Rotationally Symmetric Kähler Metrics with Extremal Condition

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Abstract of the Dissertation

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In this thesis, we study rotationally symmetric extremal Kähler metrics on $\mathbb{C}^n$ ($n \geq 2$) and $\mathbb{C}^2 \backslash \{0\}$. We provide a complete list of solutions of the extremal equation in an implicit manner. We give necessary and sufficient conditions for adding a point smoothly to the origin in $\mathbb{C}^n$. As an application, we prove that there does not exist any rotationally symmetric complete extremal Kähler metrics on $\mathbb{C}^n$ with positive bisectional curvature. We show that certain solutions on $\mathbb{C}^n$ correspond to extremal Kähler metrics with orbifold singularities, and metrics on $\mathbb{CP}^n$ with singular set $\mathbb{CP}^{n-1}$. We also show that certain solutions on $\mathbb{C}^2 \backslash \{0\}$ can be completed to give new families of cscK and strictly extremal Kähler metrics on complex line bundles over $\mathbb{CP}^1$.

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1 Introduction

1.1 $U(n)$ invariant Kähler metrics on $\mathbb{C}^n\backslash\{0\}$

Let $u(s) : (0, \infty) \to \mathbb{R}$ be a smooth function where $s = |z|^2 = |z_1|^2 + |z_2|^2 + \cdots + |z_n|^2$. Then, the real $(1,1)$-form

$$\omega = i\partial\overline{\partial}u = i\sum_{j,k=1}^{n}(\delta_{jk}u'(s) + u''(s)\overline{z}_j z_k)dz^j \wedge d\overline{z}^k \tag{1}$$

gives a positive definite Kähler metric on $\mathbb{C}^n\backslash\{0\}$ if and only if

$$u'(s) > 0, \qquad u'(s) + su''(s) > 0. \tag{2}$$

We introduce the function $g(s) = su'(s)$ and reformulate (2) as

$$g(s) > 0, \qquad g'(s) > 0. \tag{3}$$

We note that the function $g : (0, \infty) \to \mathbb{R}$ satisfying (3) is positive and strictly increasing. Therefore, $\lim_{s\to 0^+} g(s) = A$ and $\lim_{s\to\infty} g(s) = B$ always make sense. We also see that $0 \leq A < B \leq +\infty$.

Let us write the metric (1) in the form

$$\begin{aligned}\mathfrak{g} &= \left(\frac{1}{s}g(s)\delta_{jk} + \frac{1}{s^2}(sg'(s) - g(s))\overline{z}_j z_k\right) dz^j \otimes d\overline{z}^k \\ &= \left(g(s)\left(\frac{1}{s}\delta_{jk} - \frac{1}{s^2}\overline{z}_j z_k\right) + sg'(s)\left(\frac{1}{s^2}\overline{z}_j z_k\right)\right) dz^j \otimes d\overline{z}^k\end{aligned} \tag{4}$$

We will view $\mathbb{CP}^{n-1}$ as the quotient $(\mathbb{C}^n\backslash\{0\})/\mathbb{C}^*$ as well as the quotient $S^{2n-1}(1)/S^1$. Let π_1 and π_2 denote the corresponding projection maps onto $\mathbb{CP}^{n-1}$, respectively.

The real part of the standard Hermitian product on $\mathbb{C}^n$ induces the Riemannian metric on $S^{2n-1}(1)$. The standard Fubini-Study metric $\mathfrak{g}_{FS}$ on $\mathbb{CP}^{n-1}$ is induced by the Riemannian submersion $\pi_2 : S^{2n-1}(1) \to \mathbb{CP}^{n-1}$.

The metric (4) can be expressed as in [FIK03] by

$$\begin{aligned}\mathfrak{g} &= g(s)(\mathfrak{g}_{S^{2n-1}} - \eta\otimes\eta) + sg'(s)\left(\frac{1}{4s^2}ds\otimes ds + \eta\otimes\eta\right) \\ &= g(s)\pi_1^*\mathfrak{g}_{FS} + sg'(s)\mathfrak{g}_{cyl}.\end{aligned} \tag{5}$$

Here η gives the 1-form $d\theta$ when restricted to each complex line through the origin.

Let us introduce the new parameter $r = \sqrt{s}$ and write

$$sg'(s)\mathfrak{g}_{cyl} = g'(s)(dr \otimes dr + r^2 d\theta \otimes d\theta)$$

on a complex line through the origin. Note that straight lines through the origin coincide with minimal geodesics of the $U(n)$-invariant metric $\mathfrak{g}$. It follows that geodesic distance from $z = 0$ to z is given by

$$\tilde{r} = \operatorname{dist}(0, z) = \frac{1}{2}\int_0^s \sqrt{\frac{g'(s)}{s}}\, ds. \tag{6}$$

where $s = |z|^2$. We note that $g'(s)dr \otimes dr = d\tilde{r} \otimes d\tilde{r}$.

We also note that a metric $\mathfrak{g}$ on $\mathbb{C}^n$ given by (4) is complete if and only if $\displaystyle\int_0^\infty \sqrt{\frac{g'(s)}{s}}\, ds = \infty$.

1.2 Positive Bisectional Curvature Case

In [Kle77], Klembeck computed the components of the curvature tensor with respect to the orthonormal frame $\{e_1 = \frac{1}{\sqrt{g'}}\,\partial z_1, e_2 = \frac{1}{\sqrt{u'(s)}}\,\partial z_2, \ldots, e_n = \frac{1}{\sqrt{u'(s)}}\,\partial z_n\}$ at a fixed point $(z_1, 0, \ldots, 0)$.

The nonzero terms are denoted by A, B, C and are given as follows. $(2 \le i \ne j \le n)$

$$\begin{aligned} A &= R_{1\bar{1}1\bar{1}} = -\frac{1}{g'}\left(\frac{sg''}{g'}\right)' \\ B &= R_{1\bar{i}1\bar{i}} = \frac{u''}{(u')^2} - \frac{g''}{u'g'} \\ C &= R_{i\bar{i}i\bar{i}} = 2R_{i\bar{i}j\bar{j}} = -\frac{2u''}{(u')^2} \end{aligned}$$

Theorem 1.1 (Wu-Zheng [WZ11]) *Let $\mathfrak{g}$ be a complete $U(n)$ invariant Kähler metric on $\mathbb{C}^n$ $(n \ge 2)$. Then $\mathfrak{g}$ has positive bisectional curvature if and only if A, B, C are positive functions of s on $[0, \infty)$.*

Definition 1.2 *We denote by $\mathcal{M}_n$ the set of all complete $U(n)$ invariant Kähler metrics on $\mathbb{C}^n$ with positive bisectional curvature.*

In [Kle77], Klembeck constructed an explicit example of a metric in $\mathcal{M}_n$. In [Cao96, Cao97], Cao came up with two examples of Kähler Ricci soliton metrics in $\mathcal{M}_n$. In their paper [WZ11] Wu and Zheng characterized $\mathcal{M}_n$ via a function $\xi = \xi(s)$ and illustrated that the set $\mathcal{M}_n$ is actually quite large.

Definition 1.3 (Wu-Zheng [WZ11]) *The smooth function $\xi : [0, \infty) \to \mathbb{R}$ is defined by*

$$\xi(s) = -s(\log g'(s))'. \tag{7}$$

Thus, we have $g'(s) = g'(0) \exp\left(-\int_0^s \frac{\xi(s)}{s} ds\right)$.

Theorem 1.4 (Characterization of $\mathcal{M}_n$ by the function ξ, Wu-Zheng [WZ11]) *The metric given by (4) is a complete Kähler metric with positive bisectional curvature on $\mathbb{C}^n$ if and only if ξ defined by (7) satisfies*

$$\xi(0) = 0, \quad \xi' > 0, \quad \xi < 1. \tag{8}$$

If we let Ξ be the space of all $\xi \in C^\infty[0, \infty)$ satisfying (8), then Ξ is the space of all diffeomorphisms $[0, \infty) \to [0, b)$, $(0 < b \leq 1)$. The space Ξ is in one-to-one corresponence with $\mathcal{M}_n/\mathbb{R}^+$.

We will see later that no metric in $\mathcal{M}_n$ satisfies the *extremal condition.*

1.3 Extremal Condition

Definition 1.5 *We say that a Kähler metric satisfies the extremal condition if its scalar curvature R satisfies the Euler equation $R_{,\overline{\alpha}\overline{\beta}} = 0$.*

For the rotation invariant Kähler metrics on $\mathbb{C}^n \backslash \{0\}$, Calabi [Cal82] reduced the equation $R_{,\overline{\alpha}\overline{\beta}} = 0$ to a nonlinear ODE $sg'(s) = F(g(s))$ as follows.

Let us denote by $\mathcal{G} = (\mathfrak{g}_{j\overline{k}})$ the matrix of the Kähler metric. Then, as given in [HL18], we have

$$\det \mathcal{G} = (u'(s))^{n-1}(u'(s) + su''(s)) \tag{9}$$

$$\mathfrak{g}^{j\overline{k}} = e^v (u'(s))^{n-2}[(u'(s) + su''(s))\delta_{jk} - u''(s)\overline{z}_k z_j] \tag{10}$$

where $v = -\log \det \mathcal{G}$.

Moreover, by direct computation we have

$$\partial\overline{\partial} v = -\sum_{j,k=1}^{n} (\delta_{jk} v'(s) + v''(s)\overline{z}_j z_k) dz_j \wedge d\overline{z}_k. \tag{11}$$

We combine (10) and (11) to obtain

$$\begin{aligned} R &= \sum_{j,k=1}^{n} \mathfrak{g}^{j\overline{k}} \frac{\partial^2 v}{\partial z_j \partial \overline{z}_k} \\ &= s^{1-n} e^v [s^n (u'(s))^{n-1} v'(s)]'. \end{aligned}$$

We substitute the expression for $\det \mathcal{G}$ given in (9) into this equation to get

$$\begin{aligned} R(s) &= \frac{s^{1-n}[s^n (u')^{n-1} v']'}{(u')^{n-1}(u' + su'')} \\ &= \frac{nv' + s(n-1)(u')^{-1} u'' v' + sv''}{u' + su''} \\ &= \frac{v' \left(\frac{(n-1)(u'+su'')}{u'} + \frac{u'}{u'} \right) + sv''}{u' + su''} \\ &= (n-1)\frac{v'}{u'} + \frac{sv' + s^2 v''}{su' + s^2 u''}. \end{aligned} \tag{12}$$

We note that if we substitute $s = e^t$ in (12), we obtain Equation (3.9) in [Cal82].

The condition that the components of the tangent vector fields $\mathfrak{g}^{\alpha\overline{\beta}} \frac{\partial R}{\partial \overline{z}_\beta} \partial z_\alpha$ be holomorphic is equivalent to the Euler equation $R_{,\overline{\alpha}\overline{\beta}} = 0$ (see [Cal82]). This equation can be expressed in the rotationally symmetric case as follows [Cal82]:

$$\mathfrak{g}^{\alpha\overline{\beta}} \frac{\partial R}{\partial \overline{z}_\beta} = \mathfrak{g}^{\alpha\overline{\beta}} R'(s) z_\beta = z_\alpha \frac{R'(s)}{u' + su''}$$

where, in the last equality, we have used (9) and (10). The Euler equation is now equivalent to $\frac{\partial}{\partial \bar{z}_\beta}\left(z_\alpha \frac{R'}{u'+su''}\right) = 0,\ \ \beta = 1, \dots, n$; and since $\frac{R'}{u'+su''}$ is real valued, we obtain the equation

$$\frac{R'}{u'+su''} = \textit{constant}.$$

For convenience, we will set this constant to be $-(n+2)(n+1)c_4$, as in [Cal82]. We can make use of the variable change $s = e^t$ to integrate the differential equation and obtain

$$R = -(n+2)(n+1)c_4 g(s) - (n+1)nc_3 \tag{13}$$

Replacing R in (13) by its expression in term of u, v, and their derivatives, and integrating once more, we obtain Equation (3.12) in Calabi's article [Cal82]:

$$\frac{g^{n-1}g'}{c_4 g^{n+2} + c_3 g^{n+1} + g^n + c_1 g + c_0} = \frac{1}{s}. \tag{14}$$

The Euler equation $R_{,\overline{\alpha}\overline{\beta}} = 0$ has been reduced to an ODE $sg'(s) = F(g(s))$ where

$$F(g) = \frac{c_4 g^{n+2} + c_3 g^{n+1} + g^n + c_1 g + c_0}{g^{n-1}}. \tag{15}$$

After simplification of rational expression in (15) (if necessary) we will denote the polynomial in the numerator by $H(g)$. If we write $\lim_{s\to 0^+} g(s) = A$ and $\lim_{s\to+\infty} g(s) = B \leq \infty$, then we see from Lemma 4.3, that $H(A) = 0$ and $H > 0$ on (A, B). Moreover, $H(B) = 0$ whenever $B < \infty$.

