Anti-self-dual Metrics from the Geometry of Plane Conics

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Abstract of the Dissertation

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This Dissertation describes anti-self-dual Riemannian metrics on four-dimensional, oriented manifolds.

These metrics have the property that their Twistor Spaces admit a meromorphic map to the projective plane, given by sections of a square-root of the anti-canonical line bundle, whose differential is of maximal rank. Dan Moraru [Mor04] and later Maciej Dunajski and Kenneth P. Tod [DT18] described a Penrose Transform relating such anti-self-dual structures to solutions of an overdetermined system of partial differential equations on the variety of non-singular conics in the projective plane.

Dunajski and Tod described explicitly a solution to this system of PDE corresponding to the round metric on the four-sphere. In this Dissertation, we study the geometry of the cotangent bundle of the projective plane (the twistor space of the projective plane with reverse orientation, endowed with the Fubini-Study metric) to describe explicitly another solution to this system.

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Chapter 1

Introduction

Twistor Theory is a term often employed to describe the use of complex-geometric and analytic methods in the study of field theories. In this Dissertation, we limit the scope of this term to the study of certain (complex) *varieties of rational curves*, specifically those associated to Riemannian four-manifolds whose self-dual Weyl tensors vanish (more on this below).

While its roots can be dated as far back as the mid XIX century, in the work of Authur Cayley [Cay69], Julius Plücker [Plü68], and most notably Felix Klein, [Kle70], its modern formulation, however, is most appropriately attributed to Roger Penrose [Pen76] in the Lorentzian setting, and Michael Atiyah, Nigel Hitchin, and Isadore Singer[1] [AHS78], in the Riemannian context. Penrose's key observation is that the vanishing of the self-dual Weyl tensor of a (pseudo) Riemannian 4-manifold can be interpreted as the obstruction to Nijenhuis integrability of an almost-complex structure on an auxilliary manifold, the Twistor Space.

A nice feature of twistor theory is that, to a certain extent, the twistor construction is reversible. That is, there exist conditions which guarantee a complex 3-manifold is a twistor space, for *some* anti-self-dual Riemannian metric. In theory, describing the metric is simple, a basic consequence of Kodaira's deformation theory for embedded submanifolds. In practice, however, reversing the twistor construction is not as easy as it seems. This has to do with the fact that twistor spaces are rather exotic. For instance, one does not find compact twistor spaces of Kähler type, except for a pair of cases [Hit81]. This makes many of the classical techniques of algebraic geometry fail, as twistor spaces are not projective manifolds.

It is by imposing additional structure on twistor spaces that one is able to reveal their underlying Riemannian manifolds. In this thesis, we consider twistor spaces with just enough

[1]The result of Atiyah-Hitchin-Singer is stated for self-dual metrics. Here we present an adaptation, related to the original source by a change in orientation on M.

of an algebraic flavor. In short, they admit natural maps to the projective plane (coming from sections of powers of their canonical bundles). These were first considered by Dan Moraru, in his thesis [Mor04] and more recently have been revisited by Maciej Dunajski and Kenneth P. Tod [DT18], who showed that to each twistor space of this type there corresponds a function on the space of non-singular conics in $\mathbb{CP}^2$ satisfying a system of partial differential equations. We aim to explore this link between classical geometry of plane conics to describe anti-self-dual manifolds.

The dissertation is structured as follows:

1. In chapter 1, we discuss preliminaries concerning anti-self-duality, varieties of rational curves (in particular, twistor spaces) and affine line bundles, culminating in the Penrose Transform.

2. In chapter 2, we go deeper into the geometry of the space of conics, describing its realization as a symmetric space, as well as a reduction of its structure group. We conclude with a description of the range of the Penrose Transform in terms of solutions to a system of differential equations, called the Dunajski-Moraru-Tod equations (or DMT equations, for short) and characterize the differential operatros in terms of the aforementioned geometric structures on the space of conics.

3. In chapter 3 we exploit the geometry of twistor spaces with holomorphic fibrations to $\mathbb{CP}^2$ to describe solutions of the the DMT equations. We focus on two key examples: complex projective 3-space $\mathbb{CP}^3$, and the flag manifold $F_{1,2} = \mathbb{P}(\mathcal{T}^*\mathbb{CP}^2)$. The description of the solution to the Dunajski-Moraru-Tod equations associated to the latter is the main contribution of this dissertation.

A remark about reproducilibility is in order. Many of the computations needed in this dissertation were done with the aid of computer software. The results are at times too large to appropriately display on the manuscript. A webpage will be hosted at

`marlon-gomes.github.io/dissertation`

containing the Wolfram Mathematica [Wol] notebooks for these computations, as well as links to the appropriate dependencies.

1.1 Anti-self-duality in four-dimensional Riemannian Geometry

The Riemannian geometry of oriented 4-manifolds enjoys a special character, which sets it appart from other dimensions, due to the reducibility of the representation of the special orthogonal group on the space of 2-forms. The Hodge $\star$-operator, defined by metric and orientation, acts on the space of 2-forms at a point is an involution, with eigenvalues $+1$ and -1, and elements of the corresponding eigenspaces are called self-dual and anti-self-dual forms. Relative to this eigenspace decomposition, the Riemannian curvature tensor, viewed as an endomorphism on 2-forms, breaks down schematically as

$$\mathcal{R} = \left(\begin{array}{c|c} \frac{s}{12}\mathrm{Id}_{\Lambda^+} + \mathcal{W}^+ & \mathring{r} \\ \hline \mathring{r} & \frac{s}{12}\mathrm{Id}_{\Lambda^-} + \mathcal{W}^- \end{array} \right), \tag{1.1.1}$$

where s is the scalar curvature, $\mathring{r}$ the traceless Ricci endomorphism, and $\mathcal{W}^+$, $\mathcal{W}^-$ are components of the Weyl tensor relative to the self-dual/anti-self-dual decomposition.

Let (M, g) be an oriented, Riemannian 4-manifold, and denote by Z the total space of the unit sphere bundle of its bundle of self-dual 2-forms, $Z = S(\Lambda^+ M)$. The manifold Z inherits a natural Riemannian metric and orientation from $\Lambda^+ M$. The connection induced on $\Lambda^2 M$ by the Levi-Civita connection of (M, g) respects the self-dual/anti-self-dual decomposition, and induces an Ehresmann connection on Z (viewed as a sphere bundle over M), splitting its tangent bundle into horizontal and vertical components relative to the natural fibration to M.

The 6-manifold Z can be endowed with an almost-complex structure[2], depending on g, which we will denote by $\mathcal{J}_g$. Its action on the tangent bundle of Z is described in terms of the splitting discussed in the previous paragraph. On the vertical component, the fibers are two-dimensional vector spaces endowed with orientation and metric, to which we can associate a unique compatible linear complex structure, denoted by $\mathcal{J}_V$. By means of a local trivialization, a point $z \in Z$ may be identified with a pair consisting of a point p in M, and a linear almost-complex structure $J \in \mathrm{End}(\mathrm{T}_p M)$, the association between self-dual two-forms

[2]Not all 6-manifolds admit this property. For a compact 6-manifold, the obstructions to the existence of an almost-complex structure can be summarized as follows: it must be orientable; in addition, it must admit a complex line bundle whose first Chern class reduces to the second Stiefel-Whitney class modulo 2.

and almost-complex structures is obtained by *raising an index* with the metric

$$\omega_{ab} \mapsto J^a_{\ b} = g^{ac}\omega_{cb}.$$

Given a horizontal vector, we may thus project it to T_pM, act on the projection by J, an lift it back to T_zZ via the connection. The resulting endomorphism on the horizontal subbundle is trivialization-independent, and we set it as the horizontal component $\mathcal{J}_H$ of the almost complex structure $\mathcal{J}_g$ on Z.

Theorem 1.1.1. *(Atiyah-Hitchin-Singer [AHS78]) Let (M, g) be an oriented, Riemannian 4-manifold, and denote by Z the unit sphere bundle of its bundle of self-dual 2-forms, $Z = S(\Lambda^+M)$. The almost-complex structure $\mathcal{J}_g$ defined as above is integrable, in the sense of Newlander-Niremberg, if and only if $\mathcal{W}^+ = 0$.*

Definition 1.1.2. *Suppose that (M, g) is an oriented, anti-self-dual Riemannian 4-manifold. We define the Twistor Space of (M, g) as the complex 3-dimensional manifold $(Z, \mathcal{J}_g)$.*

Typically, when the metric is understood from context, we will refer to Z itself as the twistor space, omitting reference to $\mathcal{J}_g$.

We remark that the twistor spaces of conformally-related Riemannian metrics are naturally isomorphic as almost-complex manifolds, thus the construction is associated to the *conformal structure* of M, rather than a particular Riemannian metric. The details of this construction, as well as a proof of conformal invariance can be found in [dBN98].

Example 1.1.1. *The model example is $\mathbb{R}^4$, with its usual flat metric. Its twistor space is the complement of a projective line in $\mathbb{CP}^3$ (more on that on the next section). A related example (obtained by conformal compactification) is the sphere S^4, with its constant curvature metric, whose twistor space is $\mathbb{CP}^3$.*

Example 1.1.2. *The complex projective plane, $\mathbb{CP}^2$, admits a natural orientation, compatible with its complex structure. By reverse-oriented $\mathbb{CP}^2$, represented by $\overline{\mathbb{CP}^2}$ we mean the same underlying 4-manifold (forgetting about its complex structure) but with the orientation reversed. The Fubini-Study metric g_{FS} (viewed simply as a Riemannian metric, no Kähler structure assumed) is then anti-self-dual.*

To describe its twistor space, let us denote by $(\mathbb{CP}^2)^*$ the manifold parametrizing proojective lines in $\mathbb{CP}^2$. Abstractly, this manifold is isomorphic to $\mathbb{CP}^2$, but there is a duality aspect between the two that we wish to explore in subsequent chapters, hence the distinguished notation. The manifold $\mathbb{CP}^2 \times (\mathbb{CP}^2)^*$ thus parametrizes pairs of points and lines in $\mathbb{CP}^2$. We *define the flag manifold* $F_{1,2}$ *as the subvariety of* $\mathbb{CP}^2 \times (\mathbb{CP}^2)^*$ *consisting of pairs of incident points and lines in* $\mathbb{CP}^2$*. This manifold, endowed with its natural subspace complex structure, is the twistor space of* $(\overline{\mathbb{CP}^2}, g_{\mathrm{FS}})$.

Example 1.1.3. *Hyperkähler 4-manifolds are, in particular, anti-self-dual and Ricci-flat. One famous example is the* K3 *surface, whose oriented diffeomorphism type is that of a smooth, quartic hypersurface in* $\mathbb{CP}^3$*. This 4-manifold admits a plethora of hyperkähler metrics. If* (X, g_{HK}) *is a* K3 *surface endowed with a hyperkähler metric, its bundle of self-dual 2-forms is parallel, hence the twistor space* Z *is, as a smooth, real 6-dimensional manifold, simply a product* $X \times S^2$.

While both X *and* S^2 *admit complex structures, the twistor complex structure* $\mathcal{J}_{g_{\mathrm{HK}}}$ *is not the product of the complex structures in the factors. In fact, while both factors admit Kähler metrics, it is a celebrated result of Hitchin [Hit81] that there is no compact Kähler twistor space besides the two preceding examples,* $\mathbb{CP}^3$ *and* $F_{1,2}$.

1.2 Twistor space as a variety of rational curves

In what follows we will extensively use another, more geometric, interpretation of twistor spaces. Throughout this section we assume that the complex 3-manifold Z is a twistor space, but omit references to its underlying Riemannian atructure. Recall that Z naturally arises as a sphere bundle over M, the fiber over x describing linear, orientation-compatible, orthogonal almost-complex structure on $\mathrm{T}_x M$. Such spheres are embedded, *holomorphic* submanifolds of Z. Since each of them is isomorphic to $\mathbb{R}^1$, they are *rational curves* within Z.

Let $j : C \longrightarrow Z$ be one such rational curve (j is a particular embedding, which ultimately is not relevant to the construction). Its normal bundle is defined (as a complex vector bundle) as the quotient

$$\mathcal{N}_{C|Z} = j^*(\mathcal{T}Z)/\mathcal{T}C,$$

where we use the symbol $\mathcal{T}$ to denote the *holomorphic tangent bundle* of a complex manifold. The normal bundle admits a natural holomorphic structure, characterized as the unique holomorphic structure on its underlying complex vector bundle making the sequence

$$0 \longrightarrow \mathcal{T}C \longrightarrow j^*\mathcal{T}Z \longrightarrow \mathcal{N}_{C|Z} \longrightarrow 0.$$

an exact sequence of holomorphic vector bundles. Fortunately, $\mathbb{CP}^1$ is a simple enough variety that all such bundles are classified. This is a good point to fix notation. We define the tautological line bundle over the projective line, $\mathcal{O}_{\mathbb{CP}^1}(-1)$, as the subbundle of the trivial, rank 2 bundle $\mathbb{CP}^1 \times \mathbb{C}^2$ whose fiber over a point $p \in \mathbb{CP}^1$ is the line p represents in $\mathbb{C}^2$. All holomorphic vector bundles over $\mathbb{CP}^1$ are obtained by a combination of duals and tensor powers of this tautological bundle. The notation $\mathcal{O}_{\mathbb{CP}^1}(n)$ is reserved for the n-th tensor power of either $\mathcal{O}_{\mathbb{CP}^1}(-1)$ or its dual, $\mathcal{O}_{\mathbb{CP}^1}(1)$, depending on whether n is negative or positive, respectively, whereas $\mathcal{O}_{\mathbb{CP}^1}$ is used to denote the trivial line bundle.

