Variational Formulas and Strata of Abelian Differentials

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Abstract of the Thesis

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In this thesis we study the degenerations of abelian differentials. We adapt the jump problem technique developed in a recent paper [GKN17] to compute the variational formulas of any stable differential and its periods to arbitrary precision in plumbing coordinates. In particular, we give explicit variational formula for period matrices, easily reproving results of Yamada [Yam80] for nodal curves with one node and generalizing them to arbitrary stable curves. We apply the same technique to give an alternative proof of the sufficiency part of the theorem in [BCGGM18] on the closures of strata of abelian differentials with prescribed multiplicities of zeroes and poles.

We also give an explicit modular form defining the locus of quartics with a hyperflex $\Omega\mathcal{M}_3^{odd}(4)$, also known as the odd component of the minimal stratum of abelian differentials in genus 3. Using our variational formulas for the period matrices and the modular form we obtained, we provide a direct way to compute the divisor class of this locus in $\overline{\mathcal{M}}_3$.

Dedication

To my parents, Zhoujian Hu and Yihong Qiu, for raising me up and giving me all they have.

To my wife, Yang Zhou, the love of my life.

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Chapter 1

Introduction

1.1 Background and Motivations

We work over the field of complex numbers $\mathbb{C}$. Let $\mathcal{M}_g$ be the moduli space of curves. Points of $\mathcal{M}_g$ correspond to smooth Riemann surfaces of genus g. The Hodge bundle $\Omega\mathcal{M}_g$ is a rank g holomorphic vector bundle over $\mathcal{M}_g$, where the fiber over each point $[C] \in \mathcal{M}_g$ is the space of holomorphic abelian differentials (one-forms) on C.

1.1.1 Strata of Abelian Differentials

Given an abelian differential Ω on a curve C, we have $\operatorname{div}(\Omega) = m_1p_1 + m_2p_2 + \ldots + m_np_n$, where p_i are the zeroes of Ω and m_i are the multiplicities of the zeroes. We have $\sum m_i = 2g - 2$, where g is the genus of C. There is a natural stratification of the Hodge bundle by the multiplicities of zeroes.

Definition 1.1.1. Let $\mu = (m_1, \ldots, m_n)$ be an integral partition of $2g - 2$. Set-theoretically the *moduli spaces of abelian differentials* (or *stratum* of abelian differentials) is defined as

$$\Omega\mathcal{M}_{g,n}(\mu) := \left\{ (C; p_1, \ldots, p_n; \Omega) : \begin{array}{c} [C; p_1, \ldots, p_n] \in \mathcal{M}_{g,n},\ \Omega \in H^{1,0}(C, \mathbb{C}), \\ \operatorname{div}(\Omega) = \sum_{i=1}^{g} m_i p_i \end{array} \right\},$$

Each stratum $\Omega\mathcal{M}_{g,n}(\mu)$ is a complex orbitfold of dimension $d := 2g+n-1$. Away from orbitfold points, each stratum has an atlas of charts to $\mathbb{C}^d$. Local coordinates (called the *period coordinates*) on $\Omega\mathcal{M}_{g,n}(\mu)$ are defined as follows: for an abelian differential with marked zeroes $(C; p_1, \ldots, p_n; \Omega)$, a basis of the integral relative homology group $H_1(C, \Sigma; \mathbb{Z})$ is given by a symplectic basis $(A_1, \ldots, A_g; B_1, \ldots, B_g)$ of the absolute homology $H_1(C;\mathbb{Z})$, together with a choice of paths $(\gamma_1,\dots,\gamma_{n-1})$ where γ_k connects p_n and p_k. The period coordinates on the stratum are the local coordinates given by the integrals of ω over $\{A_i, B_i\}$ (these are called *absolute* period coordinates), and over $\{\gamma_k\}$ (these are called *relative* period coordinates).

The strata are intensively studied in Teichmüller dynamics. Away from the zeroes of Ω, we can find a local coordinate z on C such that $\Omega = dz$; at a zero of order m of Ω, we can find a local coordinate z such that $\Omega = z^m dz$. By identifying $\mathbb{C}$ with $\mathbb{R}^2$, namely taking the real and imaginary parts of Ω, $\mathrm{SL}_2(\mathbb{R})$ acts on the set of abelian differentials on C. The strata are preserved under this $\mathrm{SL}_2(\mathbb{R})$-action, because the action does not change the multiplicities of zeros of Ω. Details of this action and the associated flat geometry will be given in Section 2.1.1.

The guiding problem in Teichmüller dynamics is to classify the $\mathrm{SL}_2(\mathbb{R})$ orbit closures in every stratum. A complete classification of the orbit closures in genus 2 is due to McMullen [McM05a] [McM05b] [McM06] by using a very interesting connection between the orbit closures and number theory. Partial study in genus 3 is done by Bainbridge, Möller, Habegger and Zagier [BM12] [BM14] [BHM16] [MZ16].

In [Hu17], I studied the stratum $\Omega\mathcal{M}_3^{odd}(4)$. This is one of the two connected components of the stratum of the lowest dimension in genus 3. A generic point of $\Omega\mathcal{M}_3^{odd}(4)$ corresponds to a plane quartic with a hyperflex point. In [Hu17], I gave a modular form on a level cover of $\mathcal{A}_3$, the modular space of principally polarized abelian variety of dimension 3. The zero locus of my modular form is the image of $\Omega\mathcal{M}_3^{odd}(4)$ in $\mathcal{M}_3$. We will briefly introduce this result in Section 1.2.2, while the full context will be given in Chapter 5.

1.1.2 Degeneration of Abelian Differentials

It is well known that the Hodge bundle $\Omega\mathcal{M}_g$ can be extended over the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ (see [HM]). The fiber of $\Omega\overline{\mathcal{M}}_g$ over a stable nodal curve C in the boundary of $\overline{\mathcal{M}}_g$ parametrizes *stable differentials* on C, that is, meromorphic differentials on C that have at worst simple poles with opposite residues at the two pre-images of a given node.

It is natural to ask for an explicit description of how abelian differentials degenerate to stable differentials. To this end, we need local versal deformation coordinates on $\overline{\mathcal{M}}_g$. While many choices are available, we choose to use the *plumbing coordinates*, and we will introduce them briefly in Section 1.2.1 and properly in Chapter 3.

Our goal is to derive a variational formula for abelian differentials in plumbing coordinates. The term "variational formula" means an expansion in terms of both the plumbing coordinates and the logarithms of the plumbing coordinates. Such variational formulas were studied by Yamada [Yam80] and Fay [Fay73] in the case where the stable curve has only one node. Given the variational formula of abelian differentials, it is a direct computation to derive the variational formulas for integrals of abelian differentials over homology 1-cycles (periods of abelian differentials) on a Riemann surface. Via this approach, Yamada [Yam80] and Fay [Fay73] gave the variational formula for period matrices of stable curves with one node. For more general cases, Taniguchi [Tan89] computed the logarithmic term in the variational formula of the period matrices. We will review the results of Yamada, Fay and Taniguchi in Section 2.2.2.

In [HN18], Norton and I improve their results in full generality. I briefly introduce our result in Section 1.2.1, and the full context will be given in Chapter 3 and 4.

Our variational formulas are very important in the study of strata of abelian differentials. I provide two applications in this thesis. One application is to reprove the result of [BCGGM18], which I will introduce in the next subsection. Another application is that by using the degeneration of period matrices, I computed the vanishing order of my modular form in $\mathcal{A}_3$, and deduce the divisor class of the image of $\Omega\mathcal{M}_3^{odd}(4)$ in $\mathcal{M}_3$.

It is also worth mentioning that in a recent paper [AN19], Aulicino and Norton use our formula for the period matrices to show that there are no Shimura-Teichmüller curves generated by genus five surfaces. Their result completes the classification of Shimura-Teichmüller curves for all genera.

Since we are able to describe the degeneration of a general period (Section 4.1), it should not be hard to compute the degeneration of period coordinates in the compactified strata. I hope that this will shed new light on understanding the boundary of the $\mathrm{SL}_2(\mathbb{R})$-orbit closures, which in turn may provide a way towards the classification of the orbit closures.

1.1.3 Compactifications of Strata

A compactified stratum can be useful in studying the classification of the orbit closures. The question of finding a smooth compactification of the strata $\Omega\mathcal{M}_{g,n}(\mu)$ is also interesting for its own sake. The most natural and naïve option is to compactify $\Omega\mathcal{M}_g(\mu)$ by taking its closure in the compactified Hodge bundle $\Omega\overline{\mathcal{M}}_g$. However, a disadvantage of this approach is that some abelian differentials may become identically zero on some components of the limit stable curve. We then lose all information on the positions of the zeroes of these differentials on such a component.

In order to overcome this disadvantage, one can rescale the family of abelian differential with a suitable power of the local deformation coordinate, so that the limit differential does not vanish identically on a given component. In [BCGGM18], Bainbridge, Chen, Grushevsky, Gendron, and Möller define the *incidence variety compactification* (IVC) of strata by taking the closure of the strata in the projectivized compactification of the Hodge bundle $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}$. This compactification of strata is denoted by $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}^{inc}(\mu)$. In the paper they gave and proved the necessary and sufficient conditions for a stable differential to lie in the boundary of the IVC. We will give a comprehensive review of their result in Section 2.3.

In [HN18], Norton and I use the jump problem approach to give an alternative proof of the sufficiency of their conditions, which is the harder direction of the result in [BCGGM18]. Furthermore, our approach gives more information about the neighborhood of the boundary of the IVC (see Theorem 3.3.2 and Corollary 3.3.3).

1.2 Main Results

1.2.1 Degeneration of Abelian Differentials and Period Matrices.

The *plumbing coordinates* are local coordinates on $\overline{\mathcal{M}}_g$ defined as follows: Given a stable nodal curve C with m nodes, the standard plumbing construction cuts out neighborhoods of the two pre-images q_e and q_{-e} of each node q on the normalization of C, and identifies their boundaries (called seams, denoted by $\gamma_{\pm e}$). The identification map is given by $I_e : z_e \to z_{-e} := s_e/z_e^{-1}$, where s_e is called the *plumbing parameter* and z_e and z_{-e} are chosen local coordinates near two pre-images of the node respectively. The plumbing construction thus constructs a family of curves $\{C_{\underline{s}}\} =: \mathcal{C} \to \Delta$ with the central fiber C_0 identified with C, where Δ is the small poly disc neighborhood of $0 \in \mathbb{C}^m$ with coordinates given by the plumbing parameters $\underline{s} := (s_1, \ldots, s_m)$.

In [HN18], we compute the Taylor expansion of any stable differentials in plumbing coordinates, and give the explicit variational formula for the degeneration of any periods of the differential near an arbitrary stable curve. In particular we give the variational formulas for period matrices, which generalize the results of Yamada-Fay and Taniguchi.

The technique we use to construct the degenerating family $\{C_{\underline{s}}, \Omega_{\underline{s}}\}$ is called (solving) the jump problem, which was developed and used in the real-analytic setting by Grushevsky-Krichever-Norton [GKN17]. Roughly speaking, given a stable differential Ω on C, we have the mis-matches $\{\Omega|_{\gamma_e} - I_e^*(\Omega|_{\gamma_{-e}})\}$ (which we call the *jumps* of Ω) on the seams $\gamma_{\pm e}$ at opposite sides of each node. The solution to the jump problem with initial conditions from Ω is a"correction" differential η whose jumps at the seams equal to the negative of the jumps of Ω. By adding η to Ω on each irreducible component, one obtains new differentials with zero jumps. We can thus glue this newly-derived differentials to get a global meromorphic differential $\Omega_{\underline{s}}$ on $C_{\underline{s}}$. In [HN18], we construct explicitly the solution to the jump problem:

Theorem 1.2.1 (= Theorem 3.2.3)**.** *Let $(C,\Omega) \in \partial\Omega\overline{\mathcal{M}}_g$ be a stable differential. Let Ω_v be the restriction of Ω to the connected component C_v. For any $|\underline{s}|$ small enough, $\{\Omega_{v,\underline{s}} := \Omega_v + \eta_v\}$ defines a meromorphic differential $\Omega_{\underline{s}}$ on $C_{\underline{s}}$ satisfying $\Omega_v = \lim_{\underline{s}\to 0} \Omega_{\underline{s}}|_{C_v}$, where $\eta_v = \sum_{k=1}^{\infty} \eta_v^{(k)}$ is the unique solution with vanishing A-periods to the jump problem with the initial conditions from Ω. Furthermore, we have $||\eta_{v,\underline{s}}||_{L^2} = O(\sqrt{|\underline{s}|})$.*

Furthermore, we compute the leading term of the $\underline{s}$-expansion for $\eta_v^{(k)}$, which in particular gives the linear term of the $\underline{s}$-expansion for $\Omega_{\underline{s}}$ (Proposition 3.2.4). In the one node case, this expansion is the same as in [Yam80].

We denote $r_e := \operatorname{res}_{q_e} \Omega$, and have $r_e = -r_{-e}$. Let α be any oriented loop in C. Let $\{q_1, \dots, q_N\}$ be the ordered collection of nodes that α passes through (with possible repetitions). Let $\alpha_{\underline{s}}$ be a perturbation of α such that its restrictions on each C_v minus the neighborhood at each node glue correctly to give a loop on $C_{\underline{s}}$.

Theorem 1.2.2 (= Theorem 4.1.2)**.** *The variational formula of the period of $\Omega_{\underline{s}}$ over $\alpha_{\underline{s}}$ is given by:*

$$\int_{\alpha_{\underline{s}}} \Omega_{\underline{s}} = \sum_{i=1}^{N} (r_{e_i} \ln |s_{e_i}| + c_i + l_i) + O(|\underline{s}|^2),$$

here c_i and l_i are the constant and linear terms in $\underline{s}$ respectively, which are explicitly given.

The explicit expressions for the constant and the linear terms can be found in Theorem 4.1.2.

For the variational formula of the degeneration of the period matrices, we choose a suitable symplectic basis $\{A_{k,\underline{s}}, B_{k,\underline{s}}\}_{k=1}^{g}$ of $H_1(C_{\underline{s}}, \mathbb{Z})$ that depends continuously in $\underline{s}$. We take a basis $\{v_1, \dots, v_g\}$ of $H^0(C, K_C)$ dual to $\{A_1, \dots, A_g\}$, such that after applying the jump problem construction, the resulting set of differentials $\{v_{k,\underline{s}}\}_{k=1}^{g}$ is a basis of $H^0(C_{\underline{s}}, K_{C_{\underline{s}}})$ dual to $\{A_{1,\underline{s}}, \dots, A_{g,\underline{s}}\}$. We then have:

Corollary 1.2.3 (= Corollary 4.2.1). *For any fixed h, k such that $1 \leq h, k \leq g$, the expansion of $\tau_{h,k}(\underline{s})$ is given by*

$$\tau_{h,k}(\underline{s}) = \sum_{e \in E_C} \frac{N_{|e|,h} \cdot N_{|e|,k}}{2} \cdot \ln |s_e| + c_{h,k} + l_{h,k}(\underline{s}),$$

where $N_{|e|,k}$ is the intersection number of the seam $\gamma_{|e|}$ and $B_{k,\underline{s}}$, and E_C is the set of nodes of C. Moreover, the constant term $c_{h,k}$ and the linear term $l_{h,k}(\underline{s})$ are given explicitly.

The expression for the constant and the linear terms can be found in Corollary 4.2.1. This corollary in particular shows that $\tau_{h,k}(\underline{s}) - \sum_{e \in E_C} \frac{N_{|e|,h} \cdot N_{|e|,k}}{2} \cdot \ln |s_e|$ is holomorphic in $\underline{s}$, which is the main result in [Tan89].

1.2.2 A Modular Form for $\Omega\mathcal{M}_3^{odd}(4)$.

Classically, the Riemann theta constants with characteristics (or theta characteristics) $\theta[{\epsilon \atop \delta}](\tau, 0)$ are known to be *modular forms* of weight $\frac{1}{2}$. The theta-null modular form is defined as $\Theta_{null}(\tau) := \prod_{(\epsilon,\delta)\text{even}} \theta[{\epsilon \atop \delta}](\tau, 0)$, where the parity on the characteristics is given by the Weil pairing. It is known that the theta-null modular form cuts out the hyperelliptic locus in genus 3. We will discuss the theta characteristics and modular forms in more detail in Section 2.4.1.

In [Hu17], I give the counterpart of the theta-null modular form for $\mathcal{M}_3^{odd}(4)$, which is the image of the forgetful map from $\Omega\mathcal{M}_3^{odd}(4)$ to $\mathcal{M}_3$. In this thesis we also call $\mathcal{M}_3^{odd}(4)$ the *hyperflex locus*, because it parameterizes the closure of the locus of plane quartics where one of the 28 bitangent lines of the quartic is in fact a hyperflex line, i.e. its two tangent points comes together. Equivalently, it is the locus where a (generically non-hyperelliptic) curve C has a Weierstrass point p of weight 4, such that $4p \equiv K_C$.

It is known that for a given curve C, the *Riemann theta function with characteristics* (or *theta characteristics*) is a section of the line bundle $\frac{1}{2}K$ twisted by a two-torsion line bundle. One can identify the set of two-torsion points in $\mathrm{Jac}(C)$ with $(\mathbb{Z}_2)^3 \times (\mathbb{Z}_2)^3$, sending $m = (\tau\epsilon + \delta)/2$ to the characteristics (ϵ, δ), where τ is the period matrix of C. Technically our discussion depends on the choice of such an identification, and hence should be on a level cover of $\mathcal{A}_3$. In order to simplify the discussion, we choose to neglect these issues here in the introduction; they will be discussed in detail in Chapter 5.

For simplicity, let us denote the characteristics (ϵ, δ) by (i, j), where $i = 4\epsilon_1 + 2\epsilon_2 + \epsilon_3, j = 4\delta_1 + 2\delta_2 + \delta_3$. For instance, $([1, 1, 0], [0, 1, 1])$ is denoted by $(6, 3)$.

Theorem 1.2.4 (= Theorem 5.1.5). *On $\mathcal{A}_3$, the modular form $\Omega_{77}(\tau)$ defined by*

$$\begin{aligned}\Omega_{77}(\tau) := &[\theta_{01}\theta_{10}\theta_{37}\theta_{43}\theta_{52}\theta_{75}\theta_{42}\theta_{06}\theta_{30}\theta_{21}\theta_{55} + \theta_{02}\theta_{25}\theta_{34}\theta_{40}\theta_{67}\theta_{76}\theta_{33}\theta_{05}\theta_{14}\theta_{60}\theta_{42}]^2 \\ &- 4\theta_{01}\theta_{02}\theta_{10}\theta_{25}\theta_{34}\theta_{37}\theta_{40}\theta_{43}\theta_{52}\theta_{67}\theta_{75}\theta_{76}\theta_{00}\theta_{04}\theta_{57}\theta_{70}\theta_{61}\theta_{73}\theta_{20}\theta_{07}\theta_{00}\theta_{16}\end{aligned}$$

vanishes at the period matrix τ of a smooth plane quartic C if and only if C has a Weierstrass point P of weight 4 such that the 2-torsion point $[\frac{1}{2}K_C - 2P]$ on A_τ corresponds to the characteristic $(i,j) = (7,7)$. Here $\theta_{ij} := \theta_{ij}(\tau, 0)$ is the Riemann theta constant with characteristics (i,j).

The requirement that the 2-torsion point correspond to the characteristic $(i,j) = (7,7)$ is merely a technical condition to fix the choice of the identification $A_\tau[2] \simeq (\mathbb{Z}/2\mathbb{Z})^6$. In other words, the modular form Ω_{77} cuts out a locus on the level two cover of $\mathcal{A}_3$ that maps one-to-one onto the image of $\mathcal{M}_3^{odd}(4)$ in $\mathcal{A}_3$.

We can use the modular form Ω_{77} to compute the divisor class of the closure of the locus $\mathcal{M}_3^{odd}(4)$ in $\overline{\mathcal{M}}_3$:

$$[\overline{\mathcal{M}_3^{odd}(4)}] = 308\lambda - 32\delta_0 - 76\delta_1,$$

where λ is the Hodge class on $\overline{\mathcal{M}}_3$, and δ_0, δ_1 are the classes of the boundary divisors. This class was first computed in [Cuk89] using a different method.

Generally speaking, the weight of a modular form F gives the multiplicity of the Hodge class in the class of the locus cut out by F. In order to compute the analogous coefficients of the boundary divisor classes, one computes the vanishing orders of F at the boundary components. We therefore study the degeneration of the theta constants with characteristics near the boundary of $\overline{\mathcal{M}}_3$, which requires an understanding of the degeneration of the period matrices. The computation of the degeneration of theta constants is given in detail in Section 2.4.4.

1.3 Structure of the Dissertation

In Chapter 2, we review the preliminaries needed for the two main results in the dissertation. We first introduce the moduli space of curves and the strata of abelian differentials in Section 2.1. We review the Deligne-Mumford compactification of $\mathcal{M}_g$ in Section 2.2 and the Incidence Variety compactification of strata in Section 2.3. In Section 2.4, we discuss the moduli space of principally polarized abelian varieties and theta characteristics. We also discuss in detail how to extend the theta characteristics to the boundary of $\overline{\mathcal{M}}_3$ in Section 2.4.4.

In Chapter 3, we give the main results and proofs in [GKN17] using the jump problem technique. In Chapter 4 we use the results in Chapter 3 to compute the variational formula for period matrices. We also gives examples in Section 4.3 that cover the results in [Yam80]. Furthermore, we use the jump problem technique to give an alternative proof to the main results in [BCGGM18] in Section 3.3.

In Chapter 5, we give main results and proofs in [Hu17]. We compute the modular form for $\Omega\mathcal{M}_3^{odd}(4)$. We use the extension of theta characteristics (Section 2.4.4) and the degeneration of period matrices (Section 4.3) to finish the computation of the divisor class (Section 5.2 and 5.3).

Chapter 2

Preliminaries

2.1 The Moduli Space of Curves and Strata of Abelian Differentials

In this chapter, we provide a more detailed discussion of the notions and definitions provided in Chapter 1.

A *Riemann surface* is a complex manifold of real dimension two. A connected, compact Riemann surface is equivalent to a smooth irreducible projective algebraic curve. In this text we always assume that a Riemann surface is compact and connected, so we don't distinguish these two concepts. We will use C to denote a Riemann surface throughout the text.

An *abelian differential* (or a *holomorphic differential*) Ω on a Riemann surface C is a global section of the cotangent bundle of C. In this thesis, we sometimes use the term "abelian differential" to denote the pair (C, Ω) when we need to specify the curve C. In some context, others extend the definition of abelian differentials to include meromorphic differentials, but in this thesis we only consider abelian differentials that are holomorphic.

Let g denote the genus of C. The space of abelian differentials $H^{1,0}(C)$ on C is a complex g-dimensional vector space. When $g > 0$, any abelian differential Ω has $2g-2$ zeroes, counted with multiplicities. Namely, $(\Omega)_0 = m_1p_1 + \ldots + m_np_n$ where $p_1, \ldots, p_n \in C$ and $\mu = (m_1, \ldots, m_n)$ is a partition of $2g-2$.

Let $\mathcal{M}_g$ be the *moduli space of genus g Riemann surfaces.* Points in $\mathcal{M}_g$ correspond to biholomorphic equivalence classes of smooth Riemann surfaces of genus g. One can define $\mathcal{M}_g$ by taking the quotient of the Teichüller space $\mathcal{T}_g$ by the mapping class group. The complex dimension of $\mathcal{M}_g$ is $3g-3$.

The space of all abelian differentials on genus g curves forms a rank g vector bundle $\Omega\mathcal{M}_g$ over $\mathcal{M}_g$, which is called the *Hodge bundle*. We fix a partition $\mu = (m_1, \ldots, m_n)$ of $2g-2$, then we can define the *stratum of abelian differentials* as follows:

$$\Omega\mathcal{M}_{g,n}(\mu) := \left\{ (C; p_1, \ldots, p_n; \Omega) : p_i \in C, \Omega \in H^{1,0}(C, \mathbb{C}), \operatorname{div}(\Omega) = \sum_{i=1}^{g} m_i p_i \right\}.$$

These spaces give a stratification of the Hodge bundle $\Omega\mathcal{M}_g$. Hereafter we will use the strata terminology for simplicity.

2.1.1 Action of $\mathrm{GL}_2^+(\mathbb{R})$

A *translation structure* on a Riemann surface C is an atlas of complex charts $\{(U_\alpha, f_\alpha)\}_{\alpha \in I}$, where all transition functions are translations. If $g > 1$, the flat metric introduced by the translation structure has a non-empty set of singularities, denoted by Σ. We call the singularities of the translation structure *saddle points*. We also call the pair (C, Σ) a *translation surface*.

The abelian differentials on a Riemann surface C are naturally equivalent to translation structures on C. Given an abelian differential (C, Ω), we get local charts by integrating Ω away from the zeroes of Ω. Given a translation surface (C, Σ), on each chart we have a local holomorphic differential dz, where z is the local coordinate on the chart. Since all transition functions are translations, dz is globally defined on $C - \Sigma$. Moreover, a m-th order zero of Ω can be locally written as $z^m dz = d(z^{m+1})/(m+1)$. This gives a saddle point that has cone angle $2\pi \cdot (m+1)$.

By identifying $\mathbb{C}$ with $\mathbb{R}^2$, the group $\mathrm{GL}_2^+(\mathbb{R})$ acts on the set of translation structures on C by post-composing the chart maps f_α with the linear map. Therefore the group $\mathrm{GL}_2^+(\mathbb{R})$ acts on the Hodge bundle $\Omega\mathcal{M}_g$. This action preserves the number and multiplicities of the zeroes of the 1-form, i.e., the stratum $\Omega\mathcal{M}_{g,n}(\mu)$ is $\mathrm{GL}_2^+(\mathbb{R})$-invariant. Moreover, given the differential Ω, we can compute the area of the Riemann surface $A(C) = \int_C \Omega \wedge \bar{\Omega}$. We define $\Omega\mathcal{M}_{g,n}^1(\mu)$ the strata of the unit area abelian differentials. The $\mathrm{GL}_2^+(\mathbb{R})$-action decends to an $SL_2(\mathbb{R})$-action on $\Omega\mathcal{M}_{g,n}^1(\mu)$. One can also interpret this action using the flat model of an abelian differential (see [Zor06], [Wri15]).