1.4 k-twisted (Projective) Line Bundles and Orbifolds

Calabi, in his paper [Cal82], described k-twisted projective line bundles $\mathbb{CP}^1 \hookrightarrow \mathcal{F}_k^n \overset{\pi}{\twoheadrightarrow} \mathbb{CP}^{n-1}$ for any $k = 1, 2, \dots, n \geq 2$, as follows.

We cover $\mathbb{CP}^{n-1}$ by n coordinate domains $U_\lambda = \{[z_1 : \cdots : z_n] : z_\lambda \neq 0\}$ $(1 \leq \lambda \leq n)$. On each U_λ, we have a holomorphic coordinate system $({}_\lambda z^\alpha) = \left(\frac{z_\alpha}{z_\lambda}\right)$, $(1 \leq \alpha \leq n,\ \alpha \neq \lambda)$. We introduce a projective holomorphic fiber coordinate $y_\lambda \in \mathbb{C}\cup\{\infty\}$ and trivialization $\pi^{-1}(U_\lambda) \simeq U_\lambda \times \mathbb{CP}^1$ $(1 \leq \lambda \leq n)$ on $\mathcal{F}_k^n$. Here, the transition relation on the fiber coordinate in $\pi^{-1}(U_\lambda \cap U_\mu)$ is given by

$$([z_1 : \cdots : z_n], y_\mu) = \left([z_1 : \cdots : z_n], \left(\frac{z_\mu}{z_\lambda}\right) y_\lambda\right)$$

We have two distinguished sections $s_0, s_\infty : \mathbb{CP}^{n-1} \to \mathcal{F}_k^n$ with images denoted by S_0 and S_∞, respectively. Here s_0 is the zero section given by $y_\lambda = 0$, and s_∞ is the infinity section given by $y_\lambda = \infty$ $(1 \leq \lambda \leq n)$.

We note that the complement $\mathcal{F}_k^n \backslash S_\infty$ gives us the line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(-k)$, whereas $\mathcal{F}_k^n \backslash S_0$ gives $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$. Throughout the thesis, we will denote the zero sections of the line bundles $\mathcal{O}_{\mathbb{CP}^{n-1}}(-k)$ and $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$ by S_0 and S_∞, respectively. We will write $\mathring{\mathcal{F}}_k^n$ for the complement $\mathcal{F}_k^n \backslash \{S_0 \cup S_\infty\}$. We have a $k:1$ map

$$p : \mathbb{C}^n \backslash \{0\} \to \mathring{\mathcal{F}}_k^n \tag{16}$$

which assigns to any point $(z_1, \ldots, z_n)$ with $z_\lambda \neq 0$, the point in $\mathring{\mathcal{F}}_k^n \cap \pi^{-1}(U_\lambda)$ with coordinates $\left(\left(\frac{z_\alpha}{z_\lambda}\right); (z_\lambda)^k\right)$, $(1 \leq \alpha \leq n,\ \alpha \neq \lambda)$.

The map p induces a biholomorphism

$$\tilde{p} : \mathbb{C}^n \backslash \{0\}/\mathbb{Z}_k \to \mathring{\mathcal{F}}_k^n. \tag{17}$$

Thus, $\mathcal{O}_{\mathbb{CP}^{n-1}}(-k)$ is obtained by gluing a $\mathbb{CP}^{n-1}$ smoothly into $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$ at $z = 0$. Similarly, we obtain $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$ if we glue a $\mathbb{CP}^{n-1}$ smoothly into $(\mathbb{C}^n \backslash \{0\})/\mathbb{Z}_k$ at $z = \infty$.

The map $\tilde{p} : \mathbb{C}^n \backslash \{0\}/\mathbb{Z}_k \to \mathcal{O}_{\mathbb{CP}^{n-1}}(-k) \backslash S_0$ can be written as

$$\tilde{p}(z_1, \ldots, z_n) = ([z_1 : \cdots : z_n]; (z_0, \ldots, z_n)^{\otimes k}),$$

where $(z_0, \ldots, z_n)^{\otimes k}$ denotes the generator of the fiber of $\mathcal{O}_{\mathbb{CP}^{n-1}}(-k) \to \mathbb{CP}^{n-1}$ over the point $[z_1 : \cdots : z_n]$. (See Apostolov-Rollin, [AR17] for more details).

We will denote by G_k the compact space obtained from $\mathcal{F}_k^n$ by contracting its zero section S_0 to a point. When $k \geq 2$, we have $G_k = \mathbb{CP}^n/\mathbb{Z}_k$, and it has an orbifold singularity at $z = 0$ modeled on $\mathbb{C}^n/\mathbb{Z}_k$. When $k = 1$, G_k is simply $\mathbb{CP}^n$.

1.5 Closing Conditions

Suppose that we have a $U(n)$-invariant Kähler metric $\mathfrak{g}$ on $\mathbb{C}^n\backslash\{0\}$ that satisfies the extremal condition $sg'(s) = F(g(s))$ where $F(g)$ is given by (15), $n \geq 2$.

If we have a $U(n)$-invariant Kähler metric $\mathfrak{g}$ on $\mathbb{C}^n\backslash\{0\}$ given by (5), it induces a metric on $\mathring{\mathcal{F}}_k^n$ via the map $\tilde{p}$. We will denote the induced metric on $\mathring{\mathcal{F}}_k^n$ by $\mathfrak{g}$ as well.

In [Cal82], Calabi imposed certain asymptotic conditions on Kähler potential $u(s)$ as $s \to 0^+$ and $s \to \infty$. These conditions are necessary and sufficient for the metric $\mathfrak{g}$ on $\mathring{\mathcal{F}}_k^n$ to be extandable by continuity to a smooth metric on all of $\mathcal{F}_k^n$.

Cao [Cao96] used the map $\tilde{p}$ to produce $U(n)$-invariant, complete gradient Kähler-Ricci soliton ($GKRS$) metrics on line bundles over $\mathbb{CP}^{n-1}$. Feldman-Ilmanen-Knopf [FIK03] generalized this approach by producing $U(n)$-invariant GKRS metrics on $(\mathbb{C}^n\backslash\{0\})/\mathbb{Z}_k$, where they allowed new boundary behavior at $z = 0$ and $|z| = \infty$. These behaviors are listed as follows.

1. Metric is completed at $z = 0$ by adding a smooth point.
2. Metric is completed at $z = 0$ by adding an orbifold point.
3. Metric is completed at $z = 0$ by adding a smooth or singular $\mathbb{CP}^{n-1}$.
4. Metric is complete as $|z| \to 0$.

a. Metric is completed at $z = \infty$ by adding a smooth or singular $\mathbb{CP}^{n-1}$.

b. Metric is complete as $|z| \to \infty$.

We note that 1.a. gives $\mathbb{CP}^n$ (with $k = 1$), and 1.b. gives $\mathbb{C}^n$.

The boundary conditions 2.a. correspond to G_k. Conditions 3.b. give $\mathring{\mathcal{F}}_k^n \cup S_0 = \mathcal{O}_{\mathbb{CP}^{n-1}}(-k)$, and conditions 4.a. give $\mathring{\mathcal{F}}_k^n \cup S_\infty = \mathcal{O}_{\mathbb{CP}^{n-1}}(k)$. Calabi's compact k-twisted $\mathbb{CP}^1$-bundle $\mathcal{F}_k^n$ is obtained via 3.a..

A $U(n)$ invariant Kähler metric on $\mathring{\mathcal{F}}_k^n$ induced by the map (17) cannot be completed by adding a $\mathbb{CP}^{n-1}$ at $z = 0$ and a smooth or orbifold point at $z = \infty$. This follows since $g(s)$ is a strictly increasing function of s.

In [LeB88], LeBrun explicitly constructed a scalar-flat Kähler ALE metric on $\mathcal{O}_{\mathbb{CP}^1}(-k)$ for $k = 1, 2, \ldots$ For $k = 1$ and $k = 2$, these are the Burns and the Eguchi-Hanson [EH79] metrics, respectively. He–Li [HL18] gave a complete list of $U(n)$-invariant cscK metrics on $\mathbb{C}^n$, $\mathbb{C}^2\backslash\{0\}$ and $\mathbb{C}^3\backslash\{0\}$. In this work, we give a list of $U(n)$-invariant Kähler metrics with extremal condition on $\mathbb{C}^n$ and $\mathbb{C}^2\backslash\{0\}$. We will use the generalized approach introduced by Feldman–Ilmanen–Knopf in [FIK03] to find examples of complete $U(n)$-invariant Kähler metrics with constant scalar curvature or extremal condition on G_k, $\mathcal{O}_{\mathbb{CP}^1}(k)$ and $\mathcal{O}_{\mathbb{CP}^1}(-k)$. We will also obtain a complete metric on $\mathbb{C}^2\backslash\{0\}$ with both ends left open.

We refer the reader to LeBrun's article [LeB16] for a *Bianchi IX* approach to the same problem where the general solution is displayed *explicitly*.[1]

Adding a $\mathbb{CP}^{n-1}$ smoothly to $(\mathbb{C}^n\backslash\{0\})/\mathbb{Z}_k$ corresponds to a simple zero of F [Cal82]. In what follows, we explain this correspondence as it is presented in [FIK03].

If the sign of F' at the simple root is positive (resp. negative), it means we are adding $\mathbb{CP}^{n-1}$ at $z = 0$ (resp. $|z| = \infty$). This can be seen as follows. Let $\lim\limits_{s\to 0^+} g(s) = A$, $\lim\limits_{s\to\infty} g(s) = B$ $(0 < A < B \leq \infty)$. By Lemma 4.4, we have $H(A) = 0$ and $H > 0$ on (A, B). If $A > 0$ is a simple root of F, then it is a simple root of H. In this case, we have $H'(A) > 0$, implying $F'(A) > 0$. Similarly, if $B < \infty$ is a simple root, we have $F'(B) < 0$. Therefore, the sign of F' at a simple root determines whether $\mathbb{CP}^{n-1}$ is added at $z = 0$ or at $|z| = \infty$.

Let us assume

$$F(A) = 0, \quad A > 0, \quad F'(A) = \theta > 0. \tag{18}$$

For convenience, we will switch to a new parameter $t = \log s$, $-\infty < t < \infty$, as in [FIK03]. We will obtain a specific form for $g(s)$ in a neighborhood of $s = 0$.

We write the ODE $sg'(s) = F(g(s))$ in the form $\phi'(t) = F(\phi(t))$, where $\phi(t) := g(s)$. We have $\phi'(t) = sg'(s) > 0$, hence $t = t(\phi)$ is a smooth strictly increasing function of ϕ. We have a diffeomorphism $\psi = \Phi(\phi)$, given

[1] I am deeply indebted to Claude LeBrun who shared his invaluable insight on the subject.

by $\Phi(\phi) := e^{\theta t(\phi)}$. The ODE $\phi'(t) = F(\phi(t))$ is conjugate to the equation $\psi'(t) = \theta\psi(t)$, so $\phi(t)$ has the form

$$\phi(t) = A + e^{\theta t} G_0(e^{\theta t})$$

as $t \to -\infty$. Here G_0 is a smooth function on $(-\epsilon, \epsilon)$ with $G_0(0) > 0$ [Cal82, Cao96, FIK03].

Let us switch back to the parameter $s = e^t$. We have seen that (18) implies

$$g(s) = A + s^\theta G_0(s^\theta) \tag{19}$$

where G_0 is given as above. Since $F'(A) = \theta > 0$, we are adding $\mathbb{CP}^{n-1}$ at $z = 0$. It follows from (5) and (17) that each complex line through the origin has a cone angle $2\pi\theta/k$.