Theorem 1.2.1. *(Birkhoff-Grothendieck, [Gro57]) Every holomophic vector bundle $\mathcal{E}$ over $\mathbb{CP}^1$ admits a holomorphic splitting,*

$$\mathcal{E} = \bigoplus_{k=1}^{r} \mathcal{O}_{\mathbb{CP}^1}(a_k),$$

unique up to reordering the summands.

Penrose [Pen76] observed that if $C \subset Z$ is one of the rational curves arising from the twistor construction, then its normal bundle has the decomposition $\mathcal{O}_{\mathbb{CP}^1}(1) \oplus \mathcal{O}_{\mathbb{CP}^1}(1)$, the same type of normal bundle of a projective line[3] in $\mathbb{CP}^3$, thus we call such curves *twistor lines.*

In fact, this is just one facet of the story. In the early 60s, Kodaira developed a theory of deformations of compact, complex submanifolds of a fixed ambient complex manifold (not necessarily compact)[Kod62]. We summarize the relevant findings below.

Definition 1.2.2. *Let W^{r+d} be a complex manifold. By an analytic family of compact d-submanifolds of W we mean a pair (B, Z), consisting of:*

1. *a connected complex manifold B, called the base of the family;*
2. *a codimension r submanifold Z of $W \times B$, such that the restriction of the natural projection onto the second factor*

$$\varpi \colon W \times B \longrightarrow B$$
$$(w, t) \mapsto t,$$

to Z is a holomorphic submersion with compact fibers.

If (B, Z) is a family of d-submanifolds of W, we call the submanifold

$$V_t = (\varpi|_Z)^{-1}\{t\}$$

the fiber of the family over $t \in B$.

[3] By a projective line we mean the quotient of a complex 2-plane in $\mathbb{C}^4$ under the quotient map defining $\mathbb{CP}^3$.

Definition 1.2.3. *Let W^{r+d} be a complex manifold, V^d a compact submanifold. By a deformation of V within W, we mean*

1. *A connected complex manifold B, with a distinguished point t_0;*

2. *an analytic family (B, Z) of compact d-dimensional submanifolds of W, such that the fiber over $t_0 \in B$ is $Z_{t_0} = V$.*

Definition 1.2.4. *Let W be a complex manifold, (B_i, Z_i), $i = 1, 2$, families of d-dimensional submanifolds. A morphism of families from (B_1, Z_1) to (B_2, Z_2) is a pair (f, F), consisting of*

1. *a holomorphic map between bases, $f : B_1 \longrightarrow B_2$,*

2. *a holomorphic map between the total spaces of the families,*

$$F : W \times B_1 \longrightarrow W \times B_2,$$

which maps Z_1 holomorphically into Z_2, and covers f, that is, the diagram below commutes:

$$\begin{array}{ccc} W \times B_1 & \xrightarrow{\ F\ } & W \times B_2 \\ \downarrow{\scriptstyle \varpi_1} & & \downarrow{\scriptstyle \varpi_2} \\ B_1 & \xrightarrow{\ f\ } & B_2 \end{array}$$

Definition 1.2.5. *Let W a complex manifold, V a compact submanifold, and $(B_i, t_{i,0}, Z_i)$, $i = 1, 2$, be deformations of V within W. A morphism of deformations from $(B_1, t_{1,0}, Z_1)$ to $(B_2, t_{2,0}, Z_2)$ is a morphism of families $(f, F) \in \mathrm{Mor}((B_1, Z_1), (B_2, Z_2)$ such that*

$$f(t_{1,0}) = t_{2,0}.$$

This morphism is called a monomorphism if f is injective and

$$(V_1)_q = (V_2)_{f(q)},$$

for all $q \in M_1$. It is called an epimorphism if f is surjective and if for all $s \in M_2$ there exists $q \in f^{-1}(q)$ such that

$$(V_1)_q = (V_2)_s.$$

Definition 1.2.6. *Let W be a complex manifold, V a compact submanifold, and $(M, p, \mathcal{V})$ a deformation of V within M. This deformation is called locally maximal if for any other deformation $(M', p', \mathcal{V}')$, there exists a neighborhood N' of p' in M', and a monomorphism of deformations $(f, F) \in \mathrm{Mor}((N', p', \mathcal{V}'|_{N'}), (M, p, \mathcal{V}))$.*

Given an analytic family of submanifolds of W, (B, Z), and a point $t \in B$, we denote by $\mathcal{N}_t$ the normal bundle of the fiber V_t in W. There is a natural linear map

$$\mathrm{K}_t : \mathcal{T}_t B \longrightarrow \mathrm{H}^0(V_t, \mathcal{N}_t),$$

which Kodaira called the *infinitesimal displacement*. Naively, one constructs this as follows: for a curve $\gamma : D(0, R) \subset \mathbb{C} \longrightarrow B$ with $\gamma(0) = t$, $\frac{\partial \gamma}{\partial z}(0) = v$, and a fixed point $p \in V_t$, we associate a lift to W, satisfying

$$\widehat{\gamma}_p(0) = p$$

and consider the lift's velocity modulo vertical (i.e., along the fiber direction) displacements, resulting in the equivalence class of a tangent vector to W at p, modulo directions which are tangent to V_t at p, i.e., an element of $(\mathcal{N}_t)_p$. It turns out that the particular lift is not relevant (the velocities associated to distinct choices of lift differ by a vector tangent to V_t), so this map is well-defined on $\mathcal{T}_t B$. By varying the choice of $p \in V_t$ we obtain a *section* of N_t along V_t, the image $\mathrm{K}_t(v)$ of Kodaira's infinitesimal displacement map. Some work is necessary to show that this all varies holomorphically from point to point, we refer the reader to Kodaira's paper for the details.

Kodaira showed that under a vanishing condition on the normal bundle of the subvariety, it admits a locally maximal family of deformations, *roughly* parametrized at $t \in B$ by the space of sections of N_t.

Theorem 1.2.7. *(Kodaira [Kod62]) Let W be a complex manifold, V a compact submanifold thereof. Suppose that the normal bundle of V in W satisfies the condition*

$$\mathrm{H}^1(V, \mathcal{N}_{V|W}) = \{0\}.$$

Then there exists a family (B, t_0, Z) of deformations of V within W, such that the infinitesimal displacements

$$\mathrm{K}_t : \mathcal{T}_t B \longrightarrow \mathrm{H}^0(V_t, \mathcal{N}_t),$$

are isomorphisms. This family is locally maximal at every $t \in B$.

Naively, there are two simple ways a deformation of a subvariety can come about: intrinsic deformations of its complex structure, and extrinsic deformations (which change its normal bundle). In our setting, deformations of the first kind do not occur: $\mathbb{CP}^1$ is a rigid complex manifold, any two complex structures on it are isomorphic. The second issue is a bit more delicate: there are deformations of embedded rational curves that indeed change the normal bundle. According to Kodaira, what we need to ensure the *local* rigidity of the normal bundle is another vanishing condition: $\mathrm{H}^1(V, \mathrm{End}(\mathcal{N}_{V|W}) = \{0\})$.

The normal bundle of a twistor line satisfies the necessary vanishing condition to ensure the existence of locally maximal deformations,

$$\begin{aligned}\mathrm{H}^1(\mathbb{CP}^1, [\mathcal{O}_{\mathbb{CP}^1}(1)]^{\oplus 2}) &\approx [\mathrm{H}^1(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(1))]^{\oplus 2} \\ &\approx [\mathrm{H}^0(\mathbb{CP}^1, \mathcal{K}_{\mathbb{CP}^1} \otimes \mathcal{O}_{\mathbb{CP}^1}(-1))^*]^{\oplus 2}, \\ &\approx [\mathrm{H}^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(-3))^*]^{\oplus 2} \\ &= \{0\}.\end{aligned}$$

where we applied the Serre duality isomorphism $\mathrm{H}^k(X^d, F) \approx [\mathrm{H}^{d-k}(X^d, \mathcal{K}_X \otimes F^*)]^*$, on the second line. Furthermore, the dimension of $\mathrm{H}^0(C, \mathcal{N}_{C|Z}) = 4$, we should have a (complex) 4-parameter family of nearby rational curves. Again using Serre duality,

$$\mathrm{H}^1(\mathbb{CP}^1, \mathrm{End}(\mathcal{O}_{\mathbb{CP}^1}(1)^{\oplus 2})) = \mathrm{H}^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(-2)^{\oplus 4})) = \{0\},$$

on a sufficiently small neighborhood of C, the normal bundles of such curves retain the Birkhoff-Grothedieck decomposition $[\mathcal{O}_{\mathbb{CP}^1}(1)]^{\oplus 2}$. The real four-dimensonal family of twistor lines on Z arising as fibers of the twistor construction is thus part of a complex 4-dimensional family of rational curves, which we also call twistor lines. What sets real twistor lines apart is the fact that they are invariant under the anti-holomorphic involution that acts as the antipodal map on fibers of the twistor construction. An important technical remark is that the action of the involution on real twistor lines has no fixed points.

We are now in a position to characterize Twistor Spaces among complex, 3-dimensional manifolds, following [Pen76].

Theorem 1.2.8. *(Penrose) Let Z be a 3-dimensional complex manifold such that:*

1) Z is fibered over a real, oriented 4-dimensional manifold M by a family of rational curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$;

2) Z posesses an anti-holomorphic involution σ which acts as the antipodal map on the fibers.

Then M admits an anti-self-dual Riemannian metric, with respect to which Z is its twistor space.

We shall not present a proof of the theorem here, instead referring the reader to the original source, or [Bes87], chapter 13. We will, however, say a few words about how the metric structure is encoded in the hypotheses of the theorem. According to Kodaira's deformation theory, M is but a real slice of a complex 4-dimensional family of twistor lines, $M_{\mathbb{C}}$, so it makes sense to talk about the holomorphic tangent space at a point p on M. If p corresponds to a real twistor line C, one can identify the (holomorphic) tangent space $\mathcal{T}_p M_{\mathbb{C}}$ with the space of sections of a rank 2 vector bundle,

$$\mathrm{H}^0(C, \mathcal{O}_C(1) \oplus \mathcal{O}_C(1)) = \mathrm{H}^0(C, \mathcal{O}_C(1)) \oplus \mathrm{H}^0(C, \mathcal{O}_C(1)).$$

Such a section s_v decomposes into a pair of sections $(az^0 + bz^1, cz^0 + dz^1)$ of $\mathcal{O}_C(1)$ (written in homogeneous coordinates $[z^0 : z^1]$). We declare the null cone in $\mathcal{T}_p M_{\mathbb{C}}$ by stating that a tangent vector is null if the two sections have a common zero, that is $ad - bc = 0$. This defines a complex conformal structure (the freedom in choosing the scale parameter owes to the fact that (a, b) and (c, d) can only be specified up to scale) on $\mathcal{T}_p M_{\mathbb{C}}$. A quadratic form defining this conformal structure restricts to the real slice as a positive-definite[4] quadratic form, thus defining a conformal family of Riemannian metrics as we vary p along M.

1.3 Affine line bundles as twistor spaces

In this section we describe the main idea behind the construction of anti-self-dual metrics in this thesis: that twistor spaces may be constructed as affine line bundles over a complex surface. This idea appeared in the early 80s, in the work of Hitchin [Hit82] and Phillip E. Jones and K. P. Tod [JP85], in what was called the minitwistor correspondence, Claude LeBrun [LeB91], in the *hyperbolic ansatz*, and was later revealed to be related to the construction of hyperkähler metrics by Ulf Lidnström and Martin Roček [LR88] (a discussion of the relation between affine line bundles and the generalized Legendre Transform is presented in Moraru's thesis).

Definition 1.3.1. *Let $\mathcal{A}$ denote the group of affine transformations in $\mathbb{C}$. A holomorphic, affine line bundle over a complex manifold S is a fiber bundle with fiber $\mathbb{C}$ and transition functions given by local sections of the sheaf of $\mathcal{A}$-valued holomorphic maps on S.*

[4] It is here that the behavior of the involution on twistor lines is used. If the involution behaved like conjugation, for instance, this process would result on a pseudo-riemannian metric of split signature $(2, 2)$.

