Divisor varieties and syzygies of symmetric products of curves

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Abstract of the Dissertation

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Abstract. We take two themes in algebraic geometry which are well understood and indeed classical in the case of curves — Brill–Noether theory of a general curve, and the theory of syzygies of a high degree curve in projective space — and we study analogous questions in the less well understood setting of higher dimensional varieties. Specifically, we focus on those higher dimensional varieties which are symmetric products of a curve: in the first case, we describe the geometry of parameter spaces of effective divisors ("divisor varieties") associated to these symmetric products, indicating how the properties (new in the higher dimensional setting) of singularity and reducibility of these divisor varieties reflect the geometry of the underlying curve. In the second case we study how much syzygetic information about an embedded curve — that is, information about the equations defining the curve in its projective embedding — can be transferred to its symmetric product when the latter is embedded in projective space in a natural way using secant planes of the embedded curve.

Contents

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Publications

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Chapter 1

Introduction

In this thesis, we study in higher dimensions two themes in algebraic geometry that are well-understood and indeed now classical in the case of smooth, projective curves over $\mathbb{C}$. Namely, classical Brill–Noether theory studies maps from such curves to projective space and, more generally, parameter spaces of linear systems of divisors on these curves. Our first and primary purpose is to study an instance of the analogous theory on the symmetric products of such a curve. The second theme is that of the defining equations of a variety embedded in projective space. For high degree curves in projective space, the equations defining the embedded curve are relatively well understood — we study how much of what is known in the curve case can be expected to transfer to the case of the symmetric products of curves, when the latter are embedded in a way closely related to the geometry of the embedded curve itself.

The geometry of divisors plays a central role in the theory of algebraic curves and has been studied extensively over the years. The foundational results of this Brill–Noether theory — due to Kempf [33], Kleiman–Laksov [37], Griffiths–Harris [24], Gieseker [19] and Fulton–Lazarsfeld [17] — imply that on a general genus g curve C, the varieties $G^r_d(C)$ of degree d, dimension r linear series on C are smooth, irreducible projective varieties of known dimension depending only on g, d and r. This picture on general curves is complemented by a rich collection of special examples (in particular, see Example 3.1).

In higher dimensions, the story is less well understood. Mendes Lopes–Pardini–Pirola have obtained a Kempf-type existence result for the Brill–Noether theory of divisors on surfaces in [46]. Deformations of the canonical linear series have been studied by making use of the generic vanishing theorem of Green–Lazarsfeld [22, 23] and some related foundational results on the so-called *para-canonical system* (see for example the discussion in Section 3.2 and Examples 1 and 7.13) were given by Mendes Lopes–Pardini–Pirola in [47], extending earlier results of Beauville [6] and Lazarsfeld–Popa [43] (we survey a selection of this material in Section 3.2).

Our first and primary purpose in this thesis is to study in detail one class of higher dimensional examples where one can hope for a quite detailed picture, namely (the spaces of) divisors on the symmetric product of a curve. We find a number of new phenomena that do not occur for curves.

Turning to details, given a Néron–Severi class $\lambda \in \mathrm{NS}(X)$ on a smooth projective variety X, denote by $\mathrm{Div}^\lambda(X)$ and $\mathrm{Pic}^\lambda(X)$ the spaces of effective divisors and line bundles, respectively, of class λ. The Abel–Jacobi map

$$u_\lambda : \mathrm{Div}^\lambda(X) \to \mathrm{Pic}^\lambda(X)$$

sends $D \mapsto \mathcal{O}_X(D)$. Recall that $\mathrm{Div}^\lambda(X)$ can be realized as the scheme $\mathbb{P}\mathcal{F}_\lambda$ for an appropriate *Picard sheaf* $\mathcal{F}_\lambda$ on $\mathrm{Pic}^\lambda(X)$ (see for example [35, Ex. 9.4.7], but we also study this in more detail in Chapter 2).

Now take $X = C_k$, the k^{th} symmetric product of a smooth projective curve C. The divisor classes on X that will concern us come from the "anti-symmetric" bundles N_L arising from a line bundle L on C. Specifically, denote by $L^{[k]}$ the tautological rank k bundle on X associated to L and define

$$N_L = \det(L^{[k]}).$$

We will denote the first Chern class by $n(d) := c_1(N_L)$. One has a natural indentification

$$H^0(C_k, N_L) \cong \wedge^k H^0(C, L).$$

The bundle N_L in particular has generated much interest in the literature, e.g. in [15] and [16], and we note that $N_{K_C} = K_X$.

Our first main result identifies the Picard sheaf for X in terms of that for the curve:

Theorem A. *Let $\lambda = n(d)$. If $\mathcal{F}_d$ is a Picard sheaf on $\mathrm{Pic}^d(C)$ then*

$$\mathcal{F}_\lambda := \wedge^k \mathcal{F}_d$$

is a Picard sheaf on $\mathrm{Pic}^{\lambda}(X)$.

For $\lambda = n(d)$, Theorem A allows us to get a quite precise description of $\mathrm{Div}^{\lambda}(X)$ in many cases. We will state our subsequent results shortly, but first we present an example illustrating some of the new phenomena that can occur for $\mathrm{Div}^{\lambda}(X)$.

As a matter of terminology, observe that for L a line bundle on C, a basepoint-free pencil $\pi : C \to \mathbb{P}^1$ in $|L|$ gives rise to a corresponding *trace divisor* $D_\pi := \{\xi \in C_2 : \xi \text{ is in a fiber of } \pi\}$ in $|N_L|$ as seen in Figure 1.1. As indicated, a divisor in the pencil, which is a fiber of π, determines

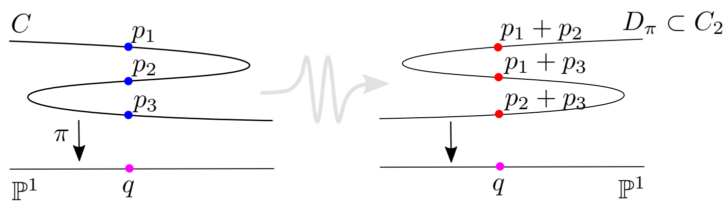

Figure 1.1: Trace divisor of a basepoint-free pencil.

points on D_π by taking unordered pairs. Varying the fiber then sweeps out all of D_π.

Example (Plane curves)**.** Let C be a smooth plane curve of degree $d \geq 5$ with $L = \mathcal{O}_C(1)$ and set $X = C_2$ and $\lambda = c_1(N_L)$. We are interested in the divisor variety $\mathrm{Div}^{\lambda}(X)$. We note first that there are two kinds of g^1_d on C:

1. Given $p \in \mathbb{P}^2$ the pencil V_p of lines through p cuts out a g^1_d on C. Of course $V_p \subset |L|$.
2. Let $L_{pq} := L \otimes \mathcal{O}_C(q - p)$. Then $|L_{pq}|$ is a *complete* g^1_d on C whenever $q \neq p$.

The first kind is parametrized by $\mathbb{P}^2$ and the second by C^2. It turns out that every divisor on X is a trace divisor of one such pencil. Hence:

$$\mathrm{Div}^{\lambda}(C_2) \cong \mathbb{P}^2 \cup C^2$$

and the intersection is simply the diagonal $\Delta \subset C^2$

$$\mathbb{P}^2 \cap C^2 = \Delta \cong C,$$

as illustrated in Figure 1.2. In this example $\mathrm{Div}^{\lambda}(X)$ has multiple irreducible components, something which never happens when X is a curve.

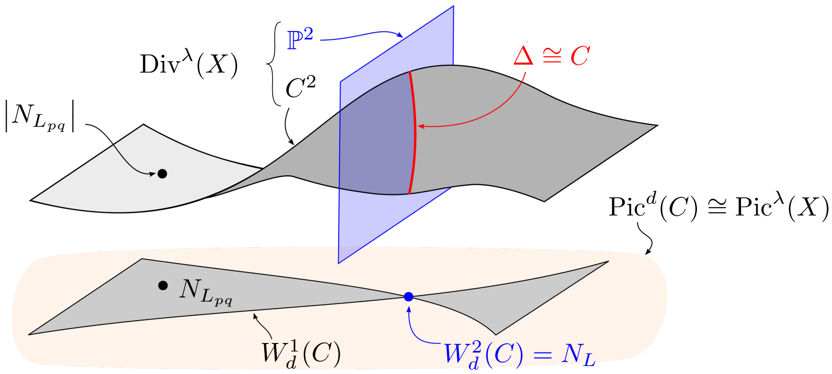

Figure 1.2: The divisor variety of the symmetric square of a plane curve.[1]

Returning to the general case with $X = C_k$ and $\lambda = n(d)$, we will need a definition from linear algebra in order to identify the irreducible components of $\mathrm{Div}^\lambda(X)$. Let V be a vector space over $\mathbb{C}$ of finite dimension and let $\eta \in \wedge^k V$.

Definition. The *enclosing dimension* of η is

$$\mathrm{enc}(\eta) := \min\{\dim W : W \subset V \text{ and } \eta \in \wedge^k W\}$$

In the setting of symmetric products $X = C_k$, for $D \in |N_L|$ we let $\mathrm{enc}(D) := \mathrm{enc}(\eta)$ when

$$\eta \in H^0(X, N_L) \cong \wedge^k H^0(C, L)$$

is a defining section of D.

The next theorem shows that *enclosing dimensions* parametrize the irreducible components of $\mathrm{Div}^\lambda(X)$ for general C:

Theorem B. *Let C be a general, smooth, projective curve of genus g over $\mathbb{C}$, let $d, e, k \in \mathbb{N}$ and set $\lambda = n(d)$ and $X = C_k$. Define*

$$N_d := \max\ \{h^0(L) : L \in \mathrm{Pic}^d(C)\}.$$

For $e \in \mathcal{E} := \{k, k+2, k+3, \ldots, N\}$ (or $\mathcal{E} \cap 2\mathbb{Z}$ if $k = 2$), set

$$\mathcal{Z}_e := \{D \in \mathrm{Div}^\lambda(X) : \mathrm{enc}(D) \le e \le h^0(L) \textit{ where } L \textit{ is such that } D \in |N_L|\}.$$

[1]see Chapter 3 for the definition of $W^r_d(C)$.

Then the $\mathcal{Z}_e$ are distinct and (except in rare cases)[2] *are the irreducible components of* $\mathrm{Div}^\lambda(X)$.

Example. Let C be a smooth projective curve of genus 4, take $X = C_2$ and $\lambda = c_1(K_X)$. Then the divisor variety $\mathrm{Div}^\lambda(X)$ has two components:

$$\mathrm{Div}^\lambda(X) \cong \Sigma \cup |K_X|$$

where Σ is a variety dominating $\mathrm{Pic}^\lambda(X)$. Its intersection with the canonical linear series on X is a Grassmannian of lines:

$$\Sigma \cap |K_X| = \mathbb{G}(1, |K_C|)$$

See Figure 1.3. In this case, one can easily distinguish the two components geometrically: the divisors $[D] \in \Sigma$ are *all* trace divisors. So any strictly paracanonical divisor on X (i.e. not canonical, but with $c_1 = \lambda$) comes from a pencil of strictly paracanonical divisors on C, whereas the canonical divisors on X arising as trace divisors fill out the Grassmannian $\mathbb{G}(1, |K_C|) \subsetneq |K_X|$ (here $\mathbb{G}(1, |K_C|)$ is naturally embedded in $|K_X|$ by the Plücker embedding since $K_X \cong N_{K_C}$ and $H^0(C_2, N_L) \cong \wedge^2 H^0(C, L)$ as mentioned above). Hence most canonical divisors on X are not traces and do not deform directly (i.e. over an irreducible base) out of their linear equivalence class.

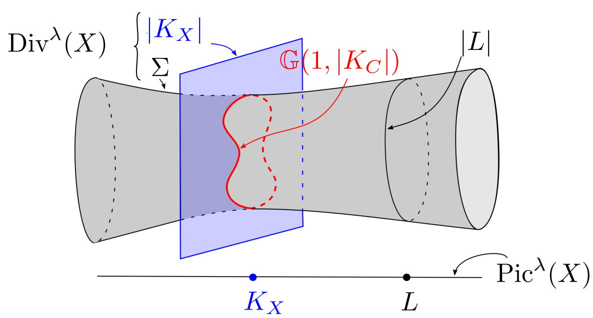

Figure 1.3: The paracanonical system of the symmetric square of a genus 4 curve.

In this example, the intersection of the irreducible components is a Grassmannian, which can be regarded as a rank-locus in $|K_X|$. In the general setting, it will again be a kind of rank-locus that forms the intersections of irreducible components. Specifically, fix $e \in \mathbb{N}$ an enclosing dimension for some $\eta \in \wedge^k V$.

[2]with $\rho(g, d, r)$ as defined in Definition 3.2, the scheme $\mathcal{Z}_{N_d}$ will be a disjoint union of projective spaces whenever $\rho(g, d, N_d - 1) = 0$.

Definition. The *subspace variety* determined by e, k and V is

$$\mathrm{Sub}_e(\wedge^k V) := \{[\eta] \in \mathbb{P}(\wedge^k V)^\vee : \mathrm{enc}(\eta) \leq e\}^3$$

Our third theorem, a more precise version of which we will give later, shows that subspace varieties form the intersections of irreducible components of $\mathrm{Div}^\lambda(X)$:

Theorem C. *Let X, λ, $\mathcal{E}$ and $\mathcal{Z}_e$ be as in Theorem B and $e < f \in \mathcal{E}$. Then for $L \in W_d^{f-1}(C)$,*

$$|N_L| \cap \mathcal{Z}_e \cap \mathcal{Z}_f \cong \mathrm{Sub}_e(\wedge^k H^0(L)).$$

We see that f determines the support *of $\mathcal{Z}_e \cap \mathcal{Z}_f$ over $\mathrm{Pic}^\lambda(X)$ while e determines the Abel–Jacobi* fibers *of it.*

Theorems B and C conclude our study of divisor varieties on symmetric products. Nonetheless, the geometry of the line bundles N_L on symmetric products C_k yields another fascinating direction of study — that of the defining equations for $C_k \subset \mathbb{P}H^0(C_k, N_L)$ whenever N_L is very ample. We pursue a conjectural relationship between the syzygies of C_k in terms of syzygies of the smooth projective curve C. The expectation there is that syzygetic behavior of C should carry over to C_k in a shifted fashion.

Consider a smooth projective variety $X \subset \mathbb{P}H^0(L) = \mathbb{P}$ embedded by a very ample line bundle L. The section ring $R(L) := \bigoplus_{n\geq 0} H^0(nL)$ is a graded module over the graded ring $S := \mathrm{Sym}(H^0(L))$ and consequently the most interesting invariant of $R(L)$ is the minimal free graded resolution:

$$E_\bullet = \cdots \to E_p \to \cdots \to E_1 \to E_0 \to R(L)$$

Of particular interest are the degrees $a_{p,j}$ of the graded components in each $E_p = \bigoplus_j S(-a_{p,j})$.

There has been a great deal of interest in this topic, largely centered around the behavior of a key property introduced in [21]:

Definition (Green–Lazarsfeld property N_p)**.** We say L has property N_0 if it is normally generated, and property N_p (for $p \geq 1$) if it has property N_{p-1} and $a_{p,j} \leq p+1$ for all j (which actually implies *equality*, by minimality of the resolution). If L has property N_p one says it has *linear syzygies to p^{th}*

[3] Here $\mathbb{P}$ denotes the projective space of 1-dimensional *quotients*.

order.

In the case of curves, we have the celebrated:

Theorem (Green, [20, Thm. 4.a.1])**.** *Let $X = C$ be a curve of genus g. If deg$(L) \geq 2g+1+p$ then L has property N_p.*

Green's theorem represents a vast generalization of the well-known fact that $\deg(L) \geq 2g+1$ implies normal generation, due to Castelnuovo [8], Mattuck [44] and Mumford [48].

In the case of symmetric products, the expectation is that the syzygies for C_k are governed by those for C, at least when one begins with a sufficiently positive line bundle L on C. This is made precise by:

Conjecture (Syzygy shifting)**.** A line bundle L on a smooth, projective curve C of genus g, with either $L = K_C$ or $\deg(L) \geq 2g+1+p$, has property N_p if and only if N_L has property N_{p-k+1} on C_k.

By way of evidence for the reverse direction of this conjecture, one has the following two propositions, the first of which is well-known (see e.g. [21])

Proposition. *Suppose $X \subset \mathbb{P}$ is a smooth projective variety embedded by a very ample line bundle L. If X admits a $(p+2)$-secant-p-plane, then property N_p fails for L.*

Proposition. *If a curve $C \subset \mathbb{P}V$, embedded by L, admits a $(p+2)$-secant-p-plane, then $C_k \subset \mathbb{P}\wedge^k V$ embedded by N_L admits a $(p-k+3)$-secant-$(p-k+1)$-plane (assuming N_L is very ample).*

Together, these propositions imply that if a certain secant plane causes N_p to fail for L on C, then a related secant plane causes N_{p-k+1} to fail for N_L on C_k. This, however, does not constitute a proof of this direction of the conjecture since failure of N_p on curves may not be caused by such secant planes in general.

In the very first case, the forward direction of the conjecture is indeed true for high degree L — we prove:

Theorem D. *If a line bundle L on a smooth, projective genus $g > 2$ curve C has degree*

$$d := \deg(L) \geq 2g+2$$

then N_L is normally generated on C_2.

Similar arguments to those used in the proof of Theorem D can be used to reduce, for example, the $k = p + 1$ case of the conjecture to a short list of necessary d-normality statements for N_L on C_{p+1}, as with 3-normality when $p = 1$ (see Section 9.2 for these details). Since $N_{K_C} = K_{C_k}$ this would have the happy consequence of generalizing Noether's theorem (on projective normality of *canonical* curves) to symmetric products. However, it is not yet clear whether or not stronger geometric conditions on the curve may be needed to conclude those missing normality statements — this is a line of investigation we continue to pursue.

1.1 Organization of the thesis

We begin in Chapter 2 by introducing Hilbert schemes of divisors in a smooth projective variety X over $\mathbb{C}$. The discussion and definitions there will introduce the focus of our main theorems in Chapter 7 concerning the structure of divisor varieties associated to symmetric products of a curve.

In Chapter 3 we recall the basic ideas from the classical Brill–Noether of a curve, introducing some useful ideas and terminology for later chapters, and also reviewing some results that reflect the current state of the art in higher dimensional Brill–Noether theory.

In Chapter 4, we recall some essential ideas concerning group actions and Künneth formulas which lay the path for the proof of our first main theorem, Theorem A, in Chapter 6.

In Chapter 5 we discuss and introduce some notions of rank, coming from linear algebra, that give rise to stratifications of certain projective spaces which — although also of independent interest — will be important for us in Chapter 7 when describing the component intersections of the divisor varieties we study.

Chapters 2-5 are mainly expository. The main new material appears in Chapters 6, 7 and 9. The material in Chapters 6 and 7 is adapted from our paper [53].

By Chapter 6 we are ready to begin focusing on our varieties of interest — symmetric products of curves. In this chapter we give an overview of the essential ideas we will need concerning their geometry and in the final section of this chapter, we prove Theorem A.

In Chapter 7 we prove Theorems B and C after some initial setup, thus establishing the desired description of the divisor varieties of symmetric products of curves. At the end of this chapter we use the description yielded by these theorems to give some interesting examples and applications of our results.

In Chapter 8 we then shift gears — we present a brief overview of the basic ideas concering the defining equations of a variety in projective space and the relations (syzygies) between those equations.

This then finally sets us up for Chapter 9 in which we discuss a conjectural relationship between resolutions of the homogeneous ideal of a curve in projective and that of its symmetric product, the latter embedded using the secant geometry of the curve. We conclude this chapter and the thesis by proving the first instance of this conjectural relationship — Theorem D.

1.2 Notation and conventions

Throughout the thesis, we will assume the following unless otherwise indicated:

- all schemes are Noetherian, separated and are over the complex numbers $\mathbb{C}$ (we will often emphasize the latter hypothesis);
- all projectivizations, denoted by $\mathbb{P}$, are parameter spaces of 1-dimensional *quotients* — if referring specifically to 1-dimensional subspaces we will use the notation $\mathbb{P}_{\text{sub}}$;
- all Grassmannians $G(V, k)$ parametrize k-dimensional *quotients* of V; however
- we will sometimes use the notation $\mathbb{G}(k, \mathbb{P})$ to denote the Grassmannian of projective k-dimensional subspaces of a projective space $\mathbb{P}$ — of course if $\mathbb{P} = \mathbb{P}V$ then we have the natural identification $\mathbb{G}(k, \mathbb{P}) \cong G(V, k+1)$;
- we use the Zariski topology throughout.

Chapter 2

Hilbert Schemes of Divisors

Let X be a smooth, projective variety over $\mathbb{C}$. In this chapter, we review the construction of the scheme $\mathrm{Div}(X)$ parametrizing effective divisors in X.

2.1 Overview

We begin by making a definition of effective divisor appropriate for our purposes:

Definition 2.1. Let Y be a scheme over $\mathbb{C}$. A subscheme $\mathcal{D} \subset Y$ is an *effective (Cartier) divisor* if the ideal sheaf $\mathcal{I}_{\mathcal{D}}$ of $\mathcal{D}$ in Y is an invertible sheaf.

Given X as above, together with a Néron–Severi class $\lambda \in \mathrm{NS}(X)$, our first goal is to construct a scheme $\mathrm{Div}^{\lambda}(X)$ whose closed point set is

$$\mathrm{Div}^{\lambda}(X) = \{\text{effective divisors } D \subset X : c_1(D) = \lambda\}$$

and which, more generally, parametrizes effective divisors in X of class λ in a functorial way — i.e. it admits a universal family $\mathcal{D}_{\lambda} \to \mathrm{Div}^{\lambda}(X)$ of effective divisors in X of class λ, and every other such family $\mathcal{D}_S \to S$ determines a classifying map $c : S \to \mathrm{Div}^{\lambda}(X)$ such that

$$\mathcal{D}_S = c^* \mathcal{D}_{\lambda}.$$

Remark 2.2. In favorable circumstances, the scheme $\mathrm{Div}^{\lambda}(X)$ will be a variety and we therefore often refer to it as a "divisor variety" — describing these divisor varieties for certain classes λ when X is a symmetric product of a curve will be our primary objective in Chapter 7.

Once constructed, the principal result of this chapter (Theorem 2.17) is to show that $\mathrm{Div}^\lambda(X)$ is actually the projectivization

$$\mathrm{Div}^\lambda(X) \cong \mathbb{P}\mathcal{F}_\lambda$$

of a sheaf $\mathcal{F}_\lambda$ (called a *Picard sheaf*) on the Picard variety

$$\mathrm{Pic}^\lambda(X) = \{\text{(isomorphism classes of) line bundles } L \to X \,:\, c_1(L) = \lambda\}.$$

It will be this fact that makes our study of $\mathrm{Div}^\lambda(X)$ tractable in Chapter 7.

Our strategy from here on will be as follows: for most of the chapter it will be preferable to *not* specify the class λ until later and deal instead with all effective divisors in X together, with the aim of parametrizing them all by a scheme $\mathrm{Div}(X)$ and then obtaining $\mathrm{Div}^\lambda(X)$ as a subscheme. To achieve this, we will introduce a corresponding functor Div_X and show that it is representable. Along the way, we introduce the Abel–Jacobi mapping $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ given by $D \mapsto \mathcal{O}_X(D)$, albeit initially at the functorial level. Finally we will introduce the notion of a *Picard sheaf* $\mathcal{F}$ on $\mathrm{Pic}(X)$ and show that its projectivization coincides with the scheme $\mathrm{Div}(X)$ that we constructed.

Our approach is largely an adaptation of the theory in [49, Lecture 15] to our particular situation.