The strata $\Omega\mathcal{M}_{g,n}(\mu)$ and the action of $\mathrm{GL}_2^+(\mathbb{R})$ are intensively studied in Teichmüller dynamics. A central open problem is the classification of $\mathrm{GL}_2^+(\mathbb{R})$-orbit closures. In genus 2, the classification is done by McMullen [McM03] [McM05a] [McM05b] [McM06]. McMullen's approach uses *Hilbert modular varieties*. These objects are of great importance in the number theory, but they are out of the range of our discussion here. A further study of the relationship between Hilbert modular varieties and Teichmüller curves in $g = 2, 3, 4$ is done by Bainbridge-Möller [BM12] [BM14], Bainbridge-Habegger-Möller [BHM16], and Möller-Zagier [MZ16]. However, given any stratum in genus ≥ 3, a complete classification of Teichmüller curves within the stratum still remains an open problem.

When the $\mathrm{GL}_2^+(\mathbb{R})$ has a closed orbit, that is, the stabilizer of the abelian differential (C, Ω) under the $\mathrm{GL}_2^+(\mathbb{R})$-action is a lattice in $SL_2(\mathbb{R})$, then the $\mathrm{GL}_2^+(\mathbb{R})$ projects to an algebraic curve in $\mathcal{M}_g$. One can show that the curve is isometrically immersed with respect to the Teichmüller metric. Such a curve is called a *Teichmüller curve*. Whether the algebraicity also holds for higher dimensional orbit closures has long been an open problem.

The recent breakthrough work of Eskin-Mirzakhani-Mohammadi [EM13] [EMM15] gives that any $\mathrm{SL}_2(\mathbb{R})$-orbit closure is locally cut out by linear equations of real coefficients in period coordinates. Filip [Fil16] further shows that all $\mathrm{SL}_2(\mathbb{R})$-orbit closures are algebraic varieties.

2.1.2 A Theorem of Kontsevich and Zorich and the Minimal Stratum in Genus 3

Since the classification of orbit closures in genus 2 case was completed by McMullen, the simplest case that remains open is in genus 3. We first introduce a general result by Kontsevich and Zorich [KZ03], who classified the connected components of $\Omega\mathcal{M}_{g,n}(\mu)$ for all g and μ. In order to state their result, we need the following definition.

Definition 2.1.1. A *spin structure* (or *theta characteristic*) on a smooth curve C is a line bundle L whose square is the canonical bundle, i.e. $L^{\otimes 2} \sim K_C$. The parity of a spin structure is given by $\dim H^0(C, L) \bmod 2$.

Theorem 2.1.2. *[KZ03] The strata of abelian differentials $\Omega\mathcal{M}_g(\mu)$ have up to three connected components, distinguished by the parity of the spin structure and by being hyperelliptic or not. For $g \geq 4$, the strata $\Omega\mathcal{M}_g(2g-2)$ and $\Omega\mathcal{M}_g(2l, 2l)$ with an integer $l = (g-1)/2$ have three components, the component of hyperelliptic flat surfaces and two components with odd or even parity of the spin structure but not consisting exclusively of hyperelliptic curves.*

In genus 3, the stratum of the lowest dimension is $\Omega\mathcal{M}_3(4)$. By [KZ03, Cor 1], it has exactly two connected components, the hyperelliptic locus $\Omega\mathcal{M}_3^{hyp}(4)$, and one having the odd parity of the spin structure $\Omega\mathcal{M}_3^{odd}(4)$. Their dimensions are 5 and

6, respectively. The generic points in $\Omega\mathcal{M}_3^{odd}(4)$ correspond to plane quartics with a hyperflex point. We hence call $\Omega\mathcal{M}_3^{odd}(4)$ the *hyperflex* locus.

It is a well-known result that in genus 3, the hyperelliptic locus is cut out by the *theta-null* modular form. The theta null modular form is given by the product of all even theta characteristics. In [Hu17], I gave the counterpart description of the hyperflex locus. Namely, I gave an explicit modular form written in terms of the theta characteristics whose zero locus is the hyperflex locus (image of $\Omega\mathcal{M}_3^{odd}(4)$ in $\mathcal{M}_3$) (see Theorem 5.1.5). In order to fully state the results, we will need the notion of abelian varieties and Riemann theta functions. We will introduce these concepts in Section 2.4.

2.2 The Deligne-Mumford Compactification of $\mathcal{M}_g$

The *Deligne-Mumford compactification* $\overline{\mathcal{M}}_g$ is defined to be the moduli space of stable genus g curves (see [HM]). A *stable curve* is a complete connected curve with finite automorphism group and at worst nodal singularities. $\overline{\mathcal{M}}_g$ is a compactification of $\mathcal{M}_g$, where the boundary points correspond to stable nodal curves.

A node p in a stable nodal curve C is called a *separating node* if $C - p$ has two connected components. And we call C of *compact type* if C has only separating nodes. We introduce the following combinatorial way to describe loci of equisingular stable curves.

Definition 2.2.1 (Dual Graphs)**.** The *dual graph* Γ_C of a stable curve C is an unoriented graph where each edge corresponds to a node of C, and each vertex v corresponds to the normalization of an irreducible component C_v of C. The edge connecting vertices corresponds to the node between components.

2.2.1 Plumbing Coordinates

Given a stable nodal curve C with m nodes, we choose local coordinates $z_{\pm e}$ near the two pre-images q_e and q_{-e} of each node $q_{|e|}$ of C. The standard plumbing construction cuts out neighborhoods $U_{\pm e} = \{|z_{\pm e}| \leq \sqrt{|s_e|}\}$ on the normalization of C, and identifies their boundaries $\gamma_{\pm e} = \{|z_e| = \sqrt{|s_e|}\}$ via a gluing map I_e sending z_e to $z_{-e} := s_e/z_e^{-1}$, where $|s_e| \ll 1$ is called the *plumbing parameter*. The irreducible components of the nodal curve C are denoted by C_v.

The plumbing construction thus constructs a family of curves $\mathcal{C} \to \Delta$ with the central fiber identified with C, where Δ is the small polydisc neighborhood of $0 \in \mathbb{C}^m$ with coordinates given by the plumbing parameters $\underline{s} := (s_1, \ldots, s_m)$. Depending on circumstances $\underline{s}$ are also called the *plumbing coordinates*, as they give versal deformation coordinates on $\overline{\mathcal{M}}_g$ to the boundary stratum containing the point C. The Riemann surface resulting from plumbing with parameters $\underline{s}$ is denoted by $C_{\underline{s}}$.

2.2.2 Compactification of the Hodge Bundle

It is a classical result that the Hodge bundle $\Omega\mathcal{M}_g$ can be extended over $\overline{\mathcal{M}}_g$ (see [HM]). The fiber of the extension $\Omega\overline{\mathcal{M}}_g$ over a nodal curve C in the boundary of $\overline{\mathcal{M}}_g$ parametrizes *stable differentials*, that is, meromorphic differentials that have at worst simple poles at the nodes with opposite residues.

A degenerating family of abelian differentials is a flat family $(\mathcal{C}, \mathcal{W})$ over a complex poly disc Δ centered at $0 \in \mathbb{C}^m$, such that the central fiber is identified with (C, Ω), where C is a stable nodal curve and Ω is a stable differential on C.

The result of Yamada [Yam80] and Fay [Fay73] gives a variational formula for degenerating families of abelian differentials in the case that the limit curve has only one node. We briefly review their results on curves of compact type here. For curves of non-compact type, the statement is of similar fashion.

In the case where the limit curve C has only one node q, we have only one plumbing parameter s. Let (C, Ω) be a stable differential, we denote C_1 and C_2 the two connected components of C, and Ω_i the restriction of Ω on C_i $(i = 1, 2)$. Then [Yam80, Theorem 1] states that there exists a meromorphic differential Ω_s on the plumbed surface C_s, which has the same singularities as Ω_i outside of U_i.

The construction of Ω_s in [Yam80] was explicit. Therefore he was able to compute the variational formula for the degeneration of period matrices in one-node case. For curves of compact type, the period matrix has the following expansion:

$$\tau_s = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix} - s \begin{bmatrix} 0 & R \\ R^T & 0 \end{bmatrix} + o(s)$$

where $\tau_1 \in \mathrm{Mat}_{g_1 \times g_1}(\mathbb{C})$ and $\tau_2 \in \mathrm{Mat}_{g_2 \times g_2}(\mathbb{C})$ satisfying $g_1 + g_2 = g$, and $R \in \mathrm{Mat}_{g_1 \times g_2}(\mathbb{C})$ is some matrix independent of s. We will use this formula for the discussion in Section 2.4.4.

For the case of multiple nodes, Taniguchi [Tan89] [Tan91] computes the first term of the variational formulas for the period matrices.

In my paper with Norton [HN18], we give a full generalization of these results. Namely, we give the complete variational formula for abelian differentials and the period matrices when the curve has any number of nodes (See Theorem 3.2.3).

Using our variational formula for abelian differentials, we compute the variational formula for any periods on the limit curve (see Theorem 4.1.2). As a corollary, we deduce the variational formula for period matrices (see Theorem 4.2.1). The special case $m = 1$ in Theorem 1.2.1 matches the main result by Yamada and Fay. Theorem 4.2.1 gives the same logarithmic term as in [Tan89]. Beyond the logarithmic term, we also give the constant and the linear term explicitly. We want to remark that via our approach one can in principle compute *all* the higher order terms in the variational formulas, both for abelian differentials and for their periods.

2.3 The Incidence Variety Compactification of Strata

In [BCGGM18], Bainbridge, Chen, Grushevsky, Gendron, and Möller define the *incidence variety compactification* (IVC) of strata $\mathbb{P}\Omega\overline{\mathcal{M}}^{inc}_{g,n}(\mu)$ by taking the closure of the strata in the projectivized compactification of the Hodge bundle $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}$.

In this section we briefly review the definitions and results in [BCGGM18].

Take a stable pointed differential (C, Ω) in the boundary of $\Omega\overline{\mathcal{M}}_{g,n}$, where C is a stable nodal curve with marked points $p_1, \ldots, p_n$, and Ω is a stable differential on C. Let $(\mathcal{C}, \mathcal{W}) \to \Delta$ be a one parameter family in the stratum $\Omega\mathcal{M}_{g,n}(\mu)$, where Δ is a disk with parameter t, such that $\mathcal{C}_0 = C$. Note that Ω may be identically zero on some irreducible component C_v of C. By an analytic argument [BCGGM18, Lemma 4.1] one can show that for each C_v there exist $l_v \in \mathbb{Z}_{\leq 0}$ such that

$$\Xi_v := \lim_{t \to 0} t^{l_v} \Omega_v$$

is non-zero and not equal to infinity. Such differentials $\{\Xi_v\}_v$ must satisfy the following conditions (see the proof of necessity of [BCGGM18, Theorem 1.3]):

(0) If $p_k \in C_v$ for some k, Ξ_v vanishes to the correct order: $\operatorname{ord}_{p_k} \Xi_v = m_k$;

(1) The only singularities of Ξ_v are (possible) poles at the nodes of C_v;

(2) For any node $q_{|e|}$ on C, $\operatorname{ord}_{q_e} \Xi_{v(e)} + \operatorname{ord}_{q_{-e}} \Xi_{v(-e)} = -2$;

(3) If $\operatorname{ord}_{q_e} \Xi_{v(e)} = \operatorname{ord}_{q_{-e}} \Xi_{v(-e)} = -1$ at some node $q_{|e|}$, then the residues are opposite at the node: $\operatorname{res}_{q_e} \Xi_{v(e)} = -\operatorname{res}_{q_{-e}} \Xi_{v(-e)}$.

Definition 2.3.1 ([BCGGM18, Def. 1.1])**.** A differential Ξ satisfying Conditions (0) $\sim$ (3) is called a *twisted differential* of type μ.

Given a one parameter family, $l : v \mapsto l_v$ gives a (full) level function on the vertices of the dual graph Γ_C. The function l makes Γ_C into a level graph, in which the order is denoted by " $\succcurlyeq$ ". Moreover, the twisted differential Ξ constructed from the one-parameter family must satisfy the following conditions (again see the proof of necessity of [BCGGM18, Theorem 1.3]):

(4) At a node e, $v(e) \succcurlyeq v(-e)$ if and only if $\mathrm{ord}_{q_e} \Xi_{v(e)} \geq \mathrm{ord}_{q_{-e}} \Xi_{v(-e)}$, and $v(e) \asymp v(-e)$ if and only if $\mathrm{ord}_{q_e} \Xi_{v(e)} = \mathrm{ord}_{q_{-e}} \Xi_{v(-e)} = -1$

(5) For any level L in the level graph, for any v such that $l_v > L$, let E_v^L be the set of all the nodes e such that $v(e) = v$, $l_{v(-e)} = L$, we have

$$\sum_{e \in E_v^L} \mathrm{res}_{q_{-e_i}} \Xi_{v(-e_i)} = 0.$$

The last condition is called the *Global residue condition* in [BCGGM18].

Definition 2.3.2 ([BCGGM18, Def. 1.2]). A twisted differential Ξ is called *compatible* with the stable differential Ω and the full level function l (or equivalently the full level graph Γ_C) if (i) Ξ and l satisfy the Conditions (0)$\sim$(5); (ii) the maxima of the level graph correspond to the components C_v where Ω_v is not identically zero, and on those components, $\Xi_v = \Omega_v$.

The main result of [BCGGM18] is that the necessary and sufficient condition for a pointed stable differential (C, Ω) to lie in the boundary of the IVC compactification of strata is the existence of a twisted differential Ξ (on C) and a full level function l (on Γ_C) such that Ξ is compatible with Ω and l.

2.4 The Moduli Space of PPAVs and Theta Characteristics

This section provides the necessary background for Chapter 5, where we give the modular form that cuts out the hyperflex locus in genus 3 and we compute its divisor class in $\overline{\mathcal{M}}_3$. To this end, we start by introducing $\mathcal{A}_g$, the moduli spaces of *principally polarized abelian varieties* (ppav) and explain how they are closely related to the moduli space of curves. We move on to discuss the Riemann theta functions with characteristics and how they extend to the boundary of the compactification of $\mathcal{A}_g$.

Our discussion on the extension of theta characteristics to the boundary will be restricted to genus 3, while our discussion on the moduli space of ppav and theta functions is for all genera.

2.4.1 The Moduli Space of Principally Polarized Abelian Varieties

An *abelian variety* is a projective algebraic variety A, with the structure of an abelian group on the set of its points, such that the group operations are morphisms. A *polarization* on A is the first Chern class of an ample line bundle L on A. A polarization $[L]$ on an abelian variety A is called *principal* if its space of sections is one-dimensional, i.e. if $h_0(A, L) = 1$.

Over the complex numbers, an abelian variety can also be seen as a complex torus $\mathbb{C}^g/\Lambda$, where Λ is a full rank integral lattice isomorphic to $\mathbb{Z}^{2g}$. Up to biholomorphisms, we can normalize the generators of Λ such that the first g generators are unit vectors. By the *Riemann bilinear relations*, in order for a complex torus to be projective, the remaining g vectors must consist a $g \times g$ matrix τ with a positive-definite imaginary part, called the *period matrix*.

The *Siegel upper half-space* of dimension g is the space of all g by g period matrices, denoted by $\mathbb{H}_g$:

$$\mathbb{H}_g := \{\tau \in \mathrm{Mat}(g \times g, \mathbb{C}) \mid \tau = \tau^t, \mathrm{Im}\,(\tau) > 0\}.$$

Definition 2.4.1 (Moduli Space of ppavs)**.** The *moduli space of principally polarized abelian varieties (ppav)* of dimension g: $\mathcal{A}_g = \Gamma_g \setminus \mathbb{H}_g$ is the quotient of $\mathbb{H}_g$ by the symplectic group $\Gamma_g := \mathrm{Sp}(2g, \mathbb{Z})$.

The moduli space of curves of genus g is closely related to $\mathcal{A}_g$. The *Jacobian* of a curve is defined by $H^{1,0}(C, \mathbb{C})^*/H_1(C, \mathbb{Z})$, where $H^{1,0}(C) \equiv \mathbb{C}^g$, $H_1(C, \mathbb{Z}) \equiv \mathbb{Z}^{2g}$ and the embedding is given by $[\gamma] : \omega \to \int_\gamma \omega$. We see that the Jacobian is a projective complex torus. We have the Torelli map $u : \mathcal{M}_g \to \mathcal{A}_g$, sending a curve to its Jacobian. The image of Torreli map is called the Schottky locus. Characterizing the Schottky locus is a classical problem in algebraic geometry.

The period matrix of the Jacobian is explicit: we take a symplectic basis $\{A_i, B_i\}_{i=1,\dots,g}$ of $H_1(C, \mathbb{Z})$. We then find a normalized basis of holomorphic differentials $\{v_1, \dots, v_g\}$ in $H^0(C, K_C)$, such that $\int_{A_i} v_j = \delta_{ij}$. Then the period matrix of C is given by $\tau_{ij} = \int_{B_i} v_j$.

The *Hodge vector bundle* over $\mathcal{A}_g$ is defined by $\mathbb{E} := \pi_*(\Omega^1_{\mathcal{X}_g/\mathcal{A}_g})$, where $\pi : \mathcal{X}_g \to \mathcal{A}_g$ is the universal family of ppavs, with the fiber over the point $[A] \in \mathcal{A}_g$ being the variety A itself. The fiber of the Hodge vector bundle over a point $[A] \in \mathcal{A}_g$ is the g-dimensional space of holomorphic one-forms on A. By pulling back via the Torelli map, we obtain the Hodge bundle $\Omega\mathcal{M}_g$ over $\mathcal{M}_g$ defined above.

2.4.2 Riemann Theta Function and Modular Forms

In Definition 2.1.1, we have seen that for a given curve C, the *Riemann theta function with characteristics* (theta characteristics for short) is a section of the line bundle $\frac{1}{2}K_C$ twisted by a two-torsion line bundle. One can identify the set of two-torsion points in $\mathrm{Jac}(C)$ with $(\mathbb{Z}_2)^g \times (\mathbb{Z}_2)^g$, sending $m = (\tau\epsilon + \delta)/2$ to the characteristics (ϵ, δ), where τ is the period matrix of C.

We can also define the theta characteristics outside of the Schottky locus. For an abelian variety A_τ, we denote its set of two-torsion points by $A_\tau[2] \simeq (\mathbb{Z}/2\mathbb{Z})^{2g}$, identifying a two-torsion point $m = (\tau\varepsilon + \delta)/2$ with a characteristic $(\varepsilon, \delta) \in (\mathbb{Z}/2\mathbb{Z})^g$. One can then define analytically the *Riemann theta function with characteristics* (ε, δ):

$$\theta \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} (\tau, z) = \sum_{m \in \mathbb{Z}^g} \exp\Big[\pi i\big((m + \frac{\epsilon}{2})^t \tau (m + \frac{\epsilon}{2}) + 2(m + \frac{\epsilon}{2})^t (z + \frac{\delta}{2})\big)\Big].$$

When $[\varepsilon, \delta] = (0, 0)$, we have the usual Riemann theta function. For a fixed τ, the theta function defines a section of a line bundle on the corresponding abelian variety A_τ, which gives a principal polarization. We define $e(m) := (-1)^{\varepsilon \cdot \delta} = \pm 1$ to be the *parity* of m. The theta function with characteristics is an odd/even function of z when $[\varepsilon, \delta]$ is odd/even. Hence as a function of τ, $\theta(\tau, 0)$ is identically zero iff $[\varepsilon, \delta]$ is odd, and $\mathrm{grad}_z\, \theta[\epsilon, \delta](\tau, z)|_{z=0}$ vanishes identically iff $[\varepsilon, \delta]$ is even.

A more general notion on the Siegel upper half space $\mathbb{H}_g$ is the *modular forms*:

Definition 2.4.2 (Modular Form)**.** Given an arithmetic subgroup $\Gamma \subset \Gamma_g$ and a representation $\rho : GL(g, \mathbb{C}) \to GL(W)$, a holomorphic function $f : \mathbb{H}_g \to W$ is called a *ρ-valued Siegel modular form* w.r.t Γ if

$$f(\gamma \circ \tau) = \rho(C\tau + D) \circ f(\tau)$$

for any $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$, and any $\tau \in \mathbb{H}_g$. For $g = 1$ we also require f to be regular at the cusps of $\Gamma \setminus \mathbb{H}_1$.

If $W = \mathbb{C}$, and $\rho(\gamma) = \det(C\tau + D)^k$, then the modular form is called a *weight* k (scalar) modular form for Γ. We recall the following transformation formula for theta functions with characteristics:

$$\theta[\epsilon, \delta](\gamma\tau, (C\tau + D)^{-1}z) = \phi \cdot \det(C\tau + D)^{1/2}\theta[\gamma \circ (\epsilon, \delta)](\tau, z)$$

for any $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_g$ acting on the characteristic (ε, δ) in the following way:

$$\gamma \circ \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix} \begin{bmatrix} \epsilon \\ \delta \end{bmatrix} + \begin{bmatrix} diag(C^t D) \\ diag(A^t B) \end{bmatrix}, \tag{2.4.1}$$

and $\phi = \phi(\varepsilon, \delta, \gamma, \tau, z)$ is some complicated function, which will become trivial if $\gamma \in \Gamma_g(4,8)$ (we will define it below). Moreover, by differentiating with respect to z we obtain:

$$\frac{\partial}{\partial z_i}\theta[\epsilon,\delta](\gamma\tau,(C\tau+D)^{-1}z) = \det(C\tau+D)^{1/2}\sum_j (C\tau+D)_{ij}\frac{\partial}{\partial z_j}\theta[\gamma\circ(\epsilon,\delta)](\tau,z)$$

for any $\gamma \in \Gamma_g(4,8)$.

This is to say that the theta constant with characteristics is a modular form of weight $\frac{1}{2}$, and the theta gradient evaluated at $z=0$ (see [SM83]) is a vector-valued modular form for the representation $\rho = \det^{\frac{1}{2}} \otimes std$ (i.e. a section of $\det \mathbb{E}^{\otimes 1/2} \otimes \mathbb{E}$) with respect to a level subgroup $\Gamma_g(4,8) \subset \Gamma_g$, which is defined in general as follows:

$$\Gamma_g(m) := \{\gamma \in \Gamma_g \mid \gamma \equiv \mathbf{1}_{2g} \mod m\},$$

$$\Gamma_g(m,2m) := \{\gamma \in \Gamma_g(m) \mid \operatorname{diag}(C^tD) \equiv \operatorname{diag}(A^tB) \equiv 0 \mod 2m\}.$$

We will call the quotient $\mathcal{A}_g(m,2m) := \Gamma_g(m,2m) \setminus \mathbb{H}_g$ a *level moduli space of ppav.* This cover of $\mathcal{A}_g$ is Galois when m is even.

From this point forward until the end of this section, we will restrict ourselves in genus 3. In genus 3, the canonical image of a non-hyperelliptic curve is a plane quartic, and the bitangent lines to the plane quartic are given by the gradients of the theta functions with odd characteristics (cf. [Dol12, ch. 5]). We will use this fact in our discussion in Chapter 5.

2.4.3 Boundary of Level Covers in Genus 3

In genus 3, the Torelli map is dominant and can be extended to a surjective morphism $\overline{u} : \overline{\mathcal{M}}_3 \to \overline{\mathcal{A}}_3$, where $\overline{\mathcal{M}}_3$ is the Deligne-Mumford compactification, and $\overline{\mathcal{A}}_3$ is the toroidal compactification (note that for g=3 the perfect cone, second Voronoi, and central cone toroidal compactifications are all the same).

Recall that $\operatorname{Pic}_{\mathbb{Q}}(\overline{\mathcal{A}}_3) = \mathbb{Q}L \oplus \mathbb{Q}D$ where L is the first Chern class of the Hodge bundle $\mathbb{E}$ on $\mathcal{A}_3$, and D is the class of the boundary divisor (See [HS02]). We further recall (see [ACG11]) that $\operatorname{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_3) = \mathbb{Q}\lambda \oplus \mathbb{Q}\delta_0 \oplus \mathbb{Q}\delta_1$, where $\lambda := u^*L$, $\delta_0 := \bar{u}^*D$ is the class of the boundary component Δ_0, the closure of the locus of irreducible curves with one node, and δ_1 is the class of the locus $\Delta_1 \simeq \mathcal{M}_{1,1} \times \mathcal{M}_{2,1}$, i.e. the closure of the locus of nodal curves of compact type. The Torelli map contracts Δ_1 onto the locus $P := \overline{\mathcal{A}}_1 \times \overline{\mathcal{A}}_2 \subset \overline{\mathcal{A}}_3$.

By definition $\mathcal{A}_3(2)$ is the moduli of ppav together with a chosen symplectic basis for the group of two torsion points. Denote $p : \mathcal{A}_3(2) \to \mathcal{A}_3$ the level map, there is a level toroidal compactification (cf. [AMRT10]) such that p can be extended to $\bar{p}: \overline{\mathcal{A}}_3(2) \to \overline{\mathcal{A}}_3$. The pre-image $\bar{p}^{-1}D$ is reducible, and its components are indexed by non-zero characteristics: $\bar{p}^{-1}D = \cup_{n\in(Z/2Z)^6-0} D_n$.

The pre-image $\bar{p}^{-1}(P)$ is also reducible, and we now recall its irreducible components: since for a generic point A in P we have $A = E \times A'$, the group of two-torsion points splits as $A[2] \simeq (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/2\mathbb{Z})^4$. Choosing such an isomorphism is the same as choosing a 2-dim symplectic subspace V in $(\mathbb{Z}/2\mathbb{Z})^6$. Hence the irreducible components of $\bar{p}^{-1}(P)$ are labeled by the choice of such subspaces, and we denote them by P_V.

