (\mathbb{R} \times S^1)$, where $W' = \widehat{W^1_\epsilon} \setminus \{R > 1 - \delta_1 + 2\delta_2\}$ is W^1_ϵ attached with a cylindrical cobordism. Now,

$$Y(F) = (r\partial_r + Y_\lambda)(r^2 - f^2) = 2(r^2 - fY_\lambda(f)) \tag{6.13}$$

Since $1 - f$ is $\mathcal{C}^2$ small, the coordinate r is nonzero in the region $W^c_0 \cap \{R > 1 - \delta\}^c$, and W' is compact, we have $2(r^2 - fY_\lambda(f)) > 0$. The conclusion follows. □

Remark 6.18. Note that $(\{F \leq 0\}, \lambda + rd\theta)$ is a $C(L)$-equivariant Liouville domain, and $C(L)$ acts freely on it. The quotient domain is Liouville homotopic to the Stein domain (M_0, J, ϕ), by Lemma 6.7. Since the properties of interest are invariant under Liouville isomorphism, we will also denote the quotient domain$(\{F \leq 0\}, \lambda_0)/C(L)$ by (M_0, λ_0).

Remark 6.19. The region $(U := \{R > 1/2\}^c \cap \{|r| \leq 1/2\}, \lambda)$ is a Liouville domain with corners. We can smooth out the corner with a $\mathcal{C}^\infty$-small perturbation. By abuse of notation, the boundary of this Liouville domain is denoted by $M = \{R = 1/2\} \times \{|r| \leq 1/2\} \cup \{R > 1/2\}^c \times \{|r| = 1/2\}$, with Liouville vector field $Y = R\partial_R + r\partial_r$. The time 1 flow of Y sends M to a new boundary $\{R = e/2\} \times \{|r| \leq e/2\} \cup \{R > e/2\}^c \times \{|r| = e/2\}$, that is, $U \cup M \times [0, 1] = \{R > e/2\}^c \cap \{|r| \leq e/2\}$. It's easy to check that $U \subset \{F \leq 0\} \subset U \cup M \times [0, 1]$.

6.4 Strongly ADC property of M_0

In this subsection, we will prove that the contact boundary of (M_0, λ_0)(as in remark 6.18) with respect to the trivialization Φ is strongly asymptotically dynamically convex. Let us first state what the framing is. Since $(\widehat{W^1_\epsilon} \times \mathbb{R} \times S^1, \lambda_0)$ is a product, it suffices to choose the G-equivariant trivialization on both components, since it descends naturally to the quotient (W_0, λ_0) (see subSection 6.2.1). We denote the boundary of (M_0, λ_0) by (Σ_0, λ_0).

Theorem 6.20. *Let F be the function of Lemma 6.17. Then (Σ_0, λ_0) satisfies the strongly ADC property with respect to a trivialization Φ, provided $\mathbf{a}$ satisfies the conditions $n \geq 3$ and $m(\mathbf{a}) \geq 2$.*

Proposition 6.21. *For any $K > 0$, there exits a $C(L)-$equivariant function F as defined in 6.10 on the Liouville domain $(\widehat{W^1_\epsilon} \times \mathbb{R} \times S^1, \lambda_0)$ with a chosen trivialization Φ such that*

(1) $\Sigma := \{F = 0\}$ is a regular level set and the Liouville vector field Y points outwards along Σ.

(2) The quotient $\Sigma_0 := \Sigma/G$ has the property that all elements of $\mathcal{P}^{<K}_\phi(\Sigma_0, \lambda_0)$ have lower SFT index at least $\min\{m(\mathbf{a}) - 3/2, n - 5/2\}$.

Proof. We will show that by choosing a proper $\mathcal{C}^2$-small function f as in Proposition 6.13, the corresponding function F satisfies the required conditions. The first condition is satisfied by the construction of F, as proved in Lemma 6.17, we only need to show the second condition is also satisfied. Recall the quotient map

$$\pi : \Sigma \to \Sigma_0$$

is an $L-$sheeted covering map. Therefore, the Reeb orbits in Σ_0 lift to fractional Reeb orbits in Σ. To be precise, if $\gamma(t), t \in [0, T]$ is a Reeb orbit in Σ_0, then the $L-$ fold Reeb orbit