2.2 The Div_X functor

For S a scheme over $\mathbb{C}$, we define the set

$$\mathrm{Div}_X(S) := \{\mathcal{D} \subset X \times S \text{ is an effective divisor } : \ \mathcal{D} \xrightarrow{p_2} S \ \text{ is flat}\}$$

(here p_2 is the second projection map restricted to $\mathcal{D}$). We note that this association of a set to a scheme S over $\mathbb{C}$

$$\begin{aligned} \mathrm{Div}_X :\ \mathrm{Sch}/\mathbb{C} &\longrightarrow \mathrm{Set} \\ S &\longmapsto \mathrm{Div}_X(S) \end{aligned}$$

naturally admits the structure of a functor: given a morphism $f : T \to S$ over $\mathbb{C}$ and an effective divisor $\mathcal{D} \subset X \times S$ one easily forms the effective divisor $(1 \times f)^*\mathcal{D} \subset X \times T$. This can be done by pulling back the locally free sheaf $\mathcal{I}_{\mathcal{D}}$ on $X \times S$ to one on $X \times T$ and then pulling back the local defining equations of $\mathcal{D}$ to the latter to get local defining equations for $(1 \times f)^*\mathcal{D}$.

Thus altogether we have:

$$\begin{aligned} \mathrm{Div}_X : \qquad \mathrm{Sch}/\mathbb{C} &\longrightarrow \mathrm{Set} \\ S &\longmapsto \mathrm{Div}_X(S) \\ (f : T \to S) &\longmapsto (\mathcal{D} \mapsto (1 \times f)^*\mathcal{D}). \end{aligned}$$

As discussed above, our goal is to find a scheme that represents Div_X — this would then constitute a true parameter space of effective divisors in X.

2.3 The Abel–Jacobi map $\mathrm{Div}_X \to \mathrm{Pic}_X$

We recall here from [35, Section 9.2] the definition of the relative Picard functor. We define

$$\mathrm{Pic}_X(S) := \mathrm{Pic}(X \times S)/\mathrm{Pic}(S)$$

where, on the right hand side, Pic refers only to the set underlying the Picard group. In our particular case, Pic_X is represented by a group scheme, locally of finite type over $\mathbb{C}$, which is non-singular (see [49, Lecture 21] — this will come up again in Section 2.5). From here on we will also denote this scheme by $\mathrm{Pic}(X)$ — there should be no ambiguity. Moreover for each $\lambda \in \mathrm{NS}(X)$ the subscheme

$$\mathrm{Pic}^{\lambda}(X) := \{[L] \in \mathrm{Pic}(X) : c_1(L) = \lambda\}$$

is, as a scheme, an isomorphic copy of the group-subscheme $\mathrm{Pic}^0(X)$, which itself is an abelian variety — the *Picard variety*.

For S a scheme over $\mathbb{C}$ as before, the association to a divisor $\mathcal{D} \subset X \times S$ of the associated line bundle $\mathcal{O}_{X\times S}(\mathcal{D})$ prompts the more general definition

$$\begin{aligned} u(S) : \mathrm{Div}_X(S) &\longrightarrow \mathrm{Pic}_X(S) \\ \mathcal{D} &\longmapsto \mathcal{O}_{X\times S}(\mathcal{D}) \bmod \mathrm{Pic}(S) \end{aligned}$$

and it turns out that, as promised, this map constitutes a natural transformation of the corresponding functors. To see this, let $f : T \to S$ be a morphism over $\mathbb{C}$. Consider the bundle $(1 \times f)^*\mathcal{O}_{X\times S}(\mathcal{D})$:

we have

$$\begin{aligned}(1\times f)^*\mathcal{O}_{X\times S}(\mathcal{D}) &= (1\times f)^*(\mathcal{I}_{\mathcal{D}}^{\vee})\\ &= ((1\times f)^*\mathcal{I}_D)^{\vee}\\ &= (\mathcal{I}_{f^*\mathcal{D}})^{\vee}.\end{aligned}$$

And evidently the diagram

$$\begin{tikzcd} \mathrm{Pic}(T) \arrow[r, hook] & \mathrm{Pic}(X\times T) \\ \mathrm{Pic}(S) \arrow[r, hook] \arrow[u, hook] & \mathrm{Pic}(X\times S) \arrow[u, hook] \end{tikzcd}$$

commutes, and thus we get a map $\mathrm{Pic}_X(S) \hookrightarrow \mathrm{Pic}_X(T)$ which takes the equivalence class $(\mathcal{O}_{X\times S}(\mathcal{D}) \bmod \mathrm{Pic}(S))$ to $((1\times f)^*\mathcal{O}_{X\times S}(\mathcal{D}) \bmod \mathrm{Pic}(T))$. Together these facts indicate that we can complete the definition for the natural transformation u on Hom-sets:

$$\begin{aligned} u:\quad \mathrm{Div}_X &\longrightarrow \mathrm{Pic}_X\\ \mathcal{D}\in\mathrm{Div}_X(S) &\longmapsto (\mathcal{O}_{X\times S}(\mathcal{D}) \bmod \mathrm{Pic}(S))\\ (\mathcal{D}\mapsto(1\times f)^*\mathcal{D}) &\longmapsto (\mathcal{O}_{X\times S}(\mathcal{D}) \bmod \mathrm{Pic}(S)) \mapsto ((1\times f)^*\mathcal{O}_{X\times S}(\mathcal{D}) \bmod \mathrm{Pic}(T)).\end{aligned}$$

2.4 Representing Div_X

Let X and $\lambda\in\mathrm{NS}(X)$ be as above and now let L denote a very ample line bundle on it. We wish to show that Div_X is representable by a scheme $\mathrm{Div}(X)$ as indicated at the beginning of the chapter, which in this case will be a countable union of schemes of finite type. For $D\subset X$ any effective divisor, let

$$P(n) := \chi(L^{\otimes n}(-D))$$

denote the *Hilbert polynomial of* $\mathcal{O}_X(-D)$ *with respect to* L. Then define the functor Div_X^P by

$$\mathrm{Div}_X^P(S) := \{\mathcal{D}\in\mathrm{Div}_X(S) : \mathcal{O}_X(-D_s) \text{ has Hilbert polynomial } P \text{ for each } s\in S\}.$$

We will show that each of the subfunctors Div_X^P is representable, and then by the decomposition

$$\mathrm{Div}_X = \bigsqcup_P \mathrm{Div}_X^P$$

the representability of Div_X, and thus the existence of the desired scheme ("divisor variety") $\mathrm{Div}^{\lambda}(X) \subset \mathrm{Div}(X)$, will follow.

Our strategy will be as follows (adapted from [49, Lecture 15]):

- For each P in the above decomposition, we will define a natural transformation $\mathrm{Div}_X^P \to h_{G(V,r)}$ to the functor of points of a Grassmannian $G(V,r)$ and argue that the natural inclusion $\mathrm{Div}_X^P \hookrightarrow \mathrm{Hilb}_X$ into the Hilbert functor of X factors through this (see Section 2.4.1 below for a definition of Hilb_X). This will allow us to construct the desired representative scheme $\mathrm{Div}^P(X)$ inside this Grassmannian.

- We will then use a result on *flattening stratifications* from [49, Lecture 8] to show that if $S \to G(V,r)$ is any S-point of the Grassmannian then there is a subscheme $Y \subset S$ such that for any morphism $g : T \to S$, we can obtain all families $\mathcal{D}_T \to T$ of effective divisors in X, with $\mathcal{O}_X(-D_t)$ having Hilbert polynomial P for all $t \in T$, by pulling back some closed subscheme $Z \subset X \times S$ along g. This will suffice for us to conclude that there is a locally closed subscheme $Y_G \subset G(V,r)$ representing Div_X^P. i.e. $Y_G = \mathrm{Div}^P(X)$.

- Finally we will show that Y_G is *closed* in $G(V,r)$, hence $\mathrm{Div}^P(X)$ is projective.

2.4.1 The Hilb_X functor

Briefly in this section, we define the Hilbert functor Hilb_X of X. For S a scheme over $\mathbb{C}$, we define

$$\mathrm{Hilb}_X(S) := \left\{ (\mathcal{F}, q) : \begin{array}{c} \mathcal{F} \text{ is a coherent sheaf on } X \times S, \\ \mathrm{Supp}(\mathcal{F}) \xrightarrow{p_2} S \text{ is proper,} \\ q : \mathcal{O}_{X \times S} \twoheadrightarrow \mathcal{F}. \end{array} \right\} \Big/ \sim$$

where $(\mathcal{F}, q) \sim (\mathcal{F}', q')$ if $\mathcal{F} \cong \mathcal{F}'$ and $\ker(q) = \ker(q')$ as subsheaves of $\mathcal{O}_{X \times S}$. Then for $f : T \to S$ over $\mathbb{C}$ we have that $(1 \times f)^* \mathcal{F}$ is coherent on $X \times T$, the diagram

$$\begin{tikzcd} \mathrm{Supp}((1 \times f)^* \mathcal{F}) \arrow[r] \arrow[d] & \mathrm{Supp}(\mathcal{F}) \arrow[d] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

is evidently Cartesian since $\mathrm{Supp}((1 \times f)^* \mathcal{F}) = (1 \times f)^{-1}(\mathrm{Supp}(\mathcal{F}))$, and of course right exactness of the pullback $(1 \times f)^*$ ensures that q pulls back to a surjection

$$(1 \times f)^* q : \mathcal{O}_{X \times T} \twoheadrightarrow (1 \times f)^* \mathcal{F}.$$

So with the above definition, Hilb_X indeed constitutes a functor

$$\mathrm{Hilb}_X : \mathrm{Sch}/\mathbb{C} \longrightarrow \mathrm{Set}$$

and the data it encapsulates corresponds precisely to the data of all families of closed subschemes in X since, if $(\mathcal{F}, q) \in \mathrm{Hilb}_X(S)$, then the kernel of the surjective morphism $q : \mathcal{O}_{X\times S} \twoheadrightarrow \mathcal{F}$ determines a family of ideal sheaves in $\mathcal{O}_X$, corresponding to a family of closed subschemes.

Since we will only refer to Hilb_X in Section 2.4.2 in the context of the inclusion $\mathrm{Div}_X \hookrightarrow \mathrm{Hilb}_X$, the outline above is all we shall need. Of course there is much more to say about this functor, particularly the problem of representing it — for many more details on these ideas we refer the reader to N. Nitsure's exposition in [50].

2.4.2 Factoring $\mathrm{Div}_X \hookrightarrow \mathrm{Hilb}_X$ through $\mathrm{h}_{G(V,r)}$

With $P(n)$ denoting the Hilbert polynomial of the ideal sheaf of an effective divisor in X, as defined above, let $m \gg 0$ so that $H^i(X, L^{\otimes m}) = 0$ for all $i > 0$, let $V := H^0(X, L^{\otimes m})$ and let $r := \chi(L^{\otimes m}) - P(m)$. We have:

Proposition 2.3. *Let P, V and r be as above. A flat family $\mathcal{D} \subset X \times S$ of effective divisors, with $\mathcal{O}_X(-\mathcal{D}_s)$ having Hilbert polynomial P for each $s \in S$, determines a natural transformation to the functor of points the Grassmannian $G(V, r)$:*

$$\mathrm{Div}_X^P \longrightarrow \mathrm{h}_{G(V,r)}.$$

Remark 2.4. In the proof of this proposition, we make use of a result due to Mumford concerning *Castelnuovo–Mumford regularity* — we refer the reader to Section 8.2 for the relevant definitions and result (Theorem 8.11), where they are introduced there in the context of syzygies.

Proof. By Mumford's theorem on Castelnuovo–Mumford regularity, if $D \subset X$ is any effective divisor such that the ideal sheaf $\mathcal{O}_X(-D)$ has Hilbert polynomial P, then that ideal sheaf will be m-regular for m large enough and depending only on P. Increasing m does not change this situation, and so by the vanishings yielded by both regularity and Serre vanishing, we have

$$H^i(L^{\otimes m}) = 0, \quad i > 0$$
$$H^i(L^{\otimes m}(-D)) = 0, \quad i > 0$$
$$H^i(L^{\otimes m}|_D) = 0, \quad i > 0$$

as well as global generation of $L^{\otimes m}(-D)$.[1]

Now let p and q denote the projections in the diagram

$$\begin{tikzcd} & X \times S \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & S \end{tikzcd}$$

and let $\mathcal{D} \subset X \times S$ be any flat family of effective divisors on X, each of whose ideal sheaf has Hilbert polynomial P. The vanishings above imply that $q_*(p^*L^{\otimes m} \otimes \mathcal{O}_{\mathcal{D}})$ is locally free on S of rank r. Note also that the formation of q_* here commutes with all base extensions $T \to S$.

The vanishings and global generation of $L^{\otimes m}(-D)$ also imply that

$$R^1 q_*(p^*L^{\otimes m} \otimes \mathcal{O}(-\mathcal{D})) = 0$$

and

$$q^*q_*(p^*L^{\otimes m} \otimes \mathcal{O}(-\mathcal{D})) \longrightarrow\!\!\!\!\!\!\rightarrow p^*L^{\otimes m} \otimes \mathcal{O}(-\mathcal{D}).$$

Combining these allows us to conclude that the following pushforward of the ideal sequence for $\mathcal{D}$ twisted by $p^*L^{\otimes m}$ is exact:

$$0 \longrightarrow q_*(p^*L^{\otimes m} \otimes \mathcal{O}(-\mathcal{D})) \longrightarrow V \otimes \mathcal{O}_S \longrightarrow q_*(p^*L^{\otimes m} \otimes \mathcal{O}_{\mathcal{D}}) \longrightarrow 0.$$

But the rightmost map here amounts to a family of r-dimensional quotients of V parametrized by S, and thus an S-valued point of the Grassmannian $G(V, r)$. Thus we have determined a natural transformation

$$\mathrm{Div}_X^P \longrightarrow \mathrm{h}_{G(V,r)}.$$

□

Now we note that, in a converse direction, given a sufficiently positive polarization on X, one obtains families of ideals in X over S in a natural way by pulling back the tautological quotient bundle from the Grassmannian to a scheme S:

[1] Although we could obtain *all* of these vanishings and the global generation using Serre vanishing alone, it would not be sufficient for our purposes since it is crucial in this proof that m be *bounded* as D varies among all effective divisors with Hilbert polynomial P.

Proposition 2.5. *Let L be a globally generated line bundle on X and $V := H^0(X, L)$. Let p and q denote the projections $X \times S \to S$ and $X \times S \to X$, respectively, as above. For $0 < r \leq \dim V$, pulling back the tautological quotient map on the Grassmannian $G(V, r)$ determines a natural transformation:*

$$h_{G(V,r)} \longrightarrow \mathrm{Hilb}_X$$

Proof. Let $S \to G(V, r)$ be an S-valued point of $G(V, r)$. This determines (via pullback) a rank r vector bundle quotient

$$\sigma : V \otimes \mathcal{O}_S \twoheadrightarrow E.$$

Let $K := \ker(\sigma)$. Pulling back along p we have

$$\begin{array}{ccccccc} p^*K & \longrightarrow & V \otimes \mathcal{O}_{X\times S} & \longrightarrow & p^*E & \longrightarrow & 0. \\ & & \downarrow & & & & \\ & & q^*L & & & & \end{array}$$

We let $\mathcal{I}$ denote the sheaf of ideals in $\mathcal{O}_{X\times S}$ that is the image of the composed map $p^*K \otimes (q^*L^{-1}) \to \mathcal{O}_{X\times S}$ obtained from the diagram above. So an S-valued point of $G(V, r)$ has produced a quotient $\mathcal{O}_{X\times S}/\mathcal{I}$ and thus we have determined a natural transformation

$$h_{G(V,r)} \longrightarrow \mathrm{Hilb}_X.$$

□

Remark 2.6. We note here that in Proposition 2.5 the family of ideals being produced will not at all be flat in general, will often be trivial and the natural transformation $\mathrm{h}_{G(V,r)} \to \mathrm{Hilb}_X$ obtained is not necessarily mapping into families of subschemes in X with any particular fixed Hilbert polynomial. However, our purpose next is to compose this with the natural transformation obtained in Proposition 2.3 in order to get a handle on Div_X^P.

So what is the composition of $\mathrm{Div}_X^P \to h_{G(V,r)}$ and $h_{G(V,r)} \to \mathrm{Hilb}_X$? Keeping track of the sheaves used in Proposition 2.3, one can see that in fact when the S-valued point of $G(V, r)$ mapping to Hilb_X in Proposition 2.5 is indeed that coming from the quotient

$$V \otimes S \twoheadrightarrow p_*(q^*L^{\otimes m_0}|_{\mathcal{D}}) \tag{2.7}$$

then the sheaf $\mathcal{I}$ of ideals on $X \times S$ constructed in Proposition 2.5 is precisely $\mathcal{O}_{X\times S}(-\mathcal{D})$. An easy way to see this is to note that this $\mathcal{I}$ is arising as the image of the diagonal map in a relativized version of the following diagram over S (where the vertical maps are evalutions):

$$\begin{tikzcd} 0 \arrow[r] & H^0(L^{\otimes m}(-D)) \otimes L^{\otimes -m} \arrow[r] \arrow[d] \arrow[dr] & H^0(L^{\otimes m}) \otimes L^{\otimes -m} \arrow[r] \arrow[d] & H^0(L|_D) \otimes L^{\otimes -m} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}_X(-D) \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_D \arrow[r] & 0 \end{tikzcd}$$

Thus the composition $\mathrm{Div}_X^P \to h_{G(V,r)} \to \mathrm{Hilb}_X$ is simply the natural inclusion of effective divisors among all subschemes of X.

2.4.3 Representing Div_X^P inside $G(V, r)$

Since in the last section we have shown that it is valid to now consider Div_X^P as a subfunctor of the functor of points $\mathrm{h}_{G(V,r)}$ of the Grassmannian, we will use this to determine a subscheme of $G(V, r)$ the represents the former.

To do this, as indicated earlier, we will need a result concerning *flattening stratifications*. We recall the following definition and theorem from [49, Lecture 8]:

Definition 2.8. A *stratification* of a scheme S is a finite set $S_1, \ldots, S_r$ of locally closed subschemes of S such that every point $s \in S$ is in exactly one subset S_i.

Now fix a scheme S, let $\mathbb{P}$ denote a projective space over $\mathbb{C}$ and suppose $\mathcal{F}$ is a coherent sheaf on $\mathbb{P} \times S$.

Theorem 2.9. *There is a stratification $S_1, \ldots, S_r$ of S such that for all morphisms $g : T \to S$ (T Noetherian), the sheaf $(\mathrm{id}_{\mathbb{P}} \times g)^*\mathcal{F}$ is flat over T if and only if the morphism g factors as*

$$T \longrightarrow \textstyle\bigsqcup_{i=1}^r S_i \longrightarrow S.$$

This is called a flattening stratification of $\mathcal{F}$*.*

Remark 2.10. Note that the analogous result holds true if we replace $\mathbb{P}$ by any projective scheme X since we can simply embed X in projective space, use the embedding to pushforward any coherent sheaf on $X \times S$ to $\mathbb{P} \times S$, and then apply the theorem to the pushforward.

With this result in hand, we now apply it to our circumstances to obtain:

Proposition 2.11. *Let V, P, m and r be as in Proposition 2.3 and let S be any scheme. Then for all closed subschemes $Z \subset X \times S$ there is a subscheme $Y \subset S$ (possibly empty) such that*

1. *if $\mathcal{D}_T \to T$ is a flat family of effective divisors in X (each of whose ideal sheaf has Hilbert polynomial P) obtained as the fiber product $\mathcal{D}_T = Z \times_S T$ along a map $g : T \to S$, then g factors through Y; and conversely*

2. *for any morphism $g : T \to S$ factoring through Y, the scheme $\mathcal{D}_T := Z \times_S T \subset X \times T$ is a flat family $\mathcal{D}_T \to T$ of effective divisors in X each of whose ideal sheaf has Hilbert polynomial P.*

The point now is that the composition $\mathrm{Div}_X^P \to \mathrm{h}_{G(V,r)} \to \mathrm{Hilb}_X$ from Section 2.4.2 has bought us *boundedness* of Div_X^P in the sense that for all schemes S parametrizing families of effective divisors in X with ideal sheaves having Hilbert polynomial P, we get a map $S \to G(V,r)$. So we can in particular take $S = G(V,r)$ and then the subscheme $Y_G \subset G(V,r)$ yielded by Proposition 2.11 is precisely the representative of Div_X^P that we wanted — the conclusions of the proposition imply that it satisfies all the desired functorial properties. Moreover, we then apply Proposition 2.5 with $S = Y_G$ to obtain a sheaf of ideals on $X \times Y_G$ which cuts out the universal family of effective divisors in X whose ideal sheaves have Hilbert polynomial P.

Proof of Proposition 2.11. Almost everything follows immediately by applying Theorem 2.9 to the sheaf $\mathcal{F} = \mathcal{O}_Z$ on $X \times S$, noting that $(\mathrm{id}_X \times g)^*\mathcal{F} = \mathcal{O}_{Z \times_S T}$ for any morphism $g : T \to S$. The only thing that remains to be verified is that in (2) the subscheme $Z \times_S T \subset X \times T$, which is already flat over T with the desired Hilbert polynomial

$$\chi(p^* L^{\otimes n} \otimes \mathcal{O}_{Z \times_S T}) = \chi(L^{\otimes n}) - P(n),$$

is indeed a Cartier divisor. To see this, we reason as in the proof of [49, Lemma, p.108]: let $t \in T$ denote a closed point such that Z_t is an effective divisor in X. We show that there is an open neighborhood U of t in T such that $Z \cap (X \times U)$ is a Cartier divisor in $X \times U$.

Since the projection $p : X \times T \to T$ is a closed map, it suffices to prove that there is an open neighborhood U of $X \times \{t\}$ in which Z is a Cartier divisor.

Let $y \in X \times T$ be a point such that $p(y) = t$, let $\mathcal{I}_{Z,y} \subset \mathcal{O}_{X \times T,y}$ be the defining ideal of Z at y and let $\mathfrak{m}_t \subset \mathcal{O}_{T,t}$ be the maximal ideal.

Since $\mathcal{O}_{X \times T,y}/\mathfrak{m}_t \cdot \mathcal{O}_{X \times T,y}$ is the local ring of y on $X \times \{t\}$, and since Z_t is a Cartier divisor,

$$\mathcal{I}_{Z,y} + \mathfrak{m}_t \cdot \mathcal{O}_{X \times T,y} = (f) + \mathfrak{m}_t \cdot \mathcal{O}_{X \times T,y}$$

for some $f \in \mathcal{O}_{X\times T,y}$. Choosing f suitably, we may assume it is in $\mathcal{I}_{Z,y}$. Then consider the exact sequence

$$0 \longrightarrow \mathcal{I}_{Z,y}/(f) \longrightarrow \mathcal{O}_{X\times T,y}/(f) \longrightarrow \mathcal{O}_{X\times T,y}/\mathcal{I}_{Z,y} \longrightarrow 0.$$

Since Z is flat over T by assumption, restricting to the fiber over $t \in T$ we have

$$\begin{array}{l} \mathrm{Tor}_1^{\mathcal{O}_{X\times T,y}}(\mathcal{O}_{X\times T,y}/\mathcal{I}_{Z,y}, \mathbb{C}(t)) \longrightarrow \quad \longrightarrow \mathcal{I}_{Z,y}/(f) + \mathfrak{m}_t \cdot \mathcal{O}_{X\times T,y} \longrightarrow \mathcal{O}_{X\times T,y}/(\mathcal{I}_{Z,y} + \mathfrak{m}_t \cdot \mathcal{O}_{X\times T,y}) \longrightarrow 0. \\ \qquad\qquad\quad \| \\ \qquad\qquad\quad 0 \end{array}$$

So we then have that $(\mathcal{I}_{Z,y}/(f)) \otimes \mathbb{C}(t) = 0$ for all $t \in T$, and thus by Nakayama's lemma, $\mathcal{I}_{Z,y}/(f) = 0$. Thus $\mathcal{I}_{Z,y} = (f)$ and hence Z is indeed a Cartier divisor at y and therefore in a neighborhood of y. $\square$

Finally, we prove:

Proposition 2.12. *$Y_G \subset G(V,r)$ is closed.*

Proof. Let $\overline{Y}_G$ denote the closure of Y_G in $G(V,r)$ and suppose for sake of contradiction that $Y_G \subsetneq \overline{Y}_G$. Choose a point $y \in \overline{Y}_G \setminus Y_G$ and an affine open neighborhood $U \subset \overline{Y}_G$ of y. Now if Y_G (and thus its closure) were 0-dimensional, there would be nothing to prove. So, assuming they are positive dimensional, we can choose an integral curve C in $\overline{Y}_G$ through y such that $C \setminus y \subset Y_G$ (possibly after shrinking U).