2.4.4 Extension of Theta Characteristics to the Boundary

In order to use a modular form to compute the corresponding divisor class in $\overline{\mathcal{M}}_3$, we need to know its vanishing order at the boundary. We will first compute the extensions of theta functions and theta gradients to the boundary.

Recall that we identify a two-torsion point $m = (\tau\varepsilon + \delta)/2$ with a characteristic $(\varepsilon, \delta) \in (\mathbb{Z}/2\mathbb{Z})^g$, and define $e(m) := (-1)^{\varepsilon\cdot\delta} = \pm 1$ to be the *parity* of m. We have the following standard definition (see [Dol12] for a more detailed discussion):

Definition 2.4.3. 1. We call a triple of characteristics m_1, m_2, m_3 *azygetic* (resp. *syzygetic*) if

$$e(m_1, m_2, m_3) := e(m_1)e(m_2)e(m_3)e(m_1 + m_2 + m_3) = -1 \text{ (resp. 1)}.$$

2. A sequence $m_1, m_2, \ldots, m_r$ is *essentially independent* if for any choice of $1 \leq i_1 < i_2 < \ldots < i_{2k} \leq r$ and $k \geq 1$ we have

$$m_{i_1} + m_{i_2} + \ldots + m_{i_{2k}} \neq 0 \mod 2.$$

Recall the notation D_n and P_V for the components of $\bar{p}^{-1}D$ and $\bar{p}^{-1}P$ in $\overline{\mathcal{A}}_3(2)$. For the purpose of computing the vanishing order of θ_m and $\operatorname{grad}_z \theta_m$, we need the characterization of the orbits of the Γ_g-action given by (2.4.1) on sets of characteristics. We have the following proposition.

Proposition 2.4.4 ([Igu72], [SM94])**.** *Two ordered sequences $m_1, m_2, \ldots, m_r$ and $n_1, n_2, \ldots, n_r$ of characteristics are conjugate under the action of Γ_g if and only if $e(m_i) = e(n_i)$, and $e(m_i, m_j, m_k) = e(n_i, n_j, n_k)$ for any $1 \leq i \leq r, 1 \leq i < j < k \leq r$, and the essentially independent subsequences correspond to each other.*

If there exists $\gamma \in \Gamma_g$ such that $\gamma(m,n) = (m',n')$, then $\operatorname{ord}_{D_n} \theta_m = \operatorname{ord}_{D_{n'}} \theta_{m'}$. Thus it suffices to compute this vanishing order for one element in each Γ_g orbit of pairs (m,n), and same argument applies to the pair (m,V) for the vanishing orders of the pulled back theta functions on Δ_1.

Since the groups Γ_g acts transitively on the set D_n of boundary components, each orbit of (m,n) under Γ_g contains all possible n. Thus we can from now on fix the boundary component D_n, and apply the proposition to find the orbit of (m,n) when m is varying: consider the set of triples $(m,n,0)$ where n is fixed and m is even (resp. odd), so that the parity of m and n remains the same, the orbits only depend on $e(m,n,0)$. By definition $e(m,n,0) = e(m)e(n)e(m+n)$, hence these orbits only depend on the parity of $m+n$.

We will also need the description of orbits of Γ_g action on pairs (m,V), where V is a symplectic 2-dim subspace of $(\mathbb{Z}/2\mathbb{Z})^{2g}$, to calculate the extension to the boundary of curves of compact type P_V.

Proposition 2.4.5. *The action of Γ_g on the set of pairs (m,V), where $V = span(n_1,n_2)$ is a symplectic 2-dim subspace of $(\mathbb{Z}/2\mathbb{Z})^{2g}$, has only two orbits, they correspond to the two cases when the number of even elements among $\{m+n_1, m+n_2, m+n_1+n_2\}$ is 1 or 3.*

Proof. Let X be the set of pairs (m,V), Y be the set of quadruples $\{m,n_1,n_2,n_1+n_2\}$. Let the map $q : Y \to X$ be the quotient under the symmetric group S_3 permuting the last three elements. Then q is Γ_g-equivariant. Denote the induced map by $q' : Y/\Gamma_g \to X/\Gamma_g$.

By the previous characterization, the Γ_g action on Y has eight orbits only depending on the parities of the triple $\{m+n_1, m+n_2, m+n_1+n_2\}$, namely $Y/\Gamma \simeq \mathbb{F}_2^3$. The map q' forgets the order of elements in the triple. Hence the orbits of σ depend only on the number of odd elements in the triple $\{m+n_1, m+n_2, m+n_1+n_2\}$.

Now by the following observation: for m odd and n_1, n_2 satisfying $\omega(n_1,n_2) \neq 0$, where ω is the standard symplectic form, we have $e(m+n_1+n_2) = e(m+n_1) \cdot e(m+n_2)$, the only possibilities for the number of even elements in the triple $\{m+n_1, m+n_2, m+n_1+n_2\}$ is then 1 and 3. ☐

The work of extension of theta constants and theta gradients to the boundary component D_n is done in [GH12], the vanishing orders are computed using Fourier-Jacobi expansion of theta function (which we write in a way that will make it easier to compute on Δ_1):

$$\theta\left[\begin{smallmatrix} \epsilon' & \epsilon \\ \delta' & \delta \end{smallmatrix}\right]\left(\left[\begin{smallmatrix} \tau' & b \\ b^t & \tau \end{smallmatrix}\right], 0\right) = \sum_{m' \in \mathbb{Z}, m'' \in \mathbb{Z}^{g-1}} \exp \pi i \left[2(m' + \frac{\epsilon'}{2}) b (m'' + \frac{\epsilon}{2})\right] A(m', m'') \qquad (2.4.2)$$

where

$$A(m',m'') = \exp \pi i \left([(m' + \frac{\epsilon'}{2})^2 \tau' + (m' + \frac{\epsilon'}{2})\delta] + [(m'' + \frac{\epsilon}{2})^t \tau (m'' + \frac{\epsilon'}{2}) + (m'' + \frac{\epsilon}{2})^t \delta] \right).$$

By the characterization of orbits of Γ_g we only need to work on a chosen boundary component D_{n_0} corresponding to $n_0 = [\begin{smallmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{smallmatrix}]$. The vanishing order of $\theta_m(\tau, 0)$ and $\mathrm{grad}_z\, \theta_m(\tau, 0)$ in τ is as follows:

Proposition 2.4.6. *We have the following:*

$$ord_{D_{n_0}} \theta_m(\tau, 0) = \begin{cases} 0 & \text{if } e(m + n_0) = 1 \\ \frac{1}{8} & \text{if } e(m + n_0) = -1 \end{cases} \tag{2.4.3}$$

$$\mathrm{ord}_{D_{n_0}}\, \mathrm{grad}_z\, \theta_m(\tau, z)|_{z=0} = \begin{cases} (\frac{1}{2}, 0, \dots, 0) & \text{if } e(m + n_0) = -1 \\ (\frac{1}{8}, \frac{1}{8}, \dots, \frac{1}{8}) & \text{if } e(m + n_0) = 1 \end{cases} \tag{2.4.4}$$

The notation above indicates the vanishing order for each partial derivative $(\frac{\partial}{\partial z_1}\theta, \frac{\partial}{\partial z_2}\theta \dots \frac{\partial}{\partial z_g}\theta)$.

For the boundary Δ_1, we can do a similar computation, which to our knowledge has not been done in literature. Following [Yam80] and [Fay73], we will consider the pinching/plumbing family of Riemann surfaces pinching a cycle homologous to zero. For a Riemann surface C of genus g, we fix an element of $\pi_1(C)$ which maps to zero in homology and is represented by a simple closed curve, and consider the plumbing family $\mathcal{C} \subset \overline{\mathcal{M}}_3$ parameterized by shrinking the length s of this curve to zero: for $s \neq 0$ the curve C_s is smooth, while for $s = 0$ the curve C_0 lies in Δ_1. We denote the period matrix of C_s by τ_s. Recall from Section 2.2.2, we have τ_s has an expansion at $s = 0$:

$$\tau_s = \begin{bmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{bmatrix} - s \begin{bmatrix} 0 & R \\ R^T & 0 \end{bmatrix} + O(s)$$

where $\tau_1 \in \mathrm{Mat}_{g_1 \times g_1}(\mathbb{C})$ and $\tau_2 \in \mathrm{Mat}_{g_2 \times g_2}(\mathbb{C})$ satisfying $g_1 + g_2 = g$, and $R \in \mathrm{Mat}_{g_1 \times g_2}(\mathbb{C})$ is some matrix independent of s. In our case $g_1 = 1$, $g_2 = 2$ and substitute into (2.4.2), so for the theta functions on the image of this degenerating family in $\overline{\mathcal{A}}_3$ we have:

$$\theta \left[\begin{smallmatrix} \epsilon' & \epsilon \\ \delta' & \delta \end{smallmatrix} \right] \left(\left[\begin{smallmatrix} \tau' & 0 \\ 0 & \tau'' \end{smallmatrix} \right], 0 \right) = \theta[\epsilon', \delta'](\tau', 0) \times \theta[\epsilon, \delta](\tau'', 0) \tag{2.4.5}$$

which vanishes if and only if $\varepsilon' \cdot \delta' = 1$ because both of the terms in the product are odd functions with respect to z. The Taylor expansion of $\theta_m(\tau, 0)$ with respect to $b = s \cdot R$ yields $\operatorname{ord}_b \theta_m(\tau, 0) = 1$ if $\varepsilon' \cdot \delta' = 1$, and it does not vanish generically otherwise.

Take the component P_{V_0} of p^*P corresponding to the following two-dimensional symplectic subspace:

$$V_0 = Span(n_1 = [\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}], n_2 = [\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}]).$$

The $\bar{u}$ pre-image of P_{V_0} is a component of $p'^*\Delta_1$ in $\overline{\mathcal{M}}_3(2)$. Then from the discussion above, one can conclude:

Proposition 2.4.7. *On the boundary component $\bar{u}^* P_{V_0}$ in $\overline{\mathcal{M}}_3(2)$, we have:*

$$\operatorname{ord}_b \theta_m(\tau, 0) = \begin{cases} 1 & \text{if } e(m+n_1) = e(m+n_2) = -1 \\ 0 & \text{otherwise} \end{cases} \tag{2.4.6}$$

$$\operatorname{ord}_b \operatorname{grad}_z \theta_m(\tau, 0) = \begin{cases} (0,1,1) & \text{if } e(m+n_1) = e(m+n_2) = 1 \\ (1,0,0) & \text{otherwise.} \end{cases} \tag{2.4.7}$$

The notation again indicates the vanishing order for each partial derivative.

Proof. We have the observation:

$$\begin{aligned} e(m+n_1) &= (-1)^{(\epsilon'+1)\delta' + \epsilon^t \delta} = (-1)^{\delta'} \cdot e(m) \\ e(m+n_2) &= (-1)^{\epsilon'} \cdot e(m). \end{aligned}$$

So the conditions in the proposition is the same as $\varepsilon' = \delta' = 1$. And the computation for gradients is parallel to the theta functions, we therefore omit it here. $\square$

Chapter 3

Degeneration of Abelian Differentials

In my paper collaborated with C.Norton [HN18], we fully generalize the results by Yamada [Yam80], Fay [Fay73] and Taniguchi [Tan89] [Tan91]. I present this result in this section, and will use the result to compute the variational formula for period matrices in Section 4.

The term "variational formula" means an expansion in terms of both s_e and $\ln|s_e|$. Note that a variational formula in this sense is *not* synonymous to a power series expansion in plumbing coordinates $\underline{s}$. We will use specifically the term "$\underline{s}$-expansion" when we mean the latter, where no logarithmic terms are involved. Moreover, throughout the section objects subscripted by "e" are indexed by the set of edges of the dual graph of the stable curve C, and those by "v" are indexed by the set of vertices of the dual graph.

3.1 Smoothing Riemann Surfaces

Let C be a stable nodal curve over the complex numbers. In this section we recall the plumbing construction and fix the notation.

We first recall Definition 2.2.1 for the definition of dual graph. For future convenience we write E_C for the set of oriented edges e of Γ_C. We will use $-e$ to denote the same edge as e but with the opposite orientation, and $|e| = |-e|$ the corresponding unoriented edge. Namely, $q_{\pm e}$ are the pre-images of the node $q_{|e|}$ in the normalization of C. We write $v(e)$ to denote the source of the oriented edge e, and write E_v for the set of edges e such that $v = v(e)$, that is, edges pointing out of the vertex v. We denote $|E|_C = \{|e|\}_{e \in E_C}$ the set of unoriented edges. The cardinality of $|E|_C$ is half of the cardinality of E_C.

3.1.1 Plumbing Construction

We now recall the local smoothing procedure of a nodal curve C via *plumbing*. There are many equivalent versions of the local plumbing procedure, we follow the one used in [Yam80], [Wol13] and [GKN17].

Definition 3.1.1. (Standard plumbing) Let q_e, q_{-e} be the two preimages of the node $q_{|e|}$ in the normalization of C. Let $z_{\pm e}$ be fixed chosen local coordinates near $q_{\pm e}$. Take a sufficiently small $s_e = s_{-e} \in \mathbb{C}$, we denote $U_{\pm e} = U_{\pm e}^{s_e} := \{|z_{\pm e}| < \sqrt{|s_e|}\} \subset C$, and denote $\gamma_{\pm e} := \partial U_{\pm e}$, which we call the *seams*. We orient each seam γ_e counter-clockwise with respect to U_e. The *standard plumbing* C_{s_e} of C is

$$C_{s_e} := [C \backslash U_e \sqcup U_{-e}]/(\gamma_e \sim \gamma_{-e}),$$

where $\gamma_e \sim \gamma_{-e}$ is identified via the diffeomorphism $I_e : \gamma_e \to \gamma_{-e}$ sending z_e to $z_{-e} = s_e/z_e$. We call the identified seam $\gamma_{|e|}$. The holomorphic structure of C_{s_e} is inherited from $C \backslash \overline{U}_e \sqcup \overline{U}_{-e}$.

Notation 3.1.2.

1. Since $s_e = s_{-e}$, we can use the notation $s_{|e|}$ and denote $\underline{s} := \{s_{|e|}\}_{|E|_C}$. In later parts of the paper we will continue to use s_e (instead of $s_{|e|}$) for simplicity.

2. We write $C_{\underline{s}}$ for the global smoothing of C by plumbing every node $q_{|e|}$ with *plumbing parameter* s_e, so that $C = C_{\underline{0}}$. Let $\widehat{C}_v := C_v \backslash \sqcup_{e \in E_v} U_e$, then $\widehat{C}_v$ has boundaries $\underline{\gamma} := \{\gamma_e\}_{e \in E_v}$. We use $\widetilde{C}_v$ to denote the interior of $\widehat{C}_v$, and $\widehat{C}_{\underline{s}}$ to denote the disjoint union of $\widehat{C}_v$ for all v. We have $C_{\underline{s}} = \widehat{C}_{\underline{s}}/\{\gamma_e \sim \gamma_{-e}\}_{|E|_C}$.

3. Throughout this paper, in a specified component C_v, the subscripted z_e is used to denote the chosen local coordinate near the node q_e for every $e \in E_v$ for the standard plumbing. The non-subscripted notation z is used to denote an arbitary local coordinate of any point in $\widetilde{C}_v$.

4. For future convenience, we denote $|\underline{s}| := \max_{|e| \in |E|_C} |s_e|$.

Remark 3.1.3. Let $u = (u_1, \ldots, u_k)$ be some coordinates along the boundary stratum of $\overline{\mathcal{M}}_g$ that C lies in. One can think of the boundary stratum as a Cartesian product of moduli of curves with marked points, and u is the combination of some coordinates chosen on each moduli space. It is a standard result in Teichmüller theory (see [IT]) that the set of plumbing parameters $\underline{s}$ together with u give local coordinates on $\overline{\mathcal{M}}_g$ near C. We denote $C_{u,\underline{s}}$ a nearby curve of C, then $C = C_{u_0,\underline{0}}$ for some u_0. Our results depend smoothly on u throughout the paper, and all the bounds we derive in this paper hold for u varying in a small neighborhood of u_0, therefore we fix the coordinate u_0 for C and consistently write $C_{\underline{s}} = C_{u_0,\underline{s}}$.

3.1.2 Conditions on Residues

Our goal is to express the variational formulas for abelian differentials with at worst simple poles on C in plumbing coordinates. Take a stable differential Ω on the stable curve C in the boundary of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$. Denote Ω_v to be the restriction of Ω to the irreducible component C_v. We require Ω_v to be a meromorphic differential whose only singularities are simple poles at the nodes of C_v. We denote the residue of Ω_v at q_e to be r_e (possibly zero). We have $r_e = -r_{-e}$ for any $e \in E_C$ by the definition of the extended Hodge bundle $\Omega\overline{\mathcal{M}}_g$ (see e.g., [HM]).

3.1.3 Jump Problem

The technique we use to construct the degenerating family of abelian differentials $\Omega_{\underline{s}}$ along the plumbing family $C_{\underline{s}}$ is called (solving) the jump problem. The main idea is that given a stable differential Ω on C, we have the mis-matches $\{\Omega|_{\gamma_e} - I_e^*(\Omega|_{\gamma_{-e}})\}$ (which we call the *jumps* of Ω) on the seams $\gamma_{\pm e}$ at opposite sides of each node $q_{|e|}$. The solution to the jump problem is a "correction" differential η that matches the jumps of Ω with opposite sign. By adding η to Ω on each irreducible component, one obtains new differentials with zero jumps, that can thus be glued to get a global holomorphic differential $\Omega_{\underline{s}}$ on $C_{\underline{s}}$.

The jump problem is a special version of the classical Dirichlet problem. It was developed and used in the real-analytic setting in a recent paper by Grushevsky, Krichever and the second author [GKN17]. In the classical approach, the Cauchy kernel on the plumbed surface is used, while in [GKN17] the fixed Cauchy kernels on the irreducible components of the nodal curve are used, which is crucial to obtain an L^2-bound of the solution to the jump problem in plumbing parameters.

Our construction of the solution to the jump problem largely follows the method in that paper. Instead of the real normalization condition used in [GKN17], we normalized the solution by requiring that it has vanishing A-periods, where A is a set of generators of a chosen Lagrangian subgroup of $H_1(C_{\underline{s}}, \mathbb{Z})$ containing the classes of the seams. This normalization condition allows us to work in the holomorphic setting (as opposed to the real-analytic setting in [GKN17]) where we can use Cauchy's integral formula. As a consequence, the construction of the solution is simpler.

Take a stable differential Ω on the stable curve C in the boundary of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$. Denote Ω_v to be the restriction of Ω to the irreducible component C_v. We require Ω_v to be a meromorphic differential whose only singularities are simple poles at the nodes of C_v. We denote the residue of Ω_v at q_e to be r_e (possibly zero). We have $r_e = -r_{-e}$ for any $e \in E_C$ by the definition of the extended Hodge bundle $\Omega\overline{\mathcal{M}}_g$ (see e.g., [HM]).

Definition 3.1.4. (Jump problem) The initial data of the jump problem is a collection $\underline{\phi}$ of complex-valued continuous 1-forms $\{\phi_e\}$ supported on the seams $\underline{\gamma}$ of $\widehat{C}_{\underline{s}}$, satisfying the conditions

$$\phi_e = -I_e^*(\phi_{-e}), \qquad \int_{\gamma_e} \phi_e = 0, \qquad \forall e \in E_C. \tag{3.1.1}$$

We call the set $\{\phi_e\}_{e\in E_C}$ *jumps*. A *solution to the jump problem* is a holomorphic differential $\eta_{\underline{s}}$ on $\widehat{C}_{\underline{s}}$ such that it is holomorphic on $\widetilde{C}_{\underline{s}}$ and continuous on the boundaries $\underline{\gamma}$, satisfying the condition

$$\eta_{\underline{s}}|_{\gamma_e} - I_e^*(\eta_{\underline{s}}|_{\gamma_{-e}}) = -\phi_e \qquad \forall e \in E_C.$$

Note that by letting $\{\phi_e\}_{E_C}$ be the mis-matches $\{\Omega_{v(e)}|_{\gamma_e} - I_e^*(\Omega_{v(-e)}|_{\gamma_{-e}})\}_{E_C}$ of Ω, one can check that they satisfy (3.1.1). Therefore $(\Omega+\eta_{\underline{s}})|_{C_{v(e)}}$ and $(\Omega+\eta_{\underline{s}})|_{C_{v(-e)}}$ have no jump along γ_e at every node e, where $\eta_{\underline{s}}$ is the solution to the jump problem with jumps the mis-matches of Ω. We can thus glue them along each seam to obtain a required global differential $\Omega_{\underline{s}}$ on $C_{\underline{s}}$.

Notation 3.1.5. For simplicity, we drop the subscript $\underline{s}$ in $\eta_{\underline{s}}$ throughout the paper. But it is important to bear in mind that the solution depends on $\underline{s}$ as the size of the seams varies with $\underline{s}$.

Note that the solution to the jump problem is never unique: adding any differential on $C_{\underline{s}}$ gives another solution. We need to impose a normalizing condition to ensure the uniqueness of the solution.

On each irreducible component C_v of the nodal curve C we choose and fix a Lagrangian subspace of $H_1(C_v, \mathbb{Z})$, and we also choose and fix a basis of the subspace. In Definition 3.1.1, the plumbed surface $\widehat{C}_{\underline{s}}$ is seen to be a subset of C. Since the seams (as boundaries of $\widehat{C}_{\underline{s}}$) are contractible on each C_v (without boundaries), we know that the classes of the seams $\{[\gamma_{|e|}]\}_{|E|_C}$ together with the union of the basis of the Lagrangian subspaces on the irreducible components span a Lagrangian subspace of $H_1(C_{\underline{s}}, \mathbb{Z})$. We can fix this Lagrangian subspace of $H_1(C_{\underline{s}}, \mathbb{Z})$ along the plumbing family $\{C_{\underline{s}}\}$. If some $\gamma_{|e|}$ is homologous to zero on $C_{\underline{s}}$, as s_e approaches 0 the Lagrangian subspace of $H_1(C_{\underline{s}}, \mathbb{Z})$ is invariant; if the class of $\gamma_{|e|}$ is non-zero, then the rank of the Lagrangian subspace drops by 1 as the corresponding element in the basis goes to zero.

We denote this choice of basis as $\{A_{1,\underline{s}}, \ldots, A_{g,\underline{s}}\}$ where the first m cycles $A_{1,\underline{s}}, \ldots, A_{m,\underline{s}}$ generate the subspace spanned by the seams $\{[\gamma_{|e|}]\}_{|E|_C}$. This choice of indexing will be used later in the computation of the period matrices in Section 4.2.

Definition 3.1.6. A solution to the jump problem is A-*normalized* if it has vanishing periods over $A_{1,\underline{s}}, \ldots, A_{g,\underline{s}}$.

Note that this definition only depends on the choice of Lagrangian subspace of $H_1(C_{\underline{s}}, \mathbb{Z})$. In particular by our choice of Lagrangian subspace, an A-normalized solution η must have vanishing periods over the seams: $\int_{\gamma_{|e|}} \eta = 0$.

It is a standard fact (see e.g., [GH]) that any holomorphic A-normalized differential is identically zero on a compact Riemann surface without boundaries. Given two A-normalized solutions η and η' on $\widehat{C}_{\underline{s}}$ which are both holomorphic by definition, the differential $\eta - \eta'$ has zero jumps on the seams and thus defines a global holomorphic A-normalized differential on $C_{\underline{s}}$, which is therefore identically zero. This shows the uniqueness of an A-normalized solution.

3.2 General Variational Formula for Abelian Differentials

In this section we construct the degenerating family $\Omega_{\underline{s}}$ in a plumbing family $C_{\underline{s}}$, and give the variational formula for $\Omega_{\underline{s}}$ in terms of $\underline{s}$. As introduced in beggining of this section, we plan to construct the solution to the jump problem that matches the jumps of $\Omega_0 = \Omega$. In the classical construction, such differentials are obtained by integrating the jumps against the *Cauchy kernel* (see the following section) on the whole $C_{\underline{s}}$. In this approach the Cauchy kernel depends on $\underline{s}$, and this dependence is implicit and hard to determine.

Alternatively, following [GKN17], we fix the Cauchy kernels on each irreducible components of the normalization of the limit stable curve C. On each component C_v we integrate the jumps $\{\Omega_{v(e)}|_{\gamma_e} - I_e^*(\Omega_{v(-e)}|_{\gamma_{-e}})\}_{e \in E_v}$ against the Cauchy kernel. In this way we obtain a differential in the classical sense on each component C_v which has jumps across the seams. We then restrict it to $\widehat{C}_v$, the component minus the "caps". In this way the original jumps are compensated by the newly-constructed differentials, but these differentials in turn produce new jumps. However, since the L_∞-norms of the newly-constructed differentials along the seams $\gamma_{|e|}$ in local coordinates z_e are controlled in an explicit way by $\underline{s}$, the new jumps are also controlled by $\underline{s}$. By iterating the process one obtain a sequence of differentials, each term controlled by a higher power of $\underline{s}$. This sequence converges to the desired solution to the jump problem.

3.2.1 The Cauchy Kernels

The construction of the A-normalized solution to the jump problem is parallel to the construction of the *almost real-normalized* solution in [GKN17], which uses a different normalizing condition, and the solution differential obtained there allows one to control the reality of periods.

Given a smooth Riemann surface C', the Cauchy kernel is the unique object on $C' \times C'$, satisfying the following properties:

1. It is a meromorphic differential of the second kind in p whose only simple poles are at $p = q$ and $p = q_0$ with residue $\pm\frac{1}{2\pi i}$;
2. It is an A-normalized differential in p: $\int_{p\in A_i} K_{C'}(p,q) = 0$, for $i = 1, \ldots, g$ and $\forall q \in C'$.