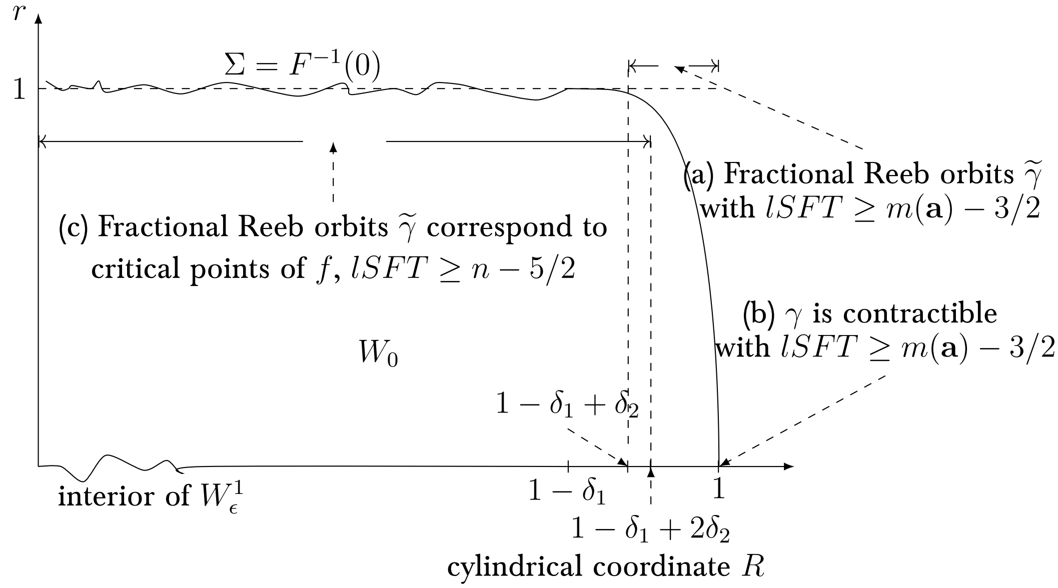

Figure 7: Lower SFT index of Fractional Reeb orbits in Σ.

$\gamma(t), t \in [0, qT]$ can be lifted to a Reeb orbit $\widetilde{\gamma(t)}, t \in [0, LT]$ in Σ. It follows that the index of γ in Σ_0 can be calculated through the index of $\widetilde{\gamma}$ in Σ_0. We will proceed by investigating the Reeb orbits in three regions:

(a) $\Sigma \cap \{1 > R > 1 - \delta_1 + \delta_2\}$, where $\widetilde{\gamma}$ has constant r, R coordinates.

(b) $\Sigma \cap \{1 = R\}$, where γ is contractible, and γ lifts to closed Reeb orbit $\widetilde{\gamma}$ in Σ.

(c) $\Sigma \cap \{R > 1 - \delta_1 + 2\delta_2\}^c$, where $\widetilde{\gamma}$ has constant coordinate in the W^1_ϵ component.

We will show that all elements of $\mathcal{P}^{<K}_\phi(\Sigma_0, \lambda_0)$(see Figure 7) can be lifted to fractional Reeb orbits either entirely contained in part (a), (b) or (c) and

(a) orbits in part(a) have lower SFT index at least $m(\mathbf{a}) - 3/2$;

(b) orbits in part (b) have lower SFT index at least $m(\mathbf{a}) - 3/2$;

(c) orbits in part (c) have lower SFT index at least $n - 5/2$.

First, in region (a), by lemma 2.23, we have

$$X_{Reeb} = \frac{X_F}{Y(F)} = \frac{2r\partial_\theta - 2fX_f}{2r^2 - 2fY_\lambda(f)} = \frac{2r\partial_\theta - 2ff'J\partial_R}{2r^2 - 2Rff'}.$$

So X_{Reeb} has no ∂_R component, and therefore the Reeb flow in the region (a) has constant R coordinate. So any Reeb orbits γ intersecting $\{R > 1 - \delta_1 + \delta_2\}$ remains entirely in region (a). Let us begin the proof with a lemma:

Lemma 6.22. *With W_0 and F defined as in Lemma 6.17, the conditions in Lemma 2.39 are satisfied.*

Proof. We will verify the conditions in three cases:

a. in the region $W_0 \setminus \{R > 1 - \delta_1 + 2\delta_2\}$, where $||1 - f||_{\mathcal{C}^2} < \epsilon$;

b. in the region $\{1 - \delta_1 + \delta_2 < R < 1\}$, where $Y = r\partial_r + R\partial_R$, and $f = f(R)$.

c. in the region $R = 1, r = 0$.

First of all, note that $b = dF(Y) = 2r^2 - 2fY_\lambda(f) > 0$ by Lemma 6.17.

Case (a): Let p be a critical point of f and define

$$A := \{(p, \sqrt{f(p)^2 + t}, \theta) \in W^1_\epsilon \times \mathbb{R} \times S^1 \,\big|\, t \in (-\epsilon_0, \epsilon_0)\}.$$

Define $C_t := F^{-1}(t)$, which is transverse to A as ∂_r is transverse to it. Let

$$A_t := C_t \cap A = \{(p, \sqrt{f(p)^2 + t}, \theta) \in W^1_\epsilon \times \mathbb{R} \times S^1\}$$

and $L_t = \sqrt{f(p)^2 + t}$, $b/L_0 = 2f(p)$. We can rescale L_t by $2f(p)$, and with $V = \frac{\partial_r}{2r}$, we have

$$db(V) = 2 > 2f(p)\frac{dL_t}{dt}\Big|_{t=0} = 1$$

Case(b): Let B be a Morse-Bott manifold of the Brieskorn manifold $(\Sigma(\mathbf{a}), \lambda)$, and $g(R) = -f^2(R)$. Then $F(t) = r(t)^2 + g(R(t)), t \in (-\epsilon_0, \epsilon_0)$. We have the following:

$$dF = 2rdr + g'(R)dR, \quad b = dF(Y) = 2r^2 + Rg'(R),$$

$$X_{Reeb} = X_F/Y(F) = (2r\partial_\theta + g'(R)J\partial_R)/b$$

Now, define for any constant $a > 0$ ($-1/a$ is the slope of tangent line of F at (r, R)),

$$A(a) := \{(q, R(t), \theta, r(t)) \in B \times (1-\delta_1+\delta_2, 1) \times S^1 \times \mathbb{R} | r^2 - f(R)^2 = t, r = ag'(R), t \in (-\epsilon_0, \epsilon_0)\}$$