Let $\widehat{C}$ denote the normalization of C, which is thus a smooth curve admitting a map $\iota : \widehat{C} \to \overline{Y}_G$. Let C° denote $\widehat{C} \setminus \iota^{-1}(y)$. Then $C^\circ \to Y_G$ determines a family $\mathcal{D}^\circ \subset X \times C^\circ$ of effective divisors in X whose ideal sheaves have Hilbert polynomial P. Moreover $\mathcal{D}^\circ$ is a Weil divisor in $X \times C^\circ$ since the latter is smooth. Thus its closure $\mathcal{D} \subset X \times \hat{C}$ is an effective divisor too, and indeed it is flat over $\widehat{C}$ since, by construction, it does not contain any of the fibers $X \times \{z\}$ for $z \in \iota^{-1}(y)$.

Thus $\mathcal{D}$ is a family of effective divisors whose ideal sheaves have Hilbert polynomial P (since $\widehat{C}$ is connected and therefore the Hilbert polynomial is constant over it) and thus determines a map $\widehat{C} \to Y_G$ that extends $C^\circ \to Y_G$. But this is a contradiction since, by our construction, the latter map cannot extend like this. $\square$

We have now completed the construction of a scheme $\mathrm{Div}^P(X) := Y_G$ which parametrizes effective divisors in X whose ideal sheaves have Hilbert polynomial P, we have seen that it admits a universal family and thus represents the functor Div_X^P, and given the above proposition we can also conclude that it is projective.

2.5 Div(X) is the projectivization of a Picard sheaf

Having established existence of the scheme $\mathrm{Div}(X)$ as that representing the functor Div_X, together with a universal family $\mathcal{D} \to \mathrm{Div}(X)$ of effective divisors on X, it is possible to use these data to then establish analogous results for the functor Pic_X. Namely, one obtains the following theorem

Theorem 2.13. *The functor* Pic_X *is representable by a scheme* $\mathrm{Pic}(X)$ *and on* $X \times \mathrm{Pic}(X)$ *there exists a universal line bundle* $\mathcal{L}$*, unique up to twisting by pullbacks of line bundles from* $\mathrm{Pic}(X)$ *itself.*

This is the content of [49, Lecture 21] (albeit generalized to dimensions higher than 2, which Mumford asserts is immediate). For the proof, we refer the reader to that lecture.

For our purposes, this theorem is essential in our aim to produce a sheaf on $\mathrm{Pic}(X)$ both which we can "get our hands on" and whose projectivization is the scheme $\mathrm{Div}(X)$ — the latter property can be thought of as "*linearizing*" the scheme $\mathrm{Div}(X)$, in a sense. With this aim in mind, we now define a *Picard sheaf*:

Definition 2.14. Let $p : X \times \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ denote the projection and let $\mathcal{L}$ be a universal line bundle. A *Picard sheaf* $\mathcal{F}_{\mathcal{L}}$ on $\mathrm{Pic}(X)$ will be an $\mathcal{O}_{\mathrm{Pic}(X)}$-module such that for any quasicoherent sheaf N on $\mathrm{Pic}(X)$, there is an isomorphism

$$q : \underline{\mathrm{Hom}}(\mathcal{F}_{\mathcal{L}}, N) \xrightarrow{\cong} p_*(\mathcal{L} \otimes p^* N). \tag{2.15}$$

Remark 2.16. Note here that in the case that N is taken to be the skyscraper sheaf $\mathcal{O}_{[L]}$ at a point $[L] \in \mathrm{Pic}(X)$ corresponding to a line bundle L, then we have

$$\underline{\mathrm{Hom}}(\mathcal{F}_{\mathcal{L}}, \mathcal{O}_{[L]}) \cong (\mathcal{F}_{\mathcal{L}})^\vee_{[L]}$$

$$p_*(\mathcal{L} \otimes p^* \mathcal{O}_{[L]}) \cong H^0(X, L)$$

and so we see that the (dual of the) isomorphism 2.15 is telling us that

$$(\mathcal{F}_{\mathcal{L}})_{[L]} \cong H^0(X, L)^\vee$$

which, after projectivizing, yields

$$\mathbb{P}(\mathcal{F}_{\mathcal{L}})_{[L]} \cong \mathbb{P}H^0(X, L)^\vee \cong |L|$$

(remember here that $\mathbb{P}$ is always denoting 1-dimensional *quotients*). So this hints that $\mathcal{F}_{\mathcal{L}}$ is playing the role of a sheaf whose projectivization parametrizes effective divisors — one just has to deal with the subtlety of working over the whole variety $\mathrm{Pic}^\lambda(X)$ at once. This is precisely the purpose of Theorem 2.17 below.

We note without proof that $\mathcal{F}_{\mathcal{L}}$ is unique up to twisting by a line bundle and that its formation commutes with base-change (from $\mathrm{Pic}(X)$ to another scheme S). Below we will indicate how to produce such a sheaf using only the existence of the universal line bundle. First however, assuming existence of Picard sheaves, we realize our stated aim:

Theorem 2.17. *Let $\mathcal{L}$ denote a universal line bundle on $X \times \mathrm{Pic}(X)$ and $\mathcal{F}_{\mathcal{L}}$ an associated Picard sheaf. Then we have an isomorphism:*

$$c : \mathrm{Div}(X) \xrightarrow{\cong} \mathbb{P}\mathcal{F}_{\mathcal{L}}$$

and the natural projection map $\mathbb{P}\mathcal{F}_{\mathcal{L}} \to \mathrm{Pic}(X)$ coincides with the Abel–Jacobi morphism $u : \mathrm{Div}(X) \to \mathrm{Pic}(X)$.

Proof of Theorem 2.17. A map $h : S \to \mathrm{Div}(X)$ corresponds to a flat family of effective divisors $\mathcal{D} \subset X \times S$. Composing with the Abel–Jacobi map $u : \mathrm{Div}(X) \to \mathrm{Pic}(X)$ yields an S-point of the latter, corresponding to the line bundle $\mathcal{O}_{X\times S}(\mathcal{D})$. Thus, letting $\mathcal{L}_S := (1 \times (u \circ h))^*\mathcal{L}$, we have:

$$\mathcal{O}_{X\times S}(\mathcal{D}) \cong \mathcal{L}_S \otimes p_2^* N$$

for some line bundle N on S (here $p_2 : X \times S \to S$ denotes projection). Moreover, N is uniquely determined since pullback $\mathrm{Pic}(S) \to \mathrm{Pic}(X \times S)$ is injective.

Now let $\sigma \in H^0(X \times S, \mathcal{L}_S \otimes p_2^* N)$ be a defining section of $\mathcal{D}$. Formation of the Picard sheaf $\mathcal{F}_{\mathcal{L}}$ commutes with base change (see Proposition 2.18 below and [36, Theorem 5]), so we have have that

$$(u \circ h)^* q : \underline{\mathrm{Hom}}((u \circ h)^* \mathcal{F}_{\mathcal{L}}, N) \xrightarrow{\cong} p_{2*}(\mathcal{L}_S \otimes p_2^* N)$$

and so σ corresponds to a morphism $f : (u \circ h)^* \mathcal{F}_{\mathcal{L}} \to N$ of sheaves on S.

In fact, f must be surjective: since $\mathcal{D}$ is a *flat* family, we have for any $t \in S$ that $\sigma \otimes \mathbb{C}(t)$ defines an effective divisor $\mathcal{D}_t \subset X$. In particular this means it is non-zero, hence $f \otimes \mathbb{C}(t)$ is surjective for all $t \in S$. So surjectivity of f then follows by Nakayama's lemma.

Surjectivity of f means, by definition of projectivization of a sheaf, we have in fact determined a map

$$S \longrightarrow \mathbb{P}\mathcal{F}_{\mathcal{L}}.$$

In fact, more importantly, one can check that the above constructions are functorial for commuting triangles

$$\begin{tikzcd} T \arrow[rr] \arrow[dr] & & \mathrm{Div}(X) \\ & S \arrow[ur] & \end{tikzcd}$$

and so we have actually determined a natural transformation

$$\lambda : \mathrm{Div}_X \longrightarrow \mathrm{h}_{\mathbb{P}\mathcal{F}_{\mathcal{L}}}$$

(where here, as in Section 2.4.3, h_Y denotes the functor of points of a scheme Y). It remains to show that λ is an isomorphism. Indeed this follows provided the maps $\lambda(S)$ are bijective for each scheme S. To see this, suppose instead that we have an element of $h_{\mathbb{P}\mathcal{F}_{\mathcal{L}}}(S)$ — that is, a map $f : S \to \mathbb{P}\mathcal{F}_{\mathcal{L}}$. This yields maps $c : S \to \mathrm{Pic}(X)$ (by composing with the projection $p : \mathbb{P}\mathcal{F}_{\mathcal{L}} \to \mathrm{Pic}(X)$) and $1_X \times f : X \times S \to X \times \mathbb{P}\mathcal{F}_{\mathcal{L}}$. Now note that in $X \times \mathbb{P}\mathcal{F}_{\mathcal{L}}$ one has the distinguished (codimension 1) incidence correspondence

$$\Phi := \{(p, [s]) : s(p) = 0\}$$

(we are thinking here of s as a section of $\mathcal{L}_s$, which is valid since $(\mathcal{F}_{\mathcal{L}})_L \cong H^0(X, L)^\vee$ and we are taking projective *quotients*). This Φ is the *universal zero locus* of sections of the line bundles $\mathcal{L}_s$ on X, for $s \in S$. Φ is determined by a section

$$\sigma \in H^0(X \times \mathbb{P}\mathcal{F}_{\mathcal{L}}, (1 \times p)^* \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}\mathcal{F}_{\mathcal{L}}}(-1))$$

which necessarily does not vanish identically on any fiber of the projection $p_2 : X \times \mathbb{P}\mathcal{F}_\mathcal{L} \to \mathbb{P}\mathcal{F}_\mathcal{L}$ since we only considered non-zero sections s in the fibers. So finally, we pull back both the line bundle $(1 \times p)^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}\mathcal{F}_\mathcal{L}}(-1)$ and the section σ along the map $1_X \times f$ — together this data yields an effective divisor $\mathcal{D} \subset X \times S$, flat over S since Φ contains no fibers of p_2.

So now we are done — one checks that the association of the map $S \to \mathbb{P}\mathcal{F}_\mathcal{L}$ to the flat family $\mathcal{D} \subset X \times S$ of effective divisors in X is inverse to the natural transformation $\lambda(S)$ above. □

Although existence of the desired sheaf is asserted above, we want to actually identify a specific instance of it (given a specific instance of the universal line bundle $\mathcal{L}$) since this will allow us to manipulate it effectively, and relate it with similar such sheaves, in calculations later in this thesis. To that end, we have

Proposition 2.18. *Let $\mathcal{L}$ denote a universal line bundle on $X \times \operatorname{Pic}(X)$ and denote by p_1, p_2 the projections. Then*

$$\mathcal{F}_\mathcal{L} := R^n p_{2*}(p_1^* K_X \otimes \mathcal{L}^\vee)$$

is a Picard sheaf for $\mathcal{L}$.

Proof. In [36], Kleiman explains a vast generalization of Serre duality, often referred to now as "relative duality". Not needing the full generality of his results, we use his [36, Corollary 24] for our purposes, which says in particular that if $f : \mathcal{X} \to \mathcal{Y}$ is a smooth map of relative dimension r between smooth projective schemes, $\mathcal{F}$ a coherent sheaf on $\mathcal{X}$ and $\mathcal{N}$ a coherent sheaf on $\mathcal{Y}$, then there is an isomorphism of sheaves

$$D^m : \underline{\operatorname{Ext}}_f^m(\mathcal{F}, K_\mathcal{X} \otimes f^*(\mathcal{N} \otimes K_\mathcal{Y}^{-1})) \xrightarrow{\cong} \underline{\operatorname{Hom}}_{\mathcal{O}_\mathcal{Y}}(R^{r-m} f_* \mathcal{F}, \mathcal{N}).$$

We now simply take:

$$\begin{aligned} \mathcal{X} &= X \times \operatorname{Pic}(X) & \qquad \mathcal{F} &= p_1^* K_X \otimes \mathcal{L}^\vee \\ \mathcal{Y} &= \operatorname{Pic}(X) & \mathcal{N} &= \mathcal{O}_\mathcal{Y} \\ f &= p_2 & m &= 0 \end{aligned}$$

then, noting that then $K_\mathcal{Y} = \mathcal{O}_{\operatorname{Pic}(X)}$ and $K_\mathcal{X} = p_1^* K_X$, the result follows immediately from the isomorphism D^0. □

Chapter 3

Brill–Noether theory

In this chapter we review the classical Brill–Noether theory for curves and indicate some analogous results that are known in higher dimensions.

3.1 Overview

Let C be a smooth projective curve of genus g. Much of the geometry of C is captured by the study of its linear series, which goes by the name of "Brill–Noether theory". More specifically, one considers the set of all divisors of given degree d on C that move in a linear series of (projective) dimension at least r.

Turning to details, we start by specializing constructions of the previous chapter to the one-dimensional setting. Thus set

$$\mathrm{Div}^d(C) := \{\text{effective divisors } D \subset C \text{ of degree } d\}$$
$$\mathrm{Pic}^d(C) := \{\text{degree } d \text{ line bundles on } C\} \cong \mathrm{Jac}(C).$$

Here the curve C and the integer d are playing the role of X and λ from Chapter 2, and all of the theory from that chapter applies in this one-dimensional setting. In fact more is true in dimension one: since C is a curve and a degree d effective divisor is simply a d-tuple of points, the variety $\mathrm{Div}^d(C)$ is isomorphic to the symmetric product $C_d := C^{\times d}/\mathfrak{S}_d$ where $\mathfrak{S}_d$ denotes the symmetric group on d elements (see [35, Exercise 9.3.8, Answer 9.3.8, pp. 260, 303] for a proof). As in the previous chapter, we have the Abel–Jacobi map

$$u_d : \mathrm{Div}^d(C) \longrightarrow \mathrm{Pic}^d(C)$$

sending a divisor D to the (isomorphism class) of $\mathcal{O}_C(D)$. Recall that the fiber of u_d over L parametrizes all effective divisors in the linear series $|L|$: indeed

$$u_d^{-1}(L) \cong \mathbb{P}_{\mathrm{sub}} H^0(C,L) \cong \mathbb{P}H^0(C,L)^\vee.$$

The Brill–Noether loci are defined to be the Zariski-closed subsets

$$W_d^r(C) := \{L \in \mathrm{Pic}^d(C) : h^0(C,L) \geq r+1\}.$$

Moreover, we introduce the auxiliary loci

$$\begin{aligned} G_d^r(C) &:= \{V \subset H^0(C,L) : \deg\, L = d, \quad \dim\, V = r+1\} \\ C_d^r &:= \{D \in \mathrm{Div}^d(C) : h^0(\mathcal{O}_C(D)) \geq r+1\}. \end{aligned}$$

In fact, these loci all carry natural scheme structures (see [5, Chapter IV] for details) and they fit into a diagram as follows:

$$\begin{tikzcd} G_d^r \arrow[dr, two heads] & C_d^r \arrow[d, two heads] & \subset & \mathrm{Div}^d(C) \arrow[d, "u_d"] \\ & W_d^r & \subset & \mathrm{Pic}^d(C) \end{tikzcd}$$

Example 3.1. One can get a sense for how the geometry of the loci W_d^r is influenced by the existence of maps from C to projective space by considering some examples:

1. If C is a hyperelliptic curve, one can pull back $\mathcal{O}_{\mathbb{P}^1}(1)$ along the hyperelliptic map $C \to \mathbb{P}^1$ to obtain a degree 2 line bundle L with two sections. This bundle and the map are unique for C and thus one has
$$W_2^1(C) = \{L\}.$$

2. Suppose this time that C is a non-hyperelliptic curve of genus 4. Then its canonical linear series embeds it as a degree 6 curve in $\mathbb{P}^3$ where it is contained in a unique quadric Q. If Q is smooth, its two rulings determine a pair of degree 3 maps $C \to \mathbb{P}^1$ and thus, by pullback of

$\mathcal{O}_{\mathbb{P}^1}(1)$, a pair of degree 3 line bundles L_1 and L_2 on C. These correspond to the points of the Brill–Noether locus $W^1_3(C)$:

$$W^1_3(C) = \{L_1, L_2\}.$$

On the other hand, if Q is singular (necessarily of rank 3 since C is non-degenerate), it has only one ruling and by similar reasoning, one finds that $W^1_3(C)$ consists of only a single point L for which $L^{\otimes 2} = K_C$. This single point $W^1_3(C)$, however, has a non-reduced scheme structure — its tangent space is naturally the one-dimensional space $\ker(H^0(L)^{\otimes 2} \to H^0(K)) \cong H^0(K_C - 2L) = H^0(\mathcal{O}_C)$ where this isomorphism can be observed using the *basepoint free pencil trick* (see e.g. [5, p. 126]).

3. Suppose this time that $C \subset \mathbb{P}^2$ is a smooth degree d curve with embedding line bundle $L = \mathcal{O}_C(1)$. For any point $p \in C$ one can project away from p onto a line and obtain a degree $d-1$ map $C \to \mathbb{P}^1$. By pullback again, one obtains a degree $d-1$ line bundle on C which coincides with $L(-p)$. Since this can be done for any point $p \in C$, one obtains a family $\{L(-p)\}_{p\in C}$ of degree $d-1$ line bundles on C and indeed this construction accounts for all such:

$$W^1_{d-1}(C) = \{L(-p)\}_{p\in C} \cong C.$$

4. For a slightly more intricate example, consider a canonical genus 5 curve $C \subset \mathbb{P}^4$ and let $l \subset \mathbb{P}^4$ denote a generically chosen secant line. Projecting away from this secant line yields a map $C \to \mathbb{P}^2$ and since l was chosen generically, this will be an immersion. By adjunction on $\mathbb{P}^2$ we know that the image must have 10 nodes, and so a further projection from any such node will determine a degree 4 map $C \to \mathbb{P}^1$. Thus, similarly to the previous examples, we have determined a line bundle $L \in W^1_4(C)$. In fact, by Riemann–Roch, any fiber of the map $C \to \mathbb{P}^1$ will span a 4-secant-2-plane of the embedding $C \subset \mathbb{P}^4$. On the other hand, by considering the restriction map

$$H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \longrightarrow H^0(C, \mathcal{O}_C(2))$$

one can see by Riemann–Roch again that C must lie in at least 3 independent quadrics in $\mathbb{P}^4$, and since C has degree 8, one uses Bézout's theorem to conclude there must be exactly 3. So the linear system of quadrics through C is a $\mathbb{P}^2$ and, with some work, one can find that the *singular* quadrics form a quintic curve $D \subset \mathbb{P}^2$. For generic C these singular quadrics will have rank 4 which means they are cones over smooth quadrics in $\mathbb{P}^3$ and therefore in particular each have two rulings by $\mathbb{P}^2$'s (away from the vertex). These rulings are precisely the 4-secant-2-planes we discovered above, but now we see that above each point in the quintic curve D of singular quadrics there are *two* such. It is verified, for example in [5, Exercises F., pp. 270-275], that this illustrates some very appealing geometry of the Brill–Noether locus $W^1_4(C)$ — it admits a morphism

$$W^1_4(C) \xrightarrow{2:1} D$$

and in the case of generic C this is an unramified cover and everything is smooth. There is much more to be said for the case of special C and we encourage the reader to visit the reference above for more details.

While special instances of curves of a fixed genus g yield varying behavior for the geometry of the W^r_d loci, as seen in the examples above, there is a range of values for d and r when one expects uniform behavior of the W^r_d's in general:

Definition 3.2. The *Brill–Noether number* is defined as

$$\rho = \rho(g, d, r) := g - (r + 1)(g - d + r).$$

The Brill–Noether number is the *expected* dimension of the loci W^r_d and G^r_d and in fact it can be seen to govern their geometry for a general curve C.

The main results from Brill–Noether theory indicate how ρ affects the geometry of G^r_d:

Theorem 3.3. *As above, let C be a smooth projective curve of genus g and, for any integers d and r, let $\rho = \rho(g, d, r)$ be the Brill–Noether number. Then*

1. *(Existence) if $\rho \geq 0$, then G^r_d is non-empty,*
2. *(Dimension) if C is general, then dim $G^r_d = \rho$,*
3. *(Connectedness) if $\rho > 0$, then G^r_d is connected, and*
4. *(Smoothness) if C is general, then G^r_d is smooth.*

Moreover, when G^r_d is smooth of dimension ρ, the variety W^r_d is Cohen–Macaulay, reduced, and normal. If $d < g + r$ then the singular locus of W^r_d is W^{r+1}_d.

Proof. The numbered results are due to Kempf [33] and Kleiman–Laksov [37], Griffiths–Harris [24], Fulton–Lazarsfeld [17], and Gieseker [19], respectively, and a systematic treatment of all of them together can be found in [5, Chapter V]. The singularity result for W^r_d is due to Kempf and a proof can be found at [5, p. 190]. □

Corollary 3.4. *For general C, if $\rho > 0$ then G^r_d is irreducible and thus so is W^r_d.*

Proof. Irreducibility of G^r_d follows by combining the smoothness and connectedness results of Theorem 3.3 and the surjection $G^r_d \twoheadrightarrow W^r_d$ then yields the conclusion for W^r_d too. □

3.1.1 A word on generality

The results of Theorem 3.3 indicate how the discrete invariants g, d and r, encapsulated by the Brill–Noether number ρ, influence the geometry of the Brill–Noether schemes G^r_d of a curve C. Note that this influence is particularly strong when C is *general* in moduli.

The notion of generality most suited to these results can in fact also be understood algebraically: for a line bundle L on C and for K the canonical bundle, the multiplication map

$$\mu_0 : H^0(C, L) \otimes H^0(C, K - L) \to H^0(K) \tag{3.5}$$

is called the *Petri map.* Injectivity of this map for all line bundles L on C suffices for statements (2) and (4) of Theorem 3.3, and the following theorem due separately to Gieseker [19] and Lazarsfeld [40] illustrates its chief consequence:

Theorem 3.6. *The set of curves C having the property that μ_0 is injective for all line bundles on C forms a non-empty open set in moduli.*

3.1.2 Useful calculations

In Chapter 7 we will study some of the divisor varieties which arise for symmetric products of a curve C. In order to parametrize the irreducible components of those divisor varieties, it will be useful to introduce now (and then refer back later to) the following terminology and ideas related to the Brill–Noether theory of C.

Definition 3.7. We define the *degree d dimension set* of C to be

$$\mathcal{R}_d := \{n \in \mathbb{Z}_{>0} : h^0(L) = n \text{ for some } L \in \mathrm{Pic}^d(C)\}.$$

Naturally,

$$|\mathcal{R}_d| = \max \mathcal{R}_d - \min \mathcal{R}_d + 1$$

and, using Theorem 3.3 and Definition 3.2, we can conclude

$$N_d := \max \mathcal{R}_d = \left\lfloor \frac{d+1-g+\sqrt{(d+1-g)^2+4g}}{2} \right\rfloor$$

$$n_d := \min \mathcal{R}_d = \begin{cases} d+1-g & \text{for } d > g \\ \\ 0 & \text{for } d \leq g \end{cases} \tag{3.8}$$

In particular here, we note that $|\mathcal{R}_d|$ is approximately *linear* in d (for fixed genus).

Remark 3.9. Although not immediately obvious from the expression for N_d, it is not hard to see that for $d \geq 2g-1$ we always have $|\mathcal{R}_d| = 1$, as expected from Riemann–Roch which in that case implies $\mathcal{R}_d = \{d+1-g\}$.

