The Charged Penrose Inequality for Manifolds with Cylindrical Ends and Related Inequalities

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Abstract of the Dissertation

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We prove the charged Penrose inequality for time symmetric initial data sets with finitely many asymptotically cylindrical ends. We do so by employing a doubling argument modeled after Weinstein and Yamada [50] and then applying the ordinary charged Penrose inequality due to Khuri, Weinstein, and Yamada [37]. We establish the appropriate rigidity result by using weak inverse mean curvature flow (IMCF) [28] which shows that equality holds if and only if our initial data is Reissner-Nordström. The techniques employed allow for a quick proof of the positive mass theorem with charge for manifolds with asymptotically cylindrical ends, giving a completely different proof of the same result found in [1]. Motivated by Bekenstein bounds [8–10] we then prove three theorems relating mass, size, charge, and angular momentum for bodies which we have published in [30]. The proofs of these three theorems are based on the generalized Jang equation proposed by Khuri and Bray [13, 14].

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1 Introduction

The study of inequalities involving physical quantities has become one of the most active areas of mathematical general relativity. These inequalities are proposed based on physical intuition, and then it is up to the mathematicians to verify them. Verification of these inequalities provides further proof of the validity of general relativity, while the fact that some inequality does not hold can provide direction for new research. The two most famous such inequalities are the positive mass theorem and the Penrose inequality. Less famous, though equally important, are Bekenstein bounds. The goal of this dissertation is to study these inequalities.

We begin with a discussion of the Penrose inequality. Conjectured by Penrose [42] in the early 1970's using a heuristic argument based on the establishment viewpoint of gravitational collapse and the assumption of cosmic censorship, it relates the mass m to the surface area A of a black hole. Defining the area radius ρ by $A = 4\pi\rho^2$, the Penrose inequality takes the form

$$m \geq \frac{1}{2}\rho. \tag{1}$$

The Riemannian Penrose inequlaity is a special case, where the mass m is the ADM mass of an asymptotically flat 3-manifold having non-negative scalar curvature and A is the area of the outermost minimal surface (with possibly multiple components). It was proven in the late 1990's by Huisken and Ilmanen using weak inverse mean curvature flow (IMCF) with the area A being that of the largest connected component of the outermost minimal surface [28], and in full generality by Bray using a novel conformal flow of metrics [12].

In the case that the outermost minimal surface has a single boundary component, inequality (1) can be extended to include charge yielding

$$m \geq \frac{1}{2}\left(\rho + \frac{q^2}{\rho}\right) \tag{2}$$

where q is the total charge enclosed by the minimal surface. One might then conjecture that (2) holds when the outermost minimal surface has multiple components. However, a time symmetric, asymptotically flat counter example was constructed in [50], by gluing two copies of the Majumdar-Papapetrou initial data sets. It is important to point out that this does not provide a counter example to cosmic censorship. As pointed out by Jang [29], (2) is equivalent to two inequalities,

$$m - \sqrt{m^2 - q^2} \leq \rho \leq m + \sqrt{m^2 - q^2} \tag{3}$$

and only the upper bound follows from Penrose's heuristic arguments. The counter example violates the lower bound. In [37] it was proven that the upper bound always holds and that the full charged Penrose inequality holds under the additional assumption $\rho \geq |q|$.

In the time symmetric case, the usual geometry associated with the Penrose inequality is that of a manifold with boundary, where the boundary is a compact outermost minimal surface. The typical examples of such geometry are the canonical slices of the Schwarzchild and ordinary Reissner-Nordström spacetimes. However, there is a second fundamental type of geometry which arises naturally and is exemplified by the canonical slice of the extreme

Reissner-Nordström spacetime. This slice is a complete manifold without boundary which possesses a region where the metric approaches a product metric on $\mathbb{R} \times S^2$. In the limit, the S^2 cross-sections have a well-defined area and vanishing mean curvature.

Hence the idea is that such "asymptotically cylindrical ends" should be able to replace minimal boundary components in the ordinary and charged Penrose inequalities. This is indeed the case and is our first theorem.

Theorem 1 *Let (M, g, E, B) be a time symmetric initial data set with a single strongly asymptotically flat end of ADM mass m, with generalized boundary consisting of finitely many asymptotically cylindrical ends and a minimal surface boundary, with generalized area radius ρ. Assume the data satisfies the charged dominant energy condition, the Maxwell constraints without charged matter, and has electric and magnetic charges q_e and q_b (with total squared charge q^2). Finally, suppose that the generalized boundary is outerminimizing, which means that any surface S enclosing the generalized boundary satisfies $|S| > 4\pi\rho^2$. Then the upper bound in (3) holds. Moreover, if $\rho \geq |q|$ then (2) holds. Equality holds in both cases if and only if the initial data is given by the canonical slice of the (possibly extreme) Reissner-Nordström spacetime with $E = (q_e/r^2)\nu_r$ and $B = (q_b/r^2)\nu_r$ where ν_r is the outward unit normal to spheres of radius r in standard coordinates.*

For the definitions of relevant terms, see Section 2. Sections 4 through 10 deal with the proof.