Again let $C_t := F^{-1}(t)$, which is transverse to $A(a)$, and

$$A_t := C_t \cap A = \{(q, R(t), \theta, r(t))\,\}$$

Then $A(t)$ is a Morse-Bott manifold in C_t. Since

$$L_t = b/2r(t) = r + \frac{Rg'(R)}{2r} = r + \frac{R}{2a}, \quad b/L_0 = 2r(0) \tag{6.14}$$

$$2r\frac{dL_t}{dt}\Big|_{t=0} = 2rr' + \frac{rR'}{a} \tag{6.15}$$

and on the other hand, $V = r'\partial_r + R'\partial_R$, $db = 4rdr + (g'(R) + Rg''(R))dR$, we have that

$$db(V) = 4rr' + R'g'(R) + RR'g''(R) \geq 2rr' + \frac{rR'}{a} = 2r\frac{dL_t}{dt}\Big|_{t=0}$$

since $r' > 0, R' > 0, g''(R) > 0$.

Case (c): Let $g = -f^2(R), B$ defined as above, define

$$A := \{(q, R(t), \theta, 0) \in B \times \mathbb{R} \times S^1 \times \mathbb{R} \,|\, g(R) = t, t \in (-\epsilon_0, \epsilon_0)\}$$

Once more, $C_t := F^{-1}(t)$, which is transverse to A as ∂_R is transverse to it, and

$$A_t := C_t \cap A = \{(q, R(t), \theta, 0) \in B \times \mathbb{R} \times S^1 \times \mathbb{R}\,\}$$

are pseudo Morse-Bott manifolds. Moreover,

$$dF = g'(R)dR,\; b = dF(Y) = Rg'(R),\; L_t = R(t),\; b/L_0 = g'(1)$$

Here,

$$g'(1)\frac{dL_t}{dt}\Big|_{t=0} = g'(R)R'|_{R=1} = \frac{d}{dt}(g(R(t)))|_{t=0} = 1$$

and with $V = \frac{\partial_R}{g'(R)}$, we have

$$db(V) = 1 + \frac{Rg''(R)}{g'(R)} > 1 = g'(1)\frac{dL_t}{dt}\Big|_{t=0}.$$

□

Now let us compute the index of the Reeb orbits in region (a). Any Reeb orbit γ can be lifted to a fractional Reeb orbit $\widetilde{\gamma}$ in the region $\Sigma \cap \{R > 1 - \delta_1 + \delta_2\}$, where $F = r^2 - f(R)^2$. The Reeb orbit can be written as $\widetilde{\gamma} = (\gamma_1, \gamma_2)$, where γ_1, γ_2 are fractional Reeb orbits of $(\Sigma(\mathbf{a}), R\lambda)$ and $(S^1, r\theta)$, for fixed r, R, so by Lemma 6.22,

$$\mu_{CZ}(\gamma, F) = \mu_{CZ}(\gamma, \lambda_0) + \frac{1}{2}.$$

Meanwhile,

$$\mu_{CZ}(\gamma, F) = \mu_{CZ}(\gamma_1, -f(R)^2) + \mu_{CZ}(\gamma_2, r^2)$$

follows the product property of Conley-Zehnder index. Note that

$$(-f(R)^2)' > 0, (-f(R)^2)'' > 0,$$

therefore we have

$$\mu_{CZ}(\gamma_1, -f(R)^2) = \mu_{CZ}(\gamma_1, c_1\lambda) + \frac{1}{2}.$$

Moreover, by Remark 2.41,

$$\mu_{CZ}(\gamma_2, r^2) = \frac{1}{2}.$$

Notice γ_1 is a fractional Reeb orbits on the Brieskorn manifold $(\Sigma(\mathbf{a}), R\lambda)$, which has the same index as $(\Sigma(\mathbf{a}), \lambda)$. By Lemma 5.5 and Lemma 5.6, we then have $\mu_{CZ}(\gamma_1) \geq m(\mathbf{a})$. Putting all equations together:

$$
\begin{aligned}
lSFT(\gamma) &= \mu_{CZ}(\gamma, \lambda_0) - \frac{1}{2}\dim B + (n+1) - 3 \\
&\geq \left(\mu_{CZ}(\gamma, F) - \frac{1}{2}\right) - n + (n+1) - 3 \\
&\geq \left(\mu_{CZ}(\gamma_1, \lambda) + \frac{1}{2}\right) + \mu_{CZ}(\gamma_2, r^2) - \frac{5}{2} \\
&\geq m(\mathbf{a}) + \frac{1}{2} + \frac{1}{2} - \frac{5}{2} \\
&= m(\mathbf{a}) - 3/2.
\end{aligned}
$$

For the region (b), the claim will be proved in Lemma 6.24.