3.2 Brill–Noether theory in higher dimensions

Now let X denote a smooth projective variety over $\mathbb{C}$, possibly of dimension higher than 1, and let $\lambda \in \mathrm{NS}(X)$ denote a Néron–Severi class. Analogously to the curve case, one can define schemes G^r_λ and W^r_λ whose closed point sets are

$$G^r_\lambda = \{V \subset H^0(X,L) : c_1(L) = \lambda \text{ and } \dim V = r+1\}$$
$$W^r_\lambda = \{L \in \mathrm{Pic}^\lambda(X) : h^0(X,L) \geq r+1\}$$

where L denotes a line bundle on X. It is natural to ask:

Question 3.10. Can we describe the geometry of G^r_λ in terms of r, λ and some simple discrete invariants of X?

Question 3.11. As with injectivity of the Petri map, can we find a simple algebraic condition for generality, which when satisfied by X yields a strengthening of any answer to Question 3.10?

These questions have been taken up in the case of surfaces by Mendes Lopes, Pardini and Pirola in [46]. They prove a Kempf-type existence result for irregular surfaces admitting no fibration over a high genus curve. Specifically, if S denotes a smooth, projective surface with irregularity $q := h^1(\mathcal{O}_S)$ and $C \subset S$ is a reduced curve with arithmetic genus $p_a(C)$, then for $r \geq 0$ define

$$\rho(C, r) := q - (r+1)(p_a(C) - C^2 + r).$$

To see where this number comes from, note its similarity to the classical Brill–Noether number $\rho(g, d, r)$ for curves, defined above — both numbers come from analogous reasoning about what the *expected dimension* of W^r_λ should be: consider $L \in W^r_\lambda$ with preimage $G^r_\lambda|_L \cong G(r+1, H^0(L))$ (a Grassmannian) under the natural map c in the following diagram

$$\begin{tikzcd} G(r+1, H^0(L)) \arrow[r, hook] & G^r_\lambda \arrow[d, "c"] \\ & W^r_\lambda. \end{tikzcd}$$

It is clear that the differential of the map $c : G^r_\lambda \to W^r_\lambda$ at a point $[W] \in G(r+1, H^0(L))$ sits in an exact sequence:

$$0 \longrightarrow \mathrm{Hom}(W, H^0(L)/W) \longrightarrow T_{[W]}G^r_\lambda \xrightarrow{c_*} T_L\mathrm{Pic}^\lambda(X).$$

In the case of curves, with $\lambda = d$, one shows that $\mathrm{im}(c_*)$ is dual (via Serre duality) to the kernel $\ker(\mu_{0,W})$ of the Petri map

$$\mu_{0,W} : W \otimes H^0(K - L) \longrightarrow H^0(K)$$

(here we have modified Definition 3.5 by using a *subspace* $W \subset H^0(L)$ in place of $H^0(L)$) and so then, after making some calculations (see e.g. [5, pp. 187-188]) one computes the dimension of $T_{[W]}G^r_d$ to be

$$\dim\, T_{[W]}G^r_d = \rho + \dim(\ker\, \mu_{0,W})$$

and the generality assumption, from Theorem 3.6 above, assures that $\mu_{0,W}$ is injective.

In any case, with the number $\rho(C,r)$ defined as shown above, the paper [46] proves the following existence result:

Theorem 3.12 (Mendes Lopes–Pardini–Pirola)**.** *Let $\lambda := [C] \in \mathrm{NS}(S)$ and suppose S admits no irrational pencils of genus > 1. If $\rho(C,r) > 1$ then W^r_λ is non-empty of dimension at least $\rho(C,r)$.*

In fact, their results are slightly more precise when one has good knowledge of the Albanese mapping of S and the (intermediate) cohomology jump loci in $\mathrm{Pic}^0(S)$.

In their paper [46, Remark 6.3] these authors remark that the condition of admitting no irrational pencils of genus > 1 can be thought of as a kind of "generality" assumption.

So Theorem 3.12 evidently addresses some of the key elements of Questions 3.10 and 3.11 and thus represents a first step towards a higher dimensional Brill–Noether theory.

In a slightly different direction, these and other authors have undertaken a study of the *paracanonical system* of a smooth, projective variety X. Recall that a paracanonical divisor $D \subset X$ is an effective divisor such that $[D] = [K] \in \mathrm{NS}(S)$ (even though D and K may or may not be *linearly* equivalent). The paracanonical system $\mathcal{P}_X$ of X is then a scheme with closed point set

$$\mathcal{P}_X = \{\text{paracanonical divisors } D \subset X\}.$$

With $\lambda := c_1(K_X)$ it is clear that

$$\mathrm{Div}^\lambda(X) = \mathcal{P}_X$$

and for higher dimensional X the paracanonical system is the natural first setting to see a new behavior exhibited by these divisor varieties:

Definition 3.13. Let $\lambda \in \mathrm{NS}(X)$ and $L \in \mathrm{Pic}^\lambda(X)$. We say that L, or equivalently the linear system $|L|$, is *exorbitant* if $|L|$ forms an irreducible component of $\mathrm{Div}^\lambda(X)$.

In [6], Beauville studies the paracanonical system of a surface S and proves a fascinating result characterizing exorbitance of the canonical bundle in terms of the *parity* of the irregularity:

Theorem 3.14. *Suppose S is a smooth projective surface of irregularity q admitting no irrational pencils of genus $> q/2$. Then when q is even K_S is exorbitant.*

Beauville then remarks that, under mild hypotheses on S, the converse is also true (but a priori $|K_S|$ could be a non-reduced component of $\mathcal{P}_S$ when q is odd). Mendes Lopes–Pardini–Pirola go on to complete this story — they confirm this converse and also sharpen the result in the even irregularity case by partly describing how the exorbitant series $|K_S|$ interacts with the remaining part of $\mathcal{P}_S$. Before indicating this sharpening, we introduce some terminology:

Definition 3.15. Let X be a smooth projective variety, $\lambda \in \mathrm{NS}(X)$ and $u_\lambda : \mathrm{Div}^\lambda(X) \to \mathrm{Pic}^\lambda(X)$ the Abel–Jacobi map. The *continuous rank* $\rho(\lambda)$ of λ is defined as

$$\rho(\lambda) := \min \left\{ h^0(X, L) : L \in \mathrm{Pic}^\lambda(X) \right\}.$$

Since the fibers of u_λ are all linear systems (i.e. projective spaces) and thus irreducible, when $\rho(\lambda) > 0$ there is precisely one irreducible component of $\mathrm{Div}^\lambda(X)$ that dominates $\mathrm{Pic}^\lambda(X)$ via u_λ.

Definition 3.16. When $\rho(\lambda) > 0$, we define the *main component* $\mathrm{Div}^\lambda(X)_{\mathrm{main}}$ to be the unique component of $\mathrm{Div}^\lambda(X)$ dominating $\mathrm{Pic}^\lambda(X)$.

Now let $\lambda = c_1(K_X)$ so that $\mathrm{Div}^\lambda(X) = \mathcal{P}_X$. It is a consequence of the generic vanishing theorem of Green–Lazarsfeld [22] that then $\rho(\lambda) = \chi(K_X)$ and so when $\chi(K_X) > 0$ the paracanonical system $\mathcal{P}_X$ has a unique component $\mathcal{P}_{\mathrm{main}}$ dominating $\mathrm{Pic}^\lambda(X)$. With this in mind, we recall

Theorem 3.17 (Mendes Lopes–Pardini–Pirola, [47, Theorem 1.3])**.** *Let S be a surface with $\chi(K_S) > 0$ and irregularity $q \geq 2$ without an irrational pencil of genus $> q/2$. Then*

1. *if q is odd, then $|K_S| \subset \mathcal{P}_{main}$;*

2. *if q is even, then $\Sigma := |K_S| \cap \mathcal{P}_{main} = |K_S| \cap \overline{\mathcal{P}_S \setminus |K_S|}$ is an integral hypersurface in $|K_S|$ of degree $q/2$. Moreover, if S has no irrational pencil of genus > 1 then*

$$\mathrm{Sing}(\Sigma) = \{[s] \in \Sigma : \mathrm{rank}(\cup s) < q - 2\}.$$

So in particular, the first statement of this Theorem indeed confirms the converse statement of Theorem 3.14 that for surfaces S of odd irregularity q admitting no irrational pencils of genus $> q/2$, K_S is *not* exorbitant. To achieve this, they use the following *derivative complex* of a line bundle L on a smooth projective variety X, introduced in [22] to study the first-order deformation theory of the cohomology groups $H^k(L)$. Given a vector $v \in H^1(\mathcal{O}_X)$ one has

$$H^{k-1}(L) \xrightarrow{\cup v} H^k(L) \xrightarrow{\cup v} H^{k+1}(L)$$

and the authors obtain geometric results by proving exactness of this complex in certain instances. When $X = S$ is a surface with the properties described above and $L = K_S$ they apply their result to obtain the theorem above by noting that non-triviality of the map $(\cup v) : H^1(K_S) \to H^2(K_S) \cong \mathbb{C}$ for *all* $v \in H^1(\mathcal{O}_S)$ depends on the parity of $h^1(\mathcal{O}_S)$.

Looking instead at the second statement of this Theorem, we see a first step in answering the very natural question:

Question 3.18. For a smooth projective variety X with $\chi(K_X) > 0$ and K_X exorbitant, what is the geometry of $|K_X| \cap \mathcal{P}_{\mathrm{main}}$?

This question is indeed a chief motivator of our broader investigation of divisor varieties on symmetric products of curves (and their components). In particular, we answer this question fully for X the symmetric product of an arbitrary smooth curve (see Example 7.13).

Chapter 4

Finite group actions, equivariant cohomology & Künneth formulae

For the proof of our Theorem A in Chapter 6 we need to recall some facts about group actions on coherent sheaves and the subsequent behavior of the sheaf cohomology. Much of the material in this chapter is known, but we include a brief review of what we need for the benefit of the reader.

4.1 Group actions on coherent sheaves

From here on, G denotes a finite group. Recall that we are working over $\mathbb{C}$ (in particular, in characteristic 0).

Let X be a normal variety admitting an algebraic action of G, and let $\mathcal{F}$ be a coherent, locally free $\mathcal{O}_X$-module admitting an action of G commuting with that on X (a G-equivariant structure).

The G-action on $\mathcal{F}$ induces a corresponding one on exterior powers $\wedge^r \mathcal{F}$: if the former is given, for $g \in G$ and $x \in X$, by

$$\mathcal{F}_{gx} \xrightarrow{\rho} \mathcal{F}_x$$

then the latter is given by

$$\begin{aligned} (\wedge^r \mathcal{F})_{gx} &\xrightarrow{\wedge^r \rho} (\wedge^r \mathcal{F})_x \\ f_1 \wedge \cdots \wedge f_r &\longmapsto \rho(f_1) \wedge \cdots \wedge \rho(f_r). \end{aligned}$$

In particular, one obtains an induced action on the determinant line bundle $\det(\mathcal{F})$. Moreover, one can "multiply" a character

$$\chi : G \longrightarrow \mathbb{C}^{\times}$$

of G by an equivariant structure ρ on $\mathcal{F}$ to obtain a new equivariant structure

$$\begin{aligned} \mathcal{F}_{gx} &\xrightarrow{\ \chi\rho\ } \mathcal{F}_x \\ f &\longmapsto \chi(g) \cdot \rho(f). \end{aligned}$$

In the case that the G-action on X is trivial, one can take the *sheaf of invariants* $\mathcal{F}^G$ of $\mathcal{F}$: for $U \subset X$ an affine open,

$$\mathcal{F}^G(U) := \mathcal{F}(U)^G.$$

If Y is another normal variety and $\pi : X \to Y$ a finite G-invariant morphism, then G acts on $\pi_*\mathcal{F}$ and we define the *equivariant pushforward*

$$\pi_*^G \mathcal{F} := (\pi_*\mathcal{F})^G.$$

When $\mathcal{F}$ is locally free, $\pi_*^G\mathcal{F}$ will be locally free too.

If instead $\mathcal{G}$ is a coherent sheaf on Y then the pullback $\pi^*\mathcal{G}$ has a natural equivariant structure on X coming from the trivial equivariant structure on $\mathcal{O}_X$ and the canonical isomorphisms

$$(f^*\mathcal{G})_{gx} \cong \mathcal{G}_{[x]} \otimes_{\mathcal{O}_Y} \mathcal{O}_X \cong (f^*\mathcal{G})_x$$

Proposition 4.1 (Equivariant rank drop)**.** *For X and G as above, let E and F be locally free G-equivariant sheaves on X of equal rank. If $u : E \to F$ is an injective G-equivariant homomorphism such that $D := \mathrm{Supp}(\mathrm{coker}\ u)$ is a G-invariant divisor, then there is a G-equivariant isomorphism:*

$$\det(u) : \det(E) \xrightarrow{\ \cong\ } \det(F) \otimes \mathcal{O}_X(-D)$$

Proof. The isomorphism alone follows by [4, Lemma 5.1]. That it is G-equivariant is clear from the setup. $\square$

Proposition 4.2 (Cohomology and invariants)**.** *For X, Y, G and π as above, let $\mathcal{F}$ be a G-equivariant coherent sheaf on X and S a normal variety fitting into the following commutative diagram (with π and τ both G-invariant, τ and $\overline{\tau}$ both flat and projective):*

$$\begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "\tau"] & \\ Y \arrow[r, "\overline{\tau}"] & S \end{tikzcd}$$

Then we can calculate higher direct images of $\pi_*^G\mathcal{F}$ along $\overline{\tau}$ by taking invariants of the corresponding higher direct images along τ upstairs:

$$R^i\overline{\tau}_*(\pi_*^G\mathcal{F}) \cong (R^i\tau_*\mathcal{F})^G$$

Proof. Since G is finite and the G-modules in which $\pi_*\mathcal{F}$ takes its values are over $\mathbb{C}$-algebras, the invariants functor $(\_)^G$ is exact in our situation (a consequence of Maschke's theorem on complete reducibility of G-representations). Therefore as a trivial special case of Grothendieck's spectral sequence (see [28, Theorem 2.4.1]) we have that $R^i\tau_*^G\mathcal{F} = (R^i\tau_*\mathcal{F})^G$ for all i (because recall that $\tau_*^G = (\_)^G \circ \tau_*$). Similarly $R^i\pi_*^G\mathcal{F} = (R^i\pi_*\mathcal{F})^G$. Since π is finite, $R^i\pi_*\mathcal{F} = 0$ for $i > 0$. Now we note that $\tau_*^G = \overline{\tau}_* \circ \pi_*^G$. We can apply the Grothendieck spectral sequence here too to conclude $R^p\overline{\tau}_*(R^q\pi_*^G\mathcal{F})$ abuts to $R^{p+q}\tau_*^G\mathcal{F} = (R^{p+q}\tau_*\mathcal{F})^G$, but since the higher direct images of π vanish, this abutment immediately reduces to the desired isomorphism. $\square$

Remark 4.3. The same result could be achieved in the above proposition with weaker hypotheses on G, on the spaces X, Y and S and for different fields. However, we will only work with the symmetric group over $\mathbb{C}$.

4.2 Künneth Formula

Suppose we have the following Cartesian diagram of schemes:

$$\begin{tikzcd} X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, "\tau"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

where X, Y and S are smooth varieties over $\mathbb{C}$ and where f and g are flat of relative dimensions m and n respectively.

Proposition 4.4 (Top degree Künneth formula)**.** *For X, Y, f, g, τ as above, suppose that $\mathcal{F}$ and $\mathcal{G}$ are locally free coherent sheaves on X and Y respectively. Then the cup-product morphism*

$$(R^m f_*\mathcal{F}) \otimes (R^n g_*\mathcal{G}) \xrightarrow{\cup} R^{m+n}\tau_*(\mathcal{F} \boxtimes_S \mathcal{G})$$

is an isomorphism.

Proof. At the level of complexes, the relative cup-product map

$$Rf_*\mathcal{F} \otimes^{\mathbf{L}}_S Rg_*\mathcal{G} = R\tau_*(\mathcal{F} \boxtimes_S \mathcal{O}_Y) \otimes^{\mathbf{L}}_S R\tau_*(\mathcal{O}_X \boxtimes_S \mathcal{G}) \longrightarrow R\tau_*(\mathcal{F} \boxtimes_S \mathcal{G})$$

(see [54, Remark 0B68]) is, in our case, a quasi-isomorphism. This can be seen by noting:

$$\begin{aligned} Rf_*\mathcal{F} \otimes^{\mathbf{L}}_S Rg_*\mathcal{G} &\simeq Rf_*(f^*Rg_*\mathcal{G} \otimes^{\mathbf{L}}_X \mathcal{F}) && \text{(projection formula)} \\ &\simeq Rf_*(R{p_1}_*{p_2}^*\mathcal{G} \otimes^{\mathbf{L}}_X \mathcal{F}) && \text{(flat base-change)} \\ &\simeq Rf_*R{p_1}_*(\mathcal{F} \boxtimes_S \mathcal{G}) && \text{(projection formula).} \end{aligned}$$

Thus, with $\mathscr{H}^i$ denoting the i^{th} cohomology sheaf, we have

$$\mathscr{H}^{m+n}(Rf_*\mathcal{F} \otimes^{\mathbf{L}}_S Rg_*\mathcal{G}) \cong R^{m+n}\tau_*(\mathcal{F} \boxtimes_S \mathcal{G}).$$

Now note that locally on S we have complexes $F_\bullet \simeq Rf_*\mathcal{F}$ and $G_\bullet \simeq Rg_*\mathcal{G}$ consisting of free, finitely generated sheaves of lengths at most m and n, respectively (see [5, Theorem 2.6, p. 175], or [49, Section 5] more generally). Since $\mathcal{F}$ and $\mathcal{G}$ are locally free, we have, locally on S, $\mathrm{Tot}_\bullet(F_\bullet \otimes G_\bullet) \simeq Rf_*\mathcal{F} \otimes^{\mathbf{L}}_S Rg_*\mathcal{G}$. Thus we are reduced to showing

$$\mathscr{H}^m(F_\bullet) \otimes_S \mathscr{H}^n(G_\bullet) \longrightarrow \mathscr{H}^{m+n}(\mathrm{Tot}_\bullet(F \otimes G)) \tag{4.5}$$

is an isomorphism. Here $\mathrm{Tot}_\bullet(F \otimes G)$ is the total tensor product complex with degree k term

$$\mathrm{Tot}_k(F \otimes G) = \bigoplus_{i+j=k} F_i \otimes G_j$$

and differential $d^{i+j}_{F\otimes G} = \Sigma_{i+j=k} d^{i,j}_{F\otimes G}$ where

$$d^{i,j}_{F\otimes G} = d^i_F \otimes \mathrm{id}_{G_j} + (-1)^i \mathrm{id}_{F_i} \otimes d^j_G.$$

The $(-1)^i$ term is sometimes called the *Koszul sign rule* and is necessary for $\mathrm{Tot}_\bullet(F \otimes G)$ to form a complex. Now the Künneth spectral sequence (see [27, 6.7.3(a)] and [27, 6.7.6]) yields the abutment

$$E^2_{p,-q} := \bigoplus\nolimits_{i+j=q} \underline{\mathrm{Tor}}^{\mathcal{O}_S}_p(\mathscr{H}^i(F_\bullet), \mathscr{H}^j(G_\bullet)) \implies \mathscr{H}^{p+q}(\mathrm{Tot}_\bullet(F \otimes G))$$

and from this the result follows since the vanishings $\mathscr{H}^i(F_\bullet) = \mathscr{H}^j(G_\bullet) = 0$ for $i > m$ and $j > n$ leave $\mathscr{H}^m(F_\bullet) \otimes_S \mathscr{H}^n(G_\bullet)$ as the lower-left-corner term (position $(p,q) = (0, -m-n)$) of a lower-right-quadrant spectral sequence, so the desired isomorphism 4.5 follows from convergence of this corner term. $\square$

Now if we take the product of $(k-1)$ copies each of X, $\mathcal{F}$ and f (relative to S) as follows:

$$Y := X \times_S X \times_S \cdots \times_S X$$
$$\mathcal{G} := \mathcal{F} \boxtimes_S \mathcal{F} \boxtimes_S \cdots \boxtimes_S \mathcal{F}$$
$$g := f \times_S f \times_S \cdots \times_S f$$

then this isomorphism becomes

$$\Phi : (R^m f_* \mathcal{F})^{\otimes k} \xrightarrow{\cong} R^{km} \tau_* (\mathcal{F}^{\boxtimes_S k})$$

which, being a cup-product map now with k isomorphic inputs, we know must be graded-commutative (a consequence of the Koszul sign rule mentioned above). Specifically, with the symmetric group $\mathfrak{S}_k$ acting by the permutation action on $(R^m f_* \mathcal{F})^{\otimes k}$, this means that for $\sigma \in \mathfrak{S}_k$,

$$\Phi \circ \sigma = \mathrm{sgn}(\sigma)^m \Phi \tag{4.6}$$

where $\mathrm{sgn} : \mathfrak{S}_k \to \mathbb{Z}/2$ the sign representation. By deducing a natural action of $\mathfrak{S}_k$ on $R^{km}\tau_*(\mathcal{F}^{\boxtimes_S k})$, we will next use 4.6 to realize Φ as an equivariant isomorphism which thus descends to invariant subsheaves.

We begin with the permutation action on the (relative) product: for $\sigma \in \mathfrak{S}_k$ we have

$$\begin{aligned} X \times_S X \times_S \cdots \times_S X &\xrightarrow{\sigma} X \times_S X \times_S \cdots \times_S X \\ (x_1, \ldots, x_k) &\longmapsto (x_{\sigma(1)}, \ldots, x_{\sigma(k)}) \end{aligned}$$

and for this action there is a natural, compatible equivariant structure ρ on $\mathcal{F} \boxtimes_S \cdots \boxtimes_S \mathcal{F}$ (k copies) determined by

$$\begin{aligned} \mathcal{F}_{x_{\sigma(1)}} \boxtimes \cdots \boxtimes \mathcal{F}_{x_{\sigma(k)}} &\xrightarrow{\rho_{(\sigma,x)}} \mathcal{F}_{x_1} \boxtimes \cdots \boxtimes \mathcal{F}_{x_k} \\ f_1 \otimes \cdots \otimes f_k &\longmapsto f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(k)} \end{aligned}$$

The equivariant structure ρ on $\mathcal{F} \boxtimes_S \cdots \boxtimes_S \mathcal{F}$ naturally determines such a structure $R^{km}\tau_*\rho$ on $R^{km}\tau_*(\mathcal{F}^{\boxtimes_S k})$ for which, by 4.6, Φ is equivariant. Since there is no action of $\mathfrak{S}_k$ on S being considered, we can take invariant subsheaves thus yielding

Proposition 4.7 (Equivariant Künneth)**.** *Letting $\mathfrak{S}_k$ act as indicated above, Φ restricts to an isomorphism between the invariant subsheaves:*

$$\wedge^k R^m f_*\mathcal{F} \xrightarrow{\cong} R^{km}\tau_*(\mathcal{F}^{\boxtimes_S k})^{\mathfrak{S}_k} \qquad \textit{for } m \textit{ odd}$$

$$S^k R^m f_*\mathcal{F} \xrightarrow{\cong} R^{km}\tau_*(\mathcal{F}^{\boxtimes_S})^{\mathfrak{S}_k} \qquad \textit{for } m \textit{ even}$$

Chapter 5

Rank loci and linear algebra

In Chapter 7 we will focus our attention on certain classes of effective divisors in the k^{th} symmetric product C_k of a curve C. The divisors in these particular classes are related to the geometry of the underlying curve in a quite direct, linear algebraic way. Consequently, the varieties parametrizing these divisors are stratified using some naturally defined notions of rank which we outline in this chapter.

We begin by reviewing some more familiar material concerning secant varieties since these do show up in the context of divisor varieties of the symmetric *square* C_2 of C. However, more important is the following section on *subspace varieties* which, though perhaps less familiar to the reader in general, are an equally naturally defined collection of rank loci in Plücker space which show more generally in the context of divisor varieties of *higher* dimensional symmetric products C_k for $k \geq 3$. We will see that subspace varieties coincide with secant varieties in a particular instance.

