Ellipticity of Bartnik Boundary Data for Vaccum Stationary Spacetimes

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Abstract of the Dissertation

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In this dissertation, we establish a moduli space of stationary vacuum spacetimes and prove it admits manifold structure. In the moduli space, we set up a well-defined boundary map, assigning a metric class with its Bartnik boundary data. Furthermore, we prove the boundary map is Fredholm, by showing ellipticity for the boundary value problem consisting of stationary vaccum equations and Bartnik boundary conditions (combined with proper gauge terms). As an application, we prove that locally, the Bartnik boundary data near the standard flat one of the unit 3-ball admits a unique (up to diffeomorphism) stationary vacuum extension.

Contents

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1 Introduction

1.1 The Bartnik quasi-local mass

A spacetime $(V^{(4)}, g^{(4)})$ is a 4-manifold with a smooth Lorentzian metric $g^{(4)}$ of signature $(-,+,+,+)$. A space-like hypersurface $(\Sigma, g) \subset (V^{(4)}, g^{(4)})$ is an embedded hypersurface such that the induced metric g is Riemannian. On the hypersurface Σ, there is the second fundamental form K of $(\Sigma, g) \subset (V^{(4)}, g^{(4)})$. The triple (Σ, g, K) is called an *initial data set* of the spacetime.

An initial data set (Σ, g, K) is called *asymptotically flat* , if there is a compact set $D \subset \Sigma$ such that $\Sigma \backslash D$ is equal to a disjoint union of ends, where every end is diffeomorphic to $\mathbb{R}^3 \backslash B$ – the 3-dimensional Euclidean space minus the unit ball. For simplicity, we assume there is only one end, i.e. there is a diffeomorphism $\phi : \Sigma \backslash D \to \mathbb{R}^3 \backslash B$. The diffeomorphism ϕ is called a *chart at infinity* for the initial data set. In this chart, there are coordinates $\{x^i\}_{i=1}^3$ and a smooth radius function r. In addition, as r increases to infinity, the metric g decays to the Euclidean metric and K decays to zero, i.e.

$$g_{ij} - \delta_{ij} = O(r^{-\delta}),$$
$$K_{ij} = O(r^{-\delta-1}),$$

where δ is called the decay rate. With suitable decay rate δ and additional decay conditions on the derivatives of g and K, one can define the ADM energy momentum (E, P_i) for the initial data set as follows:

$$E = \frac{1}{16\pi} \lim_{r\to\infty} \int_{S_r} (g_{ij,i} - g_{ii,j}) dS^j,$$
$$P_i = \frac{1}{8\pi} \lim_{r\to\infty} \int_{S_r} (K_{ij} - (trK) g_{ij}) dS^j.$$

The ADM mass is then defined as,

$$m_{ADM} = \sqrt{E^2 - \Sigma_{i=1}^3 P_i^2}.$$

Schoen-Yau (cf. [SY]) proved that under the dominant energy condition, the ADM mass is well defined, i.e. $E^2 - \Sigma_{i=1}^3 P_i^2 \geq 0$ in the expression above.

With the total mass of an asymptotically flat spacetime well understood, mathematicians and physicists try to define a local notion to measure the mass of a bounded region in the spacetime. There are now numerous proposals for a notion of local or quasi-local mass in general relativity, such as the Brown-York mass, Wang-Yau mass, and so on. See e.g.[Sz] for a detailed survey. In this thesis we focus on one of the most interesting and well-studied candidates, the Bartnik quasi-local mass.

Let $\Omega \subset \Sigma$ be a bounded smooth 3-manifold with boundary $\partial\Omega \cong S^2$. Then Ω is naturally equiped with the restricted metric $g|_\Omega$ and symmetric 2-tensor $K|_\Omega$, which will still be denoted as g, K in the following. On the boundary $\partial\Omega$, we define the Bartnik boundary data as,

$$(g_{\partial\Omega}, H_{\partial\Omega}, tr_{\partial\Omega} K, \omega_{\mathbf{n}_{\partial\Omega}}). \tag{1.1}$$

Here $g_{\partial\Omega}$ is the metric induced on the boundary $\partial\Omega$; $H_{\partial\Omega}$ is the mean curvature of $\partial\Omega \subset \Omega$; $tr_{\partial\Omega} K$ is the trace of the restriction $K|_{\partial\Omega}$ of the second fundamental form; and $\omega_{\mathbf{n}_{\partial\Omega}}$ is the connection 1-form of the spacetime normal bundle of $\partial\Omega$, which is defined as,

$$\omega_{\mathbf{n}_{\partial\Omega}}(v) = K(\mathbf{n}_{\partial\Omega}, v), \ \forall v \in T(\partial\Omega),$$

where $\mathbf{n}_{\partial\Omega}$ is the outward unit normal vector field on $\partial\Omega \subset (\Omega, g)$.

The Bartnik quasi-local mass of the data set (Ω, g, K) is defined as (cf.[B1],[B2]),

$$m_B[(\Omega, g, K)] = \inf\{m_{ADM}[(M, g, K)]\},$$

where the infimum is taken over all asymptotically falt initial data set (M, g, K) which are *admissible extensions*(cf.[B1]) such that the following geometric boundary conditions are satisfied under a certain diffeomorphism $\partial M \cong \partial\Omega$:

$$\begin{cases} g_{\partial M} = g_{\partial\Omega} \\ H_{\partial M} = H_{\partial\Omega} \\ tr_{\partial M} K = tr_{\partial\Omega} K \\ \omega_{\mathbf{n}_{\partial\Omega}} = \omega_{\mathbf{n}_{\partial M}}. \end{cases} \tag{1.2}$$

It was conjectured in [B1] that Bartnik quasi-local mass of (Ω, g, K) is realized by an admissible extension (M, g, K) which can be embedded as an initial data set into a stationary vacuum spacetime. Following the conjecture, Bartnik proposed the natural question:

$$\begin{aligned} &Given\ the\ Bartnik\ boundary\ data\ (g_{\partial\Omega}, H_{\partial\Omega}, tr_{\partial\Omega}K, \omega_{\mathbf{n}_{\partial\Omega}}), is\ there \\ &an\ asymptotically\ flat,\ stationary\ vacuum\ extension\ satisfying\ the \\ &boundary\ conditions\ (1.2)? \end{aligned} \tag{1.3}$$

This is a very difficult question and many efforts have been made to answer it. In [M1], local existence and uniqueness near the standard flat metric on the unit 3-ball was proved for the static case under a certain reflection invariance condition. This result was generalized in [A3] to the case without reflection invariance. (cf. also [J][R] for results with different boundary conditions.)

In fact, before considering the existence problem, Bartnik proposed a more basic open problem:

$$Is\ the\ Bartnik\ boundary\ data\ elliptic\ for\ stationary\ vacuum\ metrics? \tag{1.4}$$

Anderson and Khuri answered this question for the static case ([AK]), but the stationary case is very complicated and remained open. This is the main motivation of the present work, and we will give a positive answer to the question (1.4) in this dissertation.

1.2 Stationary vacuum spacetimes

To proceed, we first give a brief introduction to the stationary vacuum spacetime.

A spacetime $(V^{(4)}, g^{(4)})$ is called stationary if it has a time-like Killing vector field. A trivial example is the flat Minkowski space $(\mathbb{R}^{1,3}, g_{Min})$, where

$$g_{Min} = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2.$$

Stationary vacuum spacetimes are stationary spacetimes $(V^{(4)}, g^{(4)})$ that solve the vacuum Einstein field equations

$$Ric_{g^{(4)}} = 0. \tag{1.5}$$

They are important and much studied in general relativity. There are two famous nontrivial examples: the Schwarzschild metric,

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2);$$

and the Kerr metric,

$$\begin{aligned} ds^2 = & -(1-\frac{2Mr}{\Sigma})dt^2 - \frac{4Marsin^2\theta}{\Sigma}dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ & + (r^2+a^2+\frac{2Ma^2rsin^2\theta}{\Sigma})sin^2\theta d\phi^2, \end{aligned}$$

cf.[W].

Throughout this thesis, we assume that the spacetime $(V^{(4)}, g^{(4)})$ is *globally hyperbolic*, i.e. it admits a Cauchy surface Σ. The topology of a globally hyperbolic spacetime $V^{(4)}$ is necessarily $\Sigma \times \mathbb{R}$. In the following, we recall two well-known formulations of the stationary vacuum field equations — the hypersurface formalism and the projection formalism.

1.2.1 The hypersurface formalism

In the hypersurface formalism, one define a global time function t on $V^{(4)}$ so that ∂_t is the Killing field. The metric $g^{(4)}$ may then be written globally in the form

$$g^{(4)} = -N^2dt^2 + (g_M)_{ij}(dx^i + X^idt)(dx^j + X^jdt), \tag{1.6}$$

where $\{x^i\}(i = 1, 2, 3)$ are local coordinates of the space-like hypersurface $M = \{t = 0\}$ in $V^{(4)}$. The lapse function N, the shift vector $X = X^i\partial_{x_i}$ and the induced metric $g_M = (g_M)_{ij}dx^idx^j$ are all independent of the time t.

The stationary spacetime $(V^{(4)}, g^{(4)})$ is vacuum if and only if the following stationary vacuum field equations hold on M, cf.[M],

$$\begin{cases} 2NK - L_Xg = 0, \\ Ric_{g_M} + (trK)K - 2K^2 - \frac{1}{N}D^2N + \frac{1}{N}L_XK = 0, \\ s_{g_M} + (trK)^2 - |K|^2 = 0, \\ \delta K + d(trK) = 0. \end{cases} \tag{1.7}$$

Here D^2N denotes the Hessian of function N; s_{g_M} denotes the scalar curvature of the metric g_M on M; and K is the second fundamental form of the hypersurface $M \subset (V^{(4)}, g^{(4)})$.

It is known and easy to see that when the hypersurface M is a closed $3-$manifold, there are no non-flat stationary vacuum solutions to the field equations. Lichnerowicz (cf.[L]) proved that a geodesically complete stationary vacuum spacetime is necessarily flat Minkowski space g_{Min} when the $3-$manifold M is complete and asymptotically flat, cf. also [A2] for a generalization with no asymptotic condition.

Thus nontrivial solutions of (1.7) only exist on 3-manifolds with nonempty boundary, where the issue of boundary conditions arises. Particularly, in view of the Bartnik question (1.3), we are interested in the Bartnik bounday conditions (1.2). So questions (1.3) and (1.4) are essentially asking whether the boundary value problem consisting of (1.7) and (1.2) is solvable and elliptic. However, it is very complicated to directly work with such a boundary value problem in the hypersurface formalism. So alternatively, we will first use the projection formalism to get some insight of the problem.

1.2.2 The projection formalism

In the projection formalism, we use S to denote the collection of all trajectories of the time-like Killing field Z in $(V^{(4)}, g^{(4)})$, i.e. S is the orbit space of the action of 1-parameter group $\mathbb{R}$ generated by Z. Since the spacetime is globally hyperbolic, the qoutient space S is a smooth 3-manifold and the metric $g^{(4)}$ restricted to the horizontal distribution — the orthogonal complement of span$\{Z\}$ in $TV^{(4)}$ — induces a well-defined Riemannian metric g_S on S. Let $\pi : V^{(4)} \to S$ denote the projection map, then metric $g^{(4)}$ is globally of the form

$$g^{(4)} = -e^{2u}(dt + \theta)^2 + \pi^* g_S. \tag{1.8}$$

Here θ is a 1-form on S so that the dual of the Killing vector field Z is $\xi = -e^{2u}(dt + \theta)$. The *twist tensor* ω is defined as

$$\omega = \frac{1}{2} \star_{g^{(4)}} (\xi \wedge d\xi), \tag{1.9}$$

where $\star_{g^{(4)}}$ is the hodge star of the metric $g^{(4)}$. The twist tensor provides a measurement of the integrability of the horizontal distribution TS in $V^{(4)}$. It actually lives on the quotient manifold S, because (1.9) is equivalent to

$$\omega = -\frac{1}{2} e^{3u} \star_{g_S} d\theta.$$

It is easy to observe that under the reparametrization of time

$$t' = t + f,$$

where f is a function on S, the formula (1.8) becomes

$$g^{(4)} = -e^{2u}(dt' + \theta') + \pi^* g_S,$$

with $\theta' = \theta - df$. The twist tensor ω remains invariant under this gauge transformation. Therefore, a stationary spacetime $(V^{(4)}, g^{(4)})$ corresponds uniquely to a collection of data $(g_S, u, d\theta)$ or (g_S, u, ω) on the quotient manifold S. We refer to [K] and [CH] for more details of the projection formalism.

Notice that the restriction $(\pi|_M : M \to S)$ of the projection π, gives a diffeomorphism between the hypersurface M and the quotient manifold S. Thus boundary problems in the setting $\{M, (g_M, N, X)\}$ can be transferred to equivalent boundary problems with data $\{S, (g_S, u, \omega)\}$ via this diffeomorphism and vice versa. In certain respects, the projection formalism is more canonical, since there are many distinct hypersurfaces giving rise to the same stationary solution on the 4−manifold, but the projection data is unique.

The stationary vacuum field equations in the projection formalism, which are equivalent to (1.7) in the hypersurface formalism, are given by, cf.[H1],[H2],

$$\begin{cases} Ric_{g_S} - D^2 u - (du)^2 - 2e^{-4u}(\omega \otimes \omega - |\omega|^2 g_S) = 0, \\ \Delta_{g_S} u - |du|^2 - 2e^{-4u}|\omega|^2 = 0, \\ \delta\omega + 3\langle du, \omega\rangle = 0, \\ d\omega = 0. \end{cases}$$

Here Δ_{g_S} denotes the geometric Laplacian operator, $\Delta_{g_S} u = -tr_{g_S} D^2 g_S u$. The last equation indicates that ω is exact. In the case $S \cong \mathbb{R}^3 \setminus B^3$, we can assume $\omega = d\phi$ for some function ϕ on S. Thus the system above can be expressed equivalently as,

$$\begin{cases} Ric_{g_S} - D^2 u - (du)^2 - 2e^{-4u}(d\phi \otimes d\phi - |d\phi|^2 g_S) = 0, \\ \Delta_{g_S} u - |du|^2 - 2e^{-4u}|d\phi|^2 = 0, \\ \Delta_{g_S}\phi + 3\langle du, d\phi\rangle = 0. \end{cases} \tag{1.10}$$

Compared with the system (1.7), the work of choosing proper gauge terms and dealing with the principal symbols turns out to be much easier in (the conformal transformation of) the system above. Thus it is of interest to study the ellipticity of the system (1.10) with geometrically natural prescribed boundary conditions on the quotient manifold S.

Rather than transforming the Bartnik boundary conditions from the slice M to the quotient space S, we analyze some simpler boundary conditions arising naturally from the projection formalism. In view of the Bartnik conditions, we choose $(g_{\partial S}, H_{\partial S})$ — the induced metric on the boundary and the mean curvature of the boundary — as the boundary conditions of the metric. In addition, we pose a restriction on the twist tensor ω, by requiring $\omega(\mathbf{n}) = \mathbf{n}(\phi)$ fixed on the boundary ∂S, where $\mathbf{n}$ is the normal vector of the boundary pointing outwards. Actually the collection of boundary data,

$$\{g_{\partial S}, H_{\partial S}, \mathbf{n}(\phi)\}, \tag{1.11}$$

also arises naturally from the boundary terms in the variation of a functional on S, which is the reduction of the Einstein Hilbert action from $V^{(4)}$ to S, c.f.§2.4.2.

The first main theorem we will prove is the ellipticity of the boundary data (1.11).

Theorem 1.1. *The stationary vacuum field equations* (1.10) *and boundary conditions* (1.11) *form an elliptic boundary value problem, modulo gauge transformations.*

To prove this theorem, we first present in §2.1 the conformal transformation of the vacuum field equations, which gives an operator with simpler symbols. For the purpose of ellipticity, we modify the equations using certain gauge terms. After that, in §2.4 ellipticity for (the conformal transformation of) the boundary conditions (1.11) is proved with respect to different choices of gauge terms.

Remark. The method we use to prove ellipticity in this paper is not only valid for the boundary conditions (1.11). It can also be applied to more general boundary value problems for the stationary vacuum field equations.

In §2.5 we prove a manifold structure theorem for the moduli space $\mathcal{E}_C = \mathcal{E}_C^{m,\alpha}$ of stationary vacuum spacetimes. The space $\mathcal{E}_C$ is basically the space of all $C^{m,\alpha}$ asymptotically flat stationary vacuum solutions to the system (1.10) on S modulo the action of the group $\mathcal{D}_0^{m+1,\alpha}(S)$ of diffeomorphisms on S equal to the identity on ∂S. In addition, based on the boundary conditions (1.11), we have a natural map Π, from the moduli space $\mathcal{E}_C$ to the space of boundary data defined as follows,

$$\begin{aligned} \Pi_C : &\mathcal{E}_C \to Met^{m,\alpha}(\partial S) \times C^{m-1,\alpha}(\partial S) \times C^{m-1,\alpha}(\partial S), \\ &\Pi_C[(g_S, u, \phi)] = (g_{\partial S}, H_{\partial S}, \mathbf{n}(\phi)). \end{aligned} \tag{1.12}$$

Here $Met^{m,\alpha}(\partial S)$ is the space of $C^{m,\alpha}$ metrics on ∂S; $C^{m-1,\alpha}(\partial S)$ is the space of $C^{m-1,\alpha}$ functions on ∂S. By applying the ellipticity result, we will prove the following theorem.

Theorem 1.2. *The moduli space $\mathcal{E}_C$ is an infinite dimensional C^∞ Banach manifold, and the map Π_C is C^∞ smooth and Fredholm, of Fredholm index 0.*

With the help of the manifold structure of the moduli space $\mathcal{E}_C$ and using the idea developed in the projection formalism, we can then come back to prove ellipticity of the Bartnik boundary data.

1.3 Bartnik boundary data

The Bartnik boundary data is of crucial importance both in determining the Bartnik quasi-local mass of a bounded spacelike $3-$manifold in a spacetime, as shown in the first section, and in the variation problem of the regularized Hamiltonian. In fact, a regularization $\mathcal{H}$ of the Regge-Teitelboim Hamiltonian is constructed in [B3]. When the spacetime has empty boundary, by analyzing the functional $\mathcal{H}$ and following an approach initiated by Brill-Deser-Fadeev (cf.[BDF]), Bartnik proved that stationary metrics are critical points of the ADM energy functional on the constraint manifold. However, if the spacetime has non-empty boundary, the Bartnik boundary terms arise from the variation of $\mathcal{H}$; they were explicitly identified by Bartnik in [B1].

Come back to the Bartnik questions (1.3-4). Similarly as in the previous section, we will establish a boundary value problem (BVP) to interpret the Bartnik questions. Since the boundary conditions (1.2) are defined in the hypersurface formalism, one needs to couple system (1.7) with (1.2). However, as mentioned before, it is very complicated to work with the system (1.7) directly. So instead, we couple the equation (1.5), which is equivalent to (1.7), with the boundary conditions (1.2) to obtain a BVP given by,

$$
\begin{aligned}
&Ric_{g^{(4)}} = 0 \quad \text{on } M, \\
&\begin{cases} g_{\partial M} = \gamma \\ H_{\partial M} = H \\ tr_{\partial M} K = k \\ \omega_{\mathbf{n}_{\partial M}} = \tau, \end{cases} \quad \text{on } \partial M.
\end{aligned}
\tag{1.13}
$$

Here we use (γ, H, k, τ) to denote the prescribed boundary data for simplicity. The above system (1.13) is understood as a BVP of unknown $g^{(4)} = (g_M, X, N)$, which is a tensor field defined on M and independent of the time, as is shown in (1.6). The ellipticity/exsistence question (1.3-4) is essentially asking whether this BVP is elliptic/solvable.

Another way to formulate questions (1.3-4) is to establish a boundary map. Let $\mathbf{B}$ denote the space of Bartnik boundary data, i.e. tuples (γ, H, k, τ) on ∂M. Let $\mathbb{E}$ be the space of stationary vacuum metrics on $V^{(4)}$. Then a natural boundary map Π_1 arises as,

$$
\begin{aligned}
&\Pi_1 : \mathbb{E} \to \mathbf{B}, \\
&\Pi_1(g^{(4)}) = (g_{\partial M}, H_{\partial M}, tr_{\partial M} K, \omega_{\mathbf{n}}).
\end{aligned}
\tag{1.14}
$$

The map Π_1 being Fredholm is essentially equivalent to that BVP (1.13) is elliptic. However, it is easy to observe that (1.5) is not elliptic, since it is invariant under diffeomorphisms, i.e., if $g^{(4)}$ is a stationary metric that solves (1.5), then the pull back metric $\Phi^* g^{(4)}$ of $g^{(4)}$ under an arbitrary diffeomorphism Φ of $V^{(4)}$, gives another stationary solution. This means that we need to add gauge terms to (1.5), and at the same time, modify the domain space $\mathbb{E}$ of the boundary map Π to a moduli space.

In section 3, we first analyze how to choose the right moduli space in order to obtain a well-defined boundary map. We conclude in §3.2 that the boundary map should be established as,

$$
\begin{aligned}
&\Pi : \mathcal{E} \to \mathbf{B}, \\
&\Pi([g^{(4)}]) = (g_{\partial M}, H_{\partial M}, tr_{\partial M} K, \omega_{\mathbf{n}}).
\end{aligned}
$$

Here the moduli space $\mathcal{E}$ is the quotient of $\mathbb{E}$ by a particular diffeomorphism group $\mathcal{D}$. We refer to §3.1, for the exact definition of $\mathcal{D}$; roughly it is a natural intermediate group $\mathcal{D}_3 \subset \mathcal{D} \subset \mathcal{D}_4$ between the groups of 3-dimensional diffeomorphisms on M and 4-dimensional diffeomorphisms on $V^{(4)}$. In order to prove ellipticity of the map Π, we establish in §3.2 a BVP under an additional technical assumption (cf. Assumption 3.1). We prove this BVP is elliptic in §3.3, and from this derive the main theorem of this paper:

Theorem 1.3. *The moduli space $\mathcal{E}$ is a C^∞ smooth Banach manifold of infinite dimension and the boundary map Π is Fredholm.*

We show in §3.4 that the theorem is still true without the technical assumption in §3.3, completing the proof of Theorem 1.3.

To conclude, we apply this ellipticity result in §3.5 to show that the Bartnik boundary data near the standard flat (Minkowski) metric $\tilde{g}_0^{(4)}$ on $\mathbb{R} \times (\mathbb{R}^3 \setminus B)$ can be locally uniquely realized by a stationary vacuum metric up to diffeomorphism in $\mathcal{D}$.

Theorem 1.4. *There is a neighborhood $\mathcal{U} \subset [Met^{m,\alpha} \times C^{m-1,\alpha} \times C^{m-1,\alpha} \times (\wedge^1)^{m-1,\alpha}](S^2)$ of the standard flat boundary data $(g_0, 2, 0, 0)$ such that for any $(\gamma, H, k, \tau) \in \mathcal{U}$, there is a unique stationary vacuum metric $g^{(4)} \in \mathbb{E}$ near $\tilde{g}_0^{(4)}$ up to isometry in $\mathcal{D}$, for which*

$$\Pi(g^{(4)}) = (\gamma, H, k.\tau).$$

Remark. Throughout, we assume the hypersurface $M \cong \mathbb{R}^3 \setminus B^3$ (exterior problem), together with certain asymptotically flat assumptions on the metric $g^{(4)}$. Meanwhile, all the methods and results here can be applied equally well in the case where $M \cong B^3$ (interior problem).

Theorem 1.3 is a generalization of the results proved in [AK], where spacetimes are static. Theorem 1.4 generalizes the result in [A3] of static metrics.

The results we prove in this thesis provide a firm foundation for future work on Bartnik's conjecture about the quasi-local mass in spacetimes and the existence problem of stationary vacuum metrics that satisfy the Bartnik boundary conditions. To the author's knowledge, this is the first ellipticity result of the Bartnik boundary data for general stationary vacuum metrics.

2 Projection Formalism of stationary vacuum spacetimes

2.1 Background discussion

2.1.1 Asymptotic flatness

Throughout this section, the quotient manifold S is assumed to be diffeomorphic to $\mathbb{R}^3 \setminus B$. When it goes to the infinite end, we assume that the data (g_S, u, ϕ) is asymptotically flat, in the sense that

$$g_S - g_F \to 0,\ u \to 0,\ \phi \to 0, \quad as\ r \to \infty$$

where g_F is the flat metric on $\mathbb{R}^3 \setminus B^3$ and r is the pull back to S of the radius function on $\mathbb{R}^3 \setminus B^3$ under a fixed diffeomorphism. To describe rigorously the decay behavior above, we use the weighted Holder spaces defined as follows, cf. [B2], [LP].

Definition 2.1. We define several Banach spaces for $m \in \mathbb{N}$, and $\alpha, \delta \in \mathbb{R}$ on a general Riemannian manifold $\mathbf{M} \cong \mathbb{R}^3 \setminus B^3$:

$$
\begin{aligned}
&C_\delta^m(\mathbf{M}) = \{\text{functions } v \text{ on } \mathbf{M} : ||v||_{C_\delta^m} = \Sigma_{k=0}^m sup\ r^{k+\delta}|\nabla^k v| < \infty\},\\
&C_\delta^{m,\alpha}(\mathbf{M}) = \{\text{functions } v \text{ on } \mathbf{M} :\\
&\qquad\qquad ||v||_{C_\delta^m} + sup_{x,y}[\min(r(x), r(y))^{m+\alpha+\delta}\frac{\nabla^m v(x) - \nabla^m v(y)}{|x-y|^\alpha}] < \infty\},\\
&Met_\delta^{m,\alpha}(\mathbf{M}) = \{\text{metrics } g \text{ on } \mathbf{M} : (g_{ij} - \delta_{ij}) \in C_\delta^{m,\alpha}\},\\
&(T_p)_\delta^{m,\alpha}(\mathbf{M}) = \{p - \text{tensors } \tau \text{ on } \mathbf{M} : \tau_{i_1 i_2 .. i_p} \in C_\delta^{m,\alpha}\},\\
&(\wedge_p)_\delta^{m,\alpha}(\mathbf{M}) = \{p - \text{forms } \sigma \text{ on } \mathbf{M} : \sigma_{i_1 i_2 .. i_p} \in C_\delta^{m,\alpha}\}.
\end{aligned}
$$

Definition 2.2. The data (g_S, u, ω) is called *asymptotically flat of order* δ if

$$(g_S, u, \phi) \in [Met_\delta^{m,\alpha} \times C_\delta^{m,\alpha} \times C_\delta^{m,\alpha}](S), \tag{2.1}$$

for some m, α and δ.

Throughout the following, the orders m, α and the decay rate δ are fixed, and chosen to satisfy,

$$m \geq 2,\ 0 < \alpha < 1,\ \frac{1}{2} < \delta < 1.$$

Remark. In the previous section, we introduced the diffeomorphism $\pi|_M : M \to S$ between hypersurface M and quotient space S. In fact, under this diffeomorphism, the asymptotic flatness condition (2.1) of (g_S, u, ϕ) in S is equivalent to the asymptotic condition in M:

$$(g_M, u, Y) \in [Met_\delta^{m,\alpha} \times C_\delta^{m,\alpha} \times (\wedge_1)_\delta^{m,\alpha}](M),$$

which, furthermore, is equivalent to the decay behavior as in Bartnik's work.

2.1.2 Conformal transformation of the stationary vacuum field equations and gauge choice

To simplify the symbols of the stationary field equations (1.10), we first apply a conformal transformation on the quotient manifold S:

$$g = e^{2u} g_S. \tag{2.2}$$

Under such a transformation, the data (g_S, u, ϕ) is in 1-1 correspondence to the triple (g, u, ϕ); and if (g_S, u, ϕ) is asymptocially flat as described in (2.1), it also holds that the data (g, u, ϕ) is asymptotically flat, i.e.

$$(g, u, \phi) \in [Met_\delta^{m,\alpha} \times C_\delta^{m,\alpha} \times C_\delta^{m,\alpha}](S).$$

Furthermore, the stationary vacuum field equations (1.10), which are expressed in terms of (g_S, u, ϕ), can be simplified to the following system for (g, u, ϕ), cf. [K],

$$
(I) \quad \begin{cases}
Ric_g - 2du \otimes du - 2e^{-4u} d\phi \otimes d\phi = 0,\\
\Delta_g u - 2e^{-4u}|d\phi|^2 = 0,\\
\Delta_g \phi + 4\langle du, d\phi\rangle = 0.
\end{cases}
$$

The field equations above can be expressed in an equivalent way, where the Ricci tensor in Ric_g is replaced by the Einstein tensor $Ein_g = Ric_g - \frac{1}{2}s_g g$. In fact, the trace of the first equation is given by

$$s_g - 2|du|^2 - 2e^{-4u}|d\phi|^2.$$

Let t be the term

$$t = \frac{1}{2}(s_g - 2|du|^2 - 2e^{-4u}|d\phi|^2)g.$$

Then it is easy to see that, system (I) is equivalent to the following system (II) by inserting t into the first equation,

$$(II)\begin{cases} Ric_g - 2du\otimes du - 2e^{-4u}d\phi\otimes d\phi - t = 0, \\ \Delta_g u - 2e^{-4u}|d\phi|^2 = 0, \\ \Delta_g\phi + 4\langle du, d\phi\rangle = 0. \end{cases}$$

Notice that by rearranging the terms, the first equation in (II) can be expressed as

$$(Ric_g - \frac{1}{2}s_g g) - 2du\otimes du - 2e^{-4u}d\phi\otimes d\phi + (|du|^2 + e^{-4u}|d\phi|^2)g = 0, \tag{2.3}$$

where the leading term — the term with highest order of derivative with respect to the data (g, u, ϕ) — is exactly the Einstein tensor $(Ric_g - \frac{1}{2}s_g g)$.

Observe that the system (I) (or (II)) is not elliptic, because the full system is invariant under diffeomorphism, i.e. if (g, u, ω) is a solution of the Einstein field equations, then the pull back data $\Psi^*(g, u, \omega)$ under some diffeomorphism Ψ on S, is also a solution. So to ensure ellipticity, as is usual we modify the system using a gauge term, and obtain

$$(III)\begin{cases} Ric_g - 2du\otimes du - 2e^{-4u}d\phi\otimes d\phi + T + \delta^* G = 0, \\ \Delta_g u - 2e^{-4u}|d\phi|^2 = 0, \\ \Delta_g\phi + 4\langle du, d\phi\rangle = 0. \end{cases}$$

The pair (T, G) can be

$$\begin{cases} T_1 = 0, \\ G_1 = \beta_{\tilde{g}}(g), \end{cases}$$

where $\tilde{g}$ is a reference metric near g, and β is the Bianchi operator: $\beta_{\tilde{g}}g = \delta_{\tilde{g}}g + \frac{1}{2}dtr_{\tilde{g}}g$. This corresponds to inserting $G_1 = \beta_{\tilde{g}}(g)$ (the Bianchi gauge) into the system (II). Alternately, one can set (T, G) to be

$$\begin{cases} T_2 = -t, \\ G_2 = \delta_{\tilde{g}}(g), \end{cases}$$

which corresponds to inserting $G_2 = \delta_{\tilde{g}}(g)$ (the divergence gauge) into (II).

We will be concerned with both distinct choices of gauge terms here. In the case $(T, G) = (T_1, G_1)$, the principal symbols of the system (III) are simple and ellipticity can be proved by straightforward computation. However, such a system is not self-adjoint, which makes it not suitable for the proof of the manifold theorem in §2.5. On the other hand, the system (III) with $(T, G) = (T_2, G_2)$ is formally self-adjoint, whereas its principal symbols are much more complicated. We will use the ellipticity result of the case $(T, G) = (T_1, G_1)$ to prove the ellipticity for the gauge (T_2, G_2). We refer to §2.4 for more details.

Since the boundary ∂S is not empty, it is necessary to include a boundary condition for the gauge term G. A convinient choice is

$$G = 0 \qquad on\ \partial S. \tag{2.4}$$

Next we will prove that, equipped with this boundary restriction, solutions to the gauged system (III) when $(T, G) = (T_2, G_2)$, correspond to solutions to the stationary vacuum system (II) modulo diffeomorphisms.

2.2 Moduli space of stationary vacuum spacetimes I

We begin by defining the following subsets of the space of stationary vacuum solutions.

Definition 2.3.

$$\begin{aligned}
&\mathbb{E} = \{(g, u, \phi) \in [Met_\delta^{m,\alpha} \times C_\delta^{m,\alpha} \times C_\delta^{m,\alpha}](S) : \text{solutions of } (II)\};\\
&\mathbb{E}_C = \{(g,u, \phi) \in [Met_\delta^{m,\alpha} \times C_\delta^{m,\alpha} \times C_\delta^{m,\alpha}](S) :\\
&\qquad\qquad\qquad\qquad \text{solutions of } (II) \text{ with } \delta_{\tilde{g}} g = 0 \text{ on } S\};\\
&\mathbf{Z}_C = \{(g,u, \phi) \in [Met_\delta^{m,\alpha} \times C_\delta^{m,\alpha} \times C_\delta^{m,\alpha}](S) :\\
&\qquad \text{solutions of } (III) \text{ with } (T, G) = (T_2, G_2) \text{ and boundary condition (2.4)}\}.
\end{aligned}$$

Obviously, $\mathbb{E}_C \subset Z_C$. The following lemma shows the converse is also true, i.e. $\mathbf{Z}_C \subset \mathbb{E}_C$.