Remark 1.6 *When we have $\lim_{s\to 0^+} g(s) = A > 0$, $F(A) = 0$, $F'(A) > 0$, equation* (19) *implies that geodesic distance to $z = 0$ is finite, i.e. $\frac{1}{2}\int_0^{s_0} \sqrt{\frac{g'(s)}{s}}\, ds < \infty$.*

A similar discussion follows when we have

$$F(B) = 0, \quad B > 0, \quad F'(B) = -\theta < 0. \tag{20}$$

Let us assume $g(s)$ solves $sg'(s) = F(g(s))$, with $\lim_{s\to\infty} g(s) = B$ and (20) is satisfied. Then we have

$$g(s) = B + s^{-\theta} G_\infty(s^{-\theta}) \tag{21}$$

where G_∞ is smooth on $(-\epsilon, \epsilon)$ and $G_\infty(0) < 0$.

For the proof of the following Lemma, see [FIK03].

Lemma 1.7 (Calabi [Cal82]) *Let $n \geq 2$.*

1. *When $\theta = k$ in* (19)*, the induced Kähler metric is smooth on a neighborhood of the zero section in $\mathcal{O}_{\mathbb{CP}^{n-1}}(-k)$.*
2. *When $\theta = -k$ in* (21)*, the induced Kähler metric is smooth on a neighborhood of the zero section in $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$.*

Remark 1.8 *In Section 2 Lemma 2.5, we will see that a solution of $sg'(s) = F(g(s))$ on $\mathbb{C}^n\backslash\{0\}$ gives a smooth metric on $\mathbb{C}^n$ if and only if $F(0) = 0$. In this case, we say that we are adding a smooth point at $z = 0$.*

2 $U(n)$ invariant Kähler Metrics with Extremal Condition on $\mathbb{C}^n$

2.1 List of Solutions on $\mathbb{C}^n$ and Related Results

Lemma 2.1 *Let $u \in C^\infty(0,\infty)$ be the potential of a rotation invariant Kähler metric on $\mathbb{C}^n \backslash \{0\}$ that satisfies the extremal condition. Then, the metric extends smoothly to $\mathbb{C}^n$ if and only if $u \in C^2[0,\infty)$ and $\lim\limits_{s\to 0^+} u'(s)$ is positive.*

Proof: If we assume $u \in C^2[0,\infty)$, then we have $\lim\limits_{s\to 0^+} g(s) = \lim\limits_{s\to 0^+} su'(s) = 0$, and $\lim\limits_{s\to 0^+} sg'(s) = 0$. It follows from Equation (14) that $c_0 = c_1 = 0$. Equation (14) can be written as

$$u'' = c_4 s(u')^3 + c_3(u')^2. \tag{22}$$

Differentiating (22), we obtain $u \in C^\infty[0,\infty)$. This implies that the hypothesis of Monn's smoothness result[2] (see Proposition 4.1 below) for the corresponding radial function $u(z_1,\ldots,z_n)$ is satisfied for all $k \geq 0$. Together with the condition $\lim\limits_{s\to 0^+} u'(s) > 0$, we conclude that the metric extends smoothly to $\mathbb{C}^n$.

The converse is clear. □

He-Li [HL18] gave a complete list of rotation invariant constant-scalar-curvature Kähler ($cscK$) metrics on $\mathbb{C}^n$.

Theorem 2.2 ([HL18], Theorem 1.1) *Suppose $n \geq 2$ is an integer.*

1. *The rotation invariant Kähler metric ω with zero constant scalar curvature on $\mathbb{C}^n$ must be a multiple of the standard Euclidean metric.*

2. *The rotation invariant Kähler metric ω with constant scalar curvature $n(n+1)$ on $\mathbb{C}^n$ must be of the form*

$$\omega = i\frac{\sum_{j=1}^n dz_j \wedge d\overline{z}_j}{\sum_{j=1}^n |z_j|^2 + a} - i\frac{(\sum_{j=1}^n \overline{z}_j dz_j) \wedge \overline{(\sum_{j=1}^n \overline{z}_j dz_j)}}{(\sum_{j=1}^n |z_j|^2 + a)^2}$$

[2] One has to make a parameter change $r = \sqrt{s}$, and use Proposition 4.2.

where $a > 0$ is a constant.

3. *There does not exist rotation invariant Kähler metric with negative constant scalar curvature on $\mathbb{C}^n$.*

We solve the equation $sg'(s) = F(g(s))$ to give a complete list of rotation invariant metrics on $\mathbb{C}^n$ with extremal condition.

Theorem 2.3 *Let $n \geq 2$ and $u : [0, \infty) \to \mathbb{R}$ be a smooth function such that $u(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2) = u(s)$ is the potential of a Kähler metric with $R_{,\overline{\alpha}\overline{\beta}} = 0$ on $\mathbb{C}^n$ $(n \geq 2)$. Then, one of the following is true:*

1. *ω is a cscK metric.*

2. *There exist constants β, c with $\beta > 0$ such that $g(s) = su'(s)$ is the smooth strictly increasing function ranging from 0 to β on $(0, +\infty)$ uniquely determined by*

$$\log(g(s)) - \log(\beta - g(s)) - \beta \frac{1}{g(s) - \beta} = \log s + c.$$

3. *There exist constants γ, β, c with $\gamma < 0 < \beta$ such that $g(s) = su'(s)$ is the smooth strictly increasing function ranging from 0 to β on $(0, +\infty)$ uniquely determined by*

$$\log(g(s)) + \frac{\beta\gamma}{\beta(\beta - \gamma)} \log(\beta - g(s)) + \frac{\beta\gamma}{\gamma(\gamma - \beta)} \log(g(s) - \gamma) = \log(s) + c.$$

4. *There exist constants γ, β, c with $0 < \beta < \gamma$ such that $g(s)$ is the smooth strictly increasing function ranging from 0 to β on $(0, \infty)$ determined by*

$$\log(g(s)) + \frac{\beta\gamma}{\beta(\beta - \gamma)} \log(\beta - g(s)) + \frac{\beta\gamma}{\gamma(\gamma - \beta)} \log(\gamma - g(s)) = \log(s) + c.$$

Proof: From the proof of Lemma 2.1, if u is the potential of a smooth metric on $\mathbb{C}^n$, then we have $\lim_{s\to 0^+} g(s) = 0$ and the constants c_0 and c_1 in equation (14) vanish. Equation (14) becomes

$$\frac{g'}{c_4 g^3 + c_3 g^2 + g} = \frac{1}{s}. \tag{23}$$

In this case, the polynomial $H(x)$ in Lemma 4.3 is given by $c_4x^3 + c_3x^2 + x$, and unless $c_3 = c_4 = 0$, we have $B < \infty$ for degree reasons. By the same lemma, $H(A) = 0$ and $H > 0$ on (A, B), and all roots are real. We have $A = \lim\limits_{s\to 0^+} g(s) = 0$.

<u>*Case* (1)</u> $c_4 = 0$

We see from equation (13) that ω is a cscK metric.

<u>*Case* (2)</u> $c_4 \neq 0$ *and the polynomial* $H(x) = c_4x^3 + c_3x^2 + x$ *has roots* α, β, β *with* $\alpha < \beta$

It follows from Lemma 4.3 that $\alpha = A = 0$. Since $H(x) > 0$ in (α, β), we have $c_4 > 0$, and the equation (23) can be written as

$$g'\beta^2 \left\{ \frac{1}{\beta^2}\frac{1}{g(s)} - \frac{1}{\beta^2}\frac{1}{g(s)-\beta} + \frac{1}{\beta}\frac{1}{(g(s)-\beta)^2} \right\} = \frac{1}{s}.$$

There exists a constant c such that

$$\log(g(s)) - \log(\beta - g(s)) - \frac{\beta}{g(s)-\beta} = \log s + c. \tag{24}$$

Since $H(x) = c_4x^3 + c_3x^2 + x = c_4x(x-\beta)^2$, we have $c_4\beta^2 = 1$.

On the other hand, Lemma 4.4 implies that there exists a unique smooth strictly increasing function $g(s) = su'$, $g : (0, \infty) \to (0, \beta)$ satisfying (24).

<u>*Case* (3)</u> $c_4 \neq 0$ *and the polynomial* $H(x) = c_4x^3 + c_3x^2 + x$ *has roots* α, α, β *with* $\alpha < \beta$

It follows from Lemma 4.3 that $\alpha = A = 0$, but this polynomial cannot have a double root at 0.

Case (4) *$c_4 \neq 0$ and the polynomial $H(x) = c_4x^3 + c_3x^2 + x$ has distinct roots $\gamma < \beta < \alpha = 0$*

It follows from Lemma 4.3 that $B < \infty$, and $H(x)$ must have at least two distinct nonnegative roots. So we do not get a solution from here.

Case (5) *$c_4 \neq 0$ and the polynomial $H(x) = c_4x^3 + c_3x^2 + x$ has distinct roots $\gamma < \alpha = 0 < \beta$*

By Lemma 4.3 we have $A = \alpha = 0$, $B = \beta$, and $H(x) > 0$ on $(0, \beta)$. Then $c_4 < 0$, $c_4\beta\gamma = 1$, and the equation (23) can be written as

$$g' \left\{ \frac{1}{g(s)} + \frac{\beta\gamma}{\beta(\beta-\gamma)} \frac{1}{g(s)-\beta} + \frac{\beta\gamma}{\gamma(\gamma-\beta)} \frac{1}{g(s)-\gamma} \right\} = \frac{1}{s}. \tag{25}$$

There exists a constant c such that

$$\log(g(s)) + \frac{\beta\gamma}{\beta(\beta-\gamma)} \log(\beta - g(s)) + \frac{\beta\gamma}{\gamma(\gamma-\beta)} \log(g(s)-\gamma) = \log s + c.$$

It follows from Lemma 4.4 that there exists a unique smooth strictly increasing function $g(s) : (0, \infty) \to (0, \beta)$.

Case (6) *$c_4 \neq 0$ and the polynomial $H(x) = c_4x^3 + c_3x^2 + x$ has distinct roots $\alpha = 0 < \beta < \gamma$*

It follows from Lemma 4.3 that $A = \alpha = 0$, $B = \beta$ and $c_4 > 0$. As in the previous case, Equation (23) can be rewritten as Equation (25). Integrating both sides of (25) we get

$$\log(g(s)) + \frac{\beta\gamma}{\beta(\beta-\gamma)} \log(\beta - g(s)) + \frac{\beta\gamma}{\gamma(\gamma-\beta)} \log(\gamma - g(s)) = \log s + c.$$

If we let $h(t) : (0, \beta) \to \mathbb{R}$ be the function

$$\log(t) + \frac{\beta\gamma}{\beta(\beta-\gamma)} \log(\beta - t) + \frac{\beta\gamma}{\gamma(\gamma-\beta)} \log(\gamma - t)$$

we see that $\lim_{t \to 0^+} h(t) = -\infty$, $\lim_{t \to \beta^-} h(t) = \infty$, and $h'(t) > 0$ on $(0, \beta)$. Lemma 4.4 guarantees the unique existence of a smooth $g(s) : (0, \infty) \to (0, \beta)$ with the desired properties.

$\square$

Remark 2.4 *We note that, in the above proof, in order to obtain the implicit solutions given by (1)–(4) of Theorem 2.3, we have not used the full strength of* $u(s)$ *being in* $C^\infty[0,\infty)$. *In the proof, we have only used* $u(s) \in C^\infty(0,\infty)$, $\lim_{s\to 0^+} g(s) = 0$, *and* $c_0 = c_1 = 0$. *A careful inspection of the implicit solutions (1)–(4) shows that* $\lim_{s\to 0^+} u'(s)$ *is finite and positive, hence, equation* (22) *implies that* $u(s)$ *is in* $C^\infty[0,\infty)$. *It follows from Lemma 2.1 that such metrics can be smoothly extended to the origin.*

The following lemma tells us when a rotation invariant Kähler metric with extremal condition on $\mathbb{C}^n\backslash\{0\}$ can be smoothly extended to $\mathbb{C}^n$, $n \geq 2$.

Lemma 2.5 (Adding a smooth point at $z = 0$) *Let* $g : (0,\infty) \to (A, B)$ $(0 \leq A < B \leq \infty)$ *be a positive, strictly increasing solution of* $sg' = F(g)$. *Then the following are equivalent.*

1. g *induces a smooth metric on* $\mathbb{C}^n$.
2. $\lim_{s\to 0^+} g(s) = 0$.
3. $F(0) = 0$.

Proof: See Section 2.3. $\square$

Corollary 2.6 *There does not exist a rotation invariant extremal Kähler metric with negative scalar curvature on* $\mathbb{C}^n$.

