Singularities of Hermitian-Yang-Mills connections and the Harder-Narasimhan-Seshadri filtration

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Abstract of the Dissertation

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In this thesis, we study the analytic tangent cones of admissible Hermitian-Yang-Mills connections at an isolated singular point. When the singularity is homogeneous, we show that the tangent cone is uniquely determined by certain canonical algebraic data. In general, by assuming the existence of certain stable algebraic tangent cone, we charaterize the tangent cone connection. Furthermore, we construct some optimal algebraic tangent cone for reflexive sheaves at any singular point (not necessarily isolated), which turns out to be unique in a suitable sense.

Declaration

This thesis is based on joint work with Song Sun ([5, 6, 7]). The method for the homogeneous case is from [6] while we incorporate also more preliminary results and analytic results from [5]. For results about general point singularity, it is from [5]. The results about singularities of reflexive sheaves are from [7].

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Contents

1 Introduction

Given a holomorphic vector bundle $\mathcal{E}$ over a compact Kähler manifold (X, ω), we say $\mathcal{E}$ is stable if any subsheaf $\mathcal{S} \subset \mathcal{O}(\mathcal{E})$ with torsion free quotient satisfies

$$\frac{c_1(\mathcal{S}) \cdot [\omega]^{n-1}}{\mathrm{rank}(\mathcal{S})} < \frac{c_1(\mathcal{E}) \cdot [\omega]^{n-1}}{\mathrm{rank}(\mathcal{E})}.$$

A Hermitian metric h on E determines a unique compactible connection, called Chern connection, with curvature F_h. We say h is Hermitian-Eistein or Hermitian-Yang-Mills if $\wedge_\omega F_h$ is a constant multiple of the identity map. In 1980, Hitchin and Kobayashi conjectured that when $\mathcal{E}$ is stable, there exists a Hermitian-Yang-Mills metric on $\mathcal{E}$. Later, this was solved by Donaldson [10, 11], Uhlenbeck and Yau [39], which is now known as Donaldson-Uhlenbeck-Yau theorem. Donaldson proved the theorem in the case when (X, ω) is projective by using the heat flow method while Uhlenbeck and Yau proved the theorem using continuity method for general Kähler manifold. Combining these two methods, Simpson proved the convergence of the heat flow for stable bundles over general Kähler manifolds [37]. Later, the Donaldson-Uhlenbeck-Yau theorem was generalized by Bando and Siu to the case of stable reflexive sheaves using a notion of admissible Hermitian-Yang-Mills connections. Bando and Siu again studied the heat flow and proved the convergence of the flow. When $\mathcal{E}$ is not necessarily stable, the convergence of heat flow has been extensively studied and well understood by the work of Jacob ([21, 22]), Sibley ([34]), Wentworth and Daskalopoulos ([9]), Sibley and Wentworth ([36]). It turns out that the flow will converge to an admissible Hermitian-Yang-Mills connection on the double of the canonical graded sheaf associated to the Harder-Narasimhan-Seshadri filtration of $\mathcal{E}$.

In this thesis, we will focus on the study of admissible Hermitian-Yang-Mills connections. More specifically, let (X, ω) be an n dimensional Kähler manifold, and (E, H) be a Hermitian vector bundle over $X \setminus S$ for a closed subset $S \subset X$ with locally finite real codimension four Hausdorff measure. A smooth unitary connection A on (E, H) is called an *admissible Hermitian-Yang-Mills connection* on X if the following two conditions hold

(1) A satisfies the Hermitian-Yang-Mills equation

$$F_A^{0,2} = 0; \qquad \sqrt{-1}\Lambda_\omega F_A = \mu \cdot \mathrm{Id}_E, \tag{1.1}$$

where $\mu \in \mathbb{R}$ is a constant. In the literature, (1.1) is also usually referred to as the *Hermitian-Einstein equation* with Einstein constant μ –in this paper we will use both terminologies interchangeably;

(2) A has locally finite Yang-Mills energy, i.e. for any compact subset $K \subset X$, we have

$$\int_{K \setminus S} |F_A|^2 \frac{\omega^n}{n!} < \infty \tag{1.2}$$

In particular, $\bar{\partial}_A$ defines a holomorphic structure on E over $X \setminus S$. We denote the resulting Hermitian holomorphic vector bundle by $(\mathcal{E}, H)$. Then A is the Chern connection associated to $(\mathcal{E}, H)$. Bando and Siu [4] proved that $\mathcal{E}$ [1]

[1]Strictly speaking here $\mathcal{E}$ should be the locally free sheaf generated by local holomorphic sections of E. In this thesis, to make notations simpler, we will not distinguish between a holomorphic vector bundle and the corresponding locally free sheaf.

actually extends to be a reflexive sheaf over the whole X, and H (hence A) extends smoothly to the complement of the singular set of $\mathcal{E}$, which is a complex analytic subvariety of codimension at least three.

There are several motivations for studying admissible Hermitian-Yang-Mills connections. First, from the complex geometric point of view, as mentioned above, it is proved by Bando-Siu [4] that a polystable reflexive sheaf over a compact Kähler manifold always admits an admissible Hermitian-Yang-Mills connection, as a generalization of the Donaldson-Uhlenbeck-Yau theorem [10, 39] for holomorphic vector bundles. As a result, these connections have their relevance in algebraic geometry. Second, from the gauge theory point of view, by [27] (see also [38]) these admissible Hermitian-Yang-Mills connections naturally arise on the boundary of the moduli space of smooth Hermitian-Yang-Mills connections with bounded Yang-Mills energy, as Uhlenbeck limits, and therefore they play an important role in understanding the structure of the compactified moduli space in gauge theory over higher dimensional Kähler manifolds. The third motivation is that, in connection with gauge theory over G_2 manifolds, singularities of admissible Hermitian-Yang-Mills connections in dimension three are expected to provide one possible model for singularities of G_2 instantons (when the G_2 metric is close to the product of S^1 with a three dimensional Calabi-Yau metric) (see [31, 40, 32, 24] for recent research along this direction).

Given an admissible Hermitian-Yang-Mills connection A, a natural and interesting question is to study the behavior of A near a singular point $x \in S$. In this thesis, we will always restrict to the special case when S is discrete. This is largely due to technical reasons and we certainly hope this assumption will be removed in the future. So without loss of generality, we may assume that X is the unit ball $B = \{|z| < 1\}$ in $\mathbb{C}^n$ endowed with the standard Kähler form $\omega_0 := \sqrt{-1}\bar{\partial}\partial|z|^2$, and $S = \{0\}$. We also always assume $n \geq 3$ in this paper, since the singularity is removable if $n \leq 2$.

Our goal is to understand the infinitesimal structure of A near 0 in terms of the complex/algebraic geometric information on the stalk of $\mathcal{E}$ at 0. Loosely speaking we are searching for an *analytic/algebraic* correspondence, which can be viewed as a *local* analogue of the well-known Donaldson-Uhlenbeck-Yau theorem.

From the analytic point of view, we can take *analytic tangent cones* of A at 0, which are defined as follows. Let $\lambda : z \mapsto \lambda z$ be the rescaling map centered at the origin on $\mathbb{C}^n$. Then by Uhlenbeck's compactness result ([27, 38, 39]), we know as $\lambda \to 0$, by passing to a subsequence, the rescaled sequence of connections $A_\lambda := \lambda^* A$ converge to a smooth Hermitian-Yang-Mills connection A_∞ on $\mathbb{C}^n_* \setminus \Sigma$. Here $\mathbb{C}^n_* := \mathbb{C}^n \setminus \{0\}$, and Σ is a closed subset of $\mathbb{C}^n_*$ that has locally finite Hausdorff codimension four measure, and we may assume Σ is exactly the set where the convergence is not smooth. We call Σ the *analytic bubbling set*[2]. By Bando-Siu [4], A_∞ extends to be an admissible Hermitian-Yang-Mills connection on $\mathbb{C}^n$ and it defines a reflexive sheaf $\mathcal{E}_\infty$ on $\mathbb{C}^n$. By [38] (see also the discussion in Section 2), passing to a further subsequence we may assume the Yang-Mills energy of A_λ weakly converges to a limit Radon measure

[2]For our purpose in this paper we will always remove the point 0 and we only consider the convergence of *smooth* connections, locally away from 0, so that we can directly use the Uhlenbeck convergence theory. In general one could try to understand the bubbling set of a sequence of *admissible* Hermitian-Yang-Mills connections, which we leave for future study.

μ on $\mathbb{C}^n$. Write

$$\mu = |F_{A_\infty}|^2 \mathrm{dVol} + 8\pi^2 \nu,$$

and define the *blow-up* locus as $\Sigma_b := \mathrm{Supp}(\nu) \setminus \{0\}$. We know that Σ is always a complex-analytic subvariety of $\mathbb{C}^n_*$ and by [38], Σ_b consists of precisely the closure of the complex codimension two part of Σ, and to each irreducible component of Σ_b one can associate an *analytic multiplicity*; the lower dimensional strata corresponds to the essential singularities of the connection A_∞ which can not be removed. For more detailed discussion see Section 2.

Throughout this paper, we shall call the triple (A_∞, Σ, μ) an *analytic tangent cone* of A at 0. A priori (A_∞, Σ, μ) depends on the choice of subsequences as $\lambda \to 0$. We also know that A_∞ is a HYM cone connection in the sense of Definition 2.22. Namely, the corresponding reflexive sheaf $\mathcal{E}_\infty$ on $\mathbb{C}^n$ is isomorphic to $\psi_* \pi^* \underline{\mathcal{E}}_\infty$, where $\pi : \mathbb{C}^n_* \to \mathbb{CP}^{n-1}$ is the natural projection map and $\psi : \mathbb{C}^n_* \to \mathbb{C}^n$ is the inclusion map, and

$$\underline{\mathcal{E}}_\infty = \bigoplus_j \underline{\mathcal{F}}_j$$

where each $\underline{\mathcal{F}}_j$ is a stable reflexive sheaf. The connection A_∞ is isomorphic to the direct sum of the pull-back of the (unique) Hermitian-Yang-Mills connection on each $\underline{\mathcal{F}}_j$ under the projection map π, twisted by μ_j times the pull-back of the Chern connection associated to the Fubini-Study metric on $\mathcal{O}(1)$ (this is necessary to make the Einstein constant vanish). So in short the limit connection A_∞ is uniquely characterized by the algebraic data $\underline{\mathcal{E}}_\infty := \oplus_j \underline{\mathcal{F}}_j$. We emphasize again that the analytic tangent cone is a priori not known to be unique, since it depends on not only the connection A but also the choice of subsequences.

Theorem 1.1. *Suppose $\mathcal{E}$ is a reflexive sheaf on B with 0 as an isolated singularity, such that $\mathcal{E}$ is isomorphic to $(\psi_* \pi^* \underline{\mathcal{E}})|_B$ for some holomorphic vector bundle $\underline{\mathcal{E}}$ over $\mathbb{CP}^{n-1}$. Then for any admissible Hermitian-Yang-Mills connection A on $\mathcal{E}$,*

- *all the tangent cones at 0 have the connection A_∞. More precisely, the corresponding $\mathcal{E}_\infty$ is isomorphic to $\psi_* \pi^* (Gr^{HNS}(\underline{\mathcal{E}}))^{**}$, and A_∞ is gauge equivalent to the natural Hermitian-Yang-Mills cone connection that is induced by the admissible Hermitian-Yang-Mills connection on $(Gr^{HNS}(\underline{\mathcal{E}}))^{**}$. Furthermore, $\pi^{-1}(Sing(Gr^{HNS}(\underline{\mathcal{E}}))) \subset \Sigma$ for any tangent cone (A_∞, μ, Σ).*
- *the analytic bubbling set Σ is also independent of the choice of subsequences. Moreover, it agrees with the singular set Σ^{alg} of $\pi^*(Gr^{HNS}(\underline{\mathcal{E}}))$ as a set and for each irreducible codimension 2 component, the analytic multiplicity agrees with the algebraic multiplicity. In particular, the limit measure μ is also uniquely determined by $\underline{\mathcal{E}}$.*

For the definition of algebraic multiplicity we refer to Section 3.3. Notice $\mathrm{Sing}(\mathcal{E}_\infty) \setminus \{0\}$ is obviously a subset of Σ^{alg}, and by Theorem 1.1 the difference only appears when $Gr^{HNS}(\underline{\mathcal{E}})$ fails to be reflexive.

Remark 1.2. The theorem has been proved by [23] using PDE method when $\underline{\mathcal{E}}$ is stable.

One particular interesting fact is that there are examples where $Gr^{HNS}(\underline{\mathcal{E}})$ is not reflexive and its double dual is a direct sum of line bundles, so $\psi_* \pi^* (Gr^{HNS}(\underline{\mathcal{E}}))^{**}$ is trivial, i.e. $\psi_* \pi^* (Gr^{HNS}(\underline{\mathcal{E}}))^{**} \cong \mathcal{O}_{\mathbb{C}^n}^{\oplus \operatorname{rank}(\underline{\mathcal{E}})}$. As a result, the following interesting phenomenon can happen.

Corollary 1.3. *There exists an admissible Hermitian-Yang-Mills connection on a rank two reflexive sheaf over $\mathbb{CP}^3$, such that at all of its singular points the analytic tangent cones have trivial flat connections but non-empty bubbling sets.*

In general, we can find plenty of examples with non-homogeneous isolated singularities by studying the fitting ideal of the stalk of $\mathcal{E}$ at the singular point (see section 5.2). In the non-homogeneous case, there is no longer a canonical choice of algebraic data like $Gr^{HNS}(\underline{\mathcal{E}})$ as above. An algebraic picture in general was missing.

To solve the most general case, the natural way would be to study the reflexive extension of $p^*(\mathcal{E})|_{\hat{B}\setminus D}$ across D where $D := p^{-1}(0)$. Here $p : \hat{B} \to B$ denotes the blow-up at 0. Let $\mathcal{A}$ be the set of all such extensions modulo isomorphisms over $\hat{B}$. $\mathcal{A}$ can be easily seen to be non-empty since $(p^*\mathcal{E})^{**} \in \mathcal{A}$. We call an element $\hat{\mathcal{E}} \in \mathcal{A}$ an extension of $\mathcal{E}$ at 0 and the torsion-free sheaf $\hat{\mathcal{E}}|_D$ an algebraic tangent cone of $\mathcal{E}$ at 0. Then we can prove the following concerned about non-homogeneous singularities.

Theorem 1.4. *Suppose $\mathcal{E}$ is a reflexive sheaf on B with isolated singularity at 0, such that there is an algebraic tangent cone given by a stable vector bundle $\underline{\widehat{\mathcal{E}}}$. Then for any admissible Hermitian-Yang-Mills connection A on $\mathcal{E}$, all the tangent cone have the same tangent connection A_∞, which is gauge equivalent to the natural Hermitian-Yang-Mills cone connection that is induced by the Hermitian-Yang-Mills connection on $\underline{\widehat{\mathcal{E}}}$.*

Even though such a locally free stable algebraic tangent cone does not always exist, it is natural to try to find some algebraic tangent cones that are close to being stable.

To state the result, we set up a few notations. In the following, $\mathcal{E}$ will be any fixed reflexive sheaf over B with $0 \in Sing(\mathcal{E})$ (0 not necessarily isolated) and we let $\mathcal{A}$ be the space of extensions as above. We define a function $\Phi : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ by $\Phi(\hat{\mathcal{E}}) = \mu(\underline{\mathcal{E}}_1) - \mu(\underline{\mathcal{E}}_m/\underline{\mathcal{E}}_{m-1})$ where $0 \subset \underline{\mathcal{E}}_1 \subset \cdots \underline{\mathcal{E}}_{m-1} \subset \underline{\mathcal{E}}_m = \underline{\hat{\mathcal{E}}} := \hat{\mathcal{E}}|_D$ is the Harder-Narasimhan filtration of $\underline{\hat{\mathcal{E}}}$ with respect to the $\mathcal{O}(1)$ polarization and $\mu(\cdot)$ is the slope of the corresponding sheaf. Intuitively, $\phi(\hat{\mathcal{E}})$ measures how far $\underline{\hat{\mathcal{E}}}$ is from being semistable. We denote by $Gr^{HN}(\cdot)$ the graded sheaf associated to the Harder-Narasimhan filtration of the corresponding sheaf. We say an extension $\hat{\mathcal{E}}$ of $\mathcal{E}$ at 0 is optimal if $\Phi(\hat{\mathcal{E}}) \in [0, 1)$. Then we have

Theorem 1.5. *Given a reflexive coherent sheaf $\mathcal{E}$ over B, the following hold*

(I). (Existence)*An optimal extension always exists. More precisely, given any $\hat{\mathcal{E}} \in \mathcal{A}$, there are finitely many Hecke transforms that transform $\hat{\mathcal{E}}$ into an optimal one.*

(II). (Uniqueness) *Suppose $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2 \in \mathcal{A}$ are both optimal, then there is a $k \in \mathbb{Z}$ such that $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2([D]^{\otimes k})$ differ by a Hecke transform of special type (see Definition 6.12). In particular, $Gr^{HN}(\underline{\hat{\mathcal{E}}_1}) \sim Gr^{HN}(\underline{\hat{\mathcal{E}}_2})$.[3] Moreover, if $\phi(\hat{\mathcal{E}}_1) + \phi(\hat{\mathcal{E}}_2) < 1$, $\hat{\mathcal{E}}_1$ and $\hat{\mathcal{E}}_2$ are isomorphic up to tensoring with a power of $[D]$; if $\phi(\hat{\mathcal{E}}_1) = 0$, then all the other optimal extensions are isomorphic to $\hat{\mathcal{E}}_1$ up to tensoring with $\mathcal{O}(k)$.*

[3] $\sim$ means the two sheaves are isomorphic up to tensoring with each direct sum factor with certain powers of $\mathcal{O}(1)$.

(III). (Homogeneous case) *Suppose* $\mathcal{E}$ *is homogeneous at* 0, *i.e.* $\mathcal{E}|_{B\setminus 0} \simeq \pi^*\underline{\mathcal{E}}|_{B\setminus 0}$ *for some reflexive sheaf* $\underline{\mathcal{E}}$ *on* $\mathbb{CP}^{n-1}$, *then there exists an optimal extension* $\hat{\mathcal{E}} \in \mathcal{A}$ *with*

$$\underline{\hat{\mathcal{E}}} \sim \widetilde{Gr}(\underline{\mathcal{E}}),$$

where $\widetilde{Gr}(\underline{\mathcal{E}})$ *denotes the graded sheaf determined by the partial Harder-Narasimhan filtration of* $\underline{\mathcal{E}}$ *(see section 6.2.3 for definition). In particular,*

$$Gr^{HN}(\underline{\hat{\mathcal{E}}}) \sim Gr^{HN}(\underline{\mathcal{E}}).$$

2 Background

2.1 Harder-Narasimhan-Seshadri filtration and canonical metrics

In this section, we denote $(X, \omega) = (\mathbb{CP}^{n-1}, \omega_{FS})$ although the results apply to general compact Kähler manifolds. Recall a coherent sheaf $\mathcal{F}$ on X is *torsion free* if the natural map $\mathcal{F} \to \mathcal{F}^{**}$ is injective and *reflexive* if the map is an isomorphism. The singular set $\mathrm{Sing}(\mathcal{F})$ is the set of points $x \in X$ where $\mathcal{F}_x$ is not free over $\mathcal{O}_{X,x}$. We know that $\mathrm{Sing}(\mathcal{F})$ is always a complex analytic subset of X. It has complex co-dimension at least two if $\mathcal{F}$ is torsion free, and at least three if $\mathcal{F}$ is reflexive. A good nontrivial local example of a reflexive sheaf can be given by the sheaf $\psi_*\pi^*\underline{\mathcal{F}}$ on $\mathbb{C}^m$, where $\underline{\mathcal{F}}$ is a holomorphic vector bundle on $\mathbb{CP}^{m-1}$ and $\pi : \mathbb{C}^m \setminus \{0\} \to \mathbb{CP}^{m-1}$ and $\psi : \mathbb{C}^m \setminus \{0\} \to \mathbb{C}^m$ are the natural maps.

The *slope* of a coherent sheaf $\mathcal{F}$ is defined as

$$\mu(\mathcal{F}) := \frac{2\pi(n-1)\int_X c_1(\mathcal{F}) \wedge \omega^{n-2}}{\mathrm{rank}(\mathcal{F})\int_X \omega^{n-1}} \in \mathbb{Q} \tag{2.1}$$

Here $c_1(\mathcal{F})$ can be understood as the first Chern class of the *determinant line bundle* of $\mathcal{F}$, which is always an integer, and $\mathrm{rank}(\mathcal{F})$ denotes the rank of $\mathcal{F}$.

Definition 2.1. A torsion free sheaf $\mathcal{F}$ is

- *semistable* if for all coherent subsheaves $\mathcal{F}' \subset \mathcal{F}$ with $rank(\mathcal{F}') > 0$ we have $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$;
- *stable* if for all subsheaves $\mathcal{F}' \subset \mathcal{F}$ with $0 < rank(\mathcal{F}') < rank(\mathcal{F})$ we have $\mu(\mathcal{F}') < \mu(\mathcal{F})$;
- *polystable* if $\mathcal{F}$ is the direct sum of stable sheaves with equal slope;
- *unstable* if $\mathcal{F}$ is not semistable.

The following definition is taken from Bando-Siu [4]

Definition 2.2. An *admissible* Hermitian metric on $\mathcal{F}$ is a smooth Hermitian metric defined on $\mathcal{F}|_{X\setminus\mathrm{Sing}(\mathcal{F})}$ such that the corresponding Chern connection A satisfies $\int_{X\setminus\mathrm{Sing}(\mathcal{F})} |F_A|^2 \mathrm{dVol}_\omega < \infty$, and that $|\Lambda_\omega F_A|$ is uniformly bounded on $X \setminus \mathrm{Sing}(\mathcal{F})$; it is an *admissible Hermitian-Einstein metric* if furthermore $\sqrt{-1}\Lambda_{\omega_{FS}} F_A = \mu(\mathcal{F})\mathrm{Id}$.

By definition it follows that the Chern connection of an admissible Hermitian-Einstein metric is indeed an admissible Hermitian-Yang-Mills connection as defined in the introduction. Conversely, by [4] any admissible Hermitian-Yang-Mills connection on $\mathcal{F}$ defines a unique reflexive sheaf together with an admissible Hermitian-Einstein metric so that A is the corresponding Chern connection. From now on, we will use the two terminologies interchangeably. We also drop the word "admissible" when the meaning is clear from the context.

The following theorem was proved by Donaldson and Uhlenbeck-Yau in the case of vector bundles, and later generalized by Bando-Siu to reflexive sheaves.

Theorem 2.3 (Donaldson-Uhlenbeck-Yau [10, 11, 39], Bando-Siu [4])**.** *A reflexive sheaf $\mathcal{F}$ on (X, ω) admits an admissible Hermitian-Einstein metric if and only if it is polystable.*

For later purpose we need the following

Proposition 2.4 ([4], Proposition 3)**.** *Let H be an admissible Hermitian-Einstein metric on a reflexive sheaf $\mathcal{F}$. Suppose $\mu(\mathcal{F}) \leq 0$, then any holomorphic section s of $\mathcal{F}$ must be parallel with respect to the Chern connection. Furthermore, if $\mu(\mathcal{F}) < 0$, then the only holomorphic section of $\mathcal{F}$ is the zero section.*

This has a few consequences

Corollary 2.5. *Let $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ be a non-trivial homomorphism between a stable reflexive sheaf $\mathcal{F}_1$ and a polystable reflexive sheaf $\mathcal{F}_2$ with $\mu(\mathcal{F}_1) = \mu(\mathcal{F}_2)$, then ϕ realizes $\mathcal{F}_1$ as a direct summand of $\mathcal{F}_2$.*

Proof. We view ϕ as a holomorphic section of $Hom(\mathcal{F}_1, \mathcal{F}_2) = \mathcal{F}_1^* \otimes \mathcal{F}_2$. By Theorem 2.3 we know $\mathcal{F}_1$ and $\mathcal{F}_2$ both admit a Hermitian-Einstein metric, so we get an induced Hermitian-Einstein metric on $Hom(\mathcal{F}_1, \mathcal{F}_2)$. On the other hand, $\mu(Hom(\mathcal{F}_1, \mathcal{F}_2)) = \mu(\mathcal{F}_2) - \mu(\mathcal{F}_1) = 0$. So by Proposition 2.4 ϕ is parallel. In particular, on the complement of $\mathrm{Sing}(\mathcal{F}_1) \cup \mathrm{Sing}(\mathcal{F}_2)$, $Ker(\phi)$ defines a parallel sub-bundle of $\mathcal{F}_1$, and $Im(\phi)$ defines a parallel sub-bundle of $\mathcal{F}_2$, and both $Ker(\phi)$ and $Im(\phi)$ admits induced Hermitian-Einstein metrics induced from $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. So by [4] they extend to be polystable reflexive sheaves on X. By assumption we have $Ker(\phi) = 0$, and $\mathcal{F}' = Im(\phi)$ is a direct summand of $\mathcal{F}_2$. Hence $\phi : \mathcal{F}_1 \to \mathcal{F}'$ is an isomorphism away from $\mathrm{Sing}(\mathcal{F}_1) \cup \mathrm{Sing}(\mathcal{F}')$, so extends as an isomorphism globally by Hartogs's theorem. Indeed, by taking a locally free resolution of $\mathcal{F}_1^* \otimes \mathcal{F}'$ and taking its dual, one obtains the sheaf exact sequence $0 \to (\mathcal{F}')^* \otimes \mathcal{F}_1 \to \mathcal{G}_1 \to \mathcal{G}_2$ for locally free sheaves $\mathcal{G}_1$ and $\mathcal{G}_2$. ϕ^{-1} can be naturally seen as a section of $\mathcal{G}_1$ away from $\mathrm{Sing}(\mathcal{F}_1) \cup \mathrm{Sing}(\mathcal{F}')$ which has complex codimension at least three, and it maps to zero in $\mathcal{G}_2$. So by the usual Hartogs's theorem ϕ^{-1} extends to a global section of $\mathcal{G}_1$ that maps to zero in $\mathcal{G}_2$, thus it defines a global homomorphism from $\mathcal{F}'$ to $\mathcal{F}_1$. Clearly it is the inverse of ϕ. □

Corollary 2.6. *A stable reflexive sheaf admits a unique Hermitian-Einstein metric up to constant rescalings. In general, any two Hermitian-Einstein metrics on a polystable reflexive sheaf determine the same Chern connection and the two metrics differs by a parallel complex transform on the complement of singular set of the sheaf.*

Proof. Suppose $\mathcal{F}$ is polystable, and H_1 and H_2 are two Hermitian-Einstein metrics on $\mathcal{F}$, then by Proposition 2.4 the identity map in $End(\mathcal{F})$ is parallel, with respect to the Chern connection of the Hermtian metric $H_1^* \otimes H_2$. This implies that the Chern connections of H_1 and H_2 coincide. Suppose $H_2(\cdot, \cdot) = H_1(g\cdot, \cdot)$ for a complex gauge transformation g of $\mathcal{F}$ over $X \setminus \mathrm{Sing}(\mathcal{F})$ which is Hermitian with respect to H_1. Then it follows that g is holomorphic. Now by Proposition 2.4 we conclude that g is parallel with respect to the Hermitian-Einstein metric $H_1^* \otimes H_1$ on $End(\mathcal{F})$. Hence it decomposes $\mathcal{F}$ into the direct sum of eigenspace pieces, each of which is again polystable. If $\mathcal{F}$ is stable, then g must be a multiple of identity. □

Now we move on to discuss the case when $\mathcal{F}$ is not polystable. The following two results are well-known, see for example Page 174 in [25].

Proposition 2.7. *Suppose $\mathcal{F}$ is an unstable reflexive sheaf, then there is a* unique *filtration by reflexive subsheaves*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m = \mathcal{F},$$

such that the successive quotient $Q_i := \mathcal{F}_i/\mathcal{F}_{i-1}$ is torsion free and semistable, with $\mu(Q_{i+1}) < \mu(Q_i)$.

Remark 2.8. The construction of [25] on Page 174 only states that $\mathcal{F}_i$ is torsion-free since they state the result for $\mathcal{F}$ being torsion free, but it is easy to see each $\mathcal{F}_i$ is indeed reflexive if $\mathcal{F}$ is reflexive. By Proposition 5.22 in [25] a coherent subsheaf of a reflexive sheaf is reflexive if the corresponding quotient sheaf is torsion free. One then applies this fact inductively to $\mathcal{F}_i$ for $i = m, \cdots, 1$.

The above filtration is called the *Harder-Narasimhan filtration* of $\mathcal{F}$. It follows that the associated graded object $\bigoplus_i Q_i$, which we denote by $Gr^{HN}(\mathcal{F})$, is also uniquely determined by $\mathcal{F}$.

Proposition 2.9. *Suppose Q is a semistable torsion-free sheaf, then there is a filtration by subsheaves*

$$0 = S_0 \subset S_1 \subset \cdots \subset S_q = Q,$$

so that the quotients S_i/S_{i-1} are torsion free and stable, with $\mu(S_i) = \mu(Q)$.

Such a filtration is usually referred to as a *Seshadri filtration* of Q. Note that Seshadri filtration is in general *not* unique; however, the associated graded object $\bigoplus_i S_i/S_{i-1}$ is nevertheless uniquely determined by Q up to an isomorphism.

Combining the above two results, given any reflexive sheaf $\mathcal{F}$, there is a double filtration by reflexive subsheaves

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \mathcal{F}_m = \mathcal{F} \tag{2.2}$$

and

$$\mathcal{F}_{i-1} = \mathcal{F}_{i,0} \subset \mathcal{F}_{i,1} \subset \cdots \mathcal{F}_{i,q_i} = \mathcal{F}_i \tag{2.3}$$

such that the successive quotients $\mathcal{F}_{i,j}/\mathcal{F}_{i,j-1}$ are torsion free and stable, and moreover the slope of these quotients is constant when i is fixed, and strictly decreasing when i increases. This is called the *Harder-Narasimhan-Seshadri* *filtration* of $\mathcal{F}$, and we emphasize again that only the associated graded object $Gr^{HNS}(\mathcal{F})$ is uniquely determined by $\mathcal{F}$ up to an isomorphism.

One can ask what is the analogue of a canonical Hermitian metric structure on a general $\mathcal{F}$. For semistable vector bundles on projective manifolds, the following is proved by Kobayashi [25], using Hermitian-Yang-Mills flow. This is sufficient for our purpose, but we also mention that the result has been generalized to all compact Kähler manifolds by Jacob [21].

Theorem 2.10 (Kobayashi, Theorem 10.13 in [25])**.** *Suppose $\mathcal{F}$ is a semistable vector bundle over (X, ω). Then $\mathcal{F}$ admits approximately Hermitian-Einstein metrics. Namely, for any $\epsilon > 0$, there exists a Hermitian metric H on $\mathcal{F}$ such that the associated Chern connection A satisfies*

$$|\sqrt{-1}\Lambda_\omega F_A - \mu(\mathcal{F})Id|_{L^\infty(X)} < \epsilon. \tag{2.4}$$

In the remainder of this section, we always assume $\mathcal{F}$ is locally free which may in general be unstable. This situation is more involved. Suppose the Harder-Narasimhan filtration of $\mathcal{F}$ is given as in (2.2), and denote by $S(\mathcal{F})$ the subset of X where $Gr^{HNS}(\mathcal{F})$ is not locally free. Given any Hermitian metric H on $\mathcal{F}$, by [39], each $\mathcal{F}_i$ can be identified with a weakly holomorphic projection map $\pi_i \in W^{1,2}(\mathcal{F}^* \otimes \mathcal{F})$ which is smoothly defined outside $S(\mathcal{F})$ and satisfies the following:

(a). $\pi_i = \pi_i^* = \pi_i^2$. This means that π_i is a self-adjoint projection map.

(b). $(\mathrm{Id} - \pi_i)\bar{\partial}_{\mathcal{F}}\pi_i = 0$. This condition is equivalent to that $\mathcal{F}_i$ being a holomorphic sub-bundle outside $S(\mathcal{F})$.

In particular, $(\bar{\partial}_{\mathcal{F}}\pi_i)\pi_i = 0$, and taking adjoint we also have $\pi_i\partial_{\mathcal{F}}\pi_i = 0$, where $\partial_{\mathcal{F}}$ is the $(1, 0)$ component of the Chern connection on $End(\mathcal{F})$ determined by the chosen metric H.

Now we define

$$\psi^H = \sum_i \mu_i(\pi_i - \pi_{i-1}),$$

where $\mu_i = \mu(\mathcal{F}_i/\mathcal{F}_{i-1})$. Denote by $X_0 = X\backslash\mathcal{S}(\mathcal{F})$. Then we have an orthogonal splitting over X_0 as

$$\mathcal{F} = \bigoplus_i Q_i,$$

where $Q_i := \mathcal{F}_i/\mathcal{F}_{i-1}$ is naturally identified as a sub-bundle of $\mathcal{F}_i$, given by the orthogonal complement of $\mathcal{F}_{i-1}$ in $\mathcal{F}_i$. The splitting gives $\mathcal{F}$ another holomorphic structure $\bar{\partial}_S$ outside $S(\mathcal{F})$, and this together with the fixed Hermitian metric determines a unique Chern connection which we deonte by $A_{(H,\bar{\partial}_S)}$.

Remark 2.11. By definition,

$$\bar{\partial}_S = \sum_i (\pi_i - \pi_{i-1}) \circ \bar{\partial}_{\mathcal{F}} \circ (\pi_i - \pi_{i-1}). \tag{2.5}$$

In particular, $\bar{\partial}_{\mathcal{F}} = \bar{\partial}_S - \sum_i(\pi_i - \pi_{i-1})\bar{\partial}_{\mathcal{F}}\pi_i$.

Recall for each i, with respect to the orthogonal splitting $\mathcal{F} = \mathcal{F}_i \oplus \mathcal{F}_i^\perp$, the second fundamental form of $\mathcal{F}_i$ in $\mathcal{F}$ is a smooth section of $\Lambda_X^{1,0} \otimes Hom(\mathcal{F}_i, \mathcal{F}_i^\perp)$ over X_0, whose adjoint is given by $\beta_i = -\pi_i \bar{\partial}_\mathcal{F} \pi_i^\perp = \bar{\partial}_\mathcal{F} \pi_i$, where $\pi_i^\perp$ denotes the projection map from $\mathcal{F}$ to $\mathcal{F}_i^\perp$.

Lemma 2.12. *The following estimates hold in general*

(1)

$$|\Lambda_\omega \partial_\mathcal{F} \beta_i| = |\Lambda_\omega \partial_\mathcal{F} \bar{\partial}_\mathcal{F} \pi_i| \leq |\Lambda_\omega F_{A_{(H,\bar{\partial}_\mathcal{F})}} + \sqrt{-1}\psi^H| + 2|\beta_i|^2 \tag{2.6}$$

(2)

$$|\partial_\mathcal{F} \beta_i| \leq 2|\beta_i|^2 + |F_{A_{(H,\bar{\partial}_\mathcal{F})}}| \tag{2.7}$$

Proof. Since $\bar{\partial}_\mathcal{F} = \bar{\partial}_{S_i} - \beta_i$, where $\bar{\partial}_{S_i}$ defines the split holomorphic structure with respect to the splitting $\mathcal{F} = \mathcal{F}_i \bigoplus \mathcal{F}_i^\perp$, we have

$$\begin{aligned} F_{A_{(H,\bar{\partial}_\mathcal{F})}} &= \partial_\mathcal{F} \circ \bar{\partial}_\mathcal{F} + \bar{\partial}_\mathcal{F} \circ \partial_\mathcal{F} \\ &= F_{A_{(H,\bar{\partial}_{S_i})}} - \partial_{S_i} \beta_i + \bar{\partial}_{S_i} \beta_i^* - \beta_i \wedge \beta_i^* - \beta_i^* \wedge \beta_i. \end{aligned}$$

Notice $\Lambda_\omega \partial_{S_i} \beta_i$ is a section of $Hom(\mathcal{F}_i, \mathcal{F}_i^\perp)$ and therefore is perpendicular to the remaining terms, we have

$$|\Lambda_\omega \partial_{S_i} \beta_i| \leq |\Lambda_\omega F_{A_{(H,\bar{\partial}_\mathcal{F})}} + \sqrt{-1}\psi^H|.$$

Since $\partial_\mathcal{F} = \partial_{S_i} + \beta_i^*$, we have $|\Lambda_\omega \partial_\mathcal{F} \beta_i| \leq |\Lambda_\omega F_{A_{(H,\bar{\partial}_\mathcal{F})}} + \sqrt{-1}\psi^H| + 2|\beta_i|^2$. Similarly $|\partial_{S_i} \beta_i| \leq |F_{A_{(H,\bar{\partial}_\mathcal{F})}}|$ and thus $|\partial_\mathcal{F} \beta_i| \leq 2|\beta_i|^2 + |F_{A_{(H,\bar{\partial}_\mathcal{F})}}|$. □

Proposition 2.13. *There is a $K > 0$ such that for any $\epsilon > 0$ and $\delta > 0$, $\exists$ a smooth Hermitian metric H on $\mathcal{F}$ such that the following holds for each i*

1. $\sup_i \int_X |\beta_i|^2 \leq \epsilon$;
2. $\sup_i \int_X |\Lambda_\omega \partial_\mathcal{F} \beta_i| \leq \epsilon$;
3. $\int_X |\sqrt{-1}\Lambda_\omega F_{A_{(H,\bar{\partial}_S)}} - \psi^H| \leq \epsilon$;
4. $|\Lambda_\omega F_{A_{(H,\bar{\partial}_\mathcal{F})}}|_{L^\infty} \leq K$;
5. $\sup_{X \setminus \mathcal{S}(\mathcal{F})^\delta} (|\sqrt{-1}\Lambda_\omega F_{A_{(H,\bar{\partial}_S)}} - \psi^H| + \sup_i |\beta_i| + \sup_i |\Lambda_\omega \partial_\mathcal{F} \beta_i|) \leq \epsilon$, *where* $\mathcal{S}(\mathcal{F})^\delta$ *denotes the δ-neighborhood of $\mathcal{S}(\mathcal{F})$.*

Proof. This is indeed an easy consequence of [22] and [36]. Starting from any initial Hermitian metric H_0 on $\mathcal{F}$, let H_t be the family of Hermitian metrics on $\mathcal{F}$ evolving along the Hermitian-Yang-Mills flow. We denote by A_t the Chern connection of H_t and π_i^t the projection map determined by the Harder-Narasimhan filtration and the metric H_t. Then the following holds

(1).

$$\lim_{t \to \infty} \int_X |\sqrt{-1}\Lambda_\omega F_{A_t} - \psi^{H_t}|^2 = 0, \tag{2.8}$$

see Proposition 8 in [22].

(2).

$$\lim_{t\to\infty}\int_X |\nabla_{A_t}\pi_i^t|^2 = 0, \tag{2.9}$$

see (4.20) in [22].

(3). There exists a constant $K > 0$ independent of t such that

$$|\Lambda_\omega F_{A_t}|_{L^\infty} \leq K \tag{2.10}$$

see Lemma (8.15) on Page 220 in [25].