Now suppose $\gamma_0(t), t \in [0, T], T < K$ is a Reeb orbit in Σ_0 and can be lifted to a fractional Reeb orbit in region (c). Then the $L-$ fold Reeb orbit $\gamma(t) := \gamma_0(t), t \in [0, LT]$ can be lifted to a closed Reeb orbit $\widetilde{\gamma(t)}$ in this region. Let $g(R)$ be a smooth function defined in Lemma 6.14. So $g(R) = 1$ for $R < 1 - \delta_1 + \delta_2$ and $g(R) = 0$ for $R > 1 - \delta_1 + 2\delta_2$. By abuse of notation, g can be regarded as a function on $\Sigma \cap \{1 - \delta_1 + \delta_2 < R < 1 - \delta_1 + 2\delta_2\}$. We extend g to Σ by a constant. Now define a new vector field $X = g \cdot X_{Reeb}$. Let X_W be the projection of X to W^1_ϵ, i.e.

$$X_W = g \cdot \frac{-X_f}{f - Y_\lambda(f)} = \frac{-1}{f - Y_\lambda(f)} \cdot gX_f.$$

Since $(1-f)$ is $\mathcal{C}^2$-small, $||\frac{1}{f - Y_\lambda(f)}||_{\mathcal{C}^1} < 2$. By Lemma 6.14, X_W is $\mathcal{C}^1$-small. Then by Corollary 6.28, for f sufficiently $\mathcal{C}^2$-small, any periodic orbit of period less than LK is a constant orbit, and therefore corresponds to a critical point of f. We claim that any such Reeb orbit γ_0 has lower SFT index at least $n - 5/2$, which will be proved in Proposition 6.26. □

Remark 6.23. Reeb orbits in W_0 can be graded by their $H_1/Tors$ class. Let's have a closer look at the Reeb orbits with $H_1/Tors$ grading 0. In the proof of Proposition 6.21, the Reeb orbits in the regions (a) and (c) are never null-homologous.

Lemma 6.24. *Any Reeb orbit γ in region (b) is contractible in Σ_0, and its lower SFT index is at least $m(\mathbf{a}) - 3/2$.*

Proof. In region (b), $X_{Reeb} = J\partial_R$. In fact, $\Sigma \cap \{1 = R\} = \Sigma(\mathbf{a}) \times S^1$. The Reeb flow is stationary on S^1 and coincides with the Reeb flow on the Brieskorn manifold $\Sigma(\mathbf{a})$. Therefore, any Reeb orbit γ is contractible. Suppose $\widetilde{\gamma}$ is a lift of γ. Let $B \subset \Sigma(\mathbf{a})$ be a Morse-Bott manifold for $(\Sigma(\mathbf{a}), \lambda)$. In light of Lemma 6.22, we have

$$\mu_{CZ}(B \times S^1, F) = \mu_{CZ}(B \times S^1, \lambda_0) + \frac{1}{2}.$$

By the product property of Conley-Zehnder index,

$$\mu_{CZ}(B \times S^1, F) = \mu_{CZ}(B, -f^2(R)) + \mu_{CZ}(S^1, r^2) = \mu_{CZ}(B, -f^2(R)) + \frac{1}{2}.$$

On the other hand,

$$\mu_{CZ}(B, -f^2(R)) = \mu_{CZ}(B, \lambda) + \frac{1}{2} \geq m(\mathbf{a}) + \frac{1}{2}.$$

So we conclude that

$$\mu_{CZ}(B \times S^1, \lambda_0) = \mu_{CZ}(B \times S^1, F) - \frac{1}{2} \geq m(\mathbf{a}) + \frac{1}{2}$$

and

$$\begin{aligned} lSFT(\gamma) &= \mu_{CZ}(\gamma) - \frac{1}{2} \dim \ker(D_{\gamma(0)}\psi_T - \mathbf{id}) + (n + 1 - 3) \\ &= \mu_{CZ}(B \times S^1, \lambda_0) - \frac{1}{2}(\dim B + 1) + (n + 1 - 3) \\ &\geq m(\mathbf{a}) + 1/2 - n + n - 2 = m(\mathbf{a}) - 3/2. \end{aligned}$$

□

Remark 6.25. Let $MB(p), p \in \mathbb{Z}$ be the Morse-Bott manifold of return time $\frac{p\pi}{2}$ in the Brieskorn manifold, then $MB(p) \times S^1/C(L)$ is a Morse-Bott manifold in Σ_0. Conversely, any Morse-Bott manifold of contractible Reeb orbits in Σ_0 can be lifted to Σ. By Lemma 6.24 and Remark 6.23, the contractible Morse-Bott manifolds in Σ_0 can be lifted to $\Sigma(\mathbf{a}) \times S^1 \subset \Sigma$. Indeed, each Reeb orbit in Σ_0 has L different lifts in Σ. In terms of Morse-Bott manifolds of contractible Reeb orbits, we have a one-to-one correspondence:

$$\begin{aligned} \pi : \Sigma(\mathbf{a}) \times S^1 &\to (\Sigma(\mathbf{a}) \times S^1)/C(L) \subset \Sigma_0 \\ MB(p) \times S^1 &\mapsto (MB(P) \times S^1)/C(L). \end{aligned}$$

The group action is trivial on the first factor, therefore

$$(MB(p) \times S^1)/C(L) = MB(p) \times (S^1/C(L)) \cong MB(p) \times S^1$$

and

$$\mu_{CZ}(MB(p) \times S^1, \lambda_0) = \mu_{CZ}(MB(p), \lambda) + \frac{1}{2}.$$