The positive mass theorem states that

$$m \geq 0 \tag{4}$$

where m is the ADM mass of a (particular) end in an asymptotically flat manifold. This can be extended to the positive mass theorem with charge

$$m \geq |q|. \tag{5}$$

These were proven in [44], [45], and [22] in the case of asymptotically flat manifolds satisfying the charged dominant energy condition. It was observed that the methods used to prove Theorem 1 could be used to prove a version of the positive mass theorem with charge in the time symmetric case.

Theorem 2 *Let (M, g, E, B) be a time symmetric initial data set for the Einstein-Maxwell equations with a single asymptotically flat end and with a generalized boundary consisting of finitely many asymptotically cylindrical ends and a minimal surface boundary. Then*

$$m \geq |q|. \tag{6}$$

This result was established in [1] using spinorial techniques. Our proof uses completely different methods and has a better chance of generalizing to the non-time symmetric setting. Since the proof relies on the techniques developed for the proof of Theorem 1, it is much shorter and is confined to Section 11.

The remainder of this work deals with the so-called Bekenstein bounds. These results have already been published by the author and Marcus Khuri in [30]. The only thing to keep in mind is that in the entirety of this work we use the convention that the speed of light c and the gravitational constant G are set to $c = G = 1$. In contrast, these constant were kept in the above mentioned papaer.

In [8] Bekenstein utilized heuristic arguments involving black holes to derive an upper bound for the entropy of macroscopic bodies in terms of the total energy and radius of the smallest sphere that encloses the object. This inequality was later generalized [9, 26, 27, 51] to include contributions from the angular momentum $\mathcal{J}$ and charge q of the body

$$\sqrt{(m\mathcal{R})^2 - \mathcal{J}^2} - \frac{q^2}{2} \geq \frac{\hbar}{2\pi k_b}\mathcal{S}, \tag{7}$$

where k_b denotes Boltzmann's constant , $\mathcal{S}$ is entropy, m is total mass/energy, $\mathcal{R}$ is the radius described above, and $\hbar$ is the reduced Planck's constant. Although the original inequality [8] without angular momentum and charge has received much attention [10, 11, 49], the enhanced relation (7) has not been properly investigated. An important initial step in that direction was taken by Dain [17] who studied the inequality

$$m^2 \geq \frac{q^4}{4\mathcal{R}^2} + \frac{\mathcal{J}^2}{\mathcal{R}^2}, \tag{8}$$

which is implied by (7) since entropy is always nonnegative. He was able to establish (8) within the context of electromagnetism, and also in general relativity for bodies with zero angular momentum contained in asymptotically flat, maximal initial data which are void of black holes. In this result m is given by the ADM mass. The idea is that a proof of (8) lends indirect evidence for the full Bekenstein bound (7). Later on Dain's result was extended to include a contribution from angular momentum [3], again in the setting of asymptotically flat, maximal initial data. The inequality produced in [3] is not quite in the form of (8), and it is not clear if one implies the other. Both results [3, 17] are based on monotonicity of the Hawking mass along IMCF, which is valid in the maximal case assuming the dominant energy condition holds.

We establish a Bekenstein-like inequality closely related to (8) without the hypothesis of maximality for the initial data, and thereby generalize the works [3, 17]. Our approach relies on a coupling of the IMCF with an embellished version of the Jang equation [13, 14], which is inspired by the proof of the positive mass theorem [44]. This yields the following:

Theorem 3 *Let (M, g, k, E, B) be a complete, axisymmetric, asymptotically flat initial data set for the Einstein-Maxwell equations, satisfying the charged dominant energy condition $\mu_{EM} \geq |J_{EM}|$ and without apparent horizons. Suppose that $\Omega \subset M$ is a body outside of which there is no charge density or momentum density in the direction of axisymmetry. If the Jang/IMCF system of equations admits a solution then*

$$m \geq \frac{q^2}{2\overline{\mathcal{R}}} + \frac{\mathcal{J}^2}{2\overline{\mathcal{R}}\,\overline{\mathcal{R}}_c^2}. \tag{9}$$

For all relevant definitions and the proof see Section 12.

In addition, the techniques used to prove the above theorem naturally lend themselves to establish a version of the Penrose inequality [39] with angular momentum and charge for general axisymmetric initial data without the maximal assumption. A similar result in the maximal case was recently given in [2]. Recall that Penrose [42] proposed a sharp inequality bounding the total energy of a black hole spacetime from below in terms of the horizon area. It serves a necessary condition for the cosmic censorship conjecture. Thus, a counterexample would disprove cosmic censorship while verification of the Penrose inequality only lends credence to the conjecture's validity. In [12, 28] the Penrose inequality has been proven in the maximal case. Moreover, generalizations including angular momentum and charge have been proposed [39] motivated by Penrose's original heuristic arguments. The full Penrose inequality may then be stated as follows

$$m^2 \geq \left(\sqrt{\frac{A}{16\pi}} + \sqrt{\frac{\pi}{A}}q^2\right)^2 + \frac{4\pi\mathcal{J}^2}{A} = \frac{1}{4}\left(\rho + \frac{q^2}{\rho}\right)^2 + \frac{\mathcal{J}^2}{\rho^2}, \tag{10}$$