The Cauchy kernel can be viewed as a multi-valued meromorphic function in q whose only simple pole is at $p = q$. Let $\{A_i, B_i\}$ be a symplectic basis of $H_1(C', \mathbb{Z})$, and let $\{v_i\}$ to be the basis of holomorphic 1-form dual to the A-cycles. The multi-valuedness is precisely as follows (where $q + \gamma$ denotes the value of the kernel at q upon extension around the loop γ):

$$K(p, q + A_i) = K(p, q); \qquad K(p, q + B_i) = K(p, q) + v_i(p).$$

Note that the Cauchy kernel is a section of a line bundle on $C' \times C'$ satisfying the first two normalization conditions above, and therefore it can be written in terms of theta functions and the Abel-Jacobi map (for a reference of the theta function see [Gun76]). We also remark that $K_{C'}$ depends on the choice of the Lagrangian subspace spanned by the A cycles. For completeness below we include the explicit expression for the Cauchy kernel in terms of theta functions:

$$K_{C'}(p,q) := \frac{1}{2\pi i}\frac{\partial}{\partial p}\ln\frac{\theta(A(p) - A(q) - Z)}{\theta(A(p) - Z)},$$

where θ denotes the theta function on the Jacobian of C', Z denotes a general point of the Jacobian, and A denotes the Abel-Jacobi map with some base point $q_0 \in Z$. The expression does not depend on the choice of Z.

We call $\omega_{C'}(p,q) := 2\pi i d_q K_{C'}(p,q)$ the *fundamental normalized bidifferential of the second kind* on C' (also known as the *Bergman kernel*). Note that the term "normalized" here means A-normalization. Namely,

$$\int_{p\in A_i} \omega_{C'}(p,q) = 0, \qquad i = 1,\dots,g.$$

The fundamental normalized bidifferential has its only pole of second order at $p = q$. It is uniquely determined by its normalization along A-cycles, symmetry in the entries, and the bi-residue coefficient along $p = q$. See notes [Ber06, Ch. 6] for a review of the Cauchy kernel and fundamental bidifferential.

When C is a nodal curve with irreducible components C_v, we denote by K_v (resp. ω_v) the Cauchy kernel (resp. bidifferential) on each C_v. Recall that z_e (or w_e, when we need to distinguish between two distinct points in the same neighborhood) denotes the local coordinates in some neighborhood V_e of q_e that contains $\overline{U}_e$. We define a local holomorphic differential $\mathbf{K}_v \in \Omega^{1,0}(\sqcup_{e\in E_v} V_e \times \sqcup_{e\in E_v} V_e)$, by taking the regular part of K_v:

$$\mathbf{K}_v(z_e, w_{e'}) := \begin{cases} K_v(z_e, w_{e'}) & \text{if } e \neq e', \\ K_v(z_e, w_e) - \frac{dz_e}{2\pi i(z_e - w_e)} & \text{if } e = e'. \end{cases}$$

Define $\boldsymbol{\omega}_v(z_e, w_{e'}) = 2\pi i d_{w_{e'}} \mathbf{K}_v(z_e, w_{e'})$, then similarly we have

$$\boldsymbol{\omega}_v(z_e, w_{e'}) = \begin{cases} \omega_v(z_e, w_{e'}) & \text{if } e \neq e', \\ \omega_v(z_e, w_e) - \frac{dz_e dw_e}{(z_e - w_e)^2} & \text{if } e = e'. \end{cases}$$

For future convenience we fix the notation for the coefficients in the expansion of $\boldsymbol{\omega}_v(z_e, w_{e'})$:

$$\boldsymbol{\omega}_v(z_e, w_{e'}) =: dw_{e'} dz_e \left(\beta^v_{e,e'} + \sum_{i,j\geq 0, i+j>0} \beta^v_{i,j} z_e^i w_{e'}^j \right). \tag{3.2.1}$$

Clearly we have $\beta^v_{e,e'} = \boldsymbol{\omega}_v(q_e, q_{e'})$. When the context is clear, we drop the superscript v and write simply $\beta_{e,e'}$ instead.

3.2.2 Approach to Solving the Jump Problem

In this section we approach the jump problem directly in order to clarify the appearance of a series expression for local differentials (3.2.3) below.

We first look at the simplest example: $g = 0$. On $\mathbb{P}^1$ the Cauchy kernel is simply $K(z, w) = \frac{dz}{w-z}$. Let γ be a Jordan curve bounding a region R on $\mathbb{P}^1$. Cauchy's integral formula implies that integrating $K(z, w)$ against a differential $f(w)dw$ holomorphic inside the region R along γ (negatively oriented with respect to R) vanishes when z is in the exterior of R; it is equal to $f(z)dz$ when $z \in R$. In other words integrating $f(w)dw$ against the Cauchy kernel defines a differential on $\mathbb{P}^1 \setminus \gamma$ whose jump along γ is precise $f(z)dz$.

When replicating this idea on Riemann surfaces of higher genus, integrating a differential which is holomorphic inside a contractible loop γ against the Cauchy kernel produces a holomorphic differential on that Riemann surface whose jump across γ is given by the differential. Below z^+ is an point outside γ and z^- is inside γ:

$$\lim_{z^+ \to z' \in \gamma} \int_\gamma K(z, w) f(w) dw - \lim_{z^- \to z' \in \gamma} \int_\gamma K(z, w) f(w) dw = f(z)dz.$$

This follows directly from Cauchy's integral formula. In [GKN17], or in general when integrating against a jump which is not meromorphic, obtaining a result such as above would require the Sokhotski-Plemelj formula (for reference see [Ro88]).

We would like to explicitly analyze the dependence of the solution to the jump problem on the plumbing parameters. Solving the jump problem on $C_{\underline{s}}$ by integrating against the Cauchy kernel on $C_{\underline{s}}$, as described above is classical [Ro88], but it does not allow one to study the dependence on plumbing parameters. Therefore the approach we take, which was introduced in [GKN17], is integration against fixed Cauchy kernels defined individually on each irreducible component of the nodal curve, which are thus independent of $\underline{s}$. The result will give an explicit expansion of the solution in $\underline{s}$, and constructing this solution is more involved.

The procedure of the construction of the solution is clarified below.

Step 1. We denote the holomorphic part of the differential $\Omega_{v(e)}(z_e)$ as $\xi_e^{(0)}(z_e) := \Omega_v(z_e) - \frac{r_e dz_e}{z_e}$. It follows from the residue condition that the singular parts of the differentials on the opposite sides of the node cancel. Thus the jumps can be written as follows:

$$\{\Omega_{v(e)}|_{\gamma_e} - I_e^*(\Omega_{v(-e)}|_{\gamma_{-e}})\}_{e \in E_v} = \{\xi_e^{(0)}|_{\gamma_e} - I_e^*(\xi_{-e}^{(0)}|_{\gamma_{-e}})\}_{e \in E_v} \, .$$

Step 2. We integrate the jumps against the Cauchy kernel. This integration defines a differential on the open Riemann surface $\tilde{C}_v$,

$$\begin{aligned}\eta_v^{(1)}(z) :=& \sum_{e\in E_v}\int_{z_e\in\gamma_e} K_v(z,z_e)(\xi_e^{(0)}|_{\gamma_e} - I_e^*\xi_{-e}^{(0)}|_{\gamma_{-e}})(z_e)\\ =& \sum_{e\in E_v}\int_{z_e\in\gamma_e} K_v(z,z_e)I_e^*\xi_{-e}^{(0)}|_{\gamma_{-e}}(z_e)\end{aligned}$$

where the equality follows from Cauchy's integral formula. We also extend $\eta_v^{(1)}(z)$ continuously to the boundary of the plumbing neighborhood.

We have an important remark here: The differential $\eta_v^{(1)}(z)$ can be seen as our first attempt at solving the jump problem, but it does *not* give the solution of the desired jump problem. There is a new jump between $\Omega_{v(e)} + \eta_{v(e)}^{(1)}$ and $\Omega_{v(-e)} + \eta_{v(-e)}^{(1)}$ on each node. The "error" comes from the holomorphic part of the Cauchy kernel.

Step 3. We look at this "error" explicitly. Locally near the seam γ_{e_0}, the differential $\eta_v^{(1)}(z_{e_0})$ for $\sqrt{|s_{e_0}|} < |z_{e_0}| < 1$ has the following expression:

$$\begin{aligned}\sum_{e\in E_v}\int_{z_e\in\gamma_e} K_v(z_{e_0},z_e)I_e^*\xi_{-e}^{(0)}(z_e) =& \sum_{e\in E_v}\int_{z_e\in\gamma_e} \mathbf{K}_v(z_{e_0},z_e)I_e^*\xi_{-e}^{(0)}(z_e) +\\ & \frac{1}{2\pi i}\int_{w_{e_0}\in\gamma_{e_0}} \frac{dz_{e_0}}{z_{e_0}-w_{e_0}} I_{e_0}^*\xi_{-e_0}^{(0)}(w_{e_0}).\end{aligned}$$

Where we recall that $\mathbf{K}_v$ is the holomorphic part of K_v, and the last part involves the integral of the singular part of the Cauchy kernel. The last integral can be evaluated by Cauchy's integral formula by noting that $|s_e z_e^{-1}| < \sqrt{s_e}$ where we point out that I^* is orientation reversing,

$$\frac{1}{2\pi i}dz_e\int_{w_e\in\gamma_e}\frac{I_e^*\xi_{-e}^{(0)}(w_e)}{z_e-w_e} = -\frac{1}{2\pi i}\frac{dz_e}{z_e}\int_{w_{-e}\in\gamma_{-e}}\frac{w_{-e}\xi_{-e}^{(0)}(w_{-e})}{w_{-e}-s_e z_e^{-1}} = -s_e\frac{dz_e}{z_e^2}\tilde{\xi}_{-e}^{(0)}(\frac{s_e}{z_e}) = I^*\xi_{-e}^{(0)}(z_e)\ . \tag{3.2.2}$$

Therefore we have the following:

$$\begin{aligned}\{\eta_{v(e)}^{(1)}|_{\gamma_e} - I_e^*\eta_{v(-e)}^{(1)}|_{\gamma_{-e}}\}_{e\in E_v} =& -\{\Omega_{v(e)}|_{\gamma_e} - I_e^*\Omega_{v(-e)}|_{\gamma_{-e}}\}_{e\in E_v}\\ &+\Big(\sum_{e'\in E_v}\int_{z_{e'}\in\gamma_{e'}}\mathbf{K}_{v(e)}(z_e,z_{e'})I_{e'}^*\xi_{-e'}^{(0)}(z_{e'})\\ &-I^*\sum_{e'\in E_{v(-e)}}\int_{z_{e'}\in\gamma_{e'}}\mathbf{K}_{v(-e)}(z_{-e},z_{e'})I_{e'}^*\xi_{-e'}^{(0)}(z_{e'})\Big)_{e\in E_v}.\end{aligned}$$

Thus we see that $\eta_v^{(1)}(z)$ has the desired jump *plus* the jump of (local) holomorphic differentials $\sum_{e' \in E_v} \int_{z_{e'} \in \gamma_{e'}} \mathbf{K}_v(z_e, z_{e'}) I_{e'}^* \xi_{-e'}^{(0)}(z_{e'})$, which is exactly the "error".

Step 4. We give an estimate on the size of the "error". We show in Lemma 3.2.2 below that the L^∞-norm of the "error" is controlled by the L^∞-norm of the plumbing parameters $|\underline{s}|$. We therefore apply the jump problem *again* to further reduce the gap. Finally, our approach to solving the jump problem is by integrating against a series constructed from the recursively appearing jump problems. We prove (in Lemma 3.2.2) that the "errors" produced in the k-th step of the recursion is controlled by the k-th power of $|\underline{s}|$, and we use this to show the convergence of the desired solution of the jump problem.

In the following section we first define the "errors" $\xi_e^{(k)}(z_e)$ in each step, and then we prove Lemma 3.2.2 which bounds each by a power of the plumbing parameters, thus the series defined by adding the "errors" converges. And at last we prove that the solution to the jump problem, denoted η_v, is the result of integrating this series against the Cauchy kernel on each irreducible component.

3.2.3 Construction of the A-normalized Solution to the Jump Problem

We construct the A-normalized solution to the jump problem as suggested by the computation above, namely we define local holomorphic differentials, which can be understood as the recursively appearing jumps, and show the series converges. The resulting local differentials are such that when integrated against the Cauchy kernel on each irreducible component of the nodal curve, the jump is given by the first term in the series.

Let Ω be a stable differential on the stable curve C. We can define recursively the following collection of *holomorphic* differentials described *locally* in the neighborhood of each node :

$$\begin{aligned} k = 0 &: \xi_e^{(0)}(z_e) := \Omega_v(z_e) - \frac{r_e dz_e}{z_e}; \\ k > 0 &: \xi_e^{(k)}(z_e) := \sum_{e' \in E_v} \int_{w_{e'} \in \gamma_{e'}} \mathbf{K}_v(z_e, w_{e'}) \cdot I_{e'}^* \xi_{-e'}^{(k-1)}(w_{e'}). \end{aligned} \tag{3.2.3}$$

Note $\xi_e^{(k)}$ for $k > 0$ depend on $\underline{s}$ as γ_e and I^* depend on $\underline{s}$. We suppress this in the notation.

Let Γ_C be the dual graph of C. Let $l^k := (e_1 \ldots, e_k)$ be an oriented path of length k in the dual graph, starting from the vertex $v = v(e_1)$. We denote L_v^k the collection of all such paths starting from the vertex v. We remark that given the Cauchy kernels on each component, the differential $\xi_{e_1}^{(k)}(z_{e_1})$ is determined by the collection of local differentials $\{\xi_{-e_k}^{(0)}(w_{-e_k})\}_{L_v^k}$ where e_k is the ending edge of $l^k \in L_v^k$.

We define $\xi_e(z_e) := \sum_{k=0}^{\infty} \xi_e^{(k)}(z_e)$. Since $\mathbf{K}_v(z_e, w_{e'})$ is holomorphic in the first variable, we have $\int_{\gamma_e} \xi_e^{(k)} = 0$ for any e and k, therefore

$$\int_{\gamma_e} \xi_e = 0, \qquad \forall e \in E_C. \tag{3.2.4}$$

The convergence of this series is ensured by the following lemma, whose proof follows very much along the lines of [GKN17]. We include the proof here for completeness. The essential ingredient of the proof is the fact that our Cauchy kernels and the bidifferentials are defined on the irreducible components, thus they are independent of the plumbing parameters $\underline{s}$.

Notation 3.2.1.

1. Throughout this section and Section 4 we use the tilde notation to denote the function corresponding to a given differential in a given local coordinate chart, for instance $\mathbf{K}(z,w) =: \widetilde{\mathbf{K}}(z,w)dz$, $\boldsymbol{\omega}(z,w) =: \widetilde{\boldsymbol{\omega}}(z,w)dzdw$, and also $\xi_e^{(k)}(z_e) := \tilde{\xi}_e^{(k)}(z_e)dz_e$.

2. To simplify notation, we denote

$$\tilde{\xi}_e := \tilde{\xi}_e^{(0)}(q_e) \tag{3.2.5}$$

at every node q_e.

3. When the function $\widetilde{\omega}_v(z,w)$ of the bidifferential $\omega_v(z,w)$ is evaluated in the second variable at any node q_e, by an abuse of notation, we write $\omega_v(z,q_e) = \widetilde{\omega}_v(z,q_e)dz$ for $z \in \widetilde{C}_v$.

4. Recall that $|s| := \max_{e\in E_C} |s_e|$. For future convenience, for any collection of holomorphic functions on the unit disks neighborhood at each node $\underline{f} := \{f_e \in \mathcal{O}(V_e)\}_{e\in E_C}$, we define the following L^∞-norms:

$$|f_e|_{\underline{s}} := \sup_{z_e\in\gamma_e} |f_e(z_e)|; \qquad |\underline{f}|_{\underline{s}} := \max_{e\in E_C} |f_e|_{\underline{s}}.$$

Moreover by the Schwarz lemma on $U_e = \{|z_e| < \sqrt{|s_e|}\}$ we have that $|\underline{f}|_{\underline{s}} \leq |\underline{f}|_{\underline{1}} \sqrt{|\underline{s}|}^{\operatorname{ord} \underline{f}}$, where $\operatorname{ord} \underline{f} := \min_{e\in E_C} \operatorname{ord}_{q_e} f_e$.

Lemma 3.2.2. *For sufficiently small $\underline{s}$, there exists a constant M_1 independent of $\underline{s}$, such that the following estimate holds:*

$$|\underline{\tilde{\xi}}^{(k)}|_{\underline{s}} \leq (|\underline{s}|M_1)^k |\underline{\tilde{\xi}}^{(0)}|_{\underline{s}}. \tag{3.2.6}$$

In particular, the local differential $\xi_e(z_e)$ is a well-defined holomorphic differential at each node $e \in E_C$.

Proof. For all v and all $e, e' \in E_v$, the Cauchy kernel $\widetilde{K}_v(z_e, w_{e'})$ is analytic in both variables and independent of $\underline{s}$. Thus there exists a uniform constant M_2 (independent of $\underline{s}$) such that for any $z_e \in V_e$,

$$|\widetilde{\mathbf{K}}_v(z_e, w_{e'}) - \widetilde{\mathbf{K}}_v(z_e, 0)| < M_2 |w_{e'}|.$$

This in turn implies

$$\max_{w_{e'} \in \gamma_{e'}} \left| \frac{\widetilde{\mathbf{K}}(z_e, w_{e'}) - \widetilde{\mathbf{K}}(z_e, 0)}{w_{e'}^2} \right| < \frac{M_2}{\sqrt{|s_{e'}|}}. \tag{3.2.7}$$

By (3.2.4), we have

$$\left| \int_{w_{e'} \in \gamma_{e'}} \widetilde{\mathbf{K}}_v(z_e, w_{e'}) I_{e'}^* \xi_{-e'}^{(k-1)}(w_{e'}) \right| = \left| \int_{w_{e'} \in \gamma_{e'}} \left[\widetilde{\mathbf{K}}_v(z_e, w_{e'}) - \widetilde{\mathbf{K}}_v(z_e, 0) \right] I_{e'}^* \xi_{-e'}^{(k-1)}(w_{e'}) \right|$$

$$= |s_{e'}| \int_{w_{e'} \in \gamma_{e'}} \left| \frac{\widetilde{\mathbf{K}}(z_e, w_{e'}) - \widetilde{\mathbf{K}}(z_e, 0)}{w_{e'}^2} \right| \left| I_{e'}^* \tilde{\xi}_{-e'}^{(k-1)}(w_{e'}) \right| dw_{e'} < |s_{e'}| M_2 \cdot 2\pi |\tilde{\xi}_{-e'}^{(k-1)}|_{\underline{s}}$$

The second equality is the result of pulling back $dw_{-e'}$. Note that the last inequality is due to the fact that for $w_{e'} \in \gamma_{e'}$, we have $|I_{e'}^* \tilde{\xi}_{-e'}^{(k-1)}(w_{e'})| = |\tilde{\xi}_{-e'}^{(k-1)}(\frac{s_{e'}}{w_{e'}})| \leq |\tilde{\xi}_{-e'}^{(k-1)}|_{\underline{s}}$, therefore the integration over $\gamma_{e'}$ gives a $\sqrt{|s_{e'}|}$ that cancels the one in (3.2.7). By definition of $\tilde{\xi}_e^{(k)}$, there exists a constant M_1 independent of $\underline{s}$ and k such that,

$$|\tilde{\xi}_e^{(k)}|_{\underline{s}} \leq |\underline{s}| M_1 \max_{e' \in E_{v(e)}} |\tilde{\xi}_{-e'}^{(k-1)}|_{\underline{s}} < |\underline{s}| M_1 |\underline{\tilde{\xi}}^{(k-1)}|_{\underline{s}}.$$

Note that the RHS is independent of e and v, we can thus pass to the maximum over $e \in E_C$ of the LHS and obtain $|\underline{\tilde{\xi}}^{(k)}|_{\underline{s}} < |\underline{s}| M_1 |\underline{\tilde{\xi}}^{(k-1)}|_{\underline{s}}$. By induction, we have the desired estimate (3.2.6).

When $|\underline{s}| < 2M_1^{-1}$, the geometric series $|\underline{\tilde{\xi}}|_{\underline{s}} := \sum_{k=0}^{\infty} |\underline{\tilde{\xi}}^{(k)}|_{\underline{s}}$ converges to a limit bounded by $\left(1 + \frac{|\underline{s}| M_1}{1 - |\underline{s}| M_1}\right) |\underline{\tilde{\xi}}^{(0)}|_{\underline{s}} < 2 |\underline{\tilde{\xi}}^{(0)}|_{\underline{s}} < 2\sqrt{|\underline{s}|}^{\operatorname{ord} \underline{\tilde{\xi}}^{(0)}} |\underline{\tilde{\xi}}^{(0)}|_{\underline{1}}$. We therefore conclude that the local differential $\xi_e(z_e)$ is analytic in $\underline{s}$. $\square$

We now construct the solution to the jump problem with initial data $\{\Omega_v(e)|_{\gamma_e} - I_e^*(\Omega_{v(-e)}|_{\gamma_{-e}})\}$. We define the following differential on $\widehat{C}_v$:

$$\eta_v(z) = \sum_{e\in E_v} \int_{z_e\in\gamma_e} K_v(z, z_e) I_e^* \xi_{-e}(z_e). \tag{3.2.8}$$

where $z \in \widetilde{C}_v$. By extending continuously to the seams, the differential η_v is defined on $\widehat{C}_v$.

Recall $\xi_e(z_e) := \sum_{k=0}^{\infty} \xi_e^{(k)}(z_e)$. For future use we denote,

$$\eta_v^{(k)}(z) := \sum_{e\in E_v} \int_{z_e\in\gamma_e} K_v(z, z_e) \cdot I_e^* \xi_{-e}^{(k-1)}(z_e). \tag{3.2.9}$$

In this notation we have $\eta_v := \sum_{k=1}^{\infty} \eta_v^{(k)}$.

We claim the differentials $\eta_v(z)$ are single-valued. This follows from noticing the multi-valuedness of $K(z, z_e)$ along B_i depends exclusively on z, and thus any multi-valuedness is canceled after integration against $I_e^*\xi_{-e}$ by (3.2.4).

$$\int_{z_e\in\gamma_e} K_v(z+B_i, z_e))\cdot I_e^*\xi_{-e}(z_e) - \int_{z_e\in\gamma_e} K_v(z, z_e))\cdot I_e^*\xi_{-e}(z_e) = v_i(z) \int_{z_e\in\gamma_e} I_e^*\xi_{-e}(z_e) = 0$$

Although the Cauchy kernel K_v has a simple pole with residue $(2\pi i)^{-1}$ at the base point q_0, it follows from (3.2.4) that $\eta_v(z)$ is holomorphic at q_0 and hence defines a holomorphic differential on $\widetilde{C}_v$. Let γ_{q_0} be a small loop around the point q_0. Here we verify integrating η_v along γ_{q_0} is zero. The paths γ_{q_0} does not intersect any γ_e, and we could exchange the order of integration in z and z_e. The integral of $K_v(z, z_e)$ along $z \in \gamma_{q_0}$ is $(2\pi i)^{-1}$ for any q. Thus by (3.2.4) integrating the result times $I_e^*\xi_{-e}(z_e)$ along γ_e is zero.

We recall that the L_2-norm of a holomorphic differential ω on a smooth Riemann surface C' is given by $||\omega||_{L_2} := \frac{i}{2}\int_{C'} \omega \wedge \overline{\omega}$. Note that both $\xi_e(z_e)$ and $\eta_v(z)$ implicitly depend on $\underline{s}$. The following theorem establishes an L^2 bound on η_v, and shows that it is the desired solution to the jump problem.

Theorem 3.2.3. *Let C be a stable nodal curve with irreducible components C_v, Ω a stable differential on C. Let Ω_v be the restriction of Ω on C_v. For $|\underline{s}|$ small enough, $\{\eta_v\}$ is the unique A-normalized solution to the jump problem with jump data $\Omega_{v(e)}|_{\gamma_e} - I_e^*(\Omega_{v(-e)}|_{\gamma_{-e}})$. Moreover, there exists a constant M independent of v and $\underline{s}$, such that the following L^2-bound of the solution holds:*

$$||\eta_v||_{L^2} < \sqrt{|\underline{s}|}^{1+\operatorname{ord}\underline{\tilde{\xi}}^{(0)}} M |\underline{\tilde{\xi}}^{(0)}|_{\underline{1}}. \tag{3.2.10}$$

Therefore $\{\Omega_{v,\underline{s}} := \Omega_v + \eta_v\}$ defines a holomorphic differential when all $|s_e| > 0$, denoted $\Omega_{\underline{s}}$ on $C_{\underline{s}}$, satisfying $\Omega_v = \lim_{\underline{s}\to 0} \Omega_{\underline{s}}|_{C_v}$ uniformly on compact sets of $C_v \setminus \cup_{e\in E_v} q_e$.

Proof. **Step 1.** We first show that the solutions η_v are A-normalized. Recall that our choice of the maximal Lagrangian subspace of $H_1(C_{\underline{s}}, \mathbb{Z})$ contains the subspace generated by the classes of the seams $\gamma_{|e|}$. By the fact that $\{K_v(z, z_e)\}_{v,e}$ are A-normalized in the first variable, the integral $\int_{z\in A_{i,\underline{s}}} \eta_v^{(k)}(z)$ is zero if the class $[A_{i,\underline{s}}]$ does not belong to the span of the seams $\{[\gamma_{|e|}]\}$.