Proposition 6.26. *As defined in the proof of part (c) of Proposition 6.21, the Reeb orbit γ_0 has lower SFT index at least $n - 5/2$.*

Proof. The $L-$fold iterate $\gamma(t)$ can be lifted to a Reeb orbit $\widetilde{\gamma(t)}$ in the Region (c). Its W^1_ϵ component is a critical point p of f. Since the conditions of Lemma 2.39 are satisfied,

$$\mu_{CZ}(B_0, \lambda_0) + \frac{1}{2} = \mu_{CZ}(B_0, F).$$

Everything descends down to the quotient M_0. We will use the same notations for the quotient. We have the Hamiltonian orbit $\gamma_0 = (p, \gamma_2)$, where p is a constant orbit in W^1_ϵ while γ_2 is an orbit in $\mathbb{R} \times S^1$. The index is

$$\mu_{CZ}(B_0, F) = \mu_{CZ}(p, -f^2) + \mu_{CZ}(\gamma_2, r^2) = \mu_{CZ}(p, -f^2) + \frac{1}{2}$$

Since $f(p) \neq 0$, $Ind_p(f^2) = Ind_p(f)$, hence

$$\mu_{CZ}(p, -f^2) = Ind_p(f^2) - n = Ind_p(f) - n$$

by Corollary 2.27. Since indices of critical points of f is at least n, so $\mu_{CZ}(p) \geq 0$ (see remark 6.16). Thus lower SFT index

$$\begin{aligned} lSFT(\gamma) &= \mu_{CZ}(B_0, \lambda_0) - \frac{1}{2}\dim B_0 + (n+1) - 3 \\ &= \mu_{CZ}(B_0, F) - \frac{1}{2} - \frac{1}{2} + (n+1) - 3 \\ &\geq 0 + \frac{1}{2} + n - 3 = n - 5/2 \end{aligned}$$

where the Morse-Bott manifold $B_0 = S^1$. □

Proof of Theorem 6.20. Recall the definition of a strongly ADC contact manifold: there exists a sequence of non-increasing contact forms α_i and increasing positive numbers D_i going to infinity such that all elements of $\mathcal{P}_\Phi^{<D_i}(\Sigma, \alpha_i)$ have positive lower SFT index.

In light of Proposition 6.21, let $K_i = K^i$(K is a fixed large number, the explicit conditions will be clear later in this proof), there exists a $C(L)-$ equivariant function F_i such that all elements of $\mathcal{P}_\Phi^{<K_i}(\Sigma_i, \lambda_0|_{\Sigma_i})$ have positive lower SFT index (since $\min\{m(\mathbf{a}) - 3/2, n - 5/2\} > 0$), where $\Sigma_i := F_i^{-1}(0)/C(L)$ is the boundary of the quotient manifold.

By Remark 6.19, we notice that conditions of Corollary 6.31 are satisfied, so there exists a contactomorphism $f_i : \Sigma_0 \to \Sigma_{i+1}$ and a constant C independent of F_i, such that

$$\frac{1}{C} \cdot \lambda_0|_{\Sigma_0} < f_i^*(\lambda_0|_{\Sigma_i}) < C \cdot \lambda_0|_{\Sigma}.$$

So the non-increasing contact forms α_i can be defined as $\alpha_i = \frac{1}{C^i} f_i^*(\lambda_0|_{\Sigma_i}) < \alpha_{i-1}$, and $D_i := K_i/C^i$, which goes to infinity as long as $K > C$. Then $\mathcal{P}_\Phi^{<D_i}(\Sigma_0, \alpha_i) = \mathcal{P}_\Phi^{<L_i}(\Sigma_i, \lambda_0|_{\Sigma_i})$, which shows that all elements have positive lower SFT index. □

We follow the idea of F.laudenbach in the proof of the following lemma.

Lemma 6.27 (Proposition 6.1.5 [AD14], [Lbdm04]). *Let X be a vector field on $\mathbb{R}^{2n}$. If $||dX||_{L^2} < \frac{2\pi}{L}$, the only periodic orbits with period less than L are constant orbits.*

Proof. Consider the solution $u(t)$ of period $T \leq L$ and take its Fourier expansion as well as $\dot{u}, \ddot{u}$.

$$u(t) = \sum_k c_k(u) e^{2k\pi i t/T}, \quad \dot{u}(t) = \sum_k \frac{2k\pi i}{T} c_k(u) e^{2k\pi i/T}$$

So by Parseval's identity, we have

$$||\ddot{u}||_{L^2}^2 = \sum \frac{4k^2\pi^2}{T^2} |c_k(\dot{u})|^2 \geq \sum_{k \neq 0} \frac{4\pi^2}{T^2} |c_k(\dot{u})|^2 = \frac{4\pi^2}{T^2} ||\dot{u}(t)||_{L^2}^2$$

since $c_0(\dot{u}) = 0$. Hence,

$$||\ddot{u}||_{L^2} \geq \frac{2\pi}{T} ||\dot{u}(t)||_{L^2}.$$

On the other hand, since $\ddot{u} = (dX)(\dot{u})$, $||dX||_{L^2} < \frac{2\pi}{L}$, so

$$||\ddot{u}||_{L^2} < \frac{2\pi}{L} ||\dot{u}(t)||_{L^2} \leq \frac{2\pi}{T} ||\dot{u}(t)||_{L^2}$$

if $\dot{u} \neq 0$. Therefore $u(t)$ is a constant orbit. □

Corollary 6.28 ([Lbdm04]). *If M is a compact manifold with boundary and X is a vector field which vanishes in the neighborhood of the boundary. Then for any $L > 0$, the flow generated by X has no non-constant periodic orbit with period less than L for sufficiently $\mathcal{C}^1$-small X.*