where A is the minimum area required to enclose the outermost apparent horizon in an axisymmetric initial data set satisfying the relevant energy condition. This comes with a rigidity statement asserting that equality holds if and only if the initial data arise from an embedding into the Kerr spacetime. We also note that the Bekenstein bound (7), when applied to black holes, implies the Penrose inequality (10). To see this, simply recall that for a black hole with event horizon area A_e the radius and entropy are given by

$$\mathcal{R} = \sqrt{\frac{A_e}{4\pi}}, \qquad \mathcal{S} = \frac{k_b A_e}{4l_p^2}, \tag{11}$$

where $l_p = \sqrt{\hbar}$ is the Planck length. Inequality (10) has been established in the maximal case without the angular momentum term in a series of papers [18, 35–37]. However, there has been very little to no progress on including angular momentum. The only result known to the author in this direction is [2]. Here we will establish a version of (10) valid in the general nonmaximal setting, assuming the existence of solutions to a canonical coupling of the Jang equation to IMCF; such solutions are known to exist in spherical symmetry.

Theorem 4 *Let (M, g, k, E, B) be an axisymmetric, asymptotically flat initial data set for the Einstein-Maxwell equations, satisfying the charged dominant energy condition $\mu_{EM} \geq |J_{EM}|$ and with outermost apparent horizon boundary having one component. Suppose further that there is no charge density or momentum density in the direction of axisymmetry. If the Jang/IMCF system of equations admits a proper solution then*

$$m^2 \geq \left(\sqrt{\frac{|\partial M|}{16\pi}} + \sqrt{\frac{\pi}{|\partial M|}}q^2\right)^2 + \frac{\mathcal{J}^2}{4\overline{\mathcal{R}}_c^2}. \tag{12}$$

For all relevant definitions and the proof see Section 13.

Lastly, the methods used to study the Penrose inequality above lead to new inequalities for bodies involving size, angular momentum, mass, and charge. In this vein, our final result is

Theorem 5 *Consider two concentric bodies $\Omega_1 \subset \Omega_2$, each having the topology of a 3-dimensional ball, inside an axisymmetric asymptotically flat initial data set (M, g, k, E, B).*

Assume the outer region is untrapped, the annular region $\Omega_2 \setminus \Omega_1$ has no charge and momentum density in the Killing direction, and the initial data are void of apparent horizons. If the Jang/IMCF system of equations admits a proper solution then

$$\frac{1}{2}\overline{\mathcal{R}}_2 \geq \frac{4\pi}{3}\overline{\mathcal{R}}_1^3 \min_{\tilde{\Omega}_1}(\mu_{EM} - |J_{EM}|) + \frac{q^2}{2\overline{\mathcal{R}}_1}\left(1 - \sqrt{\frac{\overline{\mathcal{R}}_1}{\overline{\mathcal{R}}_2}}\right) + \frac{1}{2}\frac{\mathcal{J}^2}{\overline{\mathcal{R}}_1\overline{\mathcal{R}}_{ac}^2}. \tag{13}$$

We can then turn this inequality around in order to obtain a black hole existence result. Recall that Thorne's hoop conjecture [48] roughly states that if enough matter/energy is condensed in an appropriately small region, then gravitational collapse will ensue. Mathematically this assertion may be translated into a heuristic inequality

$$\mathrm{Mass}(\Omega) > \mathcal{C} \cdot \mathrm{Size}(\Omega), \tag{14}$$

which if satisfied for a body Ω, then implies that Ω must be contained within an apparent horizon; here $\mathcal{C}$ is a universal constant. One of the primary difficulties in establishing such a result is finding a proper notion of quasi-local mass to use in the left-hand side of (14). We see that if we have an initial data set which does not satisfy (13), then the data must contain an apparent horizon.

This shows that mass is not the only quantity which can appear on the left hand side of (14). Angular momentum and charge also naturally arise on the left-hand side, and thus provide extra means to satisfy (14). This will be rigorously proven in spherical symmetry, and motivation will be given to indicate why the result should hold in generality. Related results concerning black hole existence due to concentration of angular momentum or charge have been given in [32, 33, 38], using different methods. See also [5, 6].

2 Definitions and Standard Formulas

2.1 Initial Data Sets

An initial data set for the Einstein-Maxwell equations consists of a quintuple (M, g, k, E, B) where M is a smooth 3-manifold, g is a Riemannian metric, k is a symmetric 2-tensor (the extrinsic curvature) and E and B are vector fields representing the electromagnetic field. This initial data set satisfies the constraint equations

$$\begin{aligned} 16\pi\mu &= R + (Tr_g k)^2 - |k|^2 \\ 8\pi J &= \operatorname{div}(k - (Tr_g k)g) \end{aligned} \tag{1}$$

where μ and J are the energy and momentum densities of the matter fields. It is often useful to subtract off the contributions to these quantities arising from the electromagnetic field, and refer to these quantities as μ_{EM} and J_{EM} which yields

$$\begin{aligned} 16\pi\mu_{EM} &= 16\pi\mu - 2(|E|^2 + |B|^2) \\ 8\pi J_{EM} &= 8\pi J + 2(E \times B). \end{aligned} \tag{2}$$

In the time-symmetric case ($k = 0$) we will write our initial data sets as (M, g, E, B). This represents a metric which is not changing with "time". In this case, the constraint equations simplify to

$$16\pi\mu = R \tag{3}$$

and

$$16\pi\mu_{EM} = R - 2(|E|^2 + |B|^2). \tag{4}$$

2.2 Energy Conditions and the Maxwell Constraint

Based on observations and physical principles, initial data sets are assumed to satisfy what are called energy conditions. Two common such conditions are the following:

Definition 1 *An initial data set is said to satisfy the dominant energy condition (DEC) if*

$$\mu \geq |J|. \tag{5}$$

If the data is time symmetric, this is equivalent to

$$R \geq 0. \tag{6}$$

Definition 2 *An initial data set is said to satisfy the charged dominant energy condition (CDEC) if*

$$\mu_{EM} \geq |J_{EM}| \tag{7}$$

If the data is time symmetric, then it satisfies the CDEC if

$$R \geq 2(|E|^2 + |B|^2). \tag{8}$$

Notice that the CDEC in the time symmetric case is a weaker condition then the full CDEC. The latter only implies

$$R \geq 2(|E|^2 + |B|^2 + |E \times B|). \tag{9}$$

The physical interpretations of the DEC and CDEC are most apparent in the time-symmetric case. They say, respectively, that the matter and non-electromagnetic matter densities are non-negative, in compliance with everyday experience.

Finally we have:

Definition 3 *The time-symmetric initial data set* (M, g, E, B) *is said to satisfy the Maxwell constraint without charged matter if* $\operatorname{div}_g E = \operatorname{div}_g B = 0$ *everywhere on* M*.*

Physically this means that the initial data set does not contain any charged matter in the interior.

2.3 SAF Ends, the ADM Mass, and Charges at Infinity

A manifold is said to have a *strongly asymptotically flat (SAF) end* if there is an open region diffeomorphic to the complement of a closed ball in $\mathbb{R}^3$, and in the coordinates given by this diffeomorphism the following fall off conditions hold:

$$|\partial^n(g_{ij} - \delta_{ij})| = O(|x|^{-n-1}), \quad |\partial^n k_{ij}| = O(|x|^{-n-2}), \quad |\partial^n E^i| = O(|x|^{-n-2}),$$
$$|\partial^n B^i| = O(|x|^{-n-2}), \quad n = 0, 1, 2 \quad \text{as} \quad |x| \to \infty.$$

We denote the diffeomorphism by $\Psi : \Omega \to \tilde{\Omega} = \{x : |x| > r_0\}$ for some r_0. We also require that the scalar curvature R satisfies $R \in L^1(\Omega)$. Given such an end, we can compute its ADM mass which is given by

$$m = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} (g_{ij,i} - g_{ii,j}) \nu^j dS \tag{10}$$

where S_r are coordinate spheres of Euclidean radius r in Ω and ν^j is the outward unit normal [4]. It is well known that with the above fall-off conditions this quantity is a geometric invariant of the given end [7]. Also, the above fall off conditions mean that the electric and magnetic charges measured at infinity given by

$$q_e = \lim_{r \to \infty} \frac{1}{4\pi} \int_{S_r} E_j \nu^j dS, \quad q_b = \lim_{r \to \infty} \frac{1}{4\pi} \int_{S_r} B_j \nu^j dS \tag{11}$$

are well defined. We define the squared total charge by

$$q^2 = q_e^2 + q_b^2. \tag{12}$$

Given an asymptotically flat end we can add a point at ∞ and conformally compactify. Then we can define:

Definition 4 *Suppose we have two smooth compact surfaces* S_1 *and* S_2*. We say that* S_2 *encloses* S_1 *(with respect to the chosen end) if any smooth curve which passes through* ∞ *and intersects* S_1 *also intersects* S_2*.*

2.4 Schwarzchild and Reissner-Nordström Slices

The canonical (exterior) slice of the Schwarzchild spacetime is the manifold with boundary $[2m, \infty) \times S^2$ with the metric

$$\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\sigma^2 \tag{13}$$

where $d\sigma^2$ denotes the standard metric on the unit sphere. As is well known, the Schwarzchild slice is SAF with ADM mass m and the boundary surface $r = 2m$ is an outermost minimal surface with area $16\pi m^2$. Here, outermost means that it is not enclosed by any other compact minimal surface.

Similarly for $m > |q|$, the canonical (exterior) slice of the Reissner-Nordström spacetime is the manifold $[m + \sqrt{m^2 - q^2}, \infty) \times S^2$ with the metric

$$\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\sigma^2 \tag{14}$$

where m is again the ADM mass and q^2 is the squared total charge as defined above. As can be checked, the boundary surface $r = m + \sqrt{m^2 - q^2}$ is an outermost minimal surface.

However, when $m = |q|$ something interesting occurs. If we choose any point with r-coordinate larger than m, then the distance to the "surface" $r = m$ is actually infinite. Hence, the resulting manifold is a complete manifold without boundary, which does not contain any compact minimal surfaces. Using a change of coordinates, the metric can be put into the form

$$\frac{(e^{-t/m} + m)^2}{m^2} dt^2 + (e^{-t/m} + m)^2 d\sigma^2. \tag{15}$$

with $t \in \mathbb{R}$. Letting $t \to \infty$ (which corresponds to $r \to m$ in the original coordinates) the metric approaches

$$dt^2 + m^2 d\sigma^2. \tag{16}$$

This is called extreme Reissner-Nordström and serves as the motivation for asymptotically cylindrical ends in the next subsection.