In what follows, V will denote a vector space over $\mathbb{C}$ of finite dimension d.

5.1 Secant varieties

We begin with one of the more familiar types of rank variety — that of *secant varieties.* Recall that for a smooth, projective variety X embedded in a projective space $\mathbb{P}$, one has

Definition 5.1. The n^{th} *secant variety* of X is the locus in $\mathbb{P}$ given as

$$\operatorname{Sec}_n(X) := \overline{\bigcup \operatorname{Span}_{\mathbb{P}}(x_1, \ldots, x_n)},$$

where the union is taken over all n-tuples of points in X, and the closure is taken in $\mathbb{P}$. When $n = 2$ this is often called the *chordal variety* of X.

Evidently, this definition yields a stratification of $\mathbb{P}$:

$$X = \mathrm{Sec}_1(X) \subset \mathrm{Sec}_2(X) \subset \cdots \subset \mathrm{Sec}_b(X) = \mathbb{P} \tag{5.2}$$

where b, the minimal integer yielding the equality on the right, depends on the embedding of X and is sometimes called the *typical X-rank of* $\mathbb{P}$ (see [39, Section 5.2]). A first measurement to make of $\mathrm{Sec}_n(X)$ is its dimension — one could reasonably expect that in sufficiently general circumstances the following expected dimension should be achieved

$$\text{exp-dim } \mathrm{Sec}_n(X) := \min \{ n \cdot \dim\ X + (n-1), \dim \mathbb{P} \}.$$

While this is often the case, there are important exceptions (in which cases the expected dimension is larger than the true one) and it is therefore common to define, at least when $\mathrm{Sec}_{n-1}(X) \subsetneq \mathrm{Sec}_n(X)$, the *defect*

$$\delta_X(n) := \text{exp-dim } \mathrm{Sec}_n(X) - \dim\ \mathrm{Sec}_n(X).$$

When $\delta_X(n) > 0$ we say that $\mathrm{Sec}_n(X)$ is *defective*.

Of particular interest to us are the cases

1. when $X = \nu_2(\mathbb{P}V) \subset \mathbb{P}S^2V$ is the quadratic Veronese variety of $\mathbb{P}V$;

2. when $X = G(V, 2) \subset \mathbb{P} \wedge^2 V$ is the Grassmannian of lines in $\mathbb{P}V$.

In these cases, we have the following results on defectiveness:

Theorem 5.3. *Let $X = \nu_2(\mathbb{P}V)$. Then $\delta_X(n) = \dim\ V - \lceil \frac{\dim\ V+1}{2} \rceil$.*

Theorem 5.4. *Let $X = G(V, 2)$. Then $\delta_X(n) = 2n(n-1)$ except when* $\dim\ V = 4, 5$ *in which case $\delta_X(n) = 0$.*

These results are classical and are rephrasings of corresponding statements about the dimensions of symmetric, respectively skew-symmetric, endomorphisms of V of rank $2n$. See for example [51] (symmetric) and [10] (skew-symmetric) for discussion and references, and [56, Chapter III] for the general theorems.

Remark 5.5. Although we will only require the particular cases above, the following solution of the related and well-known *Waring problem for polynomials*[1] is interesting to highlight here for completeness: let $k \geq 3$ and let $X = \nu_k(\mathbb{P}V)$ be the k^{th} Veronese re-embedding of $\mathbb{P}V$. Then $\delta_X(n) = 0$, except for $(n, k, \mathit{dim}\ V) = (5, 4, 3), (9, 4, 4), (14, 4, 5)$ and $(7, 3, 5)$ in which cases $\delta_X(n) = 1$. Significant progress on this problem was made by Terracini, while the completed solution is due to Alexander–Hirschowitz [2]. A systematic treatment can be found in [31, Chapter 1] — in particular, see Theorems 1.61 and 1.62. On the other hand, the corresponding skew-symmetric problem does not yet appear to be fully worked out — a recent overview of current progress is encompassed by [7].

5.2 Subspace varieties

While the secant varieties of the previous section are more familiar in general and are relevant for our later results in the case of the symmetric *square* of a curve, a different notion of rank will actually play the more central role for higher dimensional symmetric products.

5.2.1 A geometric overview

Let V be as before and k be a positive integer. This new notion of rank is defined as follows.

Definition 5.6. Let $\eta \in \wedge^k V$. We define the *enclosing space* $\mathrm{Enc}(\eta)$ to be the smallest subspace U such that $\eta \in \wedge^k U \subset \wedge^k V$. We denote the dimension of this enclosing space by $\mathrm{enc}(\eta)$, the *enclosing dimension* of η.

This enclosing dimension leads naturally to the definition of the following rank loci in Plücker space:

Definition 5.7. Let e be an integer such that $k \leq e \leq \dim V$. We define the skew-symmetric *subspace varieties*:

$$\mathrm{Sub}_e(\wedge^k V) := \{[\eta] \in \mathbb{P}(\wedge^k V)^\vee : \mathrm{enc}(\eta) \leq e\}$$

[1]The Waring problem for polynomials seeks to establish that a general degree k homogeneous form f on V can be written as a sum of k^{th} powers $L_1^k + \cdots + L_n^k$ for $n := \lceil \frac{1}{d}\binom{d+k-1}{k} \rceil$. This is equivalent to establishing $\mathrm{Sec}_n(\nu_k(\mathbb{P}V)) = \mathbb{P}S^k V$.

Correspondingly, this definition gives rise to a stratification of Plücker space:

$$G(V,k) = \mathrm{Sub}_k(\wedge^k V) \subset \mathrm{Sub}_{k+1}(\wedge^k V) \subset \cdots \subset \mathrm{Sub}_{\dim V}(\wedge^k V) = \mathbb{P}(\wedge^k V)^\vee.$$

In fact, in the $k=2$ case, one has $\mathrm{Sub}_{2s+1}(\wedge^2 V) = \mathrm{Sub}_{2s}(\wedge^2 V) = \mathrm{Sec}_s(G(V,2))$ (see Remark 5.10 below), so this stratification in fact matches perfectly the one in 5.2 by secant varieties (with the Grassmannian $X = G(V,2)$ embedded by its Plücker embedding).

In contrast to the $k=2$ case however, when $k \geq 3$, the stratifications of $\mathbb{P}(\wedge^k V)^\vee$ given respectively by the secant varieties $\mathrm{Sec}_n(G(V,k))$ and the subspace varieties $\mathrm{Sub}_e(\wedge^k V)$ are quite different — for example, one might at first expect $\mathrm{Sec}_s(G(V,k))$ and $\mathrm{Sub}_{sk}(\wedge^k V)$ to coincide (this would indeed reduce to what is true when $k=2$), but in fact

$$\mathrm{Sec}_3(G(V,3)) \subsetneq \mathrm{Sub}_9(\wedge^3 V)$$

while

$$\mathrm{Sub}_7(\wedge^3 V) \not\subset \mathrm{Sec}_3(G(V,3)).$$

We thank Kristian Ranestad for pointing this example out to us, and refer the interested reader to [45, Theorem 2.1] and [1, Corollary 3.6].

Remark 5.8. Definitions 5.6 and 5.7 can be reworked completely analogously in the symmetric setting, with $\wedge^k V$ replaced by $S^k V$. This leads to the analogous notion of *symmetric* subspace varieties, and the corresponding stratification of $\mathbb{P}(S^k V)^\vee$ is in general different to the one given by $\mathrm{Sec}_n(\nu_k(\mathbb{P}V))$, but matches it in the case $k=2$. It should also be noted that the symmetric subspace variety stratification is longer: it begins with $\mathrm{Sub}_1(S^k V) = \nu_k(\mathbb{P}V)$.

5.2.2 Algebraic perspective

What we have said so far about subspace varieties gives a first indication of what they look like geometrically in projective space and how they relate to or deviate from the more familiar secant varieties. In what follows, we briefly introduce comultiplication maps in order to give a different, algebraic definition of enclosing dimension. This is ultimately to allow us to make useful conclusions about the dimensions and singularities of subspace varieties.

Following [55, p. 3], let $\Delta : V \to V \oplus V$ denote the diagonal map of V. With a slight abuse of notation, one similarly denotes the resulting algebra map

$$\Delta : \wedge^{\bullet} V \longrightarrow \wedge^{\bullet}(V \oplus V) \cong \wedge^{\bullet} V \otimes \wedge^{\bullet} V$$

which is often called *comultiplication*. This comultiplication respects the gradings on these algebras and so, after projection, one gets component linear mappings

$$\Delta : \wedge^{r+s} V \to \wedge^{r} V \otimes \wedge^{s} V.$$

Explicitly, these maps are given by the rule

$$\Delta(v_1 \wedge \cdots \wedge v_{r+s}) := \sum_{\sigma \in \mathfrak{S}_{r+s}^{r,s}} (-1)^{\mathrm{sgn}(\sigma)} v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(r)} \otimes v_{\sigma(r+1)} \wedge \cdots \wedge v_{\sigma(r+s)}$$

(extended linearly) where $\mathfrak{S}_{r+s}^{r,s} \subset \mathfrak{S}_{r+s}$ is the subgroup of the symmetric group defined by

$$\mathfrak{S}_{r+s}^{r,s} := \{\sigma : \sigma(1) < \cdots < \sigma(r); \quad \sigma(r+1) < \cdots < \sigma(r+s)\}.$$

With this definition of $\Delta : \wedge^{r+s} V \to \wedge^{r} V \otimes \wedge^{s} V$ in we now give the more algebraic definition of an enclosing space $\mathrm{Enc}(\eta)$ alluded to above:

Definition 5.9. For V, k and η as above, define a contraction-type map

$$\langle \_, \eta \rangle : \wedge^{k-1} V^* \to V$$

to be the composition

$$\wedge^{k-1} V^* \xrightarrow{\;-\otimes \Delta(\eta)\;} \wedge^{k-1} V^* \otimes \wedge^{k-1} V \otimes V \xrightarrow{\;\;t \otimes -\;\;} V$$

for t the trace map $\wedge^{k-1} V^* \otimes \wedge^{k-1} V \to \mathbb{C}$. Then

$$\mathrm{Enc}(\eta) := \mathrm{im}\ \langle \_, \eta \rangle.$$

The analysis at [25, p. 210-211], for example, confirms that this is a valid re-definition of $\mathrm{Enc}(\eta)$.

Remark 5.10. We have already alluded to the fact that sometimes two subspace varieties will coincide for different choices of enclosing dimension e. We record these explicitly here:

- For $k = 2$, the enclosing dimension of a symmetric (resp. skew-symmetric) 2-tensor coincides with the rank of the corresponding symmetric (resp. skew-symmetric) matrix. Thus we have

$$\mathrm{Sub}_{2s}(\wedge^2 V) = \mathrm{Sub}_{2s+1}(\wedge^2 V)$$
$$\mathrm{Sub}_{2s}(S^2 V) = \mathrm{Sub}_{2s+1}(S^2 V)$$

  for $1 \leq s \leq \lfloor n/2 \rfloor$.

- For $k \geq 3$, these coincidences happen rarely: $\mathrm{Sub}_k(\wedge^k V) = \mathrm{Sub}_{k+1}(\wedge^k V)$ but otherwise $\mathrm{Sub}_e(\wedge^k V) \subsetneq \mathrm{Sub}_{e+1}(\wedge^k V)$ for all $k+1 \leq e \leq n$. To see this, note that the equality would be true if and only if $\mathrm{Sub}_e(\wedge^k W) = \mathbb{P}(\wedge^k W)^\vee$ for all $W \in G(e+1, V)$ — this cannot happen unless $e = k$ since for any basis $\{w_1, \dots, w_{e+1}\}$ of W the k-vector

$$\eta := \sum_{1 \leq i_1 < \cdots < i_k \leq e+1}^{e+1} w_{i_1} \wedge \cdots \wedge w_{i_k}$$

  can be seen to have $\mathrm{enc}(\eta) = e + 1$ using the algebraic definition above.

Remark 5.11. Note that by the algebraic definition of $\mathrm{Enc}(\eta)$ as the image of the contraction map $\langle \_, \eta \rangle$ above, the subspace variety $\mathrm{Sub}_e(\wedge^k V)$ is a degeneracy locus — specifically, allowing η to vary in $\wedge^k V$, the contraction $\langle \_, \eta \rangle$ defines a morphism between vector bundles

$$\varphi_k : \wedge^{k-1} V^\vee \otimes \mathcal{O}_{\mathbb{P}(\wedge^k V)^\vee}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}(\wedge^k V)^\vee}$$

and $\mathrm{Sub}_e(\wedge^k V)$ is precisely the locus where $\mathrm{rank}(\varphi_k) \leq e$.

Typically the subspace variety $\mathrm{Sub}_e(\wedge^k V)$ will be singular along $\mathrm{Sub}_{e-1}(\wedge^k V)$, but it admits a useful desingularization which is a particular case of a more general construction we briefly outline here. Note that the incidence correspondence

$$\Psi := \{([\eta], W) \in \mathbb{P}(\wedge^k V)^\vee \times G(e, V) : \eta \in \wedge^k W\}$$

maps surjectively to $\mathrm{Sub}_e(\wedge^k V) \subset \mathbb{P}(\wedge^k V)^\vee$. In fact, the fiber over $[\eta]$ is exactly

$$\{W \in G(e, V) : \mathrm{Enc}(\eta) \subset W\}$$

hence (when we are not in the situations of Remark 5.10) the map is an isomorphism over the open subset $\{[\eta] \in \mathrm{Sub}_e(\wedge^k V) : \mathrm{enc}(\eta) = e\}$ (since the fiber over any $[\eta]$ in this open set is simply the single point $([\eta], \mathrm{Enc}(\eta))$). Thus the map is birational. We note that in fact $\Psi = \mathbb{P}(\wedge^k \mathcal{S})^\vee$ for $\mathcal{S}$ the tautological sub-bundle on $G(e, V)$, hence it is a desingularization of $\mathrm{Sub}_e(\wedge^k V)$. This desingularization immediately implies:

Lemma 5.12. *For $k \geq 3$, the subspace varieties are irreducible and have dimensions*

$$\dim\left(\mathrm{Sub}_e(\wedge^k V)\right) = e(n-e) + \binom{e}{k} - 1$$

for $n = \dim V$ and $k \leq e \leq n$ except $e = k+1$.

The analogous irreducibility statement follows by the same argument in the case $k = 2$, but the dimension is calculated differently since the subspace varieties in that case are defective secant varieties, so the incidence correspondence above is no longer a desingularization.

Lemma 5.13. *The subspace varieties $Sub_{2s}(\wedge^2 V) = Sec_s G(2, V)$ are irreducible of dimensions*

$$\dim\left(\mathrm{Sec}_s G(2,V)\right) = \min\left\{\binom{n}{2} - 1,\ \ 2(n-2)s + s - 1\right\} - 2s(s-1)$$

This calculation is proven in [10].

Remark 5.14. One can apply [34, Prop. 1 and Thm. 3] to the map $\Psi \to \mathrm{Sub}_e(\wedge^k V)$ to conclude that the subspace variety $\mathrm{Sub}_e(\wedge^k V)$ is normal and Cohen–Macaulay, and — in the case $k \geq 3$ when this map is birational — it also has rational singularities.

Chapter 6

Symmetric products of curves

In this chapter, we introduce the key ideas that we will need for studying the varieties of interest to us — the symmetric products of a curve.

6.1 Setup

Let C be a smooth projective curve over $\mathbb{C}$, let C^k and C_k be the k^{th} Cartesian and symmetric products of C, respectively, and let

$$\pi : C^k \to C_k$$

denote the quotient map (by the action of the symmetric group $\mathfrak{S}_k$). Note that the symmetric product C_k can actually be viewed from three perspectives, each with particular value. It is simultaneously

1. the symmetric group quotient $C^k/\mathfrak{S}_k$;
2. the Hilbert scheme $\mathrm{Hilb}^k(C)$ of k points on C;
3. and the divisor variety $\mathrm{Div}^k(C)$ of effective degree k divisors on C

and, as is standard, we consider the first one to be the definition.

Of course, the coincidence of these three (particularly the latter two) is evident since the notions of effective divisors and tuples of points are the same on curves. That the quotient perspective is the same as these two follows essentially because a single point on a *curve* (unlike in higher dimensions) supports a *unique* scheme of length k.

To keep notation light, we will use the symbol Δ to denote both the (big) diagonal in C^k and its image in C_k, with the expectation that any distinction will be clear from context — as schemes they are the reduced induced schemes on the following closed-point loci:

$$C^k \supset \Delta = \{(p_1, \ldots, p_k) : p_i = p_j \text{ for some } 1 \le i < j \le k\}$$
$$C_k \supset \Delta = \{p_1 + \cdots + p_k : p_i = p_j \text{ for some } 1 \le i < j \le k\}.$$

Thinking of C_k as the divisor variety $\mathrm{Div}^k(C)$, define the *universal degree k divisor* $\mathcal{D} \subset C \times C_k$ by

$$\mathcal{D} := \{(p, D) : p \in \mathrm{Supp}(D)\}$$

and fix p and q as the obvious projection maps in the following diagram:

$$\begin{tikzcd} & C \times C_k & \\ & \mathcal{D} \arrow[u, phantom, "\cup|"] \arrow[dl, "p"'] \arrow[dr, "q"] & \\ C & & C_k. \end{tikzcd} \tag{6.1}$$

Remark 6.2. Note that, with q as in the diagram above, the diagonal $\Delta \subset C_k$ is the branch divisor of q. Moreover, one sees quite easily that the map

$$C \times C_{k-1} \to \mathcal{D}$$

given by $(p, D) \mapsto (p, p + D)$ is an isomorphism.

6.2 The bundles of interest on C_k

We now define the bundles on C_k whose geometry quite directly gives rise to the effective divisors that we intend to parametrize:

Definition 6.3. Let L denote a line bundle on C. The k^{th} *secant bundle* of L on C_k will be defined as

$$E_L := q_* p^* L$$

using the maps p and q from 6.1, and its determinant will be denoted

$$N_L := \det(E_L).$$

Moreover, if $\deg(L) = d$ then we let $n(d) := c_1(N_L)$ denote the first Chern class.

Remark 6.4. The terminology of *secant bundle* is used here because if L is very ample on C, thus inducing an embedding $C \subset \mathbb{P} = \mathbb{P}H^0(C, L)$, then the relative $\mathcal{O}(1)$ line bundle on the projective bundle $\mathbb{P}_{C_k}(E_L)$ induces a morphism

$$\mathbb{P}_{C_k}(E_L) \longrightarrow \mathbb{P}$$

which is generically finite onto its image, and its image is in fact the (embedded) secant variety $\mathrm{Sec}_k(C) \subset \mathbb{P}$ (see Chapter 5 for the definition of $\mathrm{Sec}_k(C)$). Indeed the finiteness remark indicates that this morphism is in fact a desingularization of $\mathrm{Sec}_k(C)$. With this same setup, assume moreover that L is k-very-ample (so that any $k+1$ points of C, with repetition allowed, are separated by sections of L) — then the bundle E_L determines an embedding

$$C_k \hookrightarrow \mathbb{G}(k-1, \mathbb{P}) := G(H^0(C, L), k)$$

(see for example [11, Main Theorem]) which can be viewed as sending $p_1 + \cdots + p_k$ to the secant plane $\mathrm{Span}_{\mathbb{P}}(p_1, \ldots, p_k)$. Naturally, N_L is then the pullback of the Plücker line bundle on the Grassmannian, and determines the corresponding embedding

$$C_k \hookrightarrow \mathbb{P} \wedge^k H^0(C, L)$$

into Plücker space.

Remark 6.5. In the literature (e.g. [38]), and especially when C_k is viewed as the Hilbert scheme $\mathrm{Hilb}^k(C)$, the bundle E_L is often also denoted $L^{[k]}$ and referred to as the *tautological rank-k bundle associated to L*. It is well-known both that

$$K_{C_k} \cong N_{K_C}$$

(indeed this follows from Riemann–Hurwitz together with Proposition 6.10) and that

$$H^0(C_k, N_L) \cong \wedge^k H^0(C, L) \tag{6.6}$$

(see, for example, [4, Ch. 5]).

6.3 Auxiliary bundles and facts

Divisors in the linear systems $|N_L|$, as L varies, are the focus of this thesis. To study them effectively, however, we will need to define the following auxiliary objects. Let $p_i : C^k \to C$ denote the i^{th} projection[1]. We define:

$$L^{\boxplus k} := \bigoplus_i p_i^* L$$
$$L^{\boxtimes k} := \bigotimes_i p_i^* L = \det(L^{\boxplus k}).$$

On $L^{\boxplus k}$ and $L^{\boxtimes k}$ we have equivariant structures $\rho_\boxplus$ and $\rho_\boxtimes$, respectively, given in each case by the natural permutation actions. With $L^{\boxtimes k}$ equipped with the structure $\rho_\boxtimes$, we then define the following line bundles on C_k:

$$T_L := \pi_*^{\mathfrak{S}_k} L^{\boxtimes k}$$

where $\mathfrak{S}_k$ acts on $L^{\boxtimes k}$ by the natural permutation action ρ indicated in Section 4.1. Note that

$$T_{L \otimes M} \cong T_L \otimes T_M. \tag{6.7}$$

Moreover, one has

$$H^0(C_k, T_L) \cong S^k H^0(C, L)$$

(this isomorphism is clear from the definition of T_L and in any case follows from Proposition 4.7 if one takes $S = \mathrm{Spec}(\mathbb{C})$ and then applies Serre duality while using the isomorphisms 6.6 and 6.7 and Proposition 6.10).

[1]We often use the symbol p_i to also denote a point in the i^{th} factor of C^k but, again, we expect this overloaded use will not cause any ambiguity in context.

With these definitions in place, we turn now to establishing two essential facts about the bundles N_L and T_L. Namely, just as T_L is the equivariant pushforward of an equivariant line bundle on C^k, it will be useful to realize N_L in an analogous way. Once we have established this, we prove in Proposition 6.10 an important and useful relation between the two — namely,

$$N_L \cong T_L(-\Delta/2).$$

Remark 6.8. Recall here that $\Delta \subset C_k$ is the branch divisor of the map $q : \mathcal{D} \cong C \times C_{k-1} \to C_k$ for q as in Diagram 6.1. For $f : X \to Y$ a finite surjective map between smooth varieties X and Y, the natural pullback morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is *split* by $1/d$ times the trace map (for $d = \deg(f)$). The dual E_f of the remaining summand of $f_*\mathcal{O}_X$ (known as the *Tschirnhausen bundle* of f — see e.g. [13]) has determinant $L_f := \det(E_f)$ which squares to the normal bundle of the branch locus of f though need not be effective itself. So L_π is what we really mean by $\Delta/2$. See [42, Example 6.3.54] for more details.

Since we will later want to vary the line bundle L in $\mathrm{Pic}(C)$, we prove the facts mentioned above in a relative setting.

If a family (over S) of line bundles $\{\mathscr{L}_t\}$ on C is given by a line bundle $\mathscr{L}$ on $C \times S$, we get corresponding families $\{N_{\mathscr{L}_t}\}$ and $\{T_{\mathscr{L}_t}\}$ by defining $N_{\mathscr{L}}$ and $T_{\mathscr{L}}$ on $C_k \times S$ analogously to the definitions of N_L and T_L. One simply replaces C, C_k and C^k by their products with S, and replaces $\mathcal{D}$, p, q and π by their relative counterparts.

Proposition 6.9. *Let S be any normal variety. For $\pi : C^k \times S \to C_k \times S$ the quotient map by the symmetric group $\mathfrak{S}_k$ and $\mathscr{L}$ any line bundle on $C \times S$, equip $\mathscr{L}^{\boxtimes_S k}$ with $\det(\rho_\boxplus)$. Then on $C_k \times S$ we have*

$$N_{\mathscr{L}} \cong \pi_*^{\mathfrak{S}_k} \mathscr{L}^{\boxtimes k}.$$

Proof. We write the proof for S simply a point and the line bundle L on C, but the general case is immediately obtained by taking products everywhere with S and replacing L on C by $\mathscr{L}$ on $C \times S$.