We claim for t large, H_t satisfies the desired properties in the Proposition with the choice of K as in the third item above. In fact, let $\beta_i^t = \bar\partial_{\mathcal{F}}\pi_i^t$. Then (2.9) implies

$$\lim_{t\to\infty}\int_X |\beta_i^t|^2 = 0.$$

By Lemma 2.12 and (2.8), we have

$$\lim_{t\to\infty}\int_X |\Lambda_\omega\partial_{\mathcal{F}}\beta_i^t| = \lim_{t\to\infty}\int_X |\Lambda_\omega\bar\partial_{\mathcal{F}}(\beta_i^t)^*| = 0.$$

Then by Remark 2.11, $\bar\partial_{\mathcal{F}} = \bar\partial_S - \sum_i(\pi_i^t - \pi_{i-1}^t)\beta_i^t$, which implies

$$\begin{aligned}\Lambda_\omega F_{A_t} = {} & \Lambda_\omega F_{(H_t,\bar\partial_S)} - \sum_{i,j}\Lambda_\omega(\pi_i^t - \pi_{i-1}^t)\beta_i^t\wedge(\beta_j^t)^*(\pi_j^t - \pi_{j-1}^t)\\ & - \sum_{i,j}\Lambda_\omega(\beta_j^t)^*(\pi_j^t - \pi_{j-1}^t)\wedge(\pi_i^t - \pi_{i-1}^t)\beta_i^t - \sum_i\Lambda_\omega\partial_S[(\pi_i^t - \pi_{i-1}^t)\beta_i^t]\\ & + \sum_i\Lambda_\omega\bar\partial_S[(\beta_i^t)^*(\pi_i^t - \pi_{i-1}^t)].\end{aligned}$$

From this, we get

$$|\Lambda_\omega F_{(H_t,\bar\partial_S)} + \sqrt{-1}\psi^{H_t}| \leq |\Lambda_\omega F_{(H_t,\bar\partial_S)} + \sqrt{-1}\psi^{H_t}| + C(\sup_i|\beta_i^t|^2 + \sup_i|\Lambda_\omega\partial_{\mathcal{E}}\beta_i^t|)$$

Then we have

$$\lim_{t\to\infty}\int_X |\Lambda_\omega F_{(H_t,\bar\partial_S)} + \sqrt{-1}\psi^{H_t}| = 0.$$

The last item follows from [36], where the analytic bubbling set of the Hermitian-Yang-Mills flow is identified with $\mathcal{S}(\mathcal{F})$ as a set, and outside this set A_t converges smoothly to a direct sum of Hermitian-Yang-Mills connection. □

Remark 2.14. Although not needed in this paper, we mention that in [34] Sibley proved the existence of L^p approximate critical Hermitian metrics, in the sense that for any $\delta > 0$ and $1 \leq p < \infty$ there exists a metric H_δ whose associated Chern connection A_δ satisfies

$$\|\sqrt{-1}\Lambda_\omega F_{A_\delta} - \psi^{H_\delta}\|_{L^p(X)} < \delta.$$

Now consider a complex gauge transform away from $\mathcal{S}(\mathcal{F})$ of the form

$$g = \sum_i f_i(\pi_i - \pi_{i-1}),$$

where each f_i is a smooth positive function. Denote $\beta = \bar\partial_{\mathcal{F}} - \bar\partial_S = -\sum_i(\pi_i - \pi_{i-1})\bar\partial_{\mathcal{F}}\pi_i$.

Lemma 2.15.

$$F_{(H,g\cdot\bar\partial_{\mathcal{F}})} = T_0 + T_1 + T_2$$

with

$$\begin{aligned} T_0 &= F_{(H,g\cdot\bar\partial_S)} = F_{(H,\bar\partial_S)} - \sum_i \partial\bar\partial\log(f_i^2)(\pi_i - \pi_{i-1}); \\ T_1 &= -(g\cdot\bar\partial_S)(g\beta g^{-1})^* + (g\cdot\bar\partial_S)^*(g\beta g^{-1}); \\ T_2 &= -g\beta g^{-1}\wedge(g\beta g^{-1})^* - (g\beta g^{-1})^*\wedge g\beta g^{-1}. \end{aligned}$$

Here

$$\begin{aligned} (g\cdot\bar\partial_S)(g\beta g^{-1})^* =& \sum_{i<j}\frac{f_i}{f_j}(\pi_j - \pi_{j-1})(\bar\partial_S\beta^*)(\pi_i - \pi_{i-1}) \\ & - 2\sum_{i<j}\bar\partial(\frac{f_i}{f_j})\wedge(\pi_j - \pi_{j-1})(\partial_{\mathcal{F}}\pi_i)(\pi_i - \pi_{i-1}), \end{aligned}$$

and $(g\cdot\bar\partial_S)^*$ *denotes the* $(1,0)$ *component of the Chern connection determined by* $(H, g\cdot\bar\partial_S)$.

Proof. By definition,

$$F_{(H,g\cdot\bar\partial_S)} = (g\cdot\bar\partial_S)^*\circ(g\cdot\bar\partial_S) + (g\cdot\bar\partial_S)\circ(g\cdot\bar\partial_S)^*$$

Now the first part follows from this by plugging $g\cdot\bar\partial_S = g\cdot\bar\partial_{\mathcal{F}} - g\beta g^{-1}$. As for the second part,

$$\begin{aligned} (g\cdot\bar\partial_S)(g\beta g^{-1})^* &= (\bar\partial_S - \bar\partial_S g\cdot g^{-1})(g^{-1}\beta^* g) \\ &= \bar\partial_S(g^{-1}\beta^* g) - [\bar\partial_S g\cdot g^{-1}, g^{-1}\beta^* g] \\ &= g^{-1}\bar\partial_S\beta^* g + 2(\bar\partial_S g^{-1})\wedge\beta^* g - 2g^{-1}\beta^*\wedge\bar\partial_S g \end{aligned}$$

Plugging $\beta = -\sum_i(\pi_i - \pi_{i-1})\bar\partial_{\mathcal{F}}\pi_i$ and using $\pi_j(\bar\partial_S\beta^*)(\pi_i - \pi_{i-1}) = 0$ for $j \le i$, we obtain the conclusion. □

Corollary 2.16. $|F_{(H,\bar\partial_S)}| \le |F_{(H,\bar\partial_{\mathcal{E}})}| + C\sup_i|\beta_i|^2$

Proof. This follows from choosing $g = 1$ in Lemma 2.15 and applying Equation (2.7), □

We finish this subsection with a technical result which will be used in Section 3.1.2.

Proposition 2.17. *Let* $\mathcal{G} \subset \mathcal{F}$ *be a saturated subsheaf and fix any smooth Hermitian metric* H *on* $\mathcal{F}$*, then there exists* $\delta = \delta(\mathcal{G}) > 0$ *so that*

$$\int_{X\setminus Sing(\mathcal{G})} |\bar\partial\pi_{\mathcal{G}}|^{2+\delta} < \infty,$$

where $\pi_{\mathcal{G}}$ *is the weakly holomorphic projection map defined by* $\mathcal{G}$ *with respect to* H.

Proof. By Hironaka resolution of singularities (see [4]), there is a sequence of blow-ups $p_k : X_k \to X_{k-1}$ ($k = 1, \cdots, N$) along smooth submanifolds of X_{k-1} of codimension at least two, with $X_0 = X$, such that $p = p_N \circ \cdots \circ p_1$ is biholomorphic on the complement of $\mathrm{Sing}(\mathcal{G})$, $p^{-1}(\mathrm{Sing}(\mathcal{G}))$ is a union $E = \bigcup E_j$ of simple normal crossing divisors (with possibly multiplicities), and $p^*\mathcal{G}|_{X_N \setminus E}$ extends to a holomorphic sub-bundle of $p^*\mathcal{F}$. We denote the sub-bundle by $\widetilde{\mathcal{G}}$. Pulling back the given Hermitian metric on $\mathcal{F}$ to $p^*\mathcal{F}$, we obtain a corresponding smooth projection map $\pi_{\widetilde{G}}$ defined by $\widetilde{\mathcal{G}}$. So

$$\int_{X \setminus \mathrm{Sing}(\mathcal{G})} |\bar{\partial}\pi_{\mathcal{G}}|^{2+\delta}\omega^{n-1} = \int_{X_N \setminus p^{-1}(\mathrm{Sing}(\mathcal{G}))} |\bar{\partial}\pi_{\widetilde{G}}|^{2+\delta}_{p^*\omega}\, p^*\omega^{n-1}.$$

Let ω_k be a smooth Kähler metric on X_k, where $\omega_0 = \omega$. Then we can naturally view ω_k as a smooth real valued $(1,1)$ form on X_N which are Kähler metrics outside E. On $X_N \setminus E$ we have

$$|\bar{\partial}\pi_{\widetilde{G}}|^{2+\delta}_{\omega_0} \leq (\mathrm{Tr}_{p^*\omega_0}\omega_N)^{1+\delta/2}|\bar{\partial}\pi_{\widetilde{G}}|^{2+\delta}_{\omega_N}.$$

Notice

$$\mathrm{Tr}_{p^*\omega_0}\omega_N \frac{(p^*\omega_0)^{n-1}}{\omega_N^{n-1}} = \frac{(n-1)\omega_N \wedge (p^*\omega_0)^{n-2}}{\omega_N^{n-1}}$$

is uniformly bounded, and $\bar{\partial}\pi_{\widetilde{G}}$ is smooth on X_N. Therefore to prove the conclusion it suffices to show that we can find $\delta > 0$ such that

$$\int_{X_N \setminus E} (\mathrm{Tr}_{p^*\omega_0}\omega_N)^{\delta/2}\omega_N^{n-1} < \infty.$$

To prove this, we first notice on $X_N \setminus E$ we have

$$\mathrm{Tr}_{p^*\omega_0}\omega_N \leq \Pi_{k=1}^N (p_N \cdots p_{k+1})^* \mathrm{Tr}_{p_k^*\omega_{k-1}}\omega_k$$

with $\omega_0 = \omega$. Now for each k, by fixing any smooth Hermitian metric on the corresponding line bundle associated to the exceptional divisor of p_k and doing a local calculation, one can easily check that the $\mathrm{Tr}_{p_k^*\omega_{k-1}}\omega_k \leq C|s_k|^{-2}$, where s_k is the defining section for the exceptional divisor of p_k. Then we have

$$\mathrm{Tr}_{p^*\omega_0}\omega_N \leq C\Pi_j|\sigma_j|^{-2a_j},$$

where σ_j is the defining section of E_j over X_N, and a_j is a positive integer. Since E is a union of simple normal crossing divisors, it is clear that we can find the desired $\delta > 0$, again by estimating the corresponding integral locally near any point of E.

□

2.2 PDE estimates

Let $\overline{B}^* = \overline{B} \setminus \{0\} \subset \mathbb{C}^n$. Again we always assume $n \geq 3$. For a function g, we denote $g^+ = \max\{g, 0\}$. The following lemma is crucial for us.

Lemma 2.18. *Suppose $g \in C^2(B^*) \cap C^0(\overline{B}^*)$ with $\int_{B^*} |g^+|^{\frac{n}{n-1}} < \infty$ and f is a non-negative function on B^*. If on B^* we have*

$$\Delta g(z) \geq -|z|^{-2}f(z), \tag{2.11}$$

then for all $z \in B^*$ the following hold, *(1). For all $z \in B^*$,*

$$g(z) \leq |g|_{L^\infty(\partial B)} + \int_{B^*} G(z,w)|w|^{-2} f(w)dw,$$

where $G(z,w)$ is the (positive) Green's function for $-\Delta$ on B. The inequality is only meaningful when the right hand side is finite.

(2). For all $z \in B^*$,

$$g(z) \leq C_0(|g|_{L^\infty(\partial B)} + \sup_{|w-z|\leq|z|/2} f(w) + (-\log|z|) \sup_{r\in(0,1]} r^{1-2n} \int_{\partial B_r} f),$$

where C_0 depends only on n. In particular, if $|f|_{L^\infty(B^*)} < \infty$, then

$$\limsup_{z\to 0} \frac{g(z)}{-\log|z|} \leq C_0 \sup_{r\in(0,1]} r^{1-2n} \int_{\partial B_r} f.$$

Proof. We first solve the Dirichlet problem $\Delta h = 0$, $h|_{\partial B} = g|_{\partial B}$, then $|h|_{L^\infty(\overline{B})} \leq |g|_{L^\infty(\partial B)}$. So we can reduce to the case that $g|_{\partial B} = 0$. Fix any $z \in B^*$, for $\epsilon < |z|/4$, we choose a cut-off function χ_ϵ supported in $B \setminus B_\epsilon$, and equal to 1 on $B \setminus B_{2\epsilon}$, with $|\nabla \chi_\epsilon| \leq C\epsilon^{-1}$ and $|\nabla^2 \chi_\epsilon| \leq C\epsilon^{-2}$. For $\tau > 0$ we denote $g_\tau = \frac{1}{2}(\sqrt{g^2+\tau^2} - \tau + g)$, then $g_\tau = 0$ on ∂B and one can check

$$\Delta g_\tau(z) \geq \frac{1}{2}\frac{\sqrt{g^2+\tau^2}+g}{\sqrt{g^2+\tau^2}}\Delta g \geq -\frac{1}{2}\frac{\sqrt{g^2+\tau^2}+g}{\sqrt{g^2+\tau^2}}|z|^{-2}f(z) \geq -|z|^{-2}f(z).$$

Using Green's representation formula we have

$$g_\tau(z) = \int_{B^*} 2\nabla_w G(z,w)\nabla\chi_\epsilon(w)g_\tau(w) + G(z,w)\Delta\chi_\epsilon(w)g_\tau(w) - G(z,w)\chi_\epsilon(w)\Delta g_\tau(w)$$

Let $\tau \to 0$, we get

$$g^+(z) \leq \int_{B^*} 2\nabla_w G(z,w)\nabla\chi_\epsilon(w)g^+(w) + G(z,w)\Delta\chi_\epsilon(w)g^+(w) + G(z,w)\chi_\epsilon(w)|w|^{-2}f(w).$$

Now let $\epsilon \to 0$, we claim the first two terms tend to zero. We only prove this for the second term and the first term can be dealt with similarly. We have

$$\begin{aligned} & |\int_{B^*} G(z,w)\Delta\chi_\epsilon(w)g^+(w)| \\ \leq\ & C\epsilon^{-2}\int_{B_{2\epsilon}\setminus B_\epsilon} |z-w|^{-2n+2}|g^+(w)| \\ \leq\ & C\epsilon^{-2}Vol(B_{2\epsilon})^{1/n}|z|^{-2n+2}(\int_{B_{2\epsilon}\setminus B_\epsilon} |g^+(w)|^{\frac{n}{n-1}})^{\frac{n-1}{n}}, \end{aligned}$$

and the last term tends to zero, since $\int_{B^*} |g^+|^{\frac{n}{n-1}} < \infty$. So we obtain

$$g^+(z) \leq \int_B G(z,w)|w|^{-2}f(w)dw.$$

This finishes the proof of (1).

Now we prove (2). Notice $G(z,w)\le C|z-w|^{2-2n}$ for a positive constant C, so it suffices to estimate the integral $\int_B |z-w|^{2-2n}|w|^{-2}f(w)$. We divide this into three parts. When $|w-z|\le |z|/2$ we have

$$\int_{|w-z|\le|z|/2} |z-w|^{2-2n}|w|^{-2}f(w)\le C\sup_{|w-z|\le|z|/2} f(w). \tag{2.12}$$

When $|w-z|\ge|z|/2$, we have $|w|\le 3|z-w|$. Then

$$\begin{aligned}\int_{|w|\ge\frac{|z|}{2},|z-w|\ge\frac{|z|}{2}} |z-w|^{2-2n}|w|^{-2}f(w) &\le 3^{2n-2}\int_{|w|\ge\frac{|z|}{2}}|w|^{-2n}f(w)\\ &\le 3^{2n-2}(-\log|z|)\sup_{r\in(0,1]} r^{1-2n}\int_{\partial B_r} f\end{aligned}$$

We also have

$$\begin{aligned}\int_{|w|\le|z|/2}|z-w|^{2-2n}|w|^{-2}f(w) &\le 2^{2n-2}|z|^{2-2n}\int_{|w|\le|z|/2}|w|^{-2}f(w)\\ &\le 2^{2n-1}\sup_{r\in(0,1]} r^{1-2n}\int_{\partial B_r} f.\end{aligned}$$

Combining the estimates above, we easily get the conclusion. □

Let H be an admissible Hermitian metric on a reflexive sheaf $\mathcal{E}$ defined on $\overline{B}$.

Theorem 2.19 ([4])**.** *For any holomorphic section s of $\mathcal{E}$, we have $\log^+|s|^2$ belongs to H^1_{loc}, and the following inequality holds in weak sense*

$$\Delta\log^+|s|^2\ge -2\frac{\langle\sqrt{-1}\Lambda_{\omega_0}Fs,s\rangle}{|s|^2}\ge -2|\sqrt{-1}\Lambda_{\omega_0}F|. \tag{2.13}$$

In particular, by Moser iteration, $|s|\in L^\infty_{loc}$. Moreover, if $|s|$ is in $L^2(B)$, then we have

$$|s|_{L^\infty(B_{1/2})}\le K_1|s|_{L^2(B)} \tag{2.14}$$

for K_1 depending on $|\sqrt{-1}\Lambda_{\omega_0}F|_{L^\infty(B)}$.

Now suppose $\mathcal{E}$ has an isolated singularity at 0. let H and H' be two admissible Hermitian metrics on $\mathcal{E}$, then $\mathrm{Tr}_H(H')$ and $\mathrm{Tr}_{H'}(H)$ are both the norms of the identity section of $End(\mathcal{E})$ with respect to the two admissible Hermitian metrics $H^*\otimes H'$ and $(H')^*\otimes H$ respectively. Applying (2.13) and Lemma 2.18 on the ball B_r with $r\to 0$ (notice (2.11) is scaling invariant), we obtain

$$\begin{aligned}&\limsup_{z\to 0}\frac{|\log\mathrm{Tr}_H(H')(z)|+|\log\mathrm{Tr}_{H'}(H)(z)|}{-\log|z|}\\ \le\ & C_0\limsup_{r\to 0} r^{1-2n}\int_{\partial B_r} r^2(|\Lambda_{\omega_0}F_H|+|\Lambda_{\omega_0}F_{H'}|).\end{aligned} \tag{2.15}$$

Notice by elementary means the left hand side bounds the ratio between the metrics H and H'. In particular, if both H and H' are Hermitian-Einstein with vanishing Einstein constant, then there is a constant $C > 0$ such that $C^{-1}H' \leq H \leq CH'$. This has been observed in [23].

For our later purposes we also need to deal with more general classes of Hermitian metrics which may not be admissible. The following Lemma makes it convenient to use Lemma 2.18.

Lemma 2.20. *Suppose $\mathcal{E}$ has an isolated singularity at 0, and H, H' be two smooth Hermitian metrics on $\mathcal{E}|_{\overline{B}^*}$ such that for some $\delta \in (0, 1]$,*

$$|F_H| + |F_{H'}| \in L^{1+\delta}(B).$$

Then

$$g^+ \in L^{\frac{n}{n-1}(1+\delta)}(B)$$

where g denotes either $\log Tr_H H'$ or $\log Tr_{H'} H$. In particular, by Lemma 2.18, if we further assume $r^2(|\Lambda F_H| + |\Lambda F_{H'}|) \in L^\infty(B^*)$, then (2.15) continues to hold.

Proof. The argument essentially follows from the proof of Theorem 2 in [4]. Fix any complex subspace $V \subset \mathbb{C}^n$ of dimension $n-2$, and denote by $p : B \to B \cap V$ the orthogonal projection. Let $\chi : \mathbb{C}^2 \to [0, 1]$ be a cut-off function which is equal to 1 for $|z| \leq 1/100$ and equal to zero for $|z| \geq 2/100$. For each $t \in V$ with $0 < |t| \leq 1/2$, χ defines a natural cut-off function on $p^{-1}(t)$. Since $p^{-1}(t)$ is a complex subspace, and $\mathcal{E}$ is a holomorphic vector bundle over $p^{-1}(t)$, we can apply the above discussion to $p^{-1}(t)$ and obtain

$$\Delta_t g \geq -C(|F_H| + |F_{H'}|),$$

where Δ_t is the Laplacian operator on $p^{-1}(t)$. Multiplying both sides by $\chi^2(g^+)^\delta$, and integrating by parts on $p^{-1}(t)$ we obtain (∇^t denotes the derivative on $p^{-1}(t)$)

$$\begin{aligned} & \int_{p^{-1}(t)} |\nabla^t(\chi(g^+)^{\frac{\delta+1}{2}})|^2 \\ \leq \; & C\Big(\int_{p^{-1}(t)} \chi^2(g^+)^\delta(|F_H| + |F_{H'}|) + \int_{p^{-1}(t)} |\nabla^t\chi|^2(g^+)^{1+\delta} \\ & + \int_{p^{-1}(t)} \chi|\nabla^t\chi|(g^+)^\delta|\nabla^t g^+|\Big), \end{aligned}$$

where the constant C depends on δ. Notice $\nabla^t\chi$ is supported outside the ball $|z| \leq 1/100$, and H and H' are both smooth away from zero, so the last two terms are uniformly bounded independent of t. For the first term on the right hand side we can use Young's inequality, and obtain for any $\epsilon > 0$, a number $C(\epsilon) > 0$ such that

$$\begin{aligned} & \int_{p^{-1}(t)} |\nabla^t(\chi(g^+)^{\frac{\delta+1}{2}})|^2 \\ \leq \; & \epsilon \int_{p^{-1}(t)} \chi^2(g^+)^{\delta+1} + C(\epsilon) \int_{p^{-1}(t)} \chi^2(|F_H| + |F_{H'}|)^{\delta+1} + C \end{aligned}$$

Using the Poincaré inequality on the unit ball in $\mathbb{C}^2$, and choosing ϵ sufficiently small, we conclude that

$$\int_{p^{-1}(t)} |\nabla^t(\chi(g^+)^{\frac{\delta+1}{2}})|^2 + \int_{p^{-1}(t)} \chi^2(g^+)^{\delta+1} \leq C(\epsilon) \int_{p^{-1}(t)} \chi^2(|F_H|+|F_{H'}|)^{\delta+1} + C.$$

Integrating this along V, and noticing that the inequality is uniform for all choices of complex subspaces V, one sees that $(g^+)^{\frac{\delta+1}{2}} \in W^{1,2}(B)$. Then by Sobolev embedding theorem, we get $g^+ \in L^{\frac{n}{n-1}(\delta+1)}(B)$. □

2.3 Analytic tangent Cones

We first describe some generalities on Hermitian-Yang-Mills cones. Let $\underline{A}$ be an admissible Hermitian-Yang-Mills connection on $(\underline{E}, \underline{H})$ over $(\mathbb{CP}^{n-1}, \omega_{FS})$ with singular set $\underline{\Sigma}$. Let $\underline{\mathcal{E}}$ be the corresponding reflexive sheaf on $\mathbb{CP}^{n-1}$, then the Einstein constant $\mu = \mu(\underline{\mathcal{E}})$. Let $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$ be the natural holomorphic projection, and denote $\Sigma = \pi^{-1}(\underline{\Sigma})$. On $\mathcal{E} := \pi^*\underline{\mathcal{E}}$ (since π is flat, we know $\mathcal{E}$ is reflexive), we consider the Hermitian metric $H := |z|^{2\mu}\pi^*\underline{H}$, and let A be the corresponding Chern connection, then it follows that

$$F_A = \pi^* F_{\underline{A}} + \sqrt{-1}\mu\pi^*\omega_{FS} \cdot \mathrm{Id}. \tag{2.16}$$

We first state a simple Lemma, whose proof follows easily from the fact that $(\mathbb{CP}^{n-1}, \omega_{FS})$ is the symplectic reduction of $(\mathbb{C}^n, \omega_0)$ under the natural S^1 action.

Lemma 2.21. *Let α be a two form on $\mathbb{CP}^{n-1}$, then $\Lambda_{\omega_0}\pi^*\alpha = |z|^{-2}\pi^*(\Lambda_{\omega_{FS}}\alpha)$.*

It follows from Lemma 2.21 and Equation (2.16) that A is an admissible Hermitian-Yang-Mills connection on $\mathcal{E}$ with singular set Σ and vanishing Einstein constant.

Definition 2.22. We call such a Hermitian-Yang-Mills connection $(\mathcal{E}, H, A)$ a *simple HYM cone*. When there is no confusion, sometimes we also use the notation $(\mathcal{E}, A)$ or simply A for brevity.

Remark 2.23. Strictly speaking, $\mathcal{E}$ is only defined on $\mathbb{C}^n \setminus \{0\}$, but by [4] we know $\mathcal{E}$ has a unique extension to $\mathbb{C}^n$ as a reflexive sheaf and the connection A can be viewed as an admissible Hermitian-Yang-Mills connection on the whole B. In our discussion in this paper (by abusing notation) we will not distinguish $\mathcal{E}$ and $\mathcal{E}|_{\mathbb{C}^n\setminus\{0\}}$.

Next we discuss the natural question that to what extent the connection A (on the sheaf $\mathcal{E}$ over $\mathbb{C}^n \setminus \{0\}$) determines $\underline{A}$ and $\underline{\mathcal{E}}$. Notice there is a standard S^1 action on $\mathbb{C}^n$, given by $e^{i\theta}.z = e^{i\theta}z$. Parallel transport along the S^1 orbit determines a smooth section P of the gauge group $\mathcal{G}$ of A. P can be naturally viewed as a section of $End(\mathcal{E})$, and using our definition it is easy to see that $P = e^{-2\pi\sqrt{-1}\mu}\mathrm{Id}$. It follows that μ is uniquely determined by A, modulo $\mathbb{Z}$. On the other hand, for any $m \in \mathbb{Z}$, let $\underline{A}(m)$ be the Chern-connection on $\mathcal{E} \otimes \mathcal{O}(m)$, where $\mathcal{O}(m)$ is endowed with the natural Hermitian metric whose Chern connection has curvature $-\sqrt{-1}m\omega_{FS}$, then it is easy to see that the Einstein constant of $\underline{A}(m)$ is $\mu_m = \mu + m$, and $A(m)$ also gives rise to a simple HYM cone which is isomorphic to A. On the underlying sheaf, this is just the obvious fact that $\pi^*\mathcal{O}(m)$ is trivial hence $\pi^*(\mathcal{E}\otimes\mathcal{O}(m))$ is isomorphic to $\pi^*\mathcal{E}$ and the metric then differs by a factor $|z|^{2m}$.

Now once we have chosen μ, we can then modify the Hermitian metric H to $H' := |z|^{-2\mu}H$, so that the corresponding new Chern connection A' has trivial holonomy around the S^1 orbit. Then by choosing local trivializations of $\mathcal{E}$ that is parallel along the $\mathbb{C}^*$ orbits we see that $(\mathcal{E}, A', H')$ descends naturally to an admissible Hermitian-Yang-Mills connection $\underline{A}$ on $\underline{\mathcal{E}}$ with Einstein constant μ. We summarize the above discussion into

Proposition 2.24. *A simple HYM cone A determines uniquely $(\underline{A}, \underline{\mathcal{E}})$, up to possibly tensoring with $\mathcal{O}(m)$ for some $m \in \mathbb{Z}$.*

For convenience we will simply call the matrix $e^{-2\pi\sqrt{-1}\mu}\mathrm{Id}$ the *holonomy of* A.

Remark 2.25. It follows that A is isomorphic to a pull-back connection from $\mathbb{CP}^{n-1}$ if and only if the holonomy is trivial. In general μ does not have to be an integer. For a simple example, we can take $\underline{\mathcal{E}}$ to be the tangent bundle of $\mathbb{CP}^{n-1}(n \geq 3)$. It is well-known that $\underline{\mathcal{E}}$ is stable, with the obvious Hermitian-Einstein metric, and $\mu = \frac{n}{n-1}$. The corresponding simple HYM cone would have non-trivial holonomy and hence can not be a pull-back connection.

Definition 2.26. A *HYM* cone is a direct sum of simple HYM cones.

Similar to the above discussion, we can uniquely write a HYM cone A as a direct sum of simple HYM cones $\bigoplus_j A_j$ such that each A_j has distinct holonomy $e^{-2\pi\sqrt{-1}\mu_j}$. We can similarly define the *holonomy of* A as an element of $(S^1)^k \subset U(k)$, where $k = rank(E)$. It is uniquely determined by its eigenvalues (with multiplicities). The underlying sheaf $\mathcal{E}$ is also isomorphic to $\bigoplus_j \pi^*\underline{\mathcal{E}}_j$ for reflexive sheaves $\underline{\mathcal{E}}_j$ over $\mathbb{CP}^{n-1}$, with $\mu_j = \mu(\mathcal{E}_j)$, and the corresponding Hermitian-Einstein metric on $\mathcal{E}$ can be written as $H = \bigoplus_j |z|^{2\mu_j}\pi^*\underline{H}_j$ for Hermitian-Einstein metrics $\underline{H}_j$ on $\underline{\mathcal{E}}_j$. So it is clear that $\mu_j \in (k!)^{-1}\mathbb{Z}$ for all j.

Next we give an intrinsic characterization of a HYM cone. This is also observed in [23].

Theorem 2.27. *Let A be an admissible Hermitian-Yang-Mills connection on $\mathbb{C}^n \setminus \{0\}$ with vanishing Einstein constant and with singular set Σ, then A is gauge equivalent to a HYM cone if and only if $\iota_{\partial_r}F_A = 0$ holds on $\mathbb{C}^n \setminus \Sigma$.*

Proof. The "only if" direction follows easily from definition, so it suffices to prove the "if" direction. Notice a priori we are not assuming Σ is $\mathbb{C}^*$-invariant. We let $\Sigma' = \{\lambda.x | x \in \Sigma, \lambda \in \mathbb{C}^*\}$, then since Σ is of complex codimension at least three, Σ' is of complex codimension at least two. We can use parallel transport along the S^1 orbit with respect to A to define a smooth section P of the gauge group $\mathcal{G}$ over $\mathbb{C}^n \setminus \Sigma'$. We claim P is covariantly constant, when viewed naturally as a section of $End(E)$. Notice this is a local property. To see this, we fix a point $z \in \mathbb{C}^n \setminus \{0\}$, and locally we can choose a trivialization of E under which we can write $d_A = d + A_0$ for a $\mathfrak{u}(l)$-valued 1-form A_0, where l is the rank of E. Modifying by an element of $\mathcal{G}$ we may assume locally around z that $A_0(\partial_r) = A_0(J\partial_r) = 0$. Notice since $\iota_{\partial_r}F_{\underline{A}} = 0$, and $F_A^{0,2} = 0$ we also have $\iota_{J\partial_r}F_A = 0$. It then follows from a direct computation that A_0 is invariant under the local $\mathbb{C}^*$ action, so we can write $A = \pi^* \underline{A}$ for a locally defined unitary connection $\underline{A}$ on $\mathbb{CP}^{n-1}$. It then follows easily that parallel transport along the $\mathbb{C}^*$ action orbit commutes with the co-variant derivative d_A, hence propagating along the S^1 orbit we obtain $d_A P = 0$.

Using P we obtain a parallel splitting of (E, A) over $\mathbb{C}^n \setminus \Sigma'$ into the direct sum of Hermitian-Yang-Mills connections. Since Σ' has complex codimension at least two, by [4] each direct summand extends to an admissible Hermitian-Yang-Mills connection on $\mathbb{C}^n$. Moreover, on each piece the holonomy P is given by multiplication by $e^{-2\pi\sqrt{-1}\mu}$ for some μ. Then we can follow the proof of Proposition 2.24 to conclude that each piece is indeed a simple HYM cone. It also follows from the above argument that Σ is indeed $\mathbb{C}^*$ invariant. □

Now we will apply the discussion above to our setting. We first recall known results on the convergence of a sequence of Hermitian-Yang-Mills connections with locally uniformly bounded Yang-Mills energy, adapted to our setting of getting analytic tangents cones. Let A be the admissible Hermitian-Yang-Mills connection on $\mathcal{E}$ over B^*. For any $\lambda \in (0, 1]$, we consider the rescaling map defined by

$$\lambda : B_{\lambda^{-1}} \to B; z \mapsto \lambda z$$

and denote

$$A_\lambda := \lambda^* A.$$

Given any subsequence $\lambda_i \to 0$, by Price's monotonicity formula [30] (see also Page 20, Remark 3 in [38]), for any $R > 0$, the sequence $\{A_{\lambda_i}\}_i$ has uniformly bounded Yang-Mills energy over $B_R \setminus \{0\}$. Then by Uhlenbeck's compactness result ([27, 38, 39]) after passing to a subsequence, we may assume $\{A_{\lambda_i}\}_i$ converges locally smoothly to A_∞ on $\mathbb{C}^n_* \setminus \Sigma$ modulo gauge transformations, where $\mathbb{C}^n_* = \mathbb{C}^n \setminus 0$ and Σ is a closed subset of $\mathbb{C}^n_*$ so that the Hausdorff $(2n-4)$ measure of $\Sigma \cap B_R$ is finite for any fixed $R > 0$. More explicitly, we have

$$\Sigma = \{z \in \mathbb{C}^n_* | \lim_{r\to 0} \liminf_{i\to\infty} r^{4-2n} \int_{B_r(z)} |F_{A_{\lambda_i}}|^2 \geq \epsilon_0\} \tag{2.17}$$

where $\epsilon_0 > 0$ denotes the constant in the ϵ-regularity theorem (see Equation (3.1.4) in [38]). We denote $\mathrm{Sing}(A_\infty)$ as the set of essential singularities of A_∞ on $\mathbb{C}^n_*$ i.e. where A_∞ can not be extended smoothly after a local gauge transform. Clearly $\mathrm{Sing}(A_\infty) \subset \Sigma$, but in general $\mathrm{Sing}(A_\infty)$ may be strictly smaller due to the removable singularities of A_∞. Passing to a further subsequence, we may assume that the sequence of Radon measures $\{\mu_i := |F_{A_{\lambda_i}}|^2 \mathrm{dVol}\}_i$ converge weakly to μ on $\mathbb{C}^n$. We define the triple (A_∞, Σ, μ) to an *analytic tangent cone* of A (associated to the chosen subsequence), and Σ is called the *analytic bubbling set*. For simplicity of notation, we denote

$$\lim_{i\to\infty} A_{\lambda_i} = (A_\infty, \Sigma, \mu).$$

By Fatou's lemma, there exists an nonnegative measure ν on $\mathbb{C}^n$ so that

$$\mu = |F_{A_\infty}|^2 \mathrm{dVol} + 8\pi^2 \nu.$$

By [38], $\mathrm{supp}(\nu) \setminus \{0\}$ is the *blow-up locus* Σ_b of the sequence $\{A_{\lambda_i}\}_i$ given as

$$\Sigma_b = \overline{\{x \in \mathbb{C}^n_* | \Theta(\mu, x) > 0, \lim_{r\to 0} r^{4-2n} \int_{B_r(x)} |F_{A_\infty}|^2 = 0\}}.$$

where $\Theta(\mu, x) := \lim_{r\to 0} r^{4-2n}\mu(B_r(x))$ is called the density function. It is easy to see that

$$\Sigma = \Sigma_b \cup \mathrm{Sing}(A_\infty). \tag{2.18}$$

The removable singularity theorem in [4] implies that A_∞ defines a reflexive sheaf $\mathcal{E}_\infty$ on $\mathbb{C}^n$, and we have

$$\mathrm{Sing}(A_\infty) = \mathrm{Sing}(\mathcal{E}_\infty) \setminus \{0\}.$$

In particular $\mathrm{Sing}(A_\infty)$ is a complex-analytic subvariety of $\mathbb{C}^n_*$. As a consequence of the monotonicity formula, Tian ([38], Lemma 5.3.1) proved that the connection A_∞ satisfying $\partial_r F_{A_\infty} = 0$, thus is a HYM cone by Theorem 2.27. Therefore $\mathrm{Sing}(A_\infty)$ is $\mathbb{C}^*$ invariant, which implies $\pi(\mathrm{Sing}(A_\infty))$ is an algebraic subvariety of $\mathbb{CP}^{n-1}$. Also the invariance of A_∞ implies that for any $r \in (0,1)$, the function

$$z \mapsto (|z|r)^{4-2n} \int_{B_{|z|r}(z)} |F_{A_\infty}|^2$$

is invariant under the natural $\mathbb{C}^*$ action on $\mathbb{C}^n_*$. By Theorem 4.3.3 in [38], we know[4] that Σ_b is also a complex-analytic subvariety of $\mathbb{C}^n_*$ of pure codimension two (see also Lemma 3.2.3 in [38]), with finitely many irreducible components Σ_k, and there are positive integers m_k such that the following current equation holds on $\mathbb{C}^n_*$

$$\lim_{i\to\infty} \frac{1}{8\pi^2} \mathrm{tr}(F_{A_{\lambda_i}} \wedge F_{A_{\lambda_i}}) = \frac{1}{8\pi^2} \mathrm{tr}(F_{A_\infty} \wedge F_{A_\infty}) + \sum m_k^{an}[\Sigma_k]. \tag{2.19}$$

In particular,

$$\nu = \sum_k m_k^{an}[\Sigma_k].$$

Later when talking about Σ_b, we always assume $\Sigma_b = \sum_k m_k^{an}\Sigma_k$ to include the multiplicities. (This will only be used in Section 3.3.) Again by Lemma 5.3.1 in [38], we know Σ_b is also radially invariant, hence it is also invariant under $\mathbb{C}^*$ action.

Summarizing the above we have

Lemma 2.28. *$\Sigma = \pi^{-1}(\underline{\Sigma})$ where $\underline{\Sigma}$ is a subvariety of $\mathbb{CP}^{n-1}$ of complex codimension at least* 2.

Now fix a smooth point $z \in \Sigma_k$, and let Δ be a *transverse slice* at z, i.e. Δ is a smooth complex two dimensional submanifold in B such that Δ is transversal to Σ_k. The following is proved in [36] (see Lemma 4.1) and the argument is purely local.

Lemma 2.29. *For Δ which is a transverse slice at a generic point $z \in \Sigma_k$, we have*

$$m_k^{an} = \lim_{i\to\infty} \frac{1}{8\pi^2} \int_\Delta \{\mathrm{tr}(F_{A_{\lambda_i}} \wedge F_{A_{\lambda_i}}) - \mathrm{tr}(F_{A_\infty} \wedge F_{A_\infty})\}. \tag{2.20}$$

[4]The proof in [38] is written in the case of compact manifolds, but as remarked in [38] (Remark 5), one only requires the boundedness of local Yang-Mills energy, which is valid in our case due to Price's monotonicty formula.

Remark 2.30. Lemma 2.29 holds for any irreducible component Σ_k which is not necessarily a component of Σ. Indeed, $m_k^{an} = 0$ in this case.

The radial invariance of tangent cones has a few easy consequences, which will be used frequently later.