Lemma 2.4. *Elements in $\mathbf{Z}_C$ are also in $\mathbb{E}_C$. As a consequence, $\mathbf{Z}_C = \mathbb{E}_C$.*

Proof. It suffices to prove the gauge term G_2 is zero in (III) under the boundary condition (2.4). From the equation (2.3) we know that, if $(T, G) = (T_2, G_2)$ then leading term of the first equation in (III) is the Einstein tensor Ein_g, where we have the Bianchi identity, $\delta_g Ein_g = 0$. Thus, taking the divergence (with respect to g) of the first equation in (III), we obtain

$$\delta_g\{Ein_g - 2du \otimes du - 2e^{-4u} d\phi \otimes d\phi + (|du|^2 + e^{-4u}|d\phi|^2)g + \delta^* G_2\} = 0,$$

which gives,

$$\delta_g\{-2du \otimes du - 2e^{-4u} d\phi \otimes d\phi + (|du|^2 + e^{-4u}|d\phi|^2)g\} + \delta\delta^* G_2 = 0.$$

Basic computation gives

$$\begin{aligned}
&\delta\{-2du \otimes du - 2e^{-4u} d\phi \otimes d\phi + (|du|^2 + e^{-4u}|d\phi|^2)g\}\\
&= -2(\Delta_g u)du - 8e^{-4u}\langle du, d\phi\rangle d\phi - 2e^{-4u}(\Delta_g \phi)d\phi + 4e^{-4u} du|d\phi|^2.
\end{aligned}$$

Together with the equations: $\Delta_g u - 2e^{-4u}|d\phi|^2 = 0$ and $\Delta_g \phi + 4\langle du, d\phi\rangle = 0$ from (III), it is easy to see that the expression above is equal to zero, and consequently

$$\delta\delta^* G_2 = 0.$$

Thus we obtain the following system for G_2,

$$\begin{cases} \delta\delta^* G_2 = 0 & on\ \ S, \\ G_2 = 0 & on\ \ \partial S. \end{cases}$$

Integration by parts gives:

$$0 = \int_S \langle \delta\delta^* G_2, G_2 \rangle = \int_S |\delta^* G_2|^2 - \int_{\partial S} \delta^* G_2(\mathbf{n}, G_2) - \int_{\partial S_\infty} \delta^* G_2(\mathbf{n}, G_2).$$

Here the finite boundary term must vanish because $G_2 = 0$ on ∂S. Basic computation shows that the term $\delta^* G_2(\mathbf{n}, G_2)$ decays at the rate $r^{-2\delta-2}$, so the boundary term at infinity is also zero. It follows that $\delta^* G_2 = 0$ in S. Thus, G_2 is a Killing field vanishing on ∂S, and hence $G_2 = 0$. □

Remark. The lemma above shows that adding the divergence gauge to the system (II) preserves the stationary vacuum property of the solutions. In contrast, it is unknown in general whether adding the Bianchi gauge to system (II) will work in the same way. In the case $(T, G) = (T_1, G_1)$, the leading term in the first equation of (III) is the Ricci tensor Ric_g. Thus instead of taking divergence as in the lemma 2.4, one needs to apply the Bianchi operator to the first equation, which yields $\beta\delta^* G_1 = 0$. The operator $\beta\delta^*$ is not positive in general, so the argument above does not apply to the Bianchi gauge.

Next, we will show $\mathbb{E} = \mathbf{Z}_C$ in the sense of moduli space.

First define a Banach space $\mathcal{X}_\delta^{m,\alpha}(S)$ of asymptotically flat vector fields vanishing on ∂S:

$$\mathcal{X}_\delta^{m,\alpha}(S) = \{\text{vector fields } X \text{ on } S : X^i \in C_\delta^{m,\alpha}(S) \text{ and } X = 0 \text{ on } \partial S\}.$$

Then the following lemma holds for the space $\mathcal{X}_\delta^{m,\alpha}(S)$.

Lemma 2.5. *The map $\delta\delta^* : \mathcal{X}_\delta^{m,\alpha}(S) \to (\Lambda^1)_{\delta+2}^{m-2,\alpha}(S)$ is an isomorphism.*

Proof. From the proof of the previous lemma, one sees that kernel of $\delta\delta^* : \mathcal{X}_\delta^{m,\alpha}(S) \to (\Lambda^1)_{\delta+2}^{m-2,\alpha}(S)$ is zero. On the other hand, $\delta\delta^*$ is an elliptic operator with Fredholm index 0, thus it is an isomorphism. □

Next, let $\mathcal{D}_0^{m+1,\alpha}(S)$ be the group of $C_\delta^{m+1,\alpha}$ diffeomorphisms of S which equal to the identity map on ∂S. These are diffeomorphisms decaying asymptotically to the identity at the rate $r^{-\delta}$. The group $\mathcal{D}_0^{m+1,\alpha}(S)$ acts freely and continuously on $Met_\delta^{m,\alpha}(S)$ by pull back and one has the following local result.

Theorem 2.6. *Given any $g \in Met_\delta^{m,\alpha}(S)$ near $\tilde{g}$, there is a unique diffeomorphism $\Psi \in \mathcal{D}_0^{m+1,\alpha}(S)$ near the identity map Id such that the pull back metric $\Psi^* g$ satisfies the divergence gauge condition $\delta_{\tilde{g}}(\Psi^* g) = 0$.*

Proof. : Define a map $\mathcal{F}$ as follows,

$$\mathcal{F} : [\mathcal{D}_0^{m+1,\alpha} \times Met_\delta^{m,\alpha}](S) \to (\Lambda^1)_{\delta+1}^{m-1,\alpha}(S),$$

$$\mathcal{F}(\Psi, g) = \delta_{\tilde{g}}(\Psi^*(g)).$$

Linearization of $\mathcal{F}$ at $(Id, \tilde{g})$ with respect to (X, h) is given by

$$D_0\mathcal{F}(X, h) = \delta\delta^* X + \delta(h).$$

Here $X \in \mathcal{X}_\delta^{m+1,\alpha}(S)$, and hence $\delta\delta^*$ is an isomorphism by the previous lemma. According to the inverse function theorem, for any g in a neigbourhood of $\tilde{g}$, there exists a unique Ψ near Id such that $F(\Psi, g) = 0$, which proves the theorem. □

Now define the moduli space $\mathcal{E}_C = \mathcal{E}_C^{m,\alpha}$ to be the quotient of the space $\mathbb{E}$ by the diffeomorphism group:

$$\mathcal{E}_C = \mathbb{E}/\mathcal{D}_0^{m+1,\alpha}(S).$$

By Lemma 2.4, any element of $\mathbf{Z}_C$ is in one of the equivalence classes in $\mathcal{E}_C$. Conversely, given stationary vacuum data (g, u, ϕ) near $\tilde{g}$, according to the theorem above, one can choose a unique diffeomorphism $\Psi \in \mathcal{D}_0^{m+1,\alpha}(S)$ near Id so that $\delta_{\tilde{g}}(\Psi^* g) = 0$, i.e. the pull back data $\Psi^*(g, u, \phi)$ belongs to $\mathbf{Z}_C$. Therefore, locally elements in the set $\mathbf{Z}_C$ near $\tilde{g}$ are in 1-1 correspondence to equivalence classes in the moduli space $\mathcal{E}_C$ near $[\tilde{g}]$. Therefore, $\mathbf{Z}_C = \mathcal{E}_C$ locally near $\tilde{g}$.

2.3 Boundary value problem in the projection formalism

As in the Introduction, we pose a geometrically natural collection of boundary conditions on ∂S:

$$\begin{cases} g_S^T = \gamma, \\ H_{g_S} = \lambda, \\ \mathbf{n}_{g_S}(\phi) = f, \end{cases} \tag{2.5}$$

where $\gamma \in Met^{m,\alpha}(\partial S)$ is a fixed metric of the surface ∂S; and $\lambda, f \in C^{m-1,\alpha}(\partial S)$ are prescribed functions on ∂S. Here and in the following sections, we use g_S^T to denote the metric on the boundary ∂S induced by g_S, H_{g_S} the mean curvature of the boundary $\partial S \subset (S, g_S)$, and $\mathbf{n}_{g_S}$ the unit normal vector field of $\partial S \subset (S, g_S)$. Under the conformal transformation (2.2), these tensor fields are transformed as,

$$g_S^T = e^{-2u} g^T,\ H_{g_S} = e^u (H_g - 2\mathbf{n}_g(u)),\ \mathbf{n}_{g_S} = e^u \mathbf{n}_g.$$

Thus one can translate the boundary conditions (2.5) to the following for the data (g, u, ω),

$$\begin{cases} e^{-2u} g^T = \gamma, \\ H_g - 2\mathbf{n}_g(u) = e^{-u}\lambda, \qquad \text{on } \partial S. \\ \mathbf{n}_g(\phi) = e^{-u} f, \end{cases} \tag{2.6}$$

Pairing these boundary conditions with the gauged vacuum field equations (III), we obtain the following boundary value problem,

$$\begin{cases} Ric_g - 2du \otimes du - 2e^{-4u} d\phi \otimes d\phi + T + \delta^* G = 0 \\ \Delta_g u - 2e^{-4u}|d\phi|^2 = 0 \qquad\qquad \text{on } S, \\ \Delta_g \phi + 4 < du, d\phi >= 0 \end{cases}$$
$$\begin{cases} G = 0 \\ e^{-2u} g^T = \gamma \\ H - 2\mathbf{n}(u) = e^{-u}\lambda \qquad \text{on } \partial S. \\ \mathbf{n}(\phi) = e^{-u} f \end{cases} \tag{2.7}$$

The main step to prove Theorem 1.1 is verifying that the boundary value problem above is elliptic. To do this, we define a differential operator $\mathcal{P} = (\mathcal{L}, \mathcal{B})$ based on it, where

$\mathcal{L}$ denotes the interior operator and $\mathcal{B}$ the boundary operator. The interior operator $\mathcal{L}$, mapping the data (g, u, ϕ) to the interior equations of (2.7), is defined as follows,

$$\begin{aligned}
\mathcal{L} : [Met_\delta^{m,\alpha} \times C_\delta^{m,\alpha} \times C_\delta^{m,\alpha}](S) &\to [(S_2)_{\delta+2}^{m-2,\alpha} \times C_{\delta+2}^{m-2,\alpha} \times C_{\delta+2}^{m-2,\alpha}](S),\\
\mathcal{L}(g, u, \phi) = \{ \ &2(Ric_g - 2du \otimes du - 2e^{-4u} d\phi \otimes d\phi + \delta^* G + T),\\
&8(\Delta_g u - 2e^{-4u}|d\phi|^2),\\
&8e^{-4u}(\Delta_g \phi + 4\langle du, d\phi\rangle) \ \}.
\end{aligned}$$

Here S_2 denotes the bundle of symmetric 2-tensors and $(S_2)_\delta^{m,\alpha}$ is the space of $C^{m,\alpha}$ asymptotically flat symmetric 2-tensors which is defined similarly as the spaces of tensor fields in Definition 2.1. The extra scalar factors 2, 8 and function $8e^{-4u}$ are for later use when proving self-adjointness below. They do not affect the ellipticity, but are necessary for the self-adjointness of the operator.

The boundary operator $\mathcal{B}$, mapping the data (g, u, ϕ) to the boundary terms in (2.7), is given by,

$$\begin{aligned}
\mathcal{B} : \ &[Met_\delta^{m,\alpha} \times C_\delta^{m,\alpha} \times C_\delta^{m,\alpha}](S)\\
&\qquad \to [(T_1^{m-1,\alpha} \times C^{m-1,\alpha}) \times S_2^{m,\alpha} \times C^{m-1,\alpha} \times C^{m-1,\alpha}](\partial S),\\
\mathcal{B}(g, u, \phi) = \{ \ &G,\\
&e^{-2u} g^T - \gamma,\\
&H - 2\mathbf{n}(u) - e^{-u}\lambda,\\
&\mathbf{n}(\phi) - e^{-u} f \ \},
\end{aligned}$$

where we write $G \in [T_1^{m-1,\alpha} \times C^{m-1,\alpha}](\partial S)$ because the gauge term G is a 1-tensor on S, and when restricted to ∂S, it induces a tangential 1-tensor G^T and a $C^{m-1,\alpha}$ function $G(\mathbf{n})$ on ∂S. For simplicity of notation, we will use $\mathbf{B}^{m,\alpha}(S)$ to denote the target space of $\mathcal{B}$, i.e.

$$\mathbf{B}^{m,\alpha}(S) = [(T_1^{m-1,\alpha} \times C^{m-1,\alpha}) \times S_2^{m,\alpha} \times C^{m-1,\alpha} \times C^{m-1,\alpha}](\partial S).$$

In the following, $\mathcal{P}$ will be written as $\mathcal{P}_1 = (\mathcal{L}_1, \mathcal{B}_1)$ if the gauge terms in $\mathcal{L}, \mathcal{B}$ correspond to the Bianchi gauge, and $\mathcal{P}_2 = (\mathcal{L}_2, \mathcal{B}_2)$ if the divergence gauge is applied.

Let (g, u, ϕ) be a fixed element in the zero set $\mathcal{P}^{-1}(0)$, and choose $\tilde{g} = g$ in the gauge term. The linearization of $\mathcal{P}$ at (g, u, ϕ) is given by

$$D\mathcal{P}(h, v, \sigma) = (D\mathcal{L}(h, v, \sigma),\ D\mathcal{B}(h, v, \sigma)),$$

where (h, v, σ) is an infinitesimal deformation of the data (g, u, ϕ), and $D\mathcal{L}$, $D\mathcal{B}$ are linearizations of the operators $\mathcal{L}$ and $\mathcal{B}$, expressed as follows,

$$\begin{aligned}
D\mathcal{L} : \ [(S_2)_\delta^{m,\alpha} \times C_\delta^{m,\alpha} \times C_\delta^{m,\alpha}](S) &\to [(S_2)_{\delta+2}^{m-2,\alpha} \times C_{\delta+2}^{m-2,\alpha} \times C_{\delta+2}^{m-2,\alpha}](S),\\
D\mathcal{L}(h, v, \sigma) = \{ \ &D^* D h - Z(h) + O_1,\\
&8\Delta_g v + O_1,\\
&8e^{-4u}(\Delta_g \sigma + 4\langle du, d\sigma\rangle) + O_0 \ \},
\end{aligned}$$

and

$$\begin{aligned} D\mathcal{B}:\ & [(S_2)_\delta^{m,\alpha}\times C_\delta^{m,\alpha}\times C_\delta^{m,\alpha}](S)\to \mathbf{B}^{m,\alpha}(S),\\ & D\mathcal{B}(h,v,\sigma)=\{\ G'_h,\\ & \qquad e^{-2u}(-2vg+h)|_{\partial S},\\ & \qquad H'_h-2\mathbf{n}(v)+O_0,\\ & \qquad \mathbf{n}(\sigma)+O_0\ \}. \end{aligned}$$

In the expression above, $D^*Dh=-\nabla^i\nabla_i h$. The terms Z and G'_h depend on the choice of gauge terms. They are of the form

$$\begin{cases} Z_1(h)=0,\\ (G_1)'_h=\beta_{\tilde g}h, \end{cases} \tag{2.8}$$

when the Bianchi gauge is choosen, and

$$\begin{cases} Z_2(h)=D^2(trh)+\Delta_g(trh)g+(\delta\delta h)g,\\ (G_2)'_h=\delta_{\tilde g}h, \end{cases} \tag{2.9}$$

when the divergence gauge is used.
The expressions O_1 and O_0 stand for terms which involve the derivative of (h,v,σ) with order not higher than 1 and 0. In the linearization $D\mathcal{B}$, the term H'_h denotes the variation of the mean curvature. We refer to §4.1 for the details of the calculation.

Since ellipticity only depends on the principal part of the operator, we can remove the lower order terms O_1 and O_0 in $D\mathcal{P}$ and study the simplified operator $P(h,v,\sigma)=(L(h,v,\sigma),B(h,v,\sigma))$, where L and B are as follows:

$$\begin{aligned} L:\ & [(S_2)_\delta^{m,\alpha}\times C_\delta^{m,\alpha}\times C_\delta^{m,\alpha}](S)\to[(S_2)_{\delta+2}^{m-2,\alpha}\times C_{\delta+2}^{m-2,\alpha}\times C_{\delta+2}^{m-2,\alpha}](S),\\ & L(h,v,\sigma)=\{\ D^*Dh-Z(h),\\ & \qquad 8\Delta_g v,\\ & \qquad 8e^{-4u}(\Delta_g\sigma+4\langle du,d\sigma\rangle)\ \}, \end{aligned}$$

and

$$\begin{aligned} B:\ & [(S_2)_\delta^{m,\alpha}\times C_\delta^{m,\alpha}\times C_\delta^{m,\alpha}](S)\to \mathbf{B}^{m,\alpha}(S),\\ & B(h,v,\sigma)=\{\ G'_h,\\ & \qquad e^{-2u}(-2vg+h)|_{\partial S},\\ & \qquad H'_h-2\mathbf{n}(v),\\ & \qquad \mathbf{n}(\sigma)\ \}. \end{aligned}$$

In the last component of L, we keep the lower order term $4\langle du,d\sigma\rangle$ for the purpose of self-adjointness discussed later; again this does not affect the ellipticity.

In the following section, if the pair (Z,G'_h) takes the values in (2.8), the operator P will be denoted by $P_1=(L_1,B_1)$; and if equipped with the divergence gauge (2.9), P will be written as $P_2=(L_2,B_2)$. We will prove the ellipticity of both operators P_1 and P_2. As a consequence, the boundary value problem (2.7) is elliptic with respect to both choices of gauges.

2.4 Ellipticity of the BVP I

Throughout this section, we use ξ to denote a generic $1-$form on S, η to denote a nonzero $1-$form tangent to the boundary ∂S, i.e. $\eta(\mathbf{n}) = 0$, and μ a nonzero $1-$form normal to the boundary ∂S, i.e. $\mu^T = 0$.

To check ellipticity, we will follow [ADN]: Let $P = (L, B)$ be a differential operator consisting of an interior operator L and a boundary operator B. Denote the matrix of principal symbols of the interior operator at ξ as $L(\xi)$ and the matrix of principal symbols of the boundary operator as $B(\xi)$. The operator P forms an elliptic boundary value problem if and only if the following two conditions hold.
(A) (properly elliptic condition): determinant $l(\xi)$ of $L(\xi)$ has no nontrivial real root;
(B) (complementing boundary condition): Take the adjoint matrix $L^*(\xi)$ of $L(\xi)$. Let $\xi = (\eta + z\mu)$. The rows of $B \cdot L^*(\eta + z\mu)$ are linearly independent modulo $l^+(z)$, where $l^+(z) = \prod(z - z_k)$ and $\{z_k\}$ are the roots of $l(\eta + z\mu) = 0$ having positive imaginary parts.

2.4.1 Ellipticity with the Bianchi gauge

Theorem 2.7. *P_1 is an elliptic operator.*

Proof. It is easy to observe that the matrix of principal symbols for L_1 at ξ is given by

$$L_1(\xi) = \begin{bmatrix} |\xi|^2 I_{6\times 6} & 0 & 0 \\ 0 & 8|\xi|^2 & 0 \\ 0 & 0 & 8e^{-4u}|\xi|^2 \end{bmatrix}.$$

The adjoint matrix of $L(\xi)$ is then given by

$$L_1^*(\xi) = \begin{bmatrix} 64e^{-4u}|\xi|^{14} I_{6\times 6} & 0 & 0 \\ 0 & 8e^{-4u}|\xi|^{14} & 0 \\ 0 & 0 & 8|\xi|^{14} \end{bmatrix}.$$

The determinant of $L_1(\xi)$ is $l(\xi) = 64e^{-4u}|\xi|^{16}$. So it is obvious that the interior operator is properly elliptic.
The root of $l(\eta + z\mu)$ with positive imaginary part is $z = i|\eta|$, and this implies

$$l^+(z) = (z - i|\eta|)^8.$$

Let C be a general vector in $\mathbb{C}^8$. The complementing boundary condition holds if the equation below has no nontrivial solution in $\mathbb{C}^8$:

$$C \cdot B_1(\eta + z\mu) \cdot L_1^*(\eta + z\mu) = 0 \pmod{l^+(z)}. \tag{2.10}$$

One sees easily that equation (3.1) is equivalent to the following,

$$(z - i|\eta|) \;\;|\;\; C \cdot B_1(\eta + z\mu) \cdot \begin{bmatrix} 64e^{-4u} I_{6\times 6} & 0 & 0 \\ 0 & 8e^{-4u} & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

And furthermore, this holds if and only if the following is true,

$$C \cdot B_1(\eta + i|\eta|\mu) \cdot \begin{bmatrix} 64e^{-4u} I_{6\times 6} & 0 & 0 \\ 0 & 8e^{-4u} & 0 \\ 0 & 0 & 8 \end{bmatrix} = 0.$$

So to prove the condition (B), it suffices to prove that the matrix of principal symbol $B_1(\xi)$, has trivial kernel, when $\xi = \eta + i|\eta|\mu$. In the following, the subscript 0 represents the direction normal to ∂S, while indices $1, 2$ represent the directions tangent to ∂S. Write the nonzero tangential 1-form $\eta = (\eta_1, \eta_2)$, so the root of the interior operator is $\xi = (i|\eta|, \eta_1, \eta_2)$. Basic computation (cf.§4.1) shows that the principal symbols of the boundary operator B_1 are given by,

$$|\eta|h_{00} - i\eta_1 h_{10} - i\eta_2 h_{20} - \frac{1}{2}|\eta|(h_{00} + h_{11} + h_{22}) = 0 \tag{2.11}$$

$$|\eta|h_{01} - i\eta_1 h_{11} - i\eta_2 h_{21} + \frac{1}{2}i\eta_1(h_{00} + h_{11} + h_{22}) = 0 \tag{2.12}$$

$$|\eta|h_{02} - i\eta_1 h_{12} - i\eta_2 h_{22} + \frac{1}{2}i\eta_2(h_{00} + h_{11} + h_{22}) = 0 \tag{2.13}$$

$$-2v + h_{11} = 0 \tag{2.14}$$

$$h_{12} = 0 \tag{2.15}$$

$$-2v + h_{22} = 0 \tag{2.16}$$

$$-\frac{1}{2}|\eta|(h_{11} + h_{22}) - i\eta_1 h_{10} - i\eta_2 h_{20} + 2|\eta|v = 0 \tag{2.17}$$

$$-|\eta|\sigma = 0 \tag{2.18}$$

According to equations (2.14),(2.15) and (2.16), we can replace h_{11} and h_{22} by $2v$ and h_{21} by 0. Then equation (2.17) gives

$$2|\eta|v - (i\eta_1 h_{10} + i\eta_2 h_{20}) - 2|\eta|v = 0,$$

i.e.

$$(i\eta_1 h_{10} + i\eta_2 h_{20}) = 0.$$

Equation (2.11) gives:

$$\frac{1}{2}|\eta|h_{00} - (i\eta_1 h_{10} + i\eta_2 h_{20}) - 2|\eta|v = 0.$$

It follows that

$$h_{00} = 4v.$$

Multiplying (2.12) by $(i\eta_1)$ and (2.13) by $(i\eta_2)$, and then summing gives:

$$2|\eta|^2 v + 4|\eta|^2 v = 0.$$

Thus $v = 0$ and consequently $h_{ij} = 0$ for all $0 \leq i, j \leq 2$.
Finally, it's obvious from equation (2.18) that $\sigma = 0$. This completes the proof. □

Remark. One can see from the proof above that principal symbols of the operator P_1 are simple so that ellipticity follows from a direct verification of the conditions (A) and (B). However, in the divergence-gauge case, principal symbols of the operator P_2 are too complicated for us to carry out the same computation as above. In the following, we will use an intermediate operator which has Bianchi gauge term G_1 in the interior and divergence gauge G_2 on the boundary, to prove the ellipticity of the operator P_2.

2.4.2 ellipticity for the divergence gauge

We begin with the following lemma:

Lemma 2.8. *If we replace $(G'_1)_h$ by $(G'_2)_h$ in the boundary part B_1 of P_1, the operator is still elliptic.*

Proof. This can be proved by a slight modification of the previous proof. After changing $\beta(h)$ to $\delta(h)$ (they only differ by a trace term) in the boundary operator, the new principal symbols of the boundary operator at $\xi = (i|\eta|, \eta_1, \eta_2)$ become

$$|\eta|h_{00} - i\eta_1 h_{10} - i\eta_2 h_{20} = 0 \tag{2.19}$$
$$|\eta|h_{01} - i\eta_1 h_{11} - i\eta_2 h_{21} = 0 \tag{2.20}$$
$$|\eta|h_{02} - i\eta_1 h_{12} - i\eta_2 h_{22} = 0 \tag{2.21}$$
$$-2v + h_{11} = 0 \tag{2.22}$$
$$h_{12} = 0 \tag{2.23}$$
$$-2v + h_{22} = 0 \tag{2.24}$$
$$-\frac{1}{2}|\eta|(h_{11} + h_{22}) - i\eta_1 h_{10} - i\eta_2 h_{20} + 2|\eta|v = 0 \tag{2.25}$$
$$-|\eta|\sigma = 0 \tag{2.26}$$

By equations (2.22),(2.23) and (2.24), we can replace h_{11} and h_{22} by $2v$ and h_{21} by 0. Then (2.25) gives

$$2|\eta|v - (i\eta_1 h_{10} + i\eta_2 h_{20}) - 2|\eta|v = 0,$$

i.e.

$$(i\eta_1 h_{10} + i\eta_2 h_{20}) = 0.$$

Equation (2.19) gives: $h_{00} = 0$.
Multiplying (2.20) by $(i\eta_1)$ and (2.21) by $(i\eta_2)$, and then summing gives

$$2|\eta|^2 v = 0.$$

Thus $v = 0$ and consequently $h_{ij} = 0$ for all $0 \leq i, j \leq 2$. □

Next, we follow the idea in $[AK]$ to prove ellipticity for the operator P_2.

Theorem 2.9. *The operator P_2 is elliptic.*

Proof. The ellipticity of a general operator $P = (L, B)$ is equivalent to the existence of a uniform estimate:

$$||(h, v, \sigma)||_{C^{m,\alpha}} \leq C(||L(h, v, \sigma)||_{C^{m-k,\alpha}} + ||B(h, v, \sigma)||_{C^{m-j,\alpha}} + ||(h, v, \sigma)||_{C^0}), \tag{2.27}$$

together with such an estimate for the adjoint operator. In the expression above, k and j denote the order of derivative in the principal parts of the operators L and B.

The operator P_2 is then elliptic as a consequence of the following two facts, which are proved in Lemma 2.10 and Proposition 2.11 below.
(1) The inequality (2.27) holds for P_2;
(2) The operator P_2 is formally self-adjoint. □

Lemma 2.10. *Inequality* (2.27) *holds for* P_2*, i.e.*

$$||(h,v,\sigma)||_{C^{m,\alpha}} \leq C(||L_2(h,v,\sigma)||_{C^{m-2,\alpha}} + ||B_2(h,v,\sigma)||_{C^{m-j,\alpha}} + ||(h,v,\sigma)||_{C^0}). \quad (2.28)$$

Proof. By Lemma 2.8, the inequality (2.28) must hold if L_2 is replaced by L_1, i.e.

$$||(h,v,\sigma)||_{C^{m,\alpha}} \leq C(||L_1(h,v,\sigma)||_{C^{m-2,\alpha}} + ||B_2(h,v,\sigma)||_{C^{m-j,\alpha}} + ||(h,v,\sigma)||_{C^0}).$$

Observe that $L_1(h,v,\sigma) = L_2(h,v,\sigma) + (D^2(trh) + \Delta_g(trh)g + \delta\delta hg, 0, 0)$. So by the interpolation inequality, $||h||_{C^{m-1,\alpha}} \leq \epsilon||h||_{C^{m,\alpha}} + \epsilon^{-1}||h||_{C^0}$, it suffices to prove

$$||\delta h||_{C^{m-1,\alpha}} \leq C(||L_2(h,v,\sigma)||_{C^{m-2,\alpha}} + ||B_2(h,v,\sigma)||_{C^{m-j,\alpha}} + ||(h,v,\sigma)||_{C^0}, \quad (2.29)$$

and

$$||D^2trh||_{C^{m-2,\alpha}} \leq C(||L_2(h,v,\sigma)||_{C^{m-2,\alpha}} + ||B_2(h,v,\sigma)||_{C^{m-j,\alpha}} + ||(h,v,\sigma)||_{C^0}). \quad (2.30)$$

First notice that, $L_2(h)$ is the $2^{nd}-$order part of the linearization of the map:

$$\Phi(g) = Ric_g + \delta^* G_2 + T_2 - 2du \otimes du - 2e^{-4u} d\phi \otimes d\phi.$$

From the proof of Lemma 2.4, one sees that

$$\delta\Phi(g) + 2(\Delta_g u - 2e^{-4u}|d\phi|^2)du + 2e^{-4u}(\Delta_g\phi + 4\langle du, d\phi\rangle)d\phi = \delta\delta^*\delta_{\tilde{g}}g.$$

Assume g is a zero of Φ. Linearizing the above equation at $\tilde{g} = g$ with respect to h gives

$$\delta D\Phi(h) + O_0 = \delta\delta^*\delta(h),$$

where O_0 denotes terms of 0-derivative order with respect to h. It is of derivative order 3 on the right hand side of the equation above, so the left hand side must be also of order 3, hence we obtain,

$$\delta L_2(h) = \delta\delta^*\delta(h).$$

The operator $\delta\delta^*$ is elliptic with respect to Dirichlet boundary conditions, and δh is included in the boundary operator B_2. Thus inequality (2.29) holds.

To prove inequality (2.30), we use the Gauss equation at ∂S :

$$|A_g|^2 - H_g + s_{g^T} = s_g - 2Ric_g(\mathbf{n},\mathbf{n}),$$

where A_g is the second fundamental form of $\partial M \subset (M,g)$ and s_{g^T} is the scalar curvature of the metric g^T on ∂M. It follows that,

$$(|A_g|^2 - H_g + s_{g^T})'_h = -L_2(\mathbf{n},\mathbf{n}) + 2\delta^*\delta(h) + O_1,$$

where O_1 denotes terms of derivative order no higher than 1 with respect to h. Observe that $s'_{g^T} = \Delta_{g^T}(trh^T) + \delta\delta(h^T) + O_1$ and the terms A'_h, H'_h only involve first order derivatives in h so they can be ignored according to the interpolation inequality. Writing $h^T = B_{2,0} + 2vg^T$, where $B_{2,0} = h^T - 2vg^T$ is one of the boundary conditions for P_2, it following that

$$s'_{g^T} = \Delta_{g^T}(trh^T - 2v) + \delta\delta(B_{2,0}) + O_1$$

Therefore, by the ellipticity of the Laplace operator on ∂S, and together with (2.29) being true, we obtain the estimate for $(trh^T - 2v)$ along ∂S:

$$||(trh^T - 2v)|_{\partial S}||_{C^{m,\alpha}} \leq C(||L_2(h,v,\sigma)||_{C^{m-2,\alpha}} + ||B_2(h,v,\sigma)||_{C^{m-j,\alpha}} + ||h||_{C^0}). \quad (2.31)$$

Since the term $(h^T - 2vg^T)$ is included in the boundary operator, $tr(h^T - 2vg^T) = (trh^T - 4v)|_{\partial S}$ is also controlled. Comparing with (2.31), we obtain the control for v on ∂S,

$$||v|_{\partial S}||_{C^{m,\alpha}} \leq C(||L_2(h,v,\sigma)||_{C^{m-2,\alpha}} + ||B_2(h,v,\sigma)||_{C^{m-j,\alpha}} + ||h||_{C^0}).$$

In addition, $\Delta_g v$ is one of the components of the interior operator L_2, so from the ellipticity of Laplace operator with Dirichlet boundary condition, we obtain the uniform estimate for v over S,

$$||v||_{C^{m,\alpha}} \leq C(||L_2(h,v,\sigma)||_{C^{m-2,\alpha}} + ||B_2(h,v,\sigma)||_{C^{m-j,\alpha}} + ||(h,v,\sigma)||_{C^0}).$$

Furthermore, observe that $\Delta_g \sigma$ is the last component of the interior operator L_2 and $\mathbf{n}(\sigma)$ is one of the boundary terms in B_2. Thus, based on the ellipticity of Laplace operator with Neumann boundary condition, we also have the uniform estimate for σ over S:

$$||\sigma||_{C^{m,\alpha}} \leq C(||L_2(h,v,\sigma)||_{C^{m-2,\alpha}} + ||B_2(h,v,\sigma)||_{C^{m-j,\alpha}} + ||(h,v,\sigma)||_{C^0}).$$

Now with v, σ being well controlled, inequality (2.30) is equivalent to the following inequality,

$$\begin{aligned} ||D^2 trh||_{C^{m-2,\alpha}} \leq C(&||L_2(h)||_{C^{m-2,\alpha}} + ||\delta(h)|_{\partial S}||_{C^{m-1,\alpha}} + ||h^T|_{\partial S}||_{C^{m,\alpha}} \\ &+ ||H'_h|_{\partial S}||_{C^{m-1,\alpha}} + ||h||_{C^0}). \end{aligned}$$

This estimate is proved in Lemma 3.2 of $[AK]$, and this completes the proof of the uniform estimate. ☐

Proposition 2.11. *Let $\mathcal{M}_2$ be the space of data (h,v,σ) on S in the kernel of the boundary operator B_2, i.e.*

$$\begin{aligned} \mathcal{M}_2 = \{ \quad (h,v,\sigma) \in &[(S_2)^{m,\alpha}_\delta \times C^{m,\alpha}_\delta \times C^{m,\alpha}_\delta](S) : \\ &\begin{cases} \delta_g(h) = 0, \\ h^T - 2vg^T = 0, \\ H'_h - 2\mathbf{n}(v) = 0, \\ \mathbf{n}(\sigma) = 0, \end{cases} \quad on\ \partial S \quad \}. \end{aligned} \quad (2.32)$$

Then the operator $L_2 : \mathcal{M}_2 \to (S_2)^{m-2,\alpha}_\delta \times C^{m-2,\alpha}_\delta \times C^{m-2,\alpha}_\delta](S)$, given by

$$\begin{aligned} L_2(h,v,w) &= \{\ D^*Dh - Z_2(h), \quad 8\Delta_g v, \quad 8e^{-4u}[\Delta_g\sigma + 4\langle du, d\sigma\rangle]\ \} \\ &= \{\ D^*Dh - D^2(trh) - \Delta_g(trh)g - (\delta\delta h)g, \quad 8\Delta_g v, \quad 8e^{-4u}[\Delta_g\sigma + 4\langle du, d\sigma\rangle]\ \}, \end{aligned}$$

is formally self-adjoint.

Proof. We will prove this proposition by showing that L_2 arises as the 2[nd] variation of a natural variational problem on the data (g,u,ϕ).

To begin, the Einstein equation $Ein_{g^{(4)}} = 0$ is the functional derivative of the Einstein-Hilbert action

$$I_{EH} = \int_{V^{(4)}} R_{g^{(4)}} dvol_{g^{(4)}}.$$

Reducing this action from 4-dimensional spacetime $V^{(4)}$ to the 3-dimensional quotient space S, one obtains the following functional on the data (g, u, ϕ), of which the Euler-Lagrange equations are exactly the field equations (II) in §2.1,

$$I_{eff} = \int_S s - 2|du|^2 - 2e^{-4u}|d\phi|^2 dvol_g.$$

We refer to [H1][H2] for further discussion of the action I_{eff}.