Recalling the diagram 6.1, we have an evaluation map for p^*L along the fibers of q:

$$q^*E_L = q^*q_*(p^*L) \to p^*L.$$

Now for each i we can define

$$\pi_i : C^k \to C \times C_{k-1}$$

to be the quotient by the action of $\mathfrak{S}_{k-1}$ permuting all but the i^{th} factor of C^k. Identifying $C \times C_{k-1}$ with $\mathcal{D} \subset C \times C_k$, one can check that $q \circ \pi_i = \pi$ for each i. Thus pulling back the above evaluation map along π_i and summing over i, one obtains the natural evaluation map

$$\mathrm{ev} : \pi^* E_L \to L^{\boxplus k}.$$

Equipping $\pi^* E_L$ with the pullback equivariant structure (introduced in Section 4.1) and $L^{\boxplus k}$ with the natural permutation action, the morphism ev is $\mathfrak{S}_k$-equivariant by construction and coker(ev) has $\mathfrak{S}_k$-invariant support Δ. Hence by Proposition 4.1 we have an equivariant isomorphism

$$\pi^* N_L \cong L^{\boxtimes k}(-\Delta).$$

for $\pi^* N_L = \det(\pi^* E_L)$ equipped with the determinant (i.e. top exterior power) of the equivariant structure on $\pi^* E_L$ and for $L^{\boxtimes k}(-\Delta)$ equipped with the equivariant structure obtained by taking the product of the sgn representation (which recall is in fact a *character* of $\mathfrak{S}_k$) with the permutation structure $\rho_{\boxtimes}$ (again, refer back to Section 4.1 for details on these equivariant structures).

Since $\pi_*^{\mathfrak{S}_k} \pi^*$ is the identity for coherent sheaves on C_k (see e.g. [38, Lemma 2.1]), we apply $\pi_*^{\mathfrak{S}_k}$ to this isomorphism to get

$$N_L \cong \pi_*^{\mathfrak{S}_k}(L^{\boxtimes k}(-\Delta)).$$

Finally, with the equivariant structure described above, $L^{\boxtimes k}(-\Delta)$ is an equivariant subsheaf of $L^{\boxtimes k}$ — the quotient is $L^{\boxtimes k}|_{\Delta} = L^{\otimes k}$ equipped with just the sgn representation on $\Delta \cong C$ (and trivial action on C). The latter equivariant sheaf has no invariants, hence

$$\pi_*^{\mathfrak{S}_k}(L^{\boxtimes k}(-\Delta)) \cong \pi_*^{\mathfrak{S}_k} L^{\boxtimes k}.$$

□

Proposition 6.10. *For $\mathscr{L}$ on $C \times S$, the line bundles $N_{\mathscr{L}}$ and $T_{\mathscr{L}}$ satisfy the relation*

$$N_{\mathscr{L}} = T_{\mathscr{L}}(-\Delta_S/2).$$

Proof. As for the previous proposition, we prove this for S a point, but the technique generalizes immediately.

By the fact that $\pi^* T_L = L^{\boxtimes k}$ and, as we saw in the proof of Proposition 6.9, $\pi^* N_L = L^{\boxtimes k}(-\Delta)$, one has

$$\pi^* N_L \cong \pi^*(T_L(-\Delta/2))$$

and the result follows from the fact that π^* is injective for line bundles. $\square$

6.4 Picard components

Implicit in our statement of Theorem A in the Introduction is the fact that the Picard component $\mathrm{Pic}^{n(d)}(C_k)$ is isomorphic to $\mathrm{Pic}^d(C)$ in a natural way. Briefly, if $D \in C_{k-1}$ is any degree $k-1$ divisor on C, define $i : C \to C_{k-1}$ by $p \mapsto p + D$. Then the isomorphisms and their inverses are:

$$\begin{array}{rcl} \mathrm{Pic}^d(C) & \xrightarrow{\cong} & \mathrm{Pic}^{n(d)}(C_k) \\ L & \longmapsto & N_L \\ N_L(\Delta/2)|_{i(C)} & \longleftarrow & N_L \end{array}$$

(the inverse is independent of the choice of D).

So from now on, without further comment, we will identify these various corresponding Picard components and consider the Brill–Noether *loci* $W^r_d = W^r_d(C)$ introduced in Chapter 3 as subschemes of $\mathrm{Pic}^{n(d)}(C_k)$ where convenient.

6.5 Proof of Theorem A

We recall the result of Proposition 2.18 from Chapter 2, which indicates how one can produce a Picard sheaf (Definition 2.14) from a universal line bundle:

Proposition 6.11. *Let X be a smooth projective variety of dimension n, $\lambda \in NS(X)$, and $\mathscr{L}_\lambda$ a universal line bundle on $X \times \mathrm{Pic}^\lambda(X)$. Let $p : X \times \mathrm{Pic}^\lambda(X) \to X$ and $\nu : X \times \mathrm{Pic}^\lambda(X) \to \mathrm{Pic}^\lambda(X)$ denote the projections. Then*

$$\mathcal{F}_\lambda := R^n \nu_*(p^* K_X \otimes \mathscr{L}_\lambda^\vee)$$

is a Picard sheaf on $\mathrm{Pic}^\lambda(X)$.

Using this proposition, we now conclude this chapter with a proof of our first main theorem, Theorem A, which we restate here:

Theorem A. *For C a smooth projective curve and d, k positive integers, let $X = C_k$ and $\lambda = n(d)$. If $\mathcal{F}_d$ is a Picard sheaf on $\mathrm{Pic}^d(C)$ then*

$$\mathcal{F}_\lambda := \wedge^k \mathcal{F}_d$$

is a Picard sheaf on $\mathrm{Pic}^\lambda(X)$.

Proof. Let $S := \mathrm{Pic}^d(C) \cong \mathrm{Pic}^\lambda(X)$. Consider the commutative diagram

$$\begin{tikzcd} C^k \times S \arrow[r, "p_i"] \arrow[d, "\pi"] \arrow[dr, "\tau"] & C \times S \arrow[d, "\nu"] \\ C_k \times S \arrow[r, "\overline{\tau}"] & S \end{tikzcd}$$

for π the quotient map and the remaining maps just the natural projections (any of the k choices for the top map is valid).

Let $\mathscr{L}$ be a universal degree d line bundle on $C \times S$. By the discussion before Proposition 6.9, $N_{\mathscr{L}}$ will then be a universal line bundle for λ on $C_k \times S$.

Now let $q : C \times S \to C$ and $\widetilde{q} : C_k \times S \to C_k$ denote the projections. By Lemma 6.11, the sheaves

$$R^1\nu_*(q^*K_C \otimes \mathscr{L}^\vee) \quad \text{and} \quad R^k\overline{\tau}_*(\widetilde{q}^*K_X \otimes N_{\mathscr{L}}^\vee)$$

are Picard sheaves on $\mathrm{Pic}^d(C)$ and $\mathrm{Pic}^\lambda(X)$ respectively.

To lighten notation, define

$$\mathcal{A} := q^*K_C \otimes \mathscr{L}^\vee$$

and then note that using the relation from Proposition 6.10 and the isomorphisms $K_X \cong N_{K_C}$ and $(T_{\mathscr{L}})^\vee \cong T_{\mathscr{L}^\vee}$ one can compute

$$T_{\mathcal{A}} \cong \widetilde{q}^*K_X \otimes N_{\mathscr{L}}^\vee.$$

So then with $\mathfrak{S}_k$ acting on $\mathcal{A}^{\boxtimes_S k}$ by permutation, one has

$$R^k\overline{\tau}_*T_{\mathcal{A}} \cong R^k\tau_*(\mathcal{A}^{\boxtimes_S k})^{\mathfrak{S}_k}$$

and the result of the theorem follows from the equivariant Künneth formula of Proposition 4.7. $\square$

Chapter 7

Divisor varieties of symmetric products

In this chapter, we prove our main theorems concerning the structure of the divisor varieties parametrizing those effective divisors in the symmetric product of a curve which come from the linear systems $|N_L|$ (see Section 6.2). Our results are mainly for *general* curves, but see Remark 7.9 below for more details. Broadly speaking, we show that

- the enclosing dimension (see Definition 5.6) is the invariant which controls the decomposition of these divisor varieties into irreducible components (Theorem 7.3); and
- the subspace varieties (see Definition 5.7) are the model for intersections of the irreducible components (Theorem 7.4).

7.1 Setup

Fix C a smooth, projective curve of genus g over $\mathbb{C}$ as before, let $X := C_k$ denote the k^{th} symmetric product, and for d a positive integer let $\lambda := n(d) \in \mathrm{NS}(X)$ — recall the definition of $n(d)$ from Definition 6.3.

Recall from Definition 3.7 the *degree d dimension set* of C

$$\mathcal{R}_d := \{n \in \mathbb{Z}_{>0} : h^0(L) = n \text{ for some } L \in \mathrm{Pic}^d(C)\} = \{n_d, n_d + 1, \dots, N_d\}$$

and define a parameter set $\mathcal{E}$ of integers as follows

$$\mathcal{E} := \begin{cases} \mathbb{Z}_{>k} \cap \mathcal{R}_d & k \geq 3 \\ 2\mathbb{Z} \cap \mathcal{R}_d & k = 2 \end{cases}.$$

If L is a line bundle on C and $\eta \in H^0(X, N_L) \cong \wedge^k H^0(C, L)$ cuts out the divisor $D = \mathrm{Zeroes}(\eta) \subset X$ then, recalling Definition 5.6, denote

$$\mathrm{enc}(D) := \mathrm{enc}(\eta).$$

Now we want to define the subschemes of $\mathrm{Div}^\lambda(X)$ that will be the irreducible components — the enclosing dimension $\mathrm{enc}(D)$ is the essential discrete invariant will parametrize these components. Namely, for e an integer such that $k \leq e \leq N_d$, we want to define a subscheme $\mathcal{Z}_e \subset \mathrm{Div}^\lambda(X)$ such that at the level of closed points we have (as mentioned in the Introduction):

$$\mathcal{Z}_e = \{D \in \mathrm{Div}^\lambda(X) : \mathrm{enc}(D) \leq e \leq h^0(L) \text{ where } L \text{ is such that } D \in |N_L|\}.$$

Remark 7.1. Here one can think of the condition $e \leq h^0(L)$ as being used to exclude divisors $D \subset C_k$ which, although satisfying the desired bound $\mathrm{enc}(D) \leq e$ on their enclosing dimension, move in a linear system of "too small" dimension. The reason for making this exclusion is because we want $\mathcal{Z}_e$ to constitute a single irreducible component and if we allowed divisors D coming from smaller linear systems, then $\mathcal{Z}_e$ would pick up "phony" irreducible components.

We obtain $\mathcal{Z}_e$ with the desired closed-point locus above by defining

$$\mathcal{Z}_e := \overline{u_\lambda^{-1}(W_d^{e-1}(C) \setminus W_d^e(C))}$$

(where $u_\lambda : \mathrm{Div}^\lambda(X) \to \mathrm{Pic}^\lambda(X)$ is the Abel–Jacobi map from Section 4) noting that the closed point intepretation then follows from Theorem 7.4.

Remark 7.2. To be clear about the scheme structure of $\mathcal{Z}_e$ here, we remark that the notation should be interpreted as meaning the minimal scheme structure (i.e. scheme theoretic intersection of all closed subschemes) containing the scheme-theoretic pre-image $u_\lambda^{-1}(W_d^{e-1}(C) \setminus W_d^e(C))$ and supported on the Zariski-closure of this pre-image's support. In any event, our theorems will only concern the situation when C is *general* in the sense discussed in Subsection 3.1.1, in which case all these loci will be reduced.

7.2 Main theorems

We now make a modified restatement of Theorem B from the introduction, which indicates that the subschemes $\mathcal{Z}_e$ are irreducible components of $\mathrm{Div}^\lambda(X)$ (as a consequence of Theorem 7.4 they will be all the components — see Corollary 7.7):

Theorem 7.3. *If C is general and $\rho(g, d, N_d - 1) > 0$, then for $e \in \mathcal{E}$ the varieties $\mathcal{Z}_e$ are distinct and are irreducible components of $\mathrm{Div}^\lambda(X)$.*

Proof. Recall that $u_\lambda : \mathrm{Div}^\lambda(X) \to \mathrm{Pic}^\lambda(X)$ is the Abel-Jacobi map, sending $D \mapsto \mathcal{O}_X(D)$. Let $W := W_d^{e-1} \setminus W_d^e \subset \mathrm{Pic}^d(C) \cong \mathrm{Pic}^\lambda(X)$ and define

$$\mathrm{Div}^\lambda(X)|_W := u_\lambda^{-1}(W)$$

which has closure $\mathcal{Z}_e$. By Theorem A we have

$$\mathrm{Div}^\lambda(X)|_W \cong \mathbb{P}(\wedge^k \mathcal{F}_d|_W)$$

for any Picard sheaf $\mathcal{F}_d$ on $\mathrm{Pic}^d(C) \cong \mathrm{Pic}^\lambda(X)$. Since $\mathcal{F}_d$ has constant rank along W and the generality assumption on C implies W is smooth and irreducible when $\rho(g, d, e-1) > 0$, this isomorphism implies $\mathrm{Div}^\lambda(X)|_W$ is irreducible, hence so is $\mathcal{Z}_e$ (except when $e = N_d = R_d + 1$ and $\rho(g, d, R_d) = 0$).

Finally, there can be no pairwise containments among the $\mathcal{Z}_e$'s. To see this, suppose $e < f \in \mathcal{E}$, $L \in \mathrm{Pic}^d(C)$ and $D \in |N_L|$. If $h^0(C, L) = e$ then $D \in \mathcal{Z}_e \setminus \mathcal{Z}_f$. On the other hand, if $h^0(C, L) = \mathrm{enc}(D) = f$ then $D \in \mathcal{Z}_f \setminus \mathcal{Z}_e$ (because if D moves with $\mathcal{O}_X(D)$ then $\mathrm{enc}(D)$ could only possibly drop as $\mathcal{O}_X(D)$ moves from W into W_d^e). Both conditions are non-empty by Theorem 3.3 and Remark 5.10 thus both $\mathcal{Z}_e$ and $\mathcal{Z}_f$ are non-empty and neither is contained in the other.

Since for $e \in \mathcal{E}$ the subvarieties $\mathcal{Z}_e$ are each closed, irreducible, contain open sets and are pairwise distinct, they form the claimed irreducible components. $\square$

The next Theorem (a restatement of Theorem C from the introduction) describes the intersections of the components just identified in Theorem 7.3.

Theorem 7.4. *Let X, λ and $\mathcal{E}$ be as in Theorem 7.3, $e < f \in \mathcal{E}$ and recall the Abel–Jacobi map $u_\lambda : \mathrm{Div}^\lambda(X) \to \mathrm{Pic}^\lambda(X)$. Then*

$$\mathcal{Z}_e \cap \mathcal{Z}_f \xrightarrow{u_\lambda} W_d^{f-1}(C)$$

and for $L \in W_d^{f-1}(C)$,

$$|N_L| \cap \mathcal{Z}_e \cap \mathcal{Z}_f \cong \mathrm{Sub}_e(\wedge^k H^0(L)).$$

Remark 7.5. We thus see that f determines the *support* of $\mathcal{Z}_e \cap \mathcal{Z}_f$ over $\mathrm{Pic}^\lambda(X)$ while e determines the Abel–Jacobi *fibers* of it.

Remark 7.6. Note that, as can be seen simply by considering the support of the components of $\mathrm{Div}^\lambda(X)$ over $\mathrm{Pic}^\lambda(X)$, the intersection $\mathcal{Z}_{e_1} \cap \cdots \cap \mathcal{Z}_{e_m}$ for $e_1 < \cdots < e_m$ in fact coincides with the pairwise intersection $\mathcal{Z}_{e_1} \cap \mathcal{Z}_{e_m}$. So Theorem 7.4 really describes all possible component intersections.

Corollary 7.7. *Let X, λ and $\mathcal{E}$ be as in Theorem 7.3. Then*

$$\{\mathcal{Z}_e : e \in \mathcal{E}\}$$

are **all** *of the irreducible components of* $\mathrm{Div}^\lambda(X)$.

Proof. For $D \in \mathrm{Div}^\lambda(X)$ there is some degree d line bundle L on C such that $D \in |N_L|$. By Remark 5.10 we must have $e := \mathrm{enc}(D) \in \mathcal{E}$ hence $D \in \mathrm{Sub}_e(\wedge^k H^0(L))$. By Theorem 7.4 $\mathcal{Z}_e$ in particular contains all of $\mathrm{Sub}_e(\wedge^k H^0(L))$ thus $D \in \mathcal{Z}_e$. Hence $\mathrm{Div}^\lambda(X) = \bigcup_{e\in\mathcal{E}} \mathcal{Z}_e$. $\square$

Proof of Theorem 7.4. For any subscheme $W \subset \mathrm{Pic}^\lambda(X)$, let

$$\mathrm{Div}^\lambda(X)|_W := u_\lambda^{-1}(W)$$

and for any $r \geq 0$ let

$$(W_d^r)^\circ := W_d^r \setminus W_d^{r+1}.$$

For $W = (W_d^{e-1})^\circ$ and any $e \in \mathcal{E}$ we have

$$\mathcal{Z}_e|_W = \mathrm{Div}^\lambda(C_k)|_W$$

and

$$\mathcal{Z}_e = \overline{\mathcal{Z}_e|_W}$$

with the closure being taken in $\mathrm{Div}^\lambda(X)$. Note that W is irreducible by generality of C.

From these two observations, we see that for $L \in W' = (W_d^{f-1})^\circ$ we have

$$\begin{aligned} |N_L| \cap \mathcal{Z}_e \cap \mathcal{Z}_f &= |N_L| \cap \overline{\mathrm{Div}^\lambda(C_k)|_W} \cap \mathrm{Div}^\lambda(C_k)|_{W'} \\ &= (|N_L| \cap \mathrm{Div}^\lambda(C_k)|_{W'}) \cap \overline{\mathrm{Div}^\lambda(C_k)|_W} \\ &= |N_L| \cap \overline{\mathrm{Div}^\lambda(C_k)|_W} \quad (\text{since } |N_L| \subset \mathrm{Div}^\lambda(C_k)|_{W'}). \end{aligned}$$

Therefore the proof is finished once the following lemma is established. □

Lemma 7.8. *For X, λ, W, L and e as above,*

$$|N_L| \cap \overline{\mathrm{Div}^\lambda(X)|_W} = \mathrm{Sub}_e(\wedge^k H^0(C, L)).$$

Proof. The $\subseteq$ direction is clear so we focus on the $\supseteq$ direction.

Let $S \subset \mathrm{Div}^\lambda(X)$ be a locally closed integral curve. For $D \in |N_L|$ we will say S *has property* $\mathbf{C}_{D,W}$ if

$$D \in S \quad \text{and} \quad S \setminus \{D\} \subset \mathrm{Div}^\lambda(X)|_W.$$

The purpose of defining property $\mathbf{C}_{D,W}$ here is to indicate that S is an integral curve which passes through the point $D \in \mathrm{Div}^\lambda(X)$ but which does not lie in the fiber of the Abel–Jacobi map $u_\lambda : \mathrm{Div}^\lambda(X) \to \mathrm{Pic}^\lambda(X)$ over the point $\mathcal{O}_X(D) = N_L$. The idea is to use such curves to keep track of how certain divisors D of fixed enclosing dimension $\mathrm{enc}(D)$ can *move*, or not, over an irreducible parameter space S in $\mathrm{Div}^\lambda(X)$ — thus gaining insight into how the components intersect. We note:

$$|N_L| \cap \overline{\mathrm{Div}^\lambda(X)|_W} = \{D \in |N_L| : \exists \text{ a curve } S \text{ with property } \mathbf{C}_{D,W}\}.$$

Now take $D \in |N_L|$ with $\mathrm{enc}(D) = e$ and let $r := e - 1$. We will produce a locally closed integral curve $S \subset \mathrm{Div}^\lambda(X)$ with property $\mathbf{C}_{D,W}$. Define

$$G_d^r|_W := \{V \subset H^0(C, L) : L \in W\} \subset G_d^r.$$

Generality of C together with the Brill–Noether theory results of Theorem 3.3 and Corollary 3.4 imply that G^r_d is irreducible and $G^r_d|_W$ is an irreducible and open (hence dense) subset.

Let $E \in \mathbb{G}(r, |L|) = G^r_d \setminus G^r_d|_W$ denote the point corresponding to $\mathrm{Enc}(D) \subset H^0(C, L)$. Since $E \in \overline{G^r_d|_W}$ there is a locally closed integral curve $T \subset G^r_d$ such that

$$E \in T \quad \text{and} \quad T \setminus \{E\} \subset G^r_d|_W.$$

Let $\mathcal{U}$ denote the tautological rank $r+1$ bundle on G^r_d. We have a commutative diagram:

$$\begin{array}{ccc} \mathbb{P} \wedge^k \mathcal{U}^* & \xrightarrow{c} & \mathrm{Div}^\lambda(X) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle u_\lambda} \\ G^r_d & \longrightarrow & \mathrm{Pic}^d(C) \end{array}$$

Note in particular that, at the level of fibers, we have the natural containment

$$\pi^{-1}(E) = \mathbb{P} \wedge^k \mathrm{Enc}(D)^* \subset |N_L|.$$

So let $[D] \in \pi^{-1}(E)$ denote the point of $\mathbb{P} \wedge^k \mathcal{U}^*$ naturally corresponding to $D \in |N_L|$. By local triviality of $\mathbb{P} \wedge^k \mathcal{U}^*$ we can choose a local section $\widetilde{T}$ of $\mathbb{P}(\wedge^k \mathcal{U}^*|_T)$ and define

$$S := c(\widetilde{T}).$$

Then clearly $D = c([D]) \in S$ and, by the choice of T and commutativity of the diagram above,

$$S \setminus \{D\} \subset c(\mathbb{P} \wedge^k \mathcal{U}^* \setminus \pi^{-1}(\mathbb{G}(r, |L|))) = \mathrm{Div}^\lambda(X)|_W.$$

Thus S has property $\mathbf{C}_{D,W}$. Thus, since $\mathrm{enc}(D) = e$ we have shown

$$\{D \in |N_L| : \mathrm{enc}(D) = e\} \subset |N_L| \cap \overline{\mathrm{Div}^\lambda(X)|_W}.$$

But since the subspace variety $\mathrm{Sub}_e(\wedge^k H^0(C, L))$ is the closure of the left hand side, and the right hand side is closed, we have the result. $\square$

Remark 7.9. We make the following note concerning the generality assumption in Theorems 7.3 and 7.4: recalling that $\lambda = n(d)$ in both of those theorems, the generality assumption is made to ensure that all line bundles, in particular those of degree d, on the underlying curve C satisfy the Petri condition of Section 3.1.1. However, for an arbitrary curve C, even if all line bundles on C do not satisfy the Petri condition, it is possible that all line bundles of a certain fixed degree d do — in that case, Theorems 7.3 and 7.4 will still hold for $X = C_k$ and $\lambda = n(d)$ as in the case of C general. In particular, all line bundles of degree $d = 2g - 2$ on an arbitrary genus g curve will satisfy the condition, so our results hold for the paracanonical system of any symmetric product since in that case $n(d) = c_1(K_X)$.

Remark 7.10. Note that using the results of Section 3.1.2, in particular Equation 3.8, one can calculate the size of the parameter set $\mathcal{E}$ and thus determine the number of components of $\mathrm{Div}^{\lambda}(X)$. We simply remark here that, for fixed k and genus g of C, the number $\#\mathcal{E}$ grows approximately linearly in d.