Naively, one can think of an affine line bundle as a holomorphic line bundle without a preferred zero section. In fact, to each affine line bundle we can associate an underlying holomorphic line bundle as follows. Let $\underline{U} = \{U_j\}_{j\in J}$ be a covering of S, and

$$\{\phi_{jk} \in \mathcal{A}_S(U_j \cap U_k)\}$$

a Čech cocycle defining an affine line bundle. Define a 1-cochain

$$\theta_{jk} \coloneqq \Theta(\phi_{jk}) \in C^1(\underline{U}, \mathcal{O}_S)$$

by means of the homomorphism $\Theta : \mathcal{A} \longrightarrow \mathbb{C}^*$ described above. The cocycle conditions ϕ, combined with the fact that Θ is a group homomorphism, readily imply that θ is a cocycle. Thus we may assign to a Čech cocycle ϕ defining an affine line bundle another cocycle θ defining a holomorphic line bundle. One verifies by a routine computation that if ϕ is modified by a coboundary, so is θ, thus the construction is unambiguous in cohomology. If an affine line bundle A is associated to a holomorphic line bundle L by this construction, we say that A is modeled after L.

We are interested in understanding under which conditions may we reverse this construction, that is, associate an affine structure to a holomorphic vector bundle. Let L be a holomorphic line bundle with transition data $\phi_j^k \in \mathcal{O}^*(U_j \cap U_k)$ associated to an open cover $\underline{U} = \{U_j\}_{j\in J}$ of X. These transition functions satisfy the usual constraints,

$$\phi_{jk}\phi_{kj} = 1,$$
$$\phi_{jk}\phi_{kl}\phi_{lj} = 1,$$

on the respective intersections where these equalities make sense. Consider the Cartier data associated to L and $\underline{U}$, that is, a global section of the quotient sheaf $\mathcal{M}^*/\mathcal{O}^*$, represented locally by a collection of invertible meromorphic functions $s_j \in \mathcal{M}^*$ satisfying

$$s_j = \phi_{jk} s_k.$$

We denote by D the Weil divisor associated to the collection s_j, and denote by $\mathcal{O}(D)$ the corresponding invertible sheaf, whose local sections are meromorphic functions $f_p \in \mathcal{M}^*(U_p)$ such that $f_p s_p$ is holomorphic. Under the correspondence between invertible sheaves and line bundles, $\mathcal{O}(D) = L$.

If we wish to construct an affine bundle modeled after L, we need to augment its transition data to include the translation components, that is, we need sections $\psi_{jk} \in \mathcal{O}_S(U_j \cap U_k)$ so that the local sections

$$\theta_{jk} = \begin{pmatrix} \phi_{jk} & \psi_{jk} \\ 0 & 1 \end{pmatrix} \in \mathcal{A}(U_j \cap U_k)$$

satisfy the 1-cocycle condition

$$\theta_{kl}(\theta_{jl})^{-1}\theta_{jk} = 1, \tag{1.3.1}$$

where this equality makes sense. This suggests a cohomological interpretation, and the next proposition shows that this is the case.

Proposition 1.3.2. *Let S be a complex manifold and L a holomorphic line bundle on S. The space of affine line bundles modeled on L is given by the cohomology group $\mathrm{H}^1(S, L)$.*

Proof. Equation (1.3.1) amounts to 4 equations on the entries of the matrix, two of which (the bottom row) are identities, and one of which (upper diagonal) is satisfied by assumption (since ϕ satisfies its own cocycle condition). The remaining equation is

$$\phi_{kl}\phi_{lj}\psi_{jk} - \phi_{kl}\phi_{lj}\psi_{jl} + \psi_{kl} = 0.$$

The interpretation of this equation is that the $\mathcal{O}_S(D)$-valued cochain defined by

$$\xi_{jk} = \frac{\psi_{jk}}{s_j}$$

on $U_j \cap U_k$, satisfies a cocycle condition: on a triple overlap $U_j \cap U_k \cap U_l$,

$$\begin{aligned} \xi_{kl} - \xi_{jl} + \xi_{jk} &= \frac{\psi_{kl}}{s_k} - \frac{\psi_{jl}}{s_j} + \frac{\psi_{jk}}{s_j} \\ &= \frac{\psi_{kl}}{s_k} - \frac{\psi_{jl}}{\phi_{jk}s_k} + \frac{\psi_{jk}}{\phi_{jk}s_k} \\ &= \frac{\psi_{kl} - \phi_{kl}\phi_{lj}\psi_{jl} + \phi_{kl}\phi_{lj}\psi_{jk}}{s_k} \\ &= 0. \end{aligned}$$

We recall that the boundary operator acting on a cochain $\zeta \in C^0(\underline{U}, \mathcal{O}_S(D))$,

$$\zeta = \{\zeta_j \in \mathcal{O}_S(D)|_{U_j}\}$$

is defined by

$$(\partial\zeta)_{jk} = \zeta_k - \zeta_j \in \mathcal{O}_S(D)(U_j \cap U_k).$$

Suppose that ξ' differs from ξ by such a coboundary, that is,

$$\begin{aligned} \xi^{'}_{jk} &= \xi_{jk} + \zeta_k - \zeta_j. \\ \frac{\psi^{'}_{jk}}{s_j} &= \frac{\psi_{jk}}{s_j} + \zeta_k - \zeta_j \\ \psi^{'}_{jk} &= \psi_{jk} + (\zeta_k - \zeta_j)s_j \end{aligned}$$

We will try to define a cochain $\gamma \in C^0(\underline{U}, \mathcal{A})$, depending upon ζ, so that the cochain associated to

$$\theta'_{jk} = \gamma_j \theta_{jk} \gamma_k^{-1}$$

is ξ'. Write

$$\theta'_{jk} = \begin{pmatrix} \phi'_{jk} & \psi'_{jk} \\ 0 & 1 \end{pmatrix}, \ \theta_{jk} = \begin{pmatrix} \phi_{jk} & \psi_{jk} \\ 0 & 1 \end{pmatrix},$$

and

$$\gamma_p = \begin{pmatrix} a_p & b_p \\ 0 & 1 \end{pmatrix},$$

for $p = j, k$. Then the above equality implies that

$$\begin{aligned}
\begin{pmatrix} \phi'_{jk} & \psi'_{jk} \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} a_j & b_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_{jk} & \psi_{jk} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_k & b_k \\ 0 & 1 \end{pmatrix}^{-1} \\
&= \begin{pmatrix} a_j & b_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_{jk} & \psi_{jk} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_k^{-1} & -b_k a_k^{-1} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a_j \phi_{jk} & (a_j \psi_{jk} + b_j) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_k^{-1} & -b_k a_k^{-1} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a_j \phi_{jk} a_k^{-1} & (-a_j \phi_{jk} b_k a_k^{-1} + a_j \psi_{jk} + b_j) \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

from which we derive two equations for a and b,

$$\begin{aligned}
\phi'_{jk} &= a_j \phi_{jk} a_k^{-1}, \\
\psi'_{jk} &= -a_j \phi_{jk} b_k a_k^{-1} + a_j \psi_{jk} + b_j.
\end{aligned}$$

The first equation tells us the the cochains $\phi', \phi \in C^1(\underline{U}, \mathcal{O}_S^*)$ differ by a coboundary. In particular, they define the same model holomorphic line bundle, L. To interpret the second equation, we make the choice of $a_p = 1$ on U_p, for all p, reducing it to

$$\psi'_{jk} = -\phi_{jk} b_k + \psi_{jk} + b_j.$$

Comparing this equation to

$$\psi'_{jk} = \psi_{jk} + (\zeta_k - \zeta_j) s_j$$

we deduce that $b_p = -\zeta_p s_p \in \mathcal{O}_S(U_p)$ yields a cochain

$$\gamma_p = \begin{pmatrix} 1 & b_p \\ 0 & 1 \end{pmatrix} \in C^0(\underline{U}, \mathcal{A})$$

satisfying the desired criteria.

We have shown that to each cocycle $\xi \in C^1(\underline{U}, \mathcal{O}_S(D)) = C^1(\underline{U}, L)$ we can associate an affine structure on the holomorphic line bundle L, and that if two such cocycles ξ, ξ' differ by a coboundary, the resulting affine structures are isomorphic. Thus the cohomology group $\mathrm{H}^1(S, L)$ classifies affine structures on L. $\square$

Next we discuss how twistor spaces can be realized as the total spaces of affine line bundles over surfaces. Consider a complex surface S endowed with a holomorphic line bundle L for which $H^1(S, L) \neq 0$, so that L admits proper (i.e. without a zero section) affine structures. Let Z be an affine line bundle over S, modeled after L, with bundle projection $p : Z \longrightarrow S$.

Assume that S contains a rational curve C with normal bundle $\mathcal{O}_C(n)$,$n \geq 1$. Suppose this curve admits a lift to Z (an assumption we will have to come to terms with later), say $\widetilde{C}$. Consider the pull-back of the normal bundle of C in S to Z, $p^*\mathcal{O}_C(n)$. Since $\widetilde{C}$ is a lift of C through p, $p|_{\widetilde{C}}$ preserves degree, so $p^*\mathcal{O}_C(n) = \mathcal{O}_{\widetilde{C}}(n)$. The normal bundle of $\widetilde{C}$ in Z, $\mathcal{N}_{\widetilde{C}|Z}$, maps surjectively onto $\mathcal{O}_{\widetilde{C}}(n)$, with kernel consisting of *vertical sections* of $\mathcal{N}_{\widetilde{C}|Z}$ along $\widetilde{C}$, that is, sections in the kernel of the bundle projection p. The kernel subbundle can thus be identified with the pull-back of model line bundle $L|_C$ to $\widetilde{C}$, $p^*L|_{\widetilde{C}}$. We can summarize this in the following exact sequence:

$$0 \longrightarrow p^*L|_{\widetilde{C}} \xrightarrow{\iota} \mathcal{N}_{\widetilde{C}|Z} \xrightarrow{\eta} \mathcal{O}_{\widetilde{C}}(n) \longrightarrow 0. \tag{1.3.2}$$

If we wish for $\widetilde{C}$ to be a twistor line in our presumptive twistor space Z, its normal bundle should be

$$\mathcal{N}_{\widetilde{C}|Z} = \mathcal{O}_{\widetilde{C}}(1) \oplus \mathcal{O}_{\widetilde{C}}(1),$$

therefore in order for the exact sequence (1.3.2) to hold, p^*L should have degree $(2-n)$ on $\widetilde{C}$. Again, since $p|_{\widetilde{C}}$ preserves degree, this implies that our starting line bundle L must have degree $(2-n)$ when restricted to C. Readers familiar with the minitwistor construction [Hit82] will now recognize that when $n = 2$, this is the condition that L is trivial along minitwistor lines.

Let us now restrict our attention to the central theme of the thesis: conics in the projective plane. Such curves have normal bundle with degree 4, so to proceed with the construction, we need a line bundle L on $\mathbb{CP}^2$ whose restriction to every conic has degree (-2). In this case, the line bundle $L = \mathcal{O}_{\mathbb{CP}^2}(-1)$ is the right tool for the job. There is, however, one minor issue: the cohomology group $H^1(\mathbb{CP}^2, L)$ vanishes, thus violating our initial assumption that

L admits proper affine structures. We will resolve this by restricting our attention to open subsets of $\mathbb{CP}^2$ instead.

Recall we made an earlier assumption that the rational curves has lifts to Z, i.e., that $A|_C$ admitts sections. This means that the affine line bundle A, when restricted to a conic C, must be improper, isomorphic to $L|_C$. Since the latter is a negative line bundle over C, if it is indeed the case that $A|_C$ has a section, it is unique (the zero section of L). Determining whether $A|_C$ has a section or not comes down to understanding how the affine bundle A restricts to C, as an element in

$$H^1(C, L|_C) = H^1(C, \mathcal{O}_C(-2)) = [H^0(C, \mathcal{O}_C)]^* \approx \mathbb{C},$$

where in the second equality we used the Serre duality isomorphism, and in the third an identification between sections of the trivial line bundle of C and constants. As we vary the curves C in $\mathbb{CP}^2$, the latter identication in this process is not canonical, but we may fix this by using a consistent identification between the canonical bundle of C and the restriction of a globally defined bundle on $\mathbb{CP}^2$

$$\mathcal{K}_C = \mathcal{O}_{\mathbb{CP}^2}(-1)|_C,$$

a consequence of the adjunction formula. We thus obtain a function F, defined on the space of conics Y^5 taking values in $\mathbb{C}$, which characterizes whether the affine bundle A stands a chance of being a twistor space or not: if a conic $C \in \mathbb{CP}^2$ admits a lift to a twistor line in A, its image under F must be 0.