Proof. First we get rid of the boundary by doubling M (glue M with itself along the boundary). Now that X can be smoothly extended since it vanishes in a neighborhood of the boundary. Now consider the new closed manifold $\tilde{M}$. Let us fix a finite collection of compact charts K_i. Since X is $\mathcal{C}^1$-small, every closed orbit with bounded period T of the flow of X has a small diameter($D \leq ||X||_{uniform} \cdot L$), which implies the entire orbit remains in one of the charts K_i. The $\mathcal{C}^1$ norm is equivalent to the Euclidean norm so the lemma above applies. □

Lemma 6.29. *Let (U, λ) be a Liouville domain, $(\widehat{U}, \hat{\lambda})$ its completion, and $\Sigma_1 := \partial U$ be the contact boundary. Suppose we have a Liouville domain $(V, \hat{\lambda})$ such that $U \subset V \subset U \cup \Sigma_1 \times [0, M]$.Then there is a contactomorphism Ψ*

$$\Psi : (\Sigma_1, \lambda_2 = \hat{\lambda}|_{\Sigma_1}) \to (\Sigma_2 := \partial V, \lambda_2 = \hat{\lambda}|_{\Sigma_2}).$$

such that $\lambda_1 \leq \Psi^* \lambda_2 \leq e^M \lambda_1$.

Proof. Since

$$U \subset V \subset \Sigma_1 \times [0, M]$$

let ψ be the flow generated by the Liouville vector field and $t(p)$ be the time when the flow starting at $p \in \Sigma_1$ reaches Σ_2, i.e, $\psi_{t(p)}(p) \in \Sigma_2$. Then $M \geq t(p) \geq 0$. Let ρ be a function on $\widehat{U}$ supported on $(-\epsilon, M+1) \times \Sigma_1$, such that $\rho((r,p)) \equiv t(p)$ on the region $[-0, M] \times \Sigma_1$. Now consider the vector field $Y := \rho \cdot \partial_r$ and we denote by

$$\Psi : \mathbb{R} \times \widehat{U} \to \widehat{U}, \quad (t,p) \mapsto \Psi_t(p)$$

the flow generated by Y. Clearly we have $\Psi_1(\Sigma_1) = \Sigma_2$ and $\Psi^* \lambda_1 = e^{t(p)} \lambda_2$. Now the conclusion follows. □

Remark 6.30. If the Liouville domains in the Lemma above are G-equivariant, then there is G-equivariant contactomorphism satisfying the above statement.

Corollary 6.31. *Let (U, λ) be a Liouville domain, suppose we have two Liouville domains V_1, V_2 with $\Sigma_1 = \partial V_1, \Sigma_2 = \partial V_2$ such that $U \subset V_i \subset U \cup \partial U \times [0, M]$. Then there exists a contactomorphism f and a constant C independent of V_i, such that*

$$\frac{1}{C} \cdot \lambda|_{\Sigma_1} < f^* \lambda|_{\Sigma_2} < C \cdot \lambda|_{\Sigma_1}.$$

7 Finiteness of positive idempotent group

We are going to show that the positive idempotent group $I_+(\Sigma_0)$ is finite. Let's recall the definitions: for any filling W of Σ_0 such that $SH_*(W) \neq 0$, we have

$$I(W) = \{ \alpha \in SH^0_n(W) \,|\, \alpha^2 - \alpha \in H^0(W) \}$$

and $I_+(W) = I(W)/H^0(W)$, hence it suffices to prove $I(W)$ is a finite group. Indeed, for the Liouville filling (M_0, λ_0) as in remark 6.18, $SH^0_k(M_0, \mathbb{Z}_2)$ is finite. We begin by introducing a spectral sequence which converges to $SH^0_*(M_0, \mathbb{Z}_2)$:

Theorem 7.1 (Theorem 5.4 [KvK16]). *Let $(W, \omega = d\lambda)$ be a Liouville domain satisfying the assumptions:*

1 *The Reeb flow on ∂W is periodic with minimal periods $T_1 \cdot \frac{\pi}{2}, T_2 \cdot \frac{\pi}{2}, \cdots, T_k \cdot \frac{\pi}{2}$, where $T_k \cdot \frac{\pi}{2}$ is the common period, i.e. the period of a principal orbit. We assume that all T_k are integers.*

2 *The restriction of the tangent bundle to the symplectization of ∂W, $T(\mathbb{R} \times \partial W)|_{\partial W}$, is trivial as a symplectic vector bundle, $c_1(W) = 0$ and we have a choice of the trivialization of the canonical bundle.*

3 *There is a compatible complex structure J for $(\xi := \ker \lambda_{\partial W}, d\lambda_{\partial W})$ such that for every periodic Reeb orbit γ the linearized Reeb flow is complex linear with respect to some unitary trivialization of $(\xi, J, d\alpha)$ along γ.*