Note that the map c in the proof of Lemma 7.8 has image $\mathcal{Z}_e$ and is in fact a relativized version of the surjection

$$\Psi \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Sub}_e(\wedge^k V)$$

from Section 5 where $V = H^0(C, L)$ varies with $L \in W^r_d$. Thus, except in the cases discussed in Remark 5.10, c actually realizes $\mathbb{P} \wedge^k \mathcal{U}^*$ as a desingularization of $\mathcal{Z}_e$. This quickly yields:

Theorem 7.11. *For C general, $k \geq 3$ and $e \in \mathcal{E}$ the irreducible components $\mathcal{Z}_e$ have dimensions as follows:*

$$\dim\, \mathcal{Z}_e = \rho(g, e-1, d) + \binom{e}{k} - 1$$

By Remark 5.14 we know that for $k \geq 3$ the *fibers* of $u_\lambda : \mathcal{Z}_e \to W^{e-1}_d$ are normal, Cohen–Macaulay and, for $k \geq 3$, have at worst rational singularities.

Remark 7.12. Throughout this thesis, we have focused on a particular class of tautological line bundles on C_k — those of the form N_L. There is another naturally arising class of line bundles on C_k, namely those of the form T_L (defined in Section 6.1). Though we have focused on the arguably more interesting case of the former for the sake of lighter exposition, our results can be easily reformulated into analogous ones for divisor varieties arising from the latter. The only essential difference is that the role of the (skew-symmetric) subspace varieties $\mathrm{Sub}_e(\wedge^k V)$ is played instead by the (symmetric) subspace varieties $\mathrm{Sub}_e(\mathrm{Sym}^k V)$ (for an analogous definition of *enclosing dimension*). Also, there will in general be slightly more irreducible components of divisor varieties parametrizing divisors coming from $|T_L|$ linear series, but this number will still be linear in $d = \deg(L)$ and the components will again be parametrized by an enclosing dimension.

7.3 Examples

Finally we present some examples to illustrate the results above. Unless otherwise indicated, C is a smooth projective curve over $\mathbb{C}$ and $X := C_k$ its k^{th} symmetric product for some $k \in \mathbb{N}$. The interesting range is $2 \leq k < g$.

We begin by looking at the paracanonical system of X (recall the discussion in Section 3.2). Recall from Remark 7.9 that our main results on divisor varieties of symmetric products can be applied to the paracanonical system without the generality assumption on C, since paracanonical bundles always satisfy the Petri condition.

Example 7.13 (Paracanonical system of X)**.** Let $\mathcal{P}_{\text{main}}$ denote the *main paracanonical system* of X — that is, the unique component of $\text{Div}^{\lambda}(X)$ dominating $\text{Pic}^{\lambda}(X)$ when $\lambda = c_1(K_X)$. We have

$$\dim \mathcal{P}_{\text{main}} = \dim \text{Pic}^{\lambda}(X) + \dim |N_L|$$

for L a generic paracanonical bundle on C. This yields

$$\dim \mathcal{P}_{\text{main}} = g + \binom{g-1}{k} - 1.$$

On the other hand, $\dim |K_X| = \binom{g}{k} - 1$. So we have:

$$\begin{aligned} \dim |K_X| - \dim \mathcal{P}_{\text{main}} &= \binom{g}{k} - 1 - (g + \binom{g-1}{k} - 1) \\ &= \binom{g-1}{k-1} - g \end{aligned}$$

which is negative for $k = 2, g-1$ but positive for $2 < k < g-1$, which means it is impossible for the canonical linear series to be contained in $\mathcal{P}_{\text{main}}$.

Importantly, not only does this calculation guarantee exorbitance of $|K_X|$ (recall from Definition 3.13 that one says $|L|$ is exorbitant if it forms a component of $\text{Div}^{\lambda}(X)$) when $2 < k < g-1$, but our results above indicate precisely the intersection of the main paracanonical system and the canonical linear series (what Castorena and Pirola suggestively call the *locus of deformable canonical divisors* in [9, Def. 5.3]):

$$\mathcal{P}_{\text{main}} \cap |K_{C_k}| = \mathrm{Sub}_{g-1}(\wedge^k H^0(C, K_C)).$$

Non-degeneracy of these subspace varieties, easily seen since they contain the corresponding Grassmannian in its Plücker embedding, is as expected by [9, Prop. 5.4], and their dimension (calculated in Lemma 5.12) implies that the codimension of $\mathcal{P}_{\text{main}} \cap |K_X|$ in $|K_X|$ is

$$\text{codim } \mathrm{Sub}_{g-1}(\wedge^k H^0(K_C)) = \binom{g-1}{k-1} - (g-1)$$

This, in contrast to the case for $k = 2$, is independent of the parity of g and constitutes a concrete example where the intersection is not a hypersurface, contrasting with [47, Thm. 1.3(ii)].

Remark 7.14 (Base locus of $\mathcal{P}_{\text{main}}$)**.** For arbitrary smooth projective varieties X with positive irregularity, one method of learning about $\mathcal{P}_{\text{main}}$ (defined in Example 7.13) has been to compare the base loci Z_κ and $Z_{|K_X|}$ of $\mathcal{P}_{\text{main}}$ and $|K_X|$, respectively.

In [47, Corollary 1.4], Mendes Lopes–Pardini–Pirola show that on a general type surface X with no irrational pencils of large genus, one has

$$Z_\kappa \subset Z_{|K_X|}.$$

In [9, Proposition 1.3], Castorena-Pirola generalize this to higher dimensions.

Given the possibility that $|K_X|$ is exorbitant, this containment is far from obvious. One might wonder if a non-trivial instance of this phenomenon can be observed on an appropriately chosen symmetric product. In fact it cannot: specifically, the main paracanonical system on $X = C_k$ will never have a base locus for $k < g$ (the interesting range).

This is because any point $D \in X$ in the base locus would necessarily correspond to a degree k divisor on the curve C failing to impose $k < g$ independent conditions on *all* line bundles L of degree $2g - 2$. By Riemann-Roch, this would imply that all divisors of degree k are effective which is not true for any curve C since the effective line bundles of degree $k < g$ form a k-dimensional closed subvariety of $\mathrm{Pic}^k(C)$.

Example 7.15 (The Full Picture)**.** To demonstrate our full results in a relatively concrete but illustrative case, we will let C be a general curve of genus $g = 37$. We illustrate Theorems 7.3 and 7.4 by studying systems $|N_L|$ for $L \in \mathrm{Pic}^{g-1}(C)$, which is the translate of the Picard variety in which the theta divisor can be naturally thought to live. Up to translations, $\Theta = W^0_{g-1}$. We have $\rho(g, d, r) = 37 - (r + 1)^2$, hence we have nontrivial Brill-Noether loci $W_{g-1}, W^1_{g-1}, \ldots, W^5_{g-1}$ of dimensions 36, 33, 28, 21, 12 and 1 respectively.

We will describe $\mathrm{Div}^\lambda(X)$ in the cases $k = 2, 3$ and $\lambda = c_1(N_L)$ for any $L \in \mathrm{Pic}^{g-1}(C)$.

The symmetric square (case $k = 2$): in this case $\mathrm{Div}^\lambda(X)$ has three components

$$\mathscr{Z}_2, \quad \mathscr{Z}_4, \quad \mathscr{Z}_6$$

supported over W^1_{g-1}, W^3_{g-1} and W^5_{g-1}, respectively.

The component $\mathscr{Z}_2$ has dimension 33 and is:

- birational to W^1_{g-1}
- a $\mathbb{P}^2$-bundle when restricted to $(W^2_{g-1})^\circ$
- a $G(2, 4)$-bundle when restricted to $(W^3_{g-1})^\circ$
- a $G(2, 5)$-bundle when restricted to $(W^4_{g-1})^\circ$
- a $G(2, 6)$-bundle when restricted to the whole curve W^5_{g-1}.

The component $\mathscr{Z}_4$ has dimension 26 and is:

- a $\mathbb{P}^5$-bundle when restricted to $(W^3_{g-1})^\circ$
- a $\mathbb{P}^9$-bundle when restricted to $(W^4_{g-1})^\circ$
- a $\mathrm{Sec}_3\, G(2, 6)$-bundle when restricted to the whole curve W^5_{g-1}.

Lastly, the component $\mathscr{Z}_6$ is a $\mathbb{P}^{14}$-bundle over W^5_{g-1}.

The symmetric cube (case $k = 3$): in this case $\mathrm{Div}^\lambda(X)$ again has three components

$$\mathscr{Z}_3, \quad \mathscr{Z}_5, \quad \mathscr{Z}_6$$

supported over W^2_{g-1}, W^4_{g-1} and W^5_{g-1}, respecitvely.

The component $\mathcal{Z}_3$ has dimension 28 and is:

- birational to W^2_{g-1}
- a $\mathbb{P}^3$-bundle when restricted to $(W^3_{g-1})^\circ$
- a $G(3,5)$-bundle when restricted to $(W^4_{g-1})^\circ$
- a $G(3,6)$-bundle over the whole curve W^5_{g-1}.

The component $\mathcal{Z}_5$ has dimension 21 and is:

- a $\mathbb{P}^9$-bundle when restricted to $(W^4_{g-1})^\circ$
- a $\mathrm{Sub}_5(\wedge^3\mathbb{C}^6)$-bundle when restricted to the whole curve W^5_{g-1}.

Lastly, $\mathcal{Z}_6$ is a $\mathbb{P}^{19}$-bundle over W^5_{g-1}.

Note that $\mathrm{Sub}_5(\wedge^3\mathbb{C}^6)$ is a subspace variety of dimension 14 which properly contains the Grassmannian $G(3,6)$ (dimension 9) and is properly contained in the chordal (i.e. 2-secant) variety $\mathrm{Sec}_2\, G(3,6) = \mathbb{P}^{19}$ (because secant varieties of Grassmannians $G(k,n)$ for $k > 2$ are *not* deficient - see [10, Theorem 2.1]).

Example 7.16 (Resolving the singular strata of theta divisors)**.** Recall that the Brill–Noether locus $W^0_{g-1}(C)$ is naturally identified with (a translate of) the theta divisor of C in its Jacobian $J(C) \cong \mathrm{Pic}^{g-1}(C)$. As a result of the Riemann Singularity Theorem (see [5, pg. 226]) we know that for C general, the singular locus of W^0_{g-1} is W^1_{g-1} and consequently the Abel-Jacobi mapping

$$\mathrm{Div}^{g-1}(C) \cong C_{g-1} \to W^0_{g-1}$$

is a resolution of singularities for the theta divisor.

By what we have said above, a similar phenomenon occurs for the singular locus of W^1_{g-1} when using the divisor variety of the symmetric square of the curve, for the singular locus of W^2_{g-1} when using the symmetric cube, and so on. Specifically, we have:

Corollary 7.17. *Let $\lambda = c_1(N_L)$ for L any degree $g-1$ line bundle on C and $X = C_k$ for C general. The component $\mathcal{Z}_k \subset \mathrm{Div}^\lambda(X)$ is smooth and the Abel-Jacobi mapping $\mathcal{Z}_k \to W^{k-1}_{g-1}$ is a resolution of singularities for the $(k-1)^{th}$ singular stratum of the theta divisor of C.*

We conclude with an example that shows how a divisor variety of Y a surface *not* isomorphic to a symmetric product of a curve might look. In particular, one sees *joins* and *cones* show up as component intersections, in contrast with the subspace varieties in the case of symmetric products.

Example 7.18 (Paracanonical system on double covers of C_2)**.** Let $\pi : Y \to X = C_2$ be a smooth double cover, branched along some divisor B whose associated line bundle is a square (though the root, which we will still denote by $B/2$, may not be effective). By Riemann-Hurwitz, we have

$$K_Y = \pi^*(K_X + B/2)$$

so that by the projection formula, we have:

$$H^0(K_Y) = H^0(K_X) \oplus H^0(K_X + B/2)$$

(since $\pi_* \mathcal{O}_Y \cong \mathcal{O}_X \oplus \mathcal{O}_X(-B/2)$).

With a little work analogous to what has gone before in this thesis, and assuming B is chosen so that $\mathrm{Pic}^0(Y) \cong \mathrm{Pic}^0(X)$ (which is often the case if B is positive enough, but does fail if for example $B = \Delta$), this decomposition (or, more precisely, its dual) globalizes to an analogous statement for Picard sheaves, so we get:

$$\mathcal{F}_{c_1(K_Y)} = \mathcal{F}_{c_1(K_X)} \oplus \mathcal{F}_\lambda$$

where $\lambda := c_1(K_X + B/2)$. There is some care to be taken here given that Picard sheaves are only well-defined up to twisting by a line bundle, but this will not concern what we conclude here.

Given this, we can identify the paracanonical system (let $\kappa := c_1(K_Y)$):

$$\mathrm{Div}^\kappa(Y) = \mathbb{P}(\mathcal{F}_{c_1(K_X)} \oplus \mathcal{F}_\lambda)$$

and when, for example, $B \in |T_L|$ for a (even) degree d line bundle L on C, we will have $\mathcal{F}_{c_1(K_X)} = \wedge^2 \mathcal{F}_{2g-2}$ and $\mathcal{F}_\lambda = \wedge^2 \mathcal{F}_{2g-2+d/2}$ for $\mathcal{F}_{2g-2}$ and $\mathcal{F}_{2g-2+d/2}$ Picard sheaves of the curve, for appropriate degrees. Note that B cannot be in $|N_L|$ for any L on C since N_L is never divisible in $\mathrm{Pic}(X)$.

An approach similar to the proof of Theorem 7.4 can then be taken in the current setting to show that:

- For $B = T_L$ and $d > 0$ we have:

$$|K_Y| \cap \mathrm{Div}^\kappa(Y)_{\mathrm{main}} = \mathrm{Cone}_{\mathbb{P}}(\Sigma)$$

for $\Sigma = \mathrm{Sec}_{\lfloor \frac{g-1}{2} \rfloor}\mathbb{G}(1, |K_C|)$, $\mathbb{P} = \mathbb{P} \wedge^2 H^0(K_C + L/2)^\vee$ and $\mathrm{Cone}_\Lambda(V)$ denotes the projective cone, with vertex a projective subspace Λ, over an embedding of a variety V in projective space. Note here that $L/2$ is well-defined since $B/2$ is.

- For $B = 0$ (i.e. in the étale case) we have:

$$|K_Y| \cap \mathrm{Div}^\kappa(Y)_{\mathrm{main}} = \mathrm{Join}(\Sigma, \Sigma)$$

where $\mathrm{Join}(V, W)$ denotes the union of all lines meeting two varieties V and W in a fixed projective space, and these two copies of Σ lie in the non-intersecting subspaces of $\mathbb{P}H^0(K_Y)$ corresponding to the isomorphic summands $\wedge^2 H^0(K_C)$ of $H^0(K_Y)$.

Chapter 8

Syzygies and Koszul cohomology

In the previous chapters, we have focused on the issue of parametrizing effective divisors in symmetric products of a curve, motivated in large part by the successes of the Brill–Noether theory for curves themselves, particularly when the curves are general.

Another very successful theory for curves that has seen much interest in the literature over the years is that of the defining equations of a curve once it has been embedded in projective space. It is natural to consider the extent to which our knowledge of this aspect of curve geometry could be also be transferred to the setting of symmetric products.

We begin with a general review: consider a smooth projective variety $X \subset \mathbb{P}H^0(X, L) = \mathbb{P}$ embedded by a very ample line bundle L. The section ring $R(L) := \bigoplus_{n \geq 0} H^0(X, nL)$ is a graded module over the graded ring $S := \mathrm{Sym}(H^0(X, L))$ and consequently the most interesting invariant of $R(L)$ is the minimal free graded resolution:

$$E_\bullet = \cdots \to E_p \to \cdots \to E_1 \to E_0 \to R(L). \tag{8.1}$$

If one splits this resolution into its constituent short exact sequences, we have for each i

$$0 \to M_{p+1} \to E_p \to M_p \to 0 \tag{8.2}$$

where the graded S-module M_p is known as the p^{th} *syzygy module of* $R(L)$. It is the module of relations between the generators of M_{p-1}. So for example, M_1 is the module of relations between

S-generators of $R(L)$, M_2 is the module of relations between *those* relations, and so on. When L is normally generated, one (equivalently) has that $E_0 = S$ and then the resolution 8.1 immediately yields the minimal free graded resolution

$$\cdots \to E_p \to \cdots \to E_1 \to I_L \tag{8.3}$$

of the homogeneous ideal $I_L := \ker(S \to R(L))$ of $X \subset \mathbb{P}H^0(L)$. With this perspective, $M_1 = I_L$, M_2 the relations between generators of I_L, and so on.

Of particular interest are the degrees j and the so-called "betti numbers" $\beta_{p,j} := \dim(E_{p,j})$ where $E_{p,j} \subset E_p$ is the submodule consisting only of the factors $S(-j)$.

There has been a great deal of interest in this topic, largely centered around the behavior of a key property:

Definition 8.4 (Green–Lazarsfeld property N_p)**.** We say L has property N_0 if it is normally generated, and property N_p (for $p \geq 1$) if it has property N_{p-1} and

$$E_p = \bigoplus_{j \leq p+1} E_{p,j}$$

(which actually implies $E_p = E_{p,p+1}$ by minimality of the resolution). If L has property N_p one says it has *linear syzygies to* p^{th} *order*.

In the case of curves, we have the celebrated:

Theorem 8.5 (Green, [20, Thm. 4.a.1])**.** *Let $X = C$ be a curve of genus g. If deg$(L) \geq 2g + 1 + p$ then L has property N_p.*

Green's theorem represents a vast generalization of the well-known fact that $\deg(L) \geq 2g + 1$ implies normal generation, due to Castelnuovo [8], Mattuck [44] and Mumford [48].

8.1 Koszul cohomology

In general, it can be quite a tricky business to extract information about the resolution 8.1 of the section ring $R(L)$ directly. One of the chief techniques that allows progress to be made is to turn one's attention to the Koszul cohomology groups $K_{p,q}(X, L)$ of L, which we define momentarily.

These are then related back to the (graded components of the) terms of the resolution 8.1 by way of the following theorem:

Theorem 8.6. *Let X be a smooth projective variety, L a line bundle on it, and $R(L)$ the corresponding ring of sections. Recall the syzygy modules M_p from Equation 8.2. One has*

$$M_{p,p+q} \cong K_{p,q}(X, L)$$

where $M_{p,p+q} := (M_p)_{p+q}$ is standard notation for the $(p+q)$-graded component of M_p.

It remains to define these cohomology groups and remark on the proof of Theorem 8.6.

Let V denote a finite dimensional vector space over $\mathbb{C}$, let $S := \mathrm{Sym}(V)$ denote the corresponding graded ring with irrelevant ideal I, and let B denote a finitely generated $\mathbb{Z}$-graded S-module. One has a graded resolution of the quotient $\mathbb{C} := S/I$

$$\cdots \to \wedge^p V \otimes_{\mathbb{C}} S(-p) \to \cdots \to V \otimes_{\mathbb{C}} S(-1) \to S \to \mathbb{C} \to 0. \tag{8.7}$$

(this is in fact the unique minimal such resolution — see, for example, [4, Corollary 1.6]). Tensoring with B (over S now), one then obtains a graded *complex*

$$K^\bullet(B) := \cdots \to \wedge^p V \otimes_{\mathbb{C}} B(-p) \to \cdots \to V \otimes_{\mathbb{C}} B(-1) \to B \to B \otimes_S \mathbb{C} \to 0. \tag{8.8}$$

(here we index the terms of the complex $K^\bullet(B)$ beginning at 0 with the term B on the right, not the term $B \otimes_S \mathbb{C}$). At this point one could take the homology S-modules $h_p(K^\bullet(B))$ and define associated Koszul cohomology groups $K_{p,q}(B, V)$ to be their graded components, but it is more standard in the literature to use instead the graded strands

$$(K^\bullet(B))_{p+q} := \cdots \longrightarrow \wedge^{p+1} V \otimes_{\mathbb{C}} B_{q-1} \longrightarrow \wedge^p V \otimes_{\mathbb{C}} B_q \longrightarrow \wedge^{p-1} V \otimes_{\mathbb{C}} B_{q+1} \longrightarrow \cdots$$

and define the Koszul cohomology groups as the homology at the p^{th} term shown:

$$K_{p,q}(B, V) := h_p(F^\bullet(B)_{p+q}).$$

Definition 8.9. Let X be a smooth projective variety, L a line bundle on it, $R(L)$ the corresponding section ring and take $V := H^0(X, L)$. Then the Koszul cohomology groups of L are defined as

$$K_{p,q}(X, L) := K_{p,q}(R(L), V).$$

With this definition in place, we now sketch the idea of how Theorem 8.6 is proven:

Sketch of proof. Taking the point of view of homological algebra, the sequence 8.7 constitutes a resolution of the S-module $\mathbb{C}$ and thus, after tensoring with $R(L)$, one obtains the complex $K^\bullet(R(L))$ which computes the Tor groups

$$\mathrm{Tor}_p^S(R(L), \mathbb{C}).$$

Evidently, by the discussion above, the Koszul cohomology groups $K_{p,q}(R(L), V)$ are the q-graded components of these. On the other hand, the minimal graded resolution 8.1 of $R(L)$ can be tensored with the S-module $\mathbb{C}$ to obtain the complex $E^\bullet \otimes_S \mathbb{C}$ which also computes the groups $\mathrm{Tor}_p(R(L), \mathbb{C})$ (by commutativity of Tor). But minimality of $E^\bullet$ means that the differentials in $E^\bullet \otimes_S \mathbb{C}$ are zero, and thus one has

$$M_{p,p+q} \cong (E_p)_{p+q} \cong \mathrm{Tor}_p^S(R(L), \mathbb{C})_q$$

and the result follows. □

8.2 Castelnuovo–Mumford regularity

Let $\mathcal{F}$ denote a coherent sheaf on a projective space $\mathbb{P}$ (though the definition and theorem that follow are the same as stated if one replaces $\mathbb{P}$ and $\mathcal{O}_{\mathbb{P}}(1)$ by an arbitrary smooth projective variety X and a very ample line bundle L on it). In [49, Lecture 14], crediting the main results below to Castelnuovo, Mumford introduces the notion of *regularity* for $\mathcal{F}$. For an integer m, he says that $\mathcal{F}$ has the property of being *m-regular* if

$$H^i(\mathbb{P}, \mathcal{F}(m - i)) = 0$$

for all $i > 0$. He then makes the following definition.

Definition 8.10. The *regularity* of the coherent sheaf $\mathcal{F}$ is defined as

$$\operatorname{reg}(\mathcal{F}) := \min\{m : \mathcal{F} \text{ is } m\text{-regular } \}.$$

Nowadays the regularity of $\mathcal{F}$ is often called its *Castelnuovo–Mumford regularity.* The basic facts about this circle of ideas which makes them distinctly useful are encapsulated in the following theorem of Mumford:

Theorem 8.11. *If $\mathcal{F}$ is m-regular, then*

- *$\mathcal{F}$ is $(m+k)$-regular for all $k \geq 0$; and*
- *the multiplication maps*

$$H^0(\mathbb{P}, \mathcal{F}(m)) \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) \longrightarrow H^0(\mathbb{P}, \mathcal{F}(m+k))$$

  are surjective for all $k \geq 0$.

Recall that Theorem 8.11 was in fact crucial in Chapter 2 in our proof of Proposition 2.3. However, we introduce the idea here because of its relation to the circle of ideas concerning syzygies and Koszul cohomology — it turns out that when $\mathcal{F} = \mathcal{I}_X$ is the ideal sheaf of a projective variety $X \subset \mathbb{P}$ embedded by the complete linear system of a very ample line bundle L on X, then knowing the regularity of $\mathcal{F}$ already yields a lot of information about the shape of the minimal graded resolution 8.1. Namely, we have the following restatement of [4, Proposition 2.37]:

Proposition 8.12. *Let m be a positive integer and assume $\mathcal{I}_X$ is $(m+1)$-regular. Then $\mathcal{I}_X$ is, in addition, m-regular if and only if*

$$K_{p,m}(X, L) = 0$$

for all $p > 0$.