An alternative description of this function is as follows. Let an affine structure $f \in H^1(U, L)$ be interpreted as a holomorphic one-form with values in $L = \mathcal{O}_{\mathbb{CP}^2}(-1)$ defined on a neighborhood[5] U of a conic C in the projective plane. Restricting f to the conic, if becomes a holomorphic differential with values in $\mathcal{O}_C(-2)$, which by means of integration, yields the value of F at C. This makes the dependence of F on the affine structure on L clear.

The crrespondence between affine structures on L and functions on the space of conics is what we call the *Penrose Transform*. Its properties will be investigated further in subsequent chapters.

[5] As we explained above, there are no non-zero globally defined one forms

Chapter 2

Geometry on the space of plane conics

In this chapter, we describe geometric structures on the space of smooth, irreducible conics in $\mathbb{CP}^2$ which are central to the construction of anti-self-dual metrics. Upon fixing a choice of homogeneous coordinates in $\mathbb{CP}^2$, $x = [x^0 : x^1 : x^2]$, we may assign to a non-denegerate conic C a symmetric, 3x3 matrix defining it:

$$C\colon \;\; xAx^T = 0.$$

Of course, A is only defined up to an overall scale, so we fix the representation by setting $\det(A) = 1$.

We can describe this space in a coordinate-free manner as well. The group $\mathrm{SL}(3, \mathbb{C})$ acts on the space of complex-symmetric 3x3 matrices by

$$(B, A) \mapsto BAB^T,$$

for $B \in \mathrm{SL}(3, \mathbb{C})$, A a complex, symmetric 3x3 matrix. This action preserves the determinant, and the stabilizer of the identity is the subgroup of $\mathrm{SL}(3, \mathbb{C})$ consisting of matrices whose transposes and inverses coincide, i.e., the complex special orthogonal group $\mathrm{SO}(3, \mathbb{C})$. Therefore, the space of non-degenerate plane conics can be interpreted as the quotient $\mathbb{Y}_C = \mathrm{SL}(3, \mathbb{C})/\mathrm{SO}(3, \mathbb{C})$.

This complex, 5-dimensional manifold admits a real form central to the construction of anti-self-dual metrics presented here. This is the locus of real conics without real points. Upon fixing homogeneous coordinates in $\mathbb{CP}^2$, we can define an involution by conjugation,

$$\sigma([x^0 : x^1 : x^2]) = [\overline{x^0} : \overline{x^1} : \overline{x^2}],$$

whose fixed-point set is an embedded real projective plane, $\mathbb{RP}^2$. A conic is called real if it is an invariant locus of this involution, and it is said to have no real points if it does not contain fixed points of σ. The conic defined by the identity matrix is a real conic without real points, and so is any other conic in its $\mathrm{SL}(3,\mathbb{R})$-orbit. Of course, its stabilizer relative to the $\mathrm{SL}(3,\mathbb{R})$ action is $\mathrm{SO}(3,\mathbb{R})$, so the locus of real conics without real points is $Y = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$.

2.1 Conformal and SO(3) structures

The space of plane conics may be interpreted as a *family* of rational curves in $\mathbb{CP}^2$, in the sense of Kodaira's deformation theory. Such a curve C has normal bundle $\mathcal{O}_{\mathbb{CP}^2}(2)\big|_C$, a line bundle of degree 4 on C (that is, two conics intersect at 4 points, counted with multiplicity). This bundle has vanishing cohomology in degree 1, and its space of global sections has dimension 5, hence the manifold $Y_{\mathbb{C}}$ is indeed the locally maximal family whose existence is guaranteed by Kodaira's theorem. The infinitesimal displacement map provides an isomorphism between tangent spaces to $Y_{\mathbb{C}}$ at C and the space of sections of $\mathcal{O}_C(4)$, $\mathrm{H}^0(C,\mathcal{O}_C(4))$.

We will introduce a family of bilinear forms on spaces of binary quantics which will serve to streamline notation in this and the next section.

Let V_n denote the space of homogeneous binary quantics of degree n,

$$V_n = \mathrm{Sym}^{n+1}(\mathbb{C}^2) = H^0(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(n))$$

and denote by V the space formed by the direct sum of all V_n,

$$V = \bigoplus_{l=0}^{\infty} V_n.$$

The k-th order transvectant, where k is a non-negative integer, is the bilinear form

$$\langle \cdot, \cdot \rangle_k : V \times V \longrightarrow V$$

defined by

$$\langle \phi, \psi \rangle_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{\partial^k \phi}{\partial s^{k-j} \partial t^j} \frac{\partial^k \psi}{\partial s^j \partial t^{k-j}}.$$

If ϕ has degree m, ψ has degree n, and $m+n \geq 2k$, their k-th transvectant has degree $m+n-2k$. In particular, if $m=n=k$, the transvectant yields a constant. Some special cases are well-known invariants, as the example below shows.

Example 2.1.1. *The zeroth-order transvectant is simply the product of the quantics. The first-order transvectant is the Jacobian determinant:*

$$\langle \phi, \psi \rangle_1 = \frac{\partial \phi}{\partial s}\frac{\partial \psi}{\partial t} - \frac{\partial \phi}{\partial t}\frac{\partial \psi}{\partial s},$$

an operation we used in section 1.2 to describe the conformal structure on the space of twistor lines.

Example 2.1.2. *If ϕ is quadratic, $\phi = as^2 + 2bst + ct^2$, then the transvectant $\langle \phi, \phi \rangle_2$ is a multiple of the familiar discriminant:*

$$\begin{aligned}\langle \phi, \phi \rangle_2 &= \binom{2}{0} \frac{\partial^2 \phi}{\partial s^2} \frac{\partial^2 \psi}{\partial t^2} - \binom{2}{1} \frac{\partial^2 \phi}{\partial s \partial t} \frac{\partial^2 \psi}{\partial s \partial t} - \binom{2}{2} \frac{\partial^2 \phi}{\partial t^2} \frac{\partial^2 \psi}{\partial s^2} \\ &= 4ac - 8b^2 + 4ac \\ &= -8(b^2 - ac)\end{aligned}$$

The 4th-order transvectant on binary quartics yields a non-denegerate quadratic form on $\mathcal{T}_C Y_{\mathbb{C}} \approx \mathrm{H}^0(\mathbb{C}^1, \mathcal{O}_{\mathbb{C}^1}(4))$: if $[t:s]$ are homogeneous coordinates on C, and v a tangent vector to C in $Y_{\mathbb{C}}$, identified with

$$v = \alpha t^4 + 4\beta t^3 s + 6\gamma t^2 s^2 + 4\delta t s^3 + \epsilon s^4, \tag{2.1.1}$$

then we find

$$\langle v, v \rangle_4 = 1152(\alpha\epsilon - 4\beta\delta + 3\gamma^2). \tag{2.1.2}$$

This collection of quadratic forms on the tangent spaces of $Y_{\mathbb{C}}$ defines a complex conformal structure.

Next we describe the restriction of this conformal structure to the real slice Y. It is convenient to introduce coordinates at this point. On a neighborhood of the identity, any real symmetric matrix A with $\det(A) = 1$ can be represented as

$$A = BB^T,$$

where B takes the form

$$B = \begin{pmatrix} e^c & pe^b & qe^a \\ 0 & e^b & re^a \\ 0 & 0 & e^a \end{pmatrix},$$

and $a+b+c = 1$. We use (a, b, p, q, r) as coordinates on Y (on a neighborhood of the identity), so that a conic A is represented by

$$A = \begin{pmatrix} e^{-2(a+b)} + e^{2a}q^2 + e^{2b}p^2 & e^{2a}qr + e^{2b}p & e^{2a}q \\ e^{2a}qr + e^{2b}p & e^{2a}r^2 + e^{2b} & e^{2a}r \\ e^{2a}q & e^{2a}r & e^{2a} \end{pmatrix}$$

Let us define a complex coframe on Y by

$$\begin{aligned}
\eta^1 &= -4da - 2db - 2\sqrt{-1}e^{2a+b}(dq - pdr)),\\
\eta^2 &= -e^{a+2b}dp + \sqrt{-1}e^{a-b}dr,\\
\eta^3 &= 2db,\\
\eta^4 &= -\overline{e^2},\\
\eta^5 &= \overline{e^1}.
\end{aligned}$$

The dual frame will be denoted by $\mu_1, \cdots, \mu_5$. In terms of this frame, the identification between tangent vectors to a conic $C = C(A)$ (as matrices) and quartic polynomials is given by

$$\mu_1 \mapsto t^4, \mu_2 \mapsto 4t^3 s, \mu_3 \mapsto 6t^2 s^2, \mu_4 \mapsto 4s^3 t, \mu_5 \mapsto s^4$$

With respect to this coframe, the metric (2.1.2) simplifies to

$$g = 2\eta^1 \odot \eta^5 - 8\eta^2 \odot \eta^4 + 6\eta^3 \odot \eta^3, \tag{2.1.3}$$

where we divide the transvectant by a constant to simplify the expression for the metric.

Dunajski and Tod [DT18] explicitly described one representative of the conformal class of metrics defined above in these coordinates.

Proposition 2.1.1. *(Dunajski-Tod) The conformal structure on $Y_{\mathbb{C}}$ restricts to a positive-definite conformal structure on Y. In the coordinates described above, the metric* (2.1.2) *can be written at a conic A as*

$$\begin{aligned}
g &= 4\mathrm{Tr}(A^{-1}dA \cdot A^{-1}dA) &(2.1.4)\\
&= 8\left(4da^2 + 4dadb + 4db^2 + e^{2a+4b}dp^2 + e^{4a+2b}(pdr - dq)^2 + e^{2a-2b}dr^2\right), &(2.1.5)
\end{aligned}$$

or in terms of its component matrix,

$$\begin{pmatrix}
32 & 16 & 0 & 0 & 0\\
16 & 32 & 0 & 0 & 0\\
0 & 0 & 8e^{2a+4b} & 0 & 0\\
0 & 0 & 0 & 8e^{4a+2b} & -8pe^{4a+2b}\\
0 & 0 & 0 & -8pe^{4a+2b} & 8p^2e^{4a+2b} + 8e^{2a-2b}
\end{pmatrix}.$$

The group $\mathrm{SL}(3,\mathbb{R})$ acts on (Y, g) by isometries, with isotropy $\mathrm{SO}(3,\mathbb{R})$, thus g realizes Y as the symmetric space $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$. The metric is Einstein with constant $-\frac{3}{16}$.

Proof. Simply expanding the metric in terms of the coframe $\{da, db, dp, dq, dr\}$ leads to the expression (2.1.5), which clearly shows it is positive definite.

Showing that $\mathrm{SL}(3,\mathbb{R})$ acts by isometries can be done computationally, as the generators of the action

$$\begin{aligned}
X_1 &= \partial_p + r\partial_q \\
X_2 &= \partial_q \\
X_3 &= \partial_r \\
X_4 &= \partial_a - p\partial_p - 2q\partial_q \\
X_5 &= \partial_b - 2p\partial_p - q\partial_q + r\partial_r \\
X_6 &= p\partial_b - (1 + p^2 - e^{-2a-4b})\partial_p - r\partial_q + q\partial_r \\
X_7 &= r\partial_a - r\partial_b - (1 + r^2 - e^{2b-2a})\partial_r + (e^{2b-2a}p - rq)\partial_q + (pr - q)\partial_p \\
X_8 &= q\partial_a - rp\partial_b + (p^2 r - re^{-2a-4b} - qp)\partial_p \\
&\quad + (p^2 e^{2b-2a} + e^{-4a-2b} - q^2 - 1)\partial_q + (e^{2b-2a}p - rq)\partial_r
\end{aligned}$$

can be shown to be Killing fields. The latter three generate the group of rotations $\mathrm{SO}(3,\mathbb{R})$, as they satisfy the commutator relations

$$[X_6, X_7] = X_8,\ [X_6, X_8] = -X_7,\ [X_7, X_8] = X_6.$$

A verification of this assertion was done via machine-aided computation, by means of the TensoriaCalc package for Wolfram Mathematica [Wol], developed by Yi-Zen Chu [Chu].