Since the boundary ∂S is nonempty, as is well known it is necessary to add boundary terms to the action such as Gibbons-Hawking boundary terms, cf.[GH]. The proper action with respect to the boundary data in our case is given by

$$I = \int_S s - 2|du|^2 - 2e^{-4u}|d\phi|^2 dvol_g + 2\int_{\partial S} H_g dvol_{g^T} + 16\pi m_{ADM}(g).$$

Next, let $(\mathbf{E}, \mathbf{F}, \mathbf{H})$ denote the expressions in the system (II), i.e.

$$\begin{aligned} \mathbf{E}[(g,u,\phi)] &= \frac{1}{2}(s - 2|du|^2 - 2e^{-4u}|d\phi|^2)g - Ric_g + 2(du)^2 + 2e^{-4u}(d\phi)^2, \\ \mathbf{F}[(g,u,\phi)] &= -4\Delta_g u + 8e^{-4u}|d\phi|^2, \\ \mathbf{H}[(g,u,\phi)] &= -4e^{-4u}(\Delta_g \phi + 4\langle du, d\phi\rangle). \end{aligned} \tag{2.33}$$

Then the variation of I with respect to g is given by

$$\begin{aligned} I'_g(h) = \int_S \langle \mathbf{E}, h\rangle dvol_g &+ \int_{\partial S} -\langle A, h\rangle + Htrh^T dvol_{g^T} \\ &- \int_{\partial S_\infty} \mathbf{n}(trh) + \delta h(\mathbf{n}) dvol_{\partial S_\infty} + 16\pi (m_{ADM}(g))'_h. \end{aligned} \tag{2.34}$$

We refer to §4.2 for the details of the computation. To abbreviate notation, we shall omit the volume form in the following.

Notice that the terms in the second line of the equation (2.34) can be removed, because we have

$$\int_{\partial S_\infty} \mathbf{n}(trh) + \delta h(\mathbf{n}) = 16\pi (m_{ADM}(g))'_h,$$

based on the definition of ADM mass and its variation, cf.[RT],[B1].

Basic computation shows the variations of I with respect to u and ϕ are of the form,

$$I'_u(v) = \int_S \langle \mathbf{F}, v\rangle + \int_{\partial S} -4\mathbf{n}(u)v \ + \int_{\partial S_\infty} -4\mathbf{n}(u)v, \tag{2.35}$$

and

$$I'_\phi(\sigma) = \int_S \langle \mathbf{H}, \sigma\rangle + \int_{\partial S} -4e^{-4u}\mathbf{n}(\phi)\sigma \ + \int_{\partial S_\infty} -4e^{-4u}\mathbf{n}(\phi)\sigma. \tag{2.36}$$

By simply checking the decay rate, one sees easily that the boundary terms at infinity in the expressions above are both zero.

Now let (g, u, ϕ) be a triple such that $(\mathbf{E}, \mathbf{F}, \mathbf{H})[(g, u, \phi)] = 0$, and take a 2-parameter varation of data $(g_{st}, u_{st}, \phi_{st}) = (g, u, \phi) + s(h, v, \sigma) + t(k, w, \zeta)$, with infinitesimal deformations $(h, v, \sigma), (k, w, \zeta) \in \mathcal{M}_2$.

Based on the boundary conditions in the expression (2.32), we have $h^T = 2vg^T$. The equation (2.34) then becomes:

$$I'_g(h) = \int_S \langle \mathbf{E}, h \rangle + \int_{\partial S} 2vH. \tag{2.37}$$

Take one more variation of the equation (2.37) with respect to k, and we obtain

$$I''_g(h, k) = \int_S \langle \mathbf{E}'_k, h \rangle + \int_{\partial S} 2vH'_k + 4vwH. \tag{2.38}$$

Similar operation of the equations (2.35) and (2.36) yields,

$$I''_u(v, w) = \int_S \langle \mathbf{F}'_w, v \rangle + \int_{\partial S} -4\mathbf{n}(w)v, \tag{2.39}$$

and

$$I''_\phi(\sigma, \zeta) = \int_S \langle \mathbf{H}'_\zeta, \sigma \rangle + \int_{\partial S} -4e^{-4u}\mathbf{n}(\zeta)\sigma. \tag{2.40}$$

From the symmetry of second variation, we know that $I''(h, k) = I''(k, h)$, $I''(w, v) = I''(v, w)$ and $I''(\sigma, \zeta) = I''(\zeta, \sigma)$. The equations $(2.38 - 40)$ then imply that:

$$\begin{aligned}&\int_S [\langle \mathbf{E}'_k, h \rangle + \langle \mathbf{F}'_w, v \rangle + \langle \mathbf{H}'_\zeta, \sigma \rangle] + \int_{\partial S} [2vH'_k + 4vwH - 4\mathbf{n}(w)v - 4e^{-4u}\mathbf{n}(\zeta)\sigma] \\ &= \int_S [\langle \mathbf{E}'_h, k \rangle + \langle \mathbf{F}'_v, w \rangle + \langle \mathbf{H}'_\sigma, \zeta \rangle] + \int_{\partial S} [2wH'_h + 4vwH - 4\mathbf{n}(v)w - 4e^{-4u}\mathbf{n}(\sigma)\zeta].\end{aligned}$$

By the boundary condition in (2.32), $H'_h - 2\mathbf{n}(v) = 0$, $\mathbf{n}(\sigma) = 0$ and the same for (k, w, ζ). Thus we can remove the boundary terms above and obtain

$$\int_S [\langle \mathbf{E}'_k, h \rangle + \langle \mathbf{F}'_w, v \rangle + \langle \mathbf{H}'_\zeta, \sigma \rangle] = \int_S [\langle \mathbf{E}'_h, k \rangle + \langle \mathbf{F}'_v, w \rangle + \langle \mathbf{H}'_\sigma, \zeta \rangle]. \tag{2.41}$$

On the other hand, from the boundary condition $\delta h = \delta k = 0$, it follows that,

$$\int_S \langle \delta^* \delta k, h \rangle = \int_S \langle \delta k, \delta h \rangle = \int_S \langle \delta^* \delta h, k \rangle. \tag{2.42}$$

Combining equations (2.40) and (2.41), we obtain,

$$\int_S \langle (\mathbf{E}'_k - \delta^* \delta k, \mathbf{F}'_w, \mathbf{H}'_\zeta), (h, v, \sigma) \rangle = \int_S \langle (\mathbf{E}'_h - \delta^* \delta h, \mathbf{F}'_v, \mathbf{H}'_\sigma), (k, w, \zeta) \rangle. \tag{2.43}$$

Notice that the terms of second order and first order derivative in $(\mathbf{E}'_h - \delta^* \delta h, \mathbf{F}'_v, \mathbf{H}'_\alpha)$ are the same as in the operator $-\frac{1}{2}L_2$; and the zero order terms in the equation (2.43) can be removed because of symmetry. Therefore it follows that

$$\int_S \langle L_2(k, w, \zeta), (h, v, \sigma) \rangle = \int_S \langle L_2(h, v, \sigma), (k, w, \zeta) \rangle,$$

which proves the formal self-adjointness of the operator P_2. □

Ellipticity of the operator P_2 implies that the boundary value problem (2.7) with the divergence gauge is elliptic. Together with the local equivalence between the sets $\mathbf{Z}_C$ and $\mathcal{E}_C$ in §2.2, we conclude that the collection of boundary conditions (2.6) is elliptic for the stationary vacuum field equations (II) modulo diffeomorphisms in $\mathcal{D}_0^{m+1,\alpha}(S)$.

2.4.3 Back to g_S

It is now basically trivial to prove the ellipticity of the system (1.10) equipped with boundary conditions (1.11), using the result we have obtained.

First observe that, by combining the first and second equations in (1.10), the system is equivalent to the following one,

$$\begin{cases} Ric_{g_S} - D^2u - (du)^2 - 2e^{-4u}(d\phi \otimes d\phi - |d\phi|^2 g_S) \\ \qquad\qquad\qquad +(\Delta_{g_S} u - |du|^2 - 2e^{-4u}|d\phi|^2) g_S = 0, \\ \Delta_{g_S} u - |du|^2 - 2e^{-4u}|d\phi|^2 = 0, \\ \Delta_{g_S}\phi + 3\langle du, d\phi\rangle = 0. \end{cases} \tag{2.44}$$

The trace of the first equation above is given by

$$s_{g_S} + 4\Delta_{g_S} u - 4|du|^2 - 2e^{-4u}|d\phi|^2.$$

Denote T_S as the trace term

$$T_S = -\frac{1}{2}(s_{g_S} + 4\Delta_{g_S} u - 4|du|^2 - 2e^{-4u}|d\phi|^2) g_S,$$

and let G_S be the pull back by conformal transformation of the divergence gauge term $\delta_{\tilde{g}} g$, i.e.

$$G_S = \delta_{e^{2u}\tilde{g}_S}(e^{2u} g_S),$$

where $\tilde{g}_S$ is a reference metric near g_S.

Inserting $(T_S + \delta^*_{e^{2u}g_S} G_S)$ to the first equation in system (2.44), we obtain

$$\begin{cases} Ric_{g_S} - D^2u - (du)^2 - 2e^{-4u}(d\phi \otimes d\phi - |d\phi|^2 g_S) \\ \qquad +(\Delta_{g_S} u - |du|^2 - 2e^{-4u}|d\phi|^2) g_S + T_S + \delta^*_{e^{2u}g_S} G_S = 0, \\ \Delta_{g_S} u - |du|^2 - 2e^{-4u}|d\phi|^2 = 0, \\ \Delta_{g_S}\phi + 3\langle du, d\phi\rangle = 0. \end{cases} \tag{2.45}$$

According to the system above, we define a differential operator $\mathcal{P}_S = (\mathcal{L}_S, \mathcal{B}_S)$, which consists of the interior operator $\mathcal{L}_S$, mapping the data (g_S, u, ϕ) to the interior expressions in (2.45), given by

$$\begin{aligned} \mathcal{L}_S : [Met^{m,\alpha}_\delta \times C^{m,\alpha}_\delta \times C^{m,\alpha}_\delta](S) &\to [(S_2)^{m-2,\alpha}_{\delta+2} \times C^{m-2,\alpha}_{\delta+2} \times C^{m-2,\alpha}_{\delta+2}](S) \\ \mathcal{L}_S(g_S, u, \phi) = \{\ & 2[Ric_{g_S} - D^2u - (du)^2 - 2e^{-4u}(d\phi \otimes d\phi - |d\phi|^2 g_S) \\ & \qquad + (\Delta_{g_S} u - |du|^2 - 2e^{-4u}|d\phi|^2) g_S + T_S + \delta^*_{e^{2u}g_S} G_S], \\ & 8e^{-2u}(\Delta_{g_S} u - |du|^2 - 2e^{-4u}|d\phi|^2), \\ & 8e^{-6u}(\Delta_{g_S}\phi + 3\langle du, d\phi\rangle)\ \}; \end{aligned}$$

and the boundary operator $\mathcal{B}_S$, mapping the data (g_S, u, ϕ) to boundary data including the gauge term G_S and the terms in (1.11), given by

$$\begin{aligned} \mathcal{B}_S :\ & [Met^{m,\alpha}_\delta \times C^{m,\alpha}_\delta \times C^{m,\alpha}_\delta](S) \\ & \qquad \to [(T^{m-1,\alpha}_1 \times C^{m-1,\alpha}) \times S^{m,\alpha}_2 \times C^{m-1,\alpha} \times C^{m-1,\alpha}](\partial S), \\ \mathcal{B}(g, u, \omega) = \{\ & G_S, \quad g^T_S|_{\partial S} - \gamma, \quad H_{g_S} - \lambda, \quad \mathbf{n}_{g_S}(\phi) - f\ \}. \end{aligned}$$

In addition, define an operator $\mathcal{Q}$ as the conformal transformation in (2.2),

$$\mathcal{Q} : [Met^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta}](S) \to [Met^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta}](S)$$
$$\mathcal{Q}(g_S, u, \phi) = (e^{2u} g_S, u, \phi),$$

It is easy to see, by elementary computation, that the operator $\mathcal{P}_S$ is exactly the composition of $\mathcal{P}_2$ in §2.3 and $\mathcal{Q}$, i.e.

$$\mathcal{P}_S = \mathcal{P}_2 \circ \mathcal{Q}.$$

The operator $\mathcal{P}_2$ has already been proved to be elliptic and $\mathcal{Q}$ is obviously an isomorphism. As a consequence, the operator $\mathcal{P}_S$ is also elliptic. This gives the proof of Theorem 1.1.

□

In the following section, we will apply the ellipticity of $\mathcal{P}_2$ to prove the manifold theorem for the moduli space $\mathcal{E}_C$ of stationary vacuum spacetimes.

2.5 Manifold theorem for the moduli space

Throughout this section, $(\tilde{g}, \tilde{u}, \tilde{\phi})$ denotes a collection of the conformal data which solves (II). We start by defining the following Banach spaces.

Definition 2.12.

$$\mathcal{M}_S = \{\ (g, u, \phi) \in [Met^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta}](S) :$$
$$\begin{cases} \delta_{\tilde{g}} g = 0, \\ e^{-2u} g^T|_{\partial S} = \gamma \\ H - 2\mathbf{n}(u) = e^{-u}\lambda \\ \mathbf{n}(\phi) = e^{2u} f \end{cases} \quad \text{on } \partial S, \text{ for some fixed } \gamma,\ \lambda \text{ and } f.\ \};$$

$$\mathcal{M}_C = \{\ (g, u, \phi) \in [Met^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta} \times C^{m,\alpha}_{\delta}](S) : \quad \delta_{\tilde{g}} g = 0 \ on\ \partial S\ \}$$

In the definition of $\mathcal{M}_S$, we modify the previous boundary condition $\mathbf{n}(\phi) = e^{-u} f$ into $\mathbf{n}(\phi) = e^{2u} f$, to ensure that the operator $D\hat{\Phi}$ below is formally self-adjoint on the tangent space $T\mathcal{M}_S$. This does not affect the elliptic property of the operator.

Define a map:

$$\Phi :\ \mathcal{M}_C \to [(S_2)^{m-2,\alpha}_{\delta+2} \times C^{m-2,\alpha}_{\delta+2} \times C^{m-2,\alpha}_{\delta+2}](S)$$

$$\Phi(g, u, \phi) = (\ \mathbf{E} - \delta^* \delta_{\tilde{g}} g, \quad \mathbf{F}, \quad \mathbf{H}\ )$$

where the terms $\mathbf{E}, \mathbf{F}, \mathbf{H}$ are defined as in (2.33). Thus, the zero set $\Phi^{-1}(0)$ consists of stationary vacuum data (g, u, ϕ) satisfying $\delta_{\tilde{g}} g = 0$ on S, i.e.

$$\Phi^{-1}(0) = \mathbf{Z}_C,$$

where $\mathbf{Z}_C$ is as in Definition 2.3. Henceforth, based on the analysis in §2.2, to prove the moduli space $\mathcal{E}_C$ has the structure of a Banach manifold, it suffices to prove the zero set $\Phi^{-1}(0)$ is a smooth Banach manifold. The main step of that is the following theorem.

Theorem 2.13. *The map Φ is a submersion at the point $(\tilde{g}, \tilde{u}, \tilde{\phi}) \in \Phi^{-1}(0)$, i.e. the linearization $D\Phi$ is surjective and its kernel splits in $T\mathcal{M}_C$.*

2.5.1 Proof of submersion

To prove surjectivity, we follow the methods in $[A1]$ and $[AK]$. Let $D\hat{\Phi}$ be the restriction of $D\Phi$ to the subspace $T\mathcal{M}_S \subset T\mathcal{M}_C$, i.e.

$$D\hat{\Phi} = D\Phi|_{T\mathcal{M}_S}.$$

Then the operator $D\hat{\Phi}$ is elliptic by Theorem 2.9. So $Im(D\hat{\Phi})$ is closed in $[(S_2)^{m-2,\alpha}_{\delta+2} \times C^{m-2,\alpha}_{\delta+2} \times C^{m-2,\alpha}_{\delta+2}](S)$ and has finite dimensional cokernel K. If K is trivial, then $D\hat{\Phi}$ is surjective, and hence so is $D\Phi$.

If K is nontrivial, then from the self-adjoint property of $D\hat{\Phi}$ (cf. §4.3), it follows that,

$$K^{\perp} = ImD\hat{\Phi}.$$

Thus for any element $(k, w, \zeta) \in K$ and an arbitrary element $(h, u, \sigma) \in T\mathcal{M}_S$, the following equation holds,

$$\int_S \langle D\hat{\Phi}(h, v, \sigma), (k, w, \zeta)\rangle = 0.$$

To prove surjectivity of $D\Phi$, it suffices to prove that for any triple $(k, w, \zeta) \in K$, there exists an element $(h, v, \sigma) \in T\mathcal{M}_C$, such that $\int_S \langle D\Phi(h, v, \sigma), (k, w, \zeta)\rangle \neq 0$.

Assume this is not true, i.e. there exists an element $(k, w, \zeta) \in K$ such that,

$$\int_S \langle D\Phi(h, v, \sigma), (k, w, \zeta)\rangle = 0, \quad \forall (h, v, \sigma) \in T\mathcal{M}_C. \tag{2.46}$$

First choose $(h, v, \sigma) = (\delta^* X, L_X u, L_X \phi)$, for some vector field X which vanishes on ∂S. Thus we are varying the data using diffeomorphisms in $\mathcal{D}^{m+1,\alpha}_0(S)$. In this case, since the stationary vacuum field equations (II) are invariant under diffeomorphisms, it follows that

$$D\Phi(\delta^* X, L_X u, L_X \phi) = (\delta^* Y, 0, 0) \quad \text{at } (\tilde{g}, \tilde{u}, \tilde{\phi}),$$

where $Y = \delta\delta^* X$. Note that Lemma 2.5 shows the operator $\delta\delta^*$ is surjective, so Y can be arbitrarily prescribed. Moreover, the fact $(h, v, \sigma) \in T\mathcal{M}_C$ implies that $\delta h = 0$ on ∂S, so that $Y = 0$ on ∂S. It follows from the equation (2.46) that,

$$0 = \int_S \langle \delta^* Y, k\rangle = \int_S \langle Y, \delta k\rangle + \int_{\partial S} k(Y, \mathbf{n}) = \int_S \langle Y, \delta k\rangle,$$

and thus,

$$\delta k = 0 \quad \text{on } S. \tag{2.47}$$

Next applying integration by parts to (2.46), we obtain

$$\int_S \langle D\Phi(k, w, \zeta), (h, v, \sigma)\rangle + \int_{\partial S} \tilde{B}[(h, v, \sigma), (k, w, \zeta)] = 0.$$

This holds for any $(h, v, \sigma) \in T\mathcal{M}_C$, thus it implies that

$$D\Phi(k, w, \zeta) = 0 \text{ on } S, \tag{2.48}$$

and

$$\int_{\partial S} \tilde{B}[(h, v, \sigma), (k, w, \zeta)] = 0, \quad \forall (h, v, \sigma) \in T\mathcal{M}_C. \tag{2.49}$$

Here the bilinear form $\tilde{B}$ is as follows

$$\tilde{B}[(h,v,\sigma),(k,w,\zeta)] = B[(h,v,\sigma),(k,w,\zeta)] - B[(k,w,\zeta),(h,v,\sigma)],$$

where

$$\begin{aligned} B[(h,v,\sigma),(k,w,\zeta)] = &-k(\delta h,\mathbf{n}) + \frac{1}{2}\{-\langle\nabla_{\mathbf{n}}h,k\rangle - k(\mathbf{n},dtrh) + trk[\mathbf{n}(trh) + \delta h(\mathbf{n})]\} \\ &+ [4\mathbf{n}(w) - 4k(\mathbf{n},d\tilde{u}) + 2trk\mathbf{n}(\tilde{u})]v \\ &+ 4e^{-4\tilde{u}}\sigma[\mathbf{n}(\zeta) - k(\mathbf{n},d\tilde{\phi}) + \frac{1}{2}trk\mathbf{n}(\tilde{\phi}) - 4w\mathbf{n}(\tilde{\phi})]. \end{aligned}$$

On the other hand, since the operator $D\hat{\Phi}$ is formally self-adjoint in the space $T\mathcal{M}_S$, the cokernel K of $D\hat{\Phi}$ is the same as the kernel of $D\hat{\Phi}$ in $T\mathcal{M}_S$. Therefore, the element (k,w,ζ) must satisfy the following boundary conditions,

$$\begin{cases} \delta k = 0 \\ k^T - 2w\tilde{g}^T = 0 \\ H'_k - 2\mathbf{n}(w) - 2\mathbf{n}'_k(\tilde{u}) + w(H - 2\mathbf{n}(\tilde{u})) = 0 \\ \mathbf{n}(\zeta) + \mathbf{n}'_k(\tilde{\phi}) - 2w\mathbf{n}(\tilde{\phi}) = 0 \end{cases} \quad \text{on } \partial S. \tag{2.50}$$

Based on the first equation $\delta k = 0$, together with the fact that $h \in T\mathcal{M}_C$ implies $\delta h = 0$ on ∂S, the bilinear form B can be simplified by removing the divergence terms and becomes,

$$\begin{aligned} B[(h,v,\sigma),(k,w,\zeta)] = &\frac{1}{2}\{-\langle\nabla_{\mathbf{n}}h,k\rangle - k(\mathbf{n},dtrh) + trk\mathbf{n}(trh)\} \\ &+ [4\mathbf{n}(w) - 4k(\mathbf{n},d\tilde{u}) + 2trk\mathbf{n}(\tilde{u})]v \\ &+ 4e^{-4\tilde{u}}\sigma[\mathbf{n}(\zeta) - k(\mathbf{n},d\tilde{\phi}) + \frac{1}{2}trk\mathbf{n}(\tilde{\phi}) - 4w\mathbf{n}(\tilde{\phi})]. \end{aligned} \tag{2.51}$$

Taking a triple (h,v,σ) such that $h = 0,\ \nabla_{\mathbf{n}}h = 0$ and $\sigma = v = 0$ on ∂S, and inserting it into equation (2.49), we obtain,

$$\int_{\partial S} 4\mathbf{n}(v)w + 4e^{-4u}\zeta\mathbf{n}(\sigma) = 0.$$

The terms $\mathbf{n}(v)$ and $\mathbf{n}(\sigma)$ can be chosen to be arbitrary functions along ∂S, so this implies that,

$$w = \zeta = 0 \text{ on } \partial S. \tag{2.52}$$

And consequently, we obtain

$$k^T = wg^T = 0 \text{ on } \partial S, \tag{2.53}$$

according to the second equation in (2.50); and

$$\mathbf{n}(\zeta) + \mathbf{n}'_k(\tilde{\phi}) = 4w\mathbf{n}(\tilde{\phi}) = 0 \quad \text{on } \partial S, \tag{2.54}$$

according to the last equation in (2.50).

Since $k^T = 0$ on ∂S, the trace of k is $trk = k(\mathbf{n},\mathbf{n})$. Thus in the last line of equation (2.51), the term $[\mathbf{n}(\zeta) - k(\mathbf{n},d\tilde{\phi}) + \frac{1}{2}trk\mathbf{n}(\tilde{\phi}) - 4w\mathbf{n}(\tilde{\phi})]$ vanishes on ∂S, according to the following computation,

$$
\begin{aligned}
&\mathbf{n}(\zeta) - k(\mathbf{n}, d\tilde{\phi}) + \frac{1}{2} trk\mathbf{n}(\tilde{\phi}) - 4w\mathbf{n}(\tilde{\phi}) \\
&= \mathbf{n}(\zeta) - k(\mathbf{n}, d\tilde{\phi}) + \frac{1}{2} k(\mathbf{n}, \mathbf{n})\mathbf{n}(\tilde{\phi}) - 4w\mathbf{n}(\tilde{\phi}) \\
&= \mathbf{n}(\zeta) + \mathbf{n}'_k(\tilde{\phi}) - 4w\mathbf{n}(\tilde{\phi}) \\
&= 0.
\end{aligned}
$$

Here the second equality is based on the formula of the variation of $\mathbf{n}$, cf. equation (4.5):

$$\mathbf{n}'_k = -k(\mathbf{n}) + \frac{1}{2} k(\mathbf{n}, \mathbf{n})\mathbf{n}.$$

In addition, we have $\zeta = 0$ from equation (2.52). Therefore, the form B can be simplified further by removing the last line in (2.51) and becomes,

$$
\begin{aligned}
B[(h, v, \sigma), (k, w, \zeta)] = \frac{1}{2}\{-\langle \nabla_{\mathbf{n}} h, k\rangle - k(\mathbf{n}, dtrh) + trk\mathbf{n}(trh)\} \\
+ [4\mathbf{n}(w) - 4k(\mathbf{n}, d\tilde{u}) + 2trk\mathbf{n}(\tilde{u})]v
\end{aligned}
\tag{2.55}
$$

Choose a triple (h, v, σ) so that $h = 0$ and $\nabla_{\mathbf{n}} h = 0$ on ∂S for equation (2.49). Then it follows that,

$$\int_{\partial S} [4\mathbf{n}(w) - 4k(\mathbf{n}, d\tilde{u}) + 2trk\mathbf{n}(\tilde{u})]v = 0,$$

Since the term v can be arbitrarily prescribed on ∂S, one obtains

$$4\mathbf{n}(w) - 4k(\mathbf{n}, d\tilde{u}) + 2trk\mathbf{n}(\tilde{u}) = 0 \quad \text{on } \partial S, \tag{2.56}$$

which is equivalent to the following equation since $trk = k(\mathbf{n}, \mathbf{n})$,

$$\mathbf{n}(v) + \mathbf{n}'_k(\tilde{u}) = 0 \quad \text{on } \partial S. \tag{2.57}$$

Combining this with the third equation in (2.50), one obtains

$$H'_k = 0 \quad \text{on } \partial S. \tag{2.58}$$

Based on equation (2.56), we can simplify the form B further into the following expression,

$$B[(h, v, \sigma), (k, w, \zeta)] = \frac{1}{2}\{-\langle \nabla_{\mathbf{n}} h, k\rangle - k(\mathbf{n}, dtrh) + trk\mathbf{n}(trh)\} \tag{2.59}$$

Consequently (2.49) implies that the following equation holds for any $h \in T\mathcal{M}_C$,

$$
\begin{aligned}
\int_{\partial S} &\{-\langle \nabla_{\mathbf{n}} h, k\rangle - k(\mathbf{n}, dtrh) + trk\mathbf{n}(trh)\} \\
&- \{-\langle \nabla_{\mathbf{n}} k, h\rangle - h(\mathbf{n}, dtrk) + trh\mathbf{n}(trk)\} \ = 0.
\end{aligned}
\tag{2.60}
$$

It follows from equations (2.58) and (2.60) that, c.f.[AK],

$$(A'_k)^T = 0 \quad \text{on } S. \tag{2.61}$$

Summing up the equations (2.47), (2.48), (2.52), (2.53), (2.54), (2.57), and (2.61), one obtains that the element (k, w, ζ) must satisfy the following system,

$$\begin{cases} D\Phi(k, w, \zeta) = 0, \\ \delta k = 0, \end{cases} \quad \text{on } S, \tag{2.62}$$

and the boundary conditions

$$\begin{cases} k^T = 0, \\ (A'_k)^T = 0, \\ w = \zeta = 0, \\ \mathbf{n}(v) + \mathbf{n}'_k(\tilde{u}) = 0, \\ \mathbf{n}(\zeta) + \mathbf{n}'_k(\tilde{\phi}) = 0. \end{cases} \quad \text{on } \partial S. \tag{2.63}$$

Remark The first equation in (2.62) implies that the variation of $(\mathbf{E} - \delta^*\delta_{\tilde{g}}g, \mathbf{F}, \mathbf{H})$ with respect to the deformation (k, w, ζ) vanishes, i.e.

$$D(\mathbf{E} - \delta^*\delta_{\tilde{g}}g, \mathbf{F}, \mathbf{H})_{(\tilde{g},\tilde{u},\tilde{\phi})}(k, w, \zeta) = 0,$$

Combining with the second equation in (2.62), we observe that (k, w, ζ) is a vacuum deformation, i.e. it makes the first variation of $(\mathbf{E}, \mathbf{F}, \mathbf{H})$ vanish,

$$D(\mathbf{E}, \mathbf{F}, \mathbf{H})_{(\tilde{g},\tilde{u},\tilde{\phi})}(k, w, \zeta) = 0. \tag{2.64}$$

Translating to normal geodesic gauge

$$k \to \tilde{k} = k + \delta^* V,$$

where V is a vector field such that $V = 0$ and $\tilde{k}_{0i} = 0$ on ∂S, the boundary conditions in (2.63) imply that the Cauchy data for (k, w, ζ) vanishes on ∂S. To complete the proof, we will use the following unique continuation result, which is proved in §2.5.2.

Proposition 2.14. *The boundary value problem formed by equations* (2.62) *and* (2.63) *has only the trivial solution* $k = w = \zeta = 0$.

As a consequence, $D\Phi$ is sujective. It is then a standard fact that the kernel of $D\Phi$ splits, cf.[$A1$]. This completes the proof of Theorem 2.13.

2.5.2 Proof of the unique continuation

We prove the Proposition 2.14 by generalizing the unique continuation result of [AH] from Riemannian Einstein metrics to sationary Lorentzian Einstein metrics. We first formulate a local result as follows.

On the quotient manifold $(S,\ \tilde{g})$, take an embedded cylinder $C \cong I \times B^2 \subset \mathbb{R}^3$, where $I = [0, 1]$ and B^2 is the unit disk, in such a manner that the horizontal boundary $\partial_0 C = \{0\} \times B^2$ is embedded in ∂S, and the vertical boundary $\partial C = I \times S^1$ is located in the interior of S. Equip C with the induced metric $\tilde{g}$, and choose H-harmonic coordinates $\{\tau, x^i\}(\tau \geq 0, i = 1, 2)$, such that level set $\{\tau = 0\}$ coincides with the horizontal boundary $\partial_0 C$, cf. [AH] for the definition of H-harmonic coordinates.

Notice that under the H-harmonic coordinate system, a general metric g in C can be written in the form,

$$g = zd\tau^2 + \gamma(\psi^i d\tau + x^i)(\psi^j d\tau + x^j).$$

Here γ is the induced metric on the level sets of τ function, z is called the lapse function and ψ is the shift vector. In addition, by writing the Ricci tensor Ric_g in these coordinates, one can obtain the following equations on every $\tau-$level set $\{\tau = \text{constant}\}$ (cf.[AH]):

$$(\partial_\tau^2 + z^2\Delta - 2\psi^k\partial_k\partial_\tau + \psi^k\psi^l\partial_{kl}^2)\gamma_{ij} = -2z^2(Ric_g)_{ij} - 2z\nabla_i\nabla_j z + Q_{ij}(\gamma, \partial\gamma), \tag{2.65}$$

$$\Delta z + |A_\gamma|^2 z + zRic_g(N, N) - \psi(H_\gamma) = 0, \tag{2.66}$$

$$\Delta\psi^i + 2z\langle D^2x^i, A_\gamma\rangle + z\partial_i H_\gamma + 2[(A_\gamma)^i_j\nabla^j z - \frac{1}{2}H\nabla^i z] + 2zRic_g^{0i} = 0, \tag{2.67}$$

where the Laplace operators Δ and the connection ∇ are with respect to the induced metric γ on the level surface. In (2.67), the index $i = 1, 2$ denotes the tangential direction on the hypersurface $\{\tau = \text{constant}\} \subset C$, and the index 0 denotes the normal direction. In (2.66), N denotes the normal vector of the hypersurface $\{\tau = \text{constant}\} \subset C$ such that

$$N = \frac{1}{z}(\partial_\tau - \psi), \tag{2.68}$$

and the second fundamental form is given by

$$A_\gamma = \frac{1}{2}\mathcal{L}_N\gamma. \tag{2.69}$$

In the equation (2.65), $Q_{ij}(\gamma, \partial\gamma)$ is a term which involves at most first order derivatives of (γ, z, ψ) in all directions and the 2nd tangential derivatives of ψ.

In addition, on the vertical boundary ∂C, we have the following conditions in H-harmonic coordinates:

$$z|_{\partial C} \equiv 1, \psi|_{\partial C} \equiv 0. \tag{2.70}$$

Without loss of generality, we can assume the cylinder C is sufficiently small so that $\tilde{g}$ is $C^{m,\alpha}$ close to the standard product metric on the cylinder.

Proposition 2.15. *Let data $(\tilde{g}, \tilde{u}, \tilde{\phi})$ be a stationary vacuum solution, $\Phi(\tilde{g}, \tilde{u}, \tilde{\phi}) = 0$ in C. If (k, w, ζ) is an infinitesimal deformation of $(\tilde{g}, \tilde{u}, \tilde{\phi})$ such that it satisfies the equations* (2.62) *in C and the boundary conditions* (2.63) *on $\partial_0 C$, then there exists a vector field X with $X = 0$ on $\partial_0 C$, such that*

$$k = \delta^* X, \ w = L_X\tilde{u}, \ \text{and } \zeta = L_X\tilde{\phi}.$$

Proof. First we define a Banach space $\mathcal{M}^*$ as follows,

$$\begin{aligned}\mathcal{M}^* = \{(g, u, \phi) \in &[Met^{m,\alpha} \times C^{m,\alpha} \times C^{m,\alpha}](C) : \delta_{\tilde{g}} g = 0 \text{ on } \partial C \text{ and,} \\ &g^T, \ A, \ u, \ \phi, \ \mathbf{n}(u) \text{ and } \mathbf{n}(\phi) \text{ are all fixed on } \partial_0 C\}.\end{aligned} \tag{2.71}$$

Observe that the deformation (k, w, ζ), by the hypothesis of the Proposition, is tangent to the space $\mathcal{M}^*$, i.e. $(k, w, \zeta) \in T\mathcal{M}^*$. Thus, we can assume (k, w, ζ) is the variation of a smooth curve (g_t, u_t, ϕ_t) at $t = 0$, where $(g_t, u_t, \phi_t) \in \mathcal{M}^*$ for $t \in (-\epsilon, \epsilon)$, with some $\epsilon > 0$, and $(g_0, u_0, \phi_0) = (\tilde{g}, \tilde{u}, \tilde{\phi})$.

According to [AH], there exists a smooth curve of $C^{m+1,\alpha}$ diffeomorphisms Ψ_t of C, which equal to Id on $\partial_0 C$ for all $t \in (-\epsilon, \epsilon)$ and $\Psi_0 = Id$ in C, so that $\Psi_t^*(g_t)$ share the same H-harmonic coordinates. We denote the infinitesimal variation of the new curve $(\Psi_t^*(g_t), \Psi_t^*(u_t), \Psi_t^*(\phi_t))$ at $t = 0$ as (g', u', ϕ'). It is given by

$$(g', u', \phi') = (k + \delta_{\tilde{g}}^* X, w + L_X \tilde{u}, \zeta + L_X \tilde{\phi}),$$

for some vector field X, with $X = 0$ on $\partial_0 C$. Therefore, to prove the Proposition, it suffices to prove that $g' = u' = \phi' = 0$.