In order to compute the integrals of $\eta_v^{(k)}$ along the seams, we compute the local expression for $\eta_v^{(k)}$ in the neighborhood $\{|\sqrt{s_e}| < |z_e| < 1\}$. Note that the Cauchy kernel $K_v(z_e, w_{e'})$ is holomorphic if $e \neq e'$, and it has a singular part $\frac{dz}{2\pi i(z_e - w_e)}$ when both variables are in the neighborhood of the same nodes. Therefore we have

$$\begin{aligned}
\eta_v^{(k)}(z_e) &= \frac{dz_e}{2\pi i} \int_{w_e\in\gamma_e} \frac{1}{z_e - w_e} I_e^* \xi_{-e}^{(k-1)}(w_e) + \sum_{e'\in E_v} \int_{w_{e'}\in\gamma_{e'}} \mathbf{K}_v(z_e, w_{e'}) I_{e'}^* \xi_{-e'}(w_{e'}). \\
&= \frac{dz_e}{2\pi i} \int_{w_e\in\gamma_e} \frac{1}{z_e - w_e} I_e^* \xi_{-e}^{(k-1)}(w_e) + \xi_e^{(k)}(z_e) \\
&= \left(I_e^* \xi_{-e}^{(k-1)} + \xi_e^{(k)} \right)(z_e),
\end{aligned} \tag{3.2.11}$$

the equality follows from Cauchy's integral formula, see (3.2.2) for details. Note that $\eta_v^{(k)}$ admits continuous extension to the boundary γ_e. By this expression and property (3.2.4), we conclude that $\int_{z\in\gamma_e} \eta_v^{(k)}(z) = 0$ and hence the solution η_v is A-normalized.

Step 2. We show that the differentials $\{\Omega_{v,\underline{s}}\}$ have zero jumps among the seams $\underline{\gamma} = \{\gamma_e\}_{e\in E_C}$. It is sufficient to prove

$$\left(\Omega_{v(e)} - I_e^* \Omega_{v(-e)}\right)|_{\gamma_e}(z_e) = -\sum_{k=1}^{\infty} \left(\eta_{v(e)}^{(k)} - I_e^* \eta_{v(-e)}^{(k)}\right)|_{\gamma_e}(z_e). \tag{3.2.12}$$

First we note that by the opposite residue condition ($r_e = -r_{-e}$) the singular parts of $\Omega_{v(e)}$ and $I_e^* \Omega_{v(-e)}$ cancels, therefore we have $\left(\Omega_{v(e)} - I_e^* \Omega_{v(-e)}\right)|_{\gamma_e}(z_e) = \left(\xi_e^{(0)} - I_e^* \xi_{-e}^{(0)}|_{\gamma_e}\right)(z_e)$.

For $k \geq 1$, by (3.2.11) the jumps along the identified seams for each terms $\eta_v^{(k)}$

can be analyzed.

$$\begin{aligned}\left(\eta_v^{(k)} - I_e^*\eta_{v(-e)}^{(k)}\right)(z_e) &= \left(I_e^*\xi_{-e}^{(k-1)} + \xi_e^{(k)} - \xi_e^{(k-1)} - I_e^*\xi_{-e}^{(k)}\right)(z_e)\\ &= \left(\xi_e^{(k)} - I_e^*\xi_{-e}^{(k)}\right)(z_e) - \left(\xi_e^{(k-1)} - I_e^*\xi_{-e}^{(k-1)}\right)(z_e).\end{aligned}$$

Therefore $\sum_{k=1}^{\infty}\left(\eta_{v(e)}^{(k)} - I_e^*\eta_{v(-e)}^{(k)}\right)|_{\gamma_e}(z_e) = -\left(\xi_e^{(0)} - I_e^*\xi_{-e}^{(0)}|_{\gamma_e}\right)(z_e)$, and we have shown (3.2.12).

Step 3. We want to prove the L^2-bound (3.2.10) for the solution. We take the L^∞ norm of $\tilde{\eta}_v^{(k)}(z_e) := \eta_v^{(k)}(z_e)/dz_e$ on the seams. By (3.2.11) we have

$$|\tilde{\eta}_v^{(k)}|_{\underline{s}} := \max_{|z_e|=|\sqrt{s_e}|} |\tilde{\eta}_v^{(k)}(z_e)| \leq |I_e^*\tilde{\xi}_{-e}^{(k-1)}(z_e)|_{\underline{s}} + |\tilde{\xi}_e^{(k)}|_{\underline{s}}. \tag{3.2.13}$$

By Lemma 3.2.2, we know that for any $k \geq 1$, there exists a constant M' such that $|\tilde{\eta}_v^{(k)}|_{\underline{s}} < (M'|\underline{s}|)^{k-1}|\tilde{\xi}^{(0)}|_{\underline{s}}$. By the summing the series, we have $|\tilde{\eta}_v|_{\underline{s}} < M''|\tilde{\xi}^{(0)}|_{\underline{s}}$ for some constant M''.

Take any base point $z_0 \in C_v$, define $\pi_v(z) := \int_{z_0}^{z}\eta_v$. Then since $d\pi_v = \eta_v$, by Stokes theorem, we have

$$||\eta_v||_{L^2}^2 = \frac{i}{2}\int_{\hat{C}_v}\eta_v \wedge \overline{\eta_v} = \sum_{e\in E_v}\int_{\gamma_e}\overline{\pi}_v\eta_v < M''|\tilde{\xi}^{(0)}|_{\underline{s}}\sum_{e\in E_v}\int_{z_e\gamma_e}|\overline{\pi}_v(z_e)|dz_e.$$

Since η_v is bounded on γ_e, by taking $z_0 \in \gamma_e$ the length of arc from z_0 to $z_e \in \gamma_e$ is at most $2\pi\sqrt{|\underline{s}|}$. Therefore we can bound $|\overline{\pi}_v|_{\underline{s}} = |\pi_v|_{\underline{s}}$ by $2\pi\sqrt{|\underline{s}|}|\tilde{\eta}_v|_{\underline{s}} = 2\pi M''\sqrt{|\underline{s}|}|\tilde{\xi}^{(0)}|_{\underline{s}}$. At last we have

$$||\eta_v||_{L^2}^2 < |\underline{s}| \cdot (2\pi M''|\tilde{\xi}^{(0)}|_{\underline{s}})^2 \cdot \#E_v.$$

Thus by letting $M := 2\pi M''\sqrt{\max_v \#E_v}$, since $|\underline{\tilde{\xi}}^{(0)}|_{\underline{s}} \leq |\underline{\tilde{\xi}}^{(0)}|_{\underline{1}}\sqrt{|\underline{s}|}^{\operatorname{ord}\underline{\tilde{\xi}}^{(0)}}$ we have the required L^2-bound (3.2.10) for $||\eta_v||_{L^2}$.

Note that for holomorphic differentials, convergence in L_2 sense implies uniform convergence on compact sets. Therefore we conclude that $\Omega_v = \lim_{\underline{s}\to 0}\Omega_{\underline{s}}|_{C_v}$ uniformly on compact sets of $C_v \setminus \cup_{e\in E_v} q_e$.

Lastly, the holomorphicity of $\Omega_{v,\underline{s}}$ for $\underline{s} > 0$ follows from the holomorphicity of Ω_v away from the nodes and the holomorphicity of η_v on $\tilde{C}_v$. Recall the Cauchy kernel is holomorphic in $\tilde{C}_v$ except at q_0, and we've verified that η_v does not have a pole at q_0. □

In fact, by construction (3.2.9) we can compute the $\underline{s}$-expansion of each summand $\eta_v^{(k)}$ explicitly, and thus the $\underline{s}$-expansion of the differential $\Omega_{\underline{s}}$. For future applications and comparisons to earlier works, we compute the first term in the $\underline{s}$-expansion for each summand.

Proposition 3.2.4. *Let $l^k = (e_1, \dots, e_k) \in L_v^k$ be a path of length k in Γ_C starting from a given vertex $v = v(e_1)$. Denote $s(l^k) = \prod_{i=1}^k s_{e_i}$, and $\beta(l^k) = \prod_{j=1}^{k-1} \beta_{-e_j, e_{j+1}}$. Then the expansion of $\eta_v^{(k)}$ is given by*

$$\eta_v^{(k)}(z) = (-1)^k \sum_{l^k \in L_v^k} s(l^k) \cdot \omega_v(z, q_{e_1}) \beta(l^k) \tilde{\xi}_{-e_k} + O(|\underline{s}|^{k+1}), \tag{3.2.14}$$

where $z \in \widehat{C}_v$, $\beta_{e,e'}$ is defined in (3.2.1), and $\tilde{\xi}_e$ is defined in (3.2.5).

Proof. Fix a vertex v in Γ_C. First for a fixed $e \in E_v$, we show the following expansion for $\xi_e^{(k)}$ for $k > 0$:

$$\xi_e^{(k)}(z_e) = (-1)^k \sum_{l^k \in L_v^k} s(l^k) \cdot \boldsymbol{\omega}_v(z_e, q_{e_1}) \beta(l^k) \tilde{\xi}_{-e_k} + O(|\underline{s}|^{k+1}). \tag{3.2.15}$$

This is derived by induction. For $k = 1$, we have $L_v^1 = E_v$, and $l^1 = (e_1)$ where $e_1 \in E_v$. We compute

$$\begin{aligned} \xi_e^{(1)}(z_e) &= \sum_{e_1 \in E_v} \int_{w_{e_1} \in \gamma_{e_1}} \mathbf{K}_v(z_e, w_{e_1}) I_{e_1}^* \xi_{-e_1}^{(0)}(w_{e_1}) \\ &= -\sum_{e_1 \in E_v} \int_{w_{e_1} \in \gamma_{e_1}} \mathbf{K}_v(z_e, w_{e_1}) \frac{s_{e_1}}{w_{e_1}^2} \cdot \tilde{\xi}_{-e_1} dw_{e_1} + O(s_{e_1}^2) \\ &= -\sum_{e_1 \in E_v} s_{e_1} \boldsymbol{\omega}_v(z_e, q_{e_1}) \tilde{\xi}_{-e_1} + O(|\underline{s}|^2), \end{aligned} \tag{3.2.16}$$

where the last equality follows from Cauchy's integral formula.

For the general case, by applying the inductive assumption (3.2.15) to $I_{e_1}^* \xi_{-e_1}^{(k-1)}$, we have

$$I_{e_1}^* \xi_{-e_1}^{(k-1)} = (-1)^{k-1} I_{e_1}^* \left(\boldsymbol{\omega}_{v(-e_1)}(w_{-e_1}, q_{e_2}) \right) \cdot \sum_{l^{k-1} \in L_{v(-e_1)}^{k-1}} s(l^{k-1}) \beta(l^{k-1}) \tilde{\xi}_{-e_k} + O(|\underline{s}|^k).$$

Therefore it suffices to prove that for any $e_1 \in E_v$ we have:

$$\int_{w_{e_1} \in \gamma_{e_1}} \mathbf{K}_v(z_e, w_{e_1}) I_{e_1}^* \left(\boldsymbol{\omega}_{v(-e_1)}(w_{-e_1}, q_{e_2}) \right) = -s_{e_1} \boldsymbol{\omega}_v(z_e, q_{e_1}) \beta_{-e_1, e_2} + O(|\underline{s}|^2). \tag{3.2.17}$$

This is due to $I_{e_1}^*\left(\boldsymbol{\omega}_{v(-e_1)}(w_{-e_1}, q_{e_2})\right) = I_{e_1}^*\left((\beta_{-e_1,e_2} + o(w_{-e_1}))dw_{-e_1}\right) = -\frac{s_{e_1}\beta_{-e_1,e_2}dw_{e_1}}{w_{e_1}^2} + O(|\underline{s}|^2)$ and Cauchy's integral formula. We conclude the induction for (3.2.15).

The expansion (3.2.14) for $\eta_v^{(k)}(z)$ is obtained by integrating $\xi_{e_1}^{(k-1)}(z_{e_1})$ against $K_v(z, z_{e_1})$, and the computation is exactly the same as (3.2.17):

$$\int_{w_{e_1}\in\gamma_{e_1}} K_v(z, w_{e_1}) I_{e_1}^*\left(\boldsymbol{\omega}_{v(-e_1)}(w_{-e_1}, q_{e_2})\right) = -s_{e_1}\omega_v(z, q_{e_1})\beta_{-e_1,e_2} + O(|\underline{s}|^2),$$

where $z \in \widehat{C}_v$. The proof is thus completed.

□

Remark 3.2.5. It is important to point out that the expansion (3.2.14) is *not* the $\underline{s}$-expansion of the solution η_v, while the latter is also computable by expanding the error term in (3.2.16) using the higher order coefficients β_{ij}^v of ω_v. The explicit formula for the case where Γ_C contains only one edge is given by [Yam80], and will be recomputed (up to the second order) in Section 4.3.

However, as we highlighted by the proposition, it is often more useful and practical to consider η_v as the series $\sum_{k=1}^{\infty}\eta_v^{(k)}$, given the bound (3.2.13) and the recursive construction (3.2.9). In most cases it is already useful to know the first non-constant term of $\Omega_{v,\underline{s}}$, which the proposition suffices to give:

$$\eta_v^{(1)}(z) = -\sum_{e\in E_v} s_e\omega_v(z, q_e)\tilde{\xi}_{-e} + O(s_e^2).$$

3.3 Jump Problem for Higher Order Plumbing

In this section we construct the solution to the jump problem with the initial data arising from the jumps of an abelian differential that has higher order zeroes and poles at the nodes of the limit curve. Following the terminology in [Gen15] and [BCGGM18], we call the procedure of smoothing such a differential *higher order plumbing*. We obtain an alternative proof of the sufficiency part of the main theorem in [BCGGM18]. Moreover, our approach gives more information than the two constructions given in that paper. A brief review of definitions and results in [BCGGM18] has been given in Section 2.3.

The proof of sufficiency of this result requires a construction of a family of abelian differentials in the smooth locus of the strata that degenerates to the limit differential (C, Ω), given the compatible data (Ξ, l). In [BCGGM18], the authors give two proofs to the sufficiency by: 1) constructing a one complex parameter family using plumbing; 2) constructing a one real parameter family via a flat geometry argument.

We now give a third argument via the jump problem approach. Moreover, the number of parameters over $\mathbb{C}$ in our degenerating family is equal to the number of levels in Γ_C minus 1. Similar to the plumbing argument used in [BCGGM18], we will also use a modification differential to match up the residues. Furthermore, the original argument in [BCGGM18] on the operation merging the zeroes will also be applied here to embed the family into the stratum.

Take a plumbing family $\{C_{\underline{s}}\}$ as in Definition 3.1.1 such that $C = C_0$, and $\underline{s} = \{s_{|e|}\}_{|e| \in |E|_C}$ are the plumbing parameters. Denote the restriction of Ξ on the irreducible component C_v by Ξ_v. Let N_l be the number of levels in Γ_C. Without loss of generality, we assume the range of the level function l to be $\{0, -1, \ldots, 1 - N_l\}$.

Assume $j = l_{v(-e)}$, recall that $E_v^j = \{e \in E_v : l_{v(-e)} = j\}$ as defined in condition (5). To glue the twisted differentials Ξ_v and $\Xi_{v(-e)}$, we need to add a *modification differential* $\phi_{v,j}$ to Ξ_v in order to match the residue (denoted by r_{-e}) of $\Xi_{v(-e)}$. The modification differential $\phi_{v,j}$ is chosen to be any differential which has simple pole at q_e with residue $r_e = -r_{-e}$, where $e \in E_v^j$. The global residue condition ensures that the sum of the residues of $\phi_{v,j}$ is zero. The existence of $\phi_{v,j}$ is due to the classical Mittag-Leffler problem.

For $e \in E_v^j$, assume Ξ_v has a zero of order k_e at the node q_e, then by Condition (2), $\Xi_{v(-e)}$ has a pole of order $k_e + 2$ at the node q_{-e}. In order to apply the jump problem to obtain a global differential, the following conditions need to be imposed on the plumbing parameters $\underline{s}$:

(i) For any $e, e' \in E_v^j$, we have $s_e^{k_e+1} = s_{e'}^{k_{e'}+1}$;

(ii) For any two vertices v_0, v_1 at different levels (namely $l_{v_0} \neq l_{v_1}$), for any two paths $\{e_i\}_{i \in I}$ and $\{\tilde{e}_j\}_{j \in J}$ connecting v_0, v_1 with $l_{v(e_i)} > l_{v(-e_i)}$ ($\forall i \in I$) and the same for $\{\tilde{e}_j\}$, we have $\prod_{i \in I} s_{e_i}^{k_{e_i}+1} = \prod_{j \in J} s_{\tilde{e}_j}^{k_{\tilde{e}_j}+1} =: s(v_0, v_1)$;

(iii) If $l_{v_0} = l_{v_1}$, we require that $s(v_0, v_1) = 1$.

It is important to remark that for such a tuple of plumbing parameters one can deduce that $s(v_0, v_1)$ depends only on the levels of v_0, v_1, namely, $s(v_0, v_1) = s(v_0', v_1')$ as long as $l_{v_0} = l_{v_0'}$ and $l_{v_1} = l_{v_1'}$. We can thus choose one parameter for each level drop:

Definition 3.3.1. Let $t_{i,j} := s(v_0, v_1)$ where v_0, v_1 are two vertices at level i, j respectively. The tuple $\underline{t} := \{t_{-1}, \ldots, t_{1-N_l}\}$ where $t_i := t_{0,i}/t_{0,i+1}$ are called the *scaling parameters*.

Note that $t_{i,j} = \prod_{k=i}^{j} t_k$. The theorem below gives a degenerating family of abelian differentials parametrized by $\underline{t}$ with central fiber the differential (C, Ω) in the boundary of the IVC.

Theorem 3.3.2. *Let $(C, \Omega, p_1, \ldots, p_n)$ be a stable pointed differential in a given stratum $\Omega\mathcal{M}_{g,n}(\mu)$. Given the triple (C, Ξ, l) where Ξ is a twisted differential of type μ on C and l is a full level function on Γ_C, such that Ξ is compatible with Ω and l, there exists a degenerating family of Abelian differentials $(C_{\underline{t}}, \Xi_{\underline{t}}) \subset \Omega\mathcal{M}_{g,n}(\mu)$ such that $\lim_{\underline{t}\to 0}(C_{\underline{t}}, \Xi_{\underline{t}}) = (C, \Omega)$, where $\underline{t}$ are the scaling parameters.*

Proof. The proof is completed in three steps. Firstly we construct via the jump problem method a degenerating family of abelian differentials $(C_{\underline{t}}, \widehat{\Xi}_{\underline{t}})$ in $\Omega\mathcal{M}_{g,n}$. Then we show that the family lies sufficiently "near" the stratum, that is, we show that the solution to the jump problem is uniformly controlled by some positive power of $|\underline{t}| := \max_{1\le i\le N_l - 1} |t_i|$. Lastly we apply [BCGGM18, Lemma 4.7] to merge the zeroes of $\widehat{\Xi}_{\underline{t}}$ to obtain a family contained in the stratum.

For the jump problem construction, we only need to construct the correct initial data $\{\xi_e^{(0)}\}$, the rest of the construction is given by (3.2.3) and (3.2.8).

Assume that the vertex v lies on the i-th level. We define

$$\widehat{\Xi}_v := \Xi_v + \sum_{j<i} t_{i,j}\phi_{v,j},$$

where $\phi_{v,j}$ is the modification differential we defined earlier.

We now apply the jump problem construction to glue the differentials at the opposite sides of each node $q_{|e|}$. Assume $v(e)$ and $v(-e)$ are on the levels i and j respectively. We glue $t_{0,i} \cdot \widehat{\Xi}_v$ and $t_{0,j} \cdot \widehat{\Xi}_{v(-e)}$ from the opposite sides of the node $q_{|e|}$. Namely, let

$$\begin{aligned} \xi_e^{(0)}(z_e) &:= t_{0,i} \cdot \left(\widehat{\Xi}_v(z_e) - t_{i,j} I_e^* P(\widehat{\Xi}_{v(-e)})(z_e)\right); \\ \xi_{-e}^{(0)}(z_{-e}) &:= t_{0,j} \cdot \operatorname{hol}(\widehat{\Xi}_{v(-e)})(z_{-e}), \end{aligned}$$

where $P(\cdot)$ denotes the principal part of a differential, and $\operatorname{hol}(\cdot)$ denotes the holomophic part. Conditions $(i) \sim (iii)$ ensures that $t_{0,i}t_{i,j} = t_{0,j}$.

Note that the initial data $t_{0,i}(\widehat{\Xi}_v - I_e^*\widehat{\Xi}_{v(-e)})(z_e)$ is equal to $(\xi_e^{(0)} - I_e^*\xi_{-e}^{(0)})(z_e)$. In order to apply (3.2.3) and (3.2.8) to construct the A-normalized solution to the jump problem with this initial data, we need to show that $\xi_e^{(0)}(z_e)$ is holomorphic in z_e. It is immediate because the pair of differentials $\widehat{\Xi}_v$ and $t_{i,j}\widehat{\Xi}_{v(-e)}$ have opposite residues at the node $q_{|e|}$ and the pull-back of the principal part by I_e is holomorphic.

We recall here the construction of the A-normalized solution in (3.2.3) and (3.2.9): For $k \geq 1$, we define

$$\text{for } z_e \in U_e : \qquad \xi_e^{(k)}(z_e) := \sum_{e' \in E_v} \int_{w_{e'} \in \gamma_{e'}} \mathbf{K}_v(z_e, w_{e'}) \cdot I_{e'}^* \xi_{-e'}^{(k-1)}(w_{e'});$$

$$\text{for } z \in \widehat{C}_v : \qquad \eta_v^{(k)}(z) := \sum_{e \in E_v} \int_{z_e \in \gamma_e} K_v(z, z_e) \cdot I_e^* \xi_{-e}^{(k-1)}(z_e).$$

By Theorem 3.2.3, $\eta_v := \sum_{k \geq 1} \eta_v^{(k)}(z)$ is the A-normalized solution to the jump problem of higher order zeroes and poles.

Similar to the proof of Theorem 3.2.3, we need to show that η_v is convergent by giving a L^2-bound for the solution η_v. We can repeat the proof in Lemma 3.2.2 and Theorem 3.2.3 to get (3.2.10), which we recall as

$$||\eta_v||_{L^2} < \sqrt{|\underline{s}|}^{1+\operatorname{ord} \underline{\tilde{\xi}}^{(0)}} M |\underline{\tilde{\xi}}^{(0)}|_{\underline{1}}$$

for some constant M. We have shown above that $\xi_e^{(0)}(z_e)$ is holomorphic for every e, therefore $\operatorname{ord} \underline{\tilde{\xi}}^{(0)} = \min_e \operatorname{ord}_{q_e} \tilde{\xi}_e^{(0)} \geq 0$. The only thing left to show here is that $|\underline{\tilde{\xi}}^{(0)}|_{\underline{1}}$ is bounded by some power of $\underline{t}$, in other words, the power of $\underline{t}$ in $\xi_{\pm e}^{(0)}$ is non-negative for any e.

Note that the power of $\underline{t}$ in $\xi_{-e}^{(0)}$ is automatically non-negative, we only need to show the same holds for $\xi_e^{(0)}$. Note that when pulling back the principal part of $\widehat{\Xi}_{v(-e)}$ through I_e, its lowest order term $z_{-e}^{-k_e-2} dz_{-e}$ contributes a factor of $s_e^{-k_e-1}$, which is seen to be equal to $t_{i,j}^{-1}$ by condition (ii). Since all other terms in the principal part contribute factors of lower powers of s_e, the power of $\underline{t}$ in $\xi_e^{(0)}$ must be non-negative. We can thus apply the same argument as in the proof of Lemma 3.2.2 and Theorem 3.2.3 and achieve an L^2-bound for η_v.

Let $\widehat{\widehat{\Xi}}_{v,\underline{t}} := t_{0,i} \widehat{\Xi}_v + \eta_v$ for any v at level i, then by the argument in the proof of Theorem 3.2.3, we have that $\{\widehat{\widehat{\Xi}}_{v,\underline{t}}\}_v$ glues to a global differential $\widehat{\widehat{\Xi}}_{\underline{t}}$ on $C_{\underline{t}}$ such that $\lim_{\underline{t} \to 0}(C_{\underline{t}}, \widehat{\widehat{\Xi}}_{\underline{t}}) = (C, \Omega)$.

Note that by adding the modification differential $\phi_{v,j}$ and the solution differential η_v to Ξ_v, the zeroes of multiplicity m_i of Ξ_v at $p_i \in C_v$ are broken into m_i simple zeroes in a small neighborhood U_i of p_i. The radius of the neighborhood is controlled by the norm of the added differentials. The modification differentials $\phi_{v,j}$ are multiples of $t_{i,j}$, and the argument above gives the L^2-bound on η_v. We can thus merge the zeroes of $\widehat{\widehat{\Xi}}_{\underline{t}}$ using the arguments in [BCGGM18, Lemma 4.7] to get the wanted degenerating family $(C, \Xi_{\underline{t}})$ with $\operatorname{ord}_{p_i} \Xi_{\underline{t}} = m_i$, and $\lim_{\underline{t} \to 0}(C_{\underline{t}}, \Xi_{\underline{t}}) = (C, \Omega)$.

$\square$

Although the differential $\Xi_{\underline{t}}$ depends on the choice of the modification differentials $\{\phi_v^L\}$, the existence of such a degenerating family does not rely on the choice of $\{\phi_v^L\}$. Theorem 3.3.2 in particular implies:

Corollary 3.3.3. *[BCGGM18, Sufficiency Part of Theorem 1.3] A stable pointed differential $(C, \Omega, p_1, \ldots, p_n)$ lies in the boundary of $\mathbb{P}\Omega\overline{\mathcal{M}}_{g,n}^{inc}(\mu)$ if there exist a twisted differential Ξ of type μ and a full level function l such that Ξ is compatible with Ω and l.*

Chapter 4

Degeneration of Period Matrices

4.1 General Periods

Using the construction (3.2.8) and expansion (3.2.14) of the stable differential Ω, we can compute the variational formula of its periods.

Notation 4.1.1. For a stable curve C and its dual graph Γ_C, define the map $p : H_1(C, \mathbb{Z}) \to H_1(\Gamma_C, \mathbb{Z})$ as follows: for the class of a homological (oriented) 1-cycle $[\gamma]$ on C, $p([\gamma])$ is the class of the oriented loop in the dual graph that contains the vertices corresponding to the components that γ passes, and the edges corresponding to the nodes contained in γ. The orientation of $p([\gamma])$ is inherited from the orientation of γ. It is easy to see that the map is surjective, but not injective unless all components have genus zero. Moreover, if γ is completely contained in some component C_v, then $p([\gamma]) = 0$.