Alternatively, if one uses the condensed form of the metric, (2.1.4), showing $\mathrm{SL}(3,\mathbb{R})$-invariance is simple: if $B \in \mathrm{SL}(3,\mathbb{R})$, then

$$\begin{aligned}
g(BAB^T) &= 4\mathrm{Tr}((BAB^T)^{-1}d(BAB^T) \cdot (BAB^T)^{-1}d(BAB^T)) \\
&= 4\mathrm{Tr}((B^T)^{-1}A^{-1}B^{-1}B(dA)B^T \cdot (B^T)^{-1}A^{-1}B^{-1}B(dA)B^T) \\
&= 4\mathrm{Tr}((B^T)^{-1}[A^{-1}(dA)A^{-1}(dA)]B^T) \\
&= 4\mathrm{Tr}([A^{-1}(dA)A^{-1}(dA)](B^T)^{-1}B^T) \\
&= 4\mathrm{Tr}(A^{-1}(dA)A^{-1}(dA))
\end{aligned}$$

Direct computation of curvature would be a cumbersome task, as the Riemann curvature tensor has dozens of non-zero components. Instead, we resort to computation of the Ricci tensor via TensoriaCalc, which leads to

$$r_{ij} = \begin{pmatrix} -6 & -3 & 0 & 0 & 0 \\ -3 & -6 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2}e^{2a+4b} & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{2}e^{4a+2b} & \frac{3}{2}pe^{4a+2b} \\ 0 & 0 & 0 & \frac{3}{2}pe^{4a+2b} & -\frac{3}{2}e^{2a-2b}\left(p^2e^{2a+4b}+1\right) \end{pmatrix}.$$

In other words, the metric is Einstein

$$r_{ab} = -\frac{3}{16}g_{ab},$$

with scalar curvature $-\frac{15}{16}$. □

We define a second transvectant operation which will be useful to establish the Penrose correspondence in the next section. According to Marcin Bobieński and Pawel Nurowski [BN07], there is a symmetric, cubic 3-form which encodes the reduction of the structure group of the SO(5)-frame bundle of Y to the image of SO(3) associated to the 5-dimensional irreducible representation of the latter.

The key properties of the cubic form G are encoded in the following result of Bobieński and Nurowski.

Proposition 2.1.2 ([BN07])**.** *Suppose an oriented, Riemannian 5-manifold (M, g) admits a reduction of its $SO(5,\mathbb{R})$-frame bundle to an irreducible $SO(3,\mathbb{R})$-bundle (defined by the unique 5-dimensional representation of* $\mathrm{SO}(3,\mathbb{R})$*). Then there exists a rank 3 tensor field G satisfying the following three properties:*

1. *it is totally symmetric: $G_{abc} = G_{(abc)}$;*
2. *it is trace-free: $g^{bc}G_{abc} = 0$;*
3. *it satisfies the identity*

$$G^a{}_{(bc}G_{de)a} = g_{(bc}g_{de)}$$

Conversely, the existence of such a tensor defines a reduction of the structure group of the frame bundle to an irreducible $\mathrm{SO}(3,\mathbb{R})$.

This cubic 3-form G can be easily described in terms of transvectants, as being generated by

$$\langle\langle v, v\rangle_2, v\rangle_4 .$$

If v is written as a quartic as in (2.1.1), then G is

$$G(v) = -\gamma(\alpha\epsilon + 2\beta\delta) + \alpha\delta^2 + \beta^2\epsilon + \gamma^3$$

(here we drop the multiplicative factor of (-497664) for the sake of convenience). In terms of the complex coframe defined above, G takes the form

$$G = \eta^1 \odot \eta^4 \odot \eta^4 + \eta^2 \odot \eta^2 \odot \eta^5 + \eta^3 \odot \eta^3 \odot \eta^3 - \eta^1 \odot \eta^3 \odot \eta^5 - 2\eta^2 \odot \eta^3 \odot \eta^4,$$

which makes it evident that G is symmetric. In terms of the coordinates a, b, p, q, r

$$\begin{aligned} G = &-8p^2e^{4a+2b}dbdr^2 + 16pe^{4a+2b}dbdqdr + 8e^{4a+2b}dpdqdr - 8pe^{4a+2b}dpdr^2 - 8e^{2a+4b}dadp^2 \\ &- 8e^{4a+2b}dbdq^2 + 8e^{2a-2b}dadr^2 + 8e^{2a-2b}dbdr^2 - 32da^2db - 32dadb^2. \end{aligned}$$

It can be laboriously verified that G satisfies properties (2) and (3) above. In addition, G is invariant under the $\mathrm{SL}(3,\mathbb{R})$ isometry group of g, and G is parallel

$$\nabla_a G_{bcd} = 0,$$

a condition described as *integrability* by Bobieński and Nurowski.

2.2 The Penrose Transform on the space of conics

The link between the geometry of the space of conics and anti-self-duality is provided by the Penrose Transform. Recall from chapter 1 that this is a correspondence between affine structures on the holomorphic line bundle $\mathcal{O}_{\mathbb{CP}^2}(-1)$ (over suitable open subsets of $\mathbb{CP}^2$) and functions on the space of conics. In his thesis, Moraru [Mor04] found that functions arising from this construction are not arbitrary, but rather satisfy a system of linear, second-order differential equations on $Y = \{A \in \mathrm{M}_{3\times3}(\mathbb{R}) | A = A^t,\ \det(A) = 1\}$,

$$D_1 F = \left(\mathrm{Tr}\left(A\frac{\partial}{\partial A}\right)\right)^2 F - F = 0,$$
$$D_2 F = \left(A\frac{\partial}{\partial A}\right)^2 F = 0,$$

where by $\frac{\partial}{\partial A}$ we mean the matrix-operator

$$\left(\frac{\partial}{\partial A}\right)_{ij} = \left(\frac{1+\delta_{ij}}{2}\right)\frac{\partial}{\partial a_i j}.$$

Conversely, given a solution to $D_1F = D_2F = 0$ on Y, one induces an anti-self-dual conformal structure on the level set $F = 0$ (provided a certain set of regularity conditions is satisfied, more on that later) via restriction of the conformal structure on Y. Furthermore, Moraru proved that the solution sets to this system of equations is $\mathrm{SL}(3,\mathbb{R})$-invariant, that is, if F is any solution and $g \in \mathrm{SL}(3,\mathbb{R})$, then $F \circ g$ is also a solution. This should, in principle, lead to a great deal of solutions obtained by linearity and $\mathrm{SL}(3,\mathbb{R})$-tranlations, and this is the main motivation of the thesis.

These operators were obtained by extensive use of machine-aided computation, and their geometric nature was unclear until Dunajski and Tod [DT18] put them into the context of the metric and SO(3)-structure defined in the previous section. One is simply

$$\Delta_g F + \frac{1}{12}F$$

where Δ_g is the metric Laplacian, while the other is defined in terms of the $SO(3)$ structure as

$$\Box F - \frac{1}{24}dF,$$

where

$$\Box F = (G_a^{\ bc}\nabla_b\nabla_c F)\eta^a,$$

dF stands for the usual exterior derivative, and η is the coframe defined in the previous section. Their result is as follows.

Theorem 2.2.1. *Dunajski-Tod, [DT18] Let the function $F : Y \longrightarrow \mathbb{R}$ belong to the image of the Penrose transfrom. Then*

$$\Delta_g F + \frac{1}{12}F = 0$$
$$\Box F - \frac{1}{24}dF = 0.$$

As a result of this new interpretation, Dunajski and Tod were able to obtain explicit families of solutions (most of which are defined on non-compact manifolds, or have singularities along which the normal bundles of twistor lines degenerate into $\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(2)$).

We end this section with a summary of the results of Moraru, Dunajski and Tod concerning the relation between solutions to the DMT system and anti-self-dual metrics.

Theorem 2.2.2. *Moraru, [Mor04], Dunajski-Tod, [DT18] Let $F : Y \longrightarrow \mathbb{R}$, and let M denote its zero locus. Let ρ denote the identification between 1-forms on Y and quartic polynomias provided by the coframe η, and consider the polynomial $\rho(dF)$. Define a 1-form ω*

(identifed with a polynomial) to be null if it can be written as $\omega = \kappa \times \lambda$*, where* κ *is a cubic and* λ *is a linear form, and*

$$\langle \rho(dF), \kappa \rangle_3 = 0.$$

The quadratic cone thus defined is non-degenerate if the J*-invariant*[1] *of the quartic,*

$$J = \langle \langle \rho(dF), \rho(dF) \rangle_2 \, \rho(dF) \rangle_4$$

does not vanish. In this case, the Weyl tensor of the metric is anti-self-dual if F *satisfies the DMT system.*

[1] Observe that this is the metric on Y.

Chapter 3

From conics to anti-self-dual metrics

3.1 Complex projective 3-space

This example is due to Dunajski and Tod [DT18]. We summarize their work here as a guideline for our construction in the next section.

Consider the vector space V_3 of homogeneous binary cubics in variables $[s, t]$. This is a 4-dimensional complex vector space, endowed with an irreducible representation of $\mathrm{SL}(2, \mathbb{C})$, $V_3 = \mathrm{Sym}^3(\mathbb{C}^2)$.

There is on V_3 a *holomorphic symplectic structure*, that is, a closed holomorphic 2-form Ω such that Ω^4 is a nowhere-vanishing top degree form. In a coordinate system $p_{ABC} = p_{(ABC)}$ $(A, B, C \in \{0.1\})$, this is simply the form

$$\Omega = dp_{ABC} \wedge dp^{ABC}.$$

where indices are raised with respect to the anti-symmetric matrix ϵ^{AB} with $\epsilon_{01} = -1$.

Proposition 3.1.1. *The* $\mathrm{SL}(2, \mathbb{C})$ *action on* V_3 *given by*

$$(A, u \odot v \odot w) \in \mathrm{SL}(2, \mathbb{C}) \times V_3 \mapsto (Au \odot Av \odot Aw)$$

on decomposable elements preserves the symplectic form.

Proof. A set of infinitesimal generators of the action can be computed explicitly,

$$H_{AB} = 2p_{(A}{}^{BC}\partial_{B)CD},$$

where $\partial_{BCD} = \frac{\partial}{\partial p^{BCD}}$. The interior products of the generators with Ω are closed 1-forms,

$$\iota_{H_{AB}}\Omega = d(p_A{}^{DE}p_{BDE})$$

hence by Cartan's magic formula $\mathcal{L}_{H_{AB}}\Omega = 0$. □

The Hamiltonians $\xi_{AB} = p_A{}^{DE} p_{BDE}$ span a 3-dimensional vector space isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, namely the 3-dimensional representation of $\mathrm{SL}(2,\mathbb{C})$ on V_2, the space of binary quadratics. In fact, the image of the moment map can be described in terms of a transvectant operations: a cubic $p = as^3 + 3bs^2t + 3cst^2 + dt^3$ is mapped to the quadratic

$$\begin{aligned}\langle p, p\rangle_2 &= (as+bt)(cs+dt) - (bs+ct)^2 \\ &= (-b^2+ac)s^2 + (ad-bc)st + (bd-c^2)t^2.\end{aligned}$$

up to a multiplicative factor of 72. The image thus vanishes if and only if the polynomial p has three common zeros.

Away its zero locus, the moment map descends to a map between the projectivizations of V_3 and V_2: $Q : \mathbb{CP}^3 \setminus \mathcal{R}_3 \longrightarrow \mathbb{CP}^2$, where $\mathcal{R}_3$ is the rational normal cubic consisting of polynomials with a triple root.

The map Q is quadratic on the homogeneous coordinates $[a\colon b\colon c\colon d]$ in $\mathbb{CP}^3$, that is, it is given by sections of the $\mathcal{O}_{\mathbb{CP}^3}(2) = \mathcal{K}_{\mathbb{CP}^2}^{\frac{-1}{2}}$ line bundle, the fundamental line bundle of $\mathbb{CP}^3$ as a twistor space. It follows that Q maps twistor lines in $\mathbb{CP}^3$ (which miss the rational normal cubic, a generic condition) into conics in $\mathbb{CP}^2$. Dunajski and Tod characterized the conics in Y which are images of twistor lines under Q.

Theorem 3.1.2. *Dunajski-Tod, [DT18] Let $Q : \mathbb{CP}^3 \setminus \mathcal{R}_3 \longrightarrow \mathbb{CP}^2$ be the projectivization of the moment map for the irreducible $\mathrm{SL}(2,\mathbb{C})$ representation on V_3. A conic $C \in Y_{\mathbb{C}}$, parametrized by a symmetric, 3x3 matrix A with unit determinant, is the image of a twistor line in $\mathbb{CP}^3 \setminus \mathcal{R}_3$ if and only if*

$$\mathrm{Tr}(A^2) - 2\mathrm{Tr}(A)^2 = 0$$

Under the Penrose fibration, $\mathcal{P} : \mathbb{CP}^3 \longrightarrow S^4 = \mathbb{HP}^1$, given by

$$[z^0\colon z^1\colon z^2\colon z^3] \mapsto [z^0 + z^1 j\colon z^2 + z^3 j],$$

the rational normal cubic $\mathcal{R}_3\colon [s,t] \mapsto [s^3\colon s^2t\colon st^2\colon t^3]$ is to a sphere $\mathbb{S}^2$ in a 2-sheeted branched covering, which on the affine chart covering the north pole $[1\colon 0] \in \mathcal{R}_3$ with coordinate $u = s/t$, can be written as $\omega(u) = [1+uj\colon u^2(1+uj)] \in \mathbb{HP}^1$, or using affine coordinates on a neighborhood of $[1:0] \in \mathbb{HP}^1$, $\omega(u) = u^2$, a 2-1 cover branched at the origin in $\mathbb{C}$. A similar phenomenon is observed near the south pole $[0,1]$ of $\mathcal{R}_3$.