For each positive integer p define $C(p)$ to be the set of Morse-Bott manifolds with return time p, and for each Morse-Bott manifold $\Sigma \in C(p)$ put

$$\Delta(\Sigma) = \mu_{CZ}(\Sigma) - \frac{1}{2} \dim \Sigma / S^1,$$

where the Robbin-Salamon index is computed for a symplectic path defined on $[0, p]$. Then there is a spectral sequence converging to $SH(W; R)$, whose E^1-page is given by

$$E^1_{pq} = \begin{cases} \bigoplus_{\Sigma \in C(p)} H_{p+q-\Delta(\Sigma)}(\Sigma; R) & p > 0 \\ H_{q+n}(W, \partial W; R) & p = 0 \\ 0 & p < 0. \end{cases}$$

Remark 7.2. The above spectral sequence respects the H_1 grading. Therefore, to compute $SH^0_n(M_0)$, we only need to focus on the Morse-Bott manifolds of null-homologous Reeb orbits.

Lemma 7.3. *$SH^0_k(M_0, \mathbb{Z}_2)$ is finite for all k.*

proof of lemma 7.3. Note that it suffices to find all the Morse-Bott manifolds. By Remark 6.25, the first page of the spectral sequence which converges to $SH^0_*(M_0, \mathbb{Z}_2)$ is

$$E^1_{pq} = \begin{cases} \bigoplus H_{p+q-\Delta(MB(p))}(MB(p) \times S^1; \mathbb{Z}_2) & p > 0 \\ H_{q+n}(M_0, \partial M_0; \mathbb{Z}_2) & p = 0 \\ 0 & p < 0. \end{cases}$$

The finiteness of $SH^0_k(M_0, \mathbb{Z}_2)$ follows from the following two facts: first, there are only finitely many Morse-Bott manifolds $MB(p)$ satisfying $\Delta(MB(p)) = k$, i.e.

$$k = \mu_{CZ}(MB(p) \times S^1) - \frac{1}{2}(\dim(MB(p) \times S^1)/S^1) = f_{\mathbf{a}}(p) - \frac{1}{2}(\dim MB(p) - 1).$$

The above equation can only be satisfied by finitely many $p \in \frac{1}{2L}\mathbb{Z}$, and for any p there is at most one Morse-Bott manifold with return time $p\pi/2$ in the Brieskorn manifold $\Sigma(\mathbf{a})$. Secondly,

$$H_*(MB(p) \times S^1; \mathbb{Z}_2) = 0 \quad * < 0 \text{ or } * > 2n.$$

and $H_*(MB(p) \times S^1; \mathbb{Z}_2)$ is finite dimensional for $0 \leq * \leq 2n$. Therefore $SH^0_k(M_0, \mathbb{Z}_2)$ is finite for each k, since the dimension of $\bigoplus_{p+q=k} E^1_{pq}$ is finite for each k.

□

Now we are going to prove that $SH^0_*(M_0(\mathbf{a}), \mathbb{Z}_2) \neq 0$, where $\mathbf{a}$ is defined as in Remark ??.

Lemma 7.4. *For* $\mathbf{a} = (2, 2, 2, \cdots, p_k)$*, we have* $SH^0_*(M_0(\mathbf{a}), \mathbb{Z}_2) \neq 0$*, where* $k+3 = n, n > 8, p_i's$ *are sufficiently large integers.*

Proof. It suffices to prove that $SH^0_{n-1}(M_0, \mathbb{Z}_2) \neq 0$. To that end we will focus on the total degree $p+q = n-2, n-1, n$ in the spectral sequence above. First of all, for $p = 0$, we have

$$E^1_{0q} = H_{n+q}(M_0, \partial M_0; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & q = n \\ 0, & q \neq n. \end{cases}$$

On the other hand,

$$\begin{aligned} \Delta(MB(p) \times S^1) &= \mu_{CZ}(MB(p) \times S^1) - \frac{1}{2}(\dim(MB(p) \times S^1)/S^1) \\ &= f_{\mathbf{a}}(p) - \frac{1}{2}(\dim MB(p) - 1) \end{aligned}$$

where $p\pi/2, p \in \mathbb{Z}$ is the period. Meanwhile,

$$f_{\mathbf{a}}(p) = 3\left(\left\lfloor \frac{p}{2} \right\rfloor + \left\lceil \frac{p}{2} \right\rceil\right) + \sum \left(\left\lfloor \frac{p}{p_i} \right\rfloor + \left\lceil \frac{p}{p_i} \right\rceil\right) - \left(\lfloor p \rfloor + \lceil p \rceil\right) \quad \geq 3p + k - 2p = p + n - 3$$

so $\Delta(MB(p) \times S^1) \geq p - 4 > n + 1$ for any $p > n + 5$, that is, for any Morse-Bott manifold to contribute to the homology of degree at most n, the period of such manifold is at most $n + 5$. Thus, if we require $p_i > n + 5$, then the only Morse-Bott manifolds could possibly contribute to total degree $p + q \leq n$ is $MB(p) \times S^1$, $p = 2l, 2l < n + 5$ for some $0 < l \in \mathbb{Z}$ (see Subsection 5.5 [KvK16]). Now that $p = 2l$, $l < n$, we have $MB(p) = \Sigma(2, 2, 2) \cong \mathbb{RP}^3$.