Proof: See Section 2.3. $\square$

Theorem 2.2 and Theorem 2.3 together give a complete list of $U(n)$ invariant extremal Kähler metrics on $\mathbb{C}^n$. We note that for these metrics, $\lim_{s\to 0^+} g(s) = 0$ and $\lim_{s\to +\infty} g(s) = B \leq \infty$. It follows from Remark 1.6 that, if $B < \infty$ is a simple root of $F(g)$, then the induced metric is incomplete as $|z| \to \infty$.

We can easily check that in Theorem 2.2 and Theorem 2.3 there are only two cases where B is not a simple root of F. The first case is (1) of Theorem 2.2. In this case, we have $B = \infty$, and the metric is a multiple of the standard Euclidean metric which is complete. The second case is (2) of Theorem 2.3. We will compute the geodesic distance as $|z| \to \infty$, and see that this metric is complete as well.

Example 2.7 (A complete $U(n)$ invariant extremal Kähler metric on $\mathbb{C}^n$) We will see that (2) of Theorem 2.3 induces complete metrics on $\mathbb{C}^n$. In this case, the ODE is given by $sg'(s) = F(g(s))$ where

$$F(g) = c_4 g^3 + c_3 g^2 + g = c_4 g(g-\beta)^2.$$

Here we have $A = \lim_{s\to 0^+} g(s) = 0$, $B = \lim_{s\to +\infty} g(s) = \beta < \infty$, and $c_4 = \frac{1}{\beta^2} > 0$.

We will show that geodesic distance from a point z_0 to $|z| = \infty$ is infinite, i.e.

$$\int_{s_0}^{\infty} \sqrt{\frac{g'(s)}{s}}\, ds = \int_{s_0}^{\infty} \frac{\sqrt{F(g(s))}}{s}\, ds = \infty.$$

There exists a $d_1 > 0$ such that on (s_0, ∞) we have

$$\sqrt{F(g(s))} = \frac{1}{\beta}|g-\beta|\sqrt{g} > d_1(\beta - g)$$

and

$$\int_{s_0}^{\infty} \frac{\sqrt{F(g(s))}}{s}\, ds > d_1 \int_{s_0}^{\infty} \frac{\beta - g}{s}\, ds.$$

The solution $g(s)$ is given by Equation (24) as follows.

$$\log(g(s)) - \log(\beta - g(s)) + \frac{\beta}{\beta - g(s)} = \log s + c$$

The term $\log(g(s))$ is bounded on (s_0, ∞). We can choose s_0 large enough so that $\log(\beta - g(s)) < 0$ and $\log s - \log(g(s)) + c > 0$ on (s_0, ∞). In this case, Equation (24) implies $\frac{\beta}{\beta - g(s)} < \log s + c_1$. Therefore

$$\int_{s_0}^{\infty} \sqrt{\frac{g'(s)}{s}}\, ds > d_1 \int_{s_0}^{\infty} \frac{\beta - g(s)}{s}\, ds > \beta\, d_1 \int_{s_0}^{\infty} \frac{ds}{s(\log s + c_1)} = \infty.$$

The metric is complete on $\mathbb{C}^n$.

Proposition 2.8 *There is no metric in $\mathcal{M}_n$ that satisfies the extremal condition.*

Proof: We have seen that we have only two types of complete $U(n)$ invariant extremal Kähler metrics on $\mathbb{C}^n$. The first type is given by (1) of Theorem 2.2, namely a scalar multiple of the standard Euclidean metric on $\mathbb{C}^n$. Metrics of this type clearly do not have positive bisectional curvature.

The second type is given by (2) of Theorem 2.3. We will see that bisectional curvature is not positive in this case either. We will compute the ξ function for this metric, and show that it does not satisfy the properties given in Theorem 1.4.

By definition we have $\xi = -s(\log(g'(s)))'$. We recall that the ODE $sg'(s) = F(g(s))$ is given by

$$sg'(s) = \frac{1}{\beta^2}g(s)(g(s)-\beta)^2.$$

We compute

$$(\log(g'))' = -\frac{1}{s} + \frac{(g(s)-\beta)^2}{\beta^2 s} - \frac{2g(s)(\beta - g(s))}{\beta^2 s}.$$

Then we have

$$\begin{aligned}\xi(s) &= 1 - \frac{1}{\beta^2}(g(s)-\beta)^2 + \frac{2}{\beta^2}g(s)(\beta - g(s)) \\ &= -\frac{1}{\beta^2}g(3g-4\beta)\end{aligned}$$

$\xi(s)$ is a polynomial in g restricted to the interval $(0, \beta) \ni g$. We see that $\dfrac{d\xi}{ds} = \dfrac{d\xi}{dg}\dfrac{dg}{ds}$ is not positive on $(0, \infty) \ni s$. Therefore ξ fails to satisfy the necessary and sufficient conditions in Theorem 1.4. The metric in Case (2) of Theorem 2.3 does not have positive bisectional curvature.

□

2.2 Examples of Extremal Kähler Metrics with Singularities

Dabkowski–Lock [DL16] gave a Kähler conformal compactification of LeBrun's negative mass metric on $\mathcal{O}_{\mathbb{CP}^1}(-k)$ to obtain a Kähler orbifold metric on $\hat{\mathcal{O}}_{\mathbb{CP}^1}(-k)$. The positive line bundle $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$, $k = 1, 2, \ldots$ is obtained by gluing a $\mathbb{CP}^{n-1}$ to $(\mathbb{C}^n\backslash\{0\})/\mathbb{Z}_k$ at $|z| = \infty$. If we compactify $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$ by adding a singular point at $z = 0$, we obtain the orbifold G_k. We note that the singular point $z = 0$ is modeled on $\mathbb{C}^n/\mathbb{Z}_k$. Here, we show that Case (3) of Theorem 2.3 gives a strictly extremal metric on the orbifold space G_k ($n \geq 2$).

Example 2.9 (Strictly extremal metrics on G_k, $n \geq 2$) Let us consider Case (3) of Theorem 2.3. Since $c_4 \neq 0$, it follows from Equation (13) that this is a strictly extremal metric on $\mathbb{C}^n$. From the proof of Theorem 2.3 we have the ODE $sg'(s) = F(g(s))$ where $F(g) = c_4g^3 + c_3g^2 + g = c_4\, g(g-\beta)(g-\gamma)$. Here, we have $\gamma < 0 < \beta$, $\lim\limits_{s\to 0^+} g(s) = 0$, $\lim\limits_{s\to+\infty} g(s) = \beta$, and $c_4 = \frac{1}{\beta\gamma}$.

A $U(n)$ invariant Kähler metric on $\mathbb{C}^n$ induces a smooth orbifold metric on $G_k\backslash S_\infty$ via the $k:1$ map p given by (16). Here, S_∞ stands for the zero section of $\mathcal{O}_{\mathbb{CP}^{n-1}}(k)$. It follows from Lemma 1.7 that the induced metric can be extended smoothly to S_∞ if and only if $F(\beta) = 0$ and $F'(\beta) = -k$.

We clearly have $F(\beta) = 0$. We compute $F'(\beta) = \frac{\beta-\gamma}{\gamma}$. For every positive integer k, there exist γ, β ($\gamma < 0 < \beta$) that satisfy $F'(\beta) = -k$. Namely, let $\beta = |\gamma|(k-1)$.

Dabkowski-Lock [DL16] explicitly constructed a family of extremal Kähler edge cone metrics on $(\mathbb{CP}^2, \mathbb{CP}^1)$ with cone angles $2\pi\theta$, $\theta \geq 0$. Here, we give examples of strictly extremal metrics on $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$ with cone angles $2\pi\theta$, $0 < \theta < 1$, $n \geq 2$.

Example 2.10 (Strictly extremal metrics on $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$ with cone angles $2\pi\theta$, $0 < \theta < 1$) Let us consider Case (4) of Theorem 2.3. Since $c_4 \neq 0$, it follows from Equation (13) that this is a strictly extremal metric on $\mathbb{C}^n$. From the proof of Theorem 2.3 we have the ODE $sg'(s) = F(g(s))$ where $F(g) = c_4g^3 + c_3g^2 + g = c_4\, g(g-\beta)(g-\gamma)$. Here, we have $0 < \beta < \gamma$, $\lim\limits_{s\to 0^+} g(s) = 0$, $\lim\limits_{s\to+\infty} g(s) = \beta$, and $c_4 = \frac{1}{\beta\gamma}$.

As in the previous example we compute $F(\beta) = 0$ and $F'(\beta) = \frac{\beta-\gamma}{\gamma} = -\theta$. The inequality $0 < \beta < \gamma$ implies that $0 < \theta < 1$. Therefore, these are strictly extremal metrics on $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$ with cone angle $2\pi\theta$, $0 < \theta < 1$, along $\mathbb{CP}^{n-1}$ attached at $|z| = \infty$.

2.3 Proofs

In this section we give the proofs of Lemma 2.5 and Corollary 2.6.

Proof: [Proof of Lemma 2.5]

$\underline{(1) \Rightarrow (2)}$ Let us assume we have a smooth $U(n)$ invariant metric on $\mathbb{C}^n$. Then we have $u(s) \in C^\infty[0, \infty)$. This implies $\lim\limits_{s \to 0^+} g(s) = \lim\limits_{s \to 0^+} su'(s) = 0$.

$\underline{(2) \Rightarrow (1)}$ $\lim\limits_{s \to 0^+} g(s) = 0$ implies that the constants c_0 and c_1 in equation (14) vanish. This can be seen as follows. Assume $\lim\limits_{s \to 0^+} g(s) = A = 0$ and $c_0 \neq 0$. The condition $c_0 \neq 0$ implies

$$F(g) = \frac{c_4 g^{n+2} + c_3 g^{n+1} + g^n + c_1 g + c_0}{g^{n-1}} = \frac{H(g)}{g^{n-1}}$$

and $H(0) = c_0 \neq 0$. This contradicts (1) of Lemma 4.3 which requires $H(A) = 0$. So we must have $c_0 = 0$.

Now let us assume $\lim\limits_{s \to 0^+} g(s) = 0$, $c_0 = 0$, and $c_1 \neq 0$. Then

$$F(g) = \frac{c_4 g^{n+1} + c_3 g^n + g^{n-1} + c_1}{g^{n-2}} = \frac{H(g)}{g^{n-2}}$$

and $H(0) = c_1 \neq 0$, which contradicts (1) of Lemma 4.3 again. Therefore $\lim\limits_{s \to 0^+} g(s) = 0$ implies $c_0 = c_1 = 0$.

It follows from Remark 2.4 that, since we have $\lim\limits_{s \to 0^+} g(s) = 0$ and $c_0 = c_1 = 0$, the metric smoothly extends to the origin.

$\underline{(2) \Rightarrow (3)}$ Let us assume $\lim\limits_{s \to 0^+} g(s) = 0$. We have already seen that this implies $c_0 = c_1 = 0$. It follows from the definition of F that $F(0) = 0$.

$\underline{(3) \Rightarrow (2)}$ $F(0) = 0$ implies $c_0 = c_1 = 0$. This can be seen from the definition of F $(n \geq 2)$ and the limit

$$\lim_{x \to 0} \frac{c_1 x + c_0}{x^{n-1}} = \lim_{x \to 0} (F(x) - c_4 x^3 - c_3 x^2 - x) = 0.$$

Now, we will show that $c_0 = c_1 = 0$ implies $\lim\limits_{s \to 0^+} g(s) = 0$.

Let us assume $\lim_{s\to 0^+} g(s) = A > 0$. We will arrive at a contradiction. If $c_0 = c_1 = 0$, equation (14) becomes $sg' = c_4 g^3 + c_3 g^2 + g = H(g)$.

We have the following cases:

- $\boxed{c_4 = c_3 = 0}$
  In this case $H(g) = g$ and $H(A) \neq 0$ for $A > 0$. This contradicts (1) of Lemma 4.3.

- $\boxed{c_4 = 0,\ c_3 \neq 0}$
  We have $H(g) = g(c_3 g + 1)$. Since $A > 0$ and $H(A)$ vanishes by (1) of Lemma 4.3, we have $B = \infty$. But this contradicts (2) of Lemma 4.3 for degree reasons.