For simplicity of notation, the normalized curve $(\Psi_t^*(g_t), \Psi_t^*(u_t), \Psi_t^*(\phi_t))$ will still be denoted as (g_t, u_t, ϕ_t) in the following. Since the infinitesimal variation (g', u', ϕ') is the sum of a vacuum deformation (k, w, ζ), cf.(2.64), and a diffeomorphism deformation $\frac{d}{dt}\Psi_t^*$, it must preserve the stationary vacuum property, i.e. it satisfies the following equation:

$$\frac{d}{dt}|_{t=0}(\mathbf{E}, \mathbf{F}, \mathbf{H})[(g_t, u_t, \phi_t)] = 0 \quad \text{in} \quad C,$$

which furthermore implies that,

$$s'_{g_t} = (2|du_t|^2 + 2e^{-4u_t}|d\phi_t|^2)', \tag{2.72}$$

$$Ric'_{g_t} = (2du_t \otimes du_t + 2e^{-4u_t} d\phi_t \otimes d\phi_t)', \tag{2.73}$$

$$(\Delta u_t - 2e^{-4u_t}|d\phi_t|)' = 0, \tag{2.74}$$

$$(\Delta \phi_t + 4\langle du_t, d\phi_t \rangle)' = 0, \tag{2.75}$$

where the prime mark $'$ means $\frac{d}{dt}|_{t=0}$.

Let $\{\tau, x^i\}(i = 1, 2)$ denote the common H-harmonic coordinates for g_t, where the lapse function is z_t and the shift vector is ψ_t. Thus the metric g_t is in the form,

$$g_t = z_t d\tau^2 + \gamma_t(\psi_t^i d\tau + x^i)(\psi_t^j d\tau + x^j).$$

Write $(\gamma', z', \psi', u', \phi')$ as the infinitesimal variation of the curve $(\gamma_t, z_t, \psi_t, u_t, \phi_t)$ at $t = 0$, then by the boundary conditions in (2.71), we obtain the following equation

$$\gamma' = (A')^T = u' = \phi' = \mathbf{n}(u') + \mathbf{n}'(u_0) = \mathbf{n}(\phi') + \mathbf{n}'(\phi_0) = 0, \tag{2.76}$$

on the boundary surface $\partial_0 C = \{\tau = 0\}$.

Notice that equations (2.65−67) hold for all $(\gamma_t, z_t, \psi_t, u_t, \phi_t)$, $t \in (-\epsilon, \epsilon)$. Linearization of the equation (2.66) at $t = 0$ gives

$$\begin{aligned} 0 = &\Delta z' + |A|^2 z' + z' Ric_{g_0}(\mathbf{n}, \mathbf{n}) - \psi'(H) \\ &+ \Delta' z + (|A_t|^2)' z + z[Ric_{g_t}(\mathbf{n}, \mathbf{n})]' - \psi(H_t'). \end{aligned} \tag{2.77}$$

Here the terms $\Delta', (|A_t|^2)'$, and H_t' only involve the tangential variation of γ and A, thus they are all zero on the boundary surface $\partial_0 C$ according to (2.76). In addition, for the term $[Ric_{g_t}(\mathbf{n}, \mathbf{n})]'$, we have the following equation:

$$\begin{aligned} [Ric_{g_t}(\mathbf{n}, \mathbf{n})]' &= Ric'_{g_t}(\mathbf{n}, \mathbf{n}) + 2Ric_{g_0}(\mathbf{n}', \mathbf{n}) \\ &= (2du_t \otimes du_t + 2e^{-4u_t} d\phi_t \otimes d\phi_t)'(\mathbf{n}, \mathbf{n}) \\ &\quad + 2(2du_0 \otimes du_0 + 2e^{-4u} d\phi_0 \otimes d\phi_0)(\mathbf{n}', \mathbf{n}) \\ &= [(2du \otimes du + 2e^{-4u} d\phi \otimes d\phi)(\mathbf{n}, \mathbf{n})]'. \end{aligned} \tag{2.78}$$

The second equality above is based on the equation (2.73) and the fact that (g_0, u_0, ϕ_0) is a vacuum solution. Equation (2.78) shows that $[Ric_{g_t}(\mathbf{n}, \mathbf{n})]'$ only involves the variation of $u, \mathbf{n}(u)$ and $\mathbf{n}(\phi)$. Thus it is also zero on $\partial_0 C$ according to the boundary condition (2.76).

Henceforth, by restricting the equation (2.77) to $\partial_0 C$, we can remove those terms on the second line and obtain

$$\Delta z' + |A|^2 z' + z' Ric_{g_0}(\mathbf{n}, \mathbf{n}) - \psi'(H) = 0 \quad \text{on } \partial_0 C. \tag{2.79}$$

For the same reason, it follows from the linearization of the equation (2.67) that

$$\Delta(\psi')^i + 2z'\langle D^2 x^i, A\rangle + z'\partial_i H + 2[(A)^i_j \nabla^j z' - \frac{1}{2} H \nabla^i z'] + 2z' Ric^{0i}_{g_0} = 0 \quad \text{on } \partial_0 C. \tag{2.80}$$

Furthermore, on the boundary of the surface $\partial_0 C$, we have the Dirichlet condition for z' and ψ', because linearization of the boundary condition (2.70) gives,

$$z'|_{\partial C} \equiv 0, \psi'|_{\partial C} \equiv 0.$$

Since g_0 is assumed to be $C^{m,\alpha}$ close to the standard cylinder metric, equations (2.79) and (2.80) together with the Dirichlet boundary condition, imply that

$$z' = \psi' = 0 \quad \text{on } \partial_0 C. \tag{2.81}$$

Based on (2.68) and (2.69), we have

$$A'_{\gamma_t} = \frac{1}{2}[\mathcal{L}_{\frac{1}{z_t}(\partial_\tau - \psi_t)} \gamma_t]'.$$

On the boundary $\partial_0 C$, $(A'_\gamma)^T = z' = \psi' = 0$ by (2.76) and (2.81). Hence it follows from the equation above that,

$$\partial_\tau \gamma'_{ij} = 0 \quad \text{on } \partial_0 C. \tag{2.82}$$

Observe that $N = -\mathbf{n}$ on the boundary $\partial_0 C$. From (2.68) and (2.81), it follows that,

$$\mathbf{n}'(u_0) = -[\frac{1}{z_t}(\partial_\tau - \psi_t)]'(u_0) = 0 \quad \text{on } \partial_0 C.$$

Therefore, according to the boundary conditions in (2.76), we obtain

$$\partial_\tau u' = 0 \quad \text{on } \partial_0 C. \tag{2.83}$$

Similarly, one can derive that,

$$\partial_\tau \phi' = 0 \quad \text{on } \partial_0 C. \tag{2.84}$$

By the conditions in (2.76) and $(2.82 - 84)$, the triple (γ', u', ϕ') has trivial Cauchy data on the boundary $\partial_0 C$. In the interior of C, linearization of the equation (2.65) shows

$$(\partial^2_\tau + w^2 \gamma^{kl} \partial^2_{kl} - 2\sigma^k \partial_k \partial_\tau + \sigma^k \sigma^l \partial^2_{kl})\gamma'_{ij} = O(\gamma', w', \sigma', u', \phi'), \tag{2.85}$$

where $O(\gamma', w', \sigma', u', \phi')$ is used to denote a term when it only depends on the tangential derivatives (at most 2nd order) of w', σ' and derivatives (at most 1st order) of γ', u', ϕ'. Equations (2.74) and (2.75) gives:

$$g^{\alpha\beta} \partial^2_{\alpha\beta} u' = O(\gamma', w', \sigma', u', \phi'), \tag{2.86}$$

$$g^{\alpha\beta} \partial^2_{\alpha\beta} \phi' = O(\gamma', w', \sigma', u', \phi'). \tag{2.87}$$

which are equivalent to the following equations,

$$[\partial_\tau^2 - 2\sigma^i\partial_{0i}^2 + (w^2\gamma^{ij} + \sigma^i\sigma^j)\partial_{ij}^2]u' = O(\gamma', w', \sigma', u', \phi'), \tag{2.88}$$

$$[\partial_\tau^2 - 2\sigma^i\partial_{0i}^2 + (w^2\gamma^{ij} + \sigma^i\sigma^j)\partial_{ij}^2]\phi' = O(\gamma', w', \sigma', u', \phi'), \tag{2.89}$$

since $g^{00} = w^{-2}, g^{0i} = -w^{-2}\sigma^i$, and $g^{kl} = \gamma^{kl} + w^{-2}\sigma^k\sigma^l$, where index 0 denotes the ∂_τ direction.

Observe that equations (2.85) and (2.88 − 89) have the same principal operator on (γ', u', ϕ'). We denote it as P,

$$P = [\partial_\tau^2 - 2\sigma^i\partial_{0i}^2 + (w^2\gamma^{ij} + \sigma^i\sigma^j)\partial_{ij}^2].$$

This is the same operator as in [AH], and the same procedure there can be applied to derive that $\gamma' = u' = \phi' = 0$, on condition that the Cauchy data of (γ', u', ϕ') vanishes on the boundary $\partial_0 C$. This completes the proof. □

Proposition 2.15 implies that there exists a vector field Z, which is zero on ∂S, such that $k = \delta_{\tilde{g}}^* Z, w = L_Z\tilde{u}, \zeta = L_Z\tilde{\phi}$ in a neighborhood U of ∂S. From [A1] we known, in the case $S \cong \mathbb{R}^3 - B$, Z can be uniquely extended to the entire manifold so that $k = \delta^* Z$ holds globally. From the second equation in (2.62), it follows that,

$$\delta\delta^* Z = \delta k = 0.$$

For a fixed radius $R > 1$, let $B_R \subset S$ denote the pull back of a closed ball of radius R under a chosen diffeomorphism $S \cong \mathbb{R}^3 \setminus B^3$, and A_ϵ denote the annulus between $B_{R-\epsilon}$ and B_R. Take a cutoff function $f \in C^{m+1,\alpha}(S)$ such that $f|_{B_{R-\epsilon}} \equiv 1$ and $f|_{S\setminus B_R} \equiv 0$. Let W be the compactly supported vector field $W = fZ$. Since Z is bounded in B_R, we can take ϵ small enough such that,

$$\int_S \langle W, Z\rangle = \int_{B_{R-\epsilon}} |Z|^2 + \int_{A_\epsilon} \langle fZ, Z\rangle \geq \frac{1}{2}\int_{B_{R/2}} |Z|^2 \tag{2.90}$$

According to Lemma2.5, the map $\delta\delta^*$ is surjective, therefore there exist a vector field Y, which is asymptotically zero of decay rate $(4 + \delta)$ and $Y|_{\partial S} = 0$, such that

$$\delta\delta^* Y = W.$$

Notice that $\delta^* Z$ has the decay rate δ, since $\delta^* Z = k$. From this one can derive that Z can blow up no faster than $r^{2-\delta}$ (cf. §4.4). Therefore, applying integration by parts, one can obtain

$$\int_S \langle W, Z\rangle = \int_S \langle \delta\delta^* Y, Z\rangle = \int_S \langle Y, \delta\delta^* Z\rangle = 0. \tag{2.91}$$

From equations (2.90) and (2.91), it is easy to derive that $Z = 0$ in $B_{R/2}$, thus k, w, and ζ vanish in $B_{R/2}$, which further implies that they are vanishing globally because of ellipticity. This finishes the proof of Proposition 2.14.

In conclusion, we obtain the following result:

Theorem 2.16. *The moduli space $\mathcal{E}_C$ is an infinite dimensional C^∞ Banach manifold, with tangent space*

$$T_{[(\tilde{g},\tilde{u},\tilde{\phi})]}\mathcal{E}_C \cong Ker(D\Phi_{(\tilde{g},\tilde{u},\tilde{\phi})}).$$

Proof. This is an immediate consequence of Theorem 2.13, the fact from §2.2 that $\Phi^{-1}(0) = \mathcal{E}_C$ (locally), and the implicit function theorem in Banach spaces. □

Moreover, from the ellipticity results in §2.4, it follows that,

Theorem 2.17. *The boundary map,*

$$\Pi : \mathcal{E}_C \to [S_2^{m,\alpha} \times C^{m-1,\alpha} \times C^{m-1,\alpha}](\partial S)$$
$$\Pi[(g_S, u, \phi)] = (e^{-2u} g^T,\ e^u(H_g - 2\mathbf{n}_g(u)),\ e^u \mathbf{n}_g(\phi))$$

is a C^∞ Fredholm map, of Fredholm index 0.

Proof. The fact from §2.4 that the operator P_2 is elliptic implies that the boundary map $\tilde{\Pi}$

$$\tilde{\Pi} :\ \mathbb{E}_C \to [S_2^{m,\alpha} \times C^{m-1,\alpha} \times C^{m-1,\alpha}](\partial S),$$
$$\tilde{\Pi}(g, u, \phi) = (e^{-2u} g^T,\ e^u(H_g - 2\mathbf{n}_g(u)),\ e^u \mathbf{n}_g(\phi))$$

is smooth and Fredholm. It is of Fredholm index 0 because P_2 is formally self-adjoint. Moreover, since $\mathcal{E}_C = \mathbb{E}_C / \mathcal{D}_0^{m+1,\alpha}$, and $\tilde{\Pi}$ is invariant under the action of diffeomorphisms in $\mathcal{D}_0^{m+1,\alpha}$, so it follows that Π is also a smooth Fredholm map and of index 0. □

Now translating the results above from conformal data (g, u, ϕ) back to (g_S, u, ϕ) via the isomorphism $\mathcal{Q}$ as in §2.4, proves Theorem 1.2.

□

3 Bartnik boundary data

3.1 Moduli space of stationary vacuum spacetimes II

Fix a 3−dimensional manifold $M \cong \mathbb{R}^3 \setminus B^3$. Let $V^{(4)}$ be a spacetime such that

$$V^{(4)} \cong \mathbb{R} \times M.$$

Let $\mathcal{S}$ denote the space of spacetime metrics $g^{(4)}$ which satisfy the following conditions:

1. (globally hyperbolic) There exist a global time function t defined in $V^{(4)}$, so that M equals to the level set $\{t = 0\}$ and the metric $g^{(4)}$ can be expressed globally as

$$g^{(4)} = -N^2 dt^2 + (g_M)_{ij}(dx^i + X^i dt)(dx^j + X^j dt), \tag{3.1}$$

where $\{x^i\}(i = 1, 2, 3)$ are local coordinates on M, and g_M denotes the induced metric on M, which is Riemannian .
2. (stationary) The vector field ∂_t is a time-like Killing vector field in $(V^{(4)}, g^{(4)})$. In other words, the lapse function N, shift vector X and the induced metric g_M that appear in the expression (3.1) are all independent of the time variable t. In addition, since $g^{(4)}(\partial_t, \partial_t) = -N^2 + |X|^2_{g_M}$ must be negative, one has

$$N^2 > |X|^2_{g_M}. \tag{3.2}$$

3. (asymptotically flat) The metric $g^{(4)}$ decays to the flat (Minkowski) metric at infinity. Explicitly, N, X and g_M belong to the weighted Hölder spaces on M, given by,

$$\begin{aligned} &g_M \in Met^{m,\alpha}_\delta(M),\\ &N-1 \in C^{m,\alpha}_\delta(M),\\ &X \in T^{m,\alpha}_\delta(M), \end{aligned} \tag{3.3}$$

for some fixed number $m \geq 2,\ 0<\alpha<1$, and $\frac{1}{2}<\delta<1$.

It is obvious that an element in $\mathcal{S}$ is uniquely determined by a data set (g_M, X, N). Thus $\mathcal{S}$ admits a smooth Banach manifold structure equipped with the weighted Hölder norm.

As mentioned in the introduction, one can establish a BVP (1.13) for $g^{(4)} \in \mathcal{S}$, but in order to make it elliptic, we need to add gauge terms. A standard choice is to use the Bianchi gauge leading to a modified system with unknown $g^{(4)} \in \mathcal{S}$ as follows:

$$\begin{aligned} &Ric_{g^{(4)}} + \delta^*\beta_{\tilde{g}^{(4)}} g^{(4)} = 0 \quad \text{on} \quad M,\\ &\begin{cases} g_{\partial M} = \gamma \\ H_{\partial M} = H \\ tr_{\partial M} K = k \qquad\qquad \text{on} \quad \partial M \\ \omega_{\mathbf{n}} = \tau \\ \beta_{\tilde{g}^{(4)}} g^{(4)} = 0. \end{cases} \end{aligned} \tag{3.4}$$

Here and throughout the following we use $\omega_{\mathbf{n}}$ as the abbreviation of $\omega_{\mathbf{n}_{\partial M}}$. In the gauge term $\beta_{\tilde{g}^{(4)}} g^{(4)}$ above, $\tilde{g}^{(4)}$ is a fixed background metric that belongs to $\mathcal{S}$ and is in addition vacuum. The effect of adding the gauge term $\beta_{\tilde{g}^{(4)}} g^{(4)}$ in the vacuum equation is to give a slice to the action on the solution space of (1.13) by the group $\mathcal{D}_4$ of diffeomorphisms of the spacetime. The last boundary condition $\beta_{\tilde{g}^{(4)}} g^{(4)} = 0$ in (3.4) corresponds geometrically to the requirement that the diffeomorphisms in $\mathcal{D}_4$ fix the boundary ∂M.

However, such a modification has two issues. Firstly, it is easy to observe that (3.4) is not well posed, because there are 10 interior equations on M but 11 boundary conditions on ∂M — notice that, the gauge term $\beta_{\tilde{g}^{(4)}} g^{(4)}$ defines a vector field in $V^{(4)}$, so it contributes to 4 extra boundary equations in (3.4).

Secondly, let $\mathbb{E}$ be the space of stationary vacuum metrics, i.e.

$$\mathbb{E} = \{g^{(4)} \in \mathcal{S}:\ Ric_{g^{(4)}} = 0\}; \tag{3.5}$$

then, as is explained above, after adding the gauge term $\beta_{\tilde{g}^{(4)}} g^{(4)}$, the boundary map Π_1 defined in (1.14) should be modified to Π_2 as follows,

$$\begin{aligned} \Pi_2 :&\mathbb{E}/\mathcal{D}_4 \to \mathbf{B},\\ \Pi_2([g^{(4)}]) &= (g_{\partial M}, H_{\partial M}, tr_{\partial M} K, \omega_{\mathbf{n}}), \end{aligned}$$

where the target space $\mathbf{B}$ is given by, $\mathbf{B} = Met^{m,\alpha}(\partial M) \times [C^{m-1,\alpha}(\partial M)]^2 \times \wedge_1^{m-1,\alpha}(\partial M)$. However, this map is not well defined, because elements in $\mathcal{D}_4$ do not always preserve the Bartnik boundary data (cf. Proposition 3.1), which means that the Bartnik boundary data is not well defined for an element $[g^{(4)}]$ — an equivalence class of metrics — in the moduli space $\mathbb{E}/\mathcal{D}_4$.

Since we are working with stationary metrics, it is natural to require elements in $\mathcal{D}_4$ to be time-independent and preserve the Killing vector field ∂_t. Thus a general element in $\mathcal{D}_4$ can be decomposed into two parts — a diffeomorphism on the hypersurface M and a translation of time, i.e. $\mathcal{D}_4$ can be defined as,

$$\begin{aligned}\mathcal{D}_4 = \{\Phi_{(\psi,f)}|\ \psi \in D_\delta^{m+1,\alpha}(M) \text{ and } \psi|_{\partial M} = Id_{\partial M};\\ f \in C_\delta^{m+1,\alpha}(M) \text{ and } f|_{\partial M} = 0;\\ \Phi_{(\psi,f)} : V^{(4)} \to V^{(4)},\\ \Phi_{(\psi,f)}[t,p] = [t+f, \psi(p)], \quad \forall t \in \mathbb{R},\ p \in M.\ \},\end{aligned}$$

Here $D_\delta^{m+1,\alpha}(M)$ denotes the group of $C^{m+1,\alpha}$ diffeomorphisms of M which are asymptotically Id_M at the rate of δ.

Proposition 3.1. *If an element $\Phi_{(\psi,f)} \in \mathcal{D}_4$ has a nontrivial time translation function f, then it does not preserve the Bartnik boundary data on ∂M.*

Proof. Take an arbitrary stationary metric $g^{(4)} \in \mathcal{S}$,

$$g^{(4)} = -N^2 dt^2 + g_{ij}(dx^i + X^i dt)(dx^j + X^j dt).$$

Here we use g_{ij} to denote the induced metric on slice $M = \{t = 0\}$. Choose a function $f \in C_\delta^{m+1,\alpha}(M)$, and take the diffeomorphism $\Phi_{(Id_M,f)} \in \mathcal{D}_4$:

$$\begin{aligned}\Phi_{(Id_M,f)} : V^{(4)} \to V^{(4)}\\ \Phi_{(Id_M,f)}(t-f, x_1, x_2, x_3) = (t, x_1, x_2, x_3).\end{aligned}$$

In the following, we will use Φ_f as the abbreviation of $\Phi_{(Id_M,f)}$. Let s denote the new time function, i.e.

$$s = t - f.$$

Then the pull back metric $\Phi_f^* g^{(4)}$ can be written as

$$\begin{aligned}\Phi_f^* g^{(4)} &= -N^2[d(s+f)]^2 + g_{ij}[dx^i + X^i d(s+f)][dx^j + X^j d(s+f)]\\ &= -u^2 ds^2 - u^2 df \odot ds + X_i dx^i \odot ds - u^2(df)^2 + X_i dx^i \odot df + g_{ij} dx^i dx^j,\end{aligned}$$

where $u^2 = N^2 - |X|_g^2$. Let $\hat{M}$ denote the new slice $\hat{M} = \{s = 0\}$ in $V^{(4)}$. Then, from the expression above, one easily observes that the induced metric on $\hat{M}$ is given by,

$$\hat{g}_{ij} = -u^2(df)^2 + X_i dx^i \odot df + g_{ij} dx^i dx^j.$$

Since $\Phi_f \in \mathcal{D}_4$, we have $f|_{\partial M} = 0$, i.e. the time translation is fixing the boundary — ∂M and $\partial\hat{M}$ coincide in the spacetime. Then it is obvious that $g_{\partial M}$ in the Bartnik boundary data remains the same under such a time translation. However, this is not the case for the other data $H_{\partial M}$, $tr_{\partial M} K$ and $\omega_{\mathbf{n}_{\partial M}}$.

Let $\mathbf{N} \in TV^{(4)}$ denote the future-pointing time-like unit normal vector to the slice M and $\mathbf{n}$ denote the outward unit normal of $\partial M \subset M$. When switching to the new slice $\hat{M}$, those two normal vectors $(\mathbf{N}, \mathbf{n})$ are related to $(\hat{\mathbf{N}}, \hat{\mathbf{n}})$ on the boundary ∂M in the following way,

$$\begin{bmatrix} d\Phi_f(\hat{\mathbf{N}}) \\ d\Phi_f(\hat{\mathbf{n}}) \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} \mathbf{N} \\ \mathbf{n} \end{bmatrix},$$

where a, b are scalar fields on ∂M and $a^2 - b^2 = 1$ (cf.§4.5).

For the induced metric $\hat{g} = \Phi_f^*(g)$ and the connection $\nabla_{\hat{g}} = \Phi^*(\nabla_g)$, we obtain the following formula for the mean curvature on ∂M:

$$\begin{aligned} \hat{H}_{\partial\hat{M}} &= tr_{\partial\hat{M}}(\nabla_{\hat{g}}\hat{\mathbf{n}}) \\ &= tr_{\partial M}[\nabla_g d\Phi_f(\hat{\mathbf{n}})] \\ &= tr_{\partial M}[\nabla_g(b\mathbf{N} + a\mathbf{n})] \\ &= b tr_{\partial M}(\nabla_g \mathbf{N}) + a tr_{\partial M}(\nabla_g \mathbf{n}) \\ &= b tr_{\partial M} K + a H_{\partial M}. \end{aligned} \tag{3.6}$$

It is easy to show that $tr_{\partial M} K$ is transformed in a similar way as above, i.e.

$$tr_{\partial\hat{M}}\hat{K} = a tr_{\partial M} K + b H_{\partial M} \tag{3.7}$$

As for the last boundary term $\omega_{\mathbf{n}}$, one has $\forall v \in T(\partial M)$,

$$\begin{aligned} \hat{\omega}_{\mathbf{n}}(v) &= \Phi_f^* K(\hat{\mathbf{n}}, v) \\ &= \Phi_f^* g^{(4)}(\Phi_f^*(\nabla)_v \hat{\mathbf{N}}, \hat{\mathbf{n}}) \\ &= g^{(4)}(\nabla_{d\Phi_f(v)}(a\mathbf{N} + b\mathbf{n}),\ b\mathbf{N} + a\mathbf{n}) \\ &= -b \cdot da[d\Phi_f(v)] + a \cdot db[d\Phi_f(v)] + (a^2 - b^2) g^{(4)}(\nabla_{d\Phi_f(v)}\mathbf{N}, \mathbf{n}) \\ &= a^2[d\Phi_f(v)](b/a) + K(\mathbf{n}, d\Phi_f(v)), \\ &= a^2 v(b/a) + \omega_{\mathbf{n}}(v). \end{aligned}$$

Here the last equality is based on the observation that $d\Phi_f(v) = v\ \forall v \in T(\partial M)$, since $\Phi_f|_{\partial M} = Id_{\partial M}$. From the formula above, we conclude that,

$$\hat{\omega}_{\mathbf{n}} = a^2 d_{\partial M}(b/a) + \omega_{\mathbf{n}}, \tag{3.8}$$

where $d_{\partial M}(b/a)$ denotes the exterior derivative of the scalar field on ∂M. Along the boundary ∂M, one has

$$a = \frac{1 + \langle X, \mathbf{n}\rangle \mathbf{n}(f)}{\sqrt{[1 + \langle X, \mathbf{n}\rangle \mathbf{n}(f)]^2 - N^2 |\mathbf{n}(f)|^2}}. \tag{3.9}$$

We refer to the Appendix §4.5 for the detailed calculation of the scalar fields a, b. Therefore, if the function f is nontrivial, in the sense that $\mathbf{n}(f)|_{\partial M} \neq 0$, then $a \neq 1$ by (3.9) and hence it is easy to observe from equations (3.6-8) that the Bartnik boundary conditions are not invariant under the diffeomorphism Φ_f. □

In view of the fact above, one may suggest to reduce the diffeomorphism group $\mathcal{D}_4$ to a smaller one $\mathcal{D}_3$ consisting of only 3-dim diffeomorphism on the slice, i.e.

$$\mathcal{D}_3 = \{\psi \in D_\delta^{m+1,\alpha}(M) :\ \psi|_{\partial M} = Id_{\partial M}\}. \tag{3.10}$$

However, this approach does not work either. Let Π_3 be the boundary map as follows,

$$\Pi_3 : \mathbb{E}/\mathcal{D}_3 \to \mathbf{B}$$

$$\Pi_3([g^{(4)}]) = (g_{\partial M}, H_{\partial M}, tr_{\partial M} K, \omega_{\mathbf{n}}).$$

Given a fixed boundary condition (γ, H, k, τ), and an element $g^{(4)}$ in the pre-image set $\Pi_3^{-1}[(\gamma, H, k, \tau)]$, we can take an arbitrary function f such that $f|_{\partial M} = \mathbf{n}(f)|_{\partial M} = 0$, and make time translation Φ_f to obtain a new metric $\bar{g}^{(4)} = \Phi_f^* g^{(4)}$. Then, by the previous analysis, $\bar{g}^{(4)}$ also belongs to $\Pi_3^{-1}[(\gamma, H, k, \tau)]$.

When the metric $g^{(4)}$ is varied by a smooth curve of such time translations, the corresponding infinitesimal deformation is of the form,

$$(g^{(4)})' = L_{f\partial_t} g^{(4)} = df \odot (\partial t)^\flat = df \odot (-u^2 dt + X_i dx^i).$$

The analysis in the previous paragraph implies that $(g^{(4)})' \in \mathrm{Ker} D\Pi_3$. This contributes to a nontrivial kernel element if it is not tangent to any 3-dim diffeomorphism variation, i.e. the following equation is not solvable for $Z \in TM$,

$$df \odot (-u^2 dt + X_i dx^i) = L_Z g^{(4)}. \tag{3.11}$$

Since (3.11) is an overdetermined system for Z, it is not solvable for generic choices of f. This means that the kernel of $D\Pi_3$ should be of infinite dimension, which indicates that Π_3 is not a Fredholm map.

From all the analysis above, we notice that the Neumann data $\mathbf{n}(f)$ of the time translation function plays an important role. It suggests defining a new diffeomorphism group $\mathcal{D}$ as follows,

$$\mathcal{D} = \{\Phi_{(\psi,f)} \in \mathcal{D}_4 : \mathbf{n}_g(f) = 0 \text{ on } \partial M\}. \tag{3.12}$$

It is in fact an intermediate group in the sense that $\mathcal{D}_3 \subset \mathcal{D} \subset \mathcal{D}_4$.

Remark 3.2. The vector field $\mathbf{n}_g$ in (3.12) can be taken as the unit normal vector of ∂M with respect to any Riemannian metric g on M — the group $\mathcal{D}$ does not depend on the choice of the metric g. In fact, it is easy to observe that $\mathcal{D}$ can be defined in an equivalent way:

$$\mathcal{D} = \{\Phi_{(\psi,f)} \in \mathcal{D}_4 : df = 0 \text{ at } \partial M\}.$$

Geometrically, elements in the group $\mathcal{D}$ are diffeomorphisms of the spacetime $(V^{(4)}, g^{(4)})$ which fix the boundary ∂M and the time-like unit normal vector field $\mathbf{N}$ along ∂M, since $\mathbf{n}(f) = 0$ yields $a = 1$ in (3.9).

Define $\mathcal{E}$ to be the quotient space,

$$\mathcal{E} = \mathbb{E}/\mathcal{D}.$$

Elements in $\mathcal{E}$ are equivalence classes $[g^{(4)}]$ given by,

$$[g^{(4)}] = \{\Phi_{(\psi,f)}^* g^{(4)} : \ g^{(4)} \in \mathbb{E}, \ \Phi_{(\psi,f)} \in \mathcal{D}\}.$$

Now we can consider the natural boundary map:

$$\begin{aligned} \Pi :& \mathcal{E} \to \mathbf{B} \\ \Pi([g^{(4)}]) &= (g_{\partial M}, H_{\partial M}, tr_{\partial M} K, \omega_{\mathbf{n}}). \end{aligned} \tag{3.13}$$

This map is well defined — the Bartnik boundary data is the same for all the metrics inside one equivalence class $[g^{(4)}] \in \mathcal{E}$, because the transformation formulas $(3.6-8)$ show that Bartnik boundary data is preserved under diffeomorphisms in $\mathcal{D}$. In the following sections we will prove this boundary map Π is Fredholm.

3.2 Well-defined boundary value problem

Throughout this section, we take $\tilde{g}^{(4)} \in \mathcal{E}$ as a fixed reference metric and make the following assumption:

Assumption 3.3. The BVP with time-independent unknown $Y \in T^{m,\alpha}_{\delta}(V^{(4)})$, given by,

$$\begin{cases} \beta_{g^{(4)}}\delta^*_{g^{(4)}}Y = 0 \quad \text{on} \quad M \\ Y = 0 \quad \text{on} \quad \partial M \end{cases} \tag{3.14}$$

has only the zero solution $Y = 0$ when $g^{(4)} = \tilde{g}^{(4)}$.

Remark. Throughout this paper, we say a tensor field T in $V^{(4)}$ is time-independent if $L_{\partial_t}T = 0$. In the above, $T^{m,\alpha}_{\delta}(V^{(4)})$ denotes the space of $C^{m,\alpha}$ vector fields in $V^{(4)}$, which are in addition asymptotically zero at the rate of δ.

In the following, we call the operator $\beta_{\tilde{g}^{(4)}}\delta^*_{\tilde{g}^{(4)}}$ *invertible* if (3.14) has trivial kernel. This is an open condition, since $\beta_{g^{(4)}}\delta^*_{g^{(4)}}$ with Dirichlet boundary data is an elliptic and self-adjoint operator (cf.Lemma 3.10). Thus, if $\beta_{\tilde{g}^{(4)}}\delta^*_{\tilde{g}^{(4)}}$ is invertible, then so is the operator $\beta_{g^{(4)}}\delta^*_{g^{(4)}}$ for $g^{(4)}$ near $\tilde{g}^{(4)}$ in the space $\mathcal{S}$.

We set up a BVP with unknowns $(g^{(4)}, F) \in \mathcal{S} \times C^{m,\alpha}_{\delta}(M)$ as follows,

$$\begin{cases} Ric_{g^{(4)}} + \delta^*\beta_{\tilde{g}^{(4)}}g^{(4)} = 0 \\ \Delta F = 0 \end{cases} \quad \text{on} \quad M,$$
$$\begin{cases} g_{\partial M} = \gamma \\ aH_{\partial M} + btr_{\partial M}K = H \\ atr_{\partial M}K + bH_{\partial M} = k \\ \omega_{\mathbf{n}} + a^2 d_{\partial M}(a/b) = \tau \\ \beta_{\tilde{g}^{(4)}}g^{(4)} = 0 \end{cases} \quad \text{on} \quad \partial M, \tag{3.15}$$

where

$$a = \frac{1 + \langle X, \mathbf{n}\rangle F}{\sqrt{(1 + \langle X, \mathbf{n}\rangle F)^2 - N^2F^2}}, \quad \text{and} \quad b = \sqrt{a^2 - 1}, \tag{3.16}$$

with N and X denoting the lapse function and shift vector of $g^{(4)}$. Here $\Delta = -trHess$ denotes the Laplace operator (i.e. the time-independent wave operator) with respect to the metric $g^{(4)}$. The argument to follow works in the same way if one sets Δ to be the Laplacian of the induced Riemannian metric g on the slice M. But with the former choice, the principal symbol which we will compute in §4 is simpler.