Corollary 2.31. *Given any $z \in B \setminus \{0\}$ and $r < |z|$, we always have*

(a). $\lim_{i\to\infty} \mu_i(B_r(z)) = \mu(B_r(z))$*;*

(b). $\lim_{s\to r} \mu(B_s(z)) = \mu(B_r(z))$*;*

(c). $\lim_{i\to\infty} \mu_i(B_{r_i}(z_i)) = \mu(B_r(z))$*, for* $z_i \to z, r_i \to r$*.*

Proof. For (a), by general theory on convergence of Radon measures it suffices to show that $\mu(\partial B_r(z)) = 0$. Since $\Sigma = \pi^{-1}(\underline{\Sigma})$ where $\underline{\Sigma}$ is a complex subvarierty of real codimension 4 in $\mathbb{CP}^{n-1}$, $\Sigma \cap \partial B_{r_0}(z)$ is of Hausdorff codimension at least 5, hence we have $\mu(\partial B_r(z)) = 0$. Now for (b) we notice that Σ being radially invariant implies that

$$|\mu(B_s(z)) - \mu(B_r(z))| \leq |\int_{B_s(z)} |F_{A_\infty}|^2 - \int_{B_r(z)} |F_{A_\infty}|^2| + C|s-r|$$

for some fixed constant C. So (b) follows. For (c), fix $r < r' < |z|$ and for i large one has $B_{r_i}(z_i) \subset B_{r'}(z)$. This implies

$$\mu(B_{r'}(z)) = \mu(\overline{B_{r'}(z)}) \geq \limsup_{i\to\infty} \mu_i(B_{r_i}(z_i)).$$

By letting $r' \to r$, we have

$$\mu(B_r(z)) \geq \limsup_{i\to\infty} \mu_i(B_{r_i}(z_i)).$$

Similarly one can prove $\mu(B_r(z)) \leq \liminf_{i\to\infty} \mu_i(B_{r_i}(z_i))$. This finishes the proof. □

In our definition of analytic tangent cones we always need to pass to subsequences. For our later purpose we want to restrict to a particular discrete subsequence as $\lambda \to 0$. Namely, we define $\lambda_i := 2^{-i}$ and $A_i = \lambda_i^* A$. We say two analytic tangent cones are equivalent if they have the same bubbling set and the same analytic multiplicity of each irreducible Hausdorff codimension 4 component and the corresponding connections are gauge equivalent.

Corollary 2.32. *Any analytic tangent cone (A_∞, Σ, μ) is equivalent to an analytic tangent cone arising from the limit of a subsequence of $\{A_i\}_i$.*

Later, we need to talk about convergence of holomorphic section of $\mathcal{E}$ to holomorphic sections on an tangent cone, so here we first clarify the meaning of this. Suppose

$$\lim_{j_i} A_{j_i} = (A_\infty, \Sigma, \mu).$$

Then we can write $\mathcal{E}_\infty = \pi^* \underline{\mathcal{E}}_\infty$ and $\underline{\mathcal{E}}_\infty$ is a direct sum of stable reflexive sheaves on $\mathbb{CP}^{n-1}$

$$\underline{\mathcal{E}}_\infty = \oplus_l \underline{Q}_l.$$

Then

$$A_\infty = \bigoplus_l \pi^* \underline{A}_l + \mu_l \partial \ln |z|^2 \cdot \mathrm{Id}_{\pi^* \underline{Q}_l},$$

where $\underline{A}_l$ is the unique Hemitian-Yang-Mills connection on $\underline{Q}_l$. We let $\underline{H}_l$ denote the Hermitian-Einstein metric on $\underline{Q}_l$. Then A_∞ is the Chern connection on $(\mathcal{E}_\infty, H_\infty)$ where $H_\infty := \oplus_l |z|^{2\mu(\underline{Q}_l)} \pi^* \underline{H}_l$. Fix a smooth Hermitian $\underline{H}'$ on $\underline{\mathcal{E}}$, and let $H' = \pi^* \underline{H}'$. Let $H_i = (2^{-i})^* H$ and $f_i = (H'^{-1} H_i)^{\frac{1}{2}}$ be the complex gauge transform (note f_i is Hermitian with respect to H'). Here H is the unknown Hermitian-Einstein metric on $\mathcal{E}$. Let A_i be the Chern connection given by the hermitian metric H' and the holomorphic structure $f_i \cdot \bar{\partial}_{\mathcal{E}} := f_i \circ \bar{\partial}_{\mathcal{E}} \circ f_i^{-1}$. Then $\{A_i\}_i$ is a sequence of Hermitian-Yang-Mills metrics on a fixed unitary vector bundle. Then there exists a unitary gauge isomorphism

$$P : (\mathcal{E}, H') \to (\mathcal{E}_\infty, H_\infty)$$

outside Σ and a sequence of unitary gauge transform $\{g_{j_i}\}_i$ of $(\mathcal{E}, H')$ defined outside Σ so that $\{g_{j_i} \cdot A_{j_i}\}_i$ converges to $P^* A_\infty$ smoothly outside Σ.

Now given a sequence of holomorphic sections $\{\sigma_i\}$ of $\mathcal{E}$ over B^*, we know $f_i(\sigma_i)$ is a holomorphic section of $(\mathcal{E}, f_i(\bar{\partial}_{\mathcal{E}}))$. We say $\{\sigma_i\}$ converges to a holomorphic section σ_∞ of $\mathcal{E}_\infty$, if $g_{j_i} f_i(\sigma_{j_i})$ converges smoothly to $P^{-1}\sigma_\infty$ locally away from Σ. Since $g_{j_i} \cdot f_i(A_{j_i})$ converges to $P^* A_\infty$ outside Σ, by the elliptic regularity of $\bar{\partial}$-operator, we know that for any sequence of holomorphic sections $\{\sigma_i\}_i$ which are normalized suitably, by passing to subsequences, we can always obtain limit holomorphic sections of $\mathcal{E}_\infty$ in the above sense. However, the limit is not a priori nontrivial. This would rely on the convexity result that we are going to discuss.

2.4 Cut-off

We first make a few conventions. We say a subset E of an open (or closed) annulus A is *symmetric* if for any $z \in E$, then

$$\mathbb{C}^*.z \cap A \subset E.$$

For any subset $E \subset \overline{B}_{2^{-1}} \setminus B_{2^{-2}}$, we define its *symmetrization* to be the smallest symmetric subset that contains E i.e. the set $\pi^{-1}(\pi(E)) \cap (\overline{B}_{2^{-1}} \setminus B_{2^{-2}})$. Below we shall discuss convergence of compact subsets of $\overline{B}$, and it will always be with respect to the Hausdorff distance on the space of all compact subsets of $\overline{B}$.

For any $r \in (0, 10^{-3}]$ and integer $j \geq 1$, we define E_j^r to be symmetrization of the closed set

$$\{z \in \overline{B}_{2^{-1}} \setminus B_{2^{-2}} : (|z|r)^{4-2n} \int_{B_{|z|r}(z)} |F_{A_j}|^2 \geq \frac{\epsilon_0}{2}\}.$$

Given a tangent cone (A_∞, Σ, μ), we define a symmetric set

$$N^r(A_\infty, \Sigma, \mu) := \{z \in \overline{B} \setminus B_{2^{-3}} : (|z|r)^{4-2n} \mu(B_{|z|r}(z)) \geq \frac{\epsilon_0}{2}\}.$$

Furthermore, from the definition of Σ we see that for any $r > 0$,

$$\Sigma \cap (\overline{B} \setminus B_{2^{-3}}) \subset N^r(A_\infty, \Sigma, \mu).$$

For notational convenience, we will sometimes simply denote $N^r(A_\infty, \Sigma, \mu)$ by N^r if the relevant tangent cone is clear from the context. Given a subsequence $\{A_{j_i}\}_i$ converging to (A_∞, Σ, μ), we denote

$$\Sigma^r_{j_i} := 2E^r_{j_i-1} \cup E^r_{j_i} \cup 2^{-1}E^r_{j_i+1}.$$

Now we are ready to state the main theorem of this section.

Theorem 2.33. *There exists $r_0 \in (0, 10^{-3})$ such that for any $r \in (0, r_0]$, and for any given tangent cone $(A_\infty, \Sigma, \mu) = \lim_{i\to\infty} A_{j_i}$ the following holds*

(I). Suppose V_1 and V_2 are limits of $E^r_{j_i}$ and $E^r_{j_i+1}$ respectively, then

$$m(V_1 \setminus V_2) = m(V_2 \setminus V_1) = 0$$

where $m(\cdot)$ denotes the Lebesgue measure on $\mathbb{C}^n$;

(II). $N^{\frac{r}{2}} \subset \Sigma^r_{j_i} \subset N^{2r}$ for i large. Moreover,

$$d((B \setminus \overline{B}_{2^{-3}}) \setminus N^{2r}, N^{\frac{r}{2}}) > 0,$$

$$\liminf_i d(B \setminus \overline{B}_{2^{-3}}) \setminus N^{2r}, \Sigma^r_{j_i}) > 0,$$

and

$$d((B \setminus \overline{B}_{2^{-3}}) \setminus N^{\frac{r}{2}}, \Sigma) > 0;$$

(III). There exists a constant $C = C(r) > 0$ so that for any $z \in \overline{(\overline{B}_{2^{-1}} \setminus B_{2^{-2}}) \setminus N^{\frac{r}{2}}}$, there exists a flat holomorphic disk $D_z \subset B_{\frac{3}{4}} \setminus \overline{B}_{\frac{3}{16}}$ such that $D_z \cap \Sigma = \emptyset$ and $\partial D_z \subset (B_{\frac{3}{4}} \setminus \overline{B}_{\frac{3}{16}}) \setminus N^{2r}$ and $d(D_z, \Sigma) \geq C > 0$.

2.4.1 Proof of Theorem 2.33 (I).

Given a tangent cone (A_∞, Σ, μ), we consider the following function

$$f : B_{2^{-1}} \setminus \overline{B_{2^{-2}}} \times (0, 10^{-3}) \to \mathbb{R}_+, (z, r) \mapsto (|z|r)^{4-2n}\mu(B_{|z|r}(z)).$$

Lemma 2.34. $f(z, r) = \frac{\epsilon_0}{2}$ *if and only if* $\overline{B_{|z|r}(z)} \cap \Sigma = \emptyset$ *and*

$$(|z|r)^{4-2n} \int_{B_{|z|r}(z)} |F_{A_\infty}|^2 = \frac{\epsilon_0}{2}.$$

Proof. The *if* part follows directly from the definition. For the *only if* part, suppose $\lim_i A_{j_i} = (A_\infty, \Sigma, \mu)$. If $f(z, r) = \frac{\epsilon_0}{2}$, by Corollary 2.31, there exists $r' > r$ such that

$$f(z, r') \leq \frac{3\epsilon_0}{4},$$

hence for i large

$$(|z|r')^{4-2n} \int_{B_{|z|r'}(z)} |F_{A_{j_i}}|^2 \leq \frac{5}{6}\epsilon_0.$$

By the choice of ϵ_0, $\{A_{j_i}\}_i$ converge to A_∞ smoothly over $B_{|z|\frac{r'+r}{2}}(z)$ and $B_{|z|\frac{r+r'}{2}}(z) \cap \Sigma = \emptyset$. As a result,

$$(|z|r)^{4-2n}\mu(B_{|z|r}(z)) = (|z|r)^{4-2n} \int_{B_{|z|r}(z)} |F_{A_\infty}|^2 = \frac{\epsilon_0}{2},$$

and $\overline{B_{|z|r}(z)} \cap \Sigma = \emptyset$. This finishes the proof. □

Remark 2.35. The conclusion also holds if we replace $\frac{\epsilon_0}{2}$ by any $c < \epsilon_0$.

Lemma 2.36. *For any fixed $r \in (0, 10^{-3}]$, the set*

$$\{z \in B_{2^{-1}} \setminus \overline{B_{2^{-2}}} : f(z,r) = \frac{\epsilon_0}{2}\}$$

is a symmetric real analytic subvariety of $B_{2^{-1}} \setminus \overline{B_{2^{-2}}}$, which is proper if Σ is non-empty. In particular, if $\Sigma \neq \emptyset$, then

$$m(\{z \in B_{2^{-1}} \setminus \overline{B_{2^{-2}}} : f(z,r) = \frac{\epsilon_0}{2}\}) = 0.$$

Proof. Locally near any smooth point, under a holomorphic frame, the Hermitian-Einstein metric h_∞ on $\mathcal{E}_\infty$ satisfies the following elliptic equation

$$P(h_\infty) := \sqrt{-1}\Lambda_{\omega_0}\bar{\partial}(h_\infty^{-1}\partial h_\infty) = 0.$$

Since the coefficients of P are real analytic in z, it follows from Theorem 41 on page 467 in [1] that h_∞ is also real analytic in z. Therefore, the function

$$Q : \{z \in B \setminus \{0\} : B_{|z|r}(z) \cap \Sigma = \emptyset\} \to \mathbb{R}; z \mapsto (|z|r)^{4-2n} \int_{B_{|z|r}(z)} |F_{A_\infty}|^2$$

is real analytic. Now by Lemma 2.34, we know

$$\begin{aligned}&\{z \in B_{2^{-1}} \setminus \overline{B}_{2^{-2}} : f(z,r) = \frac{\epsilon_0}{2}\}\\ =&\{z \in B_{2^{-1}} \setminus \overline{B}_{2^{-2}} : (|z|r)^{4-2n} \int_{B_{|z|r}(z)} |F_{A_\infty}|^2 = \frac{\epsilon_0}{2}, B_{|z|r}(z) \cap \Sigma = \emptyset\}.\end{aligned}$$

This easily implies $\{z \in B_{2^{-1}} \setminus \overline{B_{2^{-2}}} : f(z,r) = \frac{\epsilon_0}{2}\}$ is a symmetric real analytic subvarierty of $B_{2^{-1}} \setminus \overline{B}_{2^{-2}}$. The last statement follows from well-known facts about the zero set of a real analytic function (see for example [3]). □

Remark 2.37. It also follows from the proof that for any fixed z such that $B_{|z|r}(z) \cap \Sigma = \emptyset$, the function

$$s \mapsto (|z|s)^{4-2n} \int_{B_{|z|s}(z)} |F_{A_\infty}|^2$$

is real analytic in $(0, r)$. Then given any constant C, the set

$$\{s \in (0,r) : (|z|s)^{4-2n} \int_{B_{|z|s}(z)} |F_{A_\infty}|^2 = C\}$$

is either equal to $(0, r)$ or consists of finitely many points.

Proposition 2.38. *There exists $r_0' \in (0, 10^{-3})$ such that for any $r \in (0, r_0')$ and any tangent cone (A_∞, Σ, μ),*

$$m(\{z \in B_{2^{-1}} \setminus \overline{B_{2^{-2}}} : f(z,r) = \frac{\epsilon_0}{2}\}) = 0.$$

Proof. Otherwise, by Lemma 2.36, we can find a sequence $r_i \to 0$ and for each r_i there exists a tangent cone $(A_\infty(i), \Sigma(i), \mu(i))$ with $\Sigma(i) = \emptyset$ and

$$\{z \in B_{2^{-1}} \setminus \overline{B_{2^{-2}}} : f_i(z, r_i) = \frac{\epsilon_0}{2}\} = B_{2^{-1}} \setminus \overline{B_{2^{-2}}}.$$

Taking limits, we obtain (A_∞, Σ, μ) with $B_{2^{-1}} \setminus \overline{B_{2^{-2}}} \subset \Sigma$, which is impossible. This is a contradiction. □

Now we finish the proof of (I) for all $r \in (0, r_0']$.

Proof of (I). We claim

$$(V_1 \setminus V_2) \cup (V_2 \setminus V_1) \subset \{z \in \overline{B}_{2^{-1}} \setminus B_{2^{-2}} : f(z, r) = \frac{\epsilon_0}{2}\}.$$

Given this claim, by Proposition 2.38, we have $m(V_1 \setminus V_2) = m(V_2 \setminus V_1) = 0$. We only prove the claim for $V_1 \setminus V_2$ and the proof for $V_2 \setminus V_1$ is the same. Given any $z \in V_1 \setminus V_2$, we need to show $f(z, r) = \frac{\epsilon_0}{2}$. By passing to a subsqequence, there exists a sequence of points $z_i \in E^r_{j_i} \setminus E^r_{j_i+1}$ converging to z. By definition, for each z_i, there exists $y_i \in \overline{B}_{2^{-1}} \setminus B_{2^{-2}}$ with $\pi(z_i) = \pi(y_i)$ satisfying

$$(|y_i|r)^{4-2n} \int_{B_{|y_i|r}(y_i)} |F_{A_{j_i}}|^2 \geq \frac{\epsilon_0}{2}$$

but

$$(|\frac{y_i}{2}|r)^{4-2n} \int_{B_{|\frac{y_i}{2}|r}(\frac{y_i}{2})} |F_{A_{j_i}}|^2 < \frac{\epsilon_0}{2}.$$

By passing to a subsequence, we can assume $\{y_i\}_i$ converge to $y \in \overline{B}_{2^{-1}} \setminus B_{2^{-2}}$ with $\pi(y) = \pi(z)$. By Corollary 2.31, we have

$$(|y|r)^{4-2n} \mu(B_{|y|r}(y)) \geq \frac{\epsilon_0}{2}$$

and

$$(|\frac{y}{2}|r)^{4-2n} \mu(B_{|\frac{y}{2}|r}(\frac{y}{2})) \leq \frac{\epsilon_0}{2}.$$

By Corollary 2.31, we have

$$(|z|r)^{4-2n} \mu(B_{|z|r}(z)) = (|y|r)^{4-2n} \mu(B_{|y|r}(y)) = (|\frac{y}{2}|r)^{4-2n} \mu(B_{|\frac{y}{2}|r}(\frac{y}{2})) = \frac{\epsilon_0}{2}.$$

This finishes the proof. □

2.4.2 Proof of Theorem 2.33 (II).

Again suppose we are given a tangent cone (A_∞, Σ, μ).

Lemma 2.39. *Suppose for some $0 < r_1 < r_2 < 10^{-3}$ and $z \in B \setminus \{0\}$ we have*

$$(|z|r_1)^{4-2n} \mu(B_{|z|r_1}(z)) = (|z|r_2)^{4-2n} \mu(B_{|z|r_2}(z)),$$

then exactly one of the following holds:

- $(|z|r_2)^{4-2n} \mu(B_{|z|r_2}(z)) \geq \epsilon_0$*;*

- $(|z|r)^{4-2n}\mu(B_{|z|r}(z)) \equiv 0$ *for any* $r \le r_2$.

Proof. Suppose $(|z|r_2)^{4-2n}\mu(B_{|z|r_2}(z)) < \epsilon_0$, then by Remark 2.35 we have $\overline{B_{|z|r_2}(z)} \cap \Sigma = \emptyset$. So on $B_{|z|r_2}(z)$, A_∞ is smooth and $\mu = |F_{A_\infty}|^2 dVol$. By Price's monotonicity formula, under the above assumption, the following function

$$(|z|s)^{4-2n}\int_{B_{|z|s}(z)} |F_{A_\infty}|^2$$

is constant on $[r_1, r_2]$. So by Remark 2.37, it is constant on $(0, r_2]$. The conclusion follows by letting s tend to zero.

□

Proof of (II). Suppose $\{A_{j_i}\}_i$ converges to (A_∞, Σ, μ). We first show the inclusion $N^{\frac{r}{2}} \subset \Sigma^r_{j_i}$ for i large. Otherwise, by passing to a subsequence, there exists a sequence of points $z_{j_i} \in N^{\frac{r}{2}} \setminus \Sigma^r_{j_i}$ and z_{j_i} converges to $z \in N^{\frac{r}{2}}$. In particular,

$$(|z|\frac{r}{2})^{4-2n}\mu(B_{|z|\frac{r}{2}}(z)) \ge \frac{\epsilon_0}{2}.$$

Since $z_{j_i} \notin \Sigma^r_{j_i}$, by Corollary 2.31, we must have

$$(|z|r)^{4-2n}\mu(B_{|z|r}(z)) \le \frac{\epsilon_0}{2}.$$

By Price's monotonicity formula (see Equation (5.3.4) in [38]), we have

$$(|z|\frac{r}{2})^{4-2n}\mu(B_{|z|\frac{r}{2}}(z)) = (|z|r)^{4-2n}\mu(B_{|z|r}(z)) = \frac{\epsilon_0}{2}.$$

However, by Lemma 2.39, this is impossible. Similarly, one can get the other statements in (II) by the same argument. This finishes the proof. □

2.4.3 Proof of Theorem 2.33 (III).

Given a point $p \in \mathbb{C}^n \setminus \{0\}$, we can choose a $n-2$ dimensional complex linear subspace $\mathbb{C}^{n-2}_p \subset \mathbb{C}^n$ that contains p. Then using the flat metric on $\mathbb{C}^n$, we can identify $\mathbb{C}^n$ with an orthogonal product $\mathbb{C}^2 \times \mathbb{C}^{n-2}_p$ at p.

Definition 2.40. We say a closed subset $S \subset \overline{B}$ admits a *good cover* if $S \cap (\overline{B}_{2^{-1}} \setminus B_{2^{-2}})$ can be covered by finitely many open sets $U_k \subset B_{\frac{3}{4}} \setminus \overline{B_{\frac{3}{16}}}$ such that

- $U_k = B^2_{\delta^k_2} \times B^{n-2}_{\delta^k_3} \subset \mathbb{C}^2 \times \mathbb{C}^{n-2}_{p_k}$ for some point $p_k \in \mathbb{C}^n \setminus \{0\}$, some choice of $\mathbb{C}^{n-2}_{p_k}$, and some $\delta^k_2, \delta^k_3 > 0$, where $B^2_{\delta^k_2}$ denotes the ball $\{|z| < \delta\}$ in $\mathbb{C}^2$ and $B^2_{\delta^k_3}$ denote the ball of radius δ^k_3 centered at p_k in $\mathbb{C}^{n-2}_{p_k}$;

- $\emptyset \ne \overline{U_k} \cap S \subset V_k = B^2_{\delta^k_1} \times \overline{B}^{n-2}_{\delta^k_3}$ for some $\delta^k_1 \in (0, \delta^k_2)$;

Lemma 2.41. *For any tangent cone* (A_∞, Σ, μ), Σ *admits a good cover.*

Proof. By Lemma 2.28, we know Σ is a codimension 2 complex subvarierty of $B\backslash 0$. Then given any $p \in \Sigma \cap (\overline{B_{2^{-1}}} \backslash B_{2^{-2}})$, for a generic orthogonal projection ρ_p to some $\mathbb{C}^{n-2}_p$ at p, $\rho_p^{-1}(y) \cap \Sigma$ consists of finitely many points for any $y \in B^{n-2}_{\delta^p_3}$

for some $\delta_3^p > 0$. Then near p, one can easily construct a neighborhood U_p of p so that $U_p = B^2_{\delta_2^p} \times B^{n-2}_{\delta_3^p} \subset B_{\frac{3}{4}} \setminus \overline{B_{\frac{3}{16}}}$ for some $\delta_2^p, \delta_3^p > 0$ and $\overline{U_p} \cap \Sigma \subset V_p$ where $V_p = B^2_{\delta_1^p} \times \overline{B^{n-2}_{\delta_3^p}}$ for some $\delta_1^p \in (0, \delta_2^p)$. Now we get an open cover of $\Sigma \cap (\overline{B_{2^{-1}}} \setminus B_{2^{-2}})$ given by $\cup_p U_p$. Since $\Sigma \cap (\overline{B_{2^{-1}}} \setminus B_{2^{-2}})$ is compact, one can get a finite subcover $\cup_k U_{p_k}$. □

Remark 2.42. In Lemma 2.41, we only need that Σ has locally finite Hausdorff $(2n-4)$ measure.

Proposition 2.43. *There exists $r_0 \in (0, r_0']$ such that for any $r \in (0, r_0]$, N^{2r} admits a good cover for all tangent cones (A_∞, Σ, μ).*

Proof. Otherwise, there exists a subsequence $r_i \searrow 0$ such that for each r_i there exists N^{2r_i} for some tangent cone $(A_\infty(i), \Sigma(i), \mu(i))$ which does not admit a good cover. By using part (II) of Theorem 2.33, for each i, there exists A_{j_i} so that $N^{2r_i} \subset \Sigma^{4r_i}_{j_i}$. By passing to a subsequence, we can assume $\{A_{j_i}\}_i$ converge to some tangent cone (A_∞, Σ, μ) and $\Sigma^{4r_i}_{j_i}$ converges to a closed subset of Σ. In particular, $\{N^{2r_i}\}_i$ converges to a closed subset of Σ. By Lemma 2.41, Σ admits a good cover and we let $\cup_k U_k$ be the corresponding finite cover. Now we conclude that for i large, $\cup_k U_k$ is also a good cover of N^{2r_i}, which is a contradiction. It suffices to verify the following

- $N^{2r_i} \cap (\overline{B_{2^{-1}}} \setminus B_{2^{-2}}) \subset \cup_k U_k$ for i large. This is obvious since $\cup_k U_k$ is an open cover of $\Sigma \cap \overline{B_{2^{-1}}} \setminus B_{2^{-2}}$ and $\{N^{2r_i} \cap (\overline{B_{2^{-1}}} \setminus B_{2^{-2}})\}_i$ converge to a closed subset of $\Sigma \cap \overline{B_{2^{-1}}} \setminus B_{2^{-2}}$.
- $\overline{U_k} \cap N^{2r_i} \subset V_k$ for i large. Otherwise, by passing to a subsequence and using the finiteness of $\{U_k\}_k$, we can assume for some fixed k, there always exists $z_i \in (\overline{U_k} \cap N^{2r_i}) \setminus V_k$ for each i and z_i converges to $z \in \overline{U_k} \cap \Sigma$. Then $z \in V_k$ and thus $z_i \in V_k$ for i large. Contradiction.

□

As a direct corollary, we are ready to prove (III) for all $r \in (0, r_0]$.

Proof of (III). By Proposition 2.43, N^{2r} admits a good cover and let $\{U_k\}_k$ be the corresponding cover. So for each k

$$\overline{U_k} \cap \Sigma \subset \overline{U_k} \cap N^{\frac{r}{2}} \subset \overline{U_k} \cap N^{2r} \subset V_k$$

where $\overline{U}_k = \overline{B^2_{\delta_2^k}} \times \overline{B^{n-2}_{\delta_3^k}} \subset \mathbb{C}^2 \times \mathbb{C}^{n-2}_{p_k}$ and $V_k = B^2_{\delta_1^k} \times \overline{B^{n-2}_{\delta_3^k}}$. Consider the projection $\rho_k : \mathbb{C}^2 \times \mathbb{C}^{n-2}_{p_k} \to \mathbb{C}^{n-2}_{p_k}$. By assumption, we have

$$(B^2_{\delta_2^k} \times \overline{B^{n-2}_{\delta_3^k}}) \cap \Sigma = \overline{U_k} \cap \Sigma \subset B^2_{\delta_1^k} \times \overline{B^{n-2}_{\delta_3^k}}$$

which implies $\rho_k^{-1}(y) \cap \Sigma \cap B^2_{\delta_2}$ is a compact complex analytic subvariety of $B^2_{\delta_2}$ and thus consists of finitely points for any $y \in \overline{B^{n-2}_{\delta_3}}$. For any $z \in \overline{N^{2r} \setminus N^{\frac{r}{2}}} \cap (\overline{B}_{2^{-1}} \setminus B_{2^{-2}})$, suppose $z \in U_k$, then $\rho_k^{-1}(\rho_k(z)) \cap \Sigma \cap B^2_{\delta_2}$ consists of finitely many points which lie in $B^2_{\delta_1}$. As a result, one can easily find a flat holomorphic disk $D_z \subset U_k$ containing z such that $D_z \cap \Sigma = \emptyset$ and $\partial D_z \subset U_k \setminus V_k \subset (B_{\frac{3}{4}} \setminus \overline{B}_{\frac{3}{16}}) \setminus N^{2r}$. By perturbing the disk D_z, we can find an open neighborhood V_z of z so that for each $z' \in V_z$ there exists a flat holomorphic disk $D_{z'} \subset B \setminus \overline{B}_{2^{-3}}$ so that $D_{z'} \cap \Sigma = \emptyset$ and $\partial D_z \subset (B_{\frac{3}{4}} \setminus \overline{B}_{\frac{3}{16}}) \setminus N^{2r}$. Furthermore, $\inf_{z' \in V_z} d(D_{z'}, \Sigma) > 0$. For $z \in (\overline{B}_{2^{-1}} \setminus B_{2^{-2}}) \setminus N^{2r}$, it is obvious that one can do the same thing as above. As a result, we get an open cover $\cup_{z \in \overline{(\overline{B}_{2^{-1}} \setminus B_{2^{-2}}) \setminus N^{\frac{r}{2}}}} V_z$ of $\overline{(\overline{B}_{2^{-1}} \setminus B_{2^{-2}}) \setminus N^{\frac{r}{2}}}$. Since $\overline{(\overline{B}_{2^{-1}} \setminus B_{2^{-2}}) \setminus N^{\frac{r}{2}}}$ is compact, we can find a finite subcover $\cup_{z_i} V_{z_i}$. Let $C(r) = \min_i \inf_{z \in V_{z_i}} d(D_{z_i}, \sigma)$. This finishes the proof. □

2.5 Convexity

Given a tangent cone (A_∞, Σ, μ), suppose W is a symmetric open subset of $(B \setminus \overline{B_{2^{-3}}}) \setminus \Sigma$. We say a non-zero holomorphic section s of $\mathcal{E}_\infty$ over W is *homogeneous* of degree d if

$$\nabla_{\partial_r} s = dr^{-1} s.$$

Since s is holomorphic, this is equivalent to $\nabla_{J\partial_r} s = \sqrt{-1} dr^{-1} s$. If $s(z) \neq 0$ for some $z \in W$, then $s(z)$ is an eigenvector of the holonomy of the connection A_∞ (as defined in Section 2.3), and in particular we have

$$d \in ((\operatorname{rank} \mathcal{E}_\infty)!)^{-1} \mathbb{Z}.$$

Lemma 2.44. *Given a holomorphic section s of $\mathcal{E}$ over W with $\int_W |s|^2 < \infty$, we have an orthogonal decomposition*

$$s = \sum_{d \in \Gamma} s_d,$$

where each s_d is homogeneous of degree d, and the convergence is understood as in $L^2(W)$ and $C^\infty_{loc}(W)$.

Proof. First of all suppose $(\mathcal{E}, A)$ is the direct sum of simple HYM cones $(\mathcal{E}_j, A_j)$, then on W we can naturally write $s = \sum s_j$, where s_j is a holomorphic section of $\mathcal{E}_j|_W$. Clearly, this is an L^2 orthogonal decomposition. Therefore, without loss of generality we may assume A is a simple HYM cone. Suppose $\mathcal{E} = \pi^* \underline{\mathcal{E}}$ and $H = |z|^{2\mu} \pi^* \underline{H}$. Then locally choose a small open set $U \subset \mathbb{CP}^{n-1}$ such that $\underline{\mathcal{E}}|_U$ is free and admits a trivialization by holomorphic sections $\underline{s}_j (j = 1, \cdots, m := \operatorname{rank}(\mathcal{E}))$. On $\pi^{-1}(U) \cap W$, we can write $s = \sum_{j=1}^m f_j(z) \pi^* \underline{s}_j$ for some holomorphic functions f_j on $\pi^{-1}(U) \cap W$. We can perform Taylor series expansion and write $f_j = \sum f_{j,e}$, where each $f_{j,e}$ is homogeneous of degree e under the $\mathbb{C}^*$ action. So on $\pi^{-1}(U) \cap W$ we have an expansion $s = \sum s_d$ into direct sum of homogeneous sections, which is L^2 orthogonal over $\pi^{-1}(V)$ for any $V \subset U \cap \pi(W)$. In particular, such an expansion is independent of the choice of the local trivialization $\{\underline{s}_j\}$. This implies each s_d is indeed a global holomorphic section on W. □

Proposition 2.45. *Suppose $s \in H^0(W, \mathcal{E}_\infty)$ with $\int_W |s|^2 < \infty$, then*

$$\left(\int_{W \cap (B_{2^{-1}} \setminus \overline{B_{2^{-2}}})} |s|^2\right)^2 \leq \int_{W \cap (B_{2^{-2}} \setminus \overline{B_{2^{-3}}})} |s|^2 \cdot \int_{W \cap (B \setminus \overline{B_{2^{-1}}})} |s|^2.$$

Furthermore, if s is non-zero and the equality holds, then s must be homogeneous.

Proof. Write $s = \sum_d s_d$ where s_d is a homogeneous section of $\mathcal{E}_\infty$ over W of degree $d \in ((\operatorname{rank} \mathcal{E}_\infty)!)^{-1}\mathbb{Z}$. Then we have

$$\int_{W \cap (B \setminus \overline{B_{2^{-1}}})} |s|^2 = \sum_d \int_{W \cap (B \setminus \overline{B_{2^{-1}}})} |s_d|^2,$$

and

$$\int_{W \cap (B_{2^{-1}} \setminus \overline{B_{2^{-2}}})} |s|^2 = \sum_d 2^{-2d-2n} \int_{W \cap (B \setminus \overline{B_{2^{-1}}})} |s_d|^2,$$

and

$$\int_{W \cap (B_{2^{-2}} \setminus \overline{B_{2^{-3}}})} |s|^2 = \sum_d 2^{-4d-4n} \int_{W \cap (B \setminus \overline{B_{2^{-1}}})} |s_d|^2.$$

Now the conclusion follows from the general Cauchy-Schwartz inequality. □

Now given a saturated subsheaf $\mathcal{F} \subset \mathcal{E}$, we denote by $\pi_{\mathcal{F}} : \mathcal{E} \to \mathcal{F}$ the pointwise orthogonal projection with respect to the admissible Hermitian-Einstein metric H and let $\pi_{\mathcal{F}}^{\perp} = \mathrm{Id} - \pi_{\mathcal{F}}$. Note $\pi_{\mathcal{F}}$ is only defined away from $\mathrm{Sing}(\mathcal{E}/\mathcal{F})$. In the following we shall work under the following hypothesis, and in our later application this hypothesis will always be verified.

** Given any subsequence $\{j_i\}$, by passing to a further subsequence, $\{A_{j_i}\}_i$ converges to a tangent cone (A_∞, Σ, μ), and the corresponding pull-backs of $\pi_{\mathcal{F}}$ under the map $z \mapsto 2^{-j_i} z$ converges locally smoothly to a projection map π_∞ on $\mathcal{E}_\infty$ away from Σ. Furthermore, π_∞ is exactly the orthogonal projection onto a HYM cone direct summand $\mathcal{F}_\infty \subset \mathcal{E}_\infty$ (away from $\mathrm{Sing}(\mathcal{E}_\infty/\mathcal{F}_\infty)$).

Given any fixed $r \in (0, r_0]$, for any *smooth* section σ of $\mathcal{E}$ over $(B_{2^{-j-1}} \setminus \overline{B_{2^{-j-2}}}) \setminus 2^{-j} E_j^r$, let

$$\|\sigma\|_j^r = 2^{jn} \Big(\int_{(B_{2^{-j-1}} \setminus \overline{B_{2^{-j-2}}}) \setminus 2^{-j} E_j^r} |\sigma|^2 \Big)^{\frac{1}{2}}. \tag{2.21}$$

Proposition 2.46. *Given any $r \in (0, r_0]$ and $\lambda \notin ((rank\mathcal{E})!)^{-1}\mathbb{Z}$, there exists $j_0 = j_0(r, \lambda)$ such that for all $j \geq j_0$, if $s \in H^0(B_{j-1}, \mathcal{E})$ satisfies*

$$\|\pi_{\mathcal{F}}^{\perp} s\|_j^r > 2^{-\lambda} \|\pi_{\mathcal{F}}^{\perp} s\|_{j-1}^r,$$

then

$$\|\pi_{\mathcal{F}}^{\perp} s\|_{j+1}^r \geq 2^{-\lambda} \|\pi_{\mathcal{F}}^{\perp} s\|_j^r.$$

Proof. Otherwise, there exists a sequence of holomorphic sections $s_{j_i} \in H^0(B_{j_i-1}, \mathcal{E})$ so that

$$\|\pi_{\mathcal{F}}^{\perp} s_{j_i}\|_{j_i}^r = 1,$$

and

$$\|\pi_{\mathcal{F}}^{\perp} s_{j_i}\|_{j_i-1}^r < 2^{\lambda},$$

but

$$\|\pi_{\mathcal{F}}^{\perp} s_{j_i}\|_{j_i+1}^r < 2^{-\lambda}.$$

By passing to a subsequence, we can assume $\{A_{j_i}\}_i$ converges to a tangent cone (A_∞, Σ, μ), and the statements in $(**)$ hold. By passing to a further subsequence, we may assume $\{E^r_{j_i-1}\}_i, \{E^r_{j_i}\}_i$ and $\{E^r_{j_i+1}\}_i$ converge to W^r_0, W^r_1 and W^r_2 respectively, which are all symmetric. We then denote

$$W^r := 2W^r_0 \cup W^r_1 \cup 2^{-1}W^r_2.$$

Let $\sigma_{j_i} = (2^{-j_i})^* \pi^{\perp}_{\mathcal{F}} s_{j_i}$. Then we have

$$\int_{(B\setminus \overline{B_{2^{-3}}})\setminus \Sigma^r_{j_i}} |\sigma_{j_i}|^2 < 1 + 2^{\lambda} + 2^{-\lambda},$$

and

$$((2^{-j_i})^* \pi^{\perp}_{\mathcal{F}}) \circ \bar{\partial}_{A_{j_i}} \sigma_{j_i} = 0$$

over $(B \setminus \overline{B}_{2^{-3}}) \setminus \Sigma^r_{j_i}$. Hence σ_{j_i} is a holomorphic section of $\mathcal{F}^{\perp}_{j_i} = (2^{-j_i})^* \mathcal{F}^{\perp}$ over $(B \setminus \overline{B}_{2^{-3}}) \setminus \Sigma^r_{j_i}$ with uniformly bounded L^2 norm. Since we have smooth convergence of $(2^{-j_i})^* \pi_{\mathcal{F}}$ locally away from Σ, by standard elliptic theory, after passing to a subsequence, we can assume $\{\sigma_{j_i}\}_i$ converges to σ_∞ locally smoothly over $(B \setminus \overline{B_{2^{-3}}}) \setminus W^r$. Then σ_∞ is a holomorphic section of $\mathcal{F}^{\perp}_\infty$ over $(B \setminus \overline{B_{2^{-3}}}) \setminus W^r$ satisfying

$$\int_{(B\setminus \overline{B_{2^{-1}}})\setminus 2W^r_0} |\sigma_\infty|^2 \leq 2^{\lambda},$$

and

$$\int_{(B_{2^{-1}}\setminus \overline{B_{2^{-2}}})\setminus W^r_1} |\sigma_\infty|^2 \leq 1,$$

and

$$\int_{(B_{2^{-2}}\setminus \overline{B_{2^{-3}}})\setminus 2^{-1}W^r_2} |\sigma_\infty|^2 \leq 2^{-\lambda}.$$

Let $V^r_1 := W^r_0 \cup W^r_1 \cup W^r_2$ and $V^r = 2V^r_1 \cup V^r_1 \cup 2^{-1}V^r_1$. By Theorem 2.33 (II), we have

$$m(V^r_1 \setminus W^r_l) = m(W^r_l \setminus V^r_1) = 0$$

for $l = 0, 1, 2$. Then we have

$$\int_{(B\setminus \overline{B_{2^{-1}}})\setminus 2V^r_1} |\sigma_\infty|^2 \leq 2^{\lambda},$$

and

$$\int_{(B_{2^{-1}}\setminus \overline{B_{2^{-2}}})\setminus V^r_1} |\sigma_\infty|^2 \leq 1,$$

and

$$\int_{(B_{2^{-2}}\setminus \overline{B_{2^{-3}}})\setminus 2^{-1}V^r_1} |\sigma_\infty|^2 \leq 2^{-\lambda}.$$

Claim 2.47. $\int_{(B_{2^{-1}}\setminus \overline{B_{2^{-2}}})\setminus V^r_1} |\sigma_\infty|^2 = 1.$

Given this claim, by applying Proposition 2.45 to σ_∞ over $(B\setminus \overline{B}_{2^{-3}})\setminus V^r$, we know σ_∞ is a nonzero homogeneous section of $\mathcal{E}_\infty$ of degree λ over $(B\setminus \overline{B}_{2^{-3}})\setminus V^r$. This contradicts with our hypothesis that $\lambda \notin ((\operatorname{rank} \mathcal{E})!)^{-1}\mathbb{Z}$. $\square$

Proof of Claim 2.47. By assumption we know $N^{\frac{r}{2}} \subset \Sigma^r_{j_i} \cap \Sigma^r_{j_i-1} \cap \Sigma^r_{j_i+1}$ for i large, so $N^{\frac{r}{2}} \subset V^r_1$. Then we have

$$\|\sigma_{j_i}\|_{L^\infty((\overline{B_{2^{-1}}} \setminus B_{2^{-2}}) \setminus V^r_1)} \leq \|\sigma_{j_i}\|_{L^\infty((\overline{B_{2^{-1}}} \setminus B_{2^{-2}}) \setminus N^{\frac{r}{2}})}.$$

It suffices to prove that there exists $C = C(r)$ independent of i such that for all i large

$$\|\sigma_{j_i}\|_{L^\infty((\overline{B_{2^{-1}}} \setminus B_{2^{-2}}) \setminus N^{\frac{r}{2}})} \leq C. \tag{2.22}$$

By Theorem 2.33 (III), there exists a constant $C' = C'(r) > 0$ so that for any $z \in \overline{(B_{2^{-1}} \setminus B_{2^{-2}}) \setminus N^{\frac{r}{2}}}$, there exists a flat holomorphic disk so that $D_z \subset B_{\frac{3}{4}} \setminus \overline{B}_{\frac{3}{16}}$ with $D_z \cap \Sigma = \emptyset$ and $\partial D_z \subset (B_{\frac{3}{4}} \setminus \overline{B_{\frac{3}{16}}}) \setminus N^{2r}$ and $d(D_z, \Sigma) \geq C' > 0$. Since A_{j_i} converges to A_∞ locally smoothly over $(B \setminus \{0\}) \setminus \Sigma$, there exists a constant $C_1 = C_1(r)$ so that for any $z \in B_{\frac{3}{4}} \setminus \overline{B_{\frac{3}{16}}}$ with $d(z, \Sigma) \geq C'$

$$|F_{A_{j_i}}|(z) \leq C_1 < \infty, \ |\bar{\partial}_{A_{j_i}} (2^{-j_i})^* \pi_{\mathcal{F}}| \leq C_1 < \infty. \tag{2.23}$$

Then Claim 2.47 follows from the following

Lemma 2.48. $|\sigma_{j_i}(z)| \leq C_2 \cdot (\|\sigma_{j_i}\|_{L^\infty(\partial D_z)} + 1)$ *where* $C_2 = C_2(r)$ *is a constant independent of* i.