- $\boxed{c_4 \neq 0}$
  It follows from (2) of Lemma 4.3 that $B < \infty$. We have

  $$H(g) = c_4 g^3 + c_3 g^2 + g = c_4 g(g - A)(g - B)$$

  and $H > 0$ on (A, B), $(0 < A < B < \infty)$. This implies $c_4 < 0$, which contradicts $c_4 = \frac{1}{AB} > 0$.

Therefore, if $c_0 = c_1 = 0$, we have $\lim_{s\to 0^+} g(s) = A = 0$. $\square$

Proof: [Proof of Corollary 2.6] The ODE is given by $sg'(s) = F(g(s))$ where $F(g) = c_4 g^3 + c_3 g^2 + g$.

When $c_4 = 0$, the metric is cscK, and it follows from Theorem 2.2 that we cannot have $R < 0$.

Lemma 4.3 implies that, if we have $c_4 \neq 0$, then $\lim_{s\to+\infty} g(s) = B < \infty$ for degree reasons.

The scalar curvature $R(s)$ is given by

$$R = -(n+2)(n+1)c_4 g(s) - (n+1)nc_3.$$

The condition $R < 0$ gives $-nc_3 \leq (n+2)c_4 g(s)$. Let us check (2)–(4) of Theorem 2.3 to see this is impossible.

Case (2) We have $F(g) = c_4 g(g-\beta)^2$ where $\beta = \lim_{s\to+\infty} g(s)$, $c_4 = \frac{1}{\beta^2}$, and $c_3 = -\frac{2}{\beta}$. Then, $R < 0$ implies $n\frac{2}{\beta} \le (n+2)\frac{1}{\beta^2} g(s)$, which contradicts $\lim_{s\to 0^+} g(s) = 0$.

Case (3) We have $F(g) = c_4 g(g-\beta)(g-\gamma)$ where $\gamma < 0 < \beta$, $\lim_{s\to+\infty} g(s) = \beta$, $c_4 = \frac{1}{\beta\gamma}$, and $c_3 = -\frac{\beta+\gamma}{\beta\gamma}$. Inequality $R < 0$ implies $n\frac{\beta+\gamma}{\beta\gamma} \le (n+2)\frac{1}{\beta\gamma} g(s)$. Since $\beta\gamma < 0$, we have $g(s) \le \frac{n+2}{n} g(s) \le \beta + \gamma$. This contradicts $\lim_{s\to+\infty} g(s) = \beta$.

Case (4) We have $F(g) = c_4 g(g-\beta)(g-\gamma)$ where $0 < \beta < \gamma$, $\lim_{s\to+\infty} g(s) = \beta$, $c_4 = \frac{1}{\beta\gamma}$, and $c_3 = -\frac{\beta+\gamma}{\beta\gamma}$. Inequalities $R < 0$ and $\beta\gamma > 0$ imply $\frac{n}{n+2}(\beta+\gamma) \le g(s)$. This contradicts $\lim_{s\to 0^+} g(s) = 0$.

□

3 $U(2)$ Invariant Kähler Metrics with Extremal Condition on $\mathbb{C}^2\backslash\{0\}$

3.1 List of Solutions on $\mathbb{C}^2\backslash\{0\}$

In this section, we solve the ordinary differential equation (14) for dimension $n = 2$. The solutions with constant scalar curvature were given in [HL18].

Theorem 3.1 (He-Li [HL18], Theorem 1.2) *Let $u : (0, +\infty) \to \mathbb{R}$ be a smooth function such that $u(|z_1|^2 + |z_2|^2)$ is the potential of a Kähler metric with constant scalar curvature $R = 0$ on $\mathbb{C}^2\backslash\{0\}$. Then one of the following is true:*

(1) There exist constants a, b with $a > 0$ such that

$$u(s) = as + b$$

(2) There exist constants a, b, c with $a > 0, b > 0$ such that

$$u(s) = as + b\log s + c$$

(3) *There exist constants* α, β, c *with* $\alpha \neq 0, \beta > 0, \alpha < \beta$ *such that* $g(s) = su'(s)$ *is the smooth strictly increasing function on* $(0, +\infty)$ *ranging from* β *to* $+\infty$ *determined by*

$$\frac{\beta}{\beta - \alpha} \log(g(s) - \beta) - \frac{\alpha}{\beta - \alpha} \log(g(s) - \alpha) = \log s + c$$

(4) *There exist constants* α, c *with* $\alpha > 0$ *such that* $g(s) = su'(s)$ *is the smooth strictly increasing function ranging from* α *to* $+\infty$ *on* $(0, +\infty)$ *determined by*

$$\log(g(s) - \alpha) - \frac{\alpha}{g(s) - \alpha} = \log s + c$$

Theorem 3.2 (He-Li [HL18], Theorem 1.3) *Let* $u : (0, +\infty) \to \mathbb{R}$ *be a smooth function such that* $u(|z_1|^2 + |z_2|^2)$ *is the potential of a Kähler metric with constant scalar curvature* $R = 6$ *on* $\mathbb{C}^2 \backslash \{0\}$ *and* $g(s) = su'(s)$*. Then one of the following is true:*

(1) *There exist constants* a, c *with* $a > 0$ *such that*

$$u(s) = \log(s + a) + c$$

(2) *There exist constants* a, k *with* $a > 0, 0 < k < 1$ *such that*

$$g(s) = \frac{1}{2}(k + 1) - \frac{ka}{s^k + a}$$

(3) *There exist constants* α, β, γ, c *with* $\alpha \neq 0, \beta > 0, \alpha < \beta < \gamma, \alpha + \beta + \gamma = 1$ *such that* $g(s) = su'(s)$ *is the smooth strictly increasing function ranging from* β *to* γ *on* $(0, +\infty)$ *determined by*

$$\begin{aligned} &- \alpha(\gamma - \beta) \log(g(s) - \alpha) + \beta(\gamma - \alpha) \log(g(s) - \beta) \\ &- \gamma(\beta - \alpha) \log(\gamma - g(s)) = (\beta - \alpha)(\gamma - \beta)(\gamma - \alpha) \log s + c. \end{aligned}$$

(4) *There exist constants* α, β, γ *with* $0 < \beta < \alpha, \alpha + 2\beta = 1$ *such that* $g(s) = su'(s)$ *is the smooth strictly increasing function ranging from* β *to* α *on* $(0, +\infty)$ *determined by*

$$\alpha \log(g(s) - \beta) - \alpha \log(\alpha - g(s)) + \frac{\beta(\beta - \alpha)}{g(s) - \beta} = (\beta - \alpha)^2 \log s + c$$

Proposition 3.3 *Let $u : (0,\infty) \longleftrightarrow \mathbb{R}$ be a smooth function such that $u(|z_1|^2 + |z_2|^2)$ is the potential of a Kähler metric satisfying the extremal condition on $\mathbb{C}^2\backslash\{0\}$ and $g(s) = su'(s)$. Then one of the following is true.*

1. *Metric can be extended smoothly to $\mathbb{C}^2$.*

2. *Metric is cscK with a singularity at the origin.*

3. *There exist constants α, β, c with $0 < \alpha < \beta$ such that $g(s)$ is the smooth strictly increasing function from α to β on $(0,\infty)$ determined by*

$$\frac{\beta(\beta+2\alpha)}{(\alpha-\beta)^2}\left\{\log(g(s)-\alpha) - \log(\beta - g(s)) - \frac{\beta-\alpha}{g(s)-\beta}\right\} = \log s + c. \tag{26}$$

4. *There exist constants α, β, γ, c with $\gamma < -\frac{\alpha\beta}{\alpha+\beta} < 0 < \alpha < \beta$ such that $g(s)$ is the smooth strictly increasing function ranging from α to β on $(0,+\infty)$ determined by*

$$(\alpha\beta+\alpha\gamma+\beta\gamma)\left\{\frac{\log(g(s)-\alpha)}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\log(\beta-g(s))}{(\beta-\alpha)(\beta-\gamma)} + \frac{\log|g(s)-\gamma|}{(\gamma-\alpha)(\gamma-\beta)}\right\} = \log s + c. \tag{27}$$

5. *There exist constants α, β, γ, c with $0 < \alpha < \beta < \gamma$ such that $g(s)$ is the smooth strictly increasing function ranging from α to β on $(0,+\infty)$ determined by equation (27).*

6. *There exist constants α, β, c with $0 < \alpha < \beta$ such that $g(s)$ is the smooth strictly increasing function ranging from α to β on $(0,+\infty)$ determined by*

$$\begin{aligned}((\alpha+\beta)^2+2\alpha\beta)\Bigg\{&\frac{\alpha+\beta}{(\beta-\alpha)^3}\log(g(s)-\alpha) - \frac{\alpha}{(\alpha-\beta)^2}\frac{1}{g(s)-\alpha} - \\ &\frac{\alpha+\beta}{(\beta-\alpha)^3}\log(\beta-g(s)) - \frac{\beta}{(\alpha-\beta)^2}\frac{1}{g(s)-\beta}\Bigg\} = \log s + c.\end{aligned} \tag{28}$$

7. *There exist constants* α, β, γ, c *with* $0 < \alpha < \beta < \gamma$ *such that* $g(s)$ *is the smooth strictly increasing function ranging from* α *to* β *on* $(0, +\infty)$ *determined by*

$$(\alpha^2 + 2\alpha\beta + 2\alpha\gamma + \beta\gamma)\left\{\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\log(\gamma - g(s)) - \right. \tag{29}$$
$$\frac{\alpha}{(\alpha-\gamma)(\alpha-\beta)}\frac{1}{g(s)-\alpha} + \frac{-\alpha^2+\beta\gamma}{(\alpha-\gamma)(\alpha-\beta)}\log(g(s)-\alpha) +$$
$$\left. \frac{\beta}{(\beta-\gamma)(\beta-\alpha)^2}\log|\beta - g(s)|\right\} = \log s + c.$$

8. *There exist constants* α, β, γ, c *with* $\alpha < -\frac{\alpha^2+\beta\gamma}{2(\beta+\gamma)} < 0 < \beta < \gamma$ *such that* $g(s)$ *is the smooth strictly increasing function ranging from* β *to* γ *on* $(0, +\infty)$ *determined by equation* (29).

9. *There exist constants* α, β, γ, c *with* $\alpha < -\frac{\beta^2+2\beta\gamma}{2\beta+\gamma} < 0 < \beta < \gamma$ *such that* $g(s)$ *is the smooth strictly increasing function ranging from* β *to* γ *on* $(0, +\infty)$ *determined by*

$$(\beta^2 + 2\beta\alpha + 2\beta\gamma + \alpha\gamma)\left\{\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}\log(g(s) - \alpha)\right. \tag{30}$$
$$-\frac{\beta}{(\beta-\alpha)(\beta-\gamma)}\frac{1}{g(s)-\beta} + \frac{-\beta^2+\alpha\gamma}{(\beta-\alpha)(\beta-\gamma)}\log(g(s)-\beta)$$
$$\left. +\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)^2}\log(\gamma - g(s))\right\} = \log s + c.$$

10. *There exist constants* α, β, γ, c *with* $-\frac{\gamma(\gamma+2\beta)}{2\gamma+\beta} < \alpha < \beta < \gamma$ *and* $\alpha\beta\gamma \neq 0$ *such that* $g(s)$ *is the smooth strictly increasing function ranging from* β *to* γ *on* $(0, +\infty)$ *determined by*

$$(\gamma^2 + 2\beta\gamma + 2\alpha\gamma + \alpha\beta)\left\{\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}\log(g(s) - \alpha)\right. \tag{31}$$
$$-\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\frac{1}{g(s)-\gamma} + \frac{-\gamma^2+\alpha\beta}{(\gamma-\alpha)(\gamma-\beta)}\log(\gamma-g(s))$$
$$\left. +\frac{\beta}{(\beta-\alpha)(\beta-\gamma)^2}\log(g(s)-\beta)\right\} = \log s + c.$$

11. *There exist constants* $\alpha, \beta, \gamma, \tau, c$ *with* $-\frac{\beta\gamma+\beta\tau+\gamma\tau}{\beta+\gamma+\tau} < \alpha < \beta < \gamma < \tau$ *and* $\alpha\beta\gamma\tau \neq 0$ *such that* $g(s)$ *is the smooth strictly increasing function ranging from* β *to* γ *on* $(0, +\infty)$ *determined by*

$$
\begin{aligned}
&(\alpha\beta + \alpha\gamma + \alpha\tau + \beta\gamma + \beta\tau + \gamma\tau)\left\{\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\tau)}\log(g(s)-\alpha)\right. \\
&+\frac{\beta}{(\beta-\alpha)(\beta-\gamma)(\beta-\tau)}\log|g(s)-\beta| + \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\tau)}\log|g(s)-\gamma| \\
&\left.+\frac{\tau}{(\tau-\alpha)(\tau-\beta)(\tau-\gamma)}\log|g(s)-\tau|\right\} = \log s + c.
\end{aligned}
\tag{32}
$$

12. *There exist constants* $\alpha, \beta, \gamma, \tau, c$ *with* $\alpha\beta+\alpha\gamma+\alpha\tau+\beta\gamma+\beta\tau+\gamma\tau < 0$, $\alpha\beta\gamma\tau \neq 0$ *such that* $g(s)$ *is the smooth strictly increasing function ranging from* γ *to* τ *on* $(0, +\infty)$ *determined by* (32).