In addition, we will denote the outward unit normal to spheres of constant r in these coordinates by ν_r.

2.5 Asymptotically Cylindrical Ends

Motivated by extreme Reissner-Nordström, we say a region Λ in our initial data set is an *asymptotically cylindrical (AC) end* if it is diffeomorphic to $[0, \infty) \times N$ where N is a compact, closed, orientable 2-manifold and if there exists a product metric $\tilde{g} = dt^2 + h$, with h a Riemannian metric on N, such that in the coordinates given by the diffeomorphism

$$|\partial^n (g_{ij} - \tilde{g}_{ij})| = O\left(1/t\right), \quad n = 0, 1, 2 \quad \text{as} \quad t \to \infty.$$

We denote the coordinates on Λ by (t, ω). In addition, we require that if there are vector fields E and B defined on our cylindrical end, then there exist vector fields $\tilde{E}$ and $\tilde{B}$, with components independent of t, satisfying

$$|\partial^n(E^i - \tilde{E}^i)| = O(1/t), \quad |\partial^n(B^i - \tilde{B}^i)| = O(1/t), \text{ for } n = 0, 1 \text{ as } t \to \infty.$$

Apriori, the manifold N can be arbitrary. However, for the duration of this work, we will let $N = S^2$, the two-sphere. The justification for this can be found in Section 3.

We refer to the surfaces $N_T = \{(t, \omega) : t = T\}$ as *cross-sections.* With the above asymptotics, the areas of the cross-sections approach a limiting value, $A = \lim_{T\to\infty} |N_T|$. For brevity, we will refer to this limit value as the area of the end. We also define the area radius of the particular cylindrical end by the formula $A = 4\pi\rho^2$. In addition, the mean curvatures of the cross-sections tend to 0 as $t \to \infty$. As t increases, we say that we are moving down the cylindrical end.

Obviously, there are many different choices of cylindrical coordinates on a given end. However, if we consider the areas of the cross-sections in each such coordinate system, we see that they converge to the same value.

We can also define the intuitive notion of what it means for a surface to enclose a cylindrical end. As before, we can conformally compactify a given SAF end by adding a point at ∞.

Definition 5 *We say that a smooth compact surface S encloses a given AC end (with respect to a given SAF end) if any smooth curve which passes through ∞ and has unbounded t-coordinate in the AC coordinates intersects S.*

2.6 Apparent Horizons and Minimal Surfaces

A black hole is an object which can only be detected by knowing the full long-term evolution of the initial data set, which is generally a difficult problem. So, we focus on questions which can be answered in terms of initial data alone. Given an orientable surface $S \subset M$ in our initial data set, we can compute what are called the null expansions

$$\theta_\pm = H \pm Tr_S k$$

where H is the mean curvature of S and $Tr_S k$ is the trace of k restricted to S with respect to the metric induced on S. The null expansions quantify what happens to a shell of light emitted outwards and inwards from S. If $\theta_\pm < 0$ then both shells of light are getting smaller indicating the presence of a strong gravitational field. A surface satisfying $\theta_\pm = 0$ is called an *apparent horizon* and conforms with the expected behavior at the surface of a black hole. As the above formula shows, in the time symmetric case apparent horizons coincide with minimal surfaces. See [39] for an introduction.

2.7 The Generalized Boundary and the Outer-minimizing Condition

Suppose our initial data has an apparent horizon boundary (or a minimal surface in the time-symmetric case) ∂M with finitely many connected components and area $|\partial M|$. Also suppose our initial data has finitely many cylindrical ends. Each such end has some area A_i.

Definition 6 *The union of ∂M and the cylindrical ends is referred to as the generalized boundary. We define the area of the generalized boundary to be $A = |\partial M| + \sum_{i=1}^{n} A_i$ where A_i is the area of each end. We also define the area radius of the generalized boundary by the equation $A = 4\pi\rho^2$.*

Definition 7 *A surface $S \subset M$ is said to enclose the generalized boundary if it encloses ∂M (in the sense of Section 2.2) and each of the AC ends (in the sense of Section 2.4).*

Definition 8 *The generalized boundary is said to be outerminimizing if for any S which encloses the generalized boundary $|S| > A$.*

We explain the motivation behind Definition 8. As is well known [21], the Penrose inequality can be violated if the minimal surface taken to be the boundary of the initial data is not outermost. Briefly, there can be a large minimal surface hidden behind a small one leading into an asymptotically flat end with small mass. The area of this large minimal surface then violates the Penrose inequality.

Similarly, we could hide an AC end with large area behind a small minimal surface and violate the Penrose inequality. Hence, we need some condition which is analogous to the outermost minimal boundary condition in the ordinary Penrose and charged Penrose inequalities. Noting that an outermost minimal surface is strictly outerminimizing (meaning any surface enclosing it has larger area), we see that requiring our initial data to satisfy Definition 8 is quite natural.

2.8 Weighted Hölder Spaces

First we will define weighted Hölder spaces on a manifold with finitely many SAF ends. Denote each of the SAF ends by Ω_l and make a choice of asymptotically Euclidean coordinates on each. This means each Ω_l is difeomorphic to $\tilde{\Omega}_l = \{x : |x| > r_l\}$. Consider the set $K = M \setminus \cup_l \Omega_l$. Define a smooth function $\sigma \geq 1$ by $\sigma \equiv 1$ on K, and such that sufficiently far in each Ω_l we have $\sigma = r$, the Euclidean radial coordinate in each of the chosen asymptotically Euclidean coordinates. This σ is called the weight function.