Indeed, given this, we have

$$\|\sigma_{j_i}\|_{L^\infty((\overline{B_{2^{-1}}} \setminus B_{2^{-2}}) \setminus N^{\frac{r}{2}})} \leq C \cdot (\|\sigma_{j_i}\|_{L^\infty((B_{\frac{3}{4}} \setminus \overline{B_{\frac{3}{16}}}) \setminus N^{2r})} + 1). \tag{2.24}$$

By Theorem 2.33 (II), we have

$$d((B \setminus \overline{B}_{2^{-3}}) \setminus N^{\frac{r}{2}}, \Sigma) > 0$$

which implies $\{A_{j_i}\}_i$ converge to A_∞ uniformly over a neighborhood of $\overline{(B \setminus \overline{B}_{2^{-3}}) \setminus N^{\frac{r}{2}}}$. By Theorem 2.33 (II) again, we have

$$\liminf_i d(B \setminus \overline{B}_{2^{-3}}) \setminus N^{2r}, \Sigma^r_{j_i}) > 0,$$

which implies that $(B_{\frac{3}{4}} \setminus \overline{B}_{\frac{3}{16}}) \setminus N^{2r}$ lies in the interior of $\overline{(B \setminus \overline{B_{2^{-3}}}) \setminus \Sigma^r_{j_i}}$ for j_i large. Now Equation (2.22) follows from standard elliptic interior estimate and Equation (2.23). □

Proof of Lemma 2.48. Let $\nabla := A_{\mathcal{F}^\perp_{j_i}}|_{D_z}$ which has curvature form

$$F_\nabla = (F_{A_{j_i}} - (\bar{\partial}_{A_{j_i}} (2^{-j_i})^* \pi_{\mathcal{F}})^* \wedge \bar{\partial}_{A_{j_i}} (2^{-j_i})^* \pi_{\mathcal{F}})|_{D_z}.$$

Since $\sigma_{j_i}|_{D_z}$ is a holomorphic section of $\mathcal{F}^\perp_{j_i}|_{D_z}$, we have

$$\Delta_{D_z} \log(|\sigma_{j_i}|_{D_z}|^2 + 1) \geq -|F_\nabla| \geq -2C_1 \tag{2.25}$$

where the second inequality follows from 2.24. For the first inequality, we first identify D_z with $\{t \in \mathbb{C} : |t| < \delta_z\}$ where δ_z is the radius of D_z and by a direct calculation, we have

$$\Delta \log(|\sigma|^2 + 1) = \frac{< \sigma, \nabla_{\bar{\partial}_t} \nabla_{\partial_t} \sigma >}{|\sigma|^2 + 1} + \frac{|\nabla_{\partial_t} \sigma|^2}{|\sigma|^2 + 1} - \frac{< \nabla_{\partial_t} \sigma, \sigma >< \sigma, \nabla_{\partial_t} \sigma >}{(|\sigma|^2 + 1)^2}$$

The difference of the last two terms is non-negative by Cauchy-Schwartz inequality, and for the first term we have

$$\frac{<\sigma, \nabla_{\bar{\partial}_t}\nabla_{\partial_t}\sigma>}{|\sigma|^2+1} = \frac{<\sigma, F_\nabla(\frac{\partial}{\partial \bar{t}}, \frac{\partial}{\partial t})\sigma>}{|\sigma|^2+1} \geq -|F_\nabla|.$$

As a result, we get

$$\Delta_{D_z}(\log(|\sigma_{j_i}|_{D_z}|^2+1)+2C_1|t|^2) \geq 0$$

Now the conclusion follows from the maximal principle. □

Corollary 2.49. *Given a local section s of $\mathcal{E}$ near the origin, the following is a well-defined number in $((\operatorname{rank}\mathcal{E})!)^{-1}\mathbb{Z}\cup\{\infty\}$*

$$d^r_{\mathcal{F}}(s) := \lim_{j\to\infty} \frac{\log \|\pi^\perp_{\mathcal{F}} s\|^r_j}{-j\log 2}$$

for any $r \in (0, r_0]$.

Proposition 2.50. *Given a local section s of $\mathcal{E}$ near the origin, if $d^r_{\mathcal{F}}(s)$ is finite for some $r \in (0, r_0]$, then $\{\frac{(2^{-j_i})^*\pi^\perp_{\mathcal{F}}s}{\|\pi^\perp_{\mathcal{F}}s\|^r_{j_i}}\}_i$ converges to a non-trivial homogeneous section σ_∞ of degree $d^r_{\mathcal{F}}(s)$ of $\mathcal{F}^\perp_\infty$ over $(B_{2^{-1}}\setminus \overline{B}_{2^{-2}})\setminus\Sigma$, which extends to a holomorphic section of $\mathcal{F}^\perp_\infty$ defined over $B_{2^{-1}}\setminus\overline{B}_{2^{-2}}$.*

Proof. If $d^r_{\mathcal{F}}(s) < \infty$, by passing to a subsequence, it follows from the proof of Proposition 2.46 that $\{\sigma_{j_i} := \frac{(2^{-j_i})^*\pi^\perp s}{\|\pi^\perp s\|^r_{j_i}}\}_i$ converges to a nontrivial homogeneous holomorphic section σ_∞ of degree $d^r_{\mathcal{F}}(s)$ over $(B\setminus\overline{B}_{2^{-3}})\setminus V^r$. Furthermore, we also have

$$\|\sigma_{j_i}\|_{L^\infty((B_{2^{-1}}\setminus\overline{B}_{2^{-2}})\setminus N^{\frac{r}{2}})} \leq C(\|\sigma_{j_i}\|_{L^\infty(B_{\frac{3}{4}}\setminus\overline{B}_{\frac{3}{16}})\setminus N^{2r})}+1). \tag{2.26}$$

for some $C = C(r)$. By definition, we know $d^r_{\mathcal{F}}(s)$ is decreasing when $r \to 0$. Hence we have $d^{\frac{r}{2}}_{\mathcal{F}}(s) < \infty$ which implies $\{\frac{(2^{-j_i})^*\pi^\perp s}{\|s\|^{\frac{r}{2}}_{j_i}}\}$ converges to a homogeneous section σ'_∞. Then by Equation (2.26), we can assume $\{\sigma_{j_i}\}_i$ converges to a nontrivial homogeneous holomorphic section of $\mathcal{F}^\perp_\infty$ over $(B\setminus\overline{B}_{2^{-3}})\setminus V^{\frac{r}{2}}$ which is a multiple of σ'_∞. By repeating this process for $2^{-l}r$ inductively on $l \in \mathbb{Z}_{\geq 0}$ and passing to a subsequence, we can assume $\{\sigma_{j_i}\}_i$ converges to a nontrivial homogeneous holmorphic section σ_∞ of $\mathcal{F}^\perp_\infty$ over $(B_{2^{-1}}\setminus\overline{B}_{2^{-2}})\setminus\Sigma$. Now it remains to show that σ_∞ extends to be a holomorphic section of $\mathcal{F}^\perp_\infty$ over $B_{2^{-1}}\setminus\bar{B}_{2^{-2}}$. Since $B_{2^{-1}}$ is a precompact Stein open set in B, we can find a finite resolution of $(\mathcal{F}^\perp_\infty)^*$ over $B_{2^{-1}}$ as

$$\mathcal{O}^{n_1} \to \mathcal{O}^{n_2} \to (\mathcal{F}^\perp_\infty)^* \to 0.$$

By taking the dual of the above exact sequence, we have the following exact sequence over $B_{2^{-1}}$

$$0 \to \mathcal{F}^\perp_\infty \to \mathcal{O}^{n_2} \to \mathcal{O}^{n_1}.$$

Then we can view σ_∞ as a holomorphic section of $\mathcal{O}^{n_2}$ over $(B_{2^{-1}}\setminus\overline{B}_{2^{-2}})\setminus\Sigma$. Since Σ has Hausdorff codimension 4, σ_∞ extends to be a holomorphic section of $\mathcal{O}^{n_2}$ over $B_{2^{-1}}\setminus\overline{B}_{2^{-2}}$ (see Lemma 3 in [33]). This finishes the proof. □

Remark 2.51. When $\mathcal{F} = 0$, one can repeat the argument above and show that for any nonzero holomorphic section s of $\mathcal{E}$ defined in a neighborhood of 0, the following is well-defined

$$d(s) := \frac{1}{2}\lim_{r\to 0}\frac{\log\int_{B_r}|s|^2}{\log r} - n \in (k!)^{-1}\mathbb{Z}\cup\{+\infty\}, \tag{2.27}$$

where $k = rank(\mathcal{E})$. Furthermore, when $d(s) < \infty$, the normalized rescaled sequence given by s will converge to a nontrivial homogeneous section on any tangent cone $\mathcal{E}_\infty$ with degree equal to $d(s)$ after passing to a subsequence.

3 Homogeneous Case

3.1 Uniqueness of tangent cone connections

In this section, we will prove Theorem 1.1.

3.1.1 Semistable Case

Assume $\underline{\mathcal{E}}$ is semistable and fix a Seshadri filtration for $\underline{\mathcal{E}}$ as

$$0 = \underline{\mathcal{E}}_0 \subset \underline{\mathcal{E}}_1 \subset \underline{\mathcal{E}}_2 \subset \cdots \underline{\mathcal{E}}_m = \underline{\mathcal{E}}.$$

So

$$Gr^{HNS}(\underline{\mathcal{E}}) \simeq \bigoplus_{p=1}^{m} \underline{\mathcal{E}}_p/\underline{\mathcal{E}}_{p-1}$$

Denote $\mathcal{E}_i = \pi^*\underline{\mathcal{E}}_i$. Theorem 1.1 follow from

Theorem 3.1. *$(\mathcal{E}_\infty, A_\infty)$ is isomorphic to the natural Hermitian-Yang-Mills cone connection on $\psi_*\pi^*(Gr^{HNS}(\underline{\mathcal{E}}))^{**}$. Moreover, $Sing(\pi^*(Gr^{HNS}(\underline{\mathcal{E}}))) \subset \Sigma$.*

By tensoring with $\mathcal{O}(k)$ for k large, we may assume the following for each $p \geq 1$

- $\underline{\mathcal{E}}_p$ and $\underline{\mathcal{E}}_p/\underline{\mathcal{E}}_{p-1}$ are globally generated;
- The following sequence is exact

$$0 \to H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}}_{p-1}) \to H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}}_p) \to H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}}_p/\underline{\mathcal{E}}_{p-1}) \to 0.$$

Denote $HG_p := \{\pi^*\underline{s} : \underline{s} \in H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}}_p)\}$. Then we have a filtration

$$0 \subset HG_1 \subset HG_2 \subset \cdots HG_m = HG.$$

Denote $n_p := \dim_{\mathbb{C}} HG_p/HG_{p-1}$. For $p \geq 0$, let π_p be the pointwise orthogonal projection given by $\mathcal{E}_p \subset \mathcal{E}$ with respect to the unknown metric H and

$$\pi_p^\perp = \mathrm{Id} - \pi_p.$$

Note π_p and $\pi_p^\perp$ are both defined away from $\mathrm{Sing}(\mathcal{E}/\mathcal{E}_p)$. Fix a basis $\{\sigma_{p,l} | 1 \leq p \leq m, 1 \leq l \leq n_p\}$ of HG so that $\{\sigma_{p,l} | 1 \leq l \leq n_p, 1 \leq p \leq q\}$ form a basis for HG_q for any $1 \leq q \leq m$. For each (p, l), we denote

$$\sigma_{p,l}^j := (2^{-j})^*(\pi_{p-1}^\perp \sigma_{p,l}).$$

We view these as smooth sections of $\mathcal{E}$ defined on $B \setminus \mathrm{Sing}(\mathcal{E}/\mathcal{E}_p)$.

Fix any $r \in (0, r_0]$, where r_0 is given in Theorem 2.33. Denote

$$M_p^j = \sup_{1 \le l \le n_p} \|\pi_{p-1}^{\perp} \sigma_{p,l}\|_j^r$$

where $\| \cdot \|_j^r$ is defined as in Equation (6.9) for $p \ge 2$ and

$$M_1^j = \sup_l \|\sigma_{1,l}^j\|_{L^2(B)}.$$

By Theorem 2.10, for any $\epsilon > 0$ we can find a Hermitian metric $\underline{H}_\epsilon$ on $\underline{\mathcal{E}}$ such that $|\sqrt{-1}\Lambda_{\omega_{FS}} F_{\underline{A}_\epsilon} - \mu \mathrm{Id}|_{L^\infty(\mathbb{CP}^{n-1})} < \epsilon$ with $\mu = \mu(\underline{\mathcal{E}})$. Let $H_\epsilon = |z|^{2\mu} \pi^* \underline{H}_\epsilon$. Then $|F_{(H_\epsilon, \bar\partial_{\mathcal{E}})}| = O(r^{-2})$, so $F_{(H_\epsilon, \bar\partial_{\mathcal{E}})} \in L^2(B^*)$. Furthermore, for all $r \in (0, 1]$, we have

$$r^2 \sup_{\partial B_r} |\Lambda_{\omega_0} F_{H_\epsilon}| < \epsilon.$$

Lemma 3.2. *For any $s = \pi^* \underline{s}$, where $\underline{s} \in H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}})$, we have $d(s) = \mu$.*

Proof. By definition, there is a constant $C(\epsilon) > 0$ such that for all $r \in (0, 1)$

$$C(\epsilon)^{-1} r^{2n+2\mu} \le \int_{B_r} |s|_{H_\epsilon}^2 \le C(\epsilon) r^{2n+2\mu}.$$

Applying Lemma 2.20 with $\delta = 1$, we see (2.15) holds with $H' = H_\epsilon$, so

$$\limsup_{z \to 0} \frac{|\log \mathrm{Tr}_{H_\epsilon} H(z)| + |\log \mathrm{Tr}_H H_\epsilon(z)|}{-\log |z|} \le C_0 \epsilon,$$

where C_0 is a constant *independent* of ϵ. Then we get

$$d(s) \in [\mu - C_0\epsilon, \mu + C_0\epsilon]$$

for all $\epsilon > 0$. By letting ϵ go to 0, we obtain $d(s) = \mu$. □

Now suppose (A_∞, Σ, μ) is a tangent cone of A given by the limit of a subsequence $\{A_{j_i}\}_i$. We shall prove the following statements by induction on $p \ge 1$. Theorem 3.1 is a direct corollary of these statements.

$(a)_p$. There is a simple HYM cone direct summand $\mathcal{S}_p$ of $\mathcal{E}_\infty$ which is isomorphic to $(\mathcal{E}_p/\mathcal{E}_{p-1})^{**}$ so that $\mathcal{S}_p \perp \mathcal{S}_k$ for any $k < p$ (We take $\mathcal{S}_0 = 0$ here);

$(b)_p$. $\mathrm{Sing}(\mathcal{E}_p/\mathcal{E}_{p-1}) \cup \mathrm{Sing}(\mathcal{E}/\mathcal{E}_p) \subset \Sigma$, and over $(B_{2^{-2}} \setminus B_{2^{-1}}) \setminus \Sigma$, $\{\pi_p^{j_i}\}_i$ converges locally smoothly to the limit projection $\pi_p^\infty : \mathcal{E}_\infty \to \mathcal{E}_\infty$ given by $\oplus_{k \le r} \mathcal{S}_k \subset \mathcal{E}_\infty$. Here $\pi_p^{j_i} = (2^{-j_i})^* \pi_p$. We also denote by $(\mathcal{E}_p^\infty)^\perp$ the HYM cone direct summand of $\mathcal{E}_\infty$ so that

$$\mathcal{E}_\infty = \oplus_{k \le p} \mathcal{S}_k \oplus (\mathcal{E}_p^\infty)^\perp.$$

$(c)_p$. If $p < m$ then $d_{\mathcal{E}_p}(\sigma_{p+1,l}) = \mu(\underline{\mathcal{E}})$ for any $1 \le l \le n_{p+1}$. Here $d_{\mathcal{E}_p}$ is well-defined due to $(b)_p$.

When $p = 1$, by Remark 2.51 and Lemma 3.2, after passing to further subsequence we may assume $\{\frac{\sigma_{1,l}^{j_i}}{M_1^{j_i}}\}_i$ converges to a holomorphic homogeneous section of degree $\mu(\underline{\mathcal{E}})$ away from Σ for any $1 \leq l \leq n_1$ and at least one of the limits is non-zero.

By assumption we have the following exact sequence of coherent sheaves

$$0 \to R_1 \to \mathcal{O}^{\oplus n_1} \xrightarrow{\phi_1} \mathcal{E}_1 \to 0.$$

where

$$\phi_1(z)(a_1, \cdots, a_{n_1}) = \sum_{l=1}^{n_1} a_l \sigma_{1,l}(z).$$

Away from $\mathrm{Sing}(\mathcal{E}/\mathcal{E}_1)$, $\mathcal{E}_1$ can be viewed as a vector sub-bundle of $\mathcal{E}$. For $z \notin \Sigma \cup \mathrm{Sing}(\mathcal{E}/\mathcal{E}_1)$, we define a vector bundle homomorphism

$$\phi_1^\infty : \mathcal{O}^{\oplus n_1} \to \mathcal{E}_\infty$$

by

$$\phi_1^\infty(z)(a_1, \cdots, a_{n_1}) = \lim_{i\to\infty} (M_1^{j_i})^{-1} \sum_{l=1}^{n_1} a_l \sigma_{1,l}^{j_1}(z).$$

If $(a_1, \cdots, a_{n_1})$ is in the fiber of $(R_1)_z$, then by definition, we have $\sum_{l=1}^{n_1} a_l \sigma_{1,l}(z) = 0$, hence $\sum_{l=1}^{n_1} a_l \sigma_{1,l}^{j_i}(z) = 0$, which implies $\phi_1^\infty(z)(a_1, \cdots, a_{n_1}) = 0$ and ϕ_1^∞ descends to a homomorphism away from Σ

$$\psi_1 : \mathcal{E}_1 \simeq \mathcal{O}^{n_1}/R_1 \to \mathcal{E}_\infty$$

which satisfies $\psi_1(z)(\sigma_{1,l}(z)) = \sigma_{1,l}^\infty(z)$. Let $\mathcal{S}_1$ be the minimal simple HYM cone direct summand of $\mathcal{E}_\infty$ which contains the image of ψ_1. Note $\mathcal{S}_1$ is locally free away from Σ. Since $d(\sigma_{1,l}^\infty) = \mu(\underline{\mathcal{S}}_1)$, ψ_1 descends to be a nontrivial map defined over $\mathbb{CP}^{n-1} \setminus (\pi(\Sigma) \cup \mathrm{Sing}(\underline{\mathcal{E}}/\underline{\mathcal{E}}_1))$

$$\underline{\psi}_1 : \underline{\mathcal{E}}_1 \to \underline{\mathcal{S}}_1,$$

where $\mu(\underline{\mathcal{E}}_1) = \mu(\underline{\mathcal{E}}) = \mu(\underline{\mathcal{S}}_1)$. By Lemma 3 in [33], $\underline{\psi}_1$ extends to a sheaf homomorphism over the whole $\mathbb{CP}^{n-1}$. So it has to be an isomorphism by Corollary 2.5 and 2.6, and the minimality of $\mathcal{S}_1$. This proves $(a)_1$.

For $(b)_1$, since $\mathcal{S}_1$ is locally free away from Σ and ψ_1 maps $\mathcal{E}_1$ isomorphically onto $\mathcal{S}_1$, we know in particular $\mathcal{E}_1$ must be locally free away from Σ, and ψ_1 is a vector bundle isomorphism away from Σ. By construction the bundle map ψ_1 then factors through the bundle map $\mathcal{E}_1 \to Im(\mathcal{E}_1) \subset \mathcal{E}$. Hence on $B \setminus \Sigma$, the map $\mathcal{E}_1 \to \mathcal{E}$ must be a injective vector bundle map, and so $\mathcal{E}/\mathcal{E}_1$ is locally free. This implies $\mathrm{Sing}(\mathcal{E}/\mathcal{E}_1) \subset \Sigma$.

Given any $z \notin \Sigma$, choose a local orthonormal frame $\{e_t | 1 \leq t \leq \mathrm{rank}(\underline{\mathcal{E}}_1)\}$ for $\mathcal{S}_1$ near z. Then we can write $e_t = \sum_l a_l^t(z) \sigma_{1,l}^\infty(z)$ for each t, hence $\{e_t^{j_i} = \sum_l a_l^t \frac{\sigma_{1,l}^{j_i}}{M_1^{j_i}} : 1 \leq t \leq \mathrm{rank}(\mathcal{S}_1)\}$ is an approximately orthonormal frame of $(2^{-j_i})^*(\mathcal{E}_1)$ near z which converges to $\{e_t\}$ smoothly. In particular, $\{\pi_1^{j_i}\}_i$ converges smoothly to π_1^∞ given by $S_1 \subset \mathcal{E}_\infty$.

It remains to prove $(c)_1$. By the proof of Lemma 3.2, for any $\epsilon > 0$, there exists a smooth Hermitian metric on $\underline{H}_\epsilon$ on $\underline{\mathcal{E}}$ so that

$$|\Lambda_{\omega_{FS}} F_{(\underline{H}_\epsilon, \bar{\partial}_{\underline{\mathcal{E}}})} - \mu(\underline{\mathcal{E}}) Id|_{L^\infty} \leq \epsilon$$

and

$$C_\epsilon |z|^{2\epsilon} H_\epsilon \leq H \leq C_\epsilon^{-1} |z|^{-2\epsilon} H_\epsilon,$$

where $H_\epsilon = |z|^{2\mu(\underline{\mathcal{E}})} \pi^* \underline{H}_\epsilon$, and $C_\epsilon > 0$. Let $\pi_1^{\perp_{H_\epsilon}} \sigma_{2,l}$ denote the orthogonal projection of $\sigma_{2,l}$ to $\mathcal{E}_1$ by using H_ϵ. Then we have away from Σ

$$|\pi_1^\perp \sigma_{2,l}| \leq |\pi_1^{\perp_{H_\epsilon}} \sigma_{2,l}| \leq C_\epsilon^{-1} |z|^{-2\epsilon} |\pi_1^{\perp_{H_\epsilon}} \sigma_{2,l}|_{H_\epsilon}.$$

Similarly

$$C_\epsilon |z|^{2\epsilon} |\pi_1^{\perp_{H_\epsilon}} \sigma_{2,l}|_{H_\epsilon} \leq |\pi_1^\perp \sigma_{2,l}|.$$

As a result, we have

$$d^\epsilon_{\mathcal{E}_1}(\sigma_{2,l}) - \epsilon \leq d_{\mathcal{E}_1}(\sigma_{2,l}) \leq d^\epsilon_{\mathcal{E}_1}(\sigma_{2,l}) + \epsilon,$$

where

$$d^\epsilon_{\mathcal{E}_1}(\sigma_{2,l}) = \lim_{i \to \infty} \frac{\log \int_{(B_{2^{-j_i-1}} \setminus \overline{B}_{2^{-j_i-2}}) \setminus 2^{-j_i} E^r_{j_i}} |\pi_1^{\perp_{H_\epsilon}} \sigma_{2,l}|_{H_\epsilon}}{-2 j_i \log 2} - n.$$

Since $H_\epsilon = |z|^{2\mu(\underline{\mathcal{E}})} \pi^* \underline{H}_\epsilon$, we have $\pi_1^{\perp_{H_\epsilon}} = \pi^*(\underline{\pi}_1^{\perp_{\underline{H}_\epsilon}})$. Using the fact that $\sigma_{2,l} = \pi^* \underline{\sigma}_{2,l}$, it is easy to see

$$d^\epsilon_{\mathcal{E}_1}(\sigma_{2,l}) = \mu(\underline{\mathcal{E}})$$

Then we have

$$\mu(\underline{\mathcal{E}}) - \epsilon \leq d_{\mathcal{E}_1}(\sigma_{2,l}) \leq \mu(\underline{\mathcal{E}}) + \epsilon$$

for any $\epsilon > 0$. By letting $\epsilon \to 0$, we have $d_{\mathcal{E}_1}(\sigma_{2,l}) = \mu(\underline{\mathcal{E}})$.

Now we perform the induction argument and suppose we have established the statements

$$(a)_1, (b)_1, (c)_1, \cdots, (a)_{p-1}, (b)_{p-1}, (c)_{p-1}.$$

By $(c)_{p-1}$, we have $d_{\mathcal{E}_{p-1}}(\sigma_{p,l}) = \mu(\underline{\mathcal{E}})$ for any l. By Proposition 2.49, after passing to subsequence, $\{\frac{\sigma^{j_i}_{p,l}}{M^{j_i}_p}\}_i$ converges to homogeneous sections $\sigma^\infty_{p,l}$ of $\mathcal{E}_\infty$ with degree $\mu(\underline{\mathcal{E}})$ over $(B_{2^{-1}} \setminus \overline{B}_{2^{-2}}) \setminus \Sigma$ for each l and at least one of them is non-zero. By assumption, we have the following exact sequence of coherent sheaves on $B_{2^{-1}} \setminus \overline{B}_{2^{-2}}$,

$$0 \to R_p \to \mathcal{O}^{\oplus n_p} \xrightarrow{\phi_p} \mathcal{E}_p / \mathcal{E}_{p-1} \to 0$$

where $\phi_p(a_1, \cdots, a_{n_p}) = \sum_l a_l \sigma_{p,l}$. By $(b)_{p-1}$ we know $\mathcal{E}/\mathcal{E}_{p-1}$ is locally free away from Σ. For any $z \notin \Sigma \cup \mathrm{Sing}(\mathcal{E}_p/\mathcal{E}_{p-1})$, we define

$$\phi^\infty_p : \mathcal{O}^{\oplus n_p} \to \mathcal{E}_\infty$$

by letting

$$\phi_p^\infty(z)(a_1,\cdots,a_{n_p}) = \lim_{i\to\infty} \frac{\sum_l a_l \sigma_{p,l}^{j_i}(z)}{M_p^{j_i}} = \sum_l a_l \sigma_{p,l}^\infty(z).$$

If $\phi_p(z)(\sum_l a_l\sigma_{p,l}) = 0$ in $\mathcal{E}_p/\mathcal{E}_{p-1}$, then by definition, we have $\sum_l a_l\sigma_{p,l} \in \mathcal{E}_{p-1}$, so $\sum_l a_l \sigma_{p,l}^{j_i} = 0$. Hence away from $\Sigma \cup \mathrm{Sing}(\mathcal{E}_p/\mathcal{E}_{p-1})$, ϕ_p^∞ induces a nontrivial map

$$\psi_p : \mathcal{E}_p/\mathcal{E}_{p-1} \to \mathcal{E}_\infty$$

which satisfies $\psi_p(\phi_p(\sigma_{p,l}(z))) = \sigma_{p,l}^\infty(z)$ for $z \notin \Sigma \cup \mathrm{Sing}(\mathcal{E}_p/\mathcal{E}_{p-1})$. Let $\mathcal{S}_p$ be the minimal simple HYM cone summand containing the image of ψ_p. By $(b)_{p-1}$ and using the definition, we have

$$\mathcal{S}_p \subset (\mathcal{E}_{p-1}^\infty)^\perp$$

away from Σ. Since $\sigma_{p,l}^\infty$ are all homogeneous sections of degree equal to $\mu(\underline{\mathcal{E}})$, ψ_p descends to a nontrivial holomorphic map over $\mathbb{CP}^{n-1} \setminus (\pi(\Sigma) \cup \mathrm{Sing}(\underline{\mathcal{E}}/\underline{\mathcal{E}}_p))$ as

$$\underline{\psi}_p : \underline{\mathcal{E}}_p/\underline{\mathcal{E}}_{p-1} \to \underline{\mathcal{S}}_p$$

where $\mu(\underline{S}_p) = \mu(\underline{\mathcal{E}}) = \mu(\underline{\mathcal{S}}_p)$. Then $\underline{\psi}_p$ extends to be a nontrivial holomorphic map defined over $\mathbb{CP}^{n-1}$ and induces the following isomorphism

$$\underline{\psi}_p : (\underline{\mathcal{E}}_p/\underline{\mathcal{E}}_{p-1})^{**} \to \underline{\mathcal{S}}_p.$$

This proves $(a)_p$.

By $(b)_{p-1}$, away from Σ, $\mathcal{E}/\mathcal{E}_{p-1}$ is locally free away from Σ. Since $\mathcal{E}_p/\mathcal{E}_{p-1}$ is saturated in $\mathcal{E}/\mathcal{E}_{p-1}$, we know $\mathcal{E}_p/\mathcal{E}_{p-1}$ is reflexive away from Σ by Proposition 5.22 in [25]. Then away from Σ, ψ_p is an isomorphism between $\mathcal{E}_p/\mathcal{E}_{p-1}$ and $\mathcal{S}_p$. Since $\mathcal{S}_p$ is locally free away from Σ, we know $\mathcal{E}_p/\mathcal{E}_{p-1}$ is also locally free away from Σ, and ψ_p is a vector bundle isomorphism. As in the case $p = 1$, since the map ψ_p factors through the natural map $\mathcal{E}_p/\mathcal{E}_{p-1} \to \mathcal{E}/\mathcal{E}_{p-1}$, it follows that away from Σ, $\mathcal{E}_p/\mathcal{E}_{p-1}$ is a sub-bundle of $\mathcal{E}/\mathcal{E}_{p-1}$, and hence $\mathcal{E}_p$ is a sub-bundle of $\mathcal{E}$. This is equivalent to saying that $\mathcal{E}/\mathcal{E}_p$ is locally free away from Σ.

For any $z \notin \Sigma$, we can choose $\{e'_t | 1 \leq t \leq \mathrm{rank}(\mathcal{S}_p)\}$ to be an orthonormal frame for $\mathcal{S}_p$ near z. Then we can write $e'_t = \sum_l a_{p,l}^t \sigma_{p,l}^\infty$ for each t near z and thus $\{\sum_l a_{p,l}^t \frac{\sigma_{p,l}^{j_i}}{M_p^{j_i}} | 1 \leq t \leq \mathrm{rank}(\mathcal{S}_p)\}$ is an approximately orthonormal frame for $(2^{-j_i})^*(\mathcal{E}_p) \cap ((2^{-j_i})^*(\mathcal{E}_{p-1}))^\perp$ near z which smoothly converge to $\{e'_t : 1 \leq t \leq \mathrm{rank}(\mathcal{S}_p)\}$. In particular, we have $\{\pi_p^{j_i} - \pi_{p-1}^{j_i}\}_i$ converges to π_p^∞ given by $\mathcal{S}_p \subset \mathcal{E}_\infty$. Combining this with $(b)_{p-1}$, we have $\{\pi_p^{j_i}\}_i$ converges to the projection determined by $\oplus_{1\leq l\leq p}\mathcal{S}_l \subset \mathcal{E}_\infty$. This proves $(b)_p$.

Finally $(c)_p$ follows line by line by replacing $\mathcal{E}_1$ with $\mathcal{E}_p$ in the proof $(c)_1$. So we have established $(a)_p, (b)_p, (c)_p$. This finishes the proof.

3.1.2 General Case

Now we assume $\underline{\mathcal{E}}$ is a general holomorphic vector bundle over $\mathbb{CP}^{n-1}$. Let

$$0 = \underline{\mathcal{E}}_0 \subset \underline{\mathcal{E}}_1 \subset \cdots \underline{\mathcal{E}}_m = \underline{\mathcal{E}}$$

be the Harder-Narasimhan filtration of $\underline{\mathcal{E}}$, with $\mu_p = \mu(\underline{\mathcal{E}}_p/\underline{\mathcal{E}}_{p-1})$ strictly decreasing in p, and choose a filtration

$$\underline{\mathcal{E}}_{p-1} = \underline{\mathcal{E}}_{p,0} \subset \underline{\mathcal{E}}_{p,1} \subset \cdots \underline{\mathcal{E}}_{p,q_p} = \underline{\mathcal{E}}_p$$

so that

$$0 = \underline{\mathcal{E}}_{p,1}/\mathcal{E}_{p-1} \subset \cdots \underline{\mathcal{E}}_{p,q_p}/\mathcal{E}_{p-1} = \underline{\mathcal{E}}_p/\mathcal{E}_{p-1}$$

is a Seshadri filtration of $\underline{\mathcal{E}}_p/\mathcal{E}_{p-1}$. By tensoring $\underline{\mathcal{E}}$ with $\mathcal{O}(k)$ for k large, we may assume the following for all p and q,

- $\underline{\mathcal{E}}_p$ and $\underline{\mathcal{E}}_{p,q}$ are generated by its global sections;
- we have a short exact sequence

  $$0 \to H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}}_{p,q-1}) \to H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}}_{p,q}) \to H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}}_{p,q}/\underline{\mathcal{E}}_{p,q-1}) \to 0.$$

For $p = 1, \cdots, m$, we define

$$HG_p := \{s = \pi^* \underline{s} | \underline{s} \in H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}}_p)\}$$

and

$$HG_{p,q} := \{s = \pi^* \underline{s} | \underline{s} \in H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}}_{p,q})\}.$$

Then we have a filtration

$$0 \subset HG_{1,1} \subset \cdots HG_{1,q_1} = HG_1 \subset \cdots \subset HG_m = HG.$$

Now we can repeat the proof in Section 3.1 for the semistable case here. The only difference in the proof is the calculation of the degree. Suppose $\lim_{i\to\infty} A_{j_i} = (A_\infty, \Sigma, \mu)$ and the rescaled projections $\{(2^{-j_i})^* \pi_{p,q}\}_i$ given by the orthogonal projection $\pi_{p,q} : \mathcal{E} \to \mathcal{E}_{p,q}$ converges to a projection map $\pi_{p,q}^\infty$ so that $\pi_{p,q}^\infty$ determines a direct HYM cone summand of $\mathcal{E}_\infty$. Unlike the semistable case, the construction of a family of comparison metrics become a lot more involved. To state the proposition for the existence of a family of comparison metrics, we set up a few notations first.

Let $\underline{\Sigma} \subset \mathbb{CP}^{n-1}$ be the subset where the Harder-Narasimhan filtration is not given by sub-bundles. It is of complex codimension at least 2. We will use the following notations

- $\Sigma = \pi^{-1}(\underline{\Sigma})$, where $\pi : B^* \to \mathbb{CP}^{n-1}$;
- $\Sigma_r = \Sigma \cap S(r)$ where $S(r) = \{x \in B^* : |x| = r\}$;
- $\underline{\Sigma}^s = \{x \in \mathbb{CP}^{n-1} | d(x, \underline{\Sigma}) \leq s\}$, where the distance is measured by the fixed Fubini-Study metric on $\mathbb{CP}^{n-1}$;
- $\Sigma^s = \pi^{-1}(\underline{\Sigma}^s)$;
- $\Sigma_r^s = \{x \in S^{2n-1}(r) | d(x, \Sigma_r) < s\}$, where the distance is measured with respect to the round metric on $S^{2n-1}(r)$.

Proposition 3.3. *For any $0 < \epsilon << 1$, there exists a smooth Hermitian metric H_ϵ on $\mathcal{E}|_{B^*}$ satisfying the following*

(i). $|F_{(H_\epsilon, \bar\partial_\mathcal{E})}| \in L^{1+\delta}(B^*)$ for some $\delta > 0$;

(ii). $\sup_{r\in(0,1]} r^{1-2n}\int_{S^{2n-1}(r)} r^2|\Lambda_{\omega_0}F_{(H_\epsilon,\bar\partial_{\mathcal{E}})}|\leq\epsilon$;

(iii). $|z|^2|\Lambda_{\omega_0}F_{(H_\epsilon,\bar\partial_{\mathcal{E}})}(z)|\leq\epsilon$ *for* $z\notin\Sigma^{10^{-4}}$;

(iv). *For any* $s\in HG_i\setminus HG_{i-1}$,

$$\lim_{r\to 0}\frac{1}{2}\frac{\log\int_{B_r^*\setminus\Sigma^{10^{-3}}}|z|^\epsilon|s|^2_{H_\epsilon}}{\log r}-n=\mu_i+\frac{\epsilon}{2}.$$

(v). *Away from* $\Sigma^{10^{-3}}$, *we have*

$$H_\epsilon=\sum_{p'}\pi^*\underline{H}_\epsilon(|z|^{\mu(\underline{\mathcal{E}}_{p'}/\underline{\mathcal{E}}_{p'-1})}(\pi_{p'}-\pi_{\epsilon,p'-1})\cdot,|z|^{\mu(\underline{\mathcal{E}}_{p'}/\underline{\mathcal{E}}_{p'-1})}(\pi_{p'}-\pi_{\epsilon,p'-1})\cdot)$$

where $\pi_{\epsilon,p'}$ *is the pointwise orthogonal projection given by* $\mathcal{E}_{p'}\subset\mathcal{E}$ *with respect to the metric* $\pi^*\underline{H}_\epsilon$.