In particular this means that one can "read off" the regularity of $\mathcal{I}_X$ from the betti numbers of 8.1 since this proposition implies

$$\operatorname{reg}(\mathcal{I}_X) = \min\{m : K_{p,m'}(X, L) = 0 \text{ for all } p > 0 \text{ and all } m' \geq m \}.$$

In Proposition 9.10 we will use a basic instance of the ideas here (particularly Theorem 8.11) to reduce a proof of normal generation of N_L on C_k (for L a line bundle on a curve C) to checking surjectivity of a couple of multiplication maps — a standard use of regularity in questions about syzygies more generally.

Chapter 9

Syzygy shifting for symmetric products

Theorem 8.5 of Green gives a very clean and clear indication of how positivity of a (high degree) line bundle L on a curve C translates into linearity of the syzygies of $R(L)$, expressed by L having property N_p.

In the case of symmetric products, one might reasonably expect linearity of the syzygies of N_L to also depend in a clear way on the positivity of L itself. Indeed, the expectation is that the best one can hope for is that as the dimension of the chosen symmetric product increases, one loses linearity of the syzygies by only one order at a time:

Conjecture 9.1 (Positivity shifting). If $\deg(L) \geq 2g+1+p$ then N_L on C_k has property $N_{p-(k-1)}$.

A stronger and more hopeful version of this expectation would in fact be that the grading of the resolution of the section ring $R(N_L)$ associated to C_k both determines and is determined by that of $R(L)$ associated to C, at least when L is sufficiently positive. More hopefully still, one might expect the canonical bundle K_C to fit into such a framework. In light of Theorem 8.5 and Conjecture 9.1, one might expect this dependence to be characterized by the following "shifted" relationship:

Conjecture 9.2 (Syzygy shifting). A line bundle L on a curve C, with either $L = K_C$ or $\deg(L) \geq 2g + 1 + p$, has property N_p if and only if N_L has property N_{p-k+1} on C_k.

Before presenting our current evidence and results in the direction of these conjectures, it is very interesting to note the close relationship that they have to the far-reaching Mukai conjecture.

9.1 Mukai's conjecture and syzygy shifting

Let C be a smooth projective curve of genus g and L a degree d line bundle on it. Recall that already by Riemann–Roch, any line bundle of degree at least $2g+1$ is very ample. Indeed by Green's Theorem 8.5 it will be moreover normally generated.

Now let X be a smooth projective variety of arbitrary dimension > 1. Although in this case a line bundle no longer has an intrinsic *degree*, Fujita realized that the appropriate generalization of the very-ampleness statement above for curves should take the form

Conjecture 9.3 (Fujita)**.** Let $n := \dim X$. A line bundle on X will be very ample if it has the form

$$K_X + (n+2)A + P$$

where A denotes an ample line bundle and P a nef one.

Reider's theorem ([52]) establishes Fujita's conjecture for surfaces, but — apart from some positive results such as in [14], which in particular establishes the statement when A is *very* ample — it is otherwise quite open. Though we do not mention it here, Fujita has a completely analogous conjecture concerning global generation (as opposed to very-ampleness) and in that particular setting there are many partial results — see e.g. [3, 12, 14, 32].

Nonetheless, in light of the stronger positivity statement for curves yielded by Green's Theorem 8.5, Mukai noticed that one might also aspire to a stronger Fujita-type analog in higher dimensions:

Conjecture 9.4 (Mukai)**.** Let X, A and P be as in Conjecture 9.3. A line bundle on X will have property N_p if it has the form

$$K_X + (n+2+p)A + P.$$

While certain instances of Mukai's conjecture are known (see e.g. [18, 29, 30]), in general it is wide open.

We now note the following rather intriguing relationship between Mukai's conjecture and the shifting conjectures above:

Proposition 9.5. *Let L be a degree 1 line bundle on C. Then Conjecture 9.1 is the precise instance of Mukai's conjecture when $A = T_L$ and $P = \mathcal{O}_{C_k}$.*

Proof. This follows directly by recalling that $K_{C_k} = N_{K_C}$, $N_L \cong T_L(-\Delta/2)$ and $T_{mL} = T_L^{\otimes m}$ for any m:

$$\begin{aligned} K_{C_k} + (n+2+p)T_L &= N_{K_C} + T_{(n+2+p)L} \\ &= T_{K_C}(-\Delta/2) + T_{(k+2+p)L} \\ &= N_{K_C+(k+2+p)L} \end{aligned}$$

and $\deg(K_C + (k+2+p)L) = 2g+1+p+(k-1)$. □

9.2 Towards the shifting conjectures

By way of evidence for Conjecture 9.2, one has the following two propositions, already mentioned in the introduction and the first of which is a more well-known fact (see e.g. [21]) which we prove here for completeness:

Proposition 9.6. *Suppose $X \subset \mathbb{P}$ is a smooth projective variety embedded by a very ample line bundle L. If X admits a $(p+2)$-secant-p-plane, then property N_p fails for L.*

Proof. Suppose L had property N_p. Then there is a resolution of the ideal sheaf with the following shape:

$$\cdots \to \bigoplus \mathcal{O}(-p-1) \to \cdots \to \bigoplus \mathcal{O}(-3) \to \bigoplus \mathcal{O}(-2) \to \mathcal{I}_X \to 0.$$

If Λ is a $(p+2)$-secant-p-plane of X, we can twist this resolution by $\mathcal{O}(1)$ and restrict to Λ to obtain a *complex*

$$\cdots \to \bigoplus \mathcal{O}(-p) \to \cdots \to \bigoplus \mathcal{O}(-2) \to \bigoplus \mathcal{O}(-1) \to \mathcal{I}_X(1)|_\Lambda = \mathcal{I}_{X\cap\Lambda}(1) \to 0 \tag{9.7}$$

that will at least be exact away from $X \cap \Lambda$ (since there it is the restriction of an exact sequence of vector bundles, which remains exact). This complex may, however, fail to be exact on the length-$(p+2)$ subscheme $Z := X \cap \Lambda$.

Using the vanishings $H^k(\Lambda, \mathcal{O}_\Lambda(-k-1)) = 0$ for $1 \le k \le p+1$ we can now apply the conclusion of Proposition [41, B.1.2] to the complex 9.7 to conclude that

$$H^1(\Lambda, \mathcal{I}_Z(1)) = 0.$$

However, this vanishing would imply that the restriction map

$$H^0(\Lambda, \mathcal{O}_\Lambda(1)) \longrightarrow H^0(Z, \mathcal{O}_Z(1))$$

is surjective, which cannot happen since the dimensions of these vector spaces are $p+1$ and $p+2$ respectively. This is a contradiction, and thus L fails property N_p. $\square$

We can use the proposition just proved to then establish the following:

Proposition 9.8. *If a curve $C \subset \mathbb{P}V$, embedded by L, admits a $(p+2)$-secant-p-plane, then $C_k \subset \mathbb{P} \wedge^k V$, embedded by N_L, admits a $(p-k+3)$-secant-$(p-k+1)$-plane (assuming N_L is very ample).*

Proof. We consider the $(p+2)$-secant-p-planes of $C \subset \mathbb{P}V$ — let $P = \mathbb{P}W \subset \mathbb{P}V$ be one such (with corresponding quotient $V \twoheadrightarrow W$) and choose a labeling $q_1, \ldots, q_{p+2}$ of the points in $C \cap P$.

Note that, for any $k \leq p$, these $p+2$ points determine $\binom{p+2}{k}$ projective $(k-1)$-planes in P — i.e. a copies of $G(W,k)$ in $G(V,k)$ meeting $C_k \subset G(V,k)$ in $\binom{p+2}{k}$ points.

Now we can choose $q_1, \ldots, q_{k-1}$ to be fixed, and then for each of the remaining $(p+2)-(k-1) = p+3-k$ points $q \in \{q_k, \ldots, q_{p+2}\}$ we have determined a $\mathbb{P}^{p+1-k} \subset G(W,k)$ parametrizing those $(k-1)$-planes in the p-plane P which contain the fixed $(k-2)$-plane spanned by $\{q_1, \ldots, q_{k-1}\}$.

Under the Plücker embedding $G(V,k) \hookrightarrow \mathbb{P} \wedge^k V$, this $\mathbb{P}^{p+1-k}$ is sent to a linear subspace, which is thus a $(p+3-k)$-secant-$(p+1-k)$-plane of $C_k \subset \mathbb{P} \wedge^k V \cong \mathbb{P}H^0(N_L)$. Hence, by Proposition 9.6, property N_{p+1-k} fails for the line bundle N_L on C_k. $\square$

Together, these propositions imply that if a secant plane causes N_p to fail for L on C, then a related secant plane causes N_{p-k+1} to fail for N_L on C_k. This, however, does not constitute a proof of this direction of the Conjecture 9.2 since failure of N_p on curves may not be caused by such secant planes in general.

Nonetheless, as evidence for Conjecture 9.1, we have its very first instance in the following theorem:

Theorem 9.9. *If a line bundle L on a smooth, projective genus $g > 2$ curve C has degree*

$$d := deg(L) \geq 2g + 2$$

then N_L is normally generated on C_2.

We prove the theorem in the sequence of propositions and lemmas that form the remainder of this chapter. Note first that by [11, Main Theorem] the hypotheses imply L is 2-very ample and therefore N_L induces an embedding

$$C_2 \hookrightarrow \mathbb{P} \wedge^2 V$$

where $V := H^0(C, L)$ and $\mathbb{P}$ denotes 1-dim quotients. This embedding is induced by the same map as we introduced in Remark 6.4. Recall N_L is m-normal if

$$S^m H^0(N_L) \twoheadrightarrow H^0(N_L^{\otimes m}).$$

We reduce normal generation of N_L to showing 2- and 3-normality by a regularity argument:

Proposition 9.10. *Let $\mathcal{I}$ denote the ideal sheaf of $C_2 \subset \mathbb{P}\wedge^2 V$. With $d \geq 2g+2$, if N_L is 3-normal, then $\mathcal{I}$ is 4-regular. i.e.*

$$H^i(\mathcal{I}(4-i)) = 0 \quad \textit{for } i > 0$$

Proof. Since $N_L = N_K \otimes T_{L-K}$, $N_K = K_{C_2}$ is ample (for $g > 2$) and T_{L-K} is ample, Kodaira vanishing yields $H^1(C_2, N_L^{\otimes 2}) = H^2(C_2, N_L) = 0$. These vanishings imply $H^2(\mathcal{I}(2)) = H^3(\mathcal{I}(1)) = 0$. Then 3-normality implies $H^1(\mathcal{I}(3)) = 0$. The higher vanishings are automatic since dim $C_2 = 2$. $\square$

Remark 9.11. 4-regularity is as good as we can do here because $H^2(C_2, N_L^{\otimes m}) \neq 0$ for $m \leq 0$.

Of course, if $\mathcal{I}$ is 4-regular then Mumford's theorem 8.11 tells us it is m-regular, and hence N_L is m-normal, for $m \geq 4$. So normal generation of N_L is reduced to 2- and 3-normality. Most interesting is the argument for 2-normality in Proposition 9.13 below.

Before getting to Proposition 9.13, we need to prove a lemma and take note of some calculations. By way of setting up for the lemma, we first introduce the following notation:

- Let $\mathbb{P} := \mathbb{P}V$ for any finite dimensional vector space V over $\mathbb{C}$,

- let $X := \mathrm{Bl}_{\Delta(\mathbb{P})}(\mathbb{P}\times\mathbb{P}) \xrightarrow{b} \mathbb{P}\times\mathbb{P}$ denote the blowup along the diagonal with exceptional E,

- let H_1, H_2 denote divisors in the linear systems of the respective pullbacks of $\mathcal{O}_{\mathbb{P}}(1)$ to X,

- let $\mathbb{P}^{[2]} := \mathrm{Hilb}^2(\mathbb{P})$ denote the Hilbert scheme of 2 points on $\mathbb{P}$,

- let $G := G(V, 2)$ denote the Grassmannian of 2-dimensional quotients of V, and

- let $\mathcal{S}$ denote the tautological sub-bundle on $G(V, 2)$.

Now note that we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[d, "\widetilde{\pi}"] \arrow[drr, "\widetilde{\nu}"] & & \\ \mathbb{P}^{[2]} \arrow[r, "\nu"] & G \subset & \mathbb{P}\wedge^2 V. \end{tikzcd}$$

Here $\widetilde{\nu}$ is the map determined by the line bundle $\mathcal{O}_X(H_1 + H_2 - E)$ and is the resolution of the rational map $\mathbb{P}\times\mathbb{P} \dashrightarrow \mathbb{P}\wedge^2 V$ sending an off-diagonal pair $(p, q) \in \mathbb{P}\times\mathbb{P}$ to the point in the Grassmannian corresponding to the spanning line $\overline{pq}$. The map $\widetilde{\pi}$ is the quotient by the fixed-point-free extension of the natural involution on $\mathbb{P}\times\mathbb{P}$ — it resolves the rational map $\mathbb{P}\times\mathbb{P} \dashrightarrow \mathbb{P}^{[2]}$ sending the off-diagonal pair (p, q) to $p + q$. These maps clearly commute.

With the above setup in mind, we have:

Lemma 9.12. *Let L be as in Theorem 9.9 and in the setup above let $V := H^0(C, L)$. Recall from above that 2-very ampleness of L ensures that N_L determines an embedding of C_2 in $G \subset \mathbb{P}\wedge^2 V$. Similarly, $L \boxtimes L$ determines an embedding of C^2 in $\mathbb{P}\times\mathbb{P}$ which lifts to X via its proper transform. We have that the restriction map*

$$H^0(G, \mathcal{O}_G(2)) \to H^0(C_2, N_L^{\otimes 2})$$

can be obtained by taking invariants of the $\mathfrak{S}_2$-equivariant restriction map

$$H^0(\mathcal{O}_X(2H_1 + 2H_2 - 2E)) \to H^0(C^2, (2L \boxtimes 2L)(-2\Delta)).$$

Proof. The result of this lemma will be essentially immediate once the equivariant setup is established.

We first note that the line bundles $\mathcal{O}_{\mathbb{P}}(2) \boxtimes \mathcal{O}_{\mathbb{P}}(2)$ and $2L \boxtimes 2L$ on $\mathbb{P} \times \mathbb{P}$ and C^2, respectively, admit $\mathfrak{S}_2$-equivariant structures coming from the natural permutation actions on the bases — these are the same as or analogous to those structures discussed briefly before the proof of Proposition 4.7 and in Section 6.3. The former extends, along with the extension of the involution from $\mathbb{P} \times \mathbb{P}$ to X, to an equivariant structure on $\mathcal{O}_X(2H_1 + 2H_2)$. These then descend to equivariant structures on the respective subsheaves

$$\mathcal{O}_X(2H_1 + 2H_2 - 2E) \subset \mathcal{O}_X(2H_1 + 2H_2 - E) \subset \mathcal{O}_X(2H_1 + 2H_2)$$

and

$$(2L \boxtimes 2L)(-2\Delta) \subset (2L \boxtimes 2L)(-\Delta) \subset 2L \boxtimes 2L$$

since the involutions on the bases restrict trivially to E and Δ, respectively.

In particular, the restriction map

$$\mathcal{O}_X(2H_1 + 2H_2 - 2E) \longrightarrow (2L \boxtimes 2L)(-2\Delta)$$

is $\mathfrak{S}_2$-equivariant. Next we note the following isomorphism

$$\mathbb{P}^{[2]} \cong \mathbb{P}_G \mathrm{Sym}^2 \mathcal{S}^\vee$$

which comes from realizing the linear structure on the fibers of the map $\nu : \mathbb{P}^{[2]} \to G$ sending $p + q$ to l_{pq} (see above). Over the point $[l] \in G$ corresponding to a line $l \subset \mathbb{P}$, the fiber of ν will consist of all pairs $p + q$ such that $p, q \in l$ — i.e. it will be isomorphic to $\mathbb{P}\mathrm{Sym}^2 \widetilde{l}$, for $\widetilde{l}$ the 2-dimensional vector space quotient of V determining the line l. This realization of the fiber over $[l]$ globalizes over G to the isomorphism above. Moreover, we have

$$\nu^* \mathcal{O}_G(1) \cong \widetilde{\pi}_*^{\mathfrak{S}_2} \mathcal{O}_X(H_1 + H_2 - E).$$

We note that the embedding $C \subset \mathbb{P}$ determined by L determines a corresponding *tautological* embedding $C_2 \subset \mathbb{P}^{[2]}$ which, when composed with ν, coincides with the embedding $C_2 \subset G$ determined by N_L. Since $\nu : \mathbb{P}^{[2]} \to G$ is a projective bundle with $\nu_* \mathcal{O}_{\mathbb{P}^{[2]}} = \mathcal{O}_G$, we can realize the map

$$H^0(G, \mathcal{O}_G(2)) \longrightarrow H^0(C_2, N_L^{\otimes 2})$$

as

$$H^0(\mathbb{P}^{[2]}, \nu^* \mathcal{O}_G(2)) \longrightarrow H^0(C_2, N_L^{\otimes 2})$$

which in turn can be realized as

$$\begin{tikzcd}
H^0(\mathbb{P}^{[2]}, \widetilde{\pi}_*^{\mathfrak{S}_2} \widetilde{\pi}^* \mathcal{O}_G(2)) \arrow[r] \arrow[d, equal] & H^0(C_2, \widetilde{\pi}_*^{\mathfrak{S}_2} \widetilde{\pi}^* N_L^{\otimes 2}) \arrow[d, equal] \\
H^0(\mathbb{P}^{[2]}, \widetilde{\pi}_*^{\mathfrak{S}_2} \mathcal{O}_X(2H_1 + 2H_2 - 2E)) \arrow[r] & H^0(C_2, \widetilde{\pi}_*^{\mathfrak{S}_2}((2L \boxtimes 2L)(-2\Delta)))
\end{tikzcd}$$

and since, for example by Proposition 4.2, cohomology of an equivariant pushforward is simply the invariants of the cohomology upstairs, this last map is the following one that we wanted:

$$H^0(X, \mathcal{O}_X(2H_1 + 2H_2 - 2E))^{\mathfrak{S}_2} \longrightarrow H^0(C^2, (2L \boxtimes 2L)(-2\Delta))^{\mathfrak{S}_2}.$$

Hence we have the desired coincidence of maps. $\square$

With the setup and notation of Lemma 9.12, note that in fact the following diagram of restriction maps on X has all $\mathfrak{S}_2$-equivariant maps

$$\begin{tikzcd}
\mathcal{O}_X(2H_1 + 2H_2 - aE) \arrow[r] \arrow[d] & \mathcal{O}_{2E}(2H_1 + 2H_2 - aE) \arrow[d] \\
(2L \boxtimes 2L)(-a\Delta) \arrow[r] & ((2L \boxtimes 2L)(-a\Delta))|_{2\Delta}
\end{tikzcd}$$

for $a = 0, 1$ (when the restricted bundles on the right are equipped with the trivial equivariant structures). Hence taking global sections in the above diagram will also yield $\mathfrak{S}_2$-equivariant maps. Recalling now that the exceptional divisor E is the projective bundle $\mathbb{P}\Omega^1_{\mathbb{P}}$ and that $\mathcal{O}_E(-E) \cong \mathcal{O}_{\mathbb{P}\Omega^1_{\mathbb{P}}}(1)$ note that we have:

$$\begin{aligned}
H^0(\mathcal{O}_X(2H_1 + 2H_2 - 2E))^{\mathfrak{S}_2} &= \ker(H^0(\mathcal{O}_X(2H_1 + 2H_2 - E)) \to H^0(\mathcal{O}_E(2H_1 + 2H_2 - E)))^{\mathfrak{S}_2} \\
&= \ker(\ker((S^2V)^{\otimes 2} \to S^4V) \to H^0(\mathcal{O}_{\mathbb{P}\Omega^1_{\mathbb{P}}}(1) \otimes (b|_E)^* \mathcal{O}_{\mathbb{P}}(4)))^{\mathfrak{S}_2} \\
&= \ker(S^{2,2}V \oplus S^{3,1}V \to \ker(V \otimes S^3V \to S^4V))^{\mathfrak{S}_2} \\
&= S^{2,2}V
\end{aligned}$$

where $S^{2,2}V$ and $S^{3,1}V$ denote the Schur functors corresponding to the partitions $(2,2)$ and $(3,1)$ respectively. Similarly

$$\begin{aligned} H^0((2L \boxtimes 2L)(-2\Delta))^{\mathfrak{S}_2} &= \ker(H^0((2L \boxtimes 2L)(-\Delta)) \to H^0(C, 4L+K))^{\mathfrak{S}_2} \\ &= \ker(\wedge^2 H^0(2L) \oplus I_{2L}(2) \to H^0(4L+K))^{\mathfrak{S}_2} \\ &= I_{2L}(2) \end{aligned}$$

where $I_{2L}(2)$ denotes the space of quadrics in the embedding of C determined by the line bundle $2L$.

With the above calculations in place, we are now ready to prove 2-normality and 3-normality of N_L, for L as above, and thereby complete the proof of Theorem 9.9.

Proposition 9.13. *Still with $d \geq 2g+2$, we have that N_L is 2-normal.*

Proof. We wish to show that

$$S^2H^0(N_L) \to H^0(N_L^{\otimes 2})$$

is surjective.

Observe that the relation $N_L \cong T_L(-\Delta/2)$ implies that $N_L^{\otimes 2} \cong T_{2L}(-\Delta)$ and so

$$\begin{aligned} H^0(N_L^{\otimes 2}) &\cong \ker(H^0(T_{2L}) \to H^0(T_{2L}|_\Delta) \\ &\cong \ker(S^2H^0(C,2L) \to H^0(C,4L)) \end{aligned}$$

so we can arrange a commutative diagram

$$\begin{tikzcd} & S^2 \wedge^2 V \arrow[r] \arrow[d, "a"] & S^2S^2V \arrow[r] \arrow[d, "b"] & S^4V \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & H^0(N_L^{\otimes 2}) \arrow[r] & S^2H^0(C,2L) \arrow[r] & H^0(C,4L) & . \end{tikzcd}$$

where commutativity here follows from Lemma 9.12 and the discussion before and after it. In particular, using notation from that lemma, we note that the map $S^2 \wedge^2 V \to S^2S^2V$ here comes from the composition

$$H^0(\mathbb{P} \wedge^2 V, \mathcal{O}_{\mathbb{P}\wedge^2 V}(2)) \to H^0(G, \widetilde{\pi}^*\nu^*\mathcal{O}_G(2)) \to H^0(\mathcal{O}_X(2H_1+2H_2)).$$

The others can be interpreted similarly. Now 2-normality of L on C implies that the map b above is surjective. Thus, by the Snake Lemma, surjectivity of a will follow from surjectivity of

$$\ker(b) \to \ker(c).$$

But $\ker(b)$ is a quotient of $I_L(2) \otimes S^2V$ and $\ker(c) = I_L(4)$ (here $I_L(d)$ denotes d-ics in the homogeneous ideal of C) so this surjection in fact follows from another commutative diagram:

$$\begin{tikzcd} I_L(2) \otimes S^2V \arrow[d] \arrow[dr, two heads] & \\ \ker(b) \arrow[r] & \ker(c) \end{tikzcd}$$

where surjectivity of the diagonal arrow follows by the fact that $d \geq 2g+2$ implies that the ideal of C is generated by quadrics. $\square$

Finally, 3-normality follows rather quickly from 2-normality together with an application of Koszul duality and some vanishing:

Proposition 9.14. *Still with $d \geq 2g+2$, we have that N_L is 3-normal.*

Proof. The degree assumption implies $H^1(N_L) = H^2(N_L) = 0$. With these vanishings, the Duality Theorem for Koszul cohomology ([20, 2.c.6]) implies:

$$K_{0,3}(C_2, N_L) \cong K_{h^0(N_L)-3,0}(C_2, K_{C_2}, N_L)^*.$$

By 2-normality of N_L, the group on the left is the cokernel of $S^3H^0(N_L) \to H^0(N_L^{\otimes 3})$. On the other hand, the group on the right vanishes by the so-called "Vanishing Theorem" ([20, 3.a.1]) since $\binom{g}{2} = h^0(K_{C_2}) \leq \binom{g+3}{2} - 3 \leq \binom{d+1-g}{2} = h^0(N_L) - 3$. $\square$

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