$$H_i(\mathbb{RP}^3 \times S^1) = \begin{cases} \mathbb{Z}_2, & i = 0, 4 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & i = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

In this case,

$$\Delta(MB(2l) \times S^1) = 6l + n - 3 - 4l - 1 = 2l + n - 4 = p + n - 4.$$

So for $l > 2$, $\Delta(MB(2l) \times S^1) > n$. For $l = 1, 2$, we have (see Figure 8)

$$E^1_{pq} = H_{q-(n-4)}(\mathbb{RP}^3 \times S^1; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & q = n-4, n \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2, & q = n-3,\ n-2,\ n-1 \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $E^k_{2,n-3}(M_0, \mathbb{Z}_2) \neq 0$ stabilizes at the second page, so $SH^0_{n-1}(M_0, \mathbb{Z}_2) \neq 0$. It follows that $SH^0_*(M_0, \mathbb{Z}_2) \neq 0$. In particular, $SH^0_n(M_0, \mathbb{Z}_2) \neq 0$, since the unit lives in degree n.

□

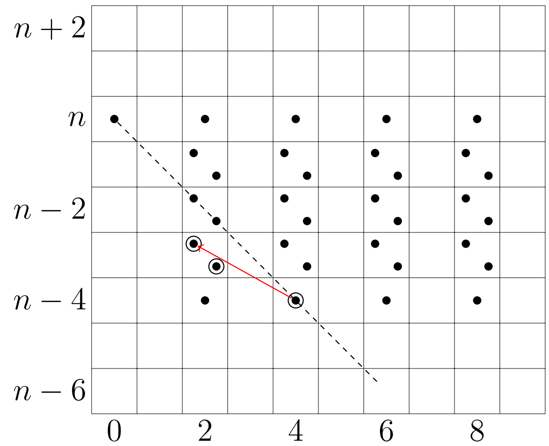

Figure 8: $E^2_{pq}(M_0, \mathbb{Z}_2) = E^1_{pq}(M_0, \mathbb{Z}_2)$: $\dim E^2_{2,n-3}(M_0, \mathbb{Z}_2) = 2$, so it can not be killed by d_2 (red arrow) since $\dim E^2_{4,n-4}(M_0, \mathbb{Z}_2) = 1$ on the second page.

Remark 7.5. Lemma 7.4 shows that $I_+(\Sigma_0(\mathbf{a}))$ is well-defined since $SH_*(M_0) \neq 0$. Furthermore, $I_+(\Sigma_0(\mathbf{a}))$ is a finite group.

Now we are ready to prove Theorem 1.5: first, we will take $\mathbf{a} = (2, 2, 2, p_1, \cdots, p_k)$ satisfying

- $p_i > k + 8$,
- $\sum \frac{1}{p_k} = \frac{1}{2}$.

Recall

$$U_{\mathbf{a}}(\epsilon) = \{\mathbf{z} \in \mathbb{C}^{n+1} | z_0^{a_0} + \cdots + z_n^{a_n} = \epsilon \cdot \beta(||\mathbf{z}||^2)\},$$

and

$$W^1_\epsilon = U_\epsilon \cap B(1), \quad \partial W^1_\epsilon = \Sigma(\mathbf{a}).$$

Let $(M_0(\mathbf{a}), \lambda_0)$ be defined as in Remark 6.18. Then we have the following facts:

1. (M_0, λ_0) is strongly ADC;
2. $SH^0_n(M_0) \neq 0$ and is finitely dimensional.
3. $H_i(M_0) = 0, i > 1, i \neq n, n+1$.

The first claim is true due to Proposition 6.20. We only need to check the condition that $m(\mathbf{a}) \geq 3$, which in turn is the result of Lemma 5.6. The second claim is proved in Lemma 7.3.

On the other hand, the Liouville vector field Y_λ is gradient-like (Lemma 6.17) for the function F which we used to define the Weinstein domain. Therefore, it is Liouville homotopic to Stein domain (M_0, J, ϕ)(Remark 6.18). $\pi_1(M_0) = \mathbb{Z}$ since M_0 is diffeomorphic to $\mathbb{C}^{n+1} \setminus V_{\mathbf{a}}(0)$ (Proposition 6.8). Let γ be an isotropic circle generating $\pi_1(M_0)$. Such γ exists by the $h-$principle(Lemma 6.11). Let M_1 be Weinstein manifold obtained from M_0 by attaching a Weinstein 2-handle with respect to the trivialization Φ (Proposition 6.10). M_1 is of finite type because M_0 is. Furthermore, attaching 2-handle along γ kills the fundamental group. Now we are going to prove that(M_1, λ_1, ψ_1) satisfies all conditions in Theorem 1.5

proof of Theorem 1.5. Indeed,we have the following facts about (M_1, λ_1, ψ_1):

1. $(\partial M_1, \lambda_1)$ is asymptotically dynamically convex;
2. $SH_*(M_1) \cong SH_*(M_0)$ as rings.
3. $\tilde{H}_i(M_1) = 0, i \neq n, n+1$

The first statement is true because subcritical surgery preserves the ADC property, by Theorem 3.9. The second statement is due to the fact that subcritical surgery doesn't change the ring structure of symplectic homology, see Theorem 2.19. The last statement on homology follows Proposition 6.9. □

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