We would like to use the definition given in [50]. There, $C^{k,\alpha}_{-\beta}$ is the set of functions ϕ on M whose k-th order derivatives are Hölder continuous and for which the norm $\|\phi\|_{C^{k,\alpha}_{-\beta}}$ defined below is finite:

$$\|\phi\|_{C^k_{-\beta}} = \sum_{i=0}^{k} \left\|\sigma^{\beta+i} D^i \phi\right\|_{C^0} \tag{17}$$

$$\left[D^k\phi\right]_{\alpha,-\beta} = \sup_{\substack{x\in M \\ 0<\mathrm{dist}(x,y)<\rho}} \sigma(x,y)^{\beta+\alpha+k} \frac{|P^x_y D^k\phi(y) - D^k\phi(x)|}{\mathrm{dist}(x,y)^\alpha} \tag{18}$$

$$\|\phi\|_{C^{k,\alpha}_{-\beta}} = \|\phi\|_{C^k_{-\beta}} + \left[D^k\phi\right]_{\alpha,-\beta}. \tag{19}$$

Here $D^i\phi$ is the tensor of i-th order derivatives of ϕ, ρ is the injectivity radius of M, $\sigma(x,y) = \max\{\sigma(x), \sigma(y)\}$, and P^x_y is parallel translation along the shortest geodesic from y to x. Notice that if M has cylindrical ends, then $\sigma = 1$ (sufficiently far down) in these ends.

This definition is very nice because it is invariantly defined, not depending on any choice of coordinates. However, it is not quite right in the case of a manifold with boundary since for points getting closer to the boundary the injectivity radius tends to 0. Instead we use the usual definition of Hölder spaces which uses a finite open cover and a partition of unity.

Let Λ_i be the i-th cylindrical end. The compact cross-section N_i can be covered by finitely many open sets $\{B_i^j\}$ diffeomorphic to Euclidean coordinate balls $\{\tilde{B}_i^j\}$. Therefore, using AC coordinates, we have that $\{V_i^j\}$ where $V_i^j = B_i^j \times (0, \infty)$ is an open cover of the region $\{t_i > 0\}$. We also have a diffeomorphism $F_{ij} : V_i^j \to \tilde{V}_i^j = \tilde{B}_i^j \times (0, \infty)$. Let $\tilde{\Lambda}_i = \{t_i > 1\}$ in the AC coordinates. Then $K_1 = K \setminus \cup_i \tilde{\Lambda}_i$ is a compact set which can be covered by finitely many open sets $\{U_m\}$ diffeomorphic to Euclidean balls and half balls $\tilde{U}_m$ Denote the diffeomorphisms by G_m. Finally, let $\Psi_l : \Omega_l \to \tilde{\Omega}_l$ be the SAF diffeomorphisms.

The collection of sets $\{\Omega_l\} \cup \{V_i^j\} \cup \{U_m\}$ is a finite open cover of M which has a partition of unity subordinate to this open cover. We write the elements of this partition of unity as ψ_l, θ_{ij}, and χ_m depending on which sets they have their support.

Now $(\chi_m \phi) \circ G^{-1}$ is a function on $\tilde{U}_m$ and thus we can compute its ordinary $C^{k,\alpha}$ norm [23]. We can do the same for $(\theta_{ij}\phi) \circ F_{ij}^{-1}$.

Given a function f defined on $\tilde{\Omega} = \{x : |x| > r_0\} \subset \mathbb{R}^3$ and a weight function (which we continue to denote by σ) we can compute

$$\|f\|_{C^{k,\alpha}_{-\beta}(\tilde{\Omega})} = \left(\sum_{i=0}^{k} \sup_{\gamma = k} \sup_{x \in \tilde{\Omega}} \sigma^{i+\beta}(x) |D^\gamma f(x)| \right) \tag{20}$$

$$+ \sup_{\gamma = k} \sup_{\substack{x,y \in \tilde{\Omega} \\ 0 < |x-y| < 1}} \sigma(x,y)^{\beta+\alpha+k} \frac{|D^\gamma f(x) - D^\gamma f(y)|}{|x-y|^\alpha} \tag{21}$$

Therefore, we finally define

$$\|\phi\|_{C^{k,\alpha}_{-\beta}(M)} = \sum_l \|(\psi_l \phi) \circ \Psi_l^{-1}\|_{C^{k,\alpha}_{-\beta}(\tilde{\Omega}_l)} + \sum_{i,j} \|(\theta_{ij}\phi) \circ F_{ij}^{-1}\|_{C^{k,\alpha}(\tilde{V}_i^j)}$$
$$+ \sum_m \|(\chi_m \phi) \circ G^{-1}\|_{C^{k,\alpha}(\tilde{U}_m)} \tag{22}$$

Even though the norm depends on the choice of open cover, the class of functions having finite Hölder norm is invariant. Given a Riemannian manifold with boundary we fix the choice of open cover and partition of unity once and for all. Furthermore, when in later sections we will double our manifolds, we will implicitly use this fixed choice of open cover on the original manifold to choose the natural open cover on the doubled manifold.