The Penrose transform thus constructs an anti-self-dual metric the complement of an S^2 in S^4. By means of a conformal change, the metric may be extended to S^4, resulting in the usual round metric which gives rise to $\mathbb{CP}^3$ as its twistor space.

3.2 The Flag variety

The flag variety $F_{1,2}(\mathbb{C}^3)$, henceforth denoted simply by $F_{1,2}$, is the variety parametrizing flags $\{l, \Pi\}$, consisting of lines, l, and planes, Π, containing the origin in $\mathbb{C}^3$, and such that the line is contained in the plane.

In what follows, we shall use two concrete models of the flag variety: as a hypersurface in $\mathbb{CP}^2 \times (\mathbb{CP}^2)^*$ (where $(\mathbb{CP}^2)^*$ denotes the projectivization of the dual space $(\mathbb{C}^3)^*$); as the projectivized cotangent bundle of $\mathbb{CP}^2$.

The first model is explained as follows. A line through the origin in $\mathbb{C}^3$ is described as a point $[p] \in \mathbb{CP}^2$, while a plane Π is the kernel of a non-zero linear functional, thus it can be described as a point $[\zeta] \in (\mathbb{CP}^2)^*$. The incidence relation $l \in \Pi$ is thus translated by the equation

$$\zeta(p) = 0, \tag{3.2.1}$$

which holds true, regardless of the chosen representatives p, ζ of the projective classes of l and Π. This describes the flag variety as a hypersurface in $\mathbb{CP}^2 \times (\mathbb{CP}^2)^*$.

Next we explain the second model. Consider the projections

$$\varpi_1 : \mathbb{CP}^2 \times (\mathbb{CP}^2)^* \longrightarrow \mathbb{CP}^2,$$

onto the first factor, and

$$\varpi_2 : \mathbb{CP}^2 \times (\mathbb{CP}^2)^* \longrightarrow (\mathbb{CP}^2)^*,$$

onto the second factor.

The projection ϖ_1 restricts to the flag variety as a holomorphic submerssion to $\mathbb{CP}^2$. The fiber over a point in $\mathbb{CP}^2$, representing a line $l \subset \mathbb{C}^3$, consists of all the planes $\Pi \subset \mathbb{C}^3$ containing this line. In what follows we describe a way to parametrize this collection of planes. Consider a plane $\Pi^{'} \subset \mathbb{C}^3$, transverse to the line l. The pair of planes $\Pi, \Pi^{'}$ intersects along a line $l' \subset \mathbb{C}^3$. This line uniquely determines the plane Π, as the plane spanned by l and l'. Thus, planes containing the line l are parametrized by their intersections with a fixed transverse plane $\Pi^{'}$. The collection of such intersections spans is the set of lines through the origin in $\Pi^{'}$, hence the set of planes containing l is parametrized by $\mathbb{P}(\Pi^{'})$, a projective line.

In fact, $\varpi_1 : F_{1,2} \longrightarrow \mathbb{CP}^2$ is a projective line bundle, i.e., the fiberwise projectivization of a rank 2 vector bundle over $\mathbb{CP}^2$. To prove this, we recall the Euler exact sequence of the holomorphic cotangent bundle of $\mathbb{CP}^2$,

$$0 \longrightarrow \Omega^1_{\mathbb{CP}^2} \stackrel{\iota}{\longrightarrow} [\mathcal{O}_{\mathbb{CP}^2}(-1)]^{\oplus 3} \stackrel{\eta}{\longrightarrow} \mathcal{O}_{\mathbb{CP}^2} \longrightarrow 0. \tag{3.2.2}$$

The map of sheaves

$$\eta : [\mathcal{O}_{\mathbb{CP}^2}(-1)]^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{CP}^2},$$

is defined as follows: a local section of $[\mathcal{O}_{\mathbb{CP}^2}(-1)]^{\oplus 3} \approx \mathcal{O}_{\mathbb{CP}^2}(-1) \otimes (\mathbb{C}^3)^*$ is an $\mathcal{O}(-1)$ twisted linear functional on $\mathbb{C}^3$, say ζ, which the map η evaluates against the tautological (Euler) section of $[\mathcal{O}_{\mathbb{CP}^2}(1)]^{\oplus 3}$, given by

$$[x^0 : x^1 : x^2] \mapsto (x^0, x^1, x^2).$$

That is,

$$[\eta(\zeta)](p) = \zeta(p),$$

in the notation of equation (3.2.1).

The map $\varpi_1 : \mathbb{CP}^2 \times (\mathbb{CP}^2)^* \longrightarrow \mathbb{CP}^2$ can be interpreted as a $\mathbb{CP}^2$-bundle over $\mathbb{CP}^2$, the projectivization of the trivial $\mathbb{C}^3$ bundle $\mathcal{O}_{\mathbb{CP}^2}^{\oplus 3} \to \mathbb{CP}^2$. Since twisting by a line bundle does not change the projectivization, we may also think of $\mathbb{CP}^2 \times (\mathbb{CP}^2)^*$ as the projectivization of the twisted bundle

$$[\mathcal{O}_{\mathbb{CP}^2}(-1)]^{\oplus 3} \to \mathbb{CP}^2.$$

We now recognize the equation characterizing points in $\mathbb{CP}^2 \times (\mathbb{CP}^2)^*$ lying on the flag manifold, (3.2.1), as the condition that the corresponding line on the vector bundle $[\mathcal{O}_{\mathbb{CP}^2}(-1)]^{\oplus 3}$ lies in the subbundle $\iota(\Omega^1_{\mathbb{CP}^2})$. Thus

$$F_{1,2} = \mathbb{P}(\Omega^1_{\mathbb{CP}^2}),$$

and the map

$$\varpi_1 : F_{1,2} \longrightarrow \mathbb{CP}^2$$

is the bundle projection.

The canonical contact structure

The previous paragraph presents the Flag variety as the projectivization of the cotangent bundle of $\mathbb{CP}^2$. Complex manifolds obtained in this way possess a canonical holomorphic contact structure, which we shall describe in the end of this subsection.

A *holomorphic contact structure* is a holomoprhic subbundle of hyperplanes in the *holomorphic* tangent bundle, satisfying the following complete non-integrability condition: for a given point on the manifold, there is no analytic hypersurface through it whose *holomorphic tangent spaces* are, on a neighborhood of this point, given by the hyperplane distribution.

One can describe the non-integrability of the hyperplane distribution in analytic terms. Let H denote the hyperplane distribution in $\mathcal{T}X$, and denote by L the quotient bundle $\mathcal{T}X/H$.

The complete non-integrability of H, is the extreme opposite of Frobenius integrability: given a local holomorphic vector field tangent to the distribution H, there exists another such vector field whose Lie bracket with the given one is transverse to H. Thus, a holomorphic contact structure defines, via the Lie Bracket, and projection to the quotient L, a nowhere-vanishing, L-valued section ω of $\Lambda^2 H^*$,

$$\omega(u, v) = [U, V] \text{ mod } H,$$

where $u, v \in H_p$ (the fiber of H over a point p in X), and U, V are holomorphic extensions to local sections of H (the particular choice of such extensions is irrelevant). In fact, this section is non-degenerate, in the sense that given u a local section of H, s a local, nowhere-vanishing section of L, there exists a local section v of H such that

$$\omega(u, v) = s.$$

Indeed, we can construct such a section explicitly by the following procedure: let w a local section of H, defined on the same neighborhood of u, such that $[u, s]$ is transverse to H. Then we can define a local, nowhere-vanishing holomorphic function f by

$$[u, w] = fs.$$

Hence $v = f^{-1}w$ is a well-defined, local holomorphic section of H, satisfying

$$\begin{aligned} [u, v] &= [u, f^{-1}w] \\ &= u(f^{-1})v + f^{-1}[u, w] \\ &= u(f^{-1})v + s, \end{aligned}$$

i.e.,

$$\omega(u, v) = s.$$

It follows that the exterior product

$$\omega^n = \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{\text{n factors}}$$

is a nowhere-vanishing section of the line bundle $\Lambda^{2n} H^* \otimes L^n$.

The section ω has a simple interpretation, locally. Consider the L-valued 1-form $\theta \in \Omega^1_M(L)$ whose value on a holomorphic tangent vector v is given by

$$\theta(v) = v \text{ mod H}.$$

It's kernel consists of the distribution H, so this is called a contact 1-form for H. In a local trivialization of L, one can differentiate θ, and by Cartan's formula, one has for any $u, v \in H$,

$$\begin{aligned} d\theta(u,v) &= \theta(u) - \theta(v) - \theta([u,v]) \\ &= -\theta([u,v]) \\ &= -[u,v] \text{ mod H} \\ &= -\omega(u,v), \end{aligned}$$

by the assumption of complete non-integrability. Making the differential of θ precise in the global context would require additional structures (namely, a connection on L). However, as seen above, this differential has an unanbigous meaning when restricted to vectors tangent to the distribution H, hence

$$\theta \wedge (d\theta)^n$$

is a nowhere-vanishing section of $\Omega_X^{2n+1} \otimes L^{n+1} = K_X \otimes L^{n+1}$, whose meaning does not depend on the choice of trivialization used to compute $d\theta$. Thus, the existence of a holomorphic contact structure on a complex, $(2n+1)$-dimensional manifold X implies that $K_X \otimes L^{n+1}$ is trivial, i.e., the anti-canonical bundle admits an $(n+1)$-th root, L, on which the contact form θ takes values.

Such structures are not easy to come by. For instance, the above restriction on the canonical bundle rules out the existence of holomorphic contact structures on all the even-degree hypersurfaces in $\mathbb{CP}^4$. However, manifolds like the flag manifold F_{12}, obtained by projectivization of the cotangent bundle of a complex manifold, there exists a canonical contact structure, which we explain next.

Let Y be an n-dimensional complex manifold, and $X = \mathbb{CP}(\Omega_Y^1)$ the projectivization of its holomorphic cotangent bundle (a complex manifold of dimension $(2n+1)$), whose bundle projection is denoted by $\varpi : X \longrightarrow Y$. We consider the pullback of the cotangent bundle of Y to X,

$$\zeta : \varpi^*(\Omega_Y^1) \longrightarrow X,$$

and define the relative tautological bundle of Y as the subbundle $\mathcal{O}_X(-1)$, whose fiber at a point $x \in X$ is the line this point represents in the vector bundle Ω_Y^1,

$$[\mathcal{O}_X(-1)]_x = \{v \in \varpi^*(\Omega_Y^1)(x) \approx \Omega_Y^1(\varpi(x)) | v \in x\}.$$

We denote the tensor powers of this line bundle by

$$\mathcal{O}_Z(n) = [\mathcal{O}_Z(-1)]^{\otimes(-n)},$$

and given a vector bundle V over X, its twisting by $\mathcal{O}_X(n)$ is

$$V(n) = V \otimes \mathcal{O}_X(n).$$

There is an induced short exact sequence of vector bundles over X,

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \varpi^*\Omega^1_Y \longrightarrow Q \longrightarrow 0,$$

where Q, the quotient bundle, is related to the relative tangent bundle of X (the kernel of the differential ϖ_*) by:

$$\mathcal{T}_{X|Y} = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-1), Q).$$

Tensoring with $\mathcal{O}_X(1)$, we obtain the relative (tangent) Euler exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\iota} \mathcal{O}_X(1) \otimes \varpi^*\Omega^1_Y \xrightarrow{\eta} \mathcal{T}_{X|Y} \longrightarrow 0, \tag{3.2.3}$$

We seek to show that X admits a contact form, i.e., a 1-form vith values in an $(n+1)$-th root of the canonical bundle K_X, satisfying the non-degeneracy condition. We will first describe the canonical bundle of X, and a natural choice of root.

The cotangent bundle of Y, Ω^1_Y, sits naturally into the cotagent bundle of Ω^1_X, via the pull-back of forms. The quotient bundle,

$$\Omega^1_{X|Y} = \Omega^1_X / \varpi^*\Omega^1_Y,$$

is called the bundle of relative 1-forms on X (its local sections are 1-forms on X modulo pullbacks of 1-forms from Y). Thus, the canonical bundle of X satisfies

$$\begin{aligned} K_X &= \det(\Omega^1_X) \\ &= \det(\varpi^*\Omega^1_Y) \otimes \det\left(\Omega^1_{X|Y}\right) \\ &= \varpi^* K_Y \otimes K_{X|Y}, \end{aligned}$$

where $K_{X|Y} = \det(\Omega^1_{X|Y})$ is the relative canonical bundle of this fibration. It follows from the relative Euler exact sequence that

$$\begin{aligned} K_{X|Y} &= \det([\mathcal{T}_{X|Y}]^*) \\ &= \det\left(\left[\mathcal{O}_X(1) \otimes \varpi^*\Omega^1_Y\right]^*\right) \\ &= \det\left(\mathcal{O}_X(-1) \otimes \varpi^*\mathcal{T}_Y\right) \\ &= \mathcal{O}_X(-n-1) \otimes \det\left(\varpi^*\mathcal{T}_Y\right) \\ &= \mathcal{O}_X(-n-1) \otimes \varpi^* K_Y^{-1}. \end{aligned}$$

It follows that

$$K_X^{-1} = \mathcal{O}_X(n+1)$$

admits a natural choice of $(n+1)$-th root, the relative hyperplane bundle $\mathcal{O}_X(1)$.