Applying the Bianchi operator to the first equation of (3.15), one obtains,

$$\beta_{g^{(4)}}\delta^*_{g^{(4)}}[\beta_{\tilde{g}^{(4)}}g^{(4)}] = 0 \quad \text{on } M. \tag{3.17}$$

In addition, the last boundary condition in (3.15) gives,

$$\beta_{\tilde{g}^{(4)}}g^{(4)} = 0 \quad \text{on } \partial M. \tag{3.18}$$

Combining (3.17) and (3.18), together with the Assumption 3.3, it follows that,

$$\beta_{\tilde{g}^{(4)}}g^{(4)} = 0,$$
$$\forall \text{ solution } g^{(4)} \text{ of (3.15) near } \tilde{g}^{(4)}.$$

Therefore, if we use $\mathcal{Q}$ to denote the solution space of (3.15), then near $\tilde{g}^{(4)}$, it is given by

$$\begin{aligned}\mathcal{Q} = \{(g^{(4)}, F) : & Ric_{g^{(4)}} = 0, \beta_{\tilde{g}^{(4)}} g^{(4)} = 0, \Delta F = 0, \text{ on } M; \\ & (g_{\partial M}, aH_{\partial M} + btr_{\partial M}K, atr_{\partial M}K + bH_{\partial M}, \omega_{\mathbf{n}} + a^2 d_{\partial M}(b/a)) = (\gamma, H, k, \tau) \text{ on } \partial M\}.\end{aligned}$$

To establish a well-defined boundary map, we first define a space $\mathcal{C}$ as follows:

$$\mathcal{C} := \{\ (g^{(4)}, F) :\ Ric_{g^{(4)}} = 0,\ \beta_{\tilde{g}^{(4)}} g^{(4)} = 0,\ \Delta F = 0 \text{ on } M\ \}.$$

Next, let $\tilde{\Pi}$ be the boundary map:

$$\begin{aligned}&\tilde{\Pi} :\ \mathcal{C} \to \mathbf{B} \\ &\tilde{\Pi}(g^{(4)}, F) = (g_{\partial M}, aH_{\partial M} + btr_{\partial M}K, atr_{\partial M}K + bH_{\partial M}, \omega_{\mathbf{n}} + a^2 d_{\partial M}(b/a)).\end{aligned}$$

This map $\tilde{\Pi}$ is closely related to the boundary map Π defined in (2.13). In fact, we have the following theorem.

Theorem 3.4. *There is a map $\mathcal{P}$ so that the space $\mathcal{C}$ is diffeomorphic to $\mathbb{E}$ via $\mathcal{P}$, and the boundary maps Π and $\tilde{\Pi}$ are related by*

$$\tilde{\Pi} = \Pi \circ \mathcal{P}.$$

Proof. Given an element $(\hat{g}^{(4)}, \hat{F}) \in \mathcal{C}$, one can take a function f on M such that $f|_{\partial M} = 0$ and $\mathbf{n}(f)|_{\partial M} = \hat{F}|_{\partial M}$, and apply the 4-dim diffeomorphism $\Phi_{(\psi,f)}$ to $\hat{g}^{(4)}$ with an arbitrary $\psi \in \mathcal{D}_3$. Thus, any element $(\hat{g}^{(4)}, \hat{F}) \in \mathcal{C}$ gives rise to a class of elements as follows,

$$\{\Phi^*_{(\psi,f)}(\hat{g}^{(4)}) :\ \Phi_{(\psi,f)} \in \mathcal{D}_4;\ \mathbf{n}(f)|_{\partial M} = \hat{F}|_{\partial M}\}. \tag{3.19}$$

It is easy to observe that the equivalence class above actually defines an element in $\mathbb{E}$. Henceforth we can define a map $\mathcal{P}$ as,

$$\begin{aligned}&\mathcal{P} : \mathcal{C} \to \mathbb{E}, \\ &\mathcal{P}(\hat{g}^{(4)}, \hat{F}) = [g^{(4)}],\end{aligned}$$

where $[g^{(4)}]$ is defined as the equivalence class (3.19).

On the other hand, consider the following map:

$$\begin{aligned}&\mathcal{G} : \mathcal{S} \times \mathcal{D}_4 \to (\wedge_1)^{m,\alpha}_{\delta} V^{(4)} \\ &\mathcal{G}(g^{(4)}, \Phi) = \beta_{\tilde{g}^{(4)}} \Phi^* g^{(4)},\end{aligned}$$

The linearization of $\mathcal{G}$ at $(\tilde{g}^{(4)}, Id_{V^{(4)}})$ is given by,

$$\begin{aligned}&D\mathcal{G}|_{(\tilde{g}^{(4)}, Id_{V^{(4)}})} : T\mathcal{S} \times T\mathcal{D}_4 \to (\wedge_1)^{m,\alpha}_{\delta} V^{(4)} \\ &D\mathcal{G}|_{(\tilde{g}^{(4)}, Id_{V^{(4)}})}[(h^{(4)}, Y)] = \beta_{\tilde{g}^{(4)}} \delta^*_{\tilde{g}^{(4)}} Y + \beta_{\tilde{g}^{(4)}} h^{(4)}.\end{aligned}$$

By the definition of $\mathcal{D}_4$, the vector field $Y \in T\mathcal{D}_4$ is time-independent, asymptotically zero and $Y = 0$ on ∂M. So the operator $\beta_{\tilde{g}^{(4)}} \delta^*_{\tilde{g}^{(4)}}$ in the linearization above is invertible by the Assumption 3.3. Therefore, by the implicit function theorem, for any $g^{(4)} \in \mathcal{S}$ near

$\tilde{g}^{(4)}$, there is a unique element $\Phi_{(\psi,f)} \in \mathcal{D}_4$ such that the pull back metric $\Phi^*_{(\psi,f)} g^{(4)}$ is gauge-free, i.e. $\beta_{\tilde{g}^{(4)}}(\Phi^*_{(\psi,f)} g^{(4)}) = 0$ in $V^{(4)}$.

Now take an arbitrary vacuum metric $g^{(4)} \in \mathcal{E} \subset \mathcal{S}$. Then the gauge-free metric $\hat{g}^{(4)} = \Phi^*_{(\psi,f)} g^{(4)}$ is also vacuum, and trivially it follows,

$$g^{(4)} = (\Phi^*_{(\psi,f)})^{-1} \hat{g}^{(4)} = \Phi^*_{(\psi^{-1},-f)} \hat{g}^{(4)}.$$

So we take $\hat{F}$ as the unique harmonic function (with respect to the metric $\hat{g}^{(4)}$) on M satisfying the Dirichlet boundary condition $\hat{F} = \mathbf{n}(-f)$ on ∂M. Pair it with $\hat{g}^{(4)}$ to obtain an element $(\hat{g}^{(4)}, \hat{F}) \in \mathcal{C}$.

Moreover, if two elements $g_1^{(4)},\ g_2^{(4)} \in \mathcal{E}$ near $\tilde{g}^{(4)}$ are equivalent under some 4-diffeomorphism $\Phi_{(\psi_0,f_0)}$, then they correspond to the same gauge-free metric $\hat{g}^{(4)}$ because of the uniqueness shown above. If, in addition, the time translation f_0 makes $\mathbf{n}(f_0) = 0$ on ∂M, then $g_1^{(4)}$ and $g_2^{(4)}$ also generate the same harmonic function $\hat{F}$ as described above. Therefore, all the metrics that belong to the same equivalence class $[g^{(4)}] \in \mathbb{E}$ give rise to a unique pair $(\hat{g}^{(4)}, \hat{F}) \in \mathcal{C}$. This implies that there is a well-defined map $\tilde{\mathcal{P}}$ given by,

$$\begin{aligned} &\tilde{\mathcal{P}} : \mathbb{E} \to \mathcal{C}, \\ &\tilde{\mathcal{P}}([g^{(4)}]) = (\hat{g}^{(4)},\ \hat{F}), \end{aligned}$$

where $(\hat{g}^{(4)},\ \hat{F})$ is obtained in the manner described above.

It is easy to check that $\mathcal{P}$ and $\tilde{\mathcal{P}}$ are the inverse map of each other. Thus, the spaces $\mathcal{C}$ and $\mathbb{E}$ are diffeomorphic via $\mathcal{P}$.

Moreover, based on the formulas $(2.6-8)$, one can easily observe that if $[g^{(4)}] = \mathcal{P}(\hat{g}^{(4)}, \hat{F})$, then their Bartnik boundary data are related in the following way,

$$\begin{aligned} (g_{\partial M}, &H_{\partial M}, tr_{\partial M} K, \omega_{\mathbf{n}}) \\ &= (\hat{g}_{\partial M}, a\hat{H}_{\partial M} + b\, tr_{\partial M}\hat{K}, a\, tr_{\partial M}\hat{K} + b\hat{H}_{\partial M}, \hat{\omega}_{\mathbf{n}} + a^2 d_{\partial M}(b/a)), \end{aligned}$$

where a, b are given by equations in (3.16) with $F = \hat{F}$. Therefore, the boundary maps $\tilde{\Pi}$ and Π are related by,

$$\tilde{\Pi} = \Pi \circ \mathcal{P}.$$

$\square$

Theorem 3.5. *The space $\mathcal{C}$ is a smooth Banach manifold, and the boundary map Π is Fredholm.*

Proof. For any stationary vacuum metric $g^{(4)}$, define $\mathcal{H}_{g^{(4)}}$ as the space of harmonic functions on M:

$$\mathcal{H}_{g^{(4)}} = \{f \in C^{m,\alpha}_\delta(M) : \Delta_{g^{(4)}} f = 0 \text{ on } M\}.$$

Since $\Delta_{g^{(4)}}$ is invertible when subjected to Dirichlet boundary conditions, it is easy to prove that,

$$\mathcal{H}_{g^{(4)}} \cong C^{m,\alpha}(\partial M).$$

Thus $\mathcal{H}$ admits a smooth Banach manifold structure.

We observe that if $(g^{(4)}, F) \in \mathcal{C}$, then $g^{(4)} \in \mathbb{E}$ and it satisfies the gauge condition $\beta_{\tilde{g}^{(4)}} g^{(4)} = 0$. By the analysis in the proof of Theorem 3.15, it is easy to see that such a metric $g^{(4)}$ is the representative of an equivalence class of metrics $[g^{(4)}] \in \mathbb{E}/\mathcal{D}_4$. Therefore, the space $\mathcal{C}$ is actually a fiber bundle over $\mathbb{E}/\mathcal{D}_4$, with the fiber at $[g^{(4)}]$ being $\mathcal{H}_{g^{(4)}}$. Thus near the reference metric $\tilde{g}^{(4)}$, we have,

$$\mathcal{C} \cong \mathbb{E}/\mathcal{D}_4 \times \mathcal{H}_{\tilde{g}^{(4)}}.$$

It is easy to observe that $\mathbb{E}/\mathcal{D}_4 = Z_C$, thus by the result from §2.5, $\mathbb{E}/\mathcal{D}_4$ is a smooth Banach manifold, and hence it follows that $\mathcal{C}$ has a smooth Banach manifold structure. To prove that the map $\tilde{\Pi}$ is Fredholm, it suffices to prove BVP (3.15) is elliptic, which will be shown in the next section. □

Combining Theorem 3.4 and Theorem 3.5 gives Theorem 1.3, modulo Assumption 3.3 (cf.§3.4).

3.3 Ellipticity of the BVP II

In this section, similar as in §2.4, we use ξ to denote a $1-$form on M, η to denote a nonzero $1-$form tangential to the boundary ∂M, and μ a nonzero $1-$form normal to the boundary ∂M. We use the index 0 to denote the direction along ∂t in $V^{(4)}$, and index $1, 2, 3$ to denote the tangential direction on M. When restricted on the boundary, index 1 denotes the (outward) normal direction to $\partial M \subset M$ and indices $2, 3$ denote directions tangent to ∂M. We use greek letters when 0 is included in the indices, and latin letters when there are only tangential components.

To prove ellipticity of the system (3.15), we define a differential operator $\mathcal{F} = (\mathcal{L}, \mathcal{B})$ with interior operator $\mathcal{L}$, mapping a pair $(g^{(4)}, F)$ to the interior equations in (3.15):

$$\begin{aligned} \mathcal{L} : \mathcal{S} \times C^{m,\alpha}_{\delta}(M) &\to S^{m-2,\alpha}_{\delta+2}(V^{(4)}) \times C^{m-2,\alpha}_{\delta+2}(M) \\ \mathcal{L}(g^{(4)}, \alpha) &= (\ Ric_{g^{(4)}} + \delta^*_{g^{(4)}} \beta_{\tilde{g}^{(4)}} g^{(4)}, \quad \Delta F \ \ ); \end{aligned}$$

and a boundary operator $\mathcal{B}$ mapping $(g^{(4)}, F)$ to the boundary terms in (3.15):

$$\begin{aligned} \mathcal{B} : \mathcal{S} \times C^{m,\alpha}_{\delta}(M) &\to \mathbb{B} \\ \mathcal{B}(g^{(4)}, F) = (\quad & g_{\partial M}, \\ & aH_{\partial M} + btr_{\partial M} K, \\ & atr_{\partial M} K + bH_{\partial M}, \\ & \omega_{\mathbf{n}} + a^2 d_{\partial M}(b/a), \\ & \beta_{\tilde{g}^{(4)}} g^{(4)} \quad). \end{aligned}$$

In the above, $S^{m-2,\alpha}_{\delta+2}(V^{(4)})$ denotes the space of symmetric 2-tensors in $V^{(4)}$, which are time independent, $C^{m-2,\alpha}$ smooth and asymptotically zero at the rate of $(\delta + 2)$; $\mathbb{B}$ is an abbreviation of the target space of $\mathcal{B}$, given by,

$$\mathbb{B} = S^{m,\alpha}(\partial M) \times [C^{m-1,\alpha}(\partial M)]^2 \times \wedge^{m-1,\alpha}(\partial M) \times [C^{m-1,\alpha}(\partial M)]^2 \times \wedge^{m-1,\alpha}(\partial M).$$

Theorem 3.6. *The linearization $D\mathcal{F}$ of $\mathcal{F}$ at $(\tilde{g}^{(4)}, 0)$ is elliptic.*

Proof. We use the characterization of ellipticity in Agmon-Douglis-Nirenberg [ADN]. We first show in §3.3.1 that $D\mathcal{F}$ is properly elliptic. In §3.3.2 we show that $D\mathcal{F}$ satisfies the complementing boundary condition.

3.3.1 Properly elliptic condition

The linearization of the interior operator at $(\tilde{g}^{(4)}, 0)$ is given by (cf.[Be])

$$\begin{gathered} D\mathcal{L} : T\mathcal{S} \times C^{m,\alpha}_{\delta}(M) \to S^{m-2,\alpha}_{\delta+2}(V^{(4)}) \times C^{m-2,\alpha}_{\delta+2}(M) \\ D\mathcal{L}(h^{(4)}, G) = (\ D^*_{\tilde{g}^{(4)}} D_{\tilde{g}^{(4)}} h^{(4)},\ \Delta G\ ). \end{gathered}$$

Here $D^*_{\tilde{g}^{(4)}} D_{\tilde{g}^{(4)}} h^{(4)}_{\alpha\beta}$ can be expressed in the $3+1$ slice formalism (2.1) of the metric as:

$$\begin{aligned} D^*_{\tilde{g}^{(4)}} D_{\tilde{g}^{(4)}} h^{(4)}_{\alpha\beta} &= -D_{\mathbf{N}} D_{\mathbf{N}} h^{(4)}_{\alpha\beta} + \Sigma^3_{i=1} D_{e_i} D_{e_i} h^{(4)}_{\alpha\beta} + O_1(h^{(4)}) \\ &= -D_{\frac{1}{N}(\partial t - X)} D_{\frac{1}{N}(\partial t - X)} h^{(4)}_{\alpha\beta} + \Sigma^3_{i=1} D_{e_i} D_{e_i} h^{(4)}_{\alpha\beta} + O_1(h^{(4)}) \\ &= -\frac{1}{N^2} \partial_X \partial_X h^{(4)}_{\alpha\beta} + \Sigma^3_{i=1} \partial_{e_i} \partial_{e_i} h^{(4)}_{\alpha\beta} + O_1(h^{(4)}), \end{aligned}$$

where $\{e_i\}$, $(i = 1, 2, 3)$ is an orthonormal basis of the tangent space on M and $O_1(h^{(4)})$ denotes those terms with lower(≤ 1) order derivatives. Recall that $\mathbf{N}$ denotes the time-like unit vector perpendicular to M. Based on (2.1),

$$\mathbf{N} = N^{-1}(\partial_t - X).$$

A similar formula holds for the term ΔG, i.e.

$$\Delta G = -\frac{1}{N^2} \partial_X \partial_X G + \Sigma^3_{i=1} \partial_{e_i} \partial_{e_i} G + O_1(G).$$

Thus, the matrix of principal symbol for $D\mathcal{L}$ is given by,

$$L(\xi) = a(\xi) I_{11\times 11} \tag{3.20}$$

with

$$a(\xi) = \xi_1^2 + \xi_2^2 + \xi_3^2 - \frac{1}{N^2}(X_i \xi^i)^2. \tag{3.21}$$

The determinant of this matrix is obviously

$$\det(L(\xi)) = [a(\xi)]^{11}.$$

Notice that $\frac{|X|^2}{N^2} < 1$ by (3.2) and hence,

$$a(\xi) = |\xi|^2 - \langle \frac{X}{N}, \xi \rangle^2 \geq |\xi|^2 - \frac{|X|^2}{N^2} |\xi|^2 > 0,$$

Therefore, the interior operator L is properly elliptic.

3.3.2 Complementing boundary condition

Recall that the complementing boundary condition is defined as (cf.[ADN]):

Let $L^*(\xi)$ be the adjoint matrix of $L(\xi)$ and set $\xi = \eta + z\mu$. The rows of the matrix $[B \cdot L^*](\eta + z\mu)$ are linearly independent modulo $l^+(z) = \prod(z - z_k)$, where $\{z_k\}$ are the roots of $detL(\eta + z\mu) = 0$ having positive imaginary parts.

Since the principal symbol of L is the identity matrix (up to a scalar) as shown in (3.20), the complementing condition will hold as long as the boundary matrix $B(\eta + z\mu)$ is not singular when z is a root of $\det L(\eta + z\mu) = 0$ with positive imaginary part.

The linearization of the boundary operator $\mathcal{B}$ at $(\tilde{g}^{(4)}, 0)$ is given by,

$$\begin{aligned}
&\mathcal{B} : T\mathcal{S} \times C^{m,\alpha}_{\delta}(M) \to \mathbb{B} \\
&D\mathcal{B}(h^{(4)}, G) = (\ h_{\partial M} \\
&\qquad\qquad DH_{\partial M}(h^{(4)}) + O_0(G) \\
&\qquad\qquad Dtr_{\partial M}K(h^{(4)}) + O_0(G) \\
&\qquad\qquad D[\omega_{\mathbf{n}}](h^{(4)}) + Nd_{\partial M}G + O_0(G) \\
&\qquad\qquad \beta_{g^{(4)}} h^{(4)}\ ).
\end{aligned} \tag{3.22}$$

Notice that at $(\tilde{g}^{(4)}, 0)$, $a = 1,\ b = 0$. The formula (3.16) of the scalar field a involves only the $0-$order information of F, thus the 2nd and 3rd boundary terms in $D\mathcal{B}$, which represent the linearization of Bartnik conditions $(aH_{\partial M} + btr_{\partial M}K)$ and $(atr_{\partial M}K + bH_{\partial M})$ at $(a = 1, b = 0)$, do not contain high order (≥ 1) derivatives of G. It is easy to check at $(a = 1, b = 0)$

$$D[a^2 d_{\partial M} b/a](G) = Nd_{\partial M}G + O_0(G),$$

which contributes to the third term in $D\mathcal{B}$.

Based on (3.22), the principal symbol of $\mathcal{B}$ is of the form:

$$B(\xi) = \begin{bmatrix} 0_{3\times 8} & \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \\ \tilde{B}_{8\times 8} & * \end{bmatrix}. \tag{3.23}$$

Here $B(\xi)$ is a 11×11 matrix, since the boundary terms in (3.22) contain 11 equations in total and 11 (ordered) unknowns

$$(G, h^{(4)}_{\alpha\beta}), \quad 0 \leq \alpha \leq \beta \leq 3.$$

Obviously, the first boundary term $h_{\partial M} = h^{(4)}_{ij}$, $(2 \leq i \leq j \leq 3)$. Thus the first three rows of B in (3.23) contain only zeros in the first eight columns and a 3×3 identity matrix at the end. The remaining eight rows of B represent the symbol of 2nd-5th boundary terms in (3.22), with $\tilde{B}$ denoting the first eight columns which are determined by the G and $h^{(4)}_{\alpha\beta}$ $(0 \leq \alpha \leq 1,\ \alpha \leq \beta \leq 3)$ components of the corresponding boundary terms. Obviously, for the complementing boundary condition, it suffices to verify that $\tilde{B}(\eta + z\mu)$ is nonsingular when z is a root of $a(\eta + z\mu)$ in (3.21) with positive imaginary part. Detailed calculation given in §4.1.2 shows that the matrix $\tilde{B}$ is given by

$$\tilde{B} = -32^{-1} N^{-11} [(\hat{B}_1)_{8\times 4}\ (\hat{B}_2)_{8\times 4}],$$

where the first four columns are given by

$$\hat{B}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_2 \\ -2N^2\xi_2 & 0 & \xi_2 & \xi_1 \\ -2N^2\xi_3 & 0 & \xi_3 & 0 \\ 0 & \xi_1 & 2S - 2\xi_1 X^1 & -2\xi_1 X^2 \\ 0 & \xi_2 & -2\xi_2 X^1 & 2S - 2\xi_2 X^2 \\ 0 & \xi_3 & -2\xi_3 X^1 & -2\xi_3 X^2 \\ 0 & S & N^2\xi_1 - SX^1 & N^2\xi_2 - SX^2 \end{bmatrix},$$

and the last four columns are given by

$$\hat{B}_2 = \begin{bmatrix} 0 & 0 & -\xi_2 & -\xi_3 \\ \xi_3 & 0 & -\xi_2 X^1 & -\xi_3 X^1 \\ 0 & -\xi_2 X^1 & \xi_3 X^3 & -\xi_2 X^3 \\ \xi_1 & -\xi_3 X^1 & -\xi_3 X^2 & \xi_2 X^2 \\ -2\xi_1 X^3 & \xi_1 X^1 X^1 + N^2\xi_1 - 2SX^1 & \xi_1 X^1 X^2 - 2SX^2 + 2N^2\xi_2 & \xi_1 X^1 X^3 - 2SX^3 + 2N^2\xi_3 \\ -2\xi_2 X^3 & \xi_2 X^1 X^1 - N^2\xi_2 & \xi_2 X^1 X^2 - 2SX^1 + 2N^2\xi_1 & \xi_2 X^1 X^3 \\ 2S - 2\xi_3 X^3 & \xi_3 X^1 X^1 - N^2\xi_3 & \xi_3 X^1 X^2 & \xi_3 X^1 X^3 - 2SX^1 + 2N^2\xi_1 \\ N^2\xi_3 - SX^3 & 0 & 0 & 0 \end{bmatrix},$$

inside which $S = \xi_1 X^1 + \xi_2 X^2 + \xi_3 X^3$. We can simplify $\hat{B}$ using elementary row and column operation of matrices (cf.§4.1.2 for the detailed calculations) and obtain an equivalent matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\xi_2 & -\xi_3 \\ 0 & 0 & 0 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ -2N^2\xi_2 & 0 & 0 & \xi_1 & 0 & 0 & \xi_1 X^1 + \xi_3 X^3 & -\xi_2 X^3 \\ -2N^2\xi_3 & 0 & 0 & 0 & \xi_1 & 0 & -\xi_3 X^2 & \xi_1 X^1 + \xi_2 X^2 \\ 0 & \xi_1 & 2S & 0 & 0 & 2N^2\xi_1 & -2SX^2 & -2SX^3 \\ 0 & \xi_2 & 0 & 2S & 0 & 0 & 2N^2\xi_1 & 0 \\ 0 & \xi_3 & 0 & 0 & 2S & 0 & 0 & 2N^2\xi_1 \\ 0 & S & N^2\xi_1 + SX^1 & SX^2 & SX^3 & N^2 S + N^2\xi_1 X^1 & 0 & 0 \end{bmatrix}. \tag{3.24}$$

Computing the determinant of the matrix above gives

$$\det(\hat{B})(\xi) = 8N^8(\xi_1^2 - \frac{S^2}{N^2})^2(\xi_1^2 + \xi_2^2)^2.$$

If $\xi = \eta + z\mu$, then

$$\det(\hat{B})(\eta + z\mu) = 8N^8(z^2 - \frac{\langle X, \eta + z\mu\rangle^2}{N^2})^2|\eta|^4.$$

If z is a complex root of $a(\eta + z\mu) = 0$, then from (4.2) it follows,

$$|\eta + z\mu|^2 - \frac{1}{N^2}\langle X, \eta + z\mu\rangle^2 = 0,$$

i.e. $|\eta|^2 + z^2 = \frac{\langle X, \eta + z\mu\rangle^2}{N^2}$, and thus

$$\begin{aligned} \det(\tilde{B})(\eta + z\mu) &= 8N^8(z^2 - \frac{\langle X, \eta + z\mu\rangle^2}{N^2})^2|\eta|^4 \\ &= 8N^8(z^2 - z^2 - |\eta|^2)^2|\eta|^4 \\ &= 8N^8|\eta|^8, \end{aligned}$$

which is obviously nonzero for $\eta \neq 0$. Thus the complementing boundary condition holds. This finishes the proof of Theorem 4.1.

$\square$

To conclude this section, recall that in §3.1 we proved the moduli space $\mathcal{E}$ is diffeomorphic to the solution space $\mathcal{C}$ constructed according to the BVP (3.15), which admits a Banach manifold structure. Thus, $\mathcal{C}$ can be interpreted geometrically as a local coordinate chart of the moduli space $\mathbb{E}$, and the map $\tilde{\Pi}$ is exactly the map Π expressed in this chart. However, such a local chart is effective only if the Assumption 3.3 holds. In the following section, we will develop an alternative local chart at a reference metric $\tilde{g}^{(4)}$ in $\mathcal{E}$ where Assumption 3.3 may not hold. Furthermore, we show that the ellipticity result still holds in this case.

Remark 3.7. The operator $\beta_{g^{(4)}}\delta^*_{g^{(4)}}$ with Dirichlet boundary condition is elliptic and self-adjoint. This is shown in §3.4 using the quotient formalism of stationary spacetimes. When expressed on the quotient manifold (S, g_S), this operator contains negative 0-order terms generated by the twist tensor of the metric. Thus if the metric is not static, these 0-order terms may create a nontrivial kernel of the operator, in which case Assumption 3.3 might fail. However, because of ellipticity and self-adjointness, this operator must be invertible at least for generic metrics in the space $\mathbb{E}$. It would be interesting to understand whether invertibility holds for all $g^{(4)} \in \mathbb{E}$.

3.4 Alternative charts

In this section, we assume that $\tilde{g}^{(4)}$ is a fixed stationary vacuum metric where the Assumption 3.3 fails.

3.4.1 Perturbation of the metric

We will use the projection formalism of stationary spacetimes in this section. Suppose that in the projection formalism, $\tilde{g}^{(4)}$ is expressed as above,

$$\tilde{g}^{(4)} = -u^2(dt+\theta)^2 + \pi^* g_S. \tag{3.25}$$

Take a smooth curve (parametrized by ϵ) of perturbations of $\tilde{g}^{(4)}$ given by,

$$g^{(4)}_\epsilon = \tilde{g}^{(4)} + \epsilon(dt+\theta)^2. \tag{3.26}$$

First we prove the following property of this family of metrics.

Proposition 3.8. *The metric g_ϵ is Bianchi-free, i.e.*

$$\beta_{\tilde{g}^{(4)}} g^{(4)}_\epsilon = 0.$$

Proof. Clearly by (3.26),

$$\beta_{\tilde{g}^{(4)}} g^{(4)}_\epsilon = \epsilon \beta_{\tilde{g}^{(4)}}(dt+\theta)^2.$$

Let

$$\alpha = (dt+\theta), \tag{3.27}$$

then obviously $\alpha(\partial_t) = 1$, $\alpha(v) = 0$, $\forall v \in TS$, and hence $tr_{g^{(4)}}\alpha^2 = -u^{-2}$. As a result,

$$\beta_{\tilde{g}^{(4)}}(\alpha^2) = \delta_{\tilde{g}^{(4)}}(\alpha^2) + \frac{1}{2}d(tr_{\tilde{g}^{(4)}}\alpha^2) = \delta_{\tilde{g}^{(4)}}(\alpha^2) + u^{-3}du. \tag{3.28}$$

For the divergence term above, we have

$$\begin{aligned}\delta_{\tilde{g}^{(4)}}(\alpha^2) &= -\frac{1}{u^2}\{-\nabla_{\partial_t}[\alpha^2(\partial_t)] + \alpha^2(\nabla_{\partial_t}\partial_t)\} \\ &= \frac{1}{u^2}\nabla_{\partial_t}\alpha = -\frac{1}{u^2}\nabla_{\partial_t}(\frac{1}{u^2}\xi) \\ &= -\frac{1}{u^4}\nabla_{\partial_t}\xi = -u^{-3}du.\end{aligned} \tag{3.29}$$

Here $\xi = -u^2(dt + \theta)$ denotes the dual of ∂_t. In the calculation above, we used the fact that $\nabla_{\partial_t}\partial_t = u\nabla u$ (cf. equation (4.18)) so that $\alpha(\nabla_{\partial_t}\partial_t) = 0$ and $\nabla_{\partial_t}\xi = udu$. Equations (3.28) and (3.29) now imply that α^2 is Bianchi-free. □

In addition to Bianchi-free, the curve $g_\epsilon^{(4)}$ possesses another property — for generic ϵ, the operator $\beta_{\tilde{g}^{(4)}}\delta^*_{g_\epsilon^{(4)}}$ is invertible in the following sense:

Proposition 3.9. *In any neighborhood I of 0, there is an $\epsilon \in I$ such that the BVP with time-independent unknown $Y \in T_\delta^{m,\alpha}(V^{(4)})$ given by,*

$$\begin{cases}\beta_{\tilde{g}^{(4)}}\delta^*_{g_\epsilon^{(4)}}Y = 0 & \text{in } V^{(4)} \\ Y = 0 & \text{on } \partial V^{(4)}\end{cases} \tag{3.30}$$

has only the trivial solution $Y = 0$.

To prove this proposition, we state the following lemma.

Lemma 3.10. *The BVP (3.30) is elliptic (for ϵ small) and formally self-adjoint.*

Proof. Since $\delta^*_{g_\epsilon^{(4)}}Y = \frac{1}{2}L_Y g_\epsilon^{(4)} = \frac{1}{2}L_Y(\tilde{g}^{(4)} + \epsilon\alpha^2) = \delta^*_{\tilde{g}^{(4)}}Y + \frac{\epsilon}{2}L_Y\alpha^2$, one has,

$$\beta_{\tilde{g}^{(4)}}\delta^*_{g_\epsilon^{(4)}}Y = \beta_{\tilde{g}^{(4)}}\delta^*_{\tilde{g}^{(4)}}Y + \frac{\epsilon}{2}\beta_{\tilde{g}^{(4)}}L_Y\alpha^2, \tag{3.31}$$

where α is as defined in (3.27). It is shown in §4.6 that in the quotient formalism (S, g_S) the operator $\beta_{\tilde{g}^{(4)}}\delta^*_{\tilde{g}^{(4)}}Y$ can be decomposed as:

$$\begin{cases}[\beta_{\tilde{g}^{(4)}}\delta^*_{\tilde{g}^{(4)}}Y]^T = (\nabla_{g_S})^*\nabla_{g_S}Y^T + u^{-2}Y^T(u)\nabla u - u^{-1}(\nabla_{g_S})_{\nabla u}Y^T \\ \qquad\qquad +2u^2 d\theta(d\theta(Y^T)) - 2u^2 d\theta(\nabla\frac{Y^\perp}{u}) \\ \\ -[\beta_{\tilde{g}^{(4)}}\delta^*_{\tilde{g}^{(4)}}Y]^\perp = -u\Delta_{g_S}(\frac{Y^\perp}{u}) + 3\langle\nabla\frac{Y^\perp}{u}, \nabla u\rangle + 2u\langle d\theta, \nabla_{g_S}Y^T\rangle,\end{cases} \tag{3.32}$$

where ∇_{g_S} (and Δ_{g_S}) denotes connection (and Laplace operator) of g_S on the quotient manifold S. We use the superscript $''T''$ to denote the restriction of a vector field in $V^{(4)}$ to the quotient manifold S, and $''\perp''$ to denote the vertical component of a vector field, i.e. $Y^\perp = u^{-1}\langle Y, \partial_t\rangle$. Notice the leading terms of the operator in (3.32) are $[(\nabla_{g_S})^*\nabla_{g_S}Y^T]$ and $[-u\Delta_{g_S}(\frac{Y^\perp}{u})]$. Thus $\beta_{\tilde{g}^{(4)}}\delta_{\tilde{g}^{(4)}}$ is an elliptic operator on S with respect to the Dirichlet boundary condition, and so is the operator $\beta_{\tilde{g}^{(4)}}\delta^*_{g_\epsilon^{(4)}}Y$ (for ϵ small) by (3.31).