Remark 1 *In the literature when dealing with Riemannian manifolds with boundary, most authors simply say the Hölder norms are defined in the "usual" way, without giving a full definition seemingly to avoid the straightforward but lengthy exposition we have given.*

We will also need a different class of functions with appropriate weights in the cylindrical ends. Given a cylindrical end and a fixed coordinate system, recall we define $\Lambda = \{(t, \omega) :$

$t \geq 0\}$. If we have finitely many such ends we can define Λ_i for each such end, with a corresponding t_i. We define the set $K_2 = M \setminus \cup_i \Lambda_i$. Now let $\mu \geq 1$ be a smooth function such that $\mu \equiv 1$ in K_2 and such that sufficiently far down each AC end $\mu = e^{\delta t_i}$. Now let $C^{k,\alpha}_{-\beta,\delta}$ be the set of functions ϕ on M whose k-th order derivatives are Hölder continuous and for which the norm $\|\phi\|_{C^{k,\alpha}_{-\beta,\delta}}$ defined below is finite:

$$\|\phi\|_{C^{k,\alpha}_{-\beta,\delta}} = \|\mu\phi\|_{C^{k,\alpha}_{-\beta}}.$$

We also define certain important closed subsets of these Banach spaces. Define

$$\tilde{C}^{k,\alpha}_{-\beta} = \left\{\phi \in C^{k,\alpha}_{-\beta} : \partial_\tau \phi|_{\partial M} = 0\right\}$$

and similarly

$$\tilde{C}^{k,\alpha}_{-\beta,\delta} = \left\{\phi \in C^{k,\alpha}_{-\beta,\delta} : \partial_\tau \phi|_{\partial M} = 0\right\}$$

where ∂_τ denotes the normal derivative. Being closed subsets of Banach spaces, these are Banach spaces themselves.

2.9 Some Standard Formulas

We recall some basic formulas. Given a smooth function $f : M \to \mathbb{R}$ defined on a Riemannian manifold we define the Laplace-Beltrami operator on f in local coordinates with respect to the metric g by

$$\Delta_g f = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f\right) \tag{23}$$

where $|g| = \det(g_{ij})$.

Given a smooth vector field V we define its divergence by the local coordinate expression

$$\operatorname{div}_g V = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} V^i\right). \tag{24}$$

Thus $\Delta_g f = \operatorname{div}_g \nabla_g f$ where $\nabla_g f$ is the gradient of f.

Given (M, g) we can define a new conformal metric by $\bar{g} = u^4 g$ where $u : M \to \mathbb{R}, u > 0$. Then the scalar curvature $\bar{R}$ with respect to this conformal metric is given by

$$\bar{R} = u^{-4}(R - u^{-1}\Delta_g u). \tag{25}$$

Also, given a surface $S \subset M$ with mean curvature H, its mean curvature in the conformal metric, denoted $\bar{H}$, is given by

$$\frac{1}{2} H u + \partial_\tau u = \frac{1}{2} \bar{H} u^3 \tag{26}$$

where ∂_τ is the normal derivative computed with respect to the unit normal os S.

2.10 A Remark on Notation

In the sequel, we will find ourselves computing quantities and objects with respect to several different metrics. To denote this dependence we will use a subscript as for example $\Delta_g f$, $\nabla_g f$, $|X|_g$, etc. In some places we might also use different expressions for the metric, as for example $g(\cdot,\cdot) = \langle \cdot,\cdot \rangle_g$ and we might drop the g-subscript when no confusion is possible.

Constants appearing in the various theorems might be different, but we usually use the same letter C for all of them. This can be justified by taking the maximum of all the constants that have appeared so far and denoting this maximum by C. Sometimes we might use different symbols C_1, C_2, etc. when we want to stress that the constants are different, or for convenience.

3 The Topology of Asymptotically Cylindrical Ends

In the definition of asymptotically cylindrical ends there is no apriori restriction on the topology of the cross-sections. However, it is well known [19, 20] that an outermost minimal surface in an asymptotically flat initial data set consists of finitely many S^2 components (this result depends chiefly on the theorems found in [40]). Therefore, it is expected that if an asymptotically cylindrical end can be observed from the asymptotically flat end, the cross-sections should be spheres. We have not been able to establish this fully. However, in the case of an asymptotically cylindrical end obeying the CDEC we have:

Theorem 6 *An asymptotically cylindrical end in a time symmetric initial data set satisfying the CDEC has cross-sections which must be either two-spheres S^2 or two-tori $\mathbb{T}^2$. If the end encloses non-zero charge the cross sections must be S^2. Furthermore, if the cross-sections are $\mathbb{T}^2$, they must be a flat torus in the limit.*