We are finally in a position to describe the canonical contact 1-form on X. This is a 1-form $\theta \in \Omega^1_X(1)$, with values in the relative hyperplane bundle $\mathcal{O}_X(1) = K_X^{-\frac{1}{n+1}}$. In fact, we've already seen it in disguise, embedded into the relative Euler sequence as the image of a trivial section of $\mathcal{O}_X$ in $\varpi^*\Omega^1_Y(1)$ by ι, but now we shall present it concretely.

Let $y_0 \in Y$, and consider a non-zero element $\vartheta \in \Omega^1_Y(y_0)$. Then, on the linear subspace spanned by ϑ, $\langle\vartheta\rangle \subset \Omega^1_Y(y_0)$, we define a linear functional, f_ϑ, by the relation

$$\varepsilon = f_\vartheta(\varepsilon)\vartheta,$$

for $\varepsilon \in \langle\vartheta\rangle$. We use such functionals to define the contact form as follows: at a point $x_0 = (y_0, [\vartheta]) \in X$, with $\vartheta \in \Omega^1_Y(y_0)$, set

$$\theta(x_0) = \varpi^*(\vartheta)(y_0) \otimes f_\vartheta \in [\Omega^1_X \otimes \mathcal{O}_X(1)]_{x_0}.$$

Neither $\varpi^*\vartheta$ nor f_ϑ are well-defined sections of the bundles $\varpi^*\Omega^1 Y$, and $\mathcal{O}_X(1)$, respectively, but their tensor product is well-defined, owing to the way they transform under a change in the representative of the class $[\vartheta] \in \mathbb{CP}(\Omega^1_Y(y_0))$, namely

$$\begin{aligned} \varpi^*(c\vartheta)(y_0) \otimes f_{c\vartheta} &= c\varpi^*(\vartheta)(y_0) \otimes c^{-1} f_\vartheta \\ &= \varpi^*(\vartheta)(y_0) \otimes f_\vartheta, \end{aligned}$$

for any non-zero constant c.

The flag variety as a twistor space

Recall the characterization of *twistor spaces* from chapter 1: a twistor space of a complex 3-manifold Z, equipped with an anti-holomorphic involution $\sigma : Z \longrightarrow Z$ (henceforth called its real structure), satisyfing the following properties:

1. Z contains a 4-parameter family of rational curves, whose normal bundles are isomorphic to $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Such curves are called twistor lines.

2. The involution maps a twistor lines into a twistor lines The twistor lines which are mapped onto themselves are called *real twistor lines.* On each real twistor line, the involution σ acts as the antipodal map.

A Hermitian inner product h on $\mathbb{C}^3$ endows it with two $\mathbb{C}^3$-antilinear isomorphisms: the index-lowering operator

$$\flat : \mathbb{C}^3 \longrightarrow (\mathbb{C}^3)^*,$$

given by

$$[\flat(v)](u) = h(u, v),$$

and its inverse, the index-raising operator

$$\sharp : (\mathbb{C}^3)^* \longrightarrow \mathbb{C}^3.$$

We shall denote the corresponding operators between projective spaces $\mathbb{CP}^2$ and $(\mathbb{CP}^2)^*$ by the same symbols, and refer to them, collectively, as "musical isomorphisms". The image of a vector x in $\mathbb{C}^3$ or a point in projective space $\mathbb{CP}^2$ under the index-lowring operator will be denoted by $x^\flat$, while the image of a linear functional in $(\mathbb{C}^3)^*$ or a point in the dual projective space $(\mathbb{CP}^2)^*$ under the index-raising isomorphisms will be denoted by $l^\sharp$.

Together, the musical isomorphisms endow $\mathbb{CP}^2 \times (\mathbb{CP}^2)^*$ with a *real structure*, that is, an anti-holomorphic involution σ, given by

$$\sigma(x, l) = (l^\sharp, x^\flat).$$

This involution leaves the flag manifold invariant, for given a point (x, l) in $F_{1,2}$, we have

$$x^\flat(l^\sharp) = \overline{(l(x))} = 0.$$

In addition, σ has no fixed points along $F_{1,2}$, hence it defines a real structure therein.

Additionally, the Hermitian form endows the flag manifold with a projection to $\mathbb{CP}^2$, which assigns to a pair consisting of a point x and a line l the intersection between the Hermitian-orthogonal complement of x and l. For clarity, we will assume that the choice of homogeneous coordinates is so that the Hermitian form is written

$$h(u, v) = v^\dagger u,$$

where $v^\dagger$ denotes the conjugate-transpose of v. Let (x, l) be a point in $F_{1,2}$. The Hermitian orthogonal complement of x is a projective line $x^\perp$ defined by the equation

$$x^\dagger z = 0.$$

Now let l be the projective line given by the equation $y^T z = 0$. Then the intersection between $x^\perp$ and l, call it $\mathcal{P}(x, l)$, is given by $x^\dagger \times y$, where $\times$ denotes the cross product. In other words,

$$\mathcal{P}([x^0 : x^1 : x^2], [y_0 : y_1 : y_2]) = [y_1\bar{x}^2 - y_2\bar{x}^0 : y_2\bar{x}^0 - y_0\bar{x}^2 : y_0\bar{x}^1 - y_1\bar{x}^0].$$

The map $\mathcal{P} : F_{1,2} \longrightarrow \overline{\mathbb{CP}^2}$ is the Penrose fibration, as we shall see. Here we regard the target $\mathbb{CP}^2$ as endowed with its non-standard orientation.

Proposition 3.2.1. *The flag variety admits a 4-paramter family of rational curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.*

Proof. We shall exhibit a 4-parameter family directly. Consider the open subset

$$U = \{(q, m) \in \mathbb{CP}^2 \times (\mathbb{CP}^2)^* |\ q \notin m\}.$$

Fix a point $(q, m) \in U$. Denote by A_q the variety parametrizing lines through q in $\mathbb{CP}^2$ (a rational curve). Concretely, if q has homogeneous coordinates $[q^0 \colon q^1 \colon q^2]$, then A_q can be parametrized as

$$[\xi, \zeta] \mapsto [\xi q^2, \zeta q^2, -\xi q^0 - \zeta q^1] \in (\mathbb{CP}^2)^*.$$

For each point $t = [\xi, \zeta] \in A_q$, consider the point r_t, the intersection of t and m. If m is described by homogeneous coordinates $[m_0 \colon m_1 \colon m_2]$, then r_t is given by the cross product,

$$r_t = [m_1\xi q^0 + m_1\zeta q^1 + m_2\zeta q^2 \colon - m_0\xi q^0 - m_0\zeta q^1 - m_2\xi q^2 \colon m_1\xi q^2 - m_0\zeta q^2] \in \mathbb{CP}^2.$$

The pair (r_t, t) belongs to $F_{1,2}$. The assignment $t \in A_q \mapsto (r_t, t) \in F_{1,2}$ defines an embedding $\phi_{q,m} : A_q \longrightarrow \mathbb{CP}^2 \times (\mathbb{CP}^2)^*$ of the rational curve A_q (parametrized by $[s, t]$) into $F_{1,2}$, depending upon the choice of m, given by

$$([m_1\xi q^0 + m_1\zeta q^1 + m_2\zeta q^2 \colon - m_0\xi q^0 - m_0\zeta q^1 - m_2\xi q^2 \colon m_1\xi q^2 - m_0\zeta q^2], [\xi q^2, \zeta q^2, -\xi q^0 - \zeta q^1]). \tag{3.2.4}$$

The image of A_q under this embedding will be denoted by $C_{q,m}$.

As (q, m) varies within U, the flag manifold is swept by the rational curves $C_{q,m}$, no two such curves being the same, thus U can be thought of as a 4-parameter family of rational curves in the flag manifold. It is clear from expression(3.2.4) that $C = C_{q,m}$ has normal bundle $\mathcal{O}_C(1) \oplus \mathcal{O}_C(1)$. □

Given integers m, n, we will use the notation

$$\mathcal{O}(m, n)$$

to denote the line bundle

$$\varpi_1^*(\mathcal{O}_{\mathbb{CP}^2}(m)) \otimes \varpi_2^*(\mathcal{O}_{(\mathbb{CP}^2)^*}(n)).$$

A section of the line bundle $\mathcal{O}(m,n)$ is said to have bi-degree (m,n). In this language, the flag variety is the zero locus of a section of bi-degree $(1,1)$, thus we can identify its normal bundle as

$$N = \mathcal{O}(1,1)\big|_{F_{1,2}}.$$

The canonical bundle of $\mathbb{CP}^2 \times (\mathbb{CP}^2)^*$ is expressed in this notation as $\mathcal{O}(-3,-3)$. The adjunction formula yields the canonical bundle of the flag variety:

$$\begin{aligned} K_{F_{1,2}} &= K_{\mathbb{CP}^2\times(\mathbb{CP}^2)^*}\big|_{F_{1,2}} \otimes N \\ &= [\mathcal{O}(-3,-3)\otimes\mathcal{O}(1,1)]\big|_{F_{1,2}} \\ &= \mathcal{O}(-2,-2)\big|_{F_{1,2}}. \end{aligned}$$

Thus, the fundamental line bundle of $F_{1,2}$ is

$$K_{F_{1,2}}^{-\frac{1}{2}} = \mathcal{O}(1,1)\Big|_{F_{1,2}}.$$

The canonical contact 1-form on $F_{1,2}$ can be written in a simple form in terms of dual homogeneous coordinates on $\mathbb{CP}^2 \times (\mathbb{CP}^2)^*$,

$$\theta = x^0 dy_0 + x^1 dy_1 + x^2 dy_2. \tag{3.2.5}$$

We will often write this form in short-hand notation as $x^i dy_i$, assuming summation over the repeated indices, or even xdy. As remarked earlier, this one-form does not have an unanbiguous differential (one must choose a connection on $\mathcal{O}(1,1)$ to differentiate it, and even doing so, the differential would depend on the chosen connection. However, a naïve computation would suggest that

$$d\theta = dx^i \wedge dy_i,$$

whence

$$\theta \wedge d\theta = x^j dy_j \wedge dx^i \wedge dy_i.$$

As it turns out, the $\mathcal{O}(2,2)$-valued 3-form $\theta \wedge d\theta$ is globally well-defined on $F_{1,2}$.

An $SL(2,\mathbb{C})$-action on the flag variety

The group $SL(2,\mathbb{C})$ admits a 3-dimensional, irreducible representation on $V = \mathrm{Sym}^2(\mathbb{C}^2)$,

$$\rho : SL(2,\mathbb{C}) \longrightarrow GL_{\mathbb{C}}(V),$$

which on decomposable elements is given by

$$A(u \odot v) = Au \odot Av,$$

where $u, v \in \mathbb{C}^2$, and $A \in SL(2,\mathbb{C})$. This induces an action on V^*, given by

$$\rho^* : SL(2,\mathbb{C}) \longrightarrow GL_{\mathbb{C}}(V^*)$$

by

$$[\rho^*(A)y](x) = y(\rho(A)^{-1}x),$$

for all $x \in V$ and $y \in V^*$.

Concretely, the action of a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}),$$

on a point $(x^0, x^1, x^2) \in V$ (thought of as a column-vector) is given by

$$A \begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & (ad+bc) & 2bd \\ c^2 & cd & d^2 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \end{pmatrix},$$

or, in other words

$$\rho(A) = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & (ad+bc) & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

We shall regard elements of V^* as column vectors too, but use subindices on the coordinates, i.e., y_1, to distinguish them from their counterparts in V. In this sense, the action on V^* is given by the inverse-traspose matrix,

$$\rho^*(A) \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} d^2 & -bd & b^2 \\ -2cd & (ab+bc) & -2ab \\ c^2 & -ac & a^2 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$$

Proposition 3.2.2. *The product action $R = \rho \times \rho^*$ preserves the flag variety*

$$F_{1,2}(V) = \{(x,y) \in V \times V^* | \langle x, y\rangle = 0\},$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between V and V^*. Furthermore, this action preserves the canonical contact form.