13. *There exist constants* α, β, a, b *with* $0 < \alpha < \beta$ *and* $a^2 + 2a(\alpha+\beta) + b^2 + \alpha\beta < 0$ *such that* $g(s)$ *is the smooth strictly increasing function ranging from* α *to* β *on* $(0, +\infty)$ *determined by*

$$
\begin{aligned}
&(a^2 + 2a(\alpha+\beta) + b^2 + \alpha\beta)\Big\{c_1\log(g(s)-\alpha) + c_2\log(\beta - g(s)) + \\
&\int_\alpha^\beta \frac{-(c_1+c_2)g(s) + 2(c_1+c_2)a - c_1\alpha - c_2\beta}{g^2(s) - 2ag(s) + a^2 + b^2} g'(s)ds\Big\} = \log s,
\end{aligned}
\tag{33}
$$

where $c_1 = \frac{\alpha}{(\alpha-\beta)(\alpha^2-2a\alpha+a^2+b^2)}$ *and* $c_2 = \frac{\beta}{(\beta-\alpha)(\beta^2-2a\beta+a^2+b^2)}$.

Proof: See Section 3.3. □

3.2 Examples of Extremal Kähler Metrics on Line Bundles over $\mathbb{CP}^1$

The family of $U(n)$ invariant extremal Kähler metrics fomulated in [Cal82] can be used to write down non-compact, constant scalar curvature Kähler metrics as in LeBrun [LeB88], Pedersen-Poon [PP91], and Simanca [Sim91] (see also Abreu [Abr10]).

Example 3.4 (Positive cscK metrics on $\mathcal{O}_{\mathbb{CP}^1}(k)$, $k \geq 1$) Let us consider the positive cscK metric on $\mathbb{C}^2\backslash\{0\}$ given by the ODE

$$\frac{gg'}{c_3 g^3 + g^2 + c_1 g + c_0} = \frac{1}{s} \tag{34}$$

where $c_3 \neq 0$, $c_0 \neq 0$. The polynomial $H(x) = c_3 x^3 + x^2 + c_1 x + c_0$ has three real roots, α, α, β, with $0 < \alpha < \beta < \infty$. It follows from Lemma 4.4 that $\lim_{s\to 0^+} g(s) = \alpha$ and $\lim_{s\to+\infty} g(s) = \beta$, and $c_3 = -\frac{1}{2\alpha+\beta}$. The ODE can be written as

$$g'(s)(-2\alpha - \beta)\left\{\frac{\alpha}{\alpha-\beta}\frac{1}{(g-\alpha)^2} - \frac{\beta}{(\beta-\alpha)^2}\frac{1}{g-\alpha} + \frac{\beta}{(\beta-\alpha)^2}\frac{1}{g-\beta}\right\} = \frac{1}{s}.$$

Therefore, there exists a constant c such that

$$-\frac{\alpha(2\alpha+\beta)}{\beta-\alpha}\frac{1}{g-\alpha} + \frac{\beta(2\alpha+\beta)}{(\beta-\alpha)^2}\log(g-\alpha) - \frac{\beta(2\alpha+\beta)}{(\beta-\alpha)^2}\log(\beta-g) = \log s + c \tag{35}$$

ODE (34) is of the form

$$sg'(s) = F(g(s)) \tag{36}$$

where

$$F(g) = \frac{c_3 g^3 + g^2 + c_1 g + c_0}{g} = \frac{c_3(g-\alpha)^2(g-\beta)}{g}.$$

We can obtain the positive line bundle $\mathcal{O}_{\mathbb{CP}^1}(k)$, $k > 0$, by gluing a $\mathbb{CP}^1$ to $(\mathbb{C}^2\backslash\{0\})/\mathbb{Z}_k$ at $|z| = \infty$. $U(n)$ invariant Kähler metric on $\mathbb{C}^2\backslash\{0\}$ determined by ODE (36) induces a metric on $\mathcal{O}_{\mathbb{CP}^1}(k)\backslash S_\infty$. Here S_∞ stands for the zero section of $\mathcal{O}_{\mathbb{CP}^1}(k)$. The induced metric can be extended by continuity to a smooth metric on $\mathcal{O}_{\mathbb{CP}^1}(k)$ if and only if $F(\beta) = 0$ and $F'(\beta) = -k$.

The condition $F(\beta) = 0$ is clearly satisfied. We compute $F'(\beta) = -\frac{(\beta-\alpha)^2}{\beta^2(\beta+2\alpha)}$. We need to show that for every positive integer k there exist constants α, β, $0 < \alpha < \beta$ which satisfy

$$\frac{(\beta-\alpha)^2}{\beta^2(\beta+2\alpha)} = k.$$

For simplicity, let us introduce new variables $x = \alpha > 0$ and $y = \beta - \alpha > 0$. Then the above equation becomes

$$y^2 - k(x+y)^2(y+3x) = 0.$$

For each positive integer k, this equation has solutions (x, y) with $x > 0$, $y > 0$.

The function $F(g)$ is strictly positive on (α, β). It follows from Calabi [Cal82] that the Kähler metric extends smoothly to $\mathcal{O}_{\mathbb{CP}^1}(k)$.

We need to show that the induced metric is complete on the total space of $\mathcal{O}_{\mathbb{CP}^1}(k)$.

The metric is complete if the improper integral that gives the geodesic distance to $z = 0$

$$\int_0^{s_0} \sqrt{\frac{g'(s)}{s}} ds = \int_0^{s_0} \frac{\sqrt{F(g(s))}}{s} ds$$

is infinite.

Since $\lim_{s \to 0^+} g(s) = \alpha > 0$ and $\lim_{s \to \infty} g(s) = \beta$, $\frac{\beta - g}{(2\alpha+\beta)g}$ is bounded on $(0, s_0)$. There exists $d_1 > 0$ such that

$$\sqrt{F(g)} = \sqrt{\left(-\frac{1}{2\alpha + \beta}\right)(g - \alpha)^2 (g - \beta)\frac{1}{g}} > d_1 (g - \alpha).$$

Then, we have

$$\int_0^{s_0} \sqrt{\frac{g'(s)}{s}} ds > d_1 \int_0^{s_0} \frac{g - \alpha}{s} ds.$$

If we choose s_0 small enough, we have $\log(g - \alpha) < 0$ on $(0, s_0)$, and $\log(\beta - g)$ is bounded. Therefore, Equation (35) implies

$$-\frac{\alpha(2\alpha + \beta)}{\beta - \alpha} \frac{1}{g(s) - \alpha} > \log s + c$$

$$\frac{g(s) - \alpha}{s} > -\frac{\beta - \alpha}{\alpha(2\alpha + \beta)} \frac{1}{s(\log s + c)}.$$

Integrating both sides of this inequality on $(0, s_0)$ we see that the integral

$$\int_0^{s_0} \frac{g(s) - \alpha}{s} ds$$

is infinite.

Example 3.5 (Strictly extremal metrics on $\mathcal{O}_{\mathbb{CP}^1}(-k)$, $k \geq 1$) Let us consider Case 10 of Proposition 3.3. Since $c_4 \neq 0$, it follows from Equation (13) that this is a strictly extremal metric on $\mathbb{C}^2 \backslash \{0\}$. We can see from the proof of Proposition 3.3 that the ODE is given by $sg'(s) = F(g(s))$ where

$$F(g) = \frac{c_4(g-\alpha)(g-\beta)(g-\gamma)^2}{g}.$$

Here, we have $\alpha < \beta < \gamma$, $\alpha\beta\gamma \neq 0$, $\lim_{s\to 0^+} g(s) = \beta$, $\lim_{s\to\infty} g(s) = \gamma$ and $c_4 = \dfrac{1}{\alpha\beta + 2\alpha\gamma + 2\beta\gamma + \gamma^2} > 0$. We note that $c_4 > 0$ implies $-\frac{\gamma(\gamma+2\beta)}{2\gamma+\beta} < \alpha$.

As in the proof of Proposition 3.3, we will rewrite the ODE as

$$\begin{aligned} g'(\gamma^2 + 2\beta\gamma + 2\alpha\gamma + \alpha\beta) \Bigg\{ &\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)} \frac{1}{g(s)-\alpha} \\ + \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)} \frac{1}{(g(s)-\gamma)^2} &+ \frac{-\gamma^2+\alpha\beta}{(\gamma-\alpha)(\gamma-\beta)} \frac{1}{g(s)-\gamma} \\ &+ \frac{\beta}{(\beta-\alpha)(\beta-\gamma)^2} \frac{1}{g(s)-\beta} \Bigg\} = \frac{1}{s}. \end{aligned}$$

Now recall that we can obtain the line bundle $\mathcal{O}_{\mathbb{CP}^1}(-k)$, $k = 1, 2, \ldots$, by gluing a $\mathbb{CP}^1$ to $(\mathbb{C}^2 \backslash \{0\})/\mathbb{Z}_k$ at $z = 0$. The $U(2)$ invariant Kähler metric on $\mathbb{C}^2 \backslash \{0\}$ determined by $sg'(s) = F(g(s))$ induces a metric on $\mathcal{O}_{\mathbb{CP}^1}(-k) \backslash S_0$. The induced metric can be extended by continuity to a smooth metric on $\mathcal{O}_{\mathbb{CP}^1}(-k)$ if and only if $F(\beta) = 0$ and $F'(\beta) = k$. The condition $F(\beta) = 0$ is clearly satisfied. We compute

$$F'(\beta) = \frac{(\beta-\alpha)(\gamma-\beta)^2}{\beta(\alpha\beta + 2\alpha\gamma + 2\beta\gamma + \gamma^2)}.$$

We need to show that for every positive integer k, there exist constants α, β, γ with

$$-\frac{\gamma(\gamma+2\beta)}{2\gamma+\beta} < \alpha < \beta < \gamma, \quad \alpha\beta\gamma \neq 0, \tag{37}$$

that satisfy

$$\frac{(\beta-\alpha)(\gamma-\beta)^2}{\beta(\alpha\beta + 2\alpha\gamma + 2\beta\gamma + \gamma^2)} = k. \tag{38}$$

For simplicity, let $\gamma = 2\beta$. Then, Equations (37) and (38) give

$$-\frac{8\beta}{5} < \alpha < \beta, \quad \alpha\beta \neq 0, \quad \frac{\beta-\alpha}{5\alpha+8\beta} = k \tag{39}$$

For each positive integer k, the pair $(\alpha, \beta) = \left(\frac{1-8k}{1+5k}\beta, \beta\right)$ satisfies (39).

We need to show that the induced metric on $\mathcal{O}_{\mathbb{CP}^1}(-k)$ is complete as $|z| \to \infty$, i.e. as $g(s) \to \gamma$. The metric is complete if the improper integral

$$\int_{s_0}^{\infty} \sqrt{\frac{g'(s)}{s}}\, ds = \int_{s_0}^{\infty} \frac{\sqrt{F(g(s))}}{s}\, ds$$

is infinite. Since $\lim_{s\to 0^+} g(s) = \beta$ and $\lim_{s\to+\infty} g(s) = \gamma > 0$, $\frac{c_4(g-\alpha)(g-\beta)}{g}$ is bounded on (s_0, ∞). There exists $d_1 > 0$ such that $\sqrt{F(g)} > d_1(\gamma - g)$. Then we have

$$\int_{s_0}^{\infty} \sqrt{\frac{g'(s)}{s}}\, ds > d_1 \int_{s_0}^{\infty} \frac{\gamma - g(s)}{s}\, ds.$$

If we choose s_0 large enough, we have $\log(\gamma - g) < 0$ on (s_0, ∞), and $\log(g-\alpha)$, $\log(g-\beta)$ are bounded. Noting that $-\gamma^2 + \alpha\beta < 0$, Equation (31) implies

$$\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\frac{1}{\gamma - g(s)} < \frac{1}{\gamma^2 + 2\beta\gamma + 2\alpha\gamma + \alpha\beta}\log s + c_1$$
$$\frac{\gamma - g(s)}{s} > \frac{c_2 \log s + c_3}{s}, \quad c_2 > 0.$$

Integrating both sides of this inequality on (s_0, ∞), we see that the integral

$$\int_{s_0}^{\infty} \frac{\gamma - g(s)}{s}\, ds$$

is infinite.