Let α be any closed oriented path on the stable curve C, such that $p([\alpha]) \neq 0$ (the zero case is trivial in our discussion below). For any small enough $\underline{s}$, there exists a small perturbation $\alpha_{\underline{s}}$ of α such that the restriction of $\alpha_{\underline{s}}$ on $\widehat{C}_{\underline{s}}$ glues to be a path on $C_{\underline{s}}$. This can be seen by requiring 1) $\alpha_{\underline{s}} \cap \gamma_e = I_e^{-1}(\alpha_{\underline{s}} \cap \gamma_{-e})$ for any seam γ_e that α passes; 2) $\alpha_{\underline{s}}$ does not totally contain any seam γ_e. By an abuse of notation, the path on $C_{\underline{s}}$ after the gluing is also denoted by $\alpha_{\underline{s}}$.

The following theorem computes the leading terms in the variational formula of $\int_{\alpha_{\underline{s}}} \Omega_{\underline{s}}$. To this end, recall that $U_e = \{|z_e| < \sqrt{|s_e|}\}$ and denote $W_e = \{|z_e| < |s_e|\}$ and $V_e = \{|z_e| < 1\}$.

Theorem 4.1.2. *For any stable differential Ω on C with residue r_e at the node q_e, let α be any closed oriented path on C such that $p([\alpha]) \neq 0$ and $\{e_0, \ldots, e_{N-1}\}$ be the* *collection of oriented edges that $p([\alpha])$ passes through (with possible repetition), such that $v(-e_{i-1}) = v(e_i)$, and let $e_N = e_0$. Then we have*

$$\int_{\alpha_{\underline{s}}} \Omega_{\underline{s}} = \sum_{i=1}^{N} (r_{e_i} \ln |s_{e_i}| + c_i + l_i) + O(|\underline{s}|^2), \tag{4.1.1}$$

where c_i and l_i are the constant and linear terms in $\underline{s}$ respectively, explicitly given as

$$c_i = \lim_{|\underline{s}|\to 0} \left(\int_{z^{-1}_{-e_{i-1}}(\sqrt{|s_{e_{i-1}}|})}^{z^{-1}_{e_i}(\sqrt{|s_{e_i}|})} \Omega_v - \frac{1}{2}(r_{e_{i-1}} \ln |s_{e_{i-1}}| + r_{e_i} \ln |s_{e_i}|) \right), \tag{4.1.2}$$

$$l_i := - \sum_{e \in E_{v(e_i)}} s_e \tilde{\xi}_{-e} \cdot \sigma_e, \tag{4.1.3}$$

where $\tilde{\xi}_e$ is defined in (3.2.5), and

$$\sigma_e := \begin{cases} \lim_{|\underline{s}|\to 0} \left(\int\limits_{q_{-e_{i-1}}}^{z^{-1}_{e_i}(s_{e_i})} \omega_{v(e_i)}(z_{e_i}, q_{e_i}) + \frac{1}{s_{e_i}} \right) & \text{if } e = e_i; \\ \lim_{|\underline{s}|\to 0} \left(\int\limits_{z^{-1}_{-e_{i-1}}(s_{e_i})}^{q_{e_i}} \omega_{v(e_i)}(z_{-e_{i-1}}, q_{-e_{i-1}}) - \frac{1}{s_{e_i}} \right) & \text{if } e = -e_{i-1}; \\ \int_{q_{-e_{i-1}}}^{q_{e_i}} \omega_{v(e_i)}(z, q_e) & \text{otherwise.} \end{cases} \tag{4.1.4}$$

Remark 4.1.3. Prior to the proof of the theorem we have two remarks. Firstly, the period integral in (4.1.1) depends not only on $p[\alpha]$, but also on the class of the actual path α. The integration over $\alpha \cap \widehat{C}_v$ gives precisely the constant term (4.1.2). Secondly, note that the limits of the integrals in (4.1.4) are singular because the integrants have a double pole on the nodes. However the singular parts are canceled by $\pm\frac{1}{s_{e_i}}$, so the limits are indeed well-defined. Computations leading to both remarks are contain in the proofs of the following lemma and the theorem.

To prove the theorem, it suffices to compute the integral on each component $C_{v(e_i)}, i = 1 \dots N$ that α passes through. To simplify notation, throughout the proof below we consider α only passing each component once, while the proof also holds for the general case. Let us denote the intersection of $\alpha_{\underline{s}}$ with $\partial U_{e_i}, \partial V_{e_i}, \partial U_{-e_{i-1}}, \partial V_{-e_{i-1}}$ respectively by $u_{e_i}, v_{e_i}, u_{-e_{i-1}}$ and $v_{-e_{i-1}}$. Then $\alpha_{\underline{s}}|_{C_{v(e_i)}}$ breaks into three pieces bounded by the four points:

1. $\alpha_{\underline{s}}|_{V_{-e_{i-1}} \setminus U_{-e_{i-1}}}$ connecting $u_{-e_{i-1}}$ and $v_{-e_{i-1}}$;

2. $\alpha_{\underline{s}}|_{\widehat{C}_{v(e_i)}\setminus V_{e_i}\cup V_{-e_{i-1}}}$ connecting v_{e_i} and $v_{-e_{i-1}}$;

3. $\alpha_{\underline{s}}|_{V_{e_i}\setminus U_{e_i}}$ connecting v_{e_i} and u_{e_i};

For convenience, by a composition of a rotation we can assume that $u_{e_i} = z_{e_i}^{-1}(\sqrt{s_{e_i}})$, $v_{e_i} = z_{e_i}^{-1}(1)$ and similarly $u_{-e_{i-1}} = z_{-e_{i-1}}^{-1}(\sqrt{s_{-e_{i-1}}})$, $v_{-e_{i-1}} = z_{-e_{i-1}}^{-1}(1)$. Therefore in the lemma and the proofs below, integrations in the local chart z_{e_i} from u_{e_i} to v_{e_i} will be written as from $\sqrt{s_{e_i}}$ to 1, for any $i = 0, \ldots, N-1$. We also remark that this assumption does not change the statement of the theorem.

The following lemma simplifies the computation:

Lemma 4.1.4. *Given an edge e, let $\Omega_{v,\underline{s}}$ and $\xi_e^{(k)}$ be defined as before. We have the following equality:*

$$\begin{aligned}\int_1^{\sqrt{s_e}} \Omega_{v(e),\underline{s}}(z_e) + \int_{\sqrt{s_e}}^1 \Omega_{v(-e),\underline{s}}(z_{-e}) =& r_e \ln|s_e| + \sum_{k=0}^{\infty} \int_1^{s_e} \xi_e^{(k)}(z_e) \\ &+ \sum_{k=0}^{\infty} \int_{s_e}^1 \xi_{-e}^{(k)}(z_{-e})\end{aligned} \tag{4.1.5}$$

Proof of Lemma 4.1.4. We recall that $\Omega_{v,\underline{s}} = \Omega_{v(e_i)} + \sum_k \eta_v^{(k)}$, and as we are concerned with the regular part of the period, locally in the annuli $V_e \setminus W_e$, we have the following expression $\Omega_{v,\underline{s}}(z_e) = r_e \frac{dz_e}{z_e} + \xi_e^{(0)}(z_e) + \sum_{k=1}^{\infty} \eta_v^{(k)}$. The logarithmic term in (4.1.5) is given by

$$\int_1^{\sqrt{s_e}} r_e \frac{dz_e}{z_e} = \frac{1}{2} r_e \ln|s_e|$$

and $r_e = -r_{-e}$. What is left to show is

$$\int_1^{\sqrt{s_e}} \xi_e^{(0)} + \int_{\sqrt{s_e}}^1 \xi_{-e}^{(0)} + \sum_{k=1}^{\infty} \int_1^{\sqrt{s_e}} \eta_v^{(k)} + \sum_{k=1}^{\infty} \int_{\sqrt{s_e}}^1 \eta_{v(-e)}^{(k)} = \sum_{k=0}^{\infty} \int_1^{s_e} \xi_e^{(k)} + \sum_{k=0}^{\infty} \int_{s_e}^1 \xi_{-e}^{(k)} \tag{4.1.6}$$

Note that for each $k \geq 0$, we have $\int_{\sqrt{s_e}}^{s_e} \xi_e^{(k)}(z_e) = \int_{\sqrt{s_e}}^1 I_e^* \xi_e^{(k)}(z_{-e})$ and $\int_{s_e}^{\sqrt{s_e}} \xi_{-e}^{(k)}(z_{-e}) = \int_1^{\sqrt{s_e}} I_e^* \xi_{-e}^{(k)}(z_e)$. This gives for $k \geq 0$:

$$\begin{aligned}\int_1^{\sqrt{s_e}} \xi_e^{(k)}(z_e) + \int_{\sqrt{s_e}}^1 \xi_{-e}^{(k)}(z_{-e}) =& \int_1^{s_e} \xi_e^{(k)}(z_e) + \int_{s_e}^1 \xi_{-e}^{(k)}(z_{-e}) \\ &- \int_{\sqrt{s_e}}^1 I_e^* \xi_e^{(k)}(z_{-e}) - \int_1^{\sqrt{s_e}} I_e^* \xi_{-e}^{(k)}(z_e).\end{aligned} \tag{4.1.7}$$

Grouping the last two terms above with the $(k+1)$ entries in $\sum_{k=1}^{\infty}\int_{1}^{\sqrt{s_e}}\eta_v^{(k)}+\sum_{k=1}^{\infty}\int_{\sqrt{s_e}}^{1}\eta_{v(-e)}^{(k)}$, and applying (3.2.11), we obtain:

$$\begin{aligned}\int_1^{\sqrt{s_e}}(\eta_{v(e)}^{(k+1)}-I_e^*\xi_{-e}^{(k)})(z_e)+\int_{\sqrt{s_e}}^1(\eta_{v(-e)}^{(k+1)}-I_e^*\xi_e^{(k)})(z_{-e})\\ =\int_1^{\sqrt{s_e}}\xi_e^{(k+1)}(z_e)+\int_{\sqrt{s_e}}^1\xi_{-e}^{(k+1)}(z_{-e})\end{aligned}\tag{4.1.8}$$

Summing up both (4.1.7) and (4.1.8) over all $k\geq 0$ and adding the two equalities together, we immediately obtain (4.1.6). The lemma follows. □

Proof of Theorem 4.1.2. Our goal is to compute the leading terms of the variational formula of $\sum_{i=0}^{N-1}\int_{\alpha_{v(e_i)}}\Omega_{v(e_i),\underline{s}}$. To this end, we rearrange the terms and compute the following integrals:

$$\int_{z_{-e_{i-1}}^{-1}(1)}^{z_{e_i}^{-1}(1)}\Omega_{v(e_i),\underline{s}}+\int_1^{\sqrt{s_{e_i}}}\Omega_{v(e_i),\underline{s}}(z_{e_i})+\int_{\sqrt{s_{e_i}}}^1\Omega_{v(-e_i),\underline{s}}(z_{-e_i})\tag{4.1.9}$$

It needs to be pointed out that the first two entries above are integrals inside C_v, while the last entry is in $C_{v(-e_i)}$. To simplify notation, in the rest of the proof we denote $v:=v(e_i)$.

Using the lemma, the last two entries of (4.1.9) are equal to $r_{e_i}\ln|s_{e_i}|+\sum_{k=0}^{\infty}\int_1^{s_{e_i}}\xi_{e_i}^{(k)}(z_{e_i})+\sum_{k=0}^{\infty}\int_{s_{e_i}}^1\xi_{-e_i}^{(k)}(z_{-e_i})$. By definition of $\Omega_{v,\underline{s}}$, the first integral in (4.1.9) is equal to $\int_{z_{-e_{i-1}}^{-1}(1)}^{z_{e_i}^{-1}(1)}\left(\Omega_v+\sum_{k\geq 1}\eta_v^{(k)}\right)$.

Note that by (3.2.14) and (3.2.15), for $k\geq 1$ the integrals of $\xi_{\pm e_i}^{(k)}$ and $\eta_v^{(k)}$ only give terms of order $\geq k$. Also observe that $\int_{v_{-e_{i-1}}}^{v_{e_i}}\Omega_v$ is a constant independent of $\underline{s}$. Thus to compute the remaining part of the constant term we only have to compute the integrals of $\int_1^{s_{e_i}}\xi_{e_i}^{(0)}(z_{e_i})+\int_{s_{e_i}}^1\xi_{-e_i}^{(0)}(z_{-e_i})$.

Since $\xi_{\pm e_i}^{(0)}(z_{\pm e_i})$ is holomorphic in $V_{\pm e_i}$, we have

$$\begin{aligned}\int_1^{s_{e_i}}\xi_{e_i}^{(0)}(z_{e_i})+\int_{s_{e_i}}^1\xi_{-e_i}^{(0)}(z_{-e_i})=\int_1^0\xi_{e_i}^{(0)}(z_{e_i})+\int_0^1\xi_{-e_i}^{(0)}(z_{-e_i})\\ +s_{e_i}\cdot\left(\tilde{\xi}_{e_i}-\tilde{\xi}_{-e_i}\right)+O(|\underline{s}|^2).\end{aligned}\tag{4.1.10}$$

Summing up the constant terms on the RHS over i, we have computed the constant term (4.1.2).

Now we compute the linear term. Note that $\xi_v^{(k)}$ are holomorphic in $\underline{s}$. Again by (3.2.15), we only need to compute the integrals of $\xi_{v(\pm e_i)}^{(1)}$, whose expansion is already given by (3.2.16). Therefore we have the following:

$$\begin{aligned}
\int_1^{s_{e_i}} \xi_{e_i}^{(1)}(z_{e_i}) =& - \sum_{e\in E_v} s_e\tilde{\xi}_{-e}\int_1^{s_{e_i}} \widetilde{\boldsymbol{\omega}}_v(z_{e_i}, q_e)dz_{e_i} + O(|\underline{s}|^2)\\
=& - \sum_{e\in E_v} s_e\tilde{\xi}_{-e}\int_1^{s_{e_i}} \widetilde{\omega}_v(z_{e_i}, q_e)dz_{e_i} + s_{e_i}\tilde{\xi}_{-e_i}\int_1^{s_{e_i}} \frac{dz_{e_i}}{z_{e_i}^2} + O(|\underline{s}|^2)\\
=& s_{e_i}\tilde{\xi}_{-e_i} - \sum_{e\in E_v, e\neq e_i} s_e\tilde{\xi}_{-e}\int_1^0 \widetilde{\omega}_v(z_{e_i}, q_e)dz_{e_i}\\
&- s_{e_i}\tilde{\xi}_{-e_i}\lim_{s_{e_i}\to 0}\left(\int_1^{s_{e_i}} \widetilde{\omega}_v(z_{e_i}, q_{e_i})dz_{e_i} + \frac{1}{s_{e_i}}\right) + O(|\underline{s}|^2).
\end{aligned}$$

The existence of the limit above can be seen by integrating the $1/z_{e_i}^2$ term in the expansion of $\widetilde{\omega}_v(z_{e_i}, q_{e_i})$.

The linear term in $\int_{s_{e_i}}^1 \xi_{-e_i}^{(1)}(z_{-e_i})$ is computed similarly:

$$\begin{aligned}
\int_{s_{e_i}}^1 \xi_{-e_i}^{(1)}(z_{-e_i}) =& - s_{e_i}\tilde{\xi}_{e_i} - \sum_{e\in E_{v(-e_i)}, e\neq -e_i} s_e\tilde{\xi}_{-e}\int_0^1 \widetilde{\omega}_v(z_{-e_i}, q_e)dz_{-e_i}\\
&- s_{e_i}\tilde{\xi}_{e_i}\lim_{s_{e_i}\to 0}\left(\int_{s_{e_i}}^1 \widetilde{\omega}_v(z_{-e_i}, q_{-e_i})dz_{-e_i} - \frac{1}{s_{e_i}}\right) + O(|\underline{s}|^2).
\end{aligned}$$

Note that the linear terms in (4.1.10) have been canceled by the linear terms produced by the singular part of ω_v. Moreover,

$$\int_{v_{-e_{i-1}}}^{v_{e_i}} \eta_v^{(1)}(z) = - \sum_{e\in E_v} s_e\cdot\tilde{\xi}_{-e}\cdot\int_{z_{-e_{i-1}}^{-1}(1)}^{z_{e_i}^{-1}(1)} \omega_v(z, q_e) + O(|\underline{s}|^2).$$

Summing up all the linear terms above, then summing up over i, we have the desired linear term. $\square$

Remark 4.1.5. Note that in the proof of the theorem, the function $h(\underline{s}) := \int_{\alpha_{\underline{s}}}\Omega_{\underline{s}} - \sum_{i=1}^N r_{e_i}\ln|s_{e_i}|$ is computed as

$$\sum_{i=1}^N\left(\sum_{k=0}^\infty\int_1^{s_{e_i}} \xi_{e_i}^{(k)}(z_{e_i}) + \sum_{k=0}^\infty\int_{s_{e_i}}^1 \xi_{-e_i}^{(k)}(z_{-e_i}) + \int_{z_{-e_{i-1}}^{-1}(1)}^{z_{e_i}^{-1}(1)} \Omega_{v(e_i),\underline{s}}\right).$$

The analyticity of $h(\underline{s})$ in $\underline{s}$ follows from the analyticity of each integrand above. The analyticity of $h(\underline{s})$ will be used in our improvement of the result of [Tan91] below. In [GKN17, Lem. 5.5], without computing any terms in $h(\underline{s})$, an estimate of $|h(\underline{s})|$ is derived in the real normalized setup.

Moreover, from the proof one can see that besides the complexity of the computation, there is no obstacle in computing every higher order terms in the expansion of the periods of $\Omega_{\underline{s}}$.

4.2 Period Matrices

Recall that in Section 3.1.3 we have chosen a basis $\{A_{i,\underline{s}}\}_{i=1}^{g}$ for a Lagrangian subspace of $H_1(C_{\underline{s}}, \mathbb{Z})$ along the plumbing family. We required that the first m A-cycles generate the span of the classes of the seams. In order to study degenerations of the period matrix, we now choose $B_{1,\underline{s}}, \dots, B_{g,\underline{s}}$ completing the A-cycles to a symplectic basis of $H_1(C_{\underline{s}}, \mathbb{Z})$. The cycles $B_{1,\underline{s}}, \dots, B_{g,\underline{s}}$ are chosen such that they vary continuously in the family.

One can easily see that for $1 \le k \le m$, $p([B_{k,\underline{0}}]) \neq 0$, while for $m+1 \le k \le g$, the map p annihilates the classes of $B_{k,\underline{0}}$. From now on we write $A_k := A_{k,\underline{0}}, B_k := B_{k,\underline{0}}$. Note that for $1 \le k \le m$, one can also see $B_{k,\underline{s}}$ as constructed from B_k by applying a small perturbation as introduced in the previous section.

By the Riemann bilinear relations, we define the following basis of abelian differential $\{v_k\}_{k=1}^{g}$ in $H^0(C, K_C)$ where $C = C_0$ is a stable curve:

1. For $m+1 \le k \le g$, B_k is contained in $\widetilde{C}_v$ for some v, thus A_k is contained in the same component. Define $v_k(z)$ to be the abelian differential dual to A_k in $H^0(C_v, K_{C_v})$.

2. For $1 \le k \le m$, $p([B_k]) \neq 0$, assume $p([B_k])$ passes the edges $e_0, \dots e_{N-1}$. Define $v_k := \sum_{i=0}^{N-1} \omega_{q_{e_i} - q_{-e_{i-1}}}$, where $\omega_{q_{e_i} - q_{-e_{i-1}}}$ denotes the A-normalized meromorphic differential of the third kind supported on $C_{v(e_i)}$ that has only simple poles at $q_{-e_{i-1}}$ and q_{e_i} with residues -1 and 1 correspondingly.

By applying the jump problem construction, we have a collection of abelian differentials $\{v_{k,\underline{s}}\}_{k=1}^{g}$ for the curve $C_{\underline{s}}$, which is seen to be a normalized basis of $H^0(C_{\underline{s}}, K_{C_{\underline{s}}})$. For every k and $|e|$, we have $\int_{\gamma_{|e|}} v_{k,\underline{s}} = \int_{\gamma_{|e|}} (v_k + \eta_{k,\underline{s}})$ on $C_{\underline{s}}$. Since the solution $\eta_{k,\underline{s}}$ to the jump problem with initial jumps of v_k is A-normalized, this is equal to the integral $\int_{\gamma_e} v_k$ on $C_{v(e)}$. Therefore by the residue theorem, we have $\int_{A_{j,\underline{s}}} v_{k,\underline{s}} = 2\pi i \cdot \delta_{jk}$. This shows that $\{v_{k,\underline{s}}\}_{k=1}^{g}$ is a normalized basis of $H^0(C_{\underline{s}}, K_{C_{\underline{s}}})$. The period matrix of $C_{\underline{s}}$ is hence defined to be $\{\tau_{h,k}(\underline{s})\}_{g\times g}$ where $\tau_{h,k}(\underline{s}) := \int_{B_{h,\underline{s}}} v_{k,\underline{s}}$.

We can apply Theorem 4.1.2 to compute the leading terms in the variational formula of $\tau_{h,k}(\underline{s})$.

Corollary 4.2.1. *For every* $|e|$ *and* k*, denote* $N_{|e|,k} := \gamma_{|e|} \cdot B_{k,\underline{s}}$ *the intersection product. For any fixed* h, k*, the expansion of* $\tau_{h,k}(\underline{s})$ *is given by*

$$\begin{aligned}\tau_{h,k}(\underline{s}) = & \sum_{|e| \in |E|_C} (N_{|e|,h} \cdot N_{|e|,k}) \cdot \ln |s_e| \\ & + \lim_{|\underline{s}| \to 0} \sum_{i=1}^{N} \left(\int_{z_{-e_{i-1}}^{-1}(\sqrt{|s_{e_{i-1}}|})}^{z_{e_i}^{-1}(\sqrt{|s_{e_i}|})} v_k - N_{|e_i|,h} N_{|e_i|,k} \ln |s_{e_i}| \right) \\ & - \sum_{e \in E_C} s_e \left(\mathrm{hol}(\tilde{v}_k)(q_e) \, \mathrm{hol}(\tilde{v}_h)(q_{-e}) \right) + O(|\underline{s}|^2),\end{aligned} \tag{4.2.1}$$

where $\{e_i\}_{i=0}^{N-1}$ *is the set of oriented edges* $p([B_h])$ *passes through, and* $\mathrm{hol}(\tilde{v}_k)$ *denotes the regular part of the Laurent expansion of the function of* v_k *near the nodes of the components where* v_k *is not identically zero. Furthermore, under our choice of the symplectic basis,* $N_{|e|,h} \cdot N_{|e|,k}$ *is equal to* 1 *if* $h = k$ *and the node* $q_{|e|}$ *lies on* B_h *and equals* 0 *otherwise.*

Remark 4.2.2. (1) For the purpose of defining the intersection product, we assign an random orientation to $\gamma_{|e|}$. We further remark that there is no canonical way to orient $\gamma_{|e|}$, and the assigned orientation does not affect the statement and the proof.

(2) The main result in [Tan91] is that $h(\underline{s}) := \tau_{h,k}(\underline{s}) - \sum_{|e| \in |E|_C} (N_{|e|,h} \cdot N_{|e|,k}) \cdot \ln |s_e|$ is holomorphic in $\underline{s}$. We can see that only the logarithmic term was computed. By Remark 4.1.5, our corollary in particular reproves his result, and we express more terms in the expansion.

(3) We want to point out that Taniguchi does not require the classes of $\gamma_{|e|}$ to be part of the symplectic basis, therefore $N_{|e|,h} \cdot N_{|e|,k}$ may be any integer. Since the A, B-cycles generate $H_1(C_{\underline{s}}, \mathbb{Z})$, the general case follows by linearity.

Proof. We first compute the logarithmic term. Note that the intersection product is independent of $\underline{s}$. When e does not lie on $p([B_{h,s}])$, we have $N_{|e|,h} = 0$, otherwise $N_{|e|,h} = \pm 1$ and the sign depends on the orientation of $[\gamma_{|e|}]$ compared to that of the corresponding generator $[A_i]$ of the symplectic basis. We now only need to prove that v_k has residue $N_{|e|,h} \cdot N_{|e|,k}$ at q_e, which is seen as follows: if $e \in p([B_k])$, then by construction of v_k, it has residue $\pm 1 = N_{|e|,k}$ at $q_{|e|}$ depending again on whether $[\gamma_{|e|}] = [A_i]$ or $-[A_i]$; if $|e|$ does not lie on $p([B_k])$, both the intersection number and the residue are 0. Note that the signs of $N_{|e|,h}$ and $N_{|e|,k}$ are always the same, therefore $N_{|e|,h} \cdot N_{|e|,k} = \delta_{i,h} \cdot \delta_{i,k}$.

Secondly, we compute the linear term. Note as can be verified from the normalization conditions of the fundamental bidifferential, $v_h(z) = \int_{B_h} \omega(w,z)$ for $m+1 \leq h \leq g$, and $v_h(z) = \sum_{i=1}^{N} \int_{q-e_{i-1}}^{q_{e_i}} \omega_{v(e_i)}(w,z)$ for $1 \leq h \leq m$, where $\{e_i\}_{i=0}^{N-1}$ is the set of edges $p([B_h])$ passes through. To compute the linear term for $\tau_{h,k}(\underline{s}) = \int_{B_{h,\underline{s}}} v_{k,\underline{s}}$, we observe that by definition of σ_e in (4.1.4), we have

$$\sigma_e = \mathrm{hol}(\tilde{v}_h)(q_e).$$

Since $\Omega := v_k$, we have $\tilde{\xi}_e = \mathrm{hol}(\widetilde{\Omega})(q_e) = \mathrm{hol}(\tilde{v}_k)(q_e)$. Note that v_h is only supported on $\cup_{v \in p([B_h])} C_v$, the sum in (4.2.1) is taken over all edges.

Lastly, the constant term follows directly from Theorem 4.1.2.