Let $Y_1, Y_2 \in T^{m,\alpha}_{\delta}(V^{(4)})$ be two time-independent vector fields which are vanishing along $\partial V^{(4)}$. Then,

$$\begin{aligned}
&\int_S \langle \beta_{\tilde{g}^{(4)}} \delta^*_{\tilde{g}^{(4)}} Y_1, Y_2 \rangle_{g^{(4)}} \cdot u \cdot dvol_{g_S} \\
=&\int_S \{ \langle [\beta_{\tilde{g}^{(4)}} \delta^*_{\tilde{g}^{(4)}} Y_1]^T, Y_2^T \rangle_{g_S} + (-[\beta_{\tilde{g}^{(4)}} \delta^*_{\tilde{g}^{(4)}} Y_1]^{\perp}) \cdot Y_2^{\perp} \} \cdot u \cdot dvol_{g_S}
\end{aligned}$$

Substituting equations in (3.32) into the integral above and then integrating by parts gives,

$$\begin{aligned}
&\int_S \{ \langle [\beta_{\tilde{g}^{(4)}} \delta^*_{\tilde{g}^{(4)}} Y_1]^T, Y_2^T \rangle_{g_S} + (-[\beta_{\tilde{g}^{(4)}} \delta^*_{\tilde{g}^{(4)}} Y_1]^{\perp}) \cdot Y_2^{\perp} \} \cdot u \cdot dvol_{g_S} \\
=&\int_S \{ \langle [\beta_{\tilde{g}^{(4)}} \delta^*_{\tilde{g}^{(4)}} Y_2]^T, Y_1^T \rangle_{g_S} + (-[\beta_{\tilde{g}^{(4)}} \delta^*_{\tilde{g}^{(4)}} Y_2]^{\perp}) \cdot Y_1^{\perp} \} \cdot u \cdot dvol_{g_S} \qquad (3.33)\\
&+ (\int_{\partial S} + \int_{\infty})[B(Y_2, Y_1) - B(Y_1, Y_2)],
\end{aligned}$$

where $B(Y_2, Y_1) = u\langle \nabla_{\mathbf{n}} Y_2^T, Y_1^T \rangle] + 2u^2 d\theta(\mathbf{n}, Y_1^T) Y_2^{\perp} + u\mathbf{n}(Y_1^{\perp}) Y_2^{\perp}$. It is obvious that the boundary integral on ∂S is zero, since Y_1, Y_2 vanish on the boundary. The boundary term at infinity $\int_\infty = \lim_{r\to\infty} \int_t \int_{S_r}$, with S_r denoting the sphere of radius r on (S, g_S), is also zero because the decay rate of the bilinear form $B(Y_1, Y_2)$ is $2\delta + 1 > 2$. Thus it follows that, the differential operator (3.32) is formally self-adjoint with respect to the measure $u \cdot dvol_{g_S}$ on S.

Remark. One has the following integration by parts formula in the spacetime $(V^{(4)}, g^{(4)})$:

$$\begin{aligned}
\int_{V^{(4)}} \langle \nabla^*_{\tilde{g}^{(4)}} \nabla_{\tilde{g}_{(4)}} Y_1, Y_2 \rangle_{\tilde{g}^{(4)}} dvol_{\tilde{g}^{(4)}} &= \int_{V^{(4)}} \langle \nabla^*_{\tilde{g}^{(4)}} \nabla_{\tilde{g}_{(4)}} Y_2, Y_1 \rangle_{\tilde{g}^{(4)}} dvol_{\tilde{g}^{(4)}} \\
&+ \int_{\partial V^{(4)}} \langle (\nabla_{\tilde{g}^{(4)}})_{\mathbf{n}} Y_2, Y_1 \rangle_{\tilde{g}^{(4)}} - \langle (\nabla_{\tilde{g}^{(4)}})_{\mathbf{n}} Y_1, Y_2 \rangle_{\tilde{g}^{(4)}}.
\end{aligned}$$

When the spacetime $(V^{(4)}, \tilde{g}^{(4)})$ is stationary, the equation above reduces to the equation (3.33) on the quotient manifold (S, g_S).

Using the same method as above, it is easy to check the following equality

$$\begin{aligned}
\int_S \langle \beta_{\tilde{g}^{(4)}} h, Y \rangle_{g^{(4)}} \cdot u \cdot dvol_{g_S} &= \int_S \langle h, \delta^*_{\tilde{g}^{(4)}} Y - \frac{1}{2}(divY) g^{(4)} \rangle_{g^{(4)}} u \cdot dvol_{g_S} \\
&+ (\int_{\partial S} + \int_{\infty}) u[-h(\partial_t, Y) + \frac{1}{2} tr_{\tilde{g}^{(4)}} h \langle \mathbf{n}, Y \rangle].
\end{aligned} \qquad (3.34)$$

holds for any time-independent symmetric 2-tensor h and vector field Y in $V^{(4)}$. Thus, as for the second term on the right side of equation (3.31), we have the following equality for all time-independent vector fields $Y_1.Y_2 \in T^{m,\alpha}_{\delta}(V^{(4)})$ which are vanishing at $\partial V^{(4)}$:

$$\begin{aligned}
&\int_S \langle \beta_{\tilde{g}^{(4)}} L_{Y_1} \alpha^2, Y_2 \rangle_{\tilde{g}^{(4)}} u \cdot dvol_{g_S} \\
=&\int_S \langle L_{Y_1} \alpha^2, \delta^* Y_2 - \frac{1}{2}(divY_2) \tilde{g}^{(4)} \rangle_{\tilde{g}^{(4)}} u \cdot dvol_{g_S} \\
=&\int_S -u^{-2} \langle\ 2d\theta(Y_1^T) - d(\frac{Y_1^{\perp}}{u}),\ \ -\frac{u^2}{2}[2d\theta(Y_2^T) - d(\frac{Y_2^{\perp}}{u})]\ \rangle_{g_S} u \cdot dvol_{g_S}
\end{aligned}$$

$$
\begin{aligned}
&= \int_S \frac{1}{2}\langle\ 2d\theta(Y_1^T) - d(\frac{Y_1^\perp}{u}),\ 2d\theta(Y_2^T) - d(\frac{Y_2^\perp}{u})\ \rangle_{g_S} u\cdot dvol_{g_S}\\
&= \int_S \langle L_{Y_2}\alpha^2, \delta^* Y_1 - \frac{1}{2}(divY_1)\tilde{g}^{(4)}\rangle_{\tilde{g}^{(4)}} u\cdot dvol_{g_S}\\
&= \int_S \langle \beta_{\tilde{g}^{(4)}} L_{Y_2}(dt+\theta)^2, Y_1\rangle_{\tilde{g}^{(4)}} u\cdot dvol_{g_S}.
\end{aligned}
$$

In the calculation above, the first equality comes from the integration by parts formula (3.34), in which the integral on the boundary ∂S is zero since $Y_2 = 0$ along ∂S, and the integral at infinity is also zero because the decay behavior of the tensor fields. Furthermore, the second equality is based on the following observations:

$$
\begin{cases}
[L_{Y_1}\alpha^2]^T = 0\\
[L_{Y_1}\alpha^2](\partial t, \partial t) = 0\\
\{[L_{Y_1}\alpha^2](\partial t)\}^T = 2d\theta(\tilde{Y}_1^T) - d(\frac{\tilde{Y}_1^\perp}{u}),
\end{cases}
\tag{3.35}
$$

and

$$
\begin{cases}
(\delta^*_{\tilde{g}^{(4)}} Y_2)^T = \delta^*_{g_S}\tilde{Y}_2^T\\
\delta^*_{\tilde{g}^{(4)}} Y_2(\partial_t,\partial_t) = -u\tilde{Y}_2^T(u)\\
[\delta^*_{\tilde{g}^{(4)}} Y_2(\partial_t)]^T = -u^2 d\theta(\tilde{Y}_2^T) + \frac{1}{2}u^2 d(\frac{\tilde{Y}_2^\perp}{u}).
\end{cases}
\tag{3.36}
$$

We refer to §4.6 for detailed proof of the equations (3.35-36).

Summing up all the facts above, we conclude that the system (3.30) is formally self-adjoint. □

Now we give proof for the Proposition 3.9.

Proof. We prove it by contradiction. Assume that the proposition is not true, so there exists an interval I which contains 0 such that for any $\epsilon \in I$, the operator $\beta_{\tilde{g}^{(4)}}\delta^*_{g_\epsilon^{(4)}}$ has a 0-eigenvector.

From Lemma 3.10, we see that system (3.30) represents a smooth curve of elliptic self-adjoint operators parametrized by ϵ on the quotient manifold (S, g_S). By the perturbation theory of self-adjoint operators (cf.[K] Theorem 3.9, [R], [W]), the eigenspaces vary smoothly with respect to ϵ. Thus, by our assumption above, there exists a smooth curve of nontrivial solutions $Y(\epsilon)$ ($\epsilon \in I$) to the system (3.30). In particular, $Y(0)$ is a nontrivial solution to (3.30) at $\epsilon = 0$. In the following we will denote it as $\tilde{Y} = Y(0)$.

Taking the linearization of (3.30) at $\epsilon = 0$, we obtain:

$$
\begin{cases}
\beta_{\tilde{g}^{(4)}}\delta^*_{\tilde{g}^{(4)}} Y' + \beta_{\tilde{g}^{(4)}}\delta^*_{g'}\tilde{Y} = 0 \quad \text{on } V^{(4)}\\
Y' = 0 \quad \text{on } \partial V^{(4)}
\end{cases}
,
\tag{3.37}
$$

where

$$
Y' = \frac{d}{d\epsilon}|_{\epsilon=0} Y(\epsilon),\ g' = \frac{d}{d\epsilon}|_{\epsilon=0} g_\epsilon^{(4)} = \alpha^2 \text{ and } \delta^*_{g'} = \frac{d}{d\epsilon}|_{\epsilon=0}\delta^*_{g(\epsilon)}.
$$

The first equation in (3.37) gives,

$$
-\beta_{\tilde{g}^{(4)}}\delta^*_{\tilde{g}^{(4)}} Y' = \beta_{\tilde{g}^{(4)}}\delta^*_{g'}\tilde{Y}.
$$

Since $\beta_{\tilde{g}^{(4)}}\delta^*_{\tilde{g}^{(4)}}$ is self-adjoint, the equation above yields that,

$$\begin{aligned}\int_{V^{(4)}} \langle \beta_{\tilde{g}^{(4)}}\delta^*_{g'}\tilde{Y}, \tilde{Y}\rangle dvol_{\tilde{g}^{(4)}} &= -\int_{V^{(4)}} \langle \beta_{\tilde{g}^{(4)}}\delta^*_{\tilde{g}^{(4)}} Y', \tilde{Y}\rangle dvol_{\tilde{g}^{(4)}} \\ &= -\int_{V^{(4)}} \langle Y', \beta_{\tilde{g}^{(4)}}\delta^*_{\tilde{g}^{(4)}}\tilde{Y}\rangle dvol_{\tilde{g}^{(4)}} \\ &= 0\end{aligned} \tag{3.38}$$

Apply integration by parts to (3.38) and obtain,

$$\int_{V^{(4)}} \langle \delta^*_{g'}\tilde{Y}, \delta^*_{\tilde{g}^{(4)}}\tilde{Y} + \frac{1}{2}(\delta\tilde{Y})\tilde{g}^{(4)}\rangle dvol_{\tilde{g}^{(4)}} = 0. \tag{3.39}$$

In the above, $\delta^*_{g'}\tilde{Y} = \frac{1}{2}L_{\tilde{Y}}g' = \frac{1}{2}L_{\tilde{Y}}\alpha^2$, since $\delta^*_{g(\epsilon)}\tilde{Y} = \frac{1}{2}L_{\tilde{Y}}g(\epsilon)$. Now apply the formulas (3.35-36) to $L_{\tilde{Y}}\alpha^2$ and $\delta^*_{\tilde{g}^{(4)}}\tilde{Y}$, and substitute them into (3.39). It follows that,

$$\int_S \frac{1}{4}u^2||2d\theta(\tilde{Y}^T) - d(\frac{\tilde{Y}^\perp}{u})||^2_{g_S} u\cdot dvol_{g_S} = 0.$$

Therefore, we have

$$2d\theta(\tilde{Y}^T) = d(\frac{\tilde{Y}^\perp}{u}). \tag{3.40}$$

Recall that $\tilde{Y}$ is a nontrivial solution to system (3.30) at $\epsilon = 0$. By applying the decomposition equations in (3.32) to the vector field $\tilde{Y}$, we express the time-independent system (3.30) (at $\epsilon = 0$) as an equivalent system:

$$\begin{cases} \nabla^*_{g_S}\nabla_{g_S}\tilde{Y}^T - \frac{1}{u}(\nabla_{g_S})_{\nabla u}\tilde{Y}^T + \frac{1}{u^2}\tilde{Y}^T(u)\nabla u \\ \qquad\qquad +2u^2 d\theta(d\theta(\tilde{Y}^T)) - 2u^2 d\theta(\nabla\frac{\tilde{Y}^\perp}{u}) = 0 \\ \\ \Delta_{g_S}(\frac{\tilde{Y}^\perp}{u}) - 3\frac{1}{u}\langle\nabla u, \nabla\frac{\tilde{Y}^\perp}{u}\rangle - 2\langle d\theta, \nabla_{g_S}\tilde{Y}^T\rangle = 0, \end{cases} \tag{3.41}$$

on (S, g_S). Observe that the last two terms in the first equation in (3.41) can be manipulated as:

$$\begin{aligned} &2u^2 d\theta(d\theta(\tilde{Y}^T)) - 2u^2 d\theta(\nabla\frac{\tilde{Y}^\perp}{u}) \\ =&2u^2 d\theta[d\theta(\tilde{Y}^T) - d(\frac{\tilde{Y}^\perp}{u})] \\ =&-2u^2 d\theta(d\theta(Y^T)), \end{aligned}$$

where the last equality is based on (3.40). Plugging this back to (3.41), we obtain

$$\nabla^*_{g_S}\nabla_{g_S}\tilde{Y}^T - \frac{1}{u}(\nabla_{g_S})_{\nabla u}\tilde{Y}^T + \frac{1}{u^2}\tilde{Y}^T(u)\nabla u - 2u^2 d\theta(d\theta(\tilde{Y}^T)) = 0$$

Pairing the equation above with $\tilde{Y}^T$ yields,

$$\frac{1}{2}\Delta_{g_S}(||\tilde{Y}^T||^2) + ||\nabla_{g_S}\tilde{Y}^T||^2 - \frac{1}{2u}(\nabla_{g_S})_{\nabla u}||\tilde{Y}^T||^2 + \frac{1}{u^2}||\tilde{Y}^T(u)||^2 + 2u^2||d\theta(\tilde{Y}^T)||^2 = 0$$

Based on this equation and the fact that $\tilde{Y}^T$ is asymptotically zero and equals to zero on ∂S, it is easy to derive by the maximum principle that $\tilde{Y}^T = 0$, and consequently $\tilde{Y}^\perp = 0$ according to the second equation in (3.41). This contradicts with the assumption that $\tilde{Y}$ is nontrivial. $\square$

Combining Propositions 3.8 and 3.9, it is straightforward to derive that,

Theorem 3.11. *In any neighborhood of $\tilde{g}^{(4)} \in \mathcal{S}$, there always exists a perturbation $g_0^{(4)}$ of $\tilde{g}^{(4)}$ such that $\beta_{\tilde{g}^{(4)}} g_0^{(4)} = 0$ (Bianchi-free) and $\beta_{\tilde{g}^{(4)}} \delta^*_{g_0^{(4)}}$ is invertible.*

3.4.2 Alternative local charts

Theorem 3.12. *Theorem 1.3 still holds without Assumption 3.3.*

Proof. In the case Assumption 3.3 fails, we take a perturbation $g_0^{(4)}$ of $\tilde{g}^{(4)}$ as described in Theorem 3.11. and modify (3.15) to a new BVP with unknowns $(g^{(4)}, F) \in \mathcal{S} \times C^{m,\alpha}_\delta(M)$ as follows:

$$\begin{cases} Ric_{g^{(4)}} - \delta^*_{g_0^{(4)}} \beta_{g^{(4)}} g_0^{(4)} = 0 \\ \Delta F = 0, \end{cases} \quad \text{on} \quad M$$
$$\begin{cases} g_{\partial M} = \gamma \\ aH_{\partial M} + btr_{\partial M} K = H \\ atr_{\partial M} K + bH_{\partial M} = k \\ \omega_{\mathbf{n}} + a^2 d_{\partial M}(a/b) = \tau \\ \beta_{g^{(4)}} g_0^{(4)} = 0. \end{cases} \quad \text{on} \quad \partial M \tag{3.42}$$

By applying Bianchi operator to the first equation above, one obtains,

$$\begin{cases} \beta_{g^{(4)}} \delta^*_{g_0^{(4)}} \beta_{g^{(4)}} g_0^{(4)} = 0 \quad \text{on } M, \\ \beta_{g^{(4)}} g_0^{(4)} = 0 \quad \text{on } \partial M. \end{cases} \tag{3.43}$$

Since the operator $\beta_{\tilde{g}^{(4)}} \delta^*_{g_0^{(4)}}$ is invertible, so is the operator $\beta_{g^{(4)}} \delta^*_{g_0^{(4)}}$ when $g^{(4)}$ is near $\tilde{g}^{(4)}$. Thus (3.43) implies that $\beta_{g^{(4)}} g_0^{(4)} = 0$. So to associate the BVP (3.42) with a natural boundary map, we first construct a solution space $\mathcal{C}_0$ near $\tilde{g}^{(4)}$ given by,

$$\mathcal{C}_0 = \{(g^{(4)}, F) \in \mathcal{S} \times C^{m,\alpha}_\delta : \ Ric_{g^{(4)}} = 0, \beta_{g^{(4)}} g_0^{(4)} = 0, \Delta F = 0 \text{ on } M \ \}.$$

Obviously, $\tilde{g}^{(4)} \in \mathcal{C}_0$ by construction. Next, as in the proof of Theorem 3.4, we need to prove that any stationary vacuum metric $g^{(4)}$ near $\tilde{g}^{(4)}$ can be transformed by a 4−dim diffeomorphism so that it satisfies the gauge condition $\beta_{g^{(4)}} g_0^{(4)} = 0$. Consider the following map:

$$\mathcal{G} : \mathcal{S} \times \mathcal{D}_4 \to (\wedge^1)^{m.\alpha}_\delta (V^{(4)})$$
$$\mathcal{G}(g^{(4)}, \Phi) = \beta_{\Phi^* g} g_0^{(4)}.$$

Notice that

$$\beta_{\Phi^* g} g_0^{(4)} = \Phi^* \{\beta_g [(\Phi^*)^{-1} g_0^{(4)}]\}.$$

Thus the linearization of $\mathcal{G}$ at $(\tilde{g}^{(4)}, Id)$ is given by,

$$D\mathcal{G}|_{(\tilde{g}^{(4)},Id)} : T\mathcal{S} \times T\mathcal{D}_4 \to (\wedge^1)^{m.\alpha}_\delta(V^{(4)})$$
$$D\mathcal{G}|_{(\tilde{g}^{(4)},Id)}[(h^{(4)}, Y)] = -\beta_{\tilde{g}^{(4)}}\delta^*_{g_0^{(4)}}Y + \beta'_{h^{(4)}}g_0^{(4)}.$$

Since in the linearization above, the operator $[-\beta_{\tilde{g}^{(4)}}\delta^*_{g_0^{(4)}}]$ is invertible, it follows by the implicit function theorem that, for any $g^{(4)}$ near $\tilde{g}^{(4)}$, there is a unique element $\Phi \in \mathcal{D}_4$ such that the gauge term $\beta_{\Phi^* g^{(4)}}g_0^{(4)}$ vanishes.

Therefore, we conclude that $\mathcal{C}_0$ is a fiber bundle over the quotient space $\mathcal{E}/\mathcal{D}_4$ with fiber being the space of harmonic functions in $C^{m,\alpha}_\delta(M)$. Furthermore, based on the Theorems 3.4 and 3.5, we conclude there exists a diffeomorphism $\mathcal{P}_0$ such that $\mathcal{C}_0 \cong \mathbb{E}$ via $\mathcal{P}_0$ and

$$\Pi_0 = \mathcal{P}_0 \circ \Pi, \tag{3.44}$$

where Π_0 is the natural boundary map defined on $\mathcal{C}_0$ given by,

$$\Pi_0 : \mathcal{C}_0 \to \mathbf{B}$$
$$\Pi_0(g^{(4)}, F) = (g_{\partial M}, aH_{\partial M} + btr_{\partial M}K, atr_{\partial M}K + bH_{\partial M}, \omega_{\mathbf{n}} + a^2 d_{\partial M}(b/a)).$$

As for ellipticity of the system (3.42), notice that since $\beta_{g_0^{(4)}}g_0^{(4)} = 0$, we have

$$(\beta_{g_0^{(4)}})'_{h^{(4)}}g_0^{(4)} = -\beta_{g_0^{(4)}}h^{(4)}.$$

Thus the linearization of the gauge term in (3.42) is given by:

$$\begin{aligned}
[-\delta^*_{g_0^{(4)}}\beta_g g_0^{(4)}]'_{h^{(4)}} &= -\delta^*_{g_0^{(4)}}(\beta_{\tilde{g}^{(4)}})'_{h^{(4)}}g_0^{(4)} \\
&= -\delta^*_{g_0^{(4)}}(\beta_{\tilde{g}^{(4)}})'_{h^{(4)}}(\tilde{g}^{(4)} + g_0^{(4)} - \tilde{g}^{(4)}) \\
&= \delta^*_{g_0^{(4)}}\beta_{\tilde{g}^{(4)}}h^{(4)} - \delta^*_{g_0^{(4)}}(\beta_{\tilde{g}^{(4)}})'_{h^{(4)}}(g_0^{(4)} - \tilde{g}^{(4)})
\end{aligned}$$

Comparing the system (3.42) with the previous one (3.15), it is easy to see that, at the reference metric $\tilde{g}^{(4)}$, the only differences between their linearizations are given by the terms

$$[\delta^*_{g_0^{(4)}}\beta_{\tilde{g}^{(4)}} - \delta^*_{\tilde{g}^{(4)}}\beta_{\tilde{g}^{(4)}}](h^{(4)}) - \delta^*_{g_0^{(4)}}(\beta_{\tilde{g}^{(4)}})'_{h^{(4)}}(g_0^{(4)} - \tilde{g}^{(4)}) \quad \text{on } M,$$
$$(\beta)'_{h^{(4)}}(g_0^{(4)} - \beta_{\tilde{g}^{(4)}}) \quad \text{on } \partial M.$$

where $h^{(4)}$ denotes the infinitesimal deformation of $g^{(4)}$. It has been proved that (3.15) is elliptic. Thus we can choose $g_0^{(4)}$ close enough to $\tilde{g}^{(4)}$ so that (3.42) is also elliptic. As a consequence Π_0 is a Fredholm map and hence so is Π because of the equivalence relation (3.44). This completes the proof.

□

3.5 Local existence and uniqueness

In this section we set the reference metric $\tilde{g}^{(4)} = \tilde{g}_0^{(4)}$, where $\tilde{g}_0^{(4)}$ is the standard flat (Minkowski) metric on $\mathbb{R} \times (\mathbb{R}^3 \setminus B)$. Since it is static, i.e. its twist tensor in the quotient formalism is zero, it is easy to verify that Assumption 3.1 holds in this case (cf.§4.6). So we can use the chart $(\mathcal{C}, \tilde{\Pi})$ in §3.2 for the Bartnik boundary map at $[\tilde{g}_0^{(4)}] \in \mathbb{E}$. Obviously, the Bartnik data of this metic is

$$\tilde{\Pi}(\tilde{g}_0^{(4)}, 0) = (g_{S^2}, 2, 0, 0), \tag{3.45}$$

where g_{S^2} is the standard metric on the unit 2-sphere S^2. In this section we apply the ellipticity result proved in the previous sections to show that in a neighborhood of the standard flat boundary data $(g_{S^2}, 2, 0, 0)$, Bartnik boundary data admits unique stationary vacuum extensions up to diffeomorphisms.

Theorem 3.13. *The kernel of* $D\tilde{\Pi}_{(\tilde{g}_0^{(4)}, 0)}$ *is trivial.*

Proof. Assume that $(h^{(4)}, G) \in \mathrm{Ker}(D\tilde{\Pi}_{(\tilde{g}_0^{(4)}, 0)})$. Since $(h^{(4)}, G) \in T\mathcal{C}$, it must be a vacuum deformation, in the sense that the following equations hold on M:

$$\begin{cases} (Ric)'_{h^{(4)}} = 0 \\ \Delta G = 0. \end{cases} \tag{3.46}$$

In addition, since elements in $\mathcal{C}$ satisfy the gauge condition $\beta_{\tilde{g}_0^{(4)}} g^{(4)} = 0$, the same equation holds for the deformation $h^{(4)}$:

$$\beta_{\tilde{g}_0^{(4)}} h^{(4)} = 0 \quad \text{on } M. \tag{3.47}$$

The vanishing of the linearized variation of the Bartnik boundary data implies:

$$\begin{cases} h_{\partial M} = 0 \\ H'_h = 0 \\ (tr_{\partial M} K)' + 2G = 0 \\ (\omega_{\mathbf{n}})' + \nabla_{\partial M} G = 0. \end{cases} \tag{3.48}$$

As we know, a stationary spacetime metric is uniquely determined by the data set (g, X, N), where g is the induced metric on the hypersurface M, X is the shift vector and N is the lapse function. For the standard metric $\tilde{g}_0^{(4)}$, the corresponding data is $(g_0, 0, 1)$ with g_0 being the flat (Euclidean) metric on $\mathbb{R}^3 \setminus B$, because $\tilde{g}_0^{(4)}$ can be expressed globally as $\tilde{g}_0^{(4)} = -dt^2 + g_0$. Thus the deformation $h^{(4)}$ can be decomposed as $h^{(4)} = (h, Y, v)$, where h is the deformation of the Riemannian metric g_0, Y is the deformation of the shift vector and v is that of the lapse function.

The vacuum condition $Ric_{g^{(4)}} = 0$ is equivalent to the following equations in terms of (g, X, N) (cf.[M]):

$$\begin{cases} K = \frac{1}{2N} L_X g \\ Ric_g + (trK)K - 2K^2 - \frac{1}{N} D^2 N + \frac{1}{N} L_X K = 0 \\ \frac{1}{N}\Delta N + |K|^2 + \frac{1}{N} tr(L_X K) = 0 \\ \delta K + d(trK) = 0. \end{cases}$$

It is easy to linearize the equations above at $(g_0, 0, 1)$ and obtain a system in terms of (h, Y, v), which is equivalent to equation (3.46), given by,

$$\begin{cases} Ric'_h - D^2 v = 0 \\ \Delta_{g_0} v = 0 \\ \delta_{g_0}\delta^*_{g_0} Y - d\delta_{g_0} Y = 0 \\ \Delta G = 0. \end{cases} \quad \text{on } M \tag{3.49}$$

The gauge equation (3.47) is equivalent to

$$\begin{cases} \delta_{g_0} Y = 0 \\ \delta_{g_0} h + \frac{1}{2} d(tr_{g_0} h - v) = 0, \end{cases} \quad \text{on } M \tag{3.50}$$

and the boundary conditions (3.48) are equivalent to:

$$\begin{cases} h_{\partial M} = 0 \\ H'_h = 0 \\ tr_{\partial M}\delta^*_{g_0} Y + 2G = 0 \\ [\delta^*_{g_0} Y(\mathbf{n})]^T + \nabla_{g_0^T} G = 0. \end{cases} \quad \text{on } \partial M \tag{3.51}$$

Here we use the superscript $''T''$ to denote the restriction of tensors to the tangent bundle of ∂M. The first two equations in (3.49) combined with the first two boundary conditions in (3.51) imply that $v = 0$ and $h = \delta^* Z$ for some vector field Z vanishing on ∂M — this is proved in the static case, cf.[A2]. Additionally, h must satisfy the gauge equation in (3.50). It follows that $h = 0$ on M.

It remains to prove $Y = 0$ and $G = 0$. The third equation in (3.49) and the first equation in (3.50) together imply:

$$\delta_{g_0}\delta^*_{g_0} Y = 0 \quad \text{on } M.$$

Pair the equation above with Y, and then integration by parts gives,

$$\begin{aligned} 0 &= \int_M \langle \delta_{g_0}\delta^*_{g_0} Y, Y\rangle_{g_0} dvol_{g_0} \\ &= \int_M |\delta^*_{g_0} Y|^2 - \int_{\partial M} \delta^*_{g_0} Y(\mathbf{n}, Y) - \int_\infty \delta^*_{g_0} Y(\mathbf{n}, Y) \\ &= \int_M |\delta^*_{g_0} Y|^2 - \int_{\partial M} \delta^*_{g_0} Y(\mathbf{n}, Y^T) - \int_{\partial M} \delta^*_{g_0} Y(\mathbf{n}, \mathbf{n})\tilde{Y}^\perp \end{aligned} \tag{3.52}$$

where $Y^\perp = \langle Y, \mathbf{n}\rangle_{g_0}$ and Y^T denotes the component of Y tangential to ∂M. In the second line, the boundary term at infinity $\int_\infty = \lim_{r\to\infty}\int_{S_r}$ is zero because the decay rate of $[\delta^*_{g_0} Y(\mathbf{n}, Y)]$ is $2\delta + 1 > 2$. For the second term in the last line, one has,

$$\begin{aligned} \delta^*_{g_0} Y(\mathbf{n}, Y^T) &= \langle [\delta^*_{g_0} Y(\mathbf{n})]^T, Y^T\rangle = -\langle \nabla_{g_0^T} G, Y^T\rangle \\ &= -div_{g_0^T}(G \cdot Y^T) + G \cdot div_{g_0^T} Y^T \\ &= -div_{g_0^T}(G \cdot Y^T) - \frac{1}{2}(tr_{\partial M}\delta_{g_0} Y) \cdot div_{g_0^T} Y^T. \end{aligned}$$

Here the second equality comes from the last boundary equation in (3.51) and the last equality is based on the third boundary equation in (3.51). As for the last term in (3.52), notice that we have the following equality on the boundary:

$$0 = \delta_{g_0} Y = -\delta^*_{g_0} Y(\mathbf{n}, \mathbf{n}) - tr_{\partial M} \delta^*_{g_0} Y,$$

so that $\delta^*_{g_0} Y(\mathbf{n}, \mathbf{n}) = -tr_{\partial M} \delta^*_{g_0} Y$. Also $tr_{\partial M} \delta^*_{g_0} Y = tr_{\partial M} \delta^*_{g_0} Y^T + tr_{\partial M} \delta^*_{g_0} (Y^\perp \mathbf{n}) = div_{g_0^T} Y^T + H_{g_0} Y^\perp = div_{g_0^T} Y^T + 2Y^\perp$ Substituting these computations into the integral equation (3.52) gives,

$$\begin{aligned}
0 = &\int_M |\delta^*_{g_0} Y|^2 \\
&+ \int_{\partial M} \frac{1}{2}(div_{g_0^T} Y^T + 2Y^\perp) \cdot div_{g_0^T} Y^T + \int_{\partial M} (div_{g_0^T} Y^T + 2Y^\perp) \tilde{Y}^\perp \\
= &\int_M |\delta^*_{g_0} Y|^2 + \frac{1}{2} \int_{\partial M} (div_{g_0^T} Y^T)^2 + 4Y^\perp \cdot div_{g_0^T} Y^T + 4(Y^\perp)^2 \\
= &\int_M |\delta^*_{g_0} Y|^2 + \frac{1}{2} \int_{\partial M} (div_{g_0^T} Y^T + 2Y^\perp)^2.
\end{aligned}$$

It immediately follows,

$$\begin{aligned}
&\delta^*_{g_0} Y = 0 \quad \text{on } M, \\
&tr_{\partial M} \delta^*_{g_0} Y = 0 \quad \text{on } \partial M.
\end{aligned} \tag{3.53}$$

The first equation above implies that Y is a Killing vector field of the flat metric g_0 on $\mathbb{R}^3 \setminus B$. In addition Y must be asymptotically zero since it comes from a deformation of the asymptotically flat metrics in $\mathcal{C}$. Thus it follows $Y = 0$ on M. The boundary equation in (3.53) implies that $G = 0$ on ∂M according to (3.51). Furthermore, G is harmonic according to (3.49). So $G = 0$ on M. □

Next, we prove that the Fredholm map $D\tilde{\Pi}_{(\tilde{g}_0^{(4)},0)}$ is of index 0 by showing the operator $D\mathcal{F} = (D\mathcal{L}, D\mathcal{B})$ defined in §4 has index 0 at $(\tilde{g}_0^{(4)}, 0)$. Here we use the idea in [A1] — the boundary data in $D\mathcal{B}$ can be continuously deformed to a collection of self-adjoint boundary data $D\tilde{\mathcal{B}}$, which is defined as follows:

$$\begin{aligned}
D\tilde{\mathcal{B}} : T_{(\tilde{g}_0^{(4)},0)}&[\mathcal{S} \times C^{m,\alpha}_\delta](M) \to \mathbb{B} \\
D\tilde{\mathcal{B}}(h^{(4)}, G) = (\quad & h_{\partial M}, \\
& \nabla_{\mathbf{n}}(h^{(4)}(\mathbf{n}, \mathbf{n})), \\
& \mathbf{n}(G), \\
& -\frac{1}{2}\nabla_{\mathbf{n}}[h^{(4)}(\partial_t)]^T, \\
& -\frac{1}{2}\nabla_{\mathbf{n}} h^{(4)}(\partial_t, \partial_t), \\
& -\nabla_{\mathbf{n}} h^{(4)}(\mathbf{n})^T, \\
& -\nabla_{\mathbf{n}} h^{(4)}(\partial_t, \mathbf{n}) \quad).
\end{aligned} \tag{3.54}$$

Let $\mathcal{N}$ denote the space of deformations $(h^{(4)}, G)$ of $(\tilde{g}_0^{(4)}, 0)$ in $\mathcal{S} \times C^{m,\alpha}_\delta(M)$ that are in the kernel of the boundary operator $D\tilde{\mathcal{B}}$, i.e.

$$\mathcal{N} = \{\ (h^{(4)}, G) \in T_{(\tilde{g}_0^{(4)},0)}[\mathcal{S} \times C^{m,\alpha}_\delta](M) : \quad D\tilde{\mathcal{B}}(h^{(4)}, G) = 0 \quad \}.$$

Lemma 3.14. *The operator $D\mathcal{L} : \mathcal{N} \to [(S_2)^{m-2,\alpha}_{\delta+2} \times C^{m-2,\alpha}_{\delta+2}](M)$, given by*

$$D\mathcal{L}(h^{(4)}, G) = (D^*_{\tilde{g}_0^{(4)}} D_{\tilde{g}_0^{(4)}} h^{(4)}, \Delta G),$$

is formally self-adjoint.