3.3 Proofs

Proof: [Proof of Proposition 3.3] It follows from equation (14) that there exists constants c_0, c_1, c_3, c_4 such that

$$\frac{gg'}{c_4 g^4 + c_3 g^3 + g^2 + c_1 g + c_0} = \frac{1}{s}. \tag{40}$$

Case (1) $c_4 = 0$.

We see from equation (13) that ω is a cscK metric. Classification of cscK metrics on $\mathbb{C}^2 \backslash 0$ is given by Theorem 1.2 in [HL18].

Case (2) $c_4 \neq 0$. $c_0 = c_1 = 0$ and $\lim_{s \to 0^+} g(s) = 0$.

It follows from Remark that, in this case, the metric can be smoothly extended to the origin, hence Theorem 2.3 applies.

Case (3) $c_4 \neq 0$, $c_0 = c_1 = 0$ and $\lim_{s \to 0^+} g(s) = A > 0$.

We have $H(x) = c_4 x^3 + c_3 x^2 + x$ and it follows from Lemma 4.3 that $H(A) = 0$, $B < \infty$, and $H(x) > 0$ in (A, B). In this case, the roots are given by $\gamma = 0 < \alpha = A < \beta = B$. But $H(x) > 0$ on (α, β), and this implies that $c_4 = \frac{1}{\alpha\beta} < 0$, which is a contradiction.

Case (4) $c_4 \neq 0$, $c_0 = 0$, $c_1 \neq 0$, and the polynomial $c_4 x^3 + c_3 x^2 + x + c_1$ has roots α, α, α.

It follows from Lemma 4.3 that $B < \infty$ for degree reasons, and this case is impossible.

Case (5) $c_4 \neq 0$, $c_0 = 0$, $c_1 \neq 0$, and the polynomial $c_4 x^3 + c_3 x^2 + x + c_1$ has roots α, α, β with $\alpha < \beta$.

It follows from Lemma 4.3 that $B < \infty$, $\alpha = A > 0$, $\beta = B$, and $H(x) > 0$ on (α, β), which implies that $c_4 < 0$.

Since $H(x) = c_4 (x - \alpha)^2 (x - \beta)$, we have $1 = c_4(\alpha^2 + 2\alpha\beta)$, which contradicts to $0 < \alpha < \beta$.

Case (6) $c_4 \neq 0$, $c_0 = 0$, $c_1 \neq 0$, and the polynomial $c_4 x^3 + c_3 x^2 + x + c_1$ has roots α, β, β with $\alpha < \beta$.

We have $\alpha \neq 0$, $\beta \neq 0$. The equation (40) can be written as

$$\frac{g'(s)}{c_4(\alpha - \beta)^2} \left\{ \frac{1}{g(s) - \alpha} - \frac{1}{g(s) - \beta} + \frac{\beta - \alpha}{(g(s) - \beta)^2} \right\} = \frac{1}{s}.$$

It follows from Lemma 4.3 that $\alpha = A$ $\beta = B$, and $H(x) > 0$ on (α, β). Hence $c_4 = \frac{1}{\beta(\beta+2\alpha)} > 0$, and there exists a constant c such that

$$\frac{\beta(\beta+2\alpha)}{(\alpha-\beta)^2}\left\{\log(g(s)-\alpha) - \log(\beta - g(s)) - \frac{\beta-\alpha}{g(s)-\beta}\right\} = \log s + c. \quad (26)$$

On the other hand, Lemma 4.4 implies that there exists a unique smooth, strictly increasing function $g(s) = su'(s)$ ranging from α to β on $(0, \infty)$.

Case (7) $c_4 \neq 0$, $c_0 = 0$, $c_1 \neq 0$, and the polynomial $c_4x^3 + c_3x^2 + x + c_1$ has real distict roots α, β, γ.

By Lemma 4.3 we have $B < \infty$ and $H(x) > 0$ on (A, B). It follows that all roots are real, and if we let $\alpha = A$, $\beta = B$, $\gamma < \alpha < \beta$, then we have $c_4 < 0$. This gives us the inequality $\gamma < -\frac{\alpha\beta}{\alpha+\beta} < 0 < \alpha < \beta$.

On the other hand, if we let $\alpha = A$, $\beta = B$, $\alpha < \beta < \gamma$, then we have $c_4 > 0$.

We can write equation (40) as

$$g'(s)(\alpha\beta+\alpha\gamma+\beta\gamma)\left\{\frac{1}{(\alpha-\beta)(\alpha-\gamma)}\frac{1}{g(s)-\alpha} + \frac{1}{(\beta-\alpha)(\beta-\gamma)}\frac{1}{g(s)-\beta}\right. \quad (41)$$
$$\left. + \frac{1}{(\gamma-\alpha)(\gamma-\beta)}\frac{1}{g(s)-\gamma}\right\} = \frac{1}{s}.$$

There exists a constant c such that

$$(\alpha\beta+\alpha\gamma+\beta\gamma)\left\{\frac{\log(g(s)-\alpha)}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\log(\beta-g(s))}{(\beta-\alpha)(\beta-\gamma)} + \frac{\log|g(s)-\gamma|}{(\gamma-\alpha)(\gamma-\beta)}\right\} = \log s + c. \quad (27)$$

If we denote the left hand side of (27) by $h(g(s))$, then we see that $\lim_{s\to 0^+} h(g(s)) = -\infty$, $\lim_{s\to+\infty} h(g(s)) > 0$, and $\frac{d}{ds}h(g(s)) > 0$ on $(0, \infty)$. It follows from Lemma 4.4 that there exists a unique smooth strictly increasing function $g(s)$ that solves the equation.

Case (8) $c_4 \neq 0$, $c_0 \neq 0$, and the polynomial $c_4x^4 + c_3x^3 + x^2 + c_1x + c_0$ has at most one real root.

By Lemma 4.3 $B < \infty$, and the equation (14) does not admit the required solution.

Case (9) $c_4 \neq 0$, $c_0 \neq 0$, and the polynomial $c_4x^4 + c_3x^3 + x^2 + c_1x + c_0$ has real roots $\alpha, \alpha, \alpha, \beta$ with $\alpha < \beta$.
It follows from Lemma 4.3 that $B < \infty$, and $H(x) > 0$ on (A, B). This implies $0 < \alpha = A < \beta = B$, and $c_4 = \frac{1}{3\alpha^2+3\alpha\beta} < 0$, which gives a contradiction.

Case (10) $c_4 \neq 0$, $c_0 \neq 0$, and the polynomial $c_4x^4 + c_3x^3 + x^2 + c_1x + c_0$ has real roots $\alpha, \beta, \beta, \beta$ with $\alpha < \beta$.
It follows from Lemma 4.3 that $0 < \alpha = A < \beta = B$, and $c_4 < 0$, which gives a contradiction.

Case (11) $c_4 \neq 0$, $c_0 \neq 0$, and the polynomial $c_4x^4 + c_3x^3 + x^2 + c_1x + c_0$ has no real roots, and has complex roots $a - ib, a - ib, a + ib, a + ib$.
It follows from Lemma 4.3 that $H(A) = 0$, which is a contradiction.

Case (12) $c_4 \neq 0$, $c_0 \neq 0$, and the polynomial $c_4x^4 + c_3x^3 + x^2 + c_1x + c_0$ has real roots $\alpha, \alpha, \beta, \beta$.
We have $c_4x^4 + c_3x^3 + x^2 + c_1x + c_0 = c_4(x-\alpha)^2(x-\beta)^2$ with $\alpha\beta \neq 0$. The equation (40) can be written as

$$g'(s)((\alpha+\beta)^2 + 2\alpha\beta)\left\{\frac{\alpha+\beta}{(\beta-\alpha)^3}\frac{1}{g(s)-\alpha} + \frac{\alpha}{(\alpha-\beta)^2}\frac{1}{(g(s)-\alpha)^2}\right.$$
$$\left. - \frac{\alpha+\beta}{(\beta-\alpha)^3}\frac{1}{g(s)-\beta} + \frac{\beta}{(\alpha-\beta)^2}\frac{1}{(g(s)-\beta)^2}\right\} = \frac{1}{s}.$$

We see from Lemma 4.3 that $\alpha = A > 0$, $\beta = B$, and $c_4 > 0$, where $c_4 = \frac{1}{(\alpha+\beta)^2+2\alpha\beta}$. We can integrate the above equation to obtain

$$((\alpha+\beta)^2 + 2\alpha\beta)\left\{\frac{\alpha+\beta}{(\beta-\alpha)^3}\log(g(s)-\alpha) - \frac{\alpha}{(\alpha-\beta)^2}\frac{1}{g(s)-\alpha} - \right. \tag{28}$$
$$\left. \frac{\alpha+\beta}{(\beta-\alpha)^3}\log(\beta - g(s)) - \frac{\beta}{(\alpha-\beta)^2}\frac{1}{g(s)-\beta}\right\} = \log s + c.$$

On the other hand, Lemma 4.4 implies that there exists a unique smooth strictly increasing function $g(s)$ ranging from α to β on $(0, \infty)$.

Case (13) $c_4 \neq 0$, $c_0 \neq 0$, and the polynomial $c_4x^4 + c_3x^3 + x^2 + c_1x + c_0$ has three distinct real roots $\alpha, \alpha, \beta, \gamma$ with $\alpha < \beta < \gamma$.

It follows from Lemma 4.3 that we can either have $\alpha = A$, $\beta = B$; or $\beta = A$, $\gamma = B$.

Let us start with the case $\alpha = A$, $\beta = B$. In this case we have $c_4x^4 + c_3x^3 + x^2 + c_1x + c_0 = c_4(x-\alpha)^2(x-\beta)(x-\gamma)$. Then, $c_4 = \frac{1}{\alpha^2+2\alpha\beta+2\alpha\gamma+\beta\gamma}$ and we can see from Lemma 4.3 that $\alpha > 0$, $c_4 > 0$.

The equation (40) can be rewritten as

$$g'(s)(\alpha^2+2\alpha\beta+2\alpha\gamma+\beta\gamma)\left\{\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\frac{1}{g(s)-\gamma}+\right. \\ \frac{\alpha}{(\alpha-\gamma)(\alpha-\beta)}\frac{1}{(g(s)-\alpha)^2}+\frac{-\alpha^2+\beta\gamma}{(\alpha-\gamma)(\alpha-\beta)}\frac{1}{g(s)-\alpha}+ \\ \left.\frac{\beta}{(\beta-\gamma)(\beta-\alpha)^2}\frac{1}{g(s)-\beta}\right\} = \frac{1}{s}. \tag{42}$$

There exists a constant c such that

$$(\alpha^2+2\alpha\beta+2\alpha\gamma+\beta\gamma)\left\{\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\log(\gamma-g(s))-\right. \\ \frac{\alpha}{(\alpha-\gamma)(\alpha-\beta)}\frac{1}{g(s)-\alpha}+\frac{-\alpha^2+\beta\gamma}{(\alpha-\gamma)(\alpha-\beta)}\log(g(s)-\alpha)+ \\ \left.\frac{\beta}{(\beta-\gamma)(\beta-\alpha)^2}\log|\beta-g(s)|\right\} = \log s + c. \tag{29}$$

Note that $-\alpha^2+\beta\gamma > 0$. By Lemma 4.4, there exists a unique smooth function $g(s) : (0,\infty) \to (\alpha,\beta)$ with $g'(s) > 0$ which solves the above equation.

On the other hand, if we assume $\beta = A$ and $\gamma = B$, then it follows from Lemma 4.3 that $c_4 = \frac{1}{\alpha^2+2\alpha\beta+2\alpha\gamma+\beta\gamma} < 0$, and $0 < \beta < \gamma$.