□

4.3 Examples

In this section we will compute four explicit examples of the variational formula for abelian differentials and for the period matrix of a stable curve C. Throughout this section the stable curve has geometric genus g, Ω is a stable differential on C. We choose the symplectic basis of holomorphic 1-cycles and its dual basis of 1-forms as in the previous sections. The notation will vary among the examples according to the structure of C.

4.3.1 One Node: Compact Type

We first deal with the case where the curve C has only one node q. In [Yam80], Yamada computed the variational formula of both abelian differentials and the period matrices to any order of the plumbing parameter s. We will reprove his result up to the second order, while the full expansion can also be found using our method.

When C is of compact type, it has two components C_1 and C_2 that meet at a single separating node q, whose pre-images are denoted by $q_1 \in C_1$ and $q_2 \in C_2$. Let z_i be the local coordinates near q_i. Denote the restriction of Ω to C_i by Ω_i $(i = 1, 2)$. The subscripts of the Cauchy kernel and its derivative are changed correspondingly.

Since the curve is of compact type, the differentials Ω_i have no residue at q_i, therefore they are holomorphic and we have $\xi_i^{(0)}(z_i) = \Omega_i(z_i)$. We denote $\tilde{\xi}_i := \tilde{\xi}_i^{(0)}(q_i)$, and Ω_s is defined on C_s by formulas (3.2.3) (3.2.9). Explicitly, by Proposition 3.2.4, the expansion of the restriction $\Omega_{i,s}$ $(i = 1, 2)$ is given by

$$\Omega_{i,s}(z) = \Omega_i(z) + \left(-s \cdot \omega_i(z, q_i)\tilde{\xi}_{i'} + s^2 \cdot \omega_i(z, q_i)\beta_{i'}\tilde{\xi}_i\right) + O(s^3) \tag{4.3.1}$$

where by convention, $i' = 2$ if $i = 1$ and vice versa, and β_i denotes the leading coefficient in the expansion of $\boldsymbol{\omega}_i$ as in (3.2.1).

Let g_i be the genus of C_i, with $g_1 + g_2 = g$. We take a normalized basis of abelian differentials $\{v_k\}_{k=1}^g$ of C such that $\{v_1, \ldots, v_{g_1}\}$ are supported on C_1, and $\{v_{g_1+1}, \ldots, v_g\}$ on C_2.

For $1 \leq k \leq g_1$, letting $\Omega_1 = v_k, \Omega_2 = 0$ in (4.3.1), we obtain

$$v_{k,s}(z) = \begin{cases} v_k(z) + s^2 \cdot \omega_1(z, q_1)\beta_2 v_k(q_1) + O(s^3) & z \in \widetilde{C}_1, \\ -s \cdot \omega_2(z, q_2) v_k(q_1) + O(s^3) & z \in \widetilde{C}_2. \end{cases}$$

For $g_1 + 1 \leq k \leq g$, the formula is symmetric. We have then reproven [Yam80, Cor. 1].

4.3.2 One Node: Non-Compact Type

In this case C is irreducible, with a single node q. We denote q_1, q_2 the pre-images of q in the normalization $\widetilde{C}$, and z_1, z_2 the corresponding local coordinates. Let Ω be a meromorphic differential on the normalization of C which has simple poles of residues r_i at q_i $(i = 1, 2)$. By the residue theorem $r_1 = -r_2$.

We now have $\xi_i^{(0)}(z_i) = \Omega(z_i) - \frac{r_i dz_i}{z_i}$. Denote $\tilde{\xi}_i = \tilde{\xi}_i^{(0)}(q_i)$. By Proposition 3.2.4, we have

$$\Omega_s(z) = \Omega(z) - s \cdot \left(\omega(z, q_1)\tilde{\xi}_2 + \omega(z, q_2)\tilde{\xi}_1\right) + O(s^2). \tag{4.3.2}$$

For a symplectic basis $\{A_{k,s}, B_{k,s}\}_{k=1}^g$, we choose $A_{1,s}$ to be the seam, whereas $B_{1,s}$ is taken to intersect $A_{1,s}$ once, oriented from the neighborhood of q_1 to the neighborhood of q_2. As in Section 4.2, we take $v_1 = \omega_{q_2 - q_1}$, and $\{v_k\}_{k=2}^g$ to be the normalized basis of $H^1(\widetilde{C}, \mathbb{C})$.

By letting $\Omega = v_k$ for $2 \leq k \leq g$, we have $r_i = 0$, and the equation (4.3.2) gives [Yam80, Cor. 4]. For the case $\Omega = v_1$, we have $r_2 = -r_1 = 1$, then (4.3.2) gives [Yam80, Cor. 5]. Moreover, we can compute the period matrix of C_s, reproving [Yam80, Cor. 6]. By Corollary 4.2.1 we have

$$\tau_{1,1}(s) = \int_{B_{1,s}} \omega_{q_2 - q_1, s} = \ln|s| + c_{1,1} + s \cdot l_{1,1} + O(s^2),$$

where $c_{1,1} = \lim_{|s| \to 0} \left(\int_{z_1^{-1}(\sqrt{|s|})}^{z_2^{-1}(\sqrt{|s|})} \omega_{q_2 - q_1} - \ln|s| \right)$, and $l_{1,1} = -2\sigma_1\sigma_2$.

We also have

$$\tau_{k,1}(s) = \tau_{1,k}(s) = \int_{B_{1,s}} v_{k,s} = c_{1,k} + s \cdot l_{1,k} + O(s^2)$$

for $2 \leq k \leq g$. Since $v_k(x)$ is holomorphic, we have $\xi_i^{(0)}(z_i) = v_k(z_i)$ and hence $\tilde{\xi}_i = v_k(p_i)$. The constant term $c_{1,k}$ is equal to $\int_{q_1}^{q_2} v_k$, and the linear term $l_{1,k}$ is seen to be $-v_k(q_1)\sigma_2 - v_k(q_2)\sigma_1$ by (4.2.1).

Finally for $2 \leq k, h \leq g$ we have

$$\tau_{k,h}(s) = \tau_{k,h} + s \cdot l_{hk},$$

where $\tau_{h,k}$ is the period matrix of the normalization of C, and $l_{h,k} = -v_k(q_1)v_h(q_2) - v_h(q_1)v_k(q_2)$.

4.3.3 Banana Curves

The second example we consider is the stable genus g curve C that has two irreducible components meeting in two distinct nodes (so-called "banana curve"). This computation has not been done in the literature before.

Let the two components of C be C_a, C_b with genera g_a and g_b where $g_a + g_b = g - 1$. The edges corresponding to the two nodes are denoted by e_1 and e_2. The preimages of the nodes and the local coordinates are denoted as $q_{\pm e_i}$ and $z_{\pm e_i}$ $(i = 1, 2)$, where "+" corresponds to the C_a side, and "−" the C_b side.

Note that in the case where C has only two components (with any number of nodes connecting them), the path l^k can only go back and forth. Therefore $\xi_{e_i}^{(k)}$ (resp. $\xi_{-e_i}^{(k)}$) $(i = 1, 2)$ and $\eta_a^{(k)}$ (resp. $\eta_b^{(k)}$) are determined by Ω_a if k is even (resp. odd), and Ω_b if k is odd (resp. even), as we can see from the terms in expansion (4.3.1). Also note that in expansion (3.2.14) of $\eta_a^{(k)}$ and $\eta_b^{(k)}$, there is no residue of Ω involved. Therefore the we can simplify our computation by assuming that $\Omega_b = 0$ and the residues of Ω_a at both nodes are zero. Under these assumptions, we have $\xi_{e_i}^{(0)}(z_{e_i}) = \Omega_a(z_{e_i})$ $(i = 1, 2)$. Furthermore, for any integer $k \geq 0$, we have

$$\xi_{-e_i}^{(2k)} = \xi_{e_i}^{(2k+1)} = 0 \qquad (i = 1, 2),$$

thus by construction (3.2.9), we have

$$\begin{aligned} \eta_a^{(2k+1)}(z) = 0 \qquad & z \in \widehat{C}_a; \\ \eta_b^{(2k)}(z) = 0 \qquad & z \in \widehat{C}_b. \end{aligned}$$

By Proposition 3.2.4, we have for $z \in \widehat{C}_b$:

$$\eta_b^{(1)}(z) = -s_1\omega_b(z, q_{-e_1})\tilde{\xi}_{e_1} - s_2\omega_b(z, q_{-e_2})\tilde{\xi}_{e_2} + O(|\underline{s}|^2),$$

and for $z \in \widehat{C}_a$:

$$\eta_a^{(2)}(z) = s_1^2\omega_a(z, q_{e_1})\beta^b_{1,1}\tilde{\xi}_{e_1} + s_2^2\omega_a(z, q_{e_2})\beta^b_{2,2}\tilde{\xi}_{e_2} \\ + s_1 s_2\left(\omega_a(z, q_{e_1})\beta^b_{1,2}\tilde{\xi}_{e_2} + \omega_a(z, q_{e_2})\beta^b_{2,1}\tilde{\xi}_{e_1}\right) + O(|\underline{s}|^3),$$

where $\beta^b_{jk} := \beta^b_{-e_j,-e_k}$ is the constant term in the expansion of $\omega_b(z_{-e_j}, z_{-e_k})$ as in (3.2.1).

Note that we can also assume $\Omega_a = 0$, and that the residues of Ω_b at both nodes are zero. The general case follows by adding the differentials in these two cases together.

We now compute the degeneration of period matrix for the banana curve. For the symplectic basis of $H_1(C_{\underline{s}}, \mathbb{Z})$, we let $A_1 := \gamma_{e_2}$, and B_1 is taken to intersect each seam once, with the orientation from q_{e_1} to q_{e_2}, then from q_{-e_2} to q_{-e_1}. Thus we let $v_1 := \omega_{q_{e_2}-q_{e_1}} + \omega_{q_{-e_1}-q_{-e_2}}$ where $\omega_{q_{e_2}-q_{e_1}}$ is supported on C_a, and $\omega_{q_{-e_1}-q_{-e_2}}$ on C_b. Take $\{A_k, B_k\}_{k=2}^{g_a+1}$ and $\{A_j, B_j\}_{j=g_a+2}^{g}$ to be the symplectic bases of $H_1(C_a, \mathbb{Z})$ and $H_1(C_b, \mathbb{Z})$ respectively. The normalized basis of holomorphic differentials $\{v_k\}_{k=2}^{g}$ on the two components are taken correspondingly, and we require that v_k is identically zero on C_b if $2 \le k \le g_a + 1$, and on C_a if $g_a + 2 \le k \le g$.

Note that v_1 has residues $r_{e_2} = r_{-e_1} = 1$, thus we have $\tau_{1,1}(\underline{s}) = \ln|s_1| + \ln|s_2| + c_{1,1} + l_{1,1} + O(|\underline{s}|^2)$. By (4.1.2), the constant term is

$$c_{1,1} = \lim_{|\underline{s}|\to 0}\left(\int_{z_{e_1}^{-1}(\sqrt{|s_1|})}^{z_{e_2}^{-1}(\sqrt{|s_2|})} \omega_{q_{e_2}-q_{e_1}} + \int_{z_{-e_2}^{-1}(\sqrt{|s_2|})}^{z_{-e_1}^{-1}(\sqrt{|s_2|})} \omega_{q_{-e_1}-q_{-e_2}} - \ln|s_1| - \ln|s_2|\right).$$

As for the linear term $l_{1,1}$, by (4.2.1) we obtain

$$l_{1,1} = -2s_1\sigma_{-e_1}\sigma_{e_1} - 2s_2\sigma_{-e_2}\sigma_{e_2}.$$

We also see that the expansion of $\tau_{k,1}(\underline{s}) = \tau_{1,k}(\underline{s})$ is given by

$$\tau_{1,k}(\underline{s}) = \begin{cases} \int_{q_{e_1}}^{q_{e_2}} v_k - s_1 v_k(q_{e_1})\sigma_{-e_1} - s_2 v_k(q_{e_2})\sigma_{-e_2} + O(|\underline{s}|^2) & \text{if } 2 \le k \le g_a + 1, \\ \int_{q_{-e_2}}^{q_{-e_1}} v_k - s_2 v_k(q_{-e_2})\sigma_{e_2} - s_1 v_k(q_{-e_1})\sigma_{e_1} + O(|\underline{s}|^2) & \text{if } g_a + 2 \le k \le g. \end{cases}$$

The remaining $(g-1)\times(g-1)$ minor $\tau_{g-1}(\underline{s}) := \{\tau_{h,k}(\underline{s})\}_{h,k=2}^{g}$ of the period matrix is computed as:

$$\tau_{g-1}(\underline{s}) = \begin{pmatrix} \tau_a & 0 \\ 0 & \tau_b \end{pmatrix} - s_1 \cdot \begin{pmatrix} 0 & {}^tR_a(q_{e_1})R_b(q_{-e_1}) \\ {}^tR_b(q_{-e_1})R_a(q_{e_1}) & 0 \end{pmatrix} \\ - s_2 \cdot \begin{pmatrix} 0 & {}^tR_a(q_{e_2})R_b(q_{-e_2}) \\ {}^tR_b(q_{-e_2})R_a(q_{e_2}) & 0 \end{pmatrix} + O(|\underline{s}|^2),$$

where τ_a (resp. τ_b) is the period matrix of C_a (resp. C_b), and $R_a := (v_2, \ldots, v_{g_a+1})$, $R_b := (v_{g_a+2}, \ldots, v_g)$.

4.3.4 Totally Degenerate Curves

It is a fact that the stable curves that lie in the intersection of the Teichmüller curve and the boundary of $\overline{\mathcal{M}}_g$ are of arithmetic genus zero. In this subsection we study the largest dimensional boundary stratum of such stable curves and give the variational formula of its period matrix, which to the knowledge of the authors is again not dealt with in literature before. The periods of totally degenerate curves has been studied by Gerritzen in his series of papers [Ger90] [Ger92a] [Ger92b]. The perspectives in those papers are algebraic, mainly by studying the theta functions, the Torreli map and the Schottky problem. No analytic construction such as plumbing is involved.

Let C be a totally degenerate stable curve, namely the normalization $\widetilde{C}$ is a $\mathbb{P}^1$ with g pairs of marked points $\{q_{\pm i}\}_{i=1}^g$. Let q_i and q_{-i} be the preimages of the i-th node on C, and $r_i = -r_{-i}$ be the residue of Ω at q_i.

Let z be the global coordinate, then the local coordinates at the pre-images of nodes are given by $z_{\pm i} := z - q_{\pm i}$, where $i = 1, \ldots, g$. We have as usual $\xi^{(0)}_{\pm i}(z) := \Omega(z) \mp \frac{r_i dz}{z - q_{\pm i}}$, and $\tilde{\xi}_{\pm i} := \tilde{\xi}^{(0)}_{\pm i}(q_{\pm i})$.

The Cauchy kernel and the fundamental bidifferential on $\mathbb{P}^1$ are given explicitly: $K(z,w) = \frac{dz}{2\pi i(z-w)}$; $\omega(z,w) = 2\pi i \partial_w K(z,w) = \frac{dzdw}{(z-w)^2}$. We compute $\widetilde{\omega}(z, q_i) = \frac{1}{(z-q_i)^2}$, for $i \in \{\pm 1, \ldots, \pm g\}$. The expansion of $\Omega_{\underline{s}}$ is thus given by

$$\Omega_{\underline{s}}(z) = \Omega(z) - dz \sum_{k=1}^{g} s_k \left(\frac{\tilde{\xi}_{-k}}{(z-q_k)^2} + \frac{\tilde{\xi}_k}{(z-q_{-k})^2} \right) + O(|\underline{s}|^2)$$

where $z \in \widehat{C}$.

The classes of the seams $\{[\gamma_i]\}_{i=1}^g$ generate the Lagrangian subgroup of $H_1(C_{\underline{s}}, \mathbb{Z})$, thus we can take $A_i := \gamma_i$ and B_i the path from q_{-i} to q_i. The corresponding normalized basis of 1-forms will be $v_i := \omega_{q_i - q_{-i}} = dz \left(\frac{1}{z-q_i} - \frac{1}{z-q_{-i}} \right)$ for $i = 1, \ldots, g$. Let $\Omega = v_i$, we have for $k \neq \pm i$, $\tilde{\xi}_k = \frac{q_i - q_{-i}}{(q_k - q_i)(q_k - q_{-i})}$ and $\tilde{\xi}_i = \tilde{\xi}_{-i} = \frac{1}{q_{-i} - q_i}$.

One thus computes the period matrix as follows.

$$\begin{aligned} i = j : \tau_{i,i} = & \ln|s_i| - 2\ln|q_i - q_{-i}| - \frac{2s_i}{(q_i - q_{-i})^2} \\ & - \sum_{k \in \{1,..\hat{i},..g\}} \frac{2s_k (q_i - q_{-i})^2}{(q_k - q_{-i})(q_k - q_i)(q_{-k} - q_{-i})(q_{-k} - q_i)} + O(|\underline{s}|^2) \end{aligned}$$

$$
\begin{aligned}
i \neq j : \tau_{i,j} = & \ln\left(q_i, q_{-i}; q_j, q_{-j}\right) - \sum_{k \neq i,j} s_k \Big(\frac{(q_i - q_{-i})(q_j - q_{-j})}{(q_k - q_i)(q_k - q_{-i})(q_{-k} - q_j)(q_{-k} - q_{-j})} \\
& + \frac{(q_i - q_{-i})(q_j - q_{-j})}{(q_k - q_j)(q_k - q_{-j})(q_{-k} - q_i)(q_{-k} - q_{-i})} \Big) \\
& - s_i \frac{q_j - q_{-j}}{q_{-i} - q_i} \Big(\frac{1}{(q_i - q_j)(q_i - q_{-j})} + \frac{1}{(q_{-i} - q_j)(q_{-i} - q_{-j})} \Big) \\
& - s_j \frac{q_i - q_{-i}}{q_{-j} - q_j} \Big(\frac{1}{(q_j - q_i)(q_j - q_{-i})} + \frac{1}{(q_{-j} - q_i)(q_{-j} - q_{-i})} \Big) + O(|\underline{s}|^2)
\end{aligned}
$$

where $(q_i, q_{-i}, q_j, q_{-j})$ stands for the cross-ratio of the (ordered) four points.

Chapter 5

The Modular Form for the Hyperflex Locus in $\mathcal{M}_3$

This section contains the main results in [Hu17]. We give a modular form for the stratum $\Omega\mathcal{M}_3^{odd}(4)$. This stratum is also known as the hyperflex locus, because the points in the stratum correspond to smooth plane quartics with a hyperflex point. We will follow the notation in [Hu17] in our discussion and denote the hyperflex locus as $\mathcal{HF}$.

5.1 The Modular Form for $\Omega\mathcal{M}_3^{odd}(4)$

It can be shown that $\mathcal{HF}$ is an irreducible divisor:

Proposition 5.1.1 ([Ver83, Ch. 1, Prop. 4.9]). *$\mathcal{HF}$ is an irreducible, five-dimensional subvariety of $\mathcal{M}_3$, and it is closed in $\mathcal{M}_3 - H_3$ where H_3 is the hyperelliptic locus.*

We denote HF the closure of $u(\mathcal{HF})$ in $\mathcal{A}_3$. We define $HF_m \subset \mathcal{A}_3(2)$ to be the set of ppav $(J(C), i)$ where the bitangent line corresponding to m under the basis defined by $i : J(C)[2] \simeq (\mathbb{Z}/2\mathbb{Z})^6$ is a hyperflex to C.

To determine the scalar modular form for $\Gamma_3(2)$ whose zero locus is HF_{77}, we need to know equation of plane quartics using their bitangents. Such a formula was known classically for an individual curve ([Dol12, Ch. 5]), but only recently Dalla Piazza, Fiorentino and Salvati Manni obtained such an expression globally [DPFSM14]. They derived an eight by eight symmetric matrix parametrizing the bitangents of a given plane quartic, such that the determinant of any four by four minors of the bitangents matrix gives the equation of the quartic. We recall their notations and results.

Definition 5.1.2. 1. We call a triple of characteristics m_1, m_2, m_3 *azygetic* (resp. *syzygetic*) if

$$e(m_1, m_2, m_3) = e(m_1)e(m_2)e(m_3)e(m_1 + m_2 + m_3) = -1 \text{ (resp. 1)}.$$

2. A $(2g + 2)$-tuple of characteristics is called a *fundamental system* if any triple within it is azygetic.

For a more detailed discussion see [Dol12]. In our case $g = 3$, any fundamental system consists of 8 characteristics, within which 3 are odd and 5 are even.

We denote:

$$b_{ij} := \operatorname{grad}_z \theta[\epsilon, \delta](\tau, z)|_{z=0},$$

where $i = 4\epsilon_1 + 2\epsilon_2 + \epsilon_3, j = 4\delta_1 + 2\delta_2 + \delta_3$, and denote the so-called Jacobian determinant by:

$$D(n_1, n_2, n_3) := b_{n_1} \wedge b_{n_2} \wedge b_{n_3}.$$

It is a scalar modular form of weight $\frac{5}{2}$ and it can be written in terms of theta constants using Jacobi's derivative formula:

Proposition 5.1.3 ([Igu81])**.** *If n_1, n_2, n_3 is an azygetic triple of odd theta characteristics, then there exists a unique quintuple of even theta characteristics m_1, m_2, m_3, m_4, m_5 such that the 8-tuple forms a fundamental system. For this fundamental system, we have*

$$D(n_1, n_2, n_3) = \pm\pi^3 \cdot \theta_{m_1}\theta_{m_2}\theta_{m_3}\theta_{m_4}\theta_{m_5}.$$

The result of Dalla Piazza, Fiorentino, Salvati Manni is then:

Proposition 5.1.4 ([DPFSM14, Cor. 6.3])**.** *Let τ be the period matrix of the Jacobian of a plane quartic, then the equation of the plane quartic is given by the determinant of the following symmetric matrix:*

$$Q(\tau, z) := \begin{pmatrix} 0 & \frac{D(31,13,26)}{D(77,31,26)}b_{77} & \frac{D(22,13,35)}{D(77,31,26)}b_{64} & \frac{D(77,64,46)}{D(77,31,26)}b_{51} \\ * & 0 & \frac{D(22,13,35)}{D(77,46,51)}b_{13} & \frac{D(77,13,31)}{D(77,31,26)}b_{26} \\ * & * & 0 & \frac{D(64,13,22)}{D(77,31,26)}b_{35} \\ * & * & * & 0 \end{pmatrix}.$$

Using this we derive the modular form Ω_{77}:

Theorem 5.1.5. *Let Ω_{77} be the following modular form with respect to $\Gamma_3(2)$:*

$$\begin{aligned}\Omega_{77} := {} & [\theta_{01}\theta_{10}\theta_{37}\theta_{43}\theta_{52}\theta_{75} \cdot D(77,64,13) + \theta_{02}\theta_{25}\theta_{34}\theta_{40}\theta_{67}\theta_{76} \cdot D(77,51,26)]^2 \\ & - 4\theta_{01}\theta_{02}\theta_{10}\theta_{25}\theta_{34}\theta_{37}\theta_{40}\theta_{43}\theta_{52}\theta_{67}\theta_{75}\theta_{76} \cdot D(77,64,51) \cdot D(77,13,26), \quad (5.1.1)\end{aligned}$$

then its zero locus in $\mathcal{A}_3(2)$ is HF_{77}.

By Proposition 5.1.3 the modular form above is the same (up to a constant) as the following:

$$\begin{aligned}\Omega_{77} = {} & [\theta_{01}\theta_{10}\theta_{37}\theta_{43}\theta_{52}\theta_{75}\theta_{42}\theta_{06}\theta_{30}\theta_{21}\theta_{55} + \theta_{02}\theta_{25}\theta_{34}\theta_{40}\theta_{67}\theta_{76}\theta_{33}\theta_{05}\theta_{14}\theta_{60}\theta_{42}]^2 \\ & - 4\theta_{01}\theta_{02}\theta_{10}\theta_{25}\theta_{34}\theta_{37}\theta_{40}\theta_{43}\theta_{52}\theta_{67}\theta_{75}\theta_{76}\theta_{00}\theta_{04}\theta_{57}\theta_{70}\theta_{61}\theta_{73}\theta_{20}\theta_{07}\theta_{00}\theta_{16}. \quad (5.1.2)\end{aligned}$$

The proof is by directly computing the formulas for the bitangents, and uses the following lemma:

Lemma 5.1.6. *Let $l = l_1x + l_2y + l_3z$ be the equation of a line in $\mathbb{P}^2$, and suppose m, n, k, s are lines written similarly. Then the two intersection points of the line $l = 0$ and the quadric $mk - ns = 0$ coincide if and only if the following expression vanishes:*

$$\Psi_{l,m,n,k,s} = \left(\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ k_1 & k_2 & k_3 \end{vmatrix} + \begin{vmatrix} l_1 & l_2 & l_3 \\ n_1 & n_2 & n_3 \\ s_1 & s_2 & s_3 \end{vmatrix} \right)^2 - 4 \cdot \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \cdot \begin{vmatrix} l_1 & l_2 & l_3 \\ k_1 & k_2 & k_3 \\ s_1 & s_2 & s_3 \end{vmatrix}. \quad (5.1.3)$$

Proof. The proof is a direct computation: we plug in the equation of l to $\{mk - ns = 0\}$ and get:

$$\begin{aligned}&[(m_1l_2 - m_2l_1)x + (m_3l_2 - m_2l_3)z] \cdot [(k_1l_2 - k_2l_1)x + (k_3l_2 - k_2l_3)z] \\ &- [(n_1l_2 - n_2l_1)x + (n_3l_2 - n_2l_3)z] \cdot [(s_1l_2 - s_2l_1)x + (s_3l_2 - s_2l_3)z] = 0.\end{aligned}$$

We will now dehomogenize at z. The discriminant of the quadric of x is a homogenous polynomial F of degree 8 in the coefficient of l, m, n, k, s. We further observe that F is divisible by l_2^2. Denote $\Psi := F/l_2^2$, we hence get the expression in the lemma. One can verify that Ψ is independent of the dehomogenization. □

Proof of Theorem 5.1.5. Using Lemma 5.1.6 we can write the coefficients of $Q(\tau, z)$ given by proposition 5.1.4 as rational functions of even theta characteristics. By clearing the denominators we have the equation of the plane quartic:

$$\det Q(\tau, 0) = (\theta_{75}\theta_{52}\theta_{43})^4 \cdot (\theta_{04}^2\theta_{73}\theta_{60})^2 \cdot [(af)^2 + (be - cd)^2 - 2(af)(be + cd)] = 0, \quad (5.1.4)$$

where

$$\begin{aligned} a &= \theta_{66}\theta_{41}\theta_{50}b_{77}, \\ b &= \theta_{70}\theta_{52}\theta_{43}b_{64}, \\ c &= \theta_{40}\theta_{76}\theta_{67}b_{51}, \\ d &= \theta_{02}\theta_{25}\theta_{34}b_{13}, \\ e &= \theta_{37}\theta_{01}\theta_{10}b_{26}, \\ f &= \theta_{24}\theta_{12}\theta_{03}b_{35}. \end{aligned}$$

Recall that on $\mathcal{A}_3$ the vanishing of theta-null defines the hyperelliptic locus, which we already know is disjoint from the hyperflex locus by Proposition 5.1.1. Thus we need to exclude the locus defined by $(\theta_{75}\theta_{52}\theta_{43})^4 \cdot (\theta_{04}^2\theta_{73}\theta_{60})^2$ from the zero locus of $\det Q(\tau, 0)$, by dividing $\det Q(\tau, 0)$ by factor $(\theta_{75}\theta_{52}\theta_{43})^4 \cdot (\theta_{04}^2\theta_{73}\theta_{60})^2$. Rewrite the remaining part:

$$(af)^2 + (be - cd)^2 - 2(af)(be + cd) = a \cdot F + (be - cd)^2$$

where F is a homogenous degree 3 polynomial in a, b, c, d, e, f. Then

$$\{a = 0\} \cap \{a \cdot F + (be - cd)^2 = 0\} = \{a = 0\} \cap 2 \cdot \{be - cd = 0\}$$

gives the two tangent points.