Proof. Let $(h^{(4)}, G), (k^{(4)}, J)$ denote two deformations in $\mathcal{N}$. Integration by parts yields:

$$\int_M \langle D\mathcal{L}(h^{(4)}, G), (k^{(4)}, J)\rangle_{\tilde{g}_0^{(4)}} dvol_{g_0} = \int_M \langle D\mathcal{L}(k^{(4)}, J), (h^{(4)}, G)\rangle_{\tilde{g}_0^{(4)}}$$
$$+ \int_{\partial M} B[(k^{(4)}, J), (h^{(4)}, G)] - B[(h^{(4)}, G), (k^{(4)}, J)].$$

Here the boundary term at infinity is zero because of the decay behavior of the the deformations. The bilinear form B is given by,

$$B[(k^{(4)}, J), (h^{(4)}, G)] = \langle \nabla_{\mathbf{n}} k^{(4)}, h^{(4)}\rangle_{\tilde{g}_0^{(4)}} + \mathbf{n}(J)G.$$

It is easy to verify that the terms above are zero because $(h^{(4)}, G)$ and $(k^{(4)}, J)$ make all the boundary terms listed in (3.54) vanish. Therefore $D\mathcal{L}$ is formally self-adjoint. $\square$

In particular, it follows that the operator $(D\mathcal{L}, D\tilde{\mathcal{B}})$ is of index 0. Next we show that the boundary data in $D\mathcal{B}$ can be deformed through elliptic boundary values to $D\tilde{\mathcal{B}}$. Define a family of boundary operator $D\mathcal{B}_t,\ t \in [0,1]$ as follows,

$$\begin{aligned}
D\mathcal{B}_t &: T_{(\tilde{g}_0^{(4)},0)}[\mathcal{S} \times C^{m,\alpha}_\delta](M) \to \mathbb{B} \\
D\mathcal{B}_t(h^{(4)}, G) = (\quad & h_{\partial M}, \\
& (1-t)(H_{\partial M})'_{h^{(4)}} + t\nabla_{\mathbf{n}}(h^{(4)}(\mathbf{n}, \mathbf{n})), \\
& (1-t)(tr_{\partial M} K)'_{h^{(4)}} + t\mathbf{n}(G), \\
& -\frac{1}{2}[\nabla_{\mathbf{n}}[h^{(4)}(\partial_t)](e_i) + (1-t)\nabla_{e_i}[h^{(4)}(\partial_t)](\mathbf{n})] + (1-t)e_i(G), \\
& -\frac{1}{2}\nabla_{\mathbf{n}} h^{(4)}(\partial_t, \partial_t) + (1-t)[\frac{1}{2}\nabla_{\mathbf{n}} tr_M h + \delta h(\mathbf{n})], \\
& -\nabla_{\mathbf{n}} h^{(4)}(\mathbf{n})^T + (1-t)[-\nabla_{e_i} h^{(4)}(e_i)^T + \frac{1}{2}\nabla_{g_0^T}(tr h^{(4)})], \\
& -\nabla_{\mathbf{n}} h^{(4)}(\mathbf{n}, \partial_t) - (1-t)\nabla_{e_i} h^{(4)}(e_i, \partial_t) \quad).
\end{aligned}$$

Here $\{e_i\},\ i = 2, 3$ denotes an orthonormal basis of $T(\partial M)$. It is easy to check that $D\mathcal{B}_1 = D\tilde{\mathcal{B}}$ and $D\mathcal{B}_0 = D\mathcal{B}$, where the last three lines above are respectively the $\mathbf{n}$, tangential (∂M) and ∂_t components of the gauge term $\beta_{\tilde{g}_0^{(4)}} h^{(4)}$ when $t = 0$.

Lemma 3.15. *The operator $(D\mathcal{L}, D\mathcal{B}_t)$ is elliptic for $t \in [0,1]$.*

Proof. One can carry out the same proof as in §4. Since the shift vector and lapse function of $\tilde{g}_0^{(4)}$ are simply $X = 0$ and $N = 1$. The interior and boundary matrices are much simpler than that in §4.2, given by,

$$L(\xi) = (\xi_1^2 + \xi_2^2 + \xi_3^2) I_{11\times 11}$$

and,

$$B_t(\xi) = \begin{bmatrix} 0_{3\times 8} & \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \\ (\tilde{B}_t)_{8\times 8} & * \end{bmatrix},$$

where $\tilde{B}_t$ is as follows,

$$\tilde{B}_t = -\frac{1}{32}\begin{bmatrix} 0 & 0 & 0 & 0 & t\xi_1 & -(1-t)\xi_2 & -(1-t)\xi_3 & 0 \\ 0 & 0 & (1-t)\xi_2 & (1-t)\xi_3 & 0 & 0 & 0 & -t\xi_1 \\ 0 & (1-t)\xi_2 & \xi_1 & 0 & 0 & 0 & 0 & -2(1-t)\xi_2 \\ 0 & (1-t)\xi_3 & 0 & \xi_1 & 0 & 0 & 0 & -2(1-t)\xi_3 \\ \xi_1 & 0 & 0 & 0 & (1-t)\xi_1 & 2(1-t)\xi_2 & 2(1-t)\xi_3 & 0 \\ (1-t)\xi_2 & 0 & 0 & 0 & -(1-t)\xi_2 & 2\xi_1 & 0 & 0 \\ (1-t)\xi_3 & 0 & 0 & 0 & -(1-t)\xi_3 & 0 & 2\xi_1 & 0 \\ 0 & \xi_1 & (1-t)\xi_2 & (1-t)\xi_3 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The determinant of $B_t(\xi)$ is

$$\det B_t(\xi) = -\frac{1}{32}[t\xi_1^4 - (2+t)(1-t)^2\xi_1^2(\xi_2^2+\xi_3^2)] \cdot [2(2+t)(1-t)^2(\xi_2^2+\xi_3^3)\xi_1^2 - 4t\xi_1^4].$$

Let $\xi = z\mu + \eta$, where $z = i|\eta|^2$, the root of $\det L(z\mu+\eta) = 0$ with positive imaginary part. Then

$$\det(B_t(z\mu+\eta)) = \frac{1}{32}[t + (2+t)(1-t)^2] \cdot [2(2+t)(1-t)^2 + 4t]|\eta|^8,$$

which obviously never vanishes for $t \in [0,1]$, $\eta \neq 0$. Thus the complementing boundary condition holds for all $t \in [0,1]$, which completes the proof. □

To conclude, we have the following theorem:

Theorem 3.16. *The boundary map $\tilde{\Pi}$ is locally a diffeomorphism near $(\tilde{g}_0^{(4)}, 0)$.*

Proof. From Lemma 3.14, 3.15 and the homotopy invariance of the index, it follows that the index of the boundary map $\tilde{\Pi}$ is 0 at $(\tilde{g}_0^{(4)}, 0)$. In addition, it is proved in Theorem 3.13 that the kernel of $D\tilde{\Pi}_{(\tilde{g}_0^{(4)},0)}$ is trivial. Thus, the linearization $D\tilde{\Pi}_{(\tilde{g}_0^{(4)},0)}$ is an isomorphism. Then the inverse function theorem in Banach spaces gives the theorem. □

Now the equivalence relation between the maps $\tilde{\Pi}$ and Π, proved in the Theorem 3.4, gives Theorem 1.4.

4 Appendix

In this section we provide the details of the computation of the differential operators and some other basic results used in the previous sections. We refer to [Be] for elementary geometric formulas.

4.1 Linearization of differential operators

4.1.1 Linearization of the differential operators in projection formalism

Let h be the infinitesimal deformation of the metric g on S. The variation of Ricci tensor is given by,

$$2Ric'_h = D^*Dh - 2\delta^*\delta h - D^2(trh) + O_0. \tag{4.1}$$

The variation of scalar curvature is given by,

$$s'_h = \Delta_g(trh) + \delta\delta h + O_0. \tag{4.2}$$

Linearization of the gauge term δ^*G in operator $\mathcal{L}$ are as follows.
For the Bianchi gauge, we have,

$$\begin{aligned} 2[\delta^*G_1]'_h &= 2[\delta^*\beta_{\tilde{g}}(g)]'_h \\ &= L_{\beta_{\tilde{g}}g}h + 2\delta^*\beta_{\tilde{g}}h \\ &= L_{\beta_{\tilde{g}}g}h + 2\delta^*\delta_{\tilde{g}}h + D^2(trh); \end{aligned} \tag{4.3}$$

and for the divergence gauge, we have,

$$2[\delta^*G_2]'_h = 2[\delta^*\delta_{\tilde{g}}g]'_h = L_{\delta_{\tilde{g}}g}h + 2\delta^*\delta_{\tilde{g}}h. \tag{4.4}$$

Combining equations (4.1) and (4.3), one can derive the linearization for $\mathcal{L}$, with the Binachi gauge, at $\tilde{g} = g$:

$$L_1(h) = D^*Dh + O_0.$$

Combining equations $(4.1-2)$ and (4.4), one can derive the linearization for $\mathcal{L}$, with the divergence gauge, at $\tilde{g} = g$:

$$L_2(h) = D^*Dh - D^2(trh) - (\Delta_g(trh) + \delta\delta h)g + O_0.$$

We know that the normal vector $\mathbf{n}$ of ∂S satisfies the following equations,

$$\begin{cases} g(\mathbf{n}, \mathbf{n}) = 1, \\ g(\mathbf{n}, T) = 0, \end{cases}$$

where T is a tangential vector. Let $\mathbf{n}'_h$ denote the variation of $\mathbf{n}$ with respect to deformation h. Then linearization of the above equations gives,

$$\begin{cases} 2g(\mathbf{n}, \mathbf{n}'_h) + h(\mathbf{n}, \mathbf{n}) = 0, \\ g(\mathbf{n}'_h, T) + h(\mathbf{n}, T) = 0, \end{cases}$$

from which, one can solve for the term $\mathbf{n}'_h$ as,

$$\mathbf{n}'_h = -\frac{1}{2}h(\mathbf{n}, \mathbf{n})\mathbf{n} - h(\mathbf{n})^T = -h(\mathbf{n}) + \frac{1}{2}h(\mathbf{n}, \mathbf{n})\mathbf{n}. \tag{4.5}$$

The variation H'_h of mean curvature H_g is given by

$$2H'_h = 2(trA)'_h = 2trA'_h - 2\langle A_g, h\rangle, \tag{4.6}$$

where A_g is the second fundamental form of $\partial S \subset (S, g)$, defined by $A_g = \frac{1}{2}L_{\mathbf{n}}g$. Linearization of A is as follows,

$$2A'_h = (L_{\mathbf{n}}h + L_{\mathbf{n}'_h}g) = \nabla_{\mathbf{n}}h + 2h \circ \nabla\mathbf{n} + L_{\mathbf{n}'_h}g.$$

Taking the trace of the equation above, we obtain,

$$2trA'_h = \nabla_{\mathbf{n}}trh + 2\langle h, A\rangle - 2\delta(\mathbf{n}'_h).$$

Pluging the expression (4.5) for $\mathbf{n}'_h$ into the equation above, we obtain,

$$\begin{aligned} 2trA'_h &= \nabla_{\mathbf{n}}trh + 2\langle h, A\rangle + 2\delta^T(h(\mathbf{n})^T) - \mathbf{n}(h(\mathbf{n},\mathbf{n})) + O_0 \\ &= \nabla_{\mathbf{n}}tr^T h + 2\langle h, A\rangle + 2\delta^T(h(\mathbf{n})^T) + O_0. \end{aligned} \tag{4.7}$$

Combining equations (4.6) and (4.7) gives,

$$H'_h = \frac{1}{2}\nabla_0(h_{11} + h_{22}) - \Sigma_{k=1}^2 \nabla^k(h_{0k}) + O_0,$$

which is the same as used in the symbol computation in §2.4.

4.1.2 Linearization of the Bartnik boundary operator

For simplicity of notation, we will write h instead of $h^{(4)}$ in this section. Subindex 1 denotes the outward normal direction to ∂M and $2, 3$ denote the tangential directions on ∂M.

1.With respect to the deformation h, linearization of $g_{\partial M}$ is easily seen to be:

$$[Dg_{\partial M}](h) = (h_{22}, h_{23}, h_{33}).$$

2.Linearization of $H_{\partial M}$:
By the formula of the linearization of mean curvature, one has

$$2DH_{\partial M}(h) = -2\partial_2 h_{12} - 2\partial_3 h_{13} + \partial_1(h_{22} + h_{33}) + O_0(h).$$

3.Linearization of the second fundamental form K:
The defining equation for K is

$$K_{ij} = -\frac{1}{2N}L_{X^\sharp}g_{ij},$$

where g_{ij} denotes the Riemannian metric induced from $g^{(4)}$ on M, and $X_i = g^{(4)}_{0i}$ denotes the shift 1−form on M. Here $X^\sharp$ (shift vector) is the dual of X with respect to the metric g on M. Thus, one obtains,

$$DK(h)_{ij} = -\frac{1}{2N}(L_{(X^\sharp)'}g_{ij} + L_{X^\sharp}h_{ij}) + O_0(h).$$

As for the variation $(X^\sharp)'$, it is given by,

$$\begin{aligned} (X^\sharp)^i &= g^{ik}g^{(4)}_{0k}, \\ [(X^\sharp)^i]' &= \tilde{h}^{ik}g^{(4)}_{0k} + g^{ik}h_{0k}, \end{aligned}$$

where $\tilde{h}$ is the variation of the inverse g^{ij}. It is easy to see that

$$\tilde{h}^{ij} g_{jk} = -g^{ij} h_{jk}.$$

Therefore,

$$\begin{aligned}
L_{[X^\sharp]'} g_{ij} &= g_{i\alpha} \nabla_j [(X^\sharp)']^\alpha + g_{j\alpha} \nabla_i [(X^\sharp)']^\alpha \\
&= \nabla_j \{g_{i\alpha} [(X^\sharp)']^\alpha\} + \nabla_i \{g_{j\alpha} [(X^\sharp)']^\alpha\} \\
&= \nabla_j \{g_{il} \tilde{h}^{lk} g^{(4)}_{0k} + g_{il} g^{lk} h_{0k}\} + \nabla_i \{g_{jl} \tilde{h}^{lk} g^{(4)}_{0k} + g_{jl} g^{lk} h_{0k}\} \\
&= \nabla_j \{-h_{il} g^{lk} g^{(4)}_{0k} + h_{0i}\} + \nabla_i \{-h_{jl} g^{lk} g^{(4)}_{0k} + h_{0j}\} \\
&= \nabla_j \{-h_{il} (X^\sharp)^l + h_{0i}\} + \nabla_i \{-h_{jl} (X^\sharp)^l + h_{0j}\},
\end{aligned}$$

and

$$L_{X^\sharp} h_{\alpha\beta} = \nabla_{X^\sharp} h_{\alpha\beta} + h_{\alpha\sigma} \nabla_\beta (X^\sharp)^\sigma + h_{\beta\sigma} \nabla_\alpha (X^\sharp)^\sigma.$$

Thus,

$$DK(h)_{ij} = -\frac{1}{2N} [\partial_i h_{0j} + \partial_j h_{0i} + \partial_{X^\sharp} h_{ij} - (X^\sharp)^l (\partial_i h_{jl} + \partial_j h_{il})] + O_0(h),$$

and consequently,

$$\begin{aligned}
[Dtr_{\partial M} K](h) &= tr_{\partial M}(DK) + O(h) \\
&= -\frac{1}{2N} [2\partial_2 h_{02} + 2\partial_3 h_{03} + \partial_{X^\sharp}(h_{22} + h_{33}) - (X^\sharp)^l (2\partial_2 h_{2l} + 2\partial_3 h_{3l})] + O_0(h), \\
[D\omega_{\mathbf{n}}](h)_k &= D[K(\mathbf{n})|_{\partial M}]_k \\
&= [DK(\mathbf{n})|_{\partial M}]_k + O_0(h) \\
&= -\frac{1}{2N} [\partial_1 h_{0k} + \partial_k h_{01} + \partial_{X^\sharp} h_{1k} - (X^\sharp)^l (\partial_1 h_{kl} + \partial_k h_{1l})] + O_0(h),
\end{aligned}$$

with $k = 2, 3$.

4. Linearization of the gauge term $\beta_{g^{(4)}} g^{(4)}$.

Obviously $D[\beta_{g^{(4)}} g^{(4)}](h) = \beta_{\tilde{g}^{(4)}} h$. For $Y \in T(V^{(4)})$,

$$\beta_{g^{(4)}} h(Y) = \delta_{g^{(4)}} h(Y) + \frac{1}{2} Y(trh),$$

and

$$\begin{aligned}
\delta_{g^{(4)}} h(Y) &= \nabla_{\mathbf{N}} h(\mathbf{N}, Y) - \nabla_k h(k, Y) + O_0(h) \\
&= \frac{1}{N^2} \nabla_{\partial_t - X} h(\partial_t - X, Y) - \nabla_k h(k, Y) + O_0(h) \\
&= -\frac{1}{N^2} \partial_X h(\partial_t - X, Y) - \partial_k h(k, Y) + O_0(h), \\
trh &= -h(\mathbf{N}, \mathbf{N}) + h_{11} + h_{22} + h_{33} \\
&= -\frac{1}{N^2} h(\partial_t - X, \partial_t - X) + h_{11} + h_{22} + h_{33} \\
&= -\frac{1}{N^2} (h_{00} + X^i X^j h_{ij} - 2X^l h_{0l}) + h_{11} + h_{22} + h_{33}.
\end{aligned}$$

Therefore, for $i = 1, 2, 3$,

$$\begin{aligned}[\beta_{\tilde{g}^{(4)}} h^{(4)}]_i = &-\frac{1}{2N^2}[\partial_i h_{00} + X^k X^j \partial_i h_{kj} - 2X^l \partial_i h_{0l}] + \frac{1}{2}\partial_i (h_{11} + h_{22} + h_{33}) \\ &- \frac{1}{N^2}[\partial_X h_{0i} - X^k \partial_X h_{ki}] - \partial_k h_{ki} + O_0(h);\end{aligned}$$

and

$$[\beta_{\tilde{g}^{(4)}} h^{(4)}]_0 = -\frac{1}{N^2}[\partial_X h_{00} - X^k \partial_X h_{k0}] - \partial_k h_{k0} + O_0(h).$$

Summing up all the computations above, we obtain the boundary symbol matrix $\tilde{B}$ in §3.3.2, given by (up to a scalar $-32^{-1}N^{-11}$),

$$\hat{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\xi_2 & -\xi_3 \\ 0 & 0 & 0 & \xi_2 & \xi_3 & 0 & -\xi_2 X^1 & -\xi_3 X^1 \\ -2N^2\xi_2 & 0 & \xi_2 & \xi_1 & 0 & -\xi_2 X^1 & \xi_3 X^3 & -\xi_2 X^3 \\ -2N^2\xi_3 & 0 & \xi_3 & 0 & \xi_1 & -\xi_3 X^1 & -\xi_3 X^2 & \xi_2 X^2 \\ 0 & \xi_1 & 2S - 2\xi_1 X^1 & -2\xi_1 X^2 & -2\xi_1 X^3 & \xi_1 X^1 X^1 + N^2\xi_1 - 2SX^1 & \xi_1 X^1 X^2 - 2SX^2 + 2N^2\xi_2 & \xi_1 X^1 X^3 - 2SX^3 + 2N^2\xi_3 \\ 0 & \xi_2 & -2\xi_2 X^1 & 2S - 2\xi_2 X^2 & -2\xi_2 X^3 & \xi_2 X^1 X^1 - N^2\xi_2 & \xi_2 X^1 X^2 - 2SX^1 + 2N^2\xi_1 & \xi_2 X^1 X^3 \\ 0 & \xi_3 & -2\xi_3 X^1 & -2\xi_3 X^2 & 2S - 2\xi_3 X^3 & \xi_3 X^1 X^1 - N^2\xi_3 & \xi_3 X^1 X^2 & \xi_3 X^1 X^3 - 2SX^1 + 2N^2\xi_1 \\ 0 & S & N^2\xi_1 - SX^1 & N^2\xi_2 - SX^2 & N^2\xi_3 - SX^3 & 0 & 0 & 0 \end{bmatrix},$$

inside which $S = \xi_1 X^1 + \xi_2 X^2 + \xi_3 X^3$.

We now carry out the following row and column operation to simplify $\hat{B}$. First, multiply the first row of $\hat{B}$ by $-X^1$ and then add it to the second row. Multiply the first row by $2N^2$ and then add it to the fifth row. The matrix becomes:

$$\hat{B}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\xi_2 & -\xi_3 \\ 0 & 0 & 0 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ -2N^2\xi_2 & 0 & \xi_2 & \xi_1 & 0 & -\xi_2 X^1 & \xi_3 X^3 & -\xi_2 X^3 \\ -2N^2\xi_3 & 0 & \xi_3 & 0 & \xi_1 & -\xi_3 X^1 & -\xi_3 X^2 & \xi_2 X^2 \\ 0 & \xi_1 & 2S - 2\xi_1 X^1 & -2\xi_1 X^2 & -2\xi_1 X^3 & \xi_1 X^1 X^1 + N^2\xi_1 - 2SX^1 & \xi_1 X^1 X^2 - 2SX^2 & \xi_1 X^1 X^3 - 2SX^3 \\ 0 & \xi_2 & -2\xi_2 X^1 & 2S - 2\xi_2 X^2 & -2\xi_2 X^3 & \xi_2 X^1 X^1 - N^2\xi_2 & \xi_2 X^1 X^2 - 2SX^1 + 2N^2\xi_1 & \xi_2 X^1 X^3 \\ 0 & \xi_3 & -2\xi_3 X^1 & -2\xi_3 X^2 & 2S - 2\xi_3 X^3 & \xi_3 X^1 X^1 - N^2\xi_3 & \xi_3 X^1 X^2 & \xi_3 X^1 X^3 - 2SX^1 + 2N^2\xi_1 \\ 0 & S & N^2\xi_1 - SX^1 & N^2\xi_2 - SX^2 & N^2\xi_3 - SX^3 & 0 & 0 & 0 \end{bmatrix}.$$

In $\hat{B}_1$, multiply the second row by $(-N^2)$ and add it to the last row:

$$\hat{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\xi_2 & -\xi_3 \\ 0 & 0 & 0 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ -2N^2\xi_2 & 0 & \xi_2 & \xi_1 & 0 & -\xi_2 X^1 & \xi_3 X^3 & -\xi_2 X^3 \\ -2N^2\xi_3 & 0 & \xi_3 & 0 & \xi_1 & -\xi_3 X^1 & -\xi_3 X^2 & \xi_2 X^2 \\ 0 & \xi_1 & 2S - 2\xi_1 X^1 & -2\xi_1 X^2 & -2\xi_1 X^3 & \xi_1 X^1 X^1 + N^2\xi_1 - 2SX^1 & \xi_1 X^1 X^2 - 2SX^2 & \xi_1 X^1 X^3 - 2SX^3 \\ 0 & \xi_2 & -2\xi_2 X^1 & 2S - 2\xi_2 X^2 & -2\xi_2 X^3 & \xi_2 X^1 X^1 - N^2\xi_2 & \xi_2 X^1 X^2 - 2SX^1 + 2N^2\xi_1 & \xi_2 X^1 X^3 \\ 0 & \xi_3 & -2\xi_3 X^1 & -2\xi_3 X^2 & 2S - 2\xi_3 X^3 & \xi_3 X^1 X^1 - N^2\xi_3 & \xi_3 X^1 X^2 & \xi_3 X^1 X^3 - 2SX^1 + 2N^2\xi_1 \\ 0 & S & N^2\xi_1 - SX^1 & -SX^2 & -SX^3 & 0 & 0 & 0 \end{bmatrix}.$$

In $\hat{B}_2$, multiply the second column by N^2 and add it to the sixth column. Then multiply the second column by X^i and add it to the $(2+i)$th column $(i = 1, 2, 3)$:

$$\hat{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\xi_2 & -\xi_3 \\ 0 & 0 & 0 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ -2N^2\xi_2 & 0 & \xi_2 & \xi_1 & 0 & -\xi_2 X^1 & \xi_3 X^3 & -\xi_2 X^3 \\ -2N^2\xi_3 & 0 & \xi_3 & 0 & \xi_1 & -\xi_3 X^1 & -\xi_3 X^2 & \xi_2 X^2 \\ 0 & \xi_1 & 2S - \xi_1 X^1 & -\xi_1 X^2 & -\xi_1 X^3 & \xi_1 X^1 X^1 + 2N^2\xi_1 - 2SX^1 & \xi_1 X^1 X^2 - 2SX^2 & \xi_1 X^1 X^3 - 2SX^3 \\ 0 & \xi_2 & -\xi_2 X^1 & 2S - \xi_2 X^2 & -\xi_2 X^3 & \xi_2 X^1 X^1 & \xi_2 X^1 X^2 - 2SX^1 + 2N^2\xi_1 & \xi_2 X^1 X^3 \\ 0 & \xi_3 & -\xi_3 X^1 & -\xi_3 X^2 & 2S - \xi_3 X^3 & \xi_3 X^1 X^1 & \xi_3 X^1 X^2 & \xi_3 X^1 X^3 - 2SX^1 + 2N^2\xi_1 \\ 0 & S & N^2\xi_1 & 0 & 0 & N^2 S & 0 & 0 \end{bmatrix}.$$

In $\hat{B}_3$, multiply the ith column by X^1 and add it to the $(i+3)$th column $(i = 3, 4, 5)$. Then multiply the second column by X^i and add it to column $(i+1)$, $(i = 1, 2, 3)$.

$$\hat{B}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\xi_2 & -\xi_3 \\ 0 & 0 & 0 & \xi_2 & \xi_3 & 0 & \xi_2 X^1 & \xi_3 X^1 \\ -2N^2\xi_2 & 0 & \xi_2 & \xi_1 & 0 & 0 & \xi_3 X^3 + \xi_1 X^1 & -\xi_2 X^3 \\ -2N^2\xi_3 & 0 & \xi_3 & 0 & \xi_1 & 0 & -\xi_3 X^2 & \xi_2 X^2 + \xi_1 X^1 \\ 0 & \xi_1 & 2S - \xi_1 X^1 & -\xi_1 X^2 & -\xi_1 X^3 & 2N^2\xi_1 & -2SX^2 & -2SX^3 \\ 0 & \xi_2 & -\xi_2 X^1 & 2S - \xi_2 X^2 & -\xi_2 X^3 & 0 & 2N^2\xi_1 & 0 \\ 0 & \xi_3 & -\xi_3 X^1 & -\xi_3 X^2 & 2S - \xi_3 X^3 & 0 & 0 & 2N^2\xi_1 \\ 0 & S & N^2\xi_1 & 0 & 0 & N^2 S + N^2\xi_1 X^1 & 0 & 0 \end{bmatrix}.$$

In $\hat{B}_4$, multiply column 2 by X^i and add it to column $(i+2)$, $(i = 1, 2, 3)$. Multiply column 1 by $(2N^2)^{-1}$ and add it to column 3. Then multiply the first row by X^1 and add it to row 2:

$$\hat{B}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -\xi_2 & -\xi_3 \\ 0 & 0 & 0 & \xi_2 & \xi_3 & 0 & 0 & 0 \\ -2N^2\xi_2 & 0 & 0 & \xi_1 & 0 & 0 & \xi_3 X^3 + \xi_1 X^1 & -\xi_2 X^3 \\ -2N^2\xi_3 & 0 & 0 & 0 & \xi_1 & 0 & -\xi_3 X^2 & \xi_2 X^2 + \xi_1 X^1 \\ 0 & \xi_1 & 2S & 0 & 0 & 2N^2\xi_1 & -2SX^2 & -2SX^3 \\ 0 & \xi_2 & 0 & 2S & 0 & 0 & 2N^2\xi_1 & 0 \\ 0 & \xi_3 & 0 & 0 & 2S & 0 & 0 & 2N^2\xi_1 \\ 0 & S & N^2\xi_1 + SX^1 & SX^2 & SX^3 & N^2 S + N^2\xi_1 X^1 & 0 & 0 \end{bmatrix}.$$

This is the matrix given in (3.24).

4.2 Variation of the Einstein Hilbert functional I in projection formalism

First, we define a functional $\tilde{I}$ as,

$$\tilde{I} = \int_S s_g - 2|du|^2 - 2e^{-4u}|d\phi|^2 dvol_g.$$

Since the variation of scalar curvature s_g is given by,

$$s'_h = \Delta_g(trh) + \delta\delta h - \langle Ric_g, h\rangle,$$

linearization of I with respect to the metric is as follows,

$$\begin{aligned} \tilde{I}'(h) &= \int_S [\Delta_g(trh) + \delta\delta h - \langle Ric_g, h\rangle + 2h(du, du) + 2e^{-4u}h(d\phi, d\phi) \\ &\qquad + \frac{1}{2}trh(s_g - 2|du|^2 - 2e^{-4u}|d\phi|^2)] \\ &= \int_S [\Delta_g(trh) + \delta\delta h \\ &\qquad + \langle -Ric_g + 2du \otimes du + 2e^{-4u}d\phi \otimes d\phi + \frac{1}{2}(s_g - 2|du|^2 - 2e^{-4u}|d\phi|^2)g, h\rangle] \\ &= \int_S [\Delta_g(trh) + \delta\delta h + \langle \mathbf{E}, h\rangle]. \end{aligned}$$

The term $\int_S[\Delta_g(trh)+\delta\delta h]$ in the expression above can be converted to a boundary term as,

$$\begin{aligned}&\int_S[\Delta_g(trh)+\delta\delta h]\\&=-\int_{\partial S}[\mathbf{n}(trh)+\delta h(\mathbf{n})]-\int_{\partial S_\infty}[\mathbf{n}(trh)+\delta h(\mathbf{n})]\\&=-\int_{\partial S}[\mathbf{n}(trh)-\mathbf{n}(h_{00})+\langle h,A\rangle]-\int_{\partial S_\infty}[\mathbf{n}(trh)+\delta h(\mathbf{n})].\end{aligned}\tag{4.8}$$

To balance the finite boundary term, we add an extra term $I_B=\int_{\partial S}2H_g$ to the functional I. Notice that the first variation of I_B with respect to the metric is given by,

$$\begin{aligned}I_B'(h)&=\int_{\partial S}[2H_h'+tr^ThH_g]\\&=\int_{\partial S}[\mathbf{n}(trh)-2\delta(\mathbf{n}_h')+tr^ThH_g].\end{aligned}$$

For a generic vector field V on ∂S, we have $\delta V=\delta^TV^T-\mathbf{n}(V_0)$. Thus, we can simplify the term $\delta(\mathbf{n}_h')$ in the boundary term above and obtain,

$$I_B'(h)=\int_{\partial S}[\mathbf{n}(trh)-\mathbf{n}(h_{00})+tr^ThH_g].\tag{4.9}$$

Combining equations (4.8) and (4.9) we can obtain the formulae of variation for the functional $\tilde{I}+I_B$:

$$(\tilde{I}+I_B)'(h)=\int_S\langle\mathbf{E},h\rangle+\int_{\partial S}[-\langle A,h\rangle+tr^ThH_g]-\int_{\partial S_\infty}[\mathbf{n}(trh)+\delta h(\mathbf{n})].$$

To remove the boundary term at infinity, we use the mass $m_{ADM}(g)$, as shown in equation (2.34).

4.3 Formal self-adjointness

To prove the self-adjointness for $D\hat{\Phi}$, we will use the functional I, as defined in §2.4,

$$I=\int_S s_g-2|du|^2-2e^{-4u}|d\phi|^2dvol_g+2\int_{\partial S}Hdvol_{g^T}+16\pi m_{ADM}(g).$$

Recall from §2.4, the first variation of I is given by,

$$\begin{aligned}I'_{(\tilde{g},\tilde{u},\tilde{\phi})}(h,v,\sigma)=&\int_S\langle(\mathbf{E},\mathbf{F},\mathbf{H}),(h,v,\sigma)\rangle\\&+\int_{\partial S}[-\langle A_{\tilde{g}},h\rangle+H_{\tilde{g}}trh^T-4\mathbf{n}(u)v-4e^{-4u}\sigma\mathbf{n}(\phi)].\end{aligned}\tag{4.10}$$

Let (h,v,σ) be in the tangent space $T\mathcal{M}_S$, which is defined by

$$\begin{aligned}T\mathcal{M}_S=\{\ &(h,v,\sigma)\in[(S_2)_\delta^{m,\alpha}\times C_\delta^{m,\alpha}\times C_\delta^{m,\alpha}](S):\\&\begin{cases}\delta_{\tilde{g}}h=0,\\h^T-2v\tilde{g}^T=0,\\ [e^u(H_{\tilde{g}}-2\mathbf{n}(u))]'_{(h,v)}=0,\\ [e^{-2u}\mathbf{n}(\phi)]'_{(h,v,\sigma)}=0,\end{cases}\quad\text{on }\partial S\ \}.\end{aligned}\tag{4.11}$$

Applying the boundary conditions in (4.11) to the equation (4.10), we obtain,

$$\begin{aligned}
&I'_{(\tilde{g},\tilde{u},\tilde{\phi})}(h,v,\sigma)\\
&= \int_S \langle (\mathbf{E},\mathbf{F},\mathbf{H}),(h,v,\sigma)\rangle + \int_{\partial S}[-\langle A_{\tilde{g}}, 2v\tilde{g}\rangle + 2vH_{\tilde{g}}tr\tilde{g}^T - 4\mathbf{n}(u)v - 4e^{-4u}\sigma\mathbf{n}(\phi)]\\
&= \int_S \langle (\mathbf{E},\mathbf{F},\mathbf{H}),(h,v,\sigma)\rangle + \int_{\partial S}[2v(H_{\tilde{g}} - 2\mathbf{n}(u)) - 4e^{-4u}\sigma\mathbf{n}(\phi)].
\end{aligned}$$

Taking the variation of I' with respect a deformation $(k,w,\zeta)\in T\mathcal{M}_S$, we obtain

$$\begin{aligned}
I''_{(\tilde{g},\tilde{u},\tilde{\phi})}[(h,v,\sigma),(k,w,\zeta)] &= \int_S \langle (\mathbf{E}',\mathbf{F}',\mathbf{H}')_{(k,w,\zeta)},(h,v,\sigma)\rangle\\
&+ \int_{\partial S}\{2v[H_{\tilde{g}} - 2\mathbf{n}(u)]'_{(k,w)} - \sigma[4e^{-4u}\mathbf{n}(\phi)]'_{(k,w,\zeta)}\}\\
&+ \int_{\partial S}\{\frac{1}{2}trk^T[2v(H_{\tilde{g}} - 2\mathbf{n}(u)) - 4e^{-4u}\sigma\mathbf{n}(\phi)]\}.
\end{aligned}$$

According to (4.11), we have $k^T = 2w\tilde{g}^T$ and $2v[H_{\tilde{g}} - 2\mathbf{n}(u)]'_{(k,w)} = -2vw(H_{\tilde{g}} - 2\mathbf{n}(u))$. Thus the boundary terms in the expression above can be simplifies as

$$\begin{aligned}
&I''_{(\tilde{g},\tilde{u},\tilde{\phi})}[(h,v,\sigma),(k,w,\zeta)]\\
&= \int_S \langle (\mathbf{E}',\mathbf{F}',\mathbf{H}')_{(k,w,\zeta)},(h,v,\sigma)\rangle + \int_{\partial S} 2wv[H_{\tilde{g}} - 2\mathbf{n}(u)]'_{(k,w)}\\
&\quad + \int_{\partial S}\{-\sigma[4e^{-4u}\mathbf{n}(\phi)]'_{(k,w,\zeta)} - 8we^{-4u}\sigma\mathbf{n}(\phi)\}\\
&= \int_S \langle (\mathbf{E}',\mathbf{F}',\mathbf{H}')_{(k,w,\zeta)},(h,v,\sigma)\rangle + \int_{\partial S} 2wv[H_{\tilde{g}} - 2\mathbf{n}(u)]'_{(k,w)}\\
&\quad + \int_{\partial S} 4e^{-4u}\sigma[2w\mathbf{n}(\phi) - (\mathbf{n}(\phi))'_{(k,\zeta)}].
\end{aligned}$$

Here the last boundary term vanishes, because

$$4e^{-4u}\sigma[2w\mathbf{n}(\phi) - (\mathbf{n}(\phi))'_{(k,\zeta)}] = -4e^{-2u}[e^{-2u}\mathbf{n}(\phi)]'_{(k,w,\zeta)} = 0,$$

based on the conditions in (4.11). Therefore, from the symmetry of the 2nd order variation of the functional I, it follows that,

$$\int_S \langle (\mathbf{E}',\mathbf{F}',\mathbf{H}')_{(k,w,\zeta)},(h,v,\sigma)\rangle = \int_S \langle (\mathbf{E}',\mathbf{F}',\mathbf{H}')_{(h,v,\sigma)},(k,w,\zeta)\rangle. \tag{4.12}$$

In addition, it is easy to derive that

$$\int_S \langle \delta^*\delta k, h\rangle = \int_S \langle \delta^*\delta h, k\rangle \quad \text{for } k,h\in T\mathcal{M}_S. \tag{4.13}$$

Combining equations (4.12) and (4.13), we obtain the formal self-adjointness for $D\hat{\Phi}$, i.e.

$$\int_S \langle D\hat{\Phi}[(k,w,\zeta)],(h,v,\sigma)\rangle = \int_S \langle D\hat{\Phi}[(h,v,\sigma)],(k,w,\zeta)\rangle,\ \forall (k,w,\zeta),(h,v,\sigma)\in T\mathcal{M}_S.$$

4.4 The blow-up rate of Z in proving the unique continuation theorem

In the proof of the unique continuation theorem (Theorem 2.14), we show that the deformation $k = \delta^* Z$ for some $(k, w, \zeta) \in K$, so that $\delta^* Z$ has the decay rate δ (denoted as $\delta^* Z \sim r^{-\delta}$), we will find an upper bound for the blow-up rate of Z in the following.