We will leave the proof of this proposition to Section 3.2. For our purpose, we will include $\pi^{-1}(Sing(\underline{\mathcal{E}}))$ in Theorem 3.5 and run the whole cut-off argument. Assuming Proposition 3.3, we have

Proposition 3.4. *We have the following*

(a). for any $s\in HG_{p,q+1}\setminus HG_{p,q}$, $d_{\mathcal{E}_{p,q}}(s)\leq\mu(\underline{\mathcal{E}}_{p,q+1}/\underline{\mathcal{E}}_{p,q})$;

(b). for any $s\in HG_{p,1}\setminus HG_{p-1}$, $d_{\mathcal{E}_{p-1}}(s)\leq\mu(\underline{\mathcal{E}}_{p,1}/\underline{\mathcal{E}}_{p-1})$.

Proof. The proof for (a) and (b) are the same. We only prove (a) here.

For any $0<\epsilon\leq 1$, let H_ϵ be the metric given in Proposition 3.3. Let $g=\log\mathrm{Tr}_H H_\epsilon$, and $f(z)=|z|^2|\Lambda_{\omega_0}F_{(H_\epsilon,\bar\partial_{\mathcal{E}})}(z)|$. As in Section 2.3, we have on B^*,

$$\Delta g\geq-|z|^{-2}f(z),$$

So by items (i), (ii), (iii) in Proposition 3.3, Lemma 2.20, and item (2) in Lemma 2.18 (replacing $|z|/2$ by $|z|/A$ for some big but fixed A), we see that there is a constant C independent of ϵ such that for any $z\notin\Sigma^{10^{-3}}$,

$$g(z)\leq C-\epsilon\log|z|.$$

In other words, we have

$$H\geq e^{-C}|z|^\epsilon H_\epsilon.$$

for any $z\notin\Sigma^{10^{-3}}$. Furthermore, away from $\Sigma^{10^{-3}}$, we have

$$H_\epsilon=\sum_{p'}\pi^*\underline{H}_\epsilon(|z|^{\mu(\underline{\mathcal{E}}_{p'}/\underline{\mathcal{E}}_{p'-1})}(\pi_{p'}-\pi_{\epsilon,p'-1})\cdot,|z|^{\mu(\underline{\mathcal{E}}_{p'}/\underline{\mathcal{E}}_{p'-1})}(\pi_{p'}-\pi_{\epsilon,p'-1})\cdot)$$

where $\pi_{\epsilon,p'}$ is the pointwise orthogonal projection given by $\mathcal{E}_{p'}\subset\mathcal{E}$ with respect to the metric $\pi^*\underline{H}_\epsilon$. Similar to the proof of $(c)_1$ in the semistable case, we have

$$d_{\mathcal{E}_{p,q}}(s)\leq\lim_{i\to\infty}\frac{\log(2^{2j_in}\int_{(B_{2^{-j_i-1}}\setminus\overline{B_{2^{-j_i-2}}})\setminus(\Sigma^{10^{-3}}\cup 2^{-j_i}E^r_{j_i})}|z|^\epsilon|(\pi^\epsilon_{p,q})^\perp s|^2_{H_\epsilon})}{-2j_i\log 2}\tag{3.1}$$

where $\pi^{\epsilon}_{p,q}$ denotes the pointwise projection given by $\mathcal{E}_{p,q} \subset \mathcal{E}$ with respect to the metric H_ϵ. However, over $(B_{2^{-j_i-1}} \setminus \overline{B_{2^{-j_i-2}}}) \setminus (\Sigma^{10^{-3}} \cup 2^{-j_i} E^r_{j_i})$, we have

$$(\pi^{\epsilon}_{p,q})^{\perp} s = (\pi_{\epsilon,(p,q)})^{\perp} s$$

where $\pi_{\epsilon,(p,q)}$ denotes the pointwise projection given by $\mathcal{E}_{p,q} \subset \mathcal{E}$ with respect to the metric $\pi^* \underline{H}_\epsilon$. Then we have

$$|(\pi^{\epsilon}_{p,q})^{\perp} s|^2_{H_\epsilon} = |z|^{2\mu(\mathcal{E}_p/\mathcal{E}_{p-1})} |(\pi_{\epsilon,(p,q)})^{\perp} s|^2_{\pi^* \underline{H}_\epsilon}.$$

By plugging this into Equation (3.1), we have

$$\begin{aligned} d_{\mathcal{E}_{p,q}}(s) &\leq \lim_{j_i} \frac{\log(2^{2j_i n} \int_{(B_{2^{-j_i-1}} \setminus \overline{B_{2^{-j_i-2}}}) \setminus (\Sigma^{10^{-3}} \cup 2^{-j_i} E^r_{j_i})} |z|^{2\mu(\mathcal{E}_p/\mathcal{E}_{p-1})+\epsilon} |(\pi_{\epsilon,(p,q)})^{\perp} s|^2_{\pi^* \underline{H}_\epsilon}}{-2j_i \log 2} \\ &= \mu(\underline{\mathcal{E}}_{p,q+1}/\underline{\mathcal{E}}_{p,q}) + \frac{\epsilon}{2}. \end{aligned}$$

By letting $\epsilon \to 0$, we have

$$d_{\mathcal{E}_{p,q}}(s) \leq \mu(\underline{\mathcal{E}}_{p,q+1}/\underline{\mathcal{E}}_{p,q}).$$

This finishes the proof. □

Given this, we can finish the proof of Theorem 1.1 by repeating what we did in the semistable case by replacing the Harder-Narasimhan filtration with a Harder-Narasimhan-Seshadri filtration. Then the induction is on (p, q) instead of p. For the base case, we can achieve $(a)_{1,1}$ and $(b)_{1,1}$ exactly the same as the semistable case by replacing $\mathcal{E}_1$ with $\mathcal{E}_{1,1}$. For $(c)_{1,1}$, otherwise, assume for some $s \in HG_{1,2}$ (without loss of generality we assume $q_1 \geq 2$), $d_{\mathcal{E}_{1,1}}(s) < \mu(\mathcal{E}_{1,2}/\mathcal{E}_{1,2})$. As $(a)_{1,1}$, we get a nontrivial map as

$$\underline{\psi}_{1,2} : (\underline{\mathcal{E}}_{1,2}/\underline{\mathcal{E}}_{1,1})^{**} \to \underline{\mathcal{S}}_{1,2}$$

where $\underline{\mathcal{S}}_{1,2}$ is polystable with $\mu(\underline{\mathcal{S}}_{1,2}) < \mu(\underline{\mathcal{E}}_{1,2}/\underline{\mathcal{E}}_{1,1}) = \mu((\underline{\mathcal{E}}_{1,2}/\underline{\mathcal{E}}_{1,1})^{**})$. This is impossile by Proposition 7.11 in [25]. When doing induction, we can achieve $(a)_{p,q}$ and $(b)_{p,q}$ exactly in the same way as the semistable case as well. For $(c)_{p,q}$, this is done exactly the same as $(c)_{1,1}$ above. This finishes the proof.

3.2 Construction of comparison metrics

Before starting the proof of Proposition 3.3, we need a lemma concerning the existence of a good cut-off function.

Lemma 3.5. *For any fixed $N >> 1$, there exists $R(N) \in (0, 10^{-7})$ which is decreasing with respect to N, and a constant $C = C(N) > 0$ so that for any $R \in (0, R(N)]$, there exists a smooth function $\chi_R : B^* \to [0, 1]$ such that the following holds on B^**

- $\chi_R|_{\Sigma^{R_r}_r} \equiv 1$;
- $\chi_R|_{S^{2n-1}(r) \setminus \Sigma^{200R_r}_r} \equiv 0$;
- $|\nabla \chi_R| \leq C R_r^{-1}$;

- $|\nabla^2 \chi_R| \leq C R_r^{-2}$, *where* $R_r = Rr^N$.

Proof. Let $\phi : \mathbb{R} \to [0,1]$ be a smooth function so that $\int_{\mathbb{R}} \phi(t)dt = 1$, $\mathrm{Supp}(\phi) = \{t \in \mathbb{R} : |t| \leq 2\}$ and ϕ is equal to 1 near 0. Define $\psi_r : S^{2n-1}(r) \to \mathbb{R}$ by setting $\psi_r = 1$ for $x \in \Sigma_r^{100R_r}$ and $\psi_r = 0$ for $x \in S^{2n-1}(r) \setminus \Sigma_r^{100R_r}$. Define $f_r : S^{2n-1}(r) \to [0,1]$ by

$$f_r(x) = c(r) \int_{S^{2n-1}(r)} \psi_r(y) \phi_{R_r}(d_{S^{2n-1}(r)}(x,y)) \mathrm{dVol}_{S^{2n-1}(r)}(y)$$

where $r = |x|$, $\phi_{R_r}(t) = \phi(\frac{t}{R_r})$ and $c(r)$ is a constant independent of x given by

$$c(r)^{-1} = \int_{S^{2n-1}(r)} \phi_{R_r}(d_{S^{2n-1}(r)}(x,y)) \mathrm{dVol}_{S^{2n-1}(r)}(y).$$

Since $\phi_{R_r}(d_{S^{2n-1}(r)}(x,y)) \equiv 1$ if $d_{S^{2n-1}(r)}(x,y)$ is small, we know f_r is indeed smooth. It is also direct to see $f_r = 0$ on $S^{2n-1}(r) \setminus \Sigma_r^{150R_r}$, $f_r = 1$ on $\Sigma_r^{50R_r}$, and furthermore $|\nabla f_r| \leq C R_r^{-1}$ and $|\nabla^2 f_r| \leq C R_r^{-2}$ for some constant $C = C(N)$. We define a smooth function $\chi_R : B^* \to [0,1]$ by

$$\chi_R(x) = \int_{\mathbb{R}} f_t(t\underline{x}) \cdot \frac{1}{R_r} \phi(\frac{1}{R_r}(r-t))dt$$

where $x = r\underline{x}$. Now we verify χ_R satisfies the desired properties for $0 < R \leq R(N)$, where

$$R(N) = \min\{\frac{1}{2}[(\frac{4}{3})^{\frac{1}{N-1}} - 1], \frac{1}{2}(1 - 50^{-\frac{1}{N-1}})\}.$$

The choice of $R(N)$ comes from the discussion below.

- If $x \in S^{2n-1}(r) \setminus \Sigma_r^{200R_r}$, then $t\underline{x} \in S^{2n-1}(t) \setminus \Sigma_t^{200\frac{tR_r}{r}}$. If $|\frac{r-t}{R_r}| \leq 2$, i.e., $(1 - 2\frac{R_r}{r})r \leq t \leq (1 + 2\frac{R_r}{r})r$, then $\Sigma_t^{150R_t} \subset \Sigma_t^{200(1+2\frac{R_r}{r})^{-N+1}R_t} \subset \Sigma_t^{200t\frac{R_r}{r}}$ for $0 < R \leq \frac{1}{2}[(\frac{4}{3})^{\frac{1}{N-1}} - 1]$, thus $t\underline{x} \in S^{2n-1}(t) \setminus \Sigma_t^{150R_t}$. This implies $f_t(t\underline{x}) = 0$, thus by definition $\chi_R(x) = 0$. So we have
$$\chi_R(x)|_{S^{2n-1}(r) \setminus \Sigma_r^{200R_r}} \equiv 0;$$
- For $x \in \Sigma_r^{R_r}$, if $|\frac{r-t}{R_r}| \leq 2$, i.e., $(1 - 2\frac{R_r}{r})r \leq t \leq (1 + 2\frac{R_r}{r})r$, then $\Sigma_t^{t\frac{R_r}{r}} \subset \Sigma_t^{(1-2\frac{R_r}{r})^{-N+1}R_t} \subset \Sigma_t^{50R_t}$ for $0 < R \leq \frac{1}{2}(1 - 50^{-\frac{1}{N-1}})$, thus $t\underline{x} \in \Sigma_t^{50R_t}$. This implies $f_t(t\underline{x}) = 1$, thus by definition $\chi_R(x) = 1$. So we have
$$\chi_R|_{\Sigma_r^{R_r}} \equiv 1;$$
- Denote by ∂_r the unit radial vector field and $r^{-1}\partial_\theta$ any unit vector field tangential to the sphere $S^{2n-1}(r)$. Then
$$\begin{aligned}|r^{-1}\partial_\theta \chi_R| &= |\int_{\mathbb{R}} r^{-1}\partial_\theta(f_t(t\underline{x})) \cdot \frac{1}{R_r}\phi(\frac{1}{R_r}(r-t))dt| \\ &\leq C \int_{\mathbb{R}} R_t^{-1} \frac{1}{R_r}\phi(\frac{1}{R_r}(r-t))dt \\ &\leq C \int_{|r-t| \leq 2R_r} R_t^{-1} R_r^{-1} dt \\ &\leq C R_r^{-1}\end{aligned}$$

and

$$
\begin{aligned}
|\partial_r \chi_R| &= \int_{\mathbb{R}} f_t(t\underline{x}) \cdot \partial_r(\frac{1}{R_r}\phi(\frac{1}{R_r}(r-t)))dt \\
&\leq C \int_{|r-t|\leq 2R_r} R_r^{-2} dt \\
&\leq C R_r^{-1}.
\end{aligned}
$$

So $|\nabla \chi_R| \leq C R_r^{-1}$;

- Similarly one can show $|\nabla^2 \chi_R| \leq C R_r^{-2}$.

This finishes the proof. □

Now we finish the proof of Proposition 3.3.

Proof of Proposition 3.3. Let K be the constant given by Proposition 2.13. For any $0 < \epsilon' << 1$, let $\underline{H}_{\epsilon'}$ be the metric on $\underline{\mathcal{E}}$ satisfying the properties listed in Proposition 2.13 with $\delta = 10^{-4}$, i.e.,

(1). $\sup_i \int_{\mathbb{CP}^{n-1}} |\underline{\beta}_i|^2_{\omega_{FS}} \leq \epsilon'$;

(2). $\sup_i \int_{\mathbb{CP}^{n-1}} |\Lambda_{\omega_{FS}} \bar{\partial}_{\underline{\mathcal{E}}} \underline{\beta}_i| \leq \epsilon'$;

(3). $\int_{\mathbb{CP}^{n-1}} |\sqrt{-1}\Lambda_{\omega_{FS}} F_{(\underline{H}_{\epsilon'}, \bar{\partial}_{\underline{S}})} - \psi^{\underline{H}_{\epsilon'}}| \leq \epsilon'$;

(4). $|\Lambda_{\omega_{FS}} F_{(\underline{H}_{\epsilon'}, \bar{\partial}_{\underline{\mathcal{E}}})}|_{L^\infty} \leq K$;

(5). $|\sqrt{-1}\Lambda_{\omega_{FS}} F_{(\underline{H}_{\epsilon'}, \bar{\partial}_{\underline{S}})} - \psi^{\underline{H}_{\epsilon'}}|(z) \leq \epsilon'$ for $z \notin \underline{\Sigma}^{10^{-4}}$;

(6). $\sup_i |\underline{\beta}_i(z)|_{\omega_{FS}} \leq \epsilon'$ for $z \notin \underline{\Sigma}^{10^{-4}}$;

(7). $\sup_i |\Lambda_{\omega_{FS}} \partial_{\underline{\mathcal{E}}} \underline{\beta}_i(z)| \leq \epsilon'$ for $z \notin \underline{\Sigma}^{10^{-4}}$.

Fix $N >> \mu_1$, and let $R(N)$ be given by Lemma 3.5. Let $R \in (0, R(N)]$ be determined later. Denote $H_{\epsilon'} = \pi^* \underline{H}_{\epsilon'}$, and apply Lemma 2.15 with $g = \sum_{i=1}^m f_i(\pi_i - \pi_{i-1})$ where

$$
f_i = (1-\chi_R)|z|^{\mu_i} + \chi_R
$$

and χ_R is given by Lemma 3.5. Since μ_i is strictly decreasing, $\sup_{i\leq j} \frac{f_i}{f_j} \leq 1$. In the following, we will estimate T_i for $i = 0, 1, 2$ given by Lemma 2.15 separately and we also use $A \lesssim B$ to denote $A \leq CB$ for some constant $C = C(n, m, N, K, \mu_1)$. For simplicity we introduce one more notation (see Figure 1)

- $\Sigma' = \bigcup_{r\in(0,1)} \Sigma_r^{200R_r}$, $\Sigma'' = \bigcup_{r\in(0,1)} \Sigma_r^{R_r}$.

(A). For $T_0 = F_{(H_{\epsilon'}, g\cdot\bar{\partial}_S)} = F_{(H_{\epsilon'}, \bar{\partial}_S)} - \sum_i \partial\bar{\partial}\log(f_i^2)(\pi_i - \pi_{i-1})$. Since $f_i = (1-\chi_R)|z|^{\mu_i} + \chi_R$, we have

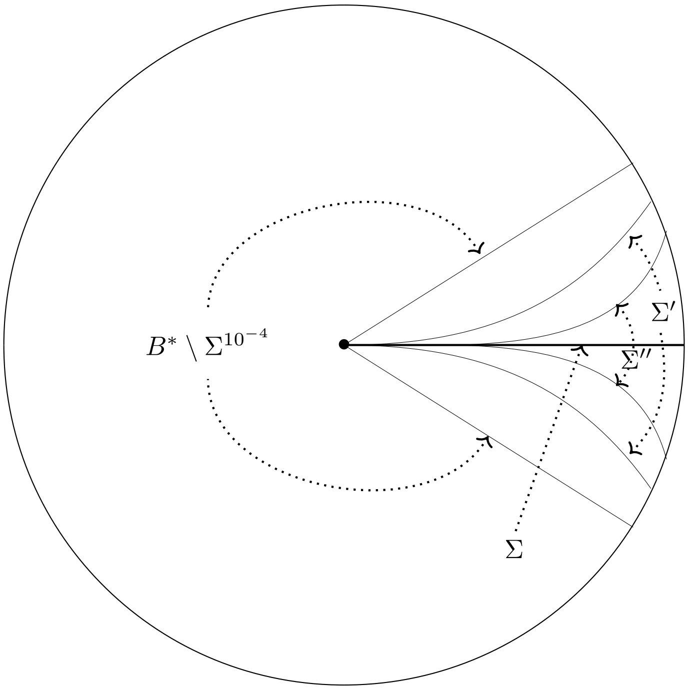

Figure 1: The cut-off

(a). For $z \notin \Sigma'$,

$$T_0 = \pi^* F_{(\underline{H}_{\epsilon'}, \bar{\partial}_{\underline{S}})} - \sum_i \mu_i \partial\bar{\partial} \log |z|^2 (\pi_i - \pi_{i-1}),$$

$$\Lambda_{\omega_0} T_0 = \frac{\pi^*(\Lambda_{\omega_{FS}} F_{(\underline{H}_{\epsilon'}, \bar{\partial}_{\underline{S}})} + \sqrt{-1}\psi^{\underline{H}_{\epsilon'}})}{r^2}$$

As a result, we have

$$|\Lambda_{\omega_0} T_0| \lesssim \frac{|\sqrt{-1}\Lambda_{\omega_{FS}} F_{(\underline{H}_{\epsilon'}, \bar{\partial}_{\underline{S}})} - \psi^{\underline{H}_{\epsilon'}}|}{r^2},$$

and

$$|T_0|_{\omega_0} \lesssim \frac{|F_{(\underline{H}_{\epsilon'}, \bar{\partial}_{\underline{\mathcal{E}}})}|_{\omega_{FS}} + \sup_i |\underline{\beta}_i|^2_{\omega_{FS}} + 1}{r^2}.$$

(b). Similarly, for $z \in \Sigma'$, using the fact that μ_i is strictly decreasing, we have

$$|T_0|_{\omega_0} \lesssim \frac{|F_{(\underline{H}_{\epsilon'}, \bar{\partial}_{\underline{\mathcal{E}}})}|_{\omega_{FS}} + \sup_i |\underline{\beta}_i|^2_{\omega_{FS}} + R_r^{-2} r^{-2\mu_1}}{r^2},$$

and

$$|\Lambda_{\omega_0} T_0| \lesssim \frac{K + \sup_i |\underline{\beta}_i|^2_{\omega_{FS}} + R_r^{-2} r^{-2\mu_1}}{r^2}.$$

Combining (a) and (b), and using Items (1) and (3) above, we get

$$\begin{aligned}
&\sup_{r\in(0,1]} r^{-(2n-1)}\int_{S^{2n-1}(r)} r^2|\Lambda_{\omega_0}T_0|\\
\lesssim&\int_{\mathbb{CP}^{n-1}}|\sqrt{-1}\Lambda_{\omega_{FS}}F_{(\underline{H}_{\epsilon'},\bar\partial_{\underline{S}})}-\psi^{\underline{H}_{\epsilon'}}|\\
&+\sup_{r\in(0,1]} r^{-(2n-1)}\int_{\Sigma'\cap S^{2n-1}(r)}(K+\sup_i|\underline{\beta}_i|^2_{\omega_{FS}}+R_r^{-2}r^{-2\mu_1})\\
\lesssim&\epsilon'+R^2,
\end{aligned}$$

where the last step we used Lemma 3.5 and the fact that $\underline{\Sigma}$ has complex codimension at least two. By Proposition 2.17 we have

$$|T_0|_{\omega_0}\in L^{1+\delta}(B^*)$$

for some $\delta>0$. Also, for $z\notin\Sigma^{10^{-4}}$, by Item (5) above,

$$r^2|\Lambda_{\omega_0}T_0|\leq|\Lambda_{\omega_{FS}}F_{(\underline{H}_{\epsilon'},\bar\partial_{\underline{S}})}+\sqrt{-1}\psi^{\underline{H}_{\epsilon'}}|\leq\epsilon'.$$

(B). For $T_1=-(g\cdot\bar\partial_S)(g\beta g^{-1})^*+(g\cdot\bar\partial_S)^*(g\beta g^{-1})$, we have

$$\begin{aligned}
(g\cdot\bar\partial_S)(g\beta g^{-1})^*=&\sum_{i<j}\frac{f_i}{f_j}(\pi_j-\pi_{j-1})(\bar\partial_S\beta^*)(\pi_i-\pi_{i-1})\\
&-2\sum_{i<j}\bar\partial(\frac{f_i}{f_j})\wedge(\pi_j-\pi_{j-1})(\partial_{\mathcal{E}}\pi_i)(\pi_i-\pi_{i-1}).
\end{aligned}$$

Plugging in $f_i=(1-\chi_R)|z|^{\mu_i}+\chi_R$, we have

(a). If $z\notin\Sigma'$, then $\frac{f_i}{f_j}=|z|^{\mu_i-\mu_j}$ where $\mu_i>\mu_j$. As a result,

$$|\Lambda_{\omega_0}(g\cdot\bar\partial_S)(g\beta g^{-1})^*|\lesssim\frac{\sup_i|\underline{\beta}_i|^2_{\omega_{FS}}+\sup_i|\Lambda_{\omega_{FS}}\partial_{\underline{\mathcal{E}}}\underline{\beta}_i|+r^{1+\mu'}\sup_i|\underline{\beta}_i|_{\omega_{FS}}}{r^2}$$

and

$$\begin{aligned}
|(g\cdot\bar\partial_S)(g\beta g^{-1})^*|_{\omega_0}\lesssim&\frac{\sup_i|\underline{\beta}_i|^2_{\omega_{FS}}+\sup_i|\partial_{\underline{\mathcal{E}}}\underline{\beta}_i|_{\omega_{FS}}+r^{1+\mu'}\sup_i|\underline{\beta}_i|_{\omega_{FS}}}{r^2}\\
\lesssim&\frac{2\sup_i|\underline{\beta}_i|^2_{\omega_{FS}}+|F_{(\underline{H}_{\epsilon'},\bar\partial_{\underline{\mathcal{E}}})}|_{\omega_{FS}}+r^{1+\mu'}\sup_i|\underline{\beta}_i|_{\omega_{FS}}}{r^2},
\end{aligned}$$

where the second inequality follows from Equation (2.7). Here $\mu'=\min\{i<j:\mu_i-\mu_j\}$.

(b). If $z\in\Sigma'$, using $|\bar\partial\frac{f_i}{f_j}|\lesssim R_r^{-1}r^{-\mu_1-1}$, we get

$$|\Lambda_{\omega_0}(g\cdot\bar\partial_S)(g\beta g^{-1})^*|\lesssim\frac{\sup_i|\underline{\beta}_i|^2_{\omega_{FS}}+\sup_i|\Lambda_{\omega_{FS}}\partial_{\underline{\mathcal{E}}}\underline{\beta}_i|_{\omega_{FS}}+R_r^{-2}r^{-2\mu_1}}{r^2}$$

and

$$\begin{aligned}
|(g\cdot\bar\partial_S)(g\beta g^{-1})^*|_{\omega_0}\lesssim&\frac{\sup_i|\underline{\beta}_i|^2_{\omega_{FS}}+\sup_i|\partial_{\underline{\mathcal{E}}}\underline{\beta}_i|_{\omega_{FS}}+R_r^{-2}r^{-2\mu_1}}{r^2}\\
\lesssim&\frac{2\sup_i|\underline{\beta}_i|^2_{\omega_{FS}}+|F_{(\underline{H},\bar\partial_{\underline{\mathcal{E}}})}|_{\omega_{FS}}+R_r^{-2}r^{-2\mu_1}}{r^2}
\end{aligned}$$

Now combining the estimates in (a) and (b), and using Items (1) and (2), we have

$$\begin{aligned}
&\sup_{r\in(0,1]} r^{-(2n-1)}\int_{S^{2n-1}(r)} r^2|\Lambda_{\omega_{FS}}T_1| \\
&\lesssim \int_{\mathbb{CP}^{n-1}} \sup_i |\underline{\beta}_i|^2_{\omega_{FS}} + \sup_i |\Lambda_{\omega_{FS}}\partial_{\underline{\mathcal{E}}}\underline{\beta}_i|_{\omega_{FS}} + r^{-(2n-1)}\int_{\Sigma'} R_r^{-2}r^{-2\mu_1} \\
&\le \epsilon' + R^2.
\end{aligned}$$

Similar to the estimate for T_0, by Proposition 2.17, we have

$$|T_1| \in L^{1+\delta}(B^*).$$

Also, for $z \notin \Sigma^{10^{-4}}$, by Items (6) and (7),

$$r^2|\Lambda_{\omega_0}T_1| \le \sup_i |\underline{\beta}_i|^2_{\omega_{FS}} + \sup_i |\Lambda_{\omega_{FS}}\partial_{\underline{\mathcal{E}}}\underline{\beta}_i| + \sup_i r^{1+\mu'}|\underline{\beta}_i|_{\omega_{FS}} \lesssim \epsilon'.$$

(C). For $T_2 = -g\beta g^{-1} \wedge (g\beta g^{-1})^* - (g\beta g^{-1})^* \wedge g\beta g^{-1}$, we have

$$|T_2|_{\omega_0} \le 2|g\beta g^{-1}|^2_{\omega_0} \lesssim (\sup_{i<j}|\frac{f_i}{f_j}|^2)\sup_i|\beta_i|^2_{\omega_0} \lesssim \frac{\sup_i|\underline{\beta}_i|^2_{\omega_{FS}}}{r^2}. \qquad (3.2)$$

Here, the second inequality follows from

$$\begin{aligned}
g\beta g^{-1} &= -\sum_{i,j,k}\frac{f_i}{f_j}(\pi_i-\pi_{i-1})(\pi_k-\pi_{k-1})\bar{\partial}_{\mathcal{E}}\pi_k(\pi_j-\pi_{j-1}) \\
&= -\sum_{i,j}\frac{f_i}{f_j}(\pi_i-\pi_{i-1})\bar{\partial}_{\mathcal{E}}\pi_i(\pi_j-\pi_{j-1}) \\
&= -\sum_{i<j}\frac{f_i}{f_j}(\pi_i-\pi_{i-1})\bar{\partial}_{\mathcal{E}}\pi_i(\pi_j-\pi_{j-1})
\end{aligned}$$

The last equality follows from $\bar{\partial}_{\mathcal{E}}\pi_i \cdot \pi_i = 0$ (π_i's are all weakly holomorphic). As a result, we have

$$\sup_{r\in(0,1]} r^{-(2n-1)}\int_{S^{2n-1}(r)} r^2|\Lambda_{\omega_0}T_2| \lesssim \epsilon'$$

and

$$|T_2|_{\omega_0} \in L^{1+\delta}(B^*)$$

for some $\delta > 0$ as $|T_0|$ and $|T_1|$. Also, for $z \notin \Sigma^{10^{-4}}$, $r^2|\Lambda_{\omega_0}T_2| \le \epsilon'$.

Now combining (A), (B), (C), we have

$$\sup_{r\in(0,1]} r^{-(2n-1)}\int_{S^{2n-1}(r)} r^2|\Lambda_{\omega_0}F_{(H_{\epsilon'},g\cdot\bar{\partial}_{\mathcal{E}})}| \lesssim \epsilon' + R^2,$$

and for $z \notin \Sigma^{10^{-4}}$,

$$r^2|\Lambda_{\omega_0}F_{(H_{\epsilon'},g\cdot\bar{\partial}_{\mathcal{E}})}| \lesssim \epsilon'.$$

Since $|F_{(H_{\epsilon'},g\cdot\bar{\partial}_\mathcal{E})}| \le |T_0|+|T_1|+|T_2|$, we have $|F_{(H_{\epsilon'},g\cdot\bar{\partial}_\mathcal{E})}| \in L^{1+\delta}(B^*)$ for some $\delta > 0$. For any $0<\epsilon << 1$, choose ϵ' and R small so that $\epsilon' + R^2 << \epsilon$ and let

$$H_\epsilon = H_{\epsilon'}(g\cdot, g\cdot) = \sum_i (1-\chi_R)|z|^{2\mu_i} H_{\epsilon'}((\pi_i - \pi_{i-1})\cdot, (\pi_i - \pi_{i-1})\cdot) + \chi_R H_{\epsilon'}.$$

The calculation above shows that H_ϵ satisfies (i), (ii) and (iii). It suffices to verify that H_ϵ satisfies (iv). For any $s \in HG_i \backslash HG_{i-1}$, we have $(\pi_i - \pi_{i-1})s \neq 0$, so

$$\begin{aligned} &\int_{B_r^*\backslash \Sigma^{10-3}} |z|^\epsilon |s|^2_{H_\epsilon} \\ =&\int_{B_r^*\backslash \Sigma^{10-3}} \sum_{j\le i} |z|^{\epsilon+2\mu_j} H_{\epsilon'}((\pi_j-\pi_{j-1})s,(\pi_j-\pi_{j-1})s) \\ =&\sum_{j\le i} a_j r^{\epsilon+2\mu_i+2n} \end{aligned}$$

where $a_i \neq 0$. Thus by taking limit $r \to 0$, we have

$$\lim_{r\to 0}\frac{1}{2}\frac{\log \int_{B_r^*\backslash\Sigma^{10-3}} |z|^\epsilon |s|^2_{H_\epsilon}}{\log r} - n = \mu_i + \frac{\epsilon}{2}.$$

This finishes the proof. □

3.3 Uniqueness of bubbling set with multiplicities

3.3.1 Chern-Simons transgression

In this section, we will collect some well-known results about the Chern-Simons transgression. Fix Δ to be smoothly isomorphic to $\{z \in \mathbb{C}^2 : |z| \le 1\}$ and let E be a complex vector bundle of rank $m \ge 2$ over Δ with a preferred smooth trivialization over $\partial\Delta$ (indeed E is always abstractly trivial). Then any connection A defined on $E|_{\partial\Delta}$ can be viewed as a smooth one form and the Chern-Simons form is defined as

$$CS(A) = \mathrm{Tr}(dA\wedge A + \frac{2}{3}A\wedge A\wedge A).$$

Given two such connections A and B, we also define the relative Chern-Simons transgression form as

$$CS(A,B) := \mathrm{Tr}(d_B a\wedge a + \frac{2}{3} a\wedge a\wedge a + 2a\wedge F_B).$$

Note $CS(A) = CS(A,0)$. Given a smooth isomorphism $g : E|_{\partial\Delta} \to E|_{\partial\Delta}$, we define the (complex) gauge transform of a connection A on $E|_{\partial\Delta}$ as

$$g\cdot A = gAg^{-1} - dg\cdot g^{-1}.$$

Lemma 3.6. *The following holds*

(a). if A extends to a smooth connection of E over the whole Δ, then

$$\int_{\partial\Delta} CS(A) = \int_\Delta Tr(F_A\wedge F_A);$$

(b). $\int_{\partial\Delta} CS(A,B) = \int_{\partial\Delta} CS(A) - \int_{\partial\Delta} CS(B)$;

(c). *For any g as above, $CS(g\cdot A, g\cdot B) = CS(A,B)$. In particular the relative Chern-Simons transgression does not depend on the choice of the common trivialization of $E|_{\partial\Delta}$;*

(d). $\deg(g) := \int_{\partial\Delta} CS(g\cdot A, A) \in 8\pi^2\mathbb{Z}$ *is independent of A and it only depends on the isotopy class of g. Moroever,* $\deg(g_1g_2) = \deg(g_1) + \deg(g_2)$;

(e). *If g extends to be an isomorphism of E over Δ, then* $\deg(g) = 0$.

Proof. (a) follows from the fact that

$$d(CS(A)) = Tr(F_A \wedge F_A).$$

(b) also follows from a direct calculation

$$CS(A) = CS(B) + CS(A,B) + d\mathrm{Tr}(a \wedge B).$$

For (c) we write $g\cdot A - g\cdot B = g(A-B)g^{-1}$. Denote $a = A-B$ and $g\cdot a = gag^{-1}$. Then we have

$$\begin{aligned}
CS(g\cdot A, g\cdot B) &= \mathrm{Tr}(d_{g\cdot B} g\cdot a \wedge g\cdot a + \frac{2}{3} g\cdot a \wedge g\cdot a \wedge g\cdot a + 2g\cdot a \wedge F_{g\cdot B}) \\
&= Tr(g(d_B a \wedge a + \frac{2}{3} a \wedge a \wedge a + 2a \wedge F_B)g^{-1}) \\
&= CS(A,B).
\end{aligned}$$

For (d), by (b) and (c), we have

$$\begin{aligned}
\int_{\partial\Delta} CS(A, g\cdot A) - CS(B, g\cdot B) &= \int_{\partial\Delta} CS(A) - CS(g\cdot A) - CS(B) + CS(g\cdot B) \\
&= \int_{\partial\Delta} CS(A,B) - CS(g\cdot A, g\cdot B) \\
&= \int_{\partial\Delta} CS(A,B) - CS(A,B) \\
&= 0.
\end{aligned}$$

To see $\deg(g) \in 8\pi^2\mathbb{Z}$, we take the trivial connection A_0 on E over Δ, so $CS(A_0) = 0$. Then we take another copy of A_0 and glue these two together along $\partial\Delta$ using g to form a connection A_1 on a bundle over S^4. Then we have

$$\deg(g) = CS(g\cdot A_0, A_0) = CS(g\cdot A_0) - CS(A_0) = \int_{S^4} Tr(F_{A_1} \wedge F_{A_1}) \in 8\pi^2\mathbb{Z}.$$

Also we have

$$\begin{aligned}
\deg(g_1g_2) &= \int_{\partial\Delta} CS(A, g_1g_2A) \\
&= \int_{\partial\Delta} CS(A, g_1A) + \int_{\partial\Delta} CS(g_1A, g_1g_2A) \\
&= \int_{\partial\Delta} CS(A, g_1A) + \int_{\partial\Delta} CS(A, g_2A) \\
&= \deg(g_1) + \deg(g_2).
\end{aligned}$$

(e) follows by using the same gluing argument. □

3.3.2 Uniqueness of bubbling set with multiplicities

Now we will finish the proof of uniqueness of bubbling sets with multiplicities. By Theorem 1.1, given any analytic tangent cone $(\mathcal{E}_\infty, A_\infty, \Sigma, \mu)$,we know that the connection A_∞ is given by the admissible Hermitian-Yang-Mills connection on $(Gr^{HNS}(\underline{\mathcal{E}}))^{**}$. More specifically, write

$$(Gr^{HNS}(\underline{\mathcal{E}}))^{**} = \oplus_l \underline{Q}_l,$$

where each $\underline{Q}_l$ is a stable reflexive sheave on $\mathbb{CP}^{n-1}$. Let $\underline{S}$ denote the set where $(Gr^{HNS}(\underline{\mathcal{E}}))^{**}$ is not locally free and μ_l denote the slope of $\underline{Q}_l$. Then Theorem 1.1 tells us that away from $\pi^{-1}(\underline{S})$, we have

$$(\mathcal{E}_\infty, A_\infty, H_\infty) = (\pi^*(Gr^{HNS}(\underline{\mathcal{E}}))^{**}, \bigoplus_l(\pi^*\underline{A}_l + \mu_l \partial \log |z|^2 \mathrm{Id}_{\pi^*\underline{Q}_l}), \bigoplus_l |z|^{2\mu_l}\pi^*\underline{H}_l)$$

where $(\underline{A}_l, \underline{H}_l)$ is the (unique) admissible Hermitian-Yang-Mills connection over $\underline{Q}_l$. In particular

$$\mathrm{Sing}(A_\infty) = \pi^{-1}(\underline{S}).$$

In the following, we denote

$$\underline{A} := \oplus_l \underline{A}_l$$

and

$$a_\infty := \oplus_l \mu_l \partial \log |z|^2 \mathrm{Id}_{\pi^*\underline{Q}_l}.$$

Let (A_∞, Σ, μ) be an analytic tangent cone associated to a subsequence $\{j_i\} \subset \{i\}$. Let $\underline{H}'$ be a fixed smooth Hermitian metric on $\underline{\mathcal{E}}$ and let $\underline{A}'$ be the Chern connection of $(\underline{H}', \bar{\partial}_{\mathcal{E}})$. Denote $H' = \pi^*\underline{H}'$. Following the convention in Section 2.3, there exits a unitary isomorphism P outside Σ

$$P : (\mathcal{E}, H') \to (\pi^*(Gr^{HNS}(\underline{\mathcal{E}}))^{**}, H_\infty)$$

and a sequence of unitary isomorphisms $\{g_i\}$ of $(\mathcal{E}, H')$ so that $\{g_i \cdot A_{j_i}\}_i$ converge to P^*A_∞ smoothly outside Σ. Here A_{j_i} denotes the Chern connection associated to $(H', f_i \circ \bar{\partial}_{\pi^*\underline{\mathcal{E}}} \circ f_i^{-1})$ where $f_i = (H'^{-1}(2^{-j_i})^*H)^{\frac{1}{2}}$. We fix a Harder-Narasimhan-Seshadri filtration for $\underline{\mathcal{E}}$ as

$$0 \subset \underline{\mathcal{E}}_1 \subset \cdots \underline{\mathcal{E}}_m = \underline{\mathcal{E}}.$$

Let $\underline{Q}_l'$ be the orthogonal complement of $\underline{\mathcal{E}}_{l-1}$ in $\underline{\mathcal{E}}_l$ with respect to $\underline{H}'$. By doing orthogonal projection, we can identify $\underline{\mathcal{E}}$ smoothly with $\oplus_l \underline{Q}_l'$ away from $\mathrm{Sing}(Gr^{HNS}(\underline{\mathcal{E}})) \subset \Sigma$

$$\underline{\rho} : \underline{\mathcal{E}} \to Gr^{HNS}(\underline{\mathcal{E}}).$$

We also denote $\rho = \pi^*\underline{\rho}$. Let $\underline{\iota} : Gr^{HNS}(\underline{\mathcal{E}}) \hookrightarrow (Gr^{HNS}(\underline{\mathcal{E}}))^{**}$ be the natural inclusion map.