Hence by the lemma, plugging a, b, c, d, e in to (5.1.3) we have

$$\Psi_{a,b,c,d,e} = \theta_{66}\theta_{73}\theta_{41}\theta_{50}\theta_{04} \cdot \Omega_{77}$$

where Ω_{77} is defined in (5.1.1). We then need to throw out the above factor for the same reason and hence have the modular form. □

Using the modular form we can now compute the class of the hyperflex locus $\mathcal{HF}$ in $\mathcal{M}_3$:

Corollary 5.1.7. *The class* $[\mathcal{HF}] \in H^2(\mathcal{M}_3, \mathbb{Q})$ *is equal to* $308 \cdot \lambda$.

Proof. First we need to compute the weight of the modular form Ω_{77}. The weight of $D(n_1, n_2, n_3)$ is $\frac{5}{2}$ and the weight of each θ_m is $\frac{1}{2}$, hence $12 \cdot \frac{1}{2} + 2 \cdot \frac{5}{2} = 11$ is the weight of the scalar modular form Ω_{77} with respect to $\Gamma_3(2)$.

Hence $[HF_{77}] = 11 \cdot p^*L$ in $\mathcal{A}_3(2)$. Set-theoretically the hyperflex locus $HF \subset \mathcal{A}_3$ is the image of $HF_{77} \subset \mathcal{A}_3(2)$ under the level two cover map p. Also for any odd characteristics m we have $p(HF_m) = HF$ for the same reason. Thus for the classes we have

$$p^*[HF] = \sum_{m \text{ odd}} [HF_m] = 28 \cdot 11 \cdot p^*L = 308 \cdot p^*L. \tag{5.1.5}$$

The second equality is due to the fact that with a fixed symplectic basis, any odd characteristics are equally likely to appear, and hence for all odd m, the class of HF_m is equal to that of HF_{77}. Pushing forward by p, by projection formula we obtain:

$$[HF] = 308 \cdot L.$$

By definition, $\lambda = u^* L$ in $\mathcal{M}_3$, we hence have the corollary claimed. $\square$

5.2 Class of the Closure of Hyperflex Locus

Let Ω_m be the image of Ω_{77} under the action of Γ_g so that it is a modular form with respect to $\Gamma_3(2)$ whose zero locus in $\mathcal{A}_3(2)$ is HF_m. As in the case of the vanishing orders of θ_m on D_n (resp. P_V), the vanishing order of Ω_m on D_n (resp. P_V) is also invariant under the action of Γ_g on the pairs (m, n) (resp. (m, V)). Denote for simplicity $d_{m,n} := \mathrm{ord}_{D_n} \Omega_m(\tau, 0)$, and $p_{m,V}$ to be the vanishing order of the pull-back of $\Omega_m(\tau, 0)$ on the component $\bar{u}^{-1} P_V$. There are only two possible values of $d_{m,n}$ corresponding to the two Γ_g orbits on (m, n), we denote the vanishing orders by d_0 and d_1 for the cases $e(m + n) = 0$ and 1. Similarly let p_1 and p_3 be the values of $p_{m,V}$ in the Γ_g orbit on the set of pairs (m, V) (subindex being the number of even elements in the triple). We have the following:

Proposition 5.2.1. *In $\overline{\mathcal{M}}_3$, we have*

$$[\overline{\mathcal{HF}}] = 308 \cdot \lambda - (16d_0 + 12d_1) \cdot \delta_0 - (10p_3 + 18p_1) \cdot \delta_1.$$

Proof. It can be concluded from a direct computation that for each $n \in (\mathbb{Z}/2\mathbb{Z})^6 - 0$, there are 16 m such that $m + n$ is even, 12 m such that $m + n$ is odd; for a fixed V, there are 18 odd theta characteristics m lies in the orbit corresponding to the case when the number of even elements in the triple $(m + n_1, m + n_2, m + n_1 + n_2)$ is 1, and 10 odd theta characteristics in the other orbit.

Consider the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{M}}_3(2) & \xrightarrow{\bar{u}'} & \overline{\mathcal{A}}_3(2) \\ \big\downarrow{\scriptstyle p'} & & \big\downarrow{\scriptstyle p} \\ \overline{\mathcal{M}}_3 & \xrightarrow{\bar{u}} & \overline{\mathcal{A}}_3 \end{array}$$

Summing up all m, we get on $\overline{\mathcal{M}}_3(2)$:

$$\bar{u}'^* \left(\sum_{m \text{ odd}} [\overline{HF}_m] \right) = 308 \cdot p'^* \lambda - \sum_{m,n} d_{m,n} \cdot \bar{u}'^* D_n - \sum_{V,n} p_{m,V} \cdot \bar{u}'^* P_V.$$

At the right hand side we have:

$$\begin{aligned}
\sum_{m,n} d_{mn} \cdot \bar{u}'^* D_n &= \bar{u}'^* \left(\sum_{m+n \text{ even}} d_0 D_n + \sum_{m+n \text{ odd}} d_1 D_n \right) \\
&= \bar{u}'^* \left(d_0 \sum_n 16 D_n + d_1 \sum_n 12 D_n \right) \\
&= \bar{u}'^* \left((16 d_0 + 12 d_1) \sum_n D_n \right) \\
&= (16 d_0 + 12 d_1) \cdot \bar{u}'^* (p^* D) \\
&= (16 d_0 + 12 d_1) \cdot p'^* \delta_0.
\end{aligned}$$

Similarly we have $\sum_{V,n} p_{m,V} \cdot \bar{u}'^* P_V = (10p_3 + 18p_1) \cdot p'^* \delta_1$. Also for the same reason as in equation (5.1.5) we have $\bar{u}'^* \left(\sum_{m \text{ odd}} [\overline{HF}_m] \right) = p'^* [\overline{\mathcal{HF}}]$. Pushing forward by p', by the projection formula both sides are multiples of $\deg(p')$. Note that the level cover map branches along the boundary components, but the projection formula applies regardless of the branching. At last we divide both sides by $\deg(p')$ and thus have the equality claimed. $\square$

We now use results from previous section to compute d_0, d_1 and p_1, p_3.

Proposition 5.2.2. *We have the following:*

$$d_{m,n} = \begin{cases} \frac{5}{4} & \text{if } m+n \text{ is even} \\ 1 & \text{otherwise} \end{cases} \tag{5.2.1}$$

$$p_{m,V} = \begin{cases} 4 & \text{all elements in the triple are even} \\ 2 & \text{otherwise.} \end{cases} \tag{5.2.2}$$

Proof. We only need to choose in each orbit a special representative to calculate, and will thus fix $m = 77$. For d_0 we choose $n = 04$ so that $m + n$ is even. We have the order at D_{04} of $\theta_{43}, \theta_{52}, \theta_{75}, \theta_{40}, \theta_{67}, \theta_{76}$ are all 1/8, while others are non-vanishing. We also have the order of $D(77, 64, 13) = D(77, 51, 26) = 1/4, D(77, 64, 51) = 3/8$, and $D(77, 13, 26) = 1/8$. Hence we have $d_0 = \min\{(3/8+1/4)\times 2, 6/8+3/8+1/8\} = 5/4$.

Similarly we choose $n = 06$ for the case $m + n$ is odd. We have the order of θ_{43}, θ_{52}, θ_{37}, θ_{40}, θ_{25}, θ_{34} is 1/8, all others are 0. And the order of $D(77, 64, 13) = 1/2, D(77, 51, 26) = 1/4, D(77, 64, 51) = D(77, 13, 26) = 1/8$, hence $d_1 = \min\{5/4, 1\} = 1$.

To compute the vanishing orders on P_V, we now choose the standard symplectic 2-dim subgroup V_0 as in section 3.2. Then $m+n_1, m+n_2$ are both even, and we can thus compute p_3. We will have $\operatorname{ord}_{V_0} D(77,64,13) = \operatorname{ord}_{V_0} D(77,64,51) = 1$, and $\operatorname{ord}_{V_0}\theta_{75} = \operatorname{ord}_{V_0}\theta_{67} = \operatorname{ord}_{V_0}\theta_{76} = 1$ and all the others are zero, hence $p_3 = \min\{(1+1)\times 2, 4\} = 4$.

Similarly we choose V_1 generated by $n_1 = [101,000], n_2 = [000,100]$ to compute p_1. We have $\operatorname{ord}_{V_1} D(77,64,51) = 1$, $\operatorname{ord}_{V_1}\theta_{43} = \operatorname{ord}_{V_1}\theta_{76} = 1$, all others are non-vanishing. We hence have $p_1 = \min\{1\times 2, 1+1+1\} = 2$.

Lastly, since the expression of the modular form is explicit, one can check by hand that the lowest order term in each case does not get cancelled, and it is indeed the case. □

Combining the results above, we can verify Cukierman's result in [Cuk89]:

Corollary 5.2.3. *In $\overline{\mathcal{M}_3}$, we have*

$$[\overline{\mathcal{HF}}] = 308\cdot\lambda - 32\cdot\delta_0 - 76\cdot\delta_1.$$

Also, the class $[\overline{HF}]$ in $\overline{\mathcal{A}}_3$ is equal to $308\cdot L - 32\cdot D$.

Proof. We only need to plug in the values $d_0 = 5/4, d_1 = 1, p_1 = 2, p_3 = 4$ in proposition 3.6. And the second claim follow easily from the discussion. □

5.3 Boundary of Higher Codimension

Using the modular form Ω_{77}, we can apply similar argument as the previous section to find out the intersection of any boundary component of $\overline{\mathcal{M}}_3$ with the closure of the hyperflex locus $\overline{\mathcal{HF}}$. As an application we consider the boundary stratum $T\subset\overline{\mathcal{M}}_3$ of stable curves which consist of two genus one curves intersecting at two nodes (so-called "banana curves"). This boundary stratum is contained in Δ_0 and is indeed an irreducible component of the self-intersection of Δ_0.

Proposition 5.3.1. *The boundary locus T is contained in the hyperflex locus $\overline{\mathcal{HF}}$.*

Remark 5.3.2. This result was recently also shown from a different approach in [Che15].

To prove the proposition, we will use Corollary 4.2.1 . We fix n elements in $\pi_1(C)$ represented by simple closed curves S_i with lengths $0\le s_i \ll 1$ for $i = 1\ldots n$. We also fix a homology basis $\{A_j, B_j\}_{j=1}^g$ such that for $1\le i\le n$, S_i is homotopic to one of the A_j possibly with a sign.

For the boundary locus T, we have $g = 3$ and $n = 2$ in Corollary 4.2.1. Furthermore, we choose the homology basis to be the standard one with intersection matrix I, so that S_1 and S_2 are both homotopic to A_1. Then the entries in the period matrix $[\tau_{h,k}]_{g\times g}$ we have

$$2\pi i\tau_{h,k} = \begin{cases} \ln s_1 + \ln s_2 + f_{h,k}(s_1, s_2) & \text{for } (h,k) = (1,1) \\ f_{h,k}(s_1, s_2) & \text{otherwise} \end{cases} \tag{5.3.1}$$

We denote $\tau = \left[\begin{smallmatrix} \tau_1 & b_1 & b_2 \\ b_1 & \tau_2 & c \\ b_2 & c & \tau_3 \end{smallmatrix}\right]$, and recall Fourier-Jacobi expansion (2.4.2), we conclude the following for $\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, 0)$:

1. If $\varepsilon_1 = 1$, then

   $$\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, 0) = \exp(\frac{1}{4}\pi i\tau_1)\cdot\exp(2\pi i\delta_1)\cdot\theta \begin{bmatrix} \varepsilon_2 & \varepsilon_3 \\ \delta_2 & \delta_3 \end{bmatrix} \Big(\begin{bmatrix} \tau_2 & c \\ c & \tau_3 \end{bmatrix}, (\frac{b_1}{2}, \frac{b_2}{2})\Big) + O(s_1) + O(s_2).$$

   Note that due to (5.3.1), we have $\exp(\pi i\tau_1) = s_1^{\frac{1}{2}} s_2^{\frac{1}{2}} \cdot \exp G(s_1, s_2)$ for some holomorphic function $G(s_1, s_2)$. Hence in this case the vanishing order of $\theta \begin{bmatrix} \varepsilon \\ \delta \end{bmatrix} (\tau, 0)$ with respect to s_1 and s_2 are $\frac{1}{8}$.

2. If $\varepsilon_1 = 0$, similarly $\theta[\begin{smallmatrix} \varepsilon \\ \delta \end{smallmatrix}](\tau, 0) = \theta[\begin{smallmatrix} \varepsilon_2 & \varepsilon_3 \\ \delta_2 & \delta_3 \end{smallmatrix}]([\begin{smallmatrix} \tau_2 & c \\ c & \tau_3 \end{smallmatrix}], 0) + O(s_1) + O(s_2)$, by definition [Tan89] of $c = f_{2,3}(s_1, s_2)$, we deduce that $c = 0$ when $s_1 = s_2 = 0$, i.e. when the curve hits boundary T. In that case, we have the constant term $\theta[\begin{smallmatrix} \varepsilon_2 & \varepsilon_3 \\ \delta_2 & \delta_3 \end{smallmatrix}]([\begin{smallmatrix} \tau_2 & 0 \\ 0 & \tau_3 \end{smallmatrix}], 0) = \theta[\begin{smallmatrix} \varepsilon_2 \\ \delta_2 \end{smallmatrix}](\tau_2, 0) \cdot \theta[\begin{smallmatrix} \varepsilon_3 \\ \delta_3 \end{smallmatrix}](\tau_3, 0) = 0$ if and only if $\varepsilon_2 = \delta_2 = 1$. Hence the only theta functions with characteristics that vanish when $s_1 = 0$ and $s_2 = 0$ are $\theta_{33}(\tau, 0)$ and $\theta_{37}(\tau, 0)$, but by taking partial derivatives one can directly show that neither is divisible by any power of $(s_1 \cdot s_2)$.

Proof of Proposition 5.3.1. As in Proposition 2.4.6 we choose the standard boundary component D_{04} so that the two cases $\varepsilon_1 = 0$ or 1 correspond to the two orbits of Γ_g action on the pair $(m, 04)$. Hence by the same discussion in the proof of Proposition 5.2.2, we have

$$\Omega_{77}(\tau, 0) = (s_1 \cdot s_2)^{\frac{5}{4}} \cdot F(s_1, s_2)$$

for some holomorphic function $F(s_1, s_2)$. Moreover, by the expression of Ω_{77} in Theorem 0.1, in each summand there is either θ_{33} or θ_{37}, which means

$$F(s_1, s_2) = \theta[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}]([\begin{smallmatrix} \tau_2 & c \\ c & \tau_3 \end{smallmatrix}], 0) + O(s_1) + O(s_2)$$

where $\tau_i = f_{i,i}(s_1, s_2)$ ($i = 2$ or 3) and $c = f_{2,3}(s_1, s_2)$ are holomorphic functions in s_1 and s_2, and $c(0, 0) = 0$. From the discussion above, $F(s_1, s_2)$ vanishes when $s_1 = 0$ and $s_2 = 0$, but is not divisible by any power of $(s_1 \cdot s_2)$.

As the normal direction of Δ_0 in the open part of $\overline{\mathcal{M}}_3$ is given by $q = \exp(\pi i \tau_{11})$, and T is the self-intersection of Δ_0 where s_1, s_2 give the two normal directions. Because the modular form Ω_{77} vanishes along T with higher order in s_1, s_2 than q, we can then conclude that the boundary stratum T is contained in the hyperflex locus $\overline{\mathcal{HF}}$.

$\square$

Bibliography

[ACG11] E. Arbarello, M. Cornalba, and P. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[AMRT10] A. Ash, D. Mumford, M. Rapoport, and Y.-S. Tai. *Smooth compactifications of locally symmetric varieties.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010. With the collaboration of Peter Scholze.

[ANW16] D. Aulicino, D. Nyugen, A. Wright *Classification of higher rank orbit closures in* $H^{odd}(4)$ J. Eur. Math. Soc. 2016

[AN18] D. Aulicino, D. Nyugen, *Rank Two Affine Manifolds in Genus 3* arXiv:1904.01625, 2019

[AN19] D. Aulicino, C. Norton *Shimura-Teichmüller Curves in Genus 5* To appear in Jour. Diff. Geom.

[BCGGM18] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky and M. Moeller. *Compactification of strata of Abelian differentials.* Duke Math. J., Vol. 167, No.12, 2347-2416, 2018.

[Ber06] M. Bertola. *Riemann Surfaces and Theta Functions.* Concordia University, 2006.

[BHM16] M. Bainbridge, P. Habeggar and M. Moeller. *Teichmller curves in genus three and just likely intersections in Gnm Gna.* Publ. Math. IHES., Vol. 124 No. 1, 1-98, 2016.

[BM12] M. Bainbridge, and M. Moeller. *The Deligne-Mumford-compactification of Hilbert modular varieties and Teichmueller curves in genus three.* Acta Math., 208,1–92, 2012

[BM14] M. Bainbridge, and M. Moeller. *The locus of real multiplication and the Schottky locus.* J. Inst. Math. Jussieu., to appear.

[Che15] D. Chen. *Degenerations of Abelian Differentials.* ArXiv e-prints, April 2015.

[Cuk89] F. Cukierman. *Families of Weierstrass points.* Duke Math. J., 58(2):317–346, 1989.

[DPFSM14] F. Dalla Piazza, A. Fiorentino, and R. Salvati Manni. *Plane quartics: the matrix of bitangents.* ArXiv e-prints, September 2014.

[Dol12] I. Dolgachev. *Classical algebraic geometry.* Cambridge University Press, Cambridge, 2012. A modern view.

[EM13] A. Eskin, M. Mirzakhani. *Teichmüller curves in genus two: torsion divisors and ratios of sines.* ArXiv e-prints, 1302.3320, 2013

[EMM15] A. Eskin and M. Mirzakhani, and A. Mohammadi. *Isolation, Equidistribution, and Orbit Closures for the SL(2,R) action on moduli space.* Ann. of Math., (2) 182 (2015), no. 2, 673–721.

[Fay73] J. Fay. *Theta functions on Riemann surfaces.* Lecture Notes in Mathematics, Vol. 352. Springer-Verlag, Berlin-New York, 1973.

[Fil16] S. Filip. Splitting mixed Hodge structures over affine invariant manifolds *Ann. of Math.*, (2) 183 (2016) 681–713.

[FM13] G. Forni and C. Matheus. *Introduction to Teichmüller Theory and Its Applications to Dynamics of Interval Exchange Transformations, Flows on Surfaces and Billiards.* arXiv:1311.2758v1, 2013.

[GH12] S. Grushevsky and K. Hulek. *The class of the locus of intermediate Jacobians of cubic threefolds.* Invent. Math., 190(1):119–168, 2012.

[Gen15] Q. Gendron *The Deligne-Mumford and the Incidence Variety Compactifications of the Strata of $\Omega\mathcal{M}_g$.* Preprint, arXiv:1503.03338, 2015.

[Ger90] L. Gerritzen *The Torelli Map at the Boundary of the Schottky Space.* J. reine angew. Math. 405(1990): 29-47.

[Ger92a] L. Gerritzen *Theta Functions of Totally Degenerate Curves.* Arch. Math 59(1992): 302-312.

[Ger92b] L. Gerritzen *Equations Defining the Periods of Totally Degenerated Curves.* Isreal Journal of Mathematics. 77(1992): 187-210.

[GH] P. Griffith, J. Harris *Principles of Algebraic Geometry.* Wiley Classics Library 52, Wiley, 2011.

[GKN17] S. Grushevsky, I. Krichever, and C. Norton. *Real-Normalized Differentials: Limits on Stable Curves.* Preprint, arXiv:1703.07806, 2017

[Gun76] R. C. Gunning *Riemann Surfaces and Generalized Theta Functions.* Springer-Verlag, 1976.

[HM] J. Harris, I. Morrison *Moduli of Curves.* Graduate Texts in Mathematics, 187. New York: Springer-Verlag, 1998.

[HS02] K. Hulek and G. Sankaran. *The geometry of Siegel modular varieties.* In Higher dimensional birational geometry (Kyoto, 1997), volume 35 of Adv. Stud. Pure Math., pages 89–156. Math. Soc. Japan, Tokyo, 2002.

[Hu17] X. Hu *Locus of Plane Quartics with A Hyperflex.* Proceedings of the American Mathematical Society 145 (2017), 1399-1413.

[HN18] X. Hu and C. Norton *General Variational Formulas for Abelian Differentials.* International Mathematics Research Notices https://doi.org/10.1093/imrn/rny106.

[Igu72] J.-I. Igusa. *Theta functions.* Springer-Verlag, New York-Heidelberg, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 194.

[Igu81] J.-I. Igusa. *On the nullwerte of Jacobians of odd theta functions.* In Symposia Mathematica, Vol. XXIV (Sympos., INDAM, Rome, 1979), pages 83–95. Academic Press, London-New York, 1981.

[IT] Y. Imayoshi, M. Taniguchi *An Introduction to Teichmüller Spaces."* Tokyo: Springer-Verlag, 1992.

[KZ03] M. Kontsevich and A. Zorich. *Connected components of the moduli spaces of Abelian differentials with prescribed singularities.* Invent. Math., 153(3):631–678, 2003.

[McM03] C. McMullen. *Billiards and Teichmüller curves on Hilbert modular surfaces. J. Amer. Math. Soc.*, 16(4):857–885, 2003.

[McM05a] C. McMullen. *Teichmüller curves in genus two: Discriminant and spin.* Math. Ann., 333(1):87–130, 2005.

[McM05b] C. McMullen. *Teichmüller curves in genus two: the decagon and beyond.* J. Reine Angew. Math., 582:173–199, 2005.

[McM06] C. McMullen. *Teichmüller curves in genus two: torsion divisors and ratios of sines.* Invent. Math., 165(3):651–672, 2006.

[MW15] M. Mirzahkani and A. Wright. *The boundary of an affine invariant submanifold.* Invent. math., 209:927–984, 2017.

[MZ16] M. Möller and D. Zagier. *Modular embeddings of Teichmller curves.* Compositio Math., 152, 2269–2349, 2016

[OS11] G. Ottaviani and E. Sernesi. *On singular Lüroth quartics.* Sci. China Math., volume 54, page 1757–1766, 2011

[Ro88] Yu. L. Rodin *The Riemann Boundary Problem on Riemann Surfaces.* Mathematics and its Applications, Vol. 16, Springer, 1988.

[SM83] R. Salvati Manni. *On the nonidentically zero Nullwerte of Jacobians of theta functions with odd characteristics.* Adv. in Math., 47(1):88–104, 1983.

[SM94] R. Salvati Manni. *Modular varieties with level* 2 *theta structure.* Amer. J. Math., 116(6):1489–1511, 1994.

[Tan89] M. Taniguchi. *Pinching deformation of arbitrary Riemann surfaces and variational formulas for Abelian differentials.* In Analytic function theory of one complex variable, volume 212 of Pitman Res. Notes Math. Ser., pages 330–345. Longman Sci. Tech., Harlow, 1989.

[Tan91] M. Taniguchi. *On the singularity of the periods of Abelian differentials with normal behavior under pinching deformation.* J. Math. Kyoto Univ., 31(4):1063–1069, 1991.

[Ver83] A. Vermeulen. *Weierstrass points of weight two on curves of genus three.* Universiteit van Amsterdam, Amsterdam, 1983. Dissertation, University of Amsterdam, Amsterdam, 1983, With a Dutch summary.

[Wri15] A. Wright. *Translation surfaces and their orbit closures: An introduction for a broad audience* EMS Surv. Math. Sci., 2015.

[Wol13] S. Wolpert *Infinitesimal Deformations of Nodal Stable Curves.* Adv. Math., 244(2013): 413–440.

[Wol15] S. Wolpert *Schiffer Variations and Abelian Differentials.* Preprint, arXiv:1508.01100, 2015

[Yam80] A. Yamada. *Precise variational formulas for Abelian differentials.* Kodai Math. J., 3(1):114–143, 1980.

[Zor06] A. Zorich. *Flat Surfaces.* Frontiers in number theory, physics, and geometry. I, 437-583, Springer, Berlin, 2006.