For a radius r large enough, let $S_r \subset S$ be the pull back of the sphere of radius r under the chosen diffeomorphism $S \cong \mathbb{R}^3 \setminus B^3$. Let N_r denote the unit normal vector of the sphere pointing outwards, then we have

$$\delta^* Z(N_r, N_r) = N_r[\tilde{g}(Z, N_r)] + \tilde{g}(Z, \nabla_{N_r} N_r).$$

One can extend N_r in a way such that $\nabla_{N_r} N_r = 0$, and thus

$$N_r[\tilde{g}(Z, N_r)] \sim r^{-\delta}.$$

Therefore, $g(Z, N_r)$ blows up no faster than $r^{1-\delta}$.
Let Z^T denote the tangential component of Z along S_r, then $Z = Z^T + g(Z, N_r)N_r$. Basic calculation shows

$$\begin{aligned} 2\delta^* Z(N_r, Z^T) &= \tilde{g}(\nabla_{N_r} Z, Z^T) + \tilde{g}(\nabla_{Z^T} Z, N_r) \\ &= \tilde{g}(\nabla_{N_r}(Z^T + g(Z, N_r)N_r), Z^T) + Z^T[\tilde{g}(Z, N_r)] - \tilde{g}(Z, \nabla_{Z^T} N_r) \\ &= \tilde{g}(\nabla_{N_r} Z^T, Z^T) + Z^T[\tilde{g}(Z, N_r)] - A(Z^T, Z^T) \\ &= |Z^T| N_r(|Z^T|) + Z^T[\tilde{g}(Z, N_r)] - A(Z^T, Z^T), \end{aligned}$$

where A denotes the second fundamental form of the hypersurface $S_r \subset (S, \tilde{g})$. Thus we obtain,

$$N_r(|Z^T|) - A(Z^T, \frac{Z^T}{|Z^T|}) = 2\delta^* Z(N_r, \frac{Z^T}{|Z^T|}) - \frac{Z^T}{|Z^T|}[\tilde{g}(Z, N_r)].$$

This implies that $[\partial_r(|Z^T|) - \frac{1}{r}|Z^T|]$ blows up no faster than $r^{1-\delta}$, and therefore the increasing rate of $|Z^T|$ is at most $r^{2-\delta}$.

4.5 Scalar fields a, b in the time translation formula of the Bartnik boundary data

As described in Proposition 3.1, since ∂M and $\partial \hat{M}$ coincide in $V^{(4)}$ under the action of diffeomorphism $\Phi_f : \hat{M} \to M$, the unit normal vectors $(\hat{\mathbf{N}}, \hat{\mathbf{n}})$ must be mapped to a pair of vectors which are perpendicular to ∂M in $V^{(4)}$. It follows that,

$$d\Phi_f(\hat{\mathbf{N}}), d\Phi_f(\hat{\mathbf{n}}) \in \operatorname{span}\{\mathbf{N}, \mathbf{n}\}$$

Therefore there exist scalar fields a, b, c, d on ∂M so that

$$\begin{cases} d\Phi_f(\hat{\mathbf{N}}) = a\mathbf{N} + b\mathbf{n}, \\ d\Phi_f(\hat{\mathbf{n}}) = c\mathbf{N} + d\mathbf{n}. \end{cases} \tag{4.14}$$

In addition, notice that

$$\begin{aligned} &\langle d\Phi_f(\hat{\mathbf{N}}), d\Phi_f(\hat{\mathbf{N}}) \rangle_{\Phi_f^* g(4)} = \langle \mathbf{N}, \mathbf{N} \rangle_{g^{(4)}} = -1; \\ &\langle d\Phi_f(\hat{\mathbf{n}}), d\Phi_f(\hat{\mathbf{n}}) \rangle_{\Phi_f^* g(4)} = \langle \mathbf{n}, \mathbf{n} \rangle_{g^{(4)}} = 1; \\ &\langle d\Phi_f(\hat{\mathbf{N}}), d\Phi_f(\hat{\mathbf{n}}) \rangle_{\Phi_f^* g(4)} = \langle \mathbf{N}, \mathbf{n} \rangle_{g^{(4)}} = 0. \end{aligned}$$

Thus,

$$\begin{cases} -a^2 + b^2 = -1, \\ -c^2 + d^2 = 1, \\ -ac + bd = 0, \end{cases} \tag{4.15}$$

which further implies that $a^2 = d^2$ and $b^2 = c^2$. Without loss of generality (up to the choice of directions), we can assume,

$$a = d > 0, \; b = c > 0.$$

From the expression (3.1) of the metric $g^{(4)}$, it is easy to see that

$$\mathbf{N} = -\frac{\partial_t - X}{||\partial_t - X||_{g^{(4)}}} = \frac{\partial_t - X}{N}.$$

As for $\hat{\mathbf{N}}$, it must be the unit vector such that the following holds,

$$\langle d\Phi_f(\hat{\mathbf{N}}), d\Phi_f(\partial_{x^i})\rangle_{g^{(4)}} = \langle \hat{\mathbf{N}}, \partial_{x^i}\rangle_{\Phi^* g^{(4)}} = 0, \; \forall i = 1,2,3. \tag{4.16}$$

It is easy to see that $d\Phi_f(\partial_{x^i}) = (\partial_i f)\partial_t + \partial_{x^i}$ and

$$\langle \partial_t - X + N^2\nabla f, (\partial_i f)\partial_t + \partial_{x^i}\rangle_{g^{(4)}} = 0, \; \forall i = 1,2,3. \tag{4.17}$$

where ∇f denotes the gradient of f with respect to the metric $g^{(4)}$. Thus, equations (4.16) and (4.17) imply that,

$$\begin{aligned} d\Phi_f(\hat{\mathbf{N}}) &= -\frac{\partial_t - X + N^2\nabla f}{||\partial_t - X + N^2\nabla f||} \\ &= \frac{\partial_t - X + N^2\nabla f}{N\sqrt{1 + 2X(f) - ||N^2\nabla f||^2}} \\ &= \frac{\partial_t - X + N^2\nabla f}{N\sqrt{1 + 2X(f) + X(f)^2 - N^2||\nabla_g f||^2}} \\ &= \frac{\partial_t - X + N^2\nabla f}{N\sqrt{(1 + X(f))^2 - N^2||\nabla_g f||^2}}, \end{aligned}$$

where $\nabla_g f$ denotes the gradient of f with respect to the induced metric g on M. Therefore, according to the first equation in (4.14), we obtain

$$\begin{aligned} a &= -g^{(4)}(\mathbf{N}, d\Phi_f(\hat{\mathbf{N}})) \\ &= \frac{g^{(4)}(\partial_t - X, \partial_t - X + N^2\nabla f)}{N^2\sqrt{(1 + X(f))^2 - N^2||\nabla_g f||^2}} \\ &= \frac{1 + X(f)}{\sqrt{(1 + X(f))^2 - N^2||\nabla_g f||^2}}. \end{aligned}$$

Moreover, since f is chosen to be vanishing on ∂M, so $\nabla_g f = \mathbf{n}(f)\cdot\mathbf{n}$ on the boundary. Thus $||\nabla_g f|||_{\partial M} = \mathbf{n}(f)$, $X(f)|_{\partial M} = \langle X, \mathbf{n}\rangle\mathbf{n}(f)$ and consequently,

$$a = \frac{1 + \langle X, \mathbf{n}\rangle\mathbf{n}(f)}{\sqrt{[1 + \langle X, \mathbf{n}\rangle\mathbf{n}(f)]^2 - N^2|\mathbf{n}(f)|^2}} \quad \text{on } \partial M,$$

which is the formula (3.9). Based on (4.15), we easily derive the formula for b as follows,

$$b = \frac{N\mathbf{n}(f)}{\sqrt{[1 + \langle X, \mathbf{n}\rangle \mathbf{n}(f)]^2 - N^2|\mathbf{n}(f)|^2}}.$$

4.6 Useful calculations in the projection formalism

Take a general stationary metric in $V^{(4)}$ expressed in the projection formalism as,

$$g^{(4)} = -u^2(dt + \theta)^2 + g_S.$$

We first state two simple facts.

1. Since ∂_t is a Killing vector field, it follows that for any vector field $Y \in TV^{(4)}$,

$$\langle \nabla_{\partial t}\partial_t, Y\rangle = -\langle \nabla_Y \partial_t, \partial_t\rangle = uY(u).$$

Thus,

$$\nabla_{\partial_t}\partial_t = u\nabla u. \tag{4.18}$$

2. For any horizontal vector fields $v, w \in TS$, one has $\langle v, \partial_t\rangle = 0$, $L_{\partial t}v = 0$, and hence

$$\langle \nabla_v w, \partial_t\rangle = -\langle w, \nabla_v \partial_t\rangle = \langle v, \nabla_w \partial_t\rangle = -\langle \nabla_w v, \partial_t\rangle.$$

It follows that,

$$\begin{aligned} &\langle \nabla_v w, \partial_t\rangle = d\xi(w, v) = -u^2 d\theta(w, v) \\ &\xi([v, w]) = \xi(\nabla_v w - \nabla_w v) = 2\langle \nabla_v w, \partial_t\rangle = -2u^2 d\theta(w, v). \end{aligned} \tag{4.19}$$

Here $\xi = -u^2(dt + \theta)$ is the dual of ∂_t.

Next we give a proof for the formula (3.35):

Let $\alpha = dt + \theta = -u^{-2}\xi$, so $\alpha(\partial_t) = 1,\ \alpha(v) = 0\ \forall v \in TS$. Then according to the the following Lie-derivative formula for time-symmetric vectors A, B in the spacetime:

$$L_Y \alpha^2(A, B) = Y[\alpha^2(A, B)] - \alpha^2([Y, A], B) - \alpha^2(A, [Y, B]),$$

it is easy to see that

$$\begin{cases} L_Y \alpha^2(\partial t, \partial t) = 0 \\ [L_Y \alpha^2]^T = 0. \end{cases}$$

As for the mixed component of $L_Y \alpha^2$, we can carry out the following computation for $v \in TS$,

$$\begin{aligned} L_Y \alpha^2(\partial_t, v) &= -\alpha^2([Y, v], \partial_t) \\ &= -\alpha([Y, v]) \\ &= u^{-2}\xi([Y, v]). \end{aligned} \tag{4.20}$$

Any vector field $Y \in TS$ can be decomposed as,

$$Y = Y^T - \frac{Y^\perp}{u}\partial t, \text{ with } Y^\perp = \frac{1}{u}\langle Y, \partial_t\rangle. \tag{4.21}$$

Thus, for $v \in TS$, one has,

$$\begin{aligned}\xi[Y,v] &= \xi([Y^T,v]) - \xi([\frac{Y^\perp}{u}\partial_t, v]) = \xi([Y^T,v]) + \xi[v(\frac{Y^\perp}{u})\partial_t]\\ &= 2u^2 d\theta(Y^T,v) - u^2 v(\frac{Y^\perp}{u}).\end{aligned}$$

Plugging this to equation (4.20) we obtain

$$[L_Y\alpha^2(\partial_t)]^T = 2d\theta(Y^T) - d(\frac{Y^\perp}{u})$$

This completes the proof of (3.35).

Using the same notation as above, we give a proof of the formula (3.36) as follows.

Based on (4.21),

$$2\delta^*_{g^{(4)}}Y = L_{Y^T}g^{(4)} - L_{\frac{Y^\perp}{u}\partial_t}g^{(4)}. \tag{4.22}$$

In the following, we assume $v, w \in TS$. For the first term in (4.22), we have

$$\begin{aligned}L_{Y^T}g^{(4)}(\partial_t,\partial_t) &= 2\langle\nabla_{\partial_t}Y^T,\partial_t\rangle = 2\langle\nabla_{Y^T}\partial_t,\partial_t\rangle = -2uY^T(u),\\ L_{Y^T}g^{(4)}(\partial_t,v) &= \langle\nabla_{\partial_t}Y^T,v\rangle + \langle\nabla_v Y^T,\partial_t\rangle\\ &= \langle\nabla_{Y^T}\partial_t,v\rangle + \langle\nabla_v Y^T,\partial_t\rangle = -\langle\nabla_{Y^T}v,\partial_t\rangle + \langle\nabla_v Y^T,\partial_t\rangle\\ &= 2\langle\nabla_v Y^T,\partial_t\rangle\\ &= -2u^2 d\theta(Y^T,v),\\ L_{Y^T}g^{(4)}(v,w) &= \langle\nabla_v Y^T,w\rangle + \langle\nabla_w Y^T,v\rangle = L_{Y^T}g_S(v,w).\end{aligned}$$

Summing up,

$$\begin{cases} L_{Y^T}g^{(4)}(\partial_t,\partial_t) = -2uY^T(u)\\ [L_{Y^T}g^{(4)}(\partial_t)]^T = -2u^2 d\theta(Y^T)\\ [L_{Y^T}g^{(4)}]^T = L_{Y^T}g_S.\end{cases} \tag{4.23}$$

As for the second term in (4.22), basic calculation yields,

$$L_{\frac{Y^\perp}{u}\partial_t}g^{(4)} = \frac{Y^\perp}{u}L_{\partial_t}g^{(4)} + d(\frac{Y^\perp}{u})\odot\xi = d(\frac{Y^\perp}{u})\odot\xi.$$

Thus,

$$\begin{cases} L_{\frac{Y^\perp}{u}\partial_t}g^{(4)}(\partial_t,\partial_t) = 0\\ [L_{\frac{Y^\perp}{u}\partial_t}g^{(4)}(\partial_t)]^T = -u^2 d(\frac{Y^\perp}{u})\\ [L_{\frac{Y^\perp}{u}\partial_t}g^{(4)}]^T = 0.\end{cases} \tag{4.24}$$

Equations (4.23) and (4.24) together give (3.36).

At last we derive the decomposition (3.32) of the Bianchi gauge operator.

We assume $g^{(4)}$ is in addition vacuum, which is equivalent to the following system in the projection formalism, (cf.[H1],[H2]),

$$\begin{cases} Ric_{g_S} = \frac{1}{u} D^2_{g_S} u + 2u^{-4}(\omega^2 - |\omega|^2_{g_S} \cdot g_S) \\ \Delta_{g_S} u = 2u^{-3}|\omega|^2_{g_S} \\ \delta\omega + 3u^{-1}\langle du, \omega\rangle = 0 \\ d\omega = 0 \end{cases}, \tag{4.25}$$

where ω is the twist tensor defined as,

$$\omega = -\frac{1}{2} u^3 \star_{g_S} d\theta.$$

Here we use subscript $''_{g_S}''$ to denote geometric operators (connection and Laplacian) of the Riemannian metric g_S on the quotient manifold S. First observe that, from the last equation in (4.25), it follows that

$$0 = d(u^3 \star_{g_S} d\theta) = d \star_{g_S} d(u^3\theta) = \delta_{g_S}(u^3 d\theta) = u^3 \delta_{g_S} d\theta - 3u^2 d\theta(\nabla u).$$

Thus, we obtain

$$u\delta_{g_S} d\theta = 3d\theta(\nabla u). \tag{4.26}$$

Moreover, based on the second equation in (4.25), one easily obtains,

$$\Delta_{g_S} u = u^3 |d\theta|^2. \tag{4.27}$$

Now we take the operator $\beta_{g^{(4)}}\delta^*_{g^{(4)}}$ acting on a time-independent vector field Y, which is decomposed as in (4.21). To begin with, because the metric $g^{(4)}$ is vacuum, a standard Bochner-Weitzenbock formula gives,

$$2\beta_{g^{(4)}}\delta^*_{g^{(4)}} Y = \nabla^*\nabla Y - Ric_{g^{(4)}}(Y) = \nabla^*\nabla Y.$$

Based on the formula of the Laplace operator, we have,

$$\nabla^*\nabla Y = \frac{1}{u^2}[\nabla_{\partial_t}\nabla_{\partial_t} Y - \nabla_{\nabla_{\partial_t}\partial_t} Y] - \Sigma_i[\nabla_{e_i}\nabla_{e_i} Y - \nabla_{\nabla_{e_i} e_i} Y], \tag{4.28}$$

where e_i $(i = 1, 2, 3)$ are taken to be geodesic normal basis on S. In the following, we compute the tensors in (4.28) term by term.

1.For the first term, since $[Y, \partial_t] = 0$, we have,

$$\nabla_{\partial_t}\nabla_{\partial_t} Y = \nabla_{\nabla_Y \partial_t}\partial_t = \nabla_{\nabla_{Y^T}\partial_t}\partial_t - \frac{Y^\perp}{u}\nabla_{\nabla_{\partial_t}\partial_t}\partial_t = \nabla_{\nabla_{Y^T}\partial_t}\partial_t - \frac{Y^\perp}{u}\nabla_{\nabla_{u\nabla u}}\partial_t.$$

Based on (7.6),

$$\nabla_v \partial_t = -u^2 d\theta(v) + u^{-1} v(u) \cdot \partial_t \ \forall v \in TS.$$

Thus, the equation above continues as,

$$\begin{aligned} &\nabla_{\partial_t}\nabla_{\partial_t} Y \\ &= \nabla_{-u^2 d\theta(Y^T) + u^{-1} Y^T(u)\cdot\partial_t}\partial_t - \frac{Y^\perp}{u}[-u^3 d\theta(\nabla u) + u^{-1} u|\nabla u|^2 \cdot \partial_t] \\ &= u^4 d\theta(d\theta(Y^T)) - u d\theta(Y^T, \nabla u) \cdot \partial_t + Y^T(u)\nabla u + Y^\perp u^2 d\theta(\nabla u) - \frac{Y^\perp}{u}|\nabla u|^2 \cdot \partial_t. \end{aligned}$$

Therefore,

$$\begin{aligned}\nabla_{\partial_t}\nabla_{\partial_t}Y = & \; u^4 d\theta(d\theta(Y^T)) + Y^T(u)\nabla u + Y^\perp u^2 d\theta(\nabla u)\\ & - [ud\theta(Y^T,\nabla u) + \frac{Y^\perp}{u}|\nabla u|^2]\cdot\partial_t.\end{aligned} \tag{4.29}$$

2.For the second term in (4.28),

$$\begin{aligned}&\nabla_{\nabla_{\partial_t}\partial_t}Y = u\nabla_{\nabla u}Y = u[\nabla_{\nabla u}Y^T - \nabla_{\nabla u}(\frac{Y^\perp}{u}\partial_t)]\\ &= u[\nabla_{\nabla u}Y^T]^T + u\langle\nabla_{\nabla u}Y^T,\partial_t\rangle\cdot\frac{\partial t}{-u^2} - u[\langle\nabla u,\nabla\frac{Y^\perp}{u}\rangle\partial_t + (\frac{Y^\perp}{u})\nabla_{\nabla u}\partial_t]\\ &= u(\nabla_{g_S})_{\nabla u}Y^T - ud\theta(\nabla u, Y^T)\cdot\partial_t - u\langle\nabla u,\nabla\frac{Y^\perp}{u}\rangle\cdot\partial_t + Y^\perp u^2 d\theta(\nabla u) - \frac{Y^\perp}{u}|\nabla u|^2\cdot\partial_t.\end{aligned} \tag{4.30}$$

3. As for the third term in (4.28), one first notices that, for two time-independent vectors $v, w \in TS$,

$$\nabla_v w = [\nabla_v w]^T + \langle\nabla_v w,\partial_t\rangle\cdot\frac{\partial_t}{-u^2} = (\nabla_{g_S})_v w + d\theta(w,v)\cdot\partial_t. \tag{4.31}$$

Applying the formula above, we can carry out the following calculation:

$$\nabla_{e_i}\nabla_{e_i}Y = \nabla_{e_i}\nabla_{e_i}Y^T - \nabla_{e_i}\nabla_{e_i}(\frac{Y^\perp}{u}\partial_t),$$

inside which we have,

$$\begin{aligned}&\nabla_{e_i}\nabla_{e_i}Y^T\\ &= \nabla_{e_i}[(\nabla_{g_S})_{e_i}Y^T + d\theta(Y^T,e_i)\cdot\partial_t]\\ &= (\nabla_{g_S})_{e_i}(\nabla_{g_S})_{e_i}Y^T + d\theta((\nabla_{g_S})_{e_i}Y^T,e_i)\cdot\partial_t + d\theta(Y^T,e_i)\cdot\nabla_{e_i}\partial_t + [\nabla_{e_i}d\theta(Y^T,e_i)]\cdot\partial_t\\ &= (\nabla_{g_S})_{e_i}(\nabla_{g_S})_{e_i}Y^T + d\theta((\nabla_{g_S})_{e_i}Y^T,e_i)\cdot\partial_t\\ &\quad + d\theta(Y^T,e_i)\cdot(-u^2 d\theta(e_i) + u^{-1}e_i(u)\cdot\partial_t) + [\nabla_{e_i}d\theta(Y^T,e_i)]\cdot\partial_t\\ &= (\nabla_{g_S})_{e_i}(\nabla_{g_S})_{e_i}Y^T - u^2 d\theta(Y^T,e_i)\cdot d\theta(e_i))\\ &\quad + [d\theta((\nabla_{g_S})_{e_i}Y^T,e_i) + u^{-1}d\theta(Y^T,e_i)e_i(u) + \nabla_{e_i}d\theta(Y^T,e_i)]\cdot\partial_t,\end{aligned}$$

and

$$
\begin{aligned}
&\nabla_{e_i}\nabla_{e_i}(\frac{Y^\perp}{u}\partial_t)\\
&= \nabla_{e_i}[e_i(\frac{Y^\perp}{u})\partial_t + \frac{Y^\perp}{u}\nabla_{e_i}\partial_t]\\
&= e_i(e_i(\frac{Y^\perp}{u}))\partial_t + e_i(\frac{Y^\perp}{u})\nabla_{e_i}\partial_t + \nabla_{e_i}[\frac{Y^\perp}{u}(-u^2)(d\theta(e_i) + u^{-1}e_i(u)\cdot\partial_t)]\\
&= e_i(e_i(\frac{Y^\perp}{u}))\partial_t + 2e_i(\frac{Y^\perp}{u})[-u^2 d\theta(e_i) + u^{-1}e_i(u)\cdot\partial_t]\\
&\quad + \frac{Y^\perp}{u}\nabla_{e_i}[(-u^2)d\theta(e_i) + u^{-1}e_i(u)\cdot\partial_t]\\
&= e_i(e_i(\frac{Y^\perp}{u}))\partial_t + 2e_i(\frac{Y^\perp}{u})(-u^2)d\theta(e_i) + 2u^{-1}e_i(\frac{Y^\perp}{u})e_i(u)\cdot\partial_t\\
&\quad + \frac{Y^\perp}{u}(\nabla_{g_S})_{e_i}[(-u^2)d\theta(e_i)] - uY^\perp d\theta(d\theta(e_i), e_i)\cdot\partial_t.\\
&\quad + \frac{Y^\perp}{u}e_i(u^{-1}e_i(u))\cdot\partial_t + \frac{Y^\perp}{u^2}e_i(u)(-u^2 d\theta(e_i) + u^{-1}e_i(u)\cdot\partial_t).
\end{aligned}
$$

Combining equations above we obtain,

$$
\begin{aligned}
&-\Sigma_i\nabla_{e_i}\nabla_{e_i}Y\\
&= (\nabla_{g_S})^*\nabla_{g_S}Y^T + u^2 d\theta(d\theta(Y^T)) - 2u^2 d\theta(\nabla\frac{Y^\perp}{u})\\
&\quad + \frac{Y^\perp}{u}\delta_{g_S}[u^2 d\theta] - Y^\perp d\theta(\nabla u)\\
&\quad + [\langle d\theta, \nabla_{g_S}Y^T\rangle + \delta_{g_S}(d\theta(Y^T)) - \Delta_{g_S}(\frac{Y^\perp}{u}) + uY^\perp|d\theta|^2]\cdot\partial_t\\
&\quad + [-u^{-1}d\theta(Y^T, \nabla u) + 2u^{-1}\langle\nabla\frac{Y^\perp}{u}, \nabla u\rangle - \frac{Y^\perp}{u^2}\Delta_{g_S}u]\cdot\partial_t.
\end{aligned}
\tag{4.32}
$$

4.The last term in (4.28) is zero because $\nabla_{e_i}e_i = 0$ based on (4.31).
Summarizing the equations (4.29-30) and (4.32), we have

$$
\begin{cases}
[\nabla^*\nabla Y]^T = (\nabla_{g_S})^*\nabla_{g_S}Y^T + u^{-2}Y^T(u)\nabla u - u^{-1}(\nabla_{g_S})_{\nabla u}Y^T\\
\qquad +2u^2 d\theta(d\theta(Y^T)) - 2u^2 d\theta(\nabla\frac{Y^\perp}{u})\\
\qquad +Y^\perp u\delta^T[d\theta] - 3Y^\perp d\theta(\nabla u)\\
\\
\langle\nabla^*\nabla Y, u^{-2}\partial_t\rangle = \Delta_{g_S}(\frac{Y^\perp}{u}) - 3u^{-1}\langle\nabla\frac{Y^\perp}{u}, \nabla u\rangle - 2\langle d\theta, \nabla_{g_S}Y^T\rangle\\
\qquad -uY^\perp|d\theta|^2 + \frac{Y^\perp}{u^2}\Delta_{g_S}u\\
\qquad -3u^{-1}d\theta(\nabla u, Y^T) + \delta^T d\theta(Y^T).
\end{cases}
$$

According to equations (4.26) and (4.27), the equations above can be simplified as,

$$
\begin{cases}
[\nabla^*\nabla Y]^T = (\nabla_{g_S})^*\nabla_{g_S}Y^T + u^{-2}Y^T(u)\nabla u - u^{-1}(\nabla_{g_S})_{\nabla u}Y^T\\
\qquad +2u^2 d\theta(d\theta(Y^T)) - 2u^2 d\theta(\nabla\frac{Y^\perp}{u})\\
\\
\langle\nabla^*\nabla Y, u^{-2}\partial_t\rangle = \Delta_{g_S}(\frac{Y^\perp}{u}) - 3u^{-1}\langle\nabla\frac{Y^\perp}{u}, \nabla u\rangle - 2\langle d\theta, \nabla_{g_S}Y^T\rangle,
\end{cases}
\tag{4.33}
$$

which directly gives the formula (3.32).

We note that in the case where $\tilde{g}^{(4)} = \tilde{g}_0^{(4)}$, the standard flat (Minkowski) metric on $\mathbb{R} \times (\mathbb{R}^3 \setminus B)$, equations in (4.33) can be simplified as

$$\begin{cases} [\nabla^*\nabla Y]^T = (\nabla_{g_0})^*\nabla_{g_0} Y^T \\ [\nabla^*\nabla Y]^\perp = \Delta_{g_0} Y^\perp, \end{cases}$$

because $\theta = 0,\ u = 1$ for $\tilde{g}_0^{(4)}$. Here g_0 denotes the flat metric in $\mathbb{R}^3 \setminus B$. Based on the decomposition above, it is easy to see that the solution to $\nabla^*\nabla Y = 0$ with trivial Dirichlet boundary condition must be $Y = 0$. Therefore, the operator $\beta_{\tilde{g}_0^{(4)}} \delta^*_{\tilde{g}_0^{(4)}}$ is invertible, i.e. the Assumption 3.3 holds for $\tilde{g}_0^{(4)}$.

References

[ADN] S. Agmon, A. Douglis, and L.Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, II, Comm. Pure App. Math., **12**, (1964) 623-727, **17** (1964), 35-92.

[AH] M.Anderson and M.Herzlich, Unique continuation results for Ricci curvature and applications, Jour. Geom. & Physics, **58**, (2008), 179-207.

[AK] M. Anderson and M. Khuri, On the Bartnik extension problem for static vacuum Einstein metrics, Classical & Quantum Gravity, **30**, (2013), 125005.

[$A1$] M. Anderson, On boundary value problems for Einstein metrics, Geom. & Topology, **12**, (2008), 2009-2015.

[$A2$] M. Anderson, On stationary vacuum solutions to the Einstein equations, M.Ann.Henri Poincare, (2000), vol. 1, 977-994.

[$A3$] M. Anderson, Local existence and uniqueness for exterior static vacuum Einstein metrics, Proc. Amer. Math. Soc., vol. 143, (2015), 3091-3096.

[Be] A.Besse, Einstein Manifolds, Springer Verlag, New York, (1987).

[BDF] D. Brill, S. Deser, and L. Fadeev. Sign of gravitational energy. Phys. Lett., 26A(11):538-539, April 1968.

[$B1$] R.Bartnik, Energy in general relativity, Tsing Hua lectures on geometry and analysis, 5-27, International Press, Cambridge, MA, (1997).

[$B2$] R.Bartnik, The mass of an asymptotically flat manifold, Comm. Pure Appl. Math., **39**, (1986), 661-693.

[$B3$] R.Bartnik, Phase space for the Einstein Equations, Communications in Analysis and Geometry, vol 13, No. 5, (2005) 845-885

[C] PT. Chrusciel On uniqueness of stationary vacuum black holes, et al. Asterisque **321**, (2008), 195-265.

[CH] PT. Chrusciel, JL. Costa, M. Heusler, Stationary Black Holes: Uniqueness and Beyond, Living Reviews in Relativity, 15, (2012), 7.

[GH] G.W. Gibbons and S.W. Hawking, Action integrals and partition functions in quantum gravity, Phys. Rev. D**15**, 2752 (1977).

[$H1$] M. Heusler, Black Hole Uniqueness Theorems, Cambridge U. Press, Cambridge, (1996).

[$H2$] M. Heusler, Stationary Black Holes: Uniqueness and Beyond, Living Reviews in Relativity, vol. 1, No. 6, (1998).

[J] J. Klenk, Existence of stationary vacuum solutions of Einstein's equations in an exterior domain, J. Austral. Math. Soc. Ser. B **41**, (1991), 231-247.
[K] D. Kramers, H. Stephani, M. MacCallum, E. Herlt, Exact Solutions of Einstein's Field Equations, Cambridge U. Press, Cambridge, (1980).
[KS] RP. Kerr, and A Schild, A new class of vacuum solutions of the Einstein field equations *Atti del Convegno sulla Relativita Generale : Problemi dell'Energia e Onde Gravitazionali* ed Barbera G, (1965), 1-12.
[L] A. Lichnerowicz, Theories Relativistes de la Gravitation et de L'Electromagnetisme, Masson and Cie., Paris, (1995).
[LP] J. Lee and T. Parker, The Yamabe problem, Bulletin Amer. Math. Soc., **17**, (1987), 37-91.
[M] V. Moncrief, Spacetime symmetries and linearization stability of the Einstein equations I, II, Journal of Mathematical Physics, **16** (1975), no.3, 493-498, **17** (1976), no.10, 1893-1902.
[$M1$] P. Miao, On existence of static metric extensions in general relativity, Comm. Math. Phys., **241**, (2003), 27-46.
[$M2$] P. Miao, A remark on boundary effects in static vacuum initial data sets, Class. Quantum Grav. **22**, (2005), 53-59
[$M3$] P. Miao, Some recent developments of the Bartnik mass, Proc. ICCM 2007, Vol. III, International Press, Boston, (2010), 331-340.
[R] O. Reula, On existence and behaviour of asymptotically flat solutions to the stationary Einstein equations, Comm. Math. Phys. vol. 112, **4**, (1989), 615-624.
[RT] T. Regge and C. Teitelboim, Role of surface integrals in the Hamiltonian formulation of general relativity, Annals of Physics, **88**, (1974), 286-318
[Sc] K. Schwarzschild, Uber das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie *Sitzungsber*. K. *Preuss. Akad. Wiss.*, (1916), 189-196.
[Sz] L.B. Szabados, Quasi-local energy-momentum and angular momentum in General Relativity, Living Rev. Relativ. (2009) 12: 4.
[SY] R.Schoen and S.-T. Yau, On the proof of the positive mass conjecture in general relativity. Comm. Math. Phys. 65 (1979), no. 1, 45-76
[T] SA. Teukolsky, The kerr metric, Class. Quant. Grav. **32**, (2015), No. 12, 124006.
[W] R. M. Wald, General Relativity, The University of Chicago Press, Chicago, (1894).
[Y] Yvonne Choquet-Bruhat, General Relativity and the Einstein Equations (Chapter XIV), Oxford University Press, 2008