Now we will follow the discussion in [36]. Let $\underline{\Sigma}^{alg}$ denote the proper analytic subvarierty in $\mathbb{CP}^{n-1}$ where $Gr^{HNS}(\underline{\mathcal{E}})$ is not locally free. Define

$$\underline{\mathcal{T}} := (Gr^{HNS}(\underline{\mathcal{E}}))^{**}/Gr^{HNS}(\underline{\mathcal{E}})$$

which is a torsion sheaf over $\mathbb{CP}^{n-1}$. Then we have

$$\underline{\Sigma}^{alg} = \mathrm{supp}(\underline{\mathcal{T}}) \cup \underline{S}.$$

By Proposition 2.3 in [36], on the complement of $\underline{S}$, $\underline{\Sigma}^{alg}$ has pure complex codimension 2. Then we define $\underline{\Sigma}_b^{alg}$ as the union of irreducible codimension 2 components in $\underline{\Sigma}^{alg}$. For each irreducible component $\underline{\Sigma}_k$ of $\underline{\Sigma}_b^{alg}$, we can associate an algebraic multiplicity m_k^{alg} to $\underline{\Sigma}_k$ by letting

$$m_k^{alg} := h^0(\underline{\Delta}, \underline{\mathcal{T}}|_{\underline{\Delta}})$$

where $\underline{\Delta}$ is a holomorphic transverse slice at a generic point of $\underline{\Sigma}_k$. We write

$$\Sigma_b^{alg} = \sum_k m_k^{alg} \Sigma_k$$

where $\Sigma_k = \pi^{-1}(\underline{\Sigma}_k)$.

Given an irreducible component Σ_k of $\Sigma_b^{an} \cup \Sigma_b^{alg}$, it has been shown how to calculate the algebraic multiplicity in [36]. More specifically, choose a class $[\underline{\Delta}]$ in $H_4(\mathbb{CP}^{n-1}, \mathbb{Q})$ whose intersection product with $\underline{\Sigma}_k$ is nonzero and $[\underline{\Delta}]$ can be represented as a codimension 2 subvarierty $\underline{\Delta}$ of $\mathbb{CP}^{n-1}$ which intersects $\underline{\Sigma}_k$ transversally and positively at points $\{\underline{z}_1, \cdots, \underline{z}_N\}$. Now we have the following (see Equation (4.5) in [36]),

$$\begin{aligned} Nm_k^{alg} = ([\underline{\Delta}] \cdot [\underline{\Sigma}_k]) m_k^{alg} = \int_{\cup_{l=1}^N \underline{\Delta} \cap \underline{B}_\sigma(z_l)} \frac{1}{8\pi^2} \{\mathrm{tr}(F_{\underline{A}'} \wedge F_{\underline{A}'}) - \mathrm{tr}(F_{\underline{\tau}^* \underline{A}} \wedge F_{\underline{\tau}^* \underline{A}})\} \\ - \int_{\cup_{l=1}^N \underline{\Delta} \cap \partial(\underline{B}_\sigma(z_l))} \frac{1}{8\pi^2} CS(\underline{A}', \underline{\tau}^* \underline{A}) \end{aligned} \tag{3.3}$$

where $\underline{\tau} = \underline{\iota} \circ \underline{\rho}$. In [36], the result is only stated for an irreducible component of Σ_b^{alg} but the calculation obviously holds for any codimension 2 subvarierty (indeed, if Σ_k is not a component of Σ_b^{alg}, then $m_k^{alg} = 0$). By choosing σ small, for each $\underline{\Delta}_l$, we can choose a holomorphic lifting Δ_l in $B_{2^{-1}} \setminus \overline{B_{2^{-2}}}$. Then Equation (3.3) can be rewritten as

$$\begin{aligned} Nm_k^{alg} = \int_{\cup_{l=1}^N \Delta_l} \frac{1}{8\pi^2} \{\mathrm{tr}(F_{\pi^* \underline{A}'} \wedge F_{\pi^* \underline{A}'}) - \mathrm{tr}(F_{\pi^* \underline{A}} \wedge F_{\pi^* \underline{A}})\} \\ - \int_{\cup_{l=1}^N \partial \Delta_l} \frac{1}{8\pi^2} CS(\pi^* \underline{A}', \tau^*(\pi^* \underline{A})). \end{aligned} \tag{3.4}$$

where $\tau = \pi^* \underline{\tau}$.

Corollary 3.7.

$$Nm_k^{alg} = Nm_k^{an} - \lim_{j_i} \int_{\cup_{l=1}^N \partial \Delta_l} \frac{1}{8\pi^2} CS(A_{j_i}, \tau^* A_\infty).$$

Proof. By Lemma 2.29 and Equation (3.4), using Lemma 3.6 we get

$$\begin{aligned}
Nm_k^{an} &= \lim_{i\to\infty}\int_{\cup_{l=1}^N \Delta_l} \frac{1}{8\pi^2}\{\mathrm{tr}(F_{A_{j_i}}\wedge F_{A_{j_i}}) - \mathrm{tr}(F_{A_\infty}\wedge F_{A_\infty})\} \\
&= Nm_k^{alg} - \lim_{i\to\infty}\frac{1}{8\pi^2}\int_{\cup_{l=1}^N \partial\Delta_l} CS(\pi^*\underline{A}', \tau^*(\pi^*\underline{A})) + CS(A_{j_i}, \pi^*\underline{A}') + CS(\pi^*\underline{A}, A_\infty) \\
&= Nm_k^{alg} - \lim_{i\to\infty}\frac{1}{8\pi^2}\int_{\cup_{l=1}^N \partial\Delta_l} CS(A_{j_i}, \tau^*(\pi^*\underline{A})) + CS(\tau^*(\pi^*\underline{A}), \tau^* A_\infty) \\
&= Nm_k^{alg} - \lim_{i\to\infty}\frac{1}{8\pi^2}\int_{U_{l=1}^N \partial\Delta_l} CS(A_{j_i}, \tau^* A_\infty).
\end{aligned}$$

□

Now the second part of Theorem 1.1 follows from the following Proposition combined with Corollary 3.7.

Proposition 3.8. *For all* $1 \le l \le N$*, we have*

$$\lim_{i\to\infty}\int_{\partial\Delta_l} CS(A_{j_i}, \tau^* A_\infty) = 0.$$

Proof. First, we have

$$\begin{aligned}
\int_{\partial\Delta_l} CS(A_{j_i}, \tau^* A_\infty) &= \int_{\partial\Delta_l} CS(g_i\cdot A_{j_i}, g_i\cdot(P^{-1}\tau)^* P^* A_\infty) \\
&= \int_{\partial\Delta_l} CS(g_i\cdot A_{j_i}, g_i\cdot(P^{-1}\tau)^{-1}\cdot P^* A_\infty)
\end{aligned}$$

We claim for i large, on $\mathcal{E}|_{\partial\Delta_l}$, we have

$$\deg(P^{-1}\tau) = \deg(g_i).$$

Given this claim, by Lemma 3.6, we have

$$\int_{\partial\Delta_l} CS(A_{j_i}, \tau^* A_\infty) = \int_{\partial\Delta_l} CS(g_i\cdot A_{j_i}, P^* A_\infty)$$

which goes to 0 since $\{g_i\cdot A_{j_i}\}_i$ converge to $P^* A_\infty$ smoothly away from Σ.

Now we prove the Claim. The key point is that in our proof of Theorem 1.1 (see Section 3.1.1), the homogeneous map ψ_l we constructed to identify $\pi^*(\underline{\mathcal{E}}_l/\underline{\mathcal{E}}_{l-1})^{**}$ with $\pi^*\underline{Q}_l$ is given by (away from Σ)

$$\psi_l = P\lim_{i\to\infty}\frac{(\pi_l^i - \pi_{l-1}^i)(g_i f_i(\pi_l - \pi_{l-1}))}{a_i}$$

Here π_l^i denotes the orthogonal projection from $\mathcal{E}$ to $(g_i f_i)(\pi^*\underline{\mathcal{E}}_l)$ with respect to the metric H', π_l denote the orthogonal projection from $\mathcal{E}$ to $\pi^*\underline{\mathcal{E}}_l$ with respect to H' and a_i is suitable normalizing constant (see the proof in Section 3.1.1). Since the map between $\underline{\mathcal{E}}_l/\underline{\mathcal{E}}_{l-1}$ and $(\underline{\mathcal{E}}_l/\underline{\mathcal{E}}_{l-1})^{**}$ which induces an isomorphism of $(\mathcal{E}_l/\mathcal{E}_{l-1})^{**}$ is unique up to rescaling, we can assume $\tau = \bigoplus_l \psi_l$ by re-normalizing a_i. Write

$$h_i := \frac{(\pi_l^i - \pi_{l-1}^i)(g_i f_i(\pi_l - \pi_{l-1}))}{a_i},$$

then it is easy to see that h_i is smoothly homotopic to $g_i f_i$ away from Σ. Indeed, consider for $t \in [0,1]$ the family

$$F_t = \frac{(\pi_l^i - \pi_{l-1}^i)(g_i f_i(\pi_l - \pi_{l-1})) + t\sum_{l_1,l_2<l}(\pi_{l_1}^i - \pi_{l_1-1}^i)(g_i f_i(\pi_{l_2} - \pi_{l_2-1}))}{(1-t)ta_i + t}$$

is a family of complex gauge transformations satisfying $F_0 = h_i$ and $F_1 = g_i f_i$. Now since f_i is defined over $B \setminus \{0\}$, we know by Lemma 3.6 that $\deg(f_i) = 0$ on $\mathcal{E}|_{\partial\Delta_l}$. So for i large, we have

$$\begin{aligned} \deg(P^{-1}\tau) &= \deg(\lim_{i\to\infty} g_i f_i) \\ &= \lim_{i\to\infty} \deg g_i. \end{aligned}$$

This finishes the proof of the claim. □

4 Examples

In this section, we will prove Corollary 1.3. We first state a lemma to construct reflexive sheaves in general. Suppose $\{f_1, \cdots f_k\}$ is a regular sequence of holomorphic function over an open subset $U \subset \mathbb{C}^n$ i.e.

$$\mathrm{Codim}_{\mathbb{C}}(\mathrm{Zero}(f_1, \cdots f_k)) = n - k.$$

Denote $u := (f_1, \cdots f_k) \in \mathcal{O}^{\oplus k}$. Consider the coherent sheaf $\mathcal{E}$ given by the following exact sequence

$$0 \to \mathcal{O} \xrightarrow{u} \mathcal{O}^{\oplus k} \to \mathcal{E} \to 0.$$

Lemma 4.1. *$\mathcal{E}$ is a reflexive sheaf over U for $k \geq 3$.*

Proof. Indeed, since $\mathrm{Codim}_{\mathbb{C}}(\mathrm{Zero}(f_1, \cdots f_k)) = n - k$, by Lemma on Page 688 in [17], the following Koszul complex given by u is exact over U

$$0 \to \mathcal{O} \xrightarrow{\wedge u} \mathcal{O}^{\oplus k} \xrightarrow{\wedge u} \wedge^2\mathcal{O}^{\oplus k} \xrightarrow{\wedge u} \wedge^3\mathcal{O}^{\oplus k} \cdots \xrightarrow{\wedge u} \mathcal{I} \to 0$$

where $\mathcal{I}$ is the ideal sheaf generated by $\{f_1, \cdots f_k\}$. By exactness of the above sequence and the definition of $\mathcal{E}$, we have the following exact sequence

$$0 \to \mathcal{E} \to \wedge^2\mathcal{O}^{\oplus k} \to \wedge^3\mathcal{O}^{\oplus k}$$

which implies $\mathcal{E}$ is reflexive by Proposition 5.22 in [25]. □

Now we discuss a class of *local* examples. Over $\mathbb{CP}^2$ we denote by $\underline{\mathcal{E}}_k$ the locally free rank 2 sheaf defined by the exact sequence

$$0 \to \mathcal{O}_{\mathbb{CP}^2} \xrightarrow{f_k} \mathcal{O}_{\mathbb{CP}^2}(1) \oplus \mathcal{O}_{\mathbb{CP}^2}(1) \oplus \mathcal{O}_{\mathbb{CP}^2}(k) \to \underline{\mathcal{E}}_k \to 0$$

where $f_k = (z_1, z_2, z_3^k)$. Let $\mathcal{E}_k = \psi_*\pi^*\underline{\mathcal{E}}_k$.

It is easy to see $c_1(\underline{\mathcal{E}}_k) = k + 2$. By using the criteria given by Lemma 1.2.5 in [29], we can easily get the following

- If $k = 1$, then $\underline{\mathcal{E}}_k$ is stable;

- If $k=2$, then $\underline{\mathcal{E}}_k$ is semistable, and a Seshadri filtration is given by

$$0 \to \mathcal{O}_{\mathbb{CP}^2}(k) \to \underline{\mathcal{E}}_k \to \mathcal{I}_{[0:0:1]} \otimes \mathcal{O}_{\mathbb{CP}^2}(2) \to 0,$$

where $\mathcal{I}_{[0:0:1]}$ is the ideal sheaf of the point $[0:0:1]$, and the first map is induced by the inclusion $\mathcal{O}_{\mathbb{CP}^2}(k) \hookrightarrow \mathcal{O}_{\mathbb{CP}^2}(1) \oplus \mathcal{O}_{\mathbb{CP}^2}(1) \oplus \mathcal{O}_{\mathbb{CP}^2}(k)$.

- If $k>2$, then $\underline{\mathcal{E}}_k$ is unstable and the Harder-Narasimhan filtration is given by

$$0 \to \mathcal{O}_{\mathbb{CP}^2}(k) \to \underline{\mathcal{E}}_k \to \mathcal{I}_{[0:0:1]} \otimes \mathcal{O}_{\mathbb{CP}^2}(2) \to 0.$$

When $k \geq 2$, we have

$$Gr^{HNS}(\underline{\mathcal{E}}_k) = \mathcal{O}_{\mathbb{CP}^2}(k) \oplus (\mathcal{I}_{[0:0:1]} \otimes \mathcal{O}_{\mathbb{CP}^2}(2)),$$

so

$$\psi_* \pi^* (Gr^{HNS}(\underline{\mathcal{E}}_k))^{**} = \mathcal{O}_{\mathbb{C}^3}^{\oplus 2},$$

and the algebraic bubbling set

$$\Sigma_b^{alg} = \{0\} \times \mathbb{C}_{z_3} \subset \mathbb{C}^3$$

with multiplicity 1.

Now suppose A is an admissible Hermitian-Yang-Mills connection on $\mathcal{E}_k|_B (k \geq 2)$, and let (A_∞, Σ, μ) be the unique analytic tangent cone of A at 0, then by Theorem 1.1, we know A_∞ is the trivial flat connection on $\mathcal{O}_{\mathbb{C}^3}^{\oplus 2}$, and the bubbling set is Σ_b, $\mu = [\Sigma_b]$. In particular, the analytic tangent cones, as defined in this paper, are all the same for all $k \geq 2$.

Remark 4.2. It is interesting to see how we can interpret the integer k here in terms of the admissible Hermitian-Yang-Mills connections on $\mathcal{E}_k$.

Finally, to prove Corollary 1.3, we consider a global example. On $\mathbb{CP}^3$, we let $\mathcal{E}$ be given as follows

$$0 \to \mathcal{O}_{\mathbb{CP}^3} \xrightarrow{\sigma} \mathcal{O}_{\mathbb{CP}^3}(2) \oplus \mathcal{O}_{\mathbb{CP}^3}(1) \oplus \mathcal{O}_{\mathbb{CP}^3}(2) \to \mathcal{E} \to 0.$$

where $\sigma = (z_1^2, z_2, z_3 z_4)$.

Lemma 4.3. *$\mathcal{E}$ is a rank 2 stable reflexive sheaf with singular set given by* $\{[0,0,0,1],[0,0,1,0]\}$.

Proof. It is obvious that $\mathcal{E}$ is locally free away from $\{[0,0,0,1],[0,0,1,0]\}$. Near $[0,0,0,1]$, since $[0,0,0,1]$ is an isolated common zero of $\{z_1^2, z_2, z_3 z_4\}$, by Lemma 4.1, we know $\mathcal{E}$ is reflexive near $[0,0,0,1]$. Similarly, $\mathcal{E}$ is also reflexive at $[0,0,1,0]$. Since $H^0(\mathbb{CP}^{n-1}, \mathcal{E} \otimes \mathcal{O}_{\mathbb{CP}^3}(-3)) = 0$, $\mathcal{E}$ is stable. □

By Theorem 2 in [4], choosing any smooth Kähler metric ω on $\mathbb{CP}^3$, there exists an admissible Hermitian-Yang-Mills connection A on $\mathcal{E}$ with singularities at $p_0 = [0,0,0,1]$ and $p_1 = [0,0,1,0]$. We will apply our Theorem 1.1 to study the local behavior of A near p_0 and p_1. Locally around p_0, $\mathcal{E}$ is isomorphic to $\mathcal{E}_2$ (as defined as above). The same is true at p_1 by symmetry. So we see $\mathcal{E}$ provides an example in Corollary 1.3. Strictly speaking, the underlying metric ω here is not flat, but as we pointed out in the beginning of Section 2, this does not cause technical difficulties.

5 General point singularities

In general, the point singularity of reflexive sheaves can be nonhomogeneous, i.e. not necessarily modelled on the pull-back of vector bundles on projective spaces (see Example 1 in Section 5.2). In this section, we will first give some general discussion about possible candidates for the algebraic data as in the homogeneous case and later will introduce an algebraic construction, called *Hecke Transform* (see section 6.1), to construct some *optimal* algebraic tangent cone for reflexive sheaves at any singular point (not necessarily isolated), which is unique in a suitable sense.

5.1 Proof of Theorem 1.4

Going back to the general setting in the introduction, we let A be an admissible Hermtian-Yang-Mills connection on B with vanishing Einstein constant, and with an isolated singularity at 0. Let $\mathcal{E}$ be the corresponding reflexive sheaf. In this section, we shall use the following notations

- $p : \widehat{B} \to B$ denotes the blow-up of B at 0. We can identify $\widehat{B}$ naturally with an open neighborhood of the zero section in the total space of the line bundle $\mathcal{O}(-1) \to \mathbb{CP}^{n-1}$;
- $i : p^{-1}(0) \simeq \mathbb{CP}^{n-1} \to \widehat{B}$ denotes the obvious inclusion map;
- $\phi : \widehat{B} \to \mathbb{CP}^{n-1}$ denotes the restriction of the projection map $\mathcal{O}(-1) \to \mathbb{CP}^{n-1}$;

Definition 5.1. An algebraic tangent cone of $\mathcal{E}$ at 0 is a coherent sheaf on $\mathbb{CP}^{n-1}$ which is given by the restriction of a reflexives sheaf $\mathcal{F}$ on $\widehat{B}$, such that $\mathcal{F}|_{\widehat{B}\setminus p^{-1}(0)}$ is isomorphic to $p^*(\mathcal{E}|_{B\setminus\{0\}})$.

For the convenience of reader we digress to discuss the notion of restriction of reflexive coherent analytic sheaves. The corresponding theory in the category of algebraic geometry is well-known. By definition for a coherent sheaf $\mathcal{F}$ on a complex manifold X and a smooth divisor D, the restriction $\mathcal{F}|_D$ is given by the pull-back of $\mathcal{F}$ under the inclusion map $i : D \to X$.

Lemma 5.2. *If $\mathcal{F}$ is reflexive then $\mathcal{F}|_D$ is torsion free.*

Proof. It suffices to prove the stalk of $\mathcal{F}|_D$ at any point p is torsion free. For this purpose we can work in the local holomorphic coordinates $z_1, z_2, \cdots, z_n$ centered p and assume D is locally given by $\{z_1 = 0\}$. Since $\mathcal{F}$ is reflexive we can find a local short exact sequence in a neighborhood U of p of the form

$$0 \to \mathcal{F} \to \mathcal{O}_U^{n_1} \xrightarrow{\phi} \mathcal{O}_U^{n_2}.$$

This can be achieved, for example, by first choosing a locally free resolution of $\mathcal{F}^*$ and then taking dual. Suppose $s \in (\mathcal{F}|_D)_p$ is a non-zero torsion. Then there is a local holomorphic function $f = f(z_2, \cdots, z_n)$ such that $f \cdot s = 0$. By definition we can write $s = [\eta]$ for an element η of $\mathcal{F}_p$ with $\eta \notin z_1 \cdot \mathcal{F}_p$. Then $f \cdot s = 0$ implies that there is a nonzero element $\lambda \in \mathcal{F}_p$ such that $f\eta = z_1\lambda$. Using the above short exact sequence we can view both η and λ as elements of $(\mathcal{O}_X^{n_1})_p$, which implies that $\eta = z_1\eta'$ for some η' in $(\mathcal{O}_X^{n_1})_p$. Since $\phi(\eta) = \phi(z_1\eta') = z_1\phi(\eta') = 0$, we know $\phi(\eta') = 0$ i.e. $\eta' \in \mathcal{F}_p$ which contradicts with $\eta \notin z_1\mathcal{F}_p$. This finishes the proof. □

Remark 5.3. Using this we can give an alternative description of $\mathcal{F}|_D$, as the subsheaf of $\mathcal{O}_D^{n_1}$ generated by the restriction of local holomorphic sections of $\mathcal{F}$, viewed as sections of $\mathcal{O}_X^{n_1}$. Indeed, if we denote the latter sheaf by $\mathcal{F}'$, then there is an obvious surjective homomorphism ψ from $\mathcal{F}|_D$ to $\mathcal{F}'$. Notice since the quotient $\mathcal{O}_X^{n_1}/\mathcal{F}$ is torsion free, we know $\mathcal{F}$ is locally free outside a codimension two complex analytic subvariety of X and the map $\mathcal{F} \to \mathcal{O}_X^{n_1}$ realizes $\mathcal{F}$ as a sub-bundle of the trivial bundle. So outside a divisor in D the restriction $\mathcal{F}|_D$ is locally free and ψ is an isomorphism. Thus the kernel of ψ is necessarily torsion and the above Lemma implies it has to be zero. This description is often helpful when dealing with explicit examples. For example, if $\mathcal{F}$ is the reflexive sheaf on $\mathbb{C}^3$ defined as the kernel of the map $\mathcal{O}_{\mathbb{C}^3} \to \mathcal{O}_{\mathbb{C}^3}$ given by (z_1, z_2, z_3), and D is the divisor $z_1 = 0$, then $\mathcal{F}|_D$ is isomorphic to $\mathcal{O}_{\mathbb{C}^2} \oplus \mathcal{I}_0$, where $\mathcal{I}_0$ is the ideal sheaf of the origin in $\mathbb{C}^2$.

Remark 5.4. Lemma 5.2 is not true if $\mathcal{F}$ is only torsion-free. For example, it is easy to see that for the ideal sheaf $\mathcal{I}_0$ of the origin in $\mathbb{C}^2$, the restriction to a line $\mathbb{C}$ through the origin indeed has torsion.

Lemma 5.2 implies that an algebraic tangent cone is always torsion free. But it is far from unique. For instance, one way to obtain an algebraic tangent cone is by first taking $(p^*\mathcal{E}^*)^*$, and then restrict to $p^{-1}(0)$. We will show how to calculate this algebraic tangent cone by examples in Section 5.2.

From now on we assume that there is an algebraic tangent cone $\underline{\widehat{\mathcal{E}}}$ which is locally free (i.e. defines a holomorphic vector bundle) on $\mathbb{CP}^{n-1}$. Denote by $\widehat{\mathcal{E}}$ the reflexive sheaf on $\widehat{B}$ that restricts to $\underline{\widehat{\mathcal{E}}}$ on $p^{-1}(0)$ and is isomorphic to the pull-back of $\mathcal{E}$ outside $p^{-1}(0)$. It is clear that $\widehat{\mathcal{E}}$ itself is also locally free.

Proposition 5.5. *For k large, the natural map*

$$r : H^0(\widehat{B}, \widehat{\mathcal{E}} \otimes \phi^*\mathcal{O}(k)) \to H^0(\mathbb{CP}^{n-1}, \underline{\widehat{\mathcal{E}}} \otimes \mathcal{O}(k))$$

is surjective.

Proof. This follows from a version of Ohsawa-Takegoshi extension theorem [28]. Fix the Kähler metric $\omega := p^*\omega_0 + \phi^*\omega_{FS}$ on $\widehat{B}$. Notice

$$\widehat{\mathcal{E}} \otimes \phi^*\mathcal{O}(k_0) = \widehat{\mathcal{E}} \otimes K_{\widehat{B}} \otimes \phi^*(\mathcal{O}(k_0 + n - 1)).$$

Fix a metric h_0 on $\widehat{\mathcal{E}} \otimes K_{\widehat{B}}$ and the metric h_{k_0} on $\phi^*(\mathcal{O}(k_0+n-1))$ given by the pull-back of the standard Hermitian metric on $\mathcal{O}(k_0+n-1) \to \mathbb{CP}^{n-1}$. Now we consider the metric on $\widehat{\mathcal{E}} \otimes \phi^*\mathcal{O}(k_0)$ given by $h = e^{-K|z|^2}h_0 \otimes h_{k_0}$. By choosing K and k_0 large, we can make the curvature operator $\Theta_h \geq 0$ in the Nakano sense. Then the claim follows from Theorem 4 in [28]. More precisely, using the notation in [28], we take the plurisubharmonic function to be $\psi = p^*(\log|z|^2)$, and take X to be the pre-image under ϕ of a hyperplane in $\mathbb{C}^{n-1}$. Then the conlusion follows if we choose $k \geq k_0$. □

Now we fix k large given by the above Proposition, replace $\widehat{\mathcal{E}}$ by $\widehat{\mathcal{E}} \otimes \phi^*\mathcal{O}(k)$, and assume $r : H^0(\widehat{B}, \widehat{\mathcal{E}}) \to H^0(\mathbb{CP}^{n-1}, \underline{\widehat{\mathcal{E}}})$ is surjective. We may assume $\underline{\widehat{\mathcal{E}}}$ is globally generated on $\mathbb{CP}^{n-1}$. Notice since $\mathcal{E}$ is reflexive, there is a natural map $\phi_*: H^0(\widehat{B}, \widehat{\mathcal{E}}) \to H^0(B, \mathcal{E})$. Denote by HG the image of ϕ_*.

Proposition 5.6. *Suppose $\underline{\widehat{\mathcal{E}}}$ is a semistable vector bundle on $\mathbb{CP}^{n-1}$, then for any $s \in HG \setminus \{0\}$, we have $d(s) = \mu(\underline{\widehat{\mathcal{E}}})$.*

Lemma 5.7. *There exists a constant $C > 0$ such that for any $\epsilon > 0$ there is a smooth Hermitian metric H_ϵ on $\mathcal{E}$ over B^* with the following properties*

(1). $\int_{B^*} |F_{(H_\epsilon, \bar{\partial}_{\mathcal{E}})}|^2 < \infty$;

(2). $|z|^2 |\Lambda_{\omega_0} F_{(H_\epsilon, \bar{\partial}_{\mathcal{E}})}(z)| \leq \epsilon + C|z|$ *for all* $z \in B^*$;

(3). For all $s \in HG \setminus \{0\}$,

$$\frac{1}{2} \lim_{r \to 0} \frac{\log \int_{B_r^*} |s|^2_{H_\epsilon}}{\log r} - n = \mu(\underline{\widehat{\mathcal{E}}}).$$

Assuming this, as in the proof of Lemma 3.2 we obtain that

$$C|z|^\epsilon H_\epsilon \leq H \leq C|z|^{-\epsilon} H_\epsilon, \tag{5.1}$$

and Proposition 5.6 follows easily.

Proof of Lemma 5.7. As in Section 3.1.1, for any $\epsilon > 0$ we can find a Hermitian metric $\underline{H}_\epsilon$ on $\underline{\widehat{\mathcal{E}}}$ such that $|\sqrt{-1}\Lambda_{\omega_{FS}} F_{\underline{A}_\epsilon} - \mu \mathrm{Id}|_{L^\infty} < \epsilon$ with $\mu = \mu(\underline{\widehat{\mathcal{E}}})$. Pulling back to $\widehat{B}$ by the map ϕ, we get a Hermitian metric H'_ϵ on $\mathcal{E}' := \phi^*(\underline{\widehat{\mathcal{E}}})$. Now by our assumption we know that $\widehat{\mathcal{E}}$ is also a vector bundle and it is isomorphic to $\mathcal{E}'$ as smooth complex vector bundles. Fixing any smooth isomorphism between these two which restricts to the natural identity map on $\underline{\widehat{\mathcal{E}}}$ over the exceptional divisor $\mathbb{CP}^{n-1}$, we may then view H'_ϵ naturally as a Hermitian metric on $\widehat{\mathcal{E}}$ too. Through this isomorphism we write $\beta = \bar{\partial}_{\widehat{\mathcal{E}}} - \bar{\partial}_{\mathcal{E}'}$, then the tangential component of the restriction of β to $\mathbb{CP}^{n-1}$ is zero. A direct computation shows

- $|\beta|_{\pi^*\omega_0} \leq C$;
- $|\partial_{\mathcal{E}'}\beta|_{\pi^*\omega_0} \leq C|z|^{-1}$.

So

$$|F_{(H'_\epsilon, \bar{\partial}_{\widehat{\mathcal{E}}})} - F_{(H'_\epsilon, \bar{\partial}_{\mathcal{E}'})}|_{\pi^*\omega_0} \leq C|z|^{-1}.$$

Now let $H_\epsilon = |z|^{2\mu} H'_\epsilon$ and using the map p we obtain a corresponding Hermitian metric on $\mathcal{E}|_{B^*}$, which we still denote by H_ϵ. Then it is clear that (1) and (2) hold. (3) follows from the fact that there exists C independent of r so that

$$C^{-1} r^{2n-1+2\mu} \leq \limsup_{r \to 0} \int_{\partial B_r} |s|^2_{H_\epsilon} \leq C r^{2n-1+2\mu}.$$

□

Now we prove Theorem 1.4. The idea is similar to that has been previously used in Section 3.1.1. Let $(\mathcal{E}_\infty, A_\infty)$ be a tangent cone of A at 0. We can build a homogeneous homomorphism $\tau: \pi^*\underline{\widehat{\mathcal{E}}} \to \mathcal{E}_\infty$ as follows. Fix a subspace V of $H^0(\widehat{B}, \widehat{\mathcal{E}})$ such that $r: V \to H^0(\mathbb{CP}^{n-1}, \underline{\widehat{\mathcal{E}}})$ is an isomorphism and we identify V with a subspace of $H^0(B, \mathcal{E})$ using the map ϕ_*. Choose a basis s_i of $H^0(\mathbb{CP}^{n-1}, \widehat{\underline{\mathcal{E}}})$ and correspondingly a basis σ_i of V. By Proposition 5.6 and by passing to a subsequence we may assume σ_i converges to homogeneous holomorphic sections $\sigma_{i,\infty}$ of $\mathcal{E}_\infty$. Let $M_j = \sup_i \|\sigma_i\|_j$, and let $\sigma'_{i,\infty}$ be the limit of $\frac{1}{M_j}\sigma_i$. Then $\sigma'_{i,\infty}$ is either zero or homogeneous of degree μ and there is at least one i such that $\sigma'_{i,\infty}$ is non-zero.

For any $x \in B^*$, and any η on the fiber $\pi^*\widehat{\underline{\mathcal{E}}}|_x$, we may write $\eta = \sum_i a_i \pi^* s_i(\pi(x))$. Then we define $\tau(\eta)$ to be $\sum_i a_i \sigma'_{i,\infty}(x)$. To see this is well-defined suppose a section $s = \sum_i a_i s_i \in H^0(\mathbb{CP}^{n-1}, \widehat{\underline{\mathcal{E}}})$ vanishes at $\pi(x)$, then we need to show the corresponding limit section $\sum_i a_i \sigma'_{i,\infty}$ vanishes at x. This follows from (5.1): let $\sigma = \sum_i a_i \sigma_i$, it is clear that $|\sigma(x)|_{H_\epsilon} \leq C|x|^{\mu+1}$, hence

$$|\sigma(x)|_H \leq C|x|^{\mu+1-\epsilon/2}.$$

On the other hand since $d(\sigma_i) = \mu$ for all i we have

$$M_j \geq C 2^{-j(\mu+\epsilon/2)}.$$

If we have chosen a priori that ϵ is sufficiently small then we know the corresponding limit of $\frac{1}{M_j}|\sigma(2^{-j}x)|$ is zero.

Now it is easy to see τ is indeed a non-trivial homogeneous homomorphism. As before using the stability of $\widehat{\underline{\mathcal{E}}}$ one can conclude that $\mathcal{E}_\infty$ is a simple HYM cone with holonomy $e^{-2\pi\sqrt{-1}\mu}$, and τ induces an isomorphism between $\widehat{\underline{\mathcal{E}}}$ and $\underline{\mathcal{E}}_\infty$. This finishes the proof of Theorem 1.4.

5.2 General discussion

Let $\mathcal{E}$ be a reflexive sheaf defined over the n-dimensional ball $B \subset \mathbb{C}^n$ with a (not necessarily isolated) singularity at 0.

Definition 5.8. We say 0 is a *homogeneous* singularity of $\mathcal{E}$ if there is a reflexive sheaf $\underline{\mathcal{E}}$ over $\mathbb{CP}^{n-1}$ such that $\iota_U^*\mathcal{E}$ is isomorphic to $\iota_U^*\iota_*\pi^*\underline{\mathcal{E}}$ for some neighborhood U of 0. Here $\iota : B^* \hookrightarrow B$ and $\iota_U : U \hookrightarrow B$ denote the inclusion maps.

We briefly recall the notion of *Fitting invariants*, following [15]. Choose a finitely generated free presentation of the stalk $\mathcal{E}_0$

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \to \mathcal{E}_0 \to 0,$$

we define the j-th *Fitting ideal* $\mathrm{Fitt}_j(\mathcal{E}, 0)$ of $\mathcal{E}$ at 0 to be the ideal of $\mathcal{O}_0$ given by the image of the $\mathcal{O}_0$-module homomorphism

$$\Lambda^{b-j}\phi : (\Lambda^{b-j}\mathcal{G})^* \otimes (\Lambda^{b-j}\mathcal{F}) \to \mathcal{O}_0,$$

where $b = rank(\mathcal{G})$. If we identify $\mathcal{F}$ with $\mathcal{O}_0^{\oplus a}$ and $\mathcal{G}$ with $\mathcal{O}_0^{\oplus b}$, and represent ϕ by a $\mathcal{O}_0$-valued matrix, then $\mathrm{Fitt}_j(\mathcal{E}, 0)$ is the ideal of $\mathcal{O}_0$ generated by all the $(b-j) \times (b-j)$ minors of the matrix. We make the convention that $\mathrm{Fitt}_j(\mathcal{E}, 0) = \mathcal{O}_0$ if $j \geq b$. It is not hard to show (see for example [15], Chapter 20) that for all j, $\mathrm{Fitt}_j(\mathcal{E}, 0)$ is a well-defined invariant of the stalk $\mathcal{E}_0$, i.e. it does not depend on the choice of the particular presentation.

The following is pointed out to us by Professor Jason Starr.

Proposition 5.9. *If $\mathcal{E}$ is homogeneous at* 0 *then all the corresponding Fitting ideals $Fitt_j(\mathcal{E}, 0)$ must be homogeneous ideals of $\mathcal{O}_0$.*

Remark 5.10. Here we say an ideal $\mathcal{I}$ of $\mathcal{O}_0$ is *homogeneous* if it is generated by homogeneous polynomials; it is not hard to see that for a homogeneous ideal $\mathcal{I}$, if a function f belongs to $\mathcal{I}$, then all the homogeneous components in the Taylor expansion of f at 0 also belong to $\mathcal{I}$.

Proof. Since the claimed property only depends on the local structure of $\mathcal{E}$ near 0, we may assume without loss of generality that on $B \setminus \{0\}$, $\mathcal{E}$ is isomorphic to $\pi^*\underline{\mathcal{E}}$ for some reflexive sheaf $\underline{\mathcal{E}}$ on $\mathbb{CP}^{n-1}$. Let l_0 be the smallest l such that $H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}}(l)) \neq 0$, and choose l_1 such that the maps $H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}} \otimes \mathcal{O}(l)) \otimes H^0(\mathbb{CP}^{n-1}, \mathcal{O}(1)) \to H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}} \otimes \mathcal{O}(l+1))$ are surjective for all $l \geq l_1$. Choosing a basis of $H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}} \otimes \mathcal{O}(l))$ for all $l \in [l_0, l_1]$ we obtain a surjective homomorphism

$$\underline{\phi} : \underline{\mathcal{F}} := \bigoplus_{l_0 \leq l \leq l_1} \mathcal{O}(-l)^{\oplus n_l} \to \underline{\mathcal{E}},$$

where $n_l = \dim H^0(\mathbb{CP}^{n-1}, \underline{\mathcal{E}} \otimes \mathcal{O}(l))$. Pulling-back to $B \setminus \{0\}$ and pushing forward to B, we obtain the corresponding map

$$\phi : \mathcal{F} \simeq \bigoplus_{l_0 \leq l \leq l_1} \mathcal{O}_B^{\oplus n_l} \to \mathcal{E}$$

We claim ϕ is surjective at 0. To see this we first notice that by definition ϕ is surjective on $B \setminus \{0\}$, so the sheaf $\mathcal{E}/\mathrm{Im}(\phi)$ is a torsion sheaf supported at the origin, hence there is an $m \geq 1$ such that $\mathrm{Im}(\phi)$ contains $\mathcal{I}_0^m \mathcal{E}$, where $\mathcal{I}_0$ is the ideal sheaf of 0. Similar to the proof of Lemma 2.44 (notice we did not use the HYM condition there), we know any local section s of $\mathcal{E}$ can be written as a Taylor series $s = \sum_{j \geq l_0} \pi^* s_j$, where s_j is a holomorphic section of $\underline{\mathcal{E}} \otimes \mathcal{O}(j)$, and we have used the natural identification $\pi^*(\mathcal{O}(-1)) \simeq \mathcal{O}_{B\setminus\{0\}}$. Now by our choice of l_1 and m it follows that $\pi^* s_j$ is a section of $\mathrm{Im}(\phi)$ for $j \leq l_1 + m$, and $\sum_{j \geq l_1+m} \pi^* s_j$ defines a germ of a section $\mathrm{Im}(\phi)$ in a neighborhood of 0. This proves the claim.