Equivalently, we can write $\alpha < -\frac{\alpha^2+\beta\gamma}{2(\beta+\gamma)} < 0 < \beta < \gamma$. Note that for any given $0 < \beta < \gamma$, such α values exist.

Equation (40) can be rewritten as equation (42) as before, however, this time we are looking for a smooth soution $g(s)$ with values in (β,γ). Keeping this in mind, we investigate (42) to obtain (29), and use Lemma 4.4 to conclude that there exists a unique smooth strictly increasing function $g : (0,\infty) \to (\beta,\gamma)$ satisfying (29).

Case (14) $c_4 \neq 0$, $c_0 \neq 0$, and the polynomial $c_4x^4 + c_3x^3 + x^2 + c_1x + c_0$ has three distinct real roots $\alpha, \beta, \beta, \gamma$ with $\alpha < \beta < \gamma$.

The equation (14) can be rewritten as

$$g'(s)(\beta^2 + 2\beta\alpha + 2\beta\gamma + \alpha\gamma)\left\{\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}\frac{1}{g(s)-\alpha}\right.$$
$$+\frac{\beta}{(\beta-\alpha)(\beta-\gamma)}\frac{1}{(g(s)-\beta)^2} + \frac{-\beta^2+\alpha\gamma}{(\beta-\alpha)(\beta-\gamma)}\frac{1}{g(s)-\beta}$$
$$\left. + \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)^2}\frac{1}{g(s)-\gamma}\right\} = \frac{1}{s}.$$

There exists a constant c such that

$$(\beta^2 + 2\beta\alpha + 2\beta\gamma + \alpha\gamma)\left\{\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}\log(g(s)-\alpha)\right. \tag{30}$$
$$-\frac{\beta}{(\beta-\alpha)(\beta-\gamma)}\frac{1}{g(s)-\beta} + \frac{-\beta^2+\alpha\gamma}{(\beta-\alpha)(\beta-\gamma)}\log|g(s)-\beta|$$
$$\left. + \frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)^2}\log(\gamma-g(s))\right\} = \log s + c.$$

It follows from Lemma 4.3 that we have $B < \infty$ for degree reasons, so we can choose either $\alpha = A$, $\beta = B$; or $\beta = A$, $\gamma = B$. By Lemma 4.3 we have $H(x) > 0$ on (A, B), which implies $c_4 < 0$ in both cases. However, since $c_4 = \frac{1}{\beta^2+2\beta\alpha+2\beta\gamma+\alpha\gamma}$ and $A > 0$, we see that the former case is impossible, leaving us with the choice $\beta = A$, $\gamma = B$. It follows from Lemma 4.4 that there exists a unique smooth strictly increasing function $g : (0, \infty) \to (\beta, \gamma)$ satisfying (30).

Case (15) $c_4 \neq 0$, $c_0 \neq 0$, and the polynomial $c_4x^4 + c_3x^3 + x^2 + c_1x + c_0$ has three distinct real roots $\alpha, \beta, \gamma, \gamma$ with $\alpha < \beta < \gamma$.

It follows from Lemma 4.3 that we can have either $\alpha = A$, $\beta = B$; or $\beta = A$, $\gamma = B$. In the former case, Lemma 4.3 implies $c_4 = \frac{1}{\alpha\beta+2\alpha\gamma+2\beta\gamma+\gamma^2} < 0$, which contradicts with our choice $0 < \alpha = A < \beta = B < \gamma$.

Let us assume $\beta = A$, $\gamma = B$. Since $H(x) > 0$ on (β, γ), we have $c_4 > 0$, which implies that $-\frac{\gamma(\gamma+2\beta)}{2\gamma+\beta} < \alpha$. We have $\alpha \neq 0$ as $c_0 \neq 0$. The equation

(40) can be written as

$$g'(\gamma^2+2\beta\gamma+2\alpha\gamma+\alpha\beta)\left\{\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}\frac{1}{g(s)-\alpha}\right.$$
$$+\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\frac{1}{(g(s)-\gamma)^2}+\frac{-\gamma^2+\alpha\beta}{(\gamma-\alpha)(\gamma-\beta)}\frac{1}{g(s)-\gamma}$$
$$\left.+\frac{\beta}{(\beta-\alpha)(\beta-\gamma)^2}\frac{1}{g(s)-\beta}\right\}=\frac{1}{s}.$$

There exists a constant c such that

$$(\gamma^2+2\beta\gamma+2\alpha\gamma+\alpha\beta)\left\{\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}\log(g(s)-\alpha)\right. \tag{31}$$
$$-\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)}\frac{1}{g(s)-\gamma}+\frac{-\gamma^2+\alpha\beta}{(\gamma-\alpha)(\gamma-\beta)}\log(\gamma-g(s))$$
$$\left.+\frac{\beta}{(\beta-\alpha)(\beta-\gamma)^2}\log(g(s)-\beta)\right\}=\log s+c.$$

It follows from Lemma 4.4 that there exists a unique smooth strictly increasing function $g(s):(0,\infty)\to(\beta,\gamma)$ that solves equation (31).

Case (16) $c_4\neq 0$, $c_0\neq 0$, and the polynomial $c_4x^4+c_3x^3+x^2+c_1x+c_0$ has four distinct real roots α,β,γ,τ with $\alpha<\beta<\gamma<\tau$, and $\alpha\beta\gamma\tau\neq 0$.
Equation (40) can be rewritten as

$$g'(\alpha\beta+\alpha\gamma+\alpha\tau+\beta\gamma+\beta\tau+\gamma\tau)\left\{\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\tau)}\frac{1}{g(s)-\alpha}\right.$$
$$+\frac{\beta}{(\beta-\alpha)(\beta-\gamma)(\beta-\tau)}\frac{1}{g(s)-\beta}+\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\tau)}\frac{1}{g(s)-\gamma}$$
$$\left.+\frac{\tau}{(\tau-\alpha)(\tau-\beta)(\tau-\gamma)}\frac{1}{g(s)-\tau}\right\}=\frac{1}{s}.$$

There exists a constant c such that

$$(\alpha\beta+\alpha\gamma+\alpha\tau+\beta\gamma+\beta\tau+\gamma\tau)\left\{\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\tau)}\log(g(s)-\alpha)\right.$$
$$+\frac{\beta}{(\beta-\alpha)(\beta-\gamma)(\beta-\tau)}\log|g(s)-\beta|+\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\tau)}\log|g(s)-\gamma| \tag{32}$$
$$\left.+\frac{\tau}{(\tau-\alpha)(\tau-\beta)(\tau-\gamma)}\log|g(s)-\tau|\right\}=\log s+c.$$

It follows from Lemma 4.3 that we have $A>0$, $B<\infty$, and $H(x)>0$ on (A,B). This implies that we have three possibilities:

(i) $\alpha=A$, $\beta=B$ and $c_4<0$.
In this case, all roots of the polynomial $H(x)$ are positive which contradicts with $c_4<0$.

(ii) $\beta=A$, $\gamma=B$, and $c_4>0$
By Lemma 4.4, there exists a unique smooth, strictly increasing function $g(s):(0,\infty)\to(\beta,\gamma)$ satisfying equation (32), whenever $\alpha>-\frac{\beta\gamma+\beta\tau+\gamma\tau}{\beta+\gamma+\tau}$.

(iii) $\gamma=A$, $\beta=\tau$, and $c_4<0$.
By Lemma 4.4, there exists a unique smooth, strictly increasing function $g(s):(0,\infty)\to(\gamma,\tau)$ satisfying equation (32), whenever $\alpha\beta+\alpha\gamma+\alpha\tau+\beta\gamma+\beta\tau+\gamma\tau<0$.

Case (17) $c_4\neq 0$, $c_0\neq 0$, and the polynomial $c_4x^4+c_3x^3+x^2+c_1x+c_0$ has four distinct roots $\alpha,\beta,a+ib,a-ib$.

It follows from Lemma 4.3 that $\alpha=A$, $\beta=B$, and $c_4<0$. If we write $H(x)=c_4(x-\alpha)(x-\beta)(x^2+2ax+a^2+b^2)$, then $c_4<0$ can be written as $a^2+2a(\alpha+\beta)+b^2+\alpha\beta<0$. This condition holds for those α,β,a,b which satisfy $b^2<\alpha^2+\alpha\beta+\beta^2$ and $-(\alpha+\beta)-\sqrt{\alpha^2+\alpha\beta+\beta^2-b^2}<a<-(\alpha+\beta)+\sqrt{\alpha^2+\alpha\beta+\beta^2-b^2}$.

The equation (40) can be rewritten as

$$(a^2+2a(\alpha+\beta)+b^2+\alpha\beta)g'\left\{c_1\frac{c_1}{g(s)-\alpha}+\frac{c_2}{g(s)-\beta}+\right.$$
$$\left.\frac{-(c_1+c_2)g(s)+2(c_1+c_2)a-c_1\alpha-c_2\beta}{g^2(s)-2ag(s)+a^2+b^2}\right\}=\frac{1}{s},$$

where $c_1 = \frac{\alpha}{(\alpha-\beta)(\alpha^2-2a\alpha+a^2+b^2)}$ and $c_2 = \frac{\beta}{(\beta-\alpha)(\beta^2-2a\beta+a^2+b^2)}$.
On the other hand, Lemma 4.4 implies that there exists a unique smooth, strictly increasing function $g(s) : (0, \infty) \to (\alpha, \beta)$ determined by

$$(a^2 + 2a(\alpha + \beta) + b^2 + \alpha\beta) \{c_1 \log(g(s) - \alpha) + c_2 \log(\beta - g(s)) + \int_\alpha^\beta \frac{-(c_1 + c_2)g(s) + 2(c_1 + c_2)a - c_1\alpha - c_2\beta}{g^2(s) - 2ag(s) + a^2 + b^2} g'(s)ds\right\} = \log s. \tag{33}$$

Here we note that the integral in equation (33) is a proper integral, since the denominator is never zero.

□

4 Technical Lemmas

Proposition 4.1 (Monn [Mon86], Proposition 2.1) *Let B be an open ball containing the origin in $\mathbb{C}^n$. Let u be a radial function on $\overline{B}$, and let $\tilde{u}(r) = u(r, 0, \ldots, 0)$. Then $u \in C^k(\overline{B})$ if and only if $\tilde{u} \in C^k[0, 1]$, and $\tilde{u}^{(\ell)}(0) = 0$ for all $\ell \leq k$, ℓ odd.*

Proposition 4.2 (Monn [Mon86], Proposition 4.1) *The k^{th} derivative of two real-valued functions, $f \circ g$, can be written as a sum of terms of the form*

$$f^{(\lambda)}(g) \cdot P(g', g'', \ldots, g^{k+1-\lambda})$$

where P is a monomial of degree $\lambda \leq k$ and of weighted degree k.

The following lemma is useful for eliminating impossible cases as solutions of the extremal equation $sg' = F(g)$.

Lemma 4.3 ([HL18], Lemma 6.2) *Suppose $H(x)$ is a polynomial of degree m and the ordinary differential equation*

$$\frac{g^k(s)g'(s)}{H(g(s))} = \frac{1}{s}$$

admits a smooth solution $g(s)$ on $(0, \infty)$ with $g(s) > 0$, $g'(s) > 0$. Denote by $A = \lim_{s\to 0^+} g(s)$, $B = \lim_{s\to+\infty} g(s)$. Then

1. *$H(A) = 0$ and $H(x) > 0$ for $x \in (A, B)$.*

2. *If $B = +\infty$, then $\deg H \leq k + 1$. Moreover, A is the largest and nonnegative real root of $H(x)$, and $H(x)$ is positive on $(A, +\infty)$.*

3. *If $B < +\infty$, then $H(B) = 0$. Moreover, A and B are two successive nonnegative real roots of the polynomial $H(x)$, and $H(x)$ is positive on the interval (A, B)*

Once the impossible cases are eliminated by the above lemma, we use the following lemma to show the existence of solutions.

Lemma 4.4 ([HL18], Lemma 6.1) *Let $h : (A, B) \to \mathbb{R}$ be a smooth, strictly increasing function with $\lim_{t \to A} h(t) = -\infty$, $\lim_{t \to B} h(t) = \infty$. Then, for any constant $a > 0$ and c, there exists a unique smooth, strictly increasing function $g : (0, \infty) \to \mathbb{R}$ such that*

$$h(g(s)) = a \log s + c$$

and $\lim_{s \to 0^+} g(s) = A$, $\lim_{s \to +\infty} g(s) = B$.

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