Proof. Showing that R preserves the flar variety is simple,

$$\begin{aligned}\langle \rho(A)x, \rho^*(A)y\rangle &= [\rho^*(A)(y)](\rho(A)x)\\ &= y(\rho(A)^{-1}\rho(A)x)\\ &= y(x)\\ &= \langle x, y\rangle\,.\end{aligned}$$

As for preservation of the contact form,

$$\begin{aligned}[R(A)]^*(x^i dy_i) &= (\rho(A)x)^i d((\rho(A)^{-1}y)_i\\ &= \rho(A)^i{}_j x^j d([\rho(A)^{-1}]^k{}_i y_k)\\ &= \rho(A)^i{}_j [\rho(A)^{-1}]^k{}_i x^j dy_k\\ &= (\rho(A)^{-1}\rho(A))^k{}_j x^j dy_k\\ &= x^j dy_j\end{aligned}$$

□

Next we describe infinitesimal generators for this action. We shall use the following generators for the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$:

$$A = \begin{pmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{pmatrix}, B = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, C = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$

with associated 1-parameter subgroups

$$e^{tA} = \begin{pmatrix}e^{\sqrt{-1}t} & 0\\ 0 & e^{-\sqrt{-1}t}\end{pmatrix}, e^{tB} = \begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}, e^{tC} = \begin{pmatrix}1 & 0\\ t & 1\end{pmatrix},$$

for $t \in \mathbb{C}$.

Proposition 3.2.3. *The contact moment map for the* $\mathrm{SL}(2,\mathbb{C})$ *action on the flag manifold can be described as* $\Theta : F_{1,2} \longrightarrow H^0(F_{1,2}, K_{F_{1,2}}^{-\frac{1}{2}}) \otimes \mathfrak{sl}(2,\mathbb{C})$

$$\Theta([x],[y]) = 2\sqrt{-1}(-x^0y_0 + x^2y_2)A^* + (-x^1y_0 - 2x^2y_1)B^* + (-2x^0y_1 - x^1y_2)C^*. \quad (3.2.6)$$

Proof. The action of the 1-parameter subgroup e^{tA} on an element $([x],[y]) \in \mathbb{CP}^2 \times (\mathbb{CP}^2)^*$ is given by

$$e^{tA}([x],[y]) = \left(\begin{bmatrix}e^{2\sqrt{-1}t}x^0\\ x^1\\ e^{-2\sqrt{-1}t}x^2\end{bmatrix}, \begin{bmatrix}e^{-2\sqrt{-1}t}y_0 & y_1 & e^{2\sqrt{-1}t}y_2\end{bmatrix}\right),$$

with infinitesimal generator

$$\begin{aligned}\chi_A([x],[y]) &= \frac{d}{dt}\left[e^{tA}([x],[y])\right]\Big|_{t=0} \\ &= 2\sqrt{-1}(x^0\partial_{x^0} - x^2\partial_{x^2} - y_0\partial_{y_0} + y_2\partial_{y_2}).\end{aligned}$$

Meanwhile, the 1-parameter subgroup e^{tB} acts by

$$e^{tB}([x],[y]) = \left(\begin{bmatrix} x^0 + tx^1 + t^2x^2 \\ x^1 + 2tx^2 \\ x^2 \end{bmatrix}, \begin{bmatrix} y_0 \\ -ty_0 + y_1 \\ t^2y_0 - 2ty_1 + y_2 \end{bmatrix}\right),$$

with infinitesimal generator

$$\begin{aligned}\chi_B([x],[y]) &= \frac{d}{dt}\left[e^{tB}([x],[y])\right]\Big|_{t=0} \\ &= x^1\partial_{x^0} + 2x^2\partial_{x^1} - y_0\partial_{y_1} - 2y_1\partial_{y_2}.\end{aligned}$$

Finally, we consider the 1-parameter subgroup e^{tC} acts by

$$e^{tC}([x],[y]) = \left(\begin{bmatrix} x^0 \\ 2tx^0 + x^1 \\ t^2x^0 + tx^1 + x^2 \end{bmatrix}, \begin{bmatrix} y_0 - 2ty_1 + t^2y_2 \\ y_1 - ty_2 \\ y_2 \end{bmatrix}\right),$$

with infinitesimal generator

$$\begin{aligned}\chi_C([x],[y]) &= \frac{d}{dt}\left[e^{tC}([x],[y])\right]\Big|_{t=0} \\ &= 2x^0\partial_{x^1} + x^1\partial_{x^2} - 2y_1\partial_{y_0} - y_2\partial_{y_1}.\end{aligned}$$

Restricting our attention to $F_{1,2}$ and applying the contact form to each of the infinitesimal generators of the action yields three sections of the fundamental line bundle $K_{F_{1,2}}^{-1/2} = \mathcal{O}(1,1)|_{F_{1,2}}$,

$$\begin{aligned}\theta(\chi_A) &= 2\sqrt{-1}(-x^0y_0 + x^2y_2), \\ \theta(\chi_B) &= -x^1y_0 - 2x^2y_1, \\ \theta(\chi_C) &= -2x^0y_1 - x^1y_2,\end{aligned}$$

and this is our promised moment map. □

The base locus

The linear system spanned by these three sections defines a map to $\mathbb{CP}^2$, away from its base locus, where the follwing equations hold:

$$x^0 y_0 - x^2 y_2 = 0, \tag{3.2.7}$$

$$x^1 y_0 + 2x^2 y_1 = 0, \tag{3.2.8}$$

$$2x^0 y_1 + x^1 y_2 = 0. \tag{3.2.9}$$

We observe that this set of equations defines a codimension 2 subvariety of $F_{1,2}$. To wit, consider the open subset of the flag variety where $x^0 \neq 0$. Therein, equations (3.2.7) and (3.2.9) imply that

$$\begin{aligned} y_0 &= \frac{x^2 y_2}{x^0}, \\ y_1 &= -\frac{x^1 y_2}{2x^0}, \end{aligned}$$

respectively, thus equation (3.2.8) is satisfied automatically:

$$\begin{aligned} x^1 y_0 + 2x^2 y_1 &= x^1 \left(\frac{x^2 y_2}{x^0} \right) - 2x^2 \left(\frac{x^1 y_2}{2x_0} \right) \\ &= \left(\frac{x^1 x^2}{x_0} - \frac{x^2 x^1}{x^0} \right) y_2 \\ &= 0. \end{aligned}$$

Let's analyze the intersection of the zero locus of the moment map with the open subset

$$U_0 = \{([x],[y]) \in \mathbb{CP}^2 \times (\mathbb{CP}^2)^* | x^0 \neq 0\}.$$

Using the descriptions of y_0 and y_1 on U into the defining equation of the flag manifold, as a subvariety of $\mathbb{CP}^2 \times (\mathbb{CP}^2)^*$, yields

$$\begin{aligned} 0 &= x^0 \left(\frac{x^2 y_2}{x^0} \right) + x^1 \left(-\frac{x^1 y_2}{2x^0} \right) + x^2 y_2 \\ &= y_2 \left[2x^2 - \frac{(x^1)^2}{2x^0} \right]. \end{aligned}$$

We remark that on U, y_2 cannot vanish, as if this were the case so would y_0 and y_1, in view of equations (3.2.7), (3.2.9), so it must be the case that

$$2x^2 - \frac{(x^1)^2}{2x^0} = 0.$$

Multiplying by $(2x_0)$ yields an equation that makes sense on all of $F_{1,2}$,

$$4x^0x^2 - (x^1)^2 = 0, \tag{3.2.10}$$

the defining equation of the zero locus of the moment map.

Let $\widetilde{C} \subset F_{1,2}$ be the subvariety defined by (3.2.10), and consider its twistor projection to $\mathbb{CP}^2$, $C = \mathcal{P}(\widetilde{C})$. The curve C is a conic, and $\widetilde{C}$ is nothing but its canonical lift to the projectivization of the holomorphic tangent bundle[1] of $\mathbb{CP}^2$, $\mathcal{T}\mathbb{CP}^2$, that is, a point in $\widetilde{C} \subset F_{1,2}$ is of the type $(p, l_p) \in \mathbb{CP}^2 \times (\mathbb{CP}^2)^*$, where p belongs to the conic C, and $l_p \in (\mathbb{CP}^2)^*$ is the tangent line to C at p. Therefore a twistor line $C_{q,m} \subset F_{1,2}$ instersects $\widetilde{C}$ if, and only if, m is tanget to the conic.

Images of twistor lines

Recall that twistor lines are parametrized by equation (3.2.4), a point $[\xi, \zeta]$ is mapped by $\phi_{q,m}$ to

$$([m_1\xi q^0 + m_1\zeta q^1 + m_2\zeta q^2 \colon -m_0\xi q^0 - m_0\zeta q^1 - m_2\xi q^2 \colon m_1\xi q^2 - m_0\zeta q^2], [\xi q^2, \zeta q^2, -\xi q^0 - \zeta q^1]).$$

We wish to understand how such curves are mapped to $\mathbb{CP}^2$ under the moment map Θ. We expect that the generic twistor line (which does not meet the base locus) will be mapped to a conic. This is indeed the case. Furthermore, we can characterize the locus of conics in Y which are images of twistor lines.

Theorem 3.2.4. *If a conic in $\mathbb{CP}^2$, represented by a symmetric 3×3 matrix A with $\det(A) = 1$, is the image of a twistor line from $F_{1,2}$ via the contact moment map, then its coefficients satisfy the cubic relation*

$$A_{11}^3 - 4A_{11}A_{12}A_{13} + 2A_{13}^2A_{22} - A_{11}^2A_{23} + (-2A_{12}^2 + A_{11}A_{22} - A_{22}A_{23})A_{33} = 0 \tag{3.2.11}$$

Proof. Let $C_{q,m}$ be a twistor line in $F_{1,2}$, where $q = [q^0 \colon q^1 \colon q^2]$, $m = [m_0 \colon m_1 \colon m_2]$. The image of $C_{q,m}$ under the moment map Θ satisfies the following equation. Applying the moment map to the coordinates of the twistor line $\phi_{q,m}([\xi, \zeta]$ (cf. (3.2.4), we obtain a point u with coordinates (u^0, u^1, u^2), where

$$u^0 = -2\sqrt{-1}(q^2\xi(q^0\xi m_1 + q_1\zeta m_1 + q^2\zeta m_2) - (-q^0\xi - q^1\zeta)(q^2\xi m_1 - q^2\zeta m_0)),$$
$$u^1 = q^2\xi(q^0\xi m_0 + q^1\zeta m_0 + q^2\xi m_2) - 2q^2\zeta(q^2\xi m_1 - q^2\zeta m_0),$$
$$u^2 = -(-q^0\xi - q^1\zeta)(-q^0\xi m_0 - q^1\zeta m_0 - q^2\xi m_2) - 2q^2\zeta(q^0\xi m_1 + q^1\zeta m_1 + q^2\zeta m_2)$$

[1]Which we identify with the holomorphic cotangent bundle, i.e. the flag variety, by means of the fixed metric on $\mathbb{CP}^2$.

By elimination of ξ, ζ, we find that u^0, u^1, u^2 safisty a quadratic equation, best expressed using matrices: $u^T A u = 0$, where

$$A = \begin{pmatrix} q^0 m_0 + q^2 m_2 & \frac{\sqrt{-1}}{2}(2q^0 m_1 - q^1 m_2) & \frac{\sqrt{-1}}{2}(q^1 m_0 - 2q^2 m_1) \\ \frac{\sqrt{-1}}{2}(2q^0 m_1 - q^1 m_2) & -2q^0 m_2 & \frac{1}{2}(2q^0 m_0 + 4q^1 m_1 + 2q^2 m_2) \\ \frac{\sqrt{-1}}{2}(q^1 m_0 - 2q^2 m_1) & \frac{1}{2}(2q^0 m_0 + 4q^1 m_1 + 2q^2 m_2) & 2q^2 m_0 \end{pmatrix}. \tag{3.2.12}$$

Again applying a sequence of eliminations, we find that the entries of Λ satisfy a cubic polynomial,

$$A_{11}^3 - 4A_{11}A_{12}A_{13} + 2A_{13}^2 A_{22} - A_{11}^2 A_{23} + (-2A_{12}^2 + A_{11}A_{22} - A_{22}A_{23})A_{33} = 0$$

□

Corollary 3.2.5. *The function $F : Y \longrightarrow \mathbb{R}$ given on a matrix A by*

$$F(A) = A_{11}^3 - 4A_{11}A_{12}A_{13} + 2A_{13}^2 A_{22} - A_{11}^2 A_{23} + (-2A_{12}^2 + A_{11}A_{22} - A_{22}A_{23})A_{33} \tag{3.2.13}$$

is a solution to the Dunajski-Moraru-Tod system. The corresponding metric is generates on the submanifold $M = \{A \in Y | F(A) = 0\}$ is anti-self-dual, and extends, via conformal compactification, to the Fubini-Study metric on $\overline{\mathbb{CP}^2}$.

In summary, we have described an explicit solution to the DMT system by exploiting the geometry of the flag variety. This system is linear, and admits a large symmetry group. We expect that careful superposition of the solutions steming from sections 3.1 and 3.2 shall result in new solutions admitting conformal compactifications. This will be subject of future research.

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