Applying similar discussion again, we get a locally free presentation of $\underline{\mathcal{E}}$

$$\underline{\mathcal{G}} \xrightarrow{\psi} \underline{\mathcal{F}} \xrightarrow{\phi} \underline{\mathcal{E}} \to 0,$$

where $\underline{\mathcal{G}}$ is also given by a direct sum of line bundles on $\mathbb{CP}^{n-1}$, and the map ψ is then represented by a matrix of homogeneous polynomials. Hence it also induces a corresponding locally free presentation of $\mathcal{E}$

$$\mathcal{G} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\phi} \mathcal{E} \to 0.$$

Then the conclusion follows directly. □

Using Proposition 5.9, one can easily find explicit examples of reflexive sheaves with non-homogeneous singularities. Now we consider the case $n = 3$. Let $\mathcal{E}_f$ be given by the short exact sequence

$$0 \to \mathcal{O}_B \xrightarrow{f} \mathcal{O}_B^{\oplus 3} \to \mathcal{E}_f \to 0, \tag{5.2}$$

where $f = (f_1, f_2, f_3)$ is a triple of holomorphic functions defined over B. We assume 0 is an isolated common zero of f. Then $\mathcal{E}_f$ is a rank two reflexive sheaf in a neighborhood of 0, by the Remark after Example 1.1.13 on Page 77, [29]. $\mathcal{E}_f$ has an isolated singularity at 0 and it follows from definition that $\mathrm{Fitt}_2(\mathcal{E}_f, 0)$ is the ideal of $\mathcal{O}_0$ generated by f_1, f_2, f_3. So if f_1, f_2, f_3 do not generate a homogeneous ideal then the corresponding $\mathcal{E}_f$ is not homogeneous.

As mentioned before, one possible algebraic tangent cone is given by $(p^*\mathcal{E}_f^*)^* \otimes \mathcal{O}_D$, which we denote by $\underline{\widehat{\mathcal{E}_f}}$. We will show that $\underline{\widehat{\mathcal{E}_f}}$ can be explicitly calculated under suitable assumption on f.

Lemma 5.11. *$\mathcal{E}_f \cong \mathcal{E}_f^*$ in a neighborhood of* 0.

Proof. This is actually true for any rank 2 reflexive sheaf $\mathcal{F}$ over B where B is the unit ball centered at 0 in $\mathbb{C}^n$. Indeed, we know $\det(\mathcal{F})$ is a holomorphic line bundle over B which has to be trivial. Let Θ be a global trivialization of $\det(\mathcal{F})$ over B. Then one can naturally define an isomorphism outside $Sing(\mathcal{F})$

$$\mathcal{F}^* \to \mathcal{F}, v \longmapsto i_v\Theta$$

where i_v denotes the contraction with v. By using $\mathcal{H}om(\mathcal{F}^*, \mathcal{F})$ and $\mathcal{H}om(\mathcal{F}, \mathcal{F}^*)$ are both reflexive, we know that the map above can be extended to be an isomorphism between $\mathcal{F}^*$ and $\mathcal{F}$. □

Using Lemma 5.11 and the fact that the pull-back functor is right-exact (See for example Page 7 in [16]), we obtain

$$\mathcal{O}_{\hat{B}} \xrightarrow{p^*f} \mathcal{O}_{\hat{B}}^{\oplus 3} \to p^*\mathcal{E}_f^* \to 0.$$

Taking dual we get

$$0 \to (p^*\mathcal{E}_f^*)^* \to \mathcal{O}_{\hat{B}}^{\oplus 3} \xrightarrow{(p^*f)^*} \mathcal{O}_{\hat{B}}.$$

Given $f = (f_1, f_2, f_3)$, we define a new triple $\widehat{f} = (\widehat{f_1}, \widehat{f_2}, \widehat{f_3})$ as follows. Let g_i be the homogeneous part of f_i which has the lowest degree in the Taylor expansion of f_i at 0 and let d be the smallest degree among the degrees of g_1, g_2, g_3. Then we define $\widehat{f_i} = g_i$ if the degree of g_i is d and $\widehat{f_i} = 0$ otherwise. Denote by $D = \mathbb{CP}^2$ the exceptional divisor of the map $p : \widehat{B} \to B$, and $[D]$ the corresponding line bundle on $\widehat{B}$. Then we can naturally view p^*f_i, $i = 1, 2, 3$ as a holomorphic section of $\mathcal{O}(-dD)$.

Lemma 5.12. *Suppose the common zero set of $(\hat{f}_1, \hat{f}_2, \hat{f}_3)$ consists of finitely many points in D, then $\underline{\widehat{\mathcal{E}_f}} \otimes \mathcal{O}(d)$ lies in the following exact sequence*

$$0 \to \underline{\widehat{\mathcal{E}_f}} \to \mathcal{O}_{\mathbb{CP}^2}^{\oplus 3} \xrightarrow{(\hat{f}_1, \hat{f}_2, \hat{f}_3)} \mathcal{O}_{\mathbb{CP}^2}(d) \to 0$$

Proof. By definition $\widehat{\mathcal{E}_f^*}$ is given by the kernel of the map $p^*f : \mathcal{O}_{\widehat{B}}^{\oplus 3} \to \mathcal{O}_{\widehat{B}}$. We work over a local chart U with coordinate $\{z_1, w_2, \cdots, w_n\}$ of $\widehat{B}$ so that the map p is given by $(z_1, w_2, \cdots, w_n) \mapsto (z_1, z_1w_2, \cdots, z_1w_n)$ and using the assumption, it follows from the assumption directly that $(z_1^{-d}f_1, z_1^{-d}f_2, z_1^{-d}f_3)$ forms a regular sequence over U since the set of their common zeros consists of isolated points. Indeed, by assumption, the only possible zeros over U must satisfy $z_1 = 0$ which implies $\hat{f}_1 = \hat{f}_2 = \hat{f}_3 = 0$. By our assumption, we know that it consists of finitely many points. Then it follows that $\widehat{\mathcal{E}_f^*}$ is generated by the sections $z_1^{-d}(f_2, -f_1, 0)$, $z_1^{-d}(f_3, 0, -f_1)$, and $z_1^{-d}(0, f_3, -f_2)$. By Remark 5.3 we know $\underline{\widehat{\mathcal{E}_f^*}} \subset \mathcal{O}_{U\cap\{z_1=0\}}^{\oplus 3}$ is generated by $z_1^{-d}(\widehat{f}_2, -\widehat{f}_1, 0)$, $z_1^{-d}(\widehat{f}_3, 0, -\widehat{f}_1)$, and $z_1^{-d}(0, \widehat{f}_3, -\widehat{f}_2)$. From this the conclusion follows easily. □

Corollary 5.13. *Suppose the common zero set of $(\hat{f}_1, \hat{f}_2, \hat{f}_3)$ is empty in D, $\underline{\hat{\mathcal{E}}_f}$ is always stable.*

Proof. Indeed, since slope stability is preserved under taking dual, it suffices to shows that $(\underline{\hat{\mathcal{E}}_f})^*$ is stable. We first assume $d = 2k$ for some $k \in \mathbb{Z}_+$. By Lemma 5.12, we know $c_1((\underline{\hat{\mathcal{E}}_f})^* \otimes \mathcal{O}(-k)) = 0$ and $(\underline{\hat{\mathcal{E}}_f})^* \otimes \mathcal{O}(-k)$ lies in the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{CP}^2}(-3k) \to (\mathcal{O}_{\mathbb{CP}^2}(-k))^{\oplus 3} \to (\underline{\hat{\mathcal{E}}_f})^* \otimes \mathcal{O}(-k) \to 0.$$

In particular, $H^0(\mathbb{CP}^2, (\underline{\hat{\mathcal{E}}_f})^* \otimes \mathcal{O}(-k)) = 0$ and the stability follows (see page 84 in [29]). When k is odd, the slope stability can be proved similarly. □

We finish this section with two examples.

Example 1. $f = (z_1^2 - z_1z_2z_3, z_2^2 - z_3^3, z_3^2 - z_1^3)$. The ideal $\mathrm{Fitt}(\mathcal{E}_f, 0)$ is not homogeneous, for otherwise the polynomials z_1^2, z_2^2 and z_3^2 must belong to the ideal generated by f_1, f_2, f_3, and it is easy to see this is impossible. Since f here satisfies the assumption in Corollary 5.13, the algebraic tangent cone $\underline{\widehat{\mathcal{E}_f}}$ defined above is a stable bundle on $\mathbb{CP}^2$. So our Theorem 1.4 applies here, yielding that any admissible Hermitian-Yang-Mills connection on the germ of $\mathcal{E}_f$ at 0 has a unique tangent cone which is a simple HYM cone defined by the Hermitian-Einstein metric on $\underline{\widehat{\mathcal{E}_f}}$.

Example 2. Let $\mathcal{E} \to \mathbb{CP}^3$ be given by the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{CP}^3} \xrightarrow{s} \mathcal{O}_{\mathbb{CP}^3}(3)^{\oplus 3} \to \mathcal{E} \to 0 \tag{5.3}$$

where $s = (z_0z_1^2 - z_1z_2z_3, z_0z_2^2 - z_3^3, z_0z_3^2 - z_1^3) \in H^0(\mathbb{CP}^3, \mathcal{O}(3)^{\oplus 3})$. We know by Bezout's theorem $\mathrm{Sing}(\mathcal{E}) = \{Z \in \mathbb{CP}^3 : s(Z) = 0\}$ which consists of 27 points (counted with multiplicities). Since $c_1(\mathcal{E}(-5)) = -1$ and $H^0(\mathbb{CP}^3, \mathcal{E}(-5)) = 0$, $\mathcal{E}$ is a stable reflexive sheaf.

So by Theorem 2.3 we know $\mathcal{E}$ admits an admissible Hermitian-Einstein metric. It is easy to see that $\mathrm{Sing}(\mathcal{E}) = \{Z_1, Z_2, \cdots, Z_{13}\}$, where $Z_1 = [1 : 0 : 0 : 0]$ is a zero of s with multiplicity 8, and locally around Z_1 the sheaf $\mathcal{E}$ is modeled exactly by Example 1; $Z_2 = [0 : 0 : 1 : 0]$ is also a zero of s with multiplicity 8, whose local model is more complicated; all the other Z_i's are simple zeroes of s locally around which $\mathcal{E}$ is homogeneous and is isomorphic to the pull-back of the tangent bundle of $\mathbb{CP}^2$. So using our results in this paper we know the tangent cones of the admissible Hermitian-Yang-Mills connection at Z_i for $i \neq 2$.

6 Optimal algebraic tangent cone

6.1 Hecke transform of reflexives sheaves

6.1.1 The case of sub-bundles

Let M be a complex manifold and D be a smooth hypersurface in M. Let E be a holomorphic vector bundle on M and denote $\underline{E} := E|_D$. Let $\underline{F}$ be a sub-bundle of $\underline{E}$. Let $\underline{Q}$ denote the quotient bundle $\underline{E}/\underline{F}$ and $\underline{p} : \underline{E} \to \underline{Q}$ the natural projection map. Then we have the following short exact sequence of vector bundles on D

$$0 \to \underline{F} \to \underline{E} \xrightarrow{p} \underline{Q} \to 0. \tag{6.1}$$

We will describe below a construction, called the *Hecke transform* along $\underline{F}$, that yields another vector bundle E' on M, which is isomorphic to E on $M \setminus D$, such that the restriction $\underline{E'} := E'|_D$ fits into an extension of the form

$$0 \to \underline{Q} \otimes N_D \to \underline{E'} \to \underline{F} \to 0, \tag{6.2}$$

where N_D is the normal bundle of D in M. In the next section we shall reinterpret it in terms of more complex-analytic language, which makes the construction more natural and generalizes to the case of coherent sheaves.

To start the construction, we choose an open cover $\{U_\alpha\}$ of a neighborhood U of D, such that $E|_{U_\alpha}$ admits a trivialization given by holomorphic sections $e_{\alpha,1}, \cdots, e_{\alpha,r}$, and such that if we denote $e_\alpha^j := e_\alpha|_{V_\alpha}$ where $V_\alpha := U_\alpha \cap D$, then $e_{\alpha,1}, \cdots, e_{\alpha,s}$ give a holomorphic trivialization of $F|_{V_\alpha}$, and $\underline{p}(e_{\alpha,s+1}), \cdots, \underline{p}(e_{\alpha,r})$ give a holomorphic trivialization of $\underline{Q}|_{V_\alpha}$. We may also assume that the divisor line bundle $[D]$ has a local trivialization t_α on each U_α. Choose a defining section s of $[D]$ so that we can write $s = s_\alpha t_\alpha$ over each U_α with s_α vanishing on D with exactly order one.

On the intersection $U_{\alpha\beta} := U_\alpha \cap U_\beta$, we can write the transition function of E as

$$\Phi_{\alpha\beta} = \begin{bmatrix} f_{\alpha\beta} & g_{\alpha\beta} \\ h_{\alpha\beta} & q_{\alpha\beta}. \end{bmatrix}$$

Denote $V_{\alpha\beta} := U_{\alpha\beta} \cap D$. Then the fact that $\underline{F}$ is a sub-bundle of $\underline{E}$ implies that $h_{\alpha\beta}|_{V_{\alpha\beta}} = 0$, and $g_{\alpha\beta}|_{V_{\alpha\beta}}$ defines the extension class in $\mathrm{Ext}^1(\underline{Q}, \underline{F})$ corresponding to the short exact sequence (6.1).

Now define a new holomorphic basis of $E|_{U_\alpha \setminus D}$ by setting $e'_{\alpha,j} = e_{\alpha,j}$ for $j \leq s$ and $e'_{\alpha,j} = s_\alpha e_{\alpha,j}$ for $j \geq s+1$. Then with respect to the new basis, the new transition matrix becomes

$$\Phi'_{\alpha\beta} = \begin{bmatrix} f_{\alpha\beta} & g_{\alpha\beta} s_\alpha \\ h_{\alpha\beta} s_\beta^{-1} & q_{\alpha\beta} s_\alpha s_\beta^{-1} \end{bmatrix}.$$

Now the entries of this matrix extend to be well-defined holomorphic functions across $V_{\alpha\beta}$. Hence it defines a holomorphic vector bundle on M, which is our desired E'. Moreover, since $s_\alpha s_\beta^{-1}$ is the transition function of the line bundle $[D]$, by adjunction formula, we see that by restricting to D, the right bottom component of $\Phi'_{\alpha\beta}$ gives the transition matrix for $\underline{Q} \otimes N_D^{-1}$. It is also clear that the whole matrix restricting to D is now a lower triangular matrix, so it is obvious that the exact sequence (6.2) holds.

One can check by definition that there is a well-defined vector bundle isomorphism from E' to E on $M \setminus D$, since by construction locally a holomorphic section of E' is a holomorphic section of E such that when restricting to D it belongs to $\underline{F}$. One can also check that the isomorphism class of E' does not depend on the choices made. It is also clear from the construction in the next subsection.

Remark 6.1. When $\dim M = 1$, $D = \{x\}$, $\underline{F}$ is a subspace of $E|_x$. In this case the above construction is usually referred to as the "*elementary modification*" or "*Hecke modification*" in the literature, and this justifies our choice of terminology.

6.1.2 General Case

Now we move on to the general case of coherent sheaves, using a more complex-algebraic language (which is kindly pointed out to us by Richard Thomas). We again suppose M is a smooth complex manifold and D is a smooth hypersurface. Let $\iota : D \to M$ be the natural inclusion map, and $\mathcal{E}$ be a reflexive sheaf on M. By lemma 5.2, we know that $\underline{\mathcal{E}} := \iota^*\mathcal{E}$ is a torsion-free coherent sheaf on D.

Let $\underline{\mathcal{F}}$ be a subsheaf of $\underline{\mathcal{E}}$ and $\underline{\mathcal{Q}}$ be the quotient sheaf. Denote $p : \mathcal{E} \to \iota_*(\underline{\mathcal{Q}})$ to be the map given by the composition of the natural surjective map $\mathcal{E} \to \iota_*\underline{\mathcal{E}}$ with the natural map $\iota_*\underline{\mathcal{E}} \to \iota_*\underline{\mathcal{Q}}$.

Lemma 6.2. *p is a surjective sheaf homomorphism.*

Proof. It suffices to show the map $\iota_*\underline{\mathcal{E}} \to \iota_*\underline{\mathcal{Q}}$ is surjective. By definition we have the following exact sequence

$$0 \to \underline{\mathcal{F}} \to \underline{\mathcal{E}} \to \underline{\mathcal{Q}} \to 0.$$

Since $\iota : D \hookrightarrow M$ is obviously *Stein*, namely, the pre-image of a Stein open set is Stein, the higher direct image $\mathcal{R}^i(\iota_*\underline{\mathcal{F}})$ vanishes for $i \geq 1$. In particular, the following is exact

$$0 \to \iota_*\underline{\mathcal{F}} \to \iota_*\underline{\mathcal{E}} \to \iota_*(\underline{\mathcal{Q}}) \to 0.$$

□

Definition 6.3. We define the *Hecke transform* $\mathcal{E}'$ of $\mathcal{E}$ along $\underline{\mathcal{F}}$ to be the kernel of the map p.

By definition, $\mathcal{E}'$ lies in the following short exact sequence

$$0 \to \mathcal{E}' \to \mathcal{E} \to \iota_*\underline{\mathcal{Q}} \to 0. \tag{6.3}$$

In particular $\mathcal{E}'$ is a subsheaf of $\mathcal{E}$ which is isomorphic to $\mathcal{E}$ over $M \setminus D$. In particular, it must be torsion-free. It is easy to check by definition that when $\mathcal{E}$ is locally free over M and $\underline{\mathcal{Q}}$ is locally free over D, this agrees with the construction in the previous subsection.

Lemma 6.4. *$\mathcal{E}'$ is reflexive if $\underline{\mathcal{F}}$ is saturated in $\underline{\mathcal{E}}$ or equivalently $\underline{\mathcal{Q}}$ is torsion-free.*

Proof. By Equation (6.3), we have the following exact sequence

$$0 \to (\mathcal{E}')^{**}/\mathcal{E}' \to \iota_* \underline{\mathcal{Q}}.$$

Since $\mathcal{I}_D \cdot \iota_* \underline{\mathcal{Q}} = 0$, we have $\mathcal{I}_D \cdot ((\mathcal{E}')^{**}/\mathcal{E}') = 0$. Then we have

$$(\mathcal{E}')^{**}/\mathcal{E}' = \iota_* \iota^* ((\mathcal{E}')^{**}/\mathcal{E}')$$

and the following exact sequence

$$0 \to \iota^*((\mathcal{E}')^{**}/\mathcal{E}') \to \iota^* \iota_* \underline{\mathcal{Q}} = \underline{\mathcal{Q}}.$$

Since $\mathcal{E}'$ is torsion-free and locally free outside D, $\mathrm{Supp}((\mathcal{E}')^{**}/\mathcal{E}')$ has codimension 1 in D, which implies $\iota^*((\mathcal{E}')^{**}/\mathcal{E}')$ is a torsion sheaf. Since $\underline{\mathcal{Q}}$ is torsion-free, by the exact sequence above, we have $\iota^*((\mathcal{E}')^{**}/\mathcal{E}') = 0$ which implies $(\mathcal{E}')^{**}/\mathcal{E}' = 0$. This finishes the proof. □

In our later applications we will always assume $\underline{\mathcal{F}}$ is saturated in $\underline{\mathcal{E}}$.

Lemma 6.5. *There exists the following exact sequence*

$$0 \to \mathcal{I}_D \cdot \mathcal{E} \to \mathcal{E}' \to \iota_* \underline{\mathcal{F}} \to 0. \tag{6.4}$$

Proof. By definition $\mathcal{E}'$ is exactly the pre-image of $\iota_* \underline{\mathcal{F}}$ under the natural map $\mathcal{E} \to \iota_* \underline{\mathcal{E}}$. So we have a natural surjective map $\mathcal{E}' \to \iota_* \underline{\mathcal{F}}$. The kernel of this map agrees with the kernel of the map $\mathcal{E} \to \iota_* \underline{\mathcal{E}}$, which is exactly $\mathcal{I}_D \cdot \mathcal{E}$. This finishes the proof. □

Denote $\underline{\mathcal{E}'} = \iota^* \mathcal{E}'$.

Proposition 6.6. *There exists the following exact sequence*

$$0 \to \underline{\mathcal{Q}} \otimes \mathcal{N}_D^* \to \underline{\mathcal{E}'} \to \underline{\mathcal{F}} \to 0,$$

where $\mathcal{N}_D^* \simeq \mathcal{I}_D / \mathcal{I}_D^2$ is the locally free sheaf associated to the co-normal bundle of D.

Proof. Applying ι^* to (6.4) we get the exact sequence

$$\iota^*(\mathcal{I}_D \cdot \mathcal{E}) \xrightarrow{\psi} \underline{\mathcal{E}'} \to \iota^* \iota_* \underline{\mathcal{F}} = \underline{\mathcal{F}} \to 0. \tag{6.5}$$

It suffices to prove $\mathrm{Ker}(\psi) = \underline{Q} \otimes \mathcal{N}_D^*$. By definition, ψ comes from the map $\mathcal{I}_D \cdot \mathcal{E} \to \mathcal{E}'$ by tensoring with $\mathcal{O}_D$, so the kernel is given by $\mathcal{I}_D \cdot \mathcal{E}' / \mathcal{I}_D^2 \cdot \mathcal{E}$. Since $\mathcal{I}_D$ is locally free, we have the following exact sequence

$$0 \to \mathcal{I}_D^2 \cdot \mathcal{E} \to \mathcal{I}_D \cdot \mathcal{E}' \to \mathcal{I}_D \otimes \iota_* \underline{\mathcal{F}} \to 0.$$

This implies that as $\mathcal{O}_M$-modules, we have

$$\mathcal{I}_D \cdot \mathcal{E}' / \mathcal{I}_D^2 \cdot \mathcal{E} = \mathcal{I}_D \otimes \iota_* \underline{\mathcal{F}} = \iota_*(\underline{\mathcal{F}} \otimes \mathcal{N}_D^*)$$

It is direct to check that the inclusion of $\mathrm{Ker}(\psi)$ in $\iota^*(\mathcal{I}_D \cdot \mathcal{E})$ is given by the natural map

$$\iota_* \underline{\mathcal{F}} \otimes \mathcal{N}_D^* \to \underline{\mathcal{E}} \otimes \mathcal{N}_D^*$$

under the natural identification $\iota^*(\mathcal{I}_D \cdot \mathcal{E}) = \underline{\mathcal{E}} \otimes \mathcal{N}_D^*$. Hence we see the image of ψ is given by

$$(\underline{\mathcal{E}} \otimes \mathcal{N}_D^*)/(\underline{\mathcal{F}} \otimes \mathcal{N}_D^*) = \underline{\mathcal{Q}} \otimes \mathcal{N}_D^*.$$

□

Now we will discuss some interesting properties of the Hecke transform. Let $\mathcal{E}''$ be the Hecke transform of $\mathcal{E}'$ along $\underline{\mathcal{Q}} \otimes \mathcal{N}_D^*$.

Lemma 6.7. *The Hecke transform is an involution up to twisting by* $[D]$ *in the sense that* $\mathcal{E}'' \cong \mathcal{E}(-[D])$.

Proof. By definition and Proposition 6.6, $\mathcal{E}''$ fits into the following exact sequence

$$0 \to \mathcal{E}'' \to \mathcal{E}' \to \iota_*(\underline{\mathcal{E}}'/(\mathcal{Q} \otimes \mathcal{N}_D^*)) = \iota_* \underline{\mathcal{F}} \to 0,$$

and the map $\mathcal{E}' \to \iota_* \underline{\mathcal{F}}$ agrees with the map in (6.4). By Lemma 6.5, $\mathcal{E}''$ is isomorphic to $\mathcal{I}_D \cdot \mathcal{E}$. □

More generally, we can take a subsheaf of $\underline{\mathcal{Q}} \otimes \mathcal{N}_D^*$ which has the form $(\underline{\mathcal{E}}_1/\underline{\mathcal{F}}) \otimes \mathcal{N}_D^*$, where $\underline{\mathcal{E}}_1 \subset \iota^* \mathcal{E}$ is a saturated subsheaf with $\underline{\mathcal{F}} \subset \underline{\mathcal{E}}_1$. Let $\mathcal{E}_1''$ be the Hecke transform of $\mathcal{E}'$ along $(\underline{\mathcal{E}}_1/\underline{\mathcal{F}}) \otimes \mathcal{N}_D^*$ and $\mathcal{E}_1'$ be the Hecke transform of $\mathcal{E}$ along $\underline{\mathcal{E}}_1$. Then the following involution property holds.

Proposition 6.8. $\mathcal{E}_1'' \simeq \mathcal{I}_D \cdot \mathcal{E}_1'$.

Proof. We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_1'' & \longrightarrow & \mathcal{E}' & \longrightarrow & \iota_*(\underline{\mathcal{E}}'/((\underline{\mathcal{E}}_1/\underline{\mathcal{F}}) \otimes \mathcal{N}_D^*)) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow = & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{E}'' = \mathcal{I}_D \cdot \mathcal{E} & \longrightarrow & \mathcal{E}' & \longrightarrow & \iota_*(\underline{\mathcal{E}}'/(\underline{\mathcal{Q}} \otimes \mathcal{N}_D^*) & \longrightarrow & 0 \end{array}$$

where the first row is by definition and the second row is by Lemma 6.7. This implies the following exact sequence

$$0 \to (\mathcal{I}_D \cdot \mathcal{E})/\mathcal{E}_1'' \to \mathcal{E}'/\mathcal{E}_1'' = \iota_*(\underline{\mathcal{E}}'/((\underline{\mathcal{E}}_1/\underline{\mathcal{F}}) \otimes \mathcal{N}_D^*)) \to \iota_*(\underline{\mathcal{E}}'/(\underline{\mathcal{Q}} \otimes \mathcal{N}_D^*) \to 0.$$

As a result, we have

$$(\mathcal{I}_D \cdot \mathcal{E})/\mathcal{E}_1'' = \iota_*(\underline{\mathcal{Q}} \otimes \mathcal{N}_D^*)/\iota_*(\underline{\mathcal{E}}_1/\underline{\mathcal{F}} \otimes \mathcal{N}_D^*) = \iota_*((\underline{\mathcal{E}}/\underline{\mathcal{E}}_1) \otimes \mathcal{N}_D^*)$$

which implies the following exact sequence

$$0 \to \mathcal{E}_1'' \to \mathcal{I}_D \cdot \mathcal{E} \to \iota_*(\underline{\mathcal{E}}/\underline{\mathcal{E}}_1 \otimes \mathcal{N}_D^*) \to 0.$$

By definition, we also have

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \iota_*(\underline{\mathcal{E}}/\underline{\mathcal{E}}_1) \to 0.$$

Since $\mathcal{I}_D$ is locally free, we have

$$0 \to \mathcal{I}_D \cdot \mathcal{E}_1 \to \mathcal{I}_D \cdot \mathcal{E} \to \iota_*(\underline{\mathcal{E}}/\underline{\mathcal{E}}_1) \otimes \mathcal{I}_D = \iota_*(\underline{\mathcal{E}}/\underline{\mathcal{E}}_1 \otimes \mathcal{N}_D^*) \to 0.$$

This finishes the proof. □

6.2 Proof of Theorem 1.5 I-III

We will apply the discussion above to our setting. Let $\mathcal{E}$ be a reflexive sheaf over B. Recall in the introduction, we denote by $\mathcal{A}$ the space of all extensions of $\mathcal{E}$ at $0 \in B$ and $\Phi : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ by $\Phi(\hat{\mathcal{E}}) = \mu(\underline{\mathcal{E}}_1) - \mu(\underline{\mathcal{E}}_m/\underline{\mathcal{E}}_{m-1})$ where $0 \subset \underline{\mathcal{E}}_1 \subset \cdots \underline{\mathcal{E}}_{m-1} \subset \underline{\mathcal{E}}_m = \underline{\hat{\mathcal{E}}} := \hat{\mathcal{E}}|_D$ is the Harder-Narasimhan filtration of $\underline{\hat{\mathcal{E}}}$ with respect to the $\mathcal{O}(1)$ polarization and $\mu(\cdot)$ is the slope of the corresponding sheaf.

6.2.1 Proof of (I)

We begin with a simple observation.

Lemma 6.9. *The image of the map $\Phi : \mathcal{A} \to \mathbb{Q}_{\geq 0}$ is discrete. In particular, a minimizer of Φ always exists.*

Proof. By definition,

$$\mu_i = \frac{\int_D c_1(\underline{\mathcal{E}}_i/\underline{\mathcal{E}}_{i-1}) \cup c_1(\mathcal{O}(1))^{n-2}}{\operatorname{rank}(\underline{\mathcal{E}}_i/\underline{\mathcal{E}}_{i-1})} \in (\operatorname{rank}(\mathcal{E})!)^{-1}\mathbb{Z}.$$

This implies for any extension $\hat{\mathcal{E}}$, $\Phi(\hat{\mathcal{E}}) \in (\operatorname{rank}(\mathcal{E})!)^{-1}\mathbb{Z}_{\geq 0}$. □

Now let $\hat{\mathcal{E}} \in \mathcal{A}$. Let $0 \subset \underline{\mathcal{E}}_1 \subset \cdots \underline{\mathcal{E}}_m = \underline{\hat{\mathcal{E}}}$ be the Harder-Narasimhan filtration of $\underline{\hat{\mathcal{E}}}$. In the following, for each $k < m$ we always denote by $\hat{\mathcal{E}}^k$ to be the Hecke transform of $\hat{\mathcal{E}}$ along $\underline{\mathcal{E}}_k$ and denote $\underline{\hat{\mathcal{E}}^k} = \iota^*\hat{\mathcal{E}}^k$. Given any sheaf $\underline{\mathcal{F}}$ over $\mathbb{CP}^{n-1}$, we also denote

$$\underline{\mathcal{F}}(j) := \underline{\mathcal{F}} \otimes \mathcal{O}(j).$$

Lemma 6.10. $\Phi(\hat{\mathcal{E}}^k) \leq \max\{\mu_{k+1} - \mu_m, \Phi(\hat{\mathcal{E}}) - 1, \mu_{k+1} - \mu_k + 1, \mu_1 - \mu_k\}$ *for any k.*

Proof. By Corollary 6.6, we have the following exact sequence

$$0 \to (\underline{\hat{\mathcal{E}}}/\underline{\mathcal{E}}_k)(1) \to \underline{\hat{\mathcal{E}}^k} \to \underline{\mathcal{E}}_k \to 0. \tag{6.6}$$

Let $0 \subset \underline{\mathcal{E}}'_1 \subset \cdots \underline{\mathcal{E}}'_{m'} = \underline{\hat{\mathcal{E}}^k}$ be the Harder-Narasimhan filtration of $\underline{\hat{\mathcal{E}}^k}$. Denote the slope of $\underline{\mathcal{E}}'_i/\underline{\mathcal{E}}'_{i-1}$ by μ'_i. By Equation (6.6), $\underline{\mathcal{E}}'_1$ fits into the following exact sequence

$$0 \to \underline{\mathcal{G}}_1 \to \underline{\mathcal{E}}'_1 \to \underline{\mathcal{G}}_2 \to 0.$$

where $\underline{\mathcal{G}}_1$ is a subsheaf $(\underline{\hat{\mathcal{E}}}/\underline{\mathcal{E}}_k)(1)$ and $\underline{\mathcal{G}}_2$ is a subsheaf of $\underline{\mathcal{E}}_k$. Since $\underline{\mathcal{E}}_{k+1}/\underline{\mathcal{E}}_k$ is the maximal destabilizing subsheaf of $\underline{\hat{\mathcal{E}}}/\underline{\mathcal{E}}_k$, we have

$$\mu(\underline{\mathcal{G}}_1) \leq \mu_{k+1} + 1$$

Similarly

$$\mu(\underline{\mathcal{G}}_2) \leq \mu_1.$$

Then one has

$$\mu'_1 \leq \max\{\mu_{k+1} + 1, \mu_1\}. \tag{6.7}$$

By taking the dual of Equation (6.6), one has the following exact sequence

$$0 \to \underline{\mathcal{E}}_k^* \to (\underline{\hat{\mathcal{E}}^k})^* \to (\underline{\hat{\mathcal{E}}}/\underline{\mathcal{E}}_k)^*(-1).$$

Similarly $(\underline{\mathcal{E}}'_{m'}/\underline{\mathcal{E}}'_{m'-1})^*$ fits into the following exact sequence

$$0 \to \underline{\mathcal{H}}_1 \to (\underline{\mathcal{E}}'_{m'}/\underline{\mathcal{E}}'_{m'-1})^* \to \underline{\mathcal{H}}_2 \to 0$$

where $\underline{\mathcal{H}}_1$ is a subsheaf of $\underline{\mathcal{E}}_k^*$ and $\underline{\mathcal{H}}_2$ is a subsheaf of $(\underline{\hat{\mathcal{E}}}/\underline{\mathcal{E}}_k)^*(-1)$. Similar to the above, we have

$$\mu(\underline{\mathcal{H}}_1) \leq -\mu_k$$

and

$$\mu(\underline{\mathcal{H}}_2) \leq -\mu_m - 1.$$

Then one has

$$-\mu'_{m'} \leq \max\{-\mu_k, -\mu_m - 1\} \tag{6.8}$$

Combining Equation (6.7) and (6.8), we get

$$\mu'_1 - \mu'_{m'} \leq \max\{\mu_{k+1} - \mu_m, \mu_1 - \mu_m - 1, \mu_{k+1} - \mu_k + 1, \mu_1 - \mu_k\}$$

This finishes the proof. □

Now we prove Theorem 1.1 (I). Since $\mathcal{A}$ is nonempty, we can fix an element $\hat{\mathcal{E}} \in \mathcal{A}$. If $\Phi(\hat{\mathcal{E}}) \geq 1$, we apply Lemma 6.10 to $\hat{\mathcal{E}}$ with $k = 1$ and get

$$\Phi(\hat{\mathcal{E}}^1) \leq \max\{\mu_2 - \mu_m, \Phi(\hat{\mathcal{E}}) - 1, \mu_2 - \mu_1 + 1\} \leq \Phi(\hat{\mathcal{E}}) - 1.$$

If $\Phi(\hat{\mathcal{E}}^1) \geq 1$, we repeat the same process for $\hat{\mathcal{E}}^1$. After finitely many steps, we can get $\hat{\mathcal{E}}' \in \mathcal{A}$ with $0 \leq \Phi(\hat{\mathcal{E}}') < 1$. The following is also clear from Lemma 6.10.

Corollary 6.11. *Suppose $\hat{\mathcal{E}} \in \mathcal{A}$ is optimal, then $\hat{\mathcal{E}}^k$ is also optimal for all k.*

Definition 6.12. We say two optimal extensions $\hat{\mathcal{E}}$ and $\hat{\mathcal{E}}'$ differ by a Hecke transform of special type if $\hat{\mathcal{E}}'$ is isomorphic $\hat{\mathcal{E}}^k$ for some k.

6.2.2 Proof of (II)

Meromorphic extension of sections

The goal in this subsection is to prove the following proposition that will be needed in our discussion later. Let $s_D \in H^0(\hat{B}, [D])$ be a defining section of D and let $\hat{\mathcal{E}}$ be any reflexive sheaf over $\hat{B}$.

Proposition 6.13. *Given any $s \in H^0(\hat{B} \setminus D, \hat{\mathcal{E}})$, there exists a k such that $s \otimes s_D^k$ extends to a holomorphic section of $\hat{\mathcal{E}}(k[D])$ over $\hat{B}$. In other words, s is a meromorphic section of $\hat{\mathcal{E}}$.*

Remark 6.14. It is a key assumption here that $[D]$ is an exceptional divisor, since otherwise the statement is false. For example, if we consider $D = \{0\} \subset \Delta$, where $\Delta = \{|z| < 1\} \subset \mathbb{C}$, and consider the trivial sheaf $\mathcal{O}$, then we have holomorphic functions on $\Delta \setminus \{0\}$ with an essential singularity at 0 which can not extend to be meromorphic functions on Δ.

Proof of the case $n = 2$. In this case $D = \mathbb{CP}^1$, and $\hat{\mathcal{E}}$ is a locally free. Denote $\hat{B}_t := p^{-1}(B_t)$ where B_t denote the ball of radius $t \in (0, 1)$ centered at 0.

It suffices to construct the following exact sequence over $\hat{B}_{\frac{1}{2}}$ for $k \in \mathbb{Z}$ large enough

$$0 \to R \to \mathcal{O}^{n_1} \to \hat{\mathcal{E}}^*(-k[D]) \to 0.$$

Indeed, given this exact sequence, by taking the double dual, we have

$$0 \to \hat{\mathcal{E}}(k[D]) \to \mathcal{O}^{n_1} \to R^* \to 0.$$

Then $s \otimes s_D^k|_{\hat{B}_{\frac{1}{2}}} \in H^0(\hat{B}_{\frac{1}{2}} \setminus D, \hat{\mathcal{E}}(k[D]))$ can be viewed as a section in $H^0(\hat{B}_{\frac{1}{2}} \setminus D, \mathcal{O}^{n_1})$. By Hartog's theorem for holomorphic functions, we know $H^0(\hat{B}_{\frac{1}{2}} \setminus$

$D, \mathcal{O}^{n_1}) = H^0(\hat{B}_{\frac{1}{2}}, \mathcal{O}^{n_1})$. Then $s \otimes s_D^k|_{\hat{B}_{\frac{1}{2}}} \in H^0(\hat{B}_{\frac{1}{2}}, \mathcal{O}^{n_1})$. By continuity, we have $s \otimes s_D^k|_{\hat{B}_{\frac{1}{2}}} \in H^0(\hat{B}_{\frac{1}{2}}, \hat{\mathcal{E}}(k[D]))$.

Now we fix a Kähler metric $\hat{\omega}$ on $\hat{B}$. In order to construct the exact sequence above, it is equivalent to constructing a set of global generators for $\hat{\mathcal{E}}^*(-k[D])$ over $\hat{B}_{\frac{1}{2}}$ for k large. This can done by the standard Hörmander technique, see for example Theorem 5.1 in [8]. Indeed, we know $\hat{B}_{\frac{1}{2}}$ is weakly pseudo-convex, and since $[D]|_D = \mathcal{O}(-1)$ is negative, one can easily construct a hermitian metric h on $\hat{\mathcal{E}}^*(-k[D])$ for k large, such that

$$\sqrt{-1}F_{h_k} \geq Ck\hat{\omega} \otimes \mathrm{Id}.$$

Now the conclusion follows from standard L^2 solution to the $\bar{\partial}$-problem, using singular weight. □

Proof of the general case. Suppose $n \geq 3$ and $\hat{\mathcal{E}}$ is a reflexive sheaf defined $\hat{B}$. Let $S = \phi(\mathrm{Sing}(\hat{\mathcal{E}})) \cap \overline{\hat{B}_{\frac{3}{4}}}$ and $\hat{S} = \phi^{-1}(S) \cap \hat{B}_{\frac{3}{4}}$. By replacing $\hat{B}_{\frac{3}{4}}$ with $\hat{B}$ which does not affect the argument, we can assume S is a closed subset in $\mathbb{CP}^{n-1}$ of Hausdorff of codimension at least 4 and so is $\hat{S}$ in $\hat{B}$. Furthermore, $\mathrm{Sing}(\hat{\mathcal{E}}) \subset \hat{S}$.

By Proposition 4 in [35], it suffices to prove that for any $z \in \mathbb{CP}^{n-1} \setminus S$, $s|_{\phi^{-1}(z)}$ is a meromorphic section of $\hat{\mathcal{E}}|_{\phi^{-1}(z)}$. Indeed, given this, by Proposition 4 in [35], we know s is a meromorphic section of $\hat{\mathcal{E}}|_{\hat{B}\setminus\hat{S}}$ which is holomorphic outside D. Then for some k, $s \otimes s_D^k$ is a holomorphic section of $\hat{\mathcal{E}}(k[D])|_{\hat{B}\setminus\hat{S}}$. Since $\hat{S}$ has Hausdorff codimension at least 4, $s \otimes s_D^k$ further extends to be a section in $H^0(\hat{B}, \hat{\mathcal{E}}(k[D]))$ (see Lemma 3 in [33]). Now we show $s|_{\phi^{-1}(z)}$ is a meromorphic section of $\hat{\mathcal{E}}|_{\phi^{-1}(z)}$ for any $z \in \mathbb{CP}^{n-1} \setminus S$. Since S has Hausdorff of codimension at least 4 in $\mathbb{CP}^{n-1}$, we can choose a complex line $\mathbb{CP}^1 \subset \mathbb{CP}^{n-1}$ which does not intersect S but contains z. Let $\hat{B}^2 = \phi^{-1}(\mathbb{CP}^1)$. Then $\hat{\mathcal{E}}|_{\hat{B}^2}$ is locally free. By the case $n = 2$ proved above, $s|_{\hat{B}^2\setminus(D\cap\hat{B}^2)}$ is a meromorphic section of $\hat{\mathcal{E}}$ over $\hat{B}^2$. In particular, $s|_{\phi^{-1}(z)}$ is a meromorphic section of $\hat{\mathcal{E}}|_{\phi^{-1}(z)}$. This finishes the proof. □

Uniqueness

We will prove (II) in this section. Suppose $\hat{\mathcal{E}}$ and $\hat{\mathcal{E}}'$ are two optimal extensions of $\mathcal{E}$ at 0. We denote $\underline{\hat{\mathcal{E}}} = \iota^*\hat{\mathcal{E}}$ and $\underline{\hat{\mathcal{E}}'} = \iota^*\hat{\mathcal{E}}'$. Let

$$0 \subset \underline{\mathcal{E}}_1 \subset \cdots \underline{\mathcal{E}}_m = \underline{\hat{\mathcal{E}}}$$

and

$$0 \subset \underline{\mathcal{E}}'_1 \subset \cdots \underline{\mathcal{E}}'_{m'} = \underline{\hat{\mathcal{E}}}'$$

be the Harder-Narasimhan filtrations of $\underline{\hat{\mathcal{E}}}$ and $\underline{\hat{\mathcal{E}}}'$ respectively. If we denote $\mu_i := \mu(\underline{\mathcal{E}}_i/\underline{\mathcal{E}}_{i-1})$ and $\mu'_i := \mu(\underline{\mathcal{E}}'_i/\underline{\mathcal{E}}'_{i-1})$, then by assumption we have

$$\mu_1 - \mu_m < 1, \mu'_1 - \mu'_m < 1,$$

and there exists a natural isomorphism $\rho : \hat{\mathcal{E}}|_{\hat{B}\setminus D} \to \hat{\mathcal{E}}'|_{\hat{B}\setminus D}$. By Proposition 6.13, ρ is a meromorphic section of $\hat{\mathcal{E}}^* \otimes \hat{\mathcal{E}}'$. Suppose $\det\rho$ has a pole of order

$k \in \mathbb{Z}$ along D. If we write $k = d \cdot \operatorname{rank}(\mathcal{E}) + k_0$ with $0 \le k_0 < \operatorname{rank}(\mathcal{E})$, then by replacing $\hat{\mathcal{E}}$ with $\hat{\mathcal{E}}(d[D])$, we may assume $0 \le k < \operatorname{rank}(\mathcal{E})$ and $\det \rho$ has a pole of order k_0 so that $0 \le k_0 < \operatorname{rank}(\mathcal{E})$. In particular, ρ^{-1} can not vanish along D. Denote $\underline{\rho} = \iota^* \rho, \underline{\rho}^{-1} = \iota^* \rho^{-1}$. Then $\underline{\rho}$ and $\underline{\rho}^{-1}$ can be viewed as two *nontrivial* holomorphic sections

$$\underline{\rho} : \underline{\hat{\mathcal{E}}} \to \underline{\hat{\mathcal{E}}}'(-l_0), \quad \underline{\rho}^{-1} : \underline{\hat{\mathcal{E}}}' \to \underline{\hat{\mathcal{E}}}(-l_0'),$$

for some $l_0, l_0' \in \mathbb{Z}_+$. Let k be the smallest integer such that $\underline{\rho}|_{\underline{\mathcal{E}}_{k+1}} \neq 0$. Then $\underline{\rho}$ descends to be a nontrivial holomorphic map $\underline{\rho} : \underline{\hat{\mathcal{E}}}/\underline{\mathcal{E}}_k \to \underline{\hat{\mathcal{E}}'}(-l_0)$ which restrict to be nonzero on $\underline{\mathcal{E}}_{k+1}/\underline{\mathcal{E}}_k$. Since $\underline{\mathcal{E}}_1'(-l_0)$ is the maximal destabilizing subsheaf of $\underline{\hat{\mathcal{E}}}'(-l_0)$, we have $\mu_1' - l_0 \ge \mu_{k+1}$. Similarly $\mu_1 - l_0' \ge \mu_j'$ for some j. Then we have

$$2 > \mu_1' - \mu_j' + \mu_1 - \mu_{k+1} \ge l_0 + l_0',$$

which implies exactly one of the following hold

(a). $l_0 = 0$;

(b). $l_0 = 1$.

Suppose first (a) holds, then by assumption, ρ can be extended to be a holomorphic section across D and thus $\det(\rho)$ is also a holomorphic section of $\det(\hat{\mathcal{E}}^*) \otimes \det(\hat{\mathcal{E}}')$ over $\hat{B}$. However, by assumption we know $\det(\rho)$ has a pole of order $k_0 \ge 0$. Then we must have $k_0 = 0$, i.e. $\det(\rho)|_D \neq 0$ which implies $\det(\rho)(z) \neq 0$ for any $z \in \hat{B} \setminus \operatorname{Sing}(\hat{\mathcal{E}}) \cup \operatorname{Sing}(\hat{\mathcal{E}}')$. In particular, ρ is an isomorphism away from complex codimension two and hence must be an isomorphism. Notice this already finishes the proof of Part (II) of Theorem 1.1 since under the assumption of (II) we know (a) must hold.

Now suppose (b) holds, i.e. $l_0 = 1$ and $l_0' = 0$. By assumption, ρ can be viewed as a holomorphic map $\rho : \hat{\mathcal{E}} \to \hat{\mathcal{E}}'([D])$ with $\underline{\rho} : \underline{\hat{\mathcal{E}}} \to \underline{\hat{\mathcal{E}}}'(-1)$ being nonzero and $\rho^{-1} : \hat{\mathcal{E}}' \to \hat{\mathcal{E}}$ is a holomorphic map with $\underline{\rho}^{-1} : \underline{\hat{\mathcal{E}}}' \to \underline{\hat{\mathcal{E}}}$ being nonzero. Then ρ^{-1} is a sheaf monomorphism since $\hat{\mathcal{E}}'$ is reflexive and $\ker(\rho^{-1})$ is supported on D. In the following, we do not distinguish between $\hat{\mathcal{E}}'$ and the image $\rho^{-1}(\hat{\mathcal{E}}')$ in $\hat{\mathcal{E}}$. Let $D' = Sing(\hat{\mathcal{E}}) \cup Sing(\hat{\mathcal{E}}') \cup Sing(\underline{\hat{\mathcal{E}}}/\underline{\mathcal{E}}_k)$.

To finish the proof of (II), it suffices to prove

Claim 6.15. $(\hat{\mathcal{E}}/\hat{\mathcal{E}}')|_{\hat{B}\setminus D'} \cong \iota_*(\underline{\hat{\mathcal{E}}}/\underline{\mathcal{E}}_k)|_{\hat{B}\setminus D'}$.

Indeed, given Claim 6.15, we have the following exact sequence outside D'

$$0 \to \hat{\mathcal{E}}' \to \hat{\mathcal{E}} \to \iota_*(\underline{\hat{\mathcal{E}}}/\underline{\mathcal{E}}_k) \to 0.$$

By definition, we have $\hat{\mathcal{E}}' = \hat{\mathcal{E}}^k$ outside D' where $\hat{\mathcal{E}}^k$ denotes the Hecke transform of $\hat{\mathcal{E}}$ along $\underline{\mathcal{E}}_k$. Since $\hat{\mathcal{E}}'$ and $\hat{\mathcal{E}}^k$ are both reflexive, they must be isomorphic.

Proof of Claim 6.15. First we prove that $\hat{\mathcal{E}}/\hat{\mathcal{E}}' = \iota_* \iota^*(\hat{\mathcal{E}}/\hat{\mathcal{E}}')$. To see this it suffices to show that for any local section s of $\hat{\mathcal{E}}$, $z_n s \in \hat{\mathcal{E}}'$. Here z_n denotes the local defining function for D after choosing a local coordinate. Indeed, by assumption, $z_n \rho(s)$ is a local holomorphic section. We also know that $\rho^{-1}(z_n \rho(s)) = z_n s$, which implies $\mathcal{I}_D \hat{\mathcal{E}} \subset \rho^{-1}(\hat{\mathcal{E}}')$. As a result, $\iota_* \iota^*(\hat{\mathcal{E}}/\hat{\mathcal{E}}') = \hat{\mathcal{E}}/\hat{\mathcal{E}}'$.

So it suffices to prove $\iota^*(\hat{\mathcal{E}}/\hat{\mathcal{E}}') = \hat{\underline{\mathcal{E}}}/\underline{\mathcal{E}}_k$ on $D \setminus D'$. Since all these sheaves are locally free away from D' this boils down to showing $\underline{\rho}^{-1}(\underline{\hat{\mathcal{E}}'}) = \underline{\mathcal{E}}_k$ on $D \setminus D'$.

We first show $\mathrm{Im}(\underline{\rho}^{-1}) \subset \underline{\mathcal{E}}_k$. If not, there exists a nontrivial map

$$\underline{\rho}^{-1} : \hat{\underline{\mathcal{E}}}' \to \hat{\underline{\mathcal{E}}}/\underline{\mathcal{E}}_k$$

which implies $\mu'_j \leq \mu_{k+1}$ for some j. Meanwhile, by assumption, $\underline{\rho}$ descends to be a nontrivial map as $\underline{\rho} : \hat{\underline{\mathcal{E}}}/\underline{\mathcal{E}}_k \to \hat{\underline{\mathcal{E}}}'(-1)$ which implies $\mu'_1 - 1 \geq \mu_{k+1}$. Then we have

$$\mu'_1 - \mu'_{m'} \geq \mu'_1 - \mu'_j \geq 1$$

which is a contradiction. Now we prove that $\mathrm{Im}(\underline{\rho}^{-1}(z)) = \underline{\mathcal{E}}_k|_z$ for $z \in D \setminus D'$. It suffices to prove

$$\mathrm{rank}(\underline{\rho}(z)) + \mathrm{rank}(\underline{\rho}^{-1}(z)) \geq \mathrm{rank}(\mathcal{E})$$

for $z \in D \setminus D'$. Now we fix $z \in D \setminus D'$ and choose local coordinates $(z_1, \cdots z_n)$ so that z_n is the local defining function for D. After choosing a local trivialization for both $\hat{\mathcal{E}}$ and $\hat{\mathcal{E}}'$ near z, we can view ρ and ρ^{-1} as a matrix. By doing Taylor expansion, we can assume

$$\rho^{-1} = A_0 + A_1 z_n + \cdots$$

and

$$z_n \rho = B_0 + B_1 z_n + \cdots$$

where A_i and B_i are matrices of holomorphic functions independent of z_n. Since $\rho^{-1} \circ (z_n \rho) = z_n \mathrm{Id}$, by comparing the coefficients in front of z_n we get

$$A_0 B_1 + A_1 B_0 = \mathrm{Id},$$

which implies

$$\begin{aligned} \mathrm{rank}(A_0) + \mathrm{rank}(B_0) &\geq \mathrm{rank}(A_0 B_1) + \mathrm{rank}(A_1 B_0) \\ &\geq \mathrm{rank}(A_0 B_1 + A_1 B_0) \\ &= \mathrm{rank}(\mathcal{E}). \end{aligned}$$

By definition, we have

$$\mathrm{rank}(\underline{\rho}(z)) + \mathrm{rank}(\underline{\rho}^{-1}(z)) = \mathrm{rank}(A_0) + \mathrm{rank}(B_0) \geq \mathrm{rank}(\mathcal{E}).$$

This then finishes the proof. □

6.2.3 Proof of (III)

Now we assume $\mathcal{E}$ is homogeneous i.e. $\mathcal{E} \simeq \psi_* \pi^* \underline{\mathcal{E}}$ for reflexive $\underline{\mathcal{E}}$ over $\mathbb{CP}^{n-1}$. Let $0 = \underline{\mathcal{E}}_0 \subset \underline{\mathcal{E}}_1 \cdots \subset \underline{\mathcal{E}}_m = \underline{\mathcal{E}}$ be the Harder-Narasimhan filtration of $\underline{\mathcal{E}}$ and denote $\mu_k = \mu(\underline{\mathcal{E}}_k/\underline{\mathcal{E}}_{k-1})$. Note $\phi^* \underline{\mathcal{E}} \in \mathcal{A}$. Let $j_0 = 0$ and define

$$j_{k+1} := \max\{s > j_k : \mu_1 - \mu_s - \lfloor \mu_1 - \mu_{j_k+1} \rfloor < 1, s \leq m\}$$

inductively for $k \geq 1$. Let l be the largest integer so that j_l is defined. Then we define the *partial* Harder-Narasimhan filtration as

$$0 = \underline{\mathcal{E}}_{j_0} \subset \underline{\mathcal{E}}_{j_1} \subset \underline{\mathcal{E}}_{j_2} \subset \cdots \underline{\mathcal{E}}_{j_l} = \underline{\mathcal{E}}.$$

Let $n_k = \lfloor \mu_1 - \mu_{j_k+1} \rfloor$ for $0 \le k \le l-1$ and define

$$\widetilde{Gr}(\underline{\mathcal{E}}) := \oplus_{i=1}^{l} (\underline{\mathcal{E}}_{j_i}/\underline{\mathcal{E}}_{j_{i-1}})(n_{i-1}).$$

Then to prove (IV), it suffices to show

Proposition 6.16. *There exists an optimal extension $\hat{\mathcal{E}} \in \mathcal{A}$ so that $\underline{\hat{\mathcal{E}}} \cong \widetilde{Gr}(\underline{\mathcal{E}})$.*

Proof. It suffices to prove the following by induction on k with $1 \le k \le l-1$. (The reason to write inductions in this way will be justified by the proof naturally.)

- $(a)_k$ there exists $\hat{\mathcal{E}}^k \in \mathcal{A}$ with $\underline{\hat{\mathcal{E}}}^k \cong \oplus_{i=1}^{k} (\underline{\mathcal{E}}_{j_i}/\underline{\mathcal{E}}_{j_{i-1}})(n_{i-1}) \oplus (\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_k})(n_k)$;
- $(b)_k$ there exists the following sheaf inclusions for $1 \le i \le k$ which are compatible with the splittings in $(a)_k$
  - $\phi^*\underline{\mathcal{E}}_{j_1} \subset \hat{\mathcal{E}}^k$;
  - Let $\hat{\mathcal{E}}_1^k := \hat{\mathcal{E}}^k$, then we can define $\hat{\mathcal{E}}_{i+1}^k = \hat{\mathcal{E}}_i^k/\phi^*((\underline{\mathcal{E}}_{j_i}/\underline{\mathcal{E}}_{j_{i-1}})(n_{i-1}))$ for $1 \le i \le k-1$ inductively and $\phi^*((\underline{\mathcal{E}}_{j_{i+1}}/\underline{\mathcal{E}}_{j_i})(n_i)) \subset \hat{\mathcal{E}}_{i+1}^k$ for $i = 1, \cdots k-1$;
  - $\hat{\mathcal{E}}_k^k/\phi^*((\underline{\mathcal{E}}_{j_k}/\underline{\mathcal{E}}_{j_k})(n_{k-1})) = \phi^*((\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_k})(n_k))$.

For $k=1$, we let $\hat{\mathcal{E}}^{1,1}$ be the Hecke transform of $\phi^*\underline{\mathcal{E}}$ along $\underline{\mathcal{E}}_{j_1}$. By Proposition 6.6, we have the following exact sequence

$$0 \to (\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1})(1) \to \underline{\hat{\mathcal{E}}}^{1,1} \to \underline{\mathcal{E}}_{j_1} \to 0.$$

By definition, there exists a natural sheaf inclusion $\phi^*\underline{\mathcal{E}}_{j_1} \subset \hat{\mathcal{E}}^{1,1}$ which restricts to be a map from $\underline{\mathcal{E}}_{j_1}$ to $\underline{\hat{\mathcal{E}}}^{1,1}$ that splits the exact sequence above i.e. $\underline{\hat{\mathcal{E}}}^{1,1} \cong \underline{\mathcal{E}}_{j_1} \oplus (\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1})(1)$. Indeed, we know that $\phi^*\underline{\mathcal{E}}_{j_1}$ lies in the kernel of the surjective map $\phi^*\underline{\mathcal{E}} \to \iota_*(\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1})$ and thus we have a natural sheaf inclusion $\phi^*\underline{\mathcal{E}}_{j_1} \subset \hat{\mathcal{E}}^{1,1}$ by definition. (This is the key difference in the homogeneous case from the general case where we have a natural inclusion $\phi^*(\underline{\mathcal{E}}_{j_1}) \subset \hat{\mathcal{E}}^{1,2}$.) The restriction map splitting the exact sequence above is tautological. Moreover, by definition, we have

$$0 \to \hat{\mathcal{E}}^{1,1}/\phi^*(\underline{\mathcal{E}}_{j_1}) \to \phi^*(\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1}) \to \iota_*(\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1}) \to 0$$

which implies $\hat{\mathcal{E}}^{1,1}/\phi^*(\underline{\mathcal{E}}_{j_1}) = \phi^*(\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1})(-[D]) = \phi^*(\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1}(1))$. (This is another key difference in the homogeneous case from the general case. That is the quotient sheaf $\hat{\mathcal{E}}^{1,2}/\phi^*\underline{\mathcal{E}}_{j_1}$ is still homogeneous , i.e. it is pulled back from the projective space.) If $n_1 > 1$, let $\hat{\mathcal{E}}^{1,2}$ be the Hecke transform of $\hat{\mathcal{E}}^{1,1}$ along $\underline{\mathcal{E}}_{j_1}$. Similarly, we have

$$0 \to (\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1})(2) \to \underline{\hat{\mathcal{E}}}^{1,2} \to \underline{\mathcal{E}}_{j_1} \to 0$$

and by definition, we have a sheaf inclusion $\phi^*\underline{\mathcal{E}}_{j_1} \subset \hat{\mathcal{E}}^{1,2}$ which restricts to be a map that splits the exact sequence above i.e. $\underline{\hat{\mathcal{E}}}^{1,2} \cong (\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1})(2) \oplus \underline{\mathcal{E}}_{j_1}$. By definition, we also have the following exact sequence

$$0 \to \hat{\mathcal{E}}^{1,2}/\phi^*\underline{\mathcal{E}}_{j_1} \to \phi^*((\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1})(1) \to \iota_*((\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1})(1)) \to 0$$

which implies $\hat{\mathcal{E}}^{1,2}/\phi^*\underline{\mathcal{E}}_{j_1} = \phi^*((\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1})(2))$. Then one can keep doing Hecke transform for $\hat{\mathcal{E}}^{1,2}$ along $\underline{\mathcal{E}}_{j_1}$ if necessary and get $\hat{\mathcal{E}}^1 := \hat{\mathcal{E}}^{1,n_1} \in \mathcal{A}$ satisfying

$(a)_1$ $\underline{\hat{\mathcal{E}}}^1 \cong \underline{\mathcal{E}}_{j_1} \oplus (\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1})(n_1)$;

$(b)_1$ there exists a sheaf inclusion $\phi^*\underline{\mathcal{E}}_{j_1} \subset \hat{\mathcal{E}}^1$ which is compatible with the splitting above and $\hat{\mathcal{E}}^1/\phi^*(\underline{\mathcal{E}}_{j_1}) = \phi^*(\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_1}(n_1))$.

Namely, after we do Hecke transform along $\underline{\mathcal{E}}_{j_1}$, $\phi^*\underline{\mathcal{E}}_{j_1}$ will always be a saturated subsheaf of the new sheaf which will give a splitting on the central fiber. And the natural quotient sheaf is still homogeneous.

To make the argument more clear, we will explain how to do $k = 2$ briefly. (Details can be found in the induction for the general case.) Given $(a)_1$ and $(b)_1$, we can keep doing Hecke transform along $\phi^*\underline{\mathcal{E}}_{j_1} \oplus \phi^*(\underline{\mathcal{E}}_{j_2}/\underline{\mathcal{E}}_{j_1}(n_1))$ to get a new sheaf $\hat{\mathcal{E}}^2$. And we have two sheaf inclusions $\phi^*\underline{\mathcal{E}}_{j_1} \subset \hat{\mathcal{E}}^2$ and $\phi^*(\underline{\mathcal{E}}_{j_2}/\mathcal{E}_{j_1}(n_1)) \subset \hat{\mathcal{E}}^2/\phi^*\underline{\mathcal{E}}_{j_1}$ which restricts to be maps that split the central fiber as we want. Furthermore, we have

$$(\hat{\mathcal{E}}^2/\phi^*(\underline{\mathcal{E}}_{j_1}))/\phi^*(\underline{\mathcal{E}}_{j_2}/\underline{\mathcal{E}}_{j_1}(n_1)) = \phi^*(\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_2}(n_2))$$

where n_2 is equal to the number of Hecke transforms along $\phi^*\underline{\mathcal{E}}_{j_1}\oplus\phi^*(\underline{\mathcal{E}}_{j_2}/\underline{\mathcal{E}}_{j_1}(n_1))$ to $\hat{\mathcal{E}}^2$.

Now we do the induction in general. Suppose we have proved $(a)_k, (b)_k$, we want to build the statements $(a)_{k+1}$ and $(b)_{k+1}$. First let $\hat{\mathcal{E}}^{k+1,1}$ to be the Heck transform of $\hat{\mathcal{E}}^k$ along $\oplus_{i=1}^k(\underline{\mathcal{E}}_{j_i}/\underline{\mathcal{E}}_{j_{i-1}})(n_{i-1}) \oplus (\underline{\mathcal{E}}_{j_{k+1}}/\underline{\mathcal{E}}_{j_k})(n_k)$. By Proposition 6.6 we have the following exact sequence

$$0 \to (\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_{k+1}})(n_k+1) \to \underline{\hat{\mathcal{E}}}^{k+1,1} \to \oplus_{i=1}^k(\underline{\mathcal{E}}_{j_i}/\underline{\mathcal{E}}_{j_{i-1}})(n_{i-1})\oplus(\underline{\mathcal{E}}_{j_{k+1}}/\underline{\mathcal{E}}_{j_k})(n_k) \to 0.$$

Then $(b)_k$ holds by replacing $\hat{\mathcal{E}}^k$ with $\hat{\mathcal{E}}^{k+1,1}$ except the last one which needs to be changed. More precisely, there exists the following sheaf inclusions for $1 \leq i \leq k$ which are compatible with the splittings in $(a)_k$

- $\phi^*\underline{\mathcal{E}}_{j_1} \subset \hat{\mathcal{E}}^{k+1,1}$;
- if we let $\hat{\mathcal{E}}_1^{k+1} := \hat{\mathcal{E}}^{k+1,1}$ and define $\hat{\mathcal{E}}_{i+1}^{k+1} = \hat{\mathcal{E}}_i^{k+1}/\phi^*((\underline{\mathcal{E}}_{j_i}/\underline{\mathcal{E}}_{j_{i-1}})(n_{i-1}))$ for $1 \leq i \leq k-1$ inductively, then $\phi^*((\underline{\mathcal{E}}_{j_{i+1}}/\underline{\mathcal{E}}_{j_i})(n_i)) \subset \hat{\mathcal{E}}_{i+1}^{k+1}$ for $i = 1, \cdots k-1$;
- $\phi^*((\underline{\mathcal{E}}_{j_{k+1}}/\underline{\mathcal{E}}_{j_k})(n_k)) \subset \hat{\mathcal{E}}_{k+1}^{k+1,1}$ and

$$\hat{\mathcal{E}}_{k+1}^{k+1,1}/\phi^*((\underline{\mathcal{E}}_{j_k+1}/\underline{\mathcal{E}}_{j_k})(n_k)) = \phi^*(\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_{k+1}}(n_k+1)).$$

Indeed, by definition, we have

$$0 \to \hat{\mathcal{E}}^{k+1,1} \to \hat{\mathcal{E}}^k \to \iota_*(\oplus_{i=1}^k(\underline{\mathcal{E}}_{j_i}/\underline{\mathcal{E}}_{j_{i-1}})(n_{i-1}) \oplus (\underline{\mathcal{E}}_{j_{k+1}}/\underline{\mathcal{E}}_{j_k})(n_k)) \to 0$$

Combining this with that $\hat{\mathcal{E}}^k$ satisfies property $(a)_k$ and $(b)_k$, we can easily get the sheaf inclusions with required properties above. Now we have

$$\underline{\hat{\mathcal{E}}}^{k+1,1} = \oplus_{i=1}^{k+1}(\underline{\mathcal{E}}_{j_i}/\underline{\mathcal{E}}_{j_{i-1}})(n_{i-1}) \oplus (\underline{\mathcal{E}}/\underline{\mathcal{E}}_{j_{k+1}})(n_k+1).$$

Indeed, the sheaf inclusion $\phi^*\underline{\mathcal{E}}_{j_1} \subset \hat{\mathcal{E}}^{k+1,1}$ restricts to be a map that gives a splitting $\hat{\mathcal{E}}^{k+1,1} = \underline{\mathcal{E}}_{j_1} \oplus \iota^*\hat{\mathcal{E}}_2^{k+1}$. For $\iota^*\hat{\mathcal{E}}_2^{k+1}$, the sheaf inclusion given by $\phi^*((\underline{\mathcal{E}}_{j_2}/\underline{\mathcal{E}}_{j_1})(n_1)) \subset \hat{\mathcal{E}}_2^{k+1}$ gives a splitting $\iota^*\hat{\mathcal{E}}_2^{k+1} = (\underline{\mathcal{E}}_{j_2}/\underline{\mathcal{E}}_{j_1})(n_1) \oplus \iota^*\hat{\mathcal{E}}_3^{k+1}$. Then one can keep doing this and finally get a splitting of $\underline{\hat{\mathcal{E}}}^{k+1,1}$ as claimed above.

Now one can repeat the process with $\hat{\mathcal{E}}^{k+1,1}$ to get $\hat{\mathcal{E}}^{k+1,2}$ by doing Hecke transform along $\oplus_{i=1}^{k}(\underline{\mathcal{E}}_{j_i}/\underline{\mathcal{E}}_{j_{i-1}})(n_{i-1}) \oplus (\underline{\mathcal{E}}_{j_{k+1}}/\underline{\mathcal{E}}_{j_k})(n_k)$ again if necessary and finally get $\hat{\mathcal{E}}^{k+1} := \hat{\mathcal{E}}^{k+1,n_{k+1}}$ satisfying properties $(a)_{k+1}$ and $(b)_{k+1}$. This finishes the proof. □

Remark 6.17. When the Harder-Narasimhan filtration of $\underline{\mathcal{E}}$ has length equal to 2, i.e.

$$0 = \underline{\mathcal{E}}_0 \subset \underline{\mathcal{E}}_1 \subset \underline{\mathcal{E}}_2 = \underline{\mathcal{E}},$$

the same argument shows that there exists an optimal extension $\hat{\mathcal{E}}$ so that $\underline{\hat{\mathcal{E}}} = \underline{\mathcal{E}}_1 \oplus (\underline{\mathcal{E}}_2/\underline{\mathcal{E}}_1)(k)$ for some integer k with $\mu_1 - 1 < \mu_2 - k \leq \mu_1$. In general, one should not expect to get an optimal extension of which the restriction splits as a direct sum of semistable torsion free sheaves by Theorem 1.1 (III) and Corollary 6.11.

6.2.4 Examples

In this section, we apply Theorem 1.5 to study the example used in Section 4.

Example 1. Consider $\underline{\mathcal{E}} \to \mathbb{CP}^2$ given by the following exact sequence

$$0 \to \mathcal{O} \xrightarrow{\sigma} \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(k) \to \underline{\mathcal{E}} \to 0,$$

where $\sigma = (z_1, z_2, z_3^k)$. Let $\mathcal{E} = \psi_*\pi^*\underline{\mathcal{E}}$. Then we have

- if $k = 1$, $\underline{\mathcal{E}}$ is stable;
- if $k = 2$, $\underline{\mathcal{E}}$ is semistable;
- if $k \geq 3$, $\underline{\mathcal{E}}$ is unstable. The Harder-Narasimhan filtration of $\underline{\mathcal{E}}$ (which is the same as the Harder-Narasimhan-Seshadri filtration in this case) is given by $0 \subset \underline{\mathcal{E}}_1 \subset \underline{\mathcal{E}}_2 = \underline{\mathcal{E}}$ where $\underline{\mathcal{E}}_1 \cong \mathcal{O}(k)$ and $\underline{\mathcal{E}}_2/\underline{\mathcal{E}}_1 \cong \mathcal{I}_{[0:0:1]}(2)$.

By Theorem 1.1, when $k \leq 2$, there exists a unique optimal extension given by $\phi^*\underline{\mathcal{E}}$ (up to equivalence). When $k \geq 3$, by Remark 6.17, there exists an optimal extension $\hat{\mathcal{E}}$ of which the restriction is given by $\mathcal{O}(2) \oplus \mathcal{I}_{[0,0,1]}(2)$. Then again by Theorem 1.1, $\hat{\mathcal{E}}$ is the unique one up to equivalence since $\mathcal{O}(2) \oplus \mathcal{I}_{[0,0,1]}(2)$ is semistable.

The next is an example where there are two optimal extensions, for which one of them has a locally free algebraic tangent cone while the other has an essential point singularity.

Example 2. Consider a vector bundle $\underline{\mathcal{E}} \to \mathbb{CP}^3$ given by the following

$$0 \to \mathcal{O} \xrightarrow{\sigma} \mathcal{O}(1)^{\oplus 3} \oplus \mathcal{O}(2) \to \underline{\mathcal{E}} \to 0. \tag{6.9}$$

where $\sigma = (z_1, z_2, z_3, z_4^2)$. Let $\mathcal{E} := \psi_*\pi^*\underline{\mathcal{E}}$. Then $\hat{\mathcal{E}} := \phi^*\underline{\mathcal{E}}$ is an optimal extension of $\mathcal{E}$ at 0 with $\Phi(\hat{\mathcal{E}}) = \frac{1}{2}$. The Harder-Narasimhan filtration of $\underline{\hat{\mathcal{E}}}$ is given by $\underline{\mathcal{E}}_1 \cong \mathcal{O}(2)$ and $\underline{\mathcal{E}}_2 = \underline{\mathcal{E}}$. Furthermore, $\underline{\mathcal{E}}_2/\underline{\mathcal{E}}_1$ fits into the following exact sequence

$$0 \to \mathcal{O} \xrightarrow{\sigma'} \mathcal{O}(1)^{\oplus 3} \to \underline{\mathcal{E}}_2/\underline{\mathcal{E}}_1 \to 0$$

where $\sigma' = (z_1, z_2, z_3)$. In particular, $\underline{\mathcal{E}}_2/\underline{\mathcal{E}}_1$ is a stable reflexive sheaf with an essential point singularity at $[0, 0, 0, 1]$. Let $\hat{\mathcal{E}}^1$ be the Hecke transform of $\hat{\mathcal{E}}$ along $\underline{\mathcal{E}}_1$ which is again an optimal extension. By Remark 6.17, $\underline{\hat{\mathcal{E}}}^1 = \underline{\mathcal{E}}_1 \oplus (\underline{\mathcal{E}}_2/\underline{\mathcal{E}}_1)(1)$. In particular, $\hat{\mathcal{E}}^1$ is an optimal extension of which the restriction splits as a direct sum of stables sheaves which has an essential point singularity.

References

[1] A. Besse. *Einstein manifolds*, Springer Science & Business Media, 2007.

[2] S. Bando. *Removable singularities for holomorphic vector bundles.* Tohoku Math. J. (2), 43(1), 61–67, 1991.

[3] B. Mityagin. *The zero set of a real analytic function.* arXiv:1512.07276 (2015).

[4] S. Bando, Y-T. Siu. *Stable sheaves and Einstein-Hermitian metrics*, Geometry and Analysis on Complex Manifolds, T. Mabuchi, J. Noguchi, and T. Ochiai, eds., World Scientific, River Edge, NJ, 1994, 39–50.

[5] X-M. Chen, S. Sun. *Singularities of Hermitian-Yang-Mills connections and the Harder-Narasimhan-Seshadri filtration* arXiv:1707.08314 (2017).

[6] X-M. Chen, S. Sun. *Analytic tangent cones of admissible Hermitian-Yang-Mills connections* arXiv:1806.11247 (2018).

[7] X-M. Chen, S.Sun. *Algebraic tangent cones of reflexive sheaves.* arXiv:1808.02172.

[8] J-P. Demailly. *Analytic methods in algebraic geometry.* Somerville, Mass: International Press, 2012.

[9] G. Daskalopoulos, R. Wentworth. *Convergence properties of the Yang-Mills flow on Kähler surfaces.* J. reine angew. Math 575 (2004): 69-99.

[10] S. Donaldson. *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles.* Proc. London Math. Soc. (3) 50.1 (1985),1-26.

[11] S. Donaldson. *Infinite determinants, stable bundles and curvature.* Duke Math. J. 54.1 (1987), 231–247.

[12] S. Donaldson. *Boundary value problems for Yang-Mills fields.* Journal of Geometry and Physics 8.1-4 (1992): 89-122.

[13] S. Donaldson, S. Sun. *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry*, Acta Math. 213 (2014), 63-106.

[14] S. Donaldson, S. Sun. *Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, II.* arXiv:1507.05082 (2015). To appear in J. Diff. Geom.

[15] D. Eisenbud. *Commutative Algebra: with a view toward algebraic geometry.* Vol. 150. Springer Science Business Media, 2013.

[16] G. Fischer. *Complex analytic geometry.* Vol. 538. Springer, 2006.

[17] P. Griffiths, J. Harris. *Principles of algebraic geometry.* John Wiley & Sons, 2014.

[18] H. Grauert, O. Remmert. *Coherent Analytic Sheaves*, Grundlehren der Mathematischen Wissenschaften, no. 265, Springer, 1984.

[19] H. Grauert, O. Riemenschneider. *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. 11 (1970), 263–292.

[20] H-J. Hein, S. Sun. *Calabi-Yau manifolds with isolated conical singularities* arXiv:1607.02940v2. To appear in Publ. Math. IHES.

[21] A. Jacob. *The limit of the Yang-Mills flow on semi-stable bundles.* Journal die reine und angewandte Mathematik (Crelles Journal) 2015.709 (2015): 1-13.

[22] A. Jacob. *The Yang-Mills flow and the Atiyah-Bott formula on compact Kähler manifolds.* American Journal of Mathematics 138.2 (2016): 329-365.

[23] A. Jacob, H. Sá Earp, T. Walpuski. *Tangent cones of Hermitian Yang-Mills connections with isolated singularities.* arXiv:1610.08283 (2016).

[24] A. Jacob, T. Walpuski. *Hermitian Yang–Mills metrics on reflexive sheaves over asymptotically cylindrical Kähler manifolds.* arXiv: 1603.07702.

[25] S. Kobayashi. *Differential geometry of complex vector bundles.* Publications of the Mathematical Society of Japan, 15. Kano Memorial Lectures, 5. Princeton University Press, Princeton, NJ (1987).

[26] R. Lazarsfeld. *Positivity in algebraic geometry I. Classical setting: line bundles and linear series.* Vol. 48. Springer Science, Business Media, 2004.

[27] H. Nakajima. *Compactness of the moduli space of Yang-Mills connections in higher dimensions.* J. Math. Soc. Japan. Vol. 40, No. 3, 1988.

[28] T. Ohsawa. *On the extension of L^2 holomorphic functions V-effects of generalization.* Nagoya Mathematical Journal 161 (2001): 1-21.

[29] C. Okonek, H. Spindler, M. Schneider. *Vector bundles on complex projective spaces.* Modern Birkhäuser Classics. (2011).

[30] P. Price. *A monotonicity formula for Yang-Mills fields*, Manuscripta Math. 43 (1983), 131–166.

[31] H. N. Sá Earp. *G_2–instantons over asymptotically cylindrical manifolds.* Geom. Topol., 19(1):61–111, 2015.

[32] H. N. Sá Earp and T. Walpuski. *G_2–instantons over twisted connected sums.* Geom. Topol., 19(3):1263–1285, 2015.

[33] B. Shiffman. *On the removal of singularities of analytic sets.* Michigan Math. J 15.1 (1968): 1.

[34] B. Sibley. *Asymptotics of the Yang-Mills flow for holomorphic vector bundles over Kähler manifolds: The canonical structure of the limit.* Journal für die reine und angewandte Mathematik (Crelles Journal) 2015.706 (2015): 123-191.

[35] Y-T. Siu. *A Hartogs type extension theorem for coherent analytic sheaves.* Annals of Mathematics 166-188 (1971).

[36] B. Sibley, R. Wentworth. *Analytic cycles, Bott-Chern forms, and singular sets for the Yang-Mills flow on Kähler manifolds.* Advances in mathematics 279 (2015), 501-531.

[37] C. Simpson *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization.* Journal of the American Mathematical Society 1.4 (1988): 867-918.

[38] G. Tian. *Gauge theory and calibrated geometry. I.* Ann. of Math. (2) 151.1 (2000), 193–268.

[39] K. Uhlenbeck and S-T. Yau. *On the existence of Hermitian-Yang-Mills connections in stable vector bundles.* Comm. Pure Appl. Math. 39.S, suppl. (1986). Frontiers of the mathematical sciences: 1985 (New York, 1985), 257-293.

[40] T. Walpuski. G_2*–instantons over twisted connected sums: an example.* Mathmatical Research Letters, 23(2):529–544, 2015.