Proof: Notice that as we go down the cylindrical end, the scalar curvature R approaches $2K$, where K is the Gaussian curvature of the cross section. Furthermore, we have $R \geq 2(|E|^2 + |B|^2)$. Letting S_T be a cross section and q^2 the enclosed charged, we have

$$\begin{aligned} q^2 = \left(\frac{1}{4\pi}\int_{N_T} E_i\nu^i\right)^2 + \left(\frac{1}{4\pi}\int_{N_T} B_i\nu^i\right)^2 &\leq \left(\frac{1}{4\pi}\int_{N_T} |E_i\nu^i|\right)^2 + \left(\frac{1}{4\pi}\int_{N_T} |B_i\nu^i|\right)^2 \\ &\leq \left(\frac{1}{4\pi}\int_{N_T} |E|\right)^2 + \left(\frac{1}{4\pi}\int_{N_T} |B|\right)^2 \leq \frac{|N_T|}{32\pi^2}\int_{N_T} 2\left(|E|^2 + |B|^2\right) \\ &\leq \frac{|N_T|}{32\pi^2}\int_{N_T} R. \end{aligned}$$

Rearranging and taking the limit $T \to \infty$ we obtain by the Gauss-Bonnet theorem

$$4\pi q^2 \leq \lim_{T\to\infty} \frac{|N_T|}{8\pi}\int_{N_T} R = \frac{A}{8\pi}\int_N 2K = A(1-\mathcal{G}) \tag{27}$$

where A is the area of the end and $\mathcal{G}$ is the genus of the cross section. Hence, $\mathcal{G} \leq 1$ meaning the cross sections must be either S^2 or $\mathbb{T}^2$, and if $q^2 \neq 0$ we must have $\mathcal{G} = 0$. If $\mathcal{G} = 1$ then $K \equiv 0$ and hence the torus is flat. □

We expect it to be possible to eliminate the special case of the flat torus at some point in the future, as was done similarly in [19, 20]. Hence for the purposes of this work we assume that all cylindrical ends have S^2 cross-sections.

4 Sketch of the Proof of Theorem 1

The main idea of the proof is to use Theorem 1.1 and Corollary 1.2 in [37] due to Khuri, Weinstein, and Yamada. For brevity we will refer to Theorem 1.1 and Corollary 1.2 as the KWY theorem. However, our initial data set currently does not satisfy all the hypotheses of this theorem. Specifically, it does not have a minimal surface boundary with area A.

We remedy this by doubling across the AC ends. Specifically, we go sufficiently far down the ends and take what we call a collar neighborhood in each end. We then chop off the ends beyond these collars, and obtain transitions to the corresponding cylindrical metrics in each collar, giving us a manifold which we denote by M^+. So, for each cylindrical end we have a new boundary component labeled by S_i^+. Now, we take a second copy of M^+ denoted M^-. We identify the boundary components S_i^+ and S_i^+ giving us a smooth Riemannian manifold, since the metric near each of these components is the appropriate cylindrical metric. We denote this smooth manifold by $\hat{M}$, and its metric by $\hat{g}$. Outside of the collars, the metric is the same as our original metric, and so $\hat{M}$ is a manifold with two SAF ends. The image under the gluing of the $S_i^{\pm}$ is now the fixed point set of an isometry with respect to $\hat{g}$ metric, and is therefore a minimal surface.

We can't directly push forward our electromagnetic fields to $\hat{M}$ since the resulting vector field won't be continuous. However, it is also obvious we don't need them to be defined on the entire manifold. We push them forward only to $\hat{M}^+$ (the image of M^+ under the gluing), so that they are defined exterior to the outermost minimal surface with respect to that end. Unfortunately, these vector fields are no longer divergence free in the transition region, and so no longer satisfy the assumptions of the KWY theorem. We then perturb the electromagnetic fields to restore the divergence constraint. This perturbation changes the charge, so we then check that the further down the AC ends we perform the chopping, the closer this new charge is to the old one.

In addition, our initial data no longer satisfies the CDEC. We restore it by a conformal change of the $\hat{g}$ metric where the conformal factor u satisfies an appropriate elliptic equation. We then check that the resulting ADM mass can be made arbitrarily close to the original mass by chopping further and further down the AC ends. Now we will have an outermost minimal surface exterior to the fixed point set of our isometry, and we will be able to apply the KWY theorem.

Finally, we establish the rigidity result by employing weak IMCF. By the arguments of [37] if equality holds the generalized boundary must consist of either an S^2 outermost minimal surface (in which case the data is Reissner-Nordström) or a single AC end. In the latter case, we prove that there exists a smooth IMCF which provides a diffeomorphism between our original data and the canonical slice of the extreme Reissner-Nordström spacetime.

Before all of this however, we show that we can conformally deform our data to satisfy the strict CDEC, with mass, charge, and generalized boundary area arbitrarily close to their original values. This is done in order to prove certain technical propositions. We then prove the charged Penrose inequality for this deformed data and then let the mass, charge, and area approach their original values, establishing the charged Penrose inequality for the original data.

5 The Initial Conformal Deformation

For technical reasons, we want our initial data to satisfy the strict CDEC where

$$R > 2\left(|E|^2 + |B|^2\right). \tag{28}$$

In order to do so, we will conformally deform our initial data set to one with mass, charge, and generalized boundary area ε-close to our original data.

Suppose we have our data set (M, g, E, B) and consider the conformal metric $\bar{g} = u^4 g$. The formula (25) suggests we solve the elliptic problem

$$\begin{aligned} \Delta_g u - \frac{1}{8} R u + \frac{1}{8}\left(R + \epsilon\varrho\right) u^{-3} = 0 \quad \text{on} \quad M \\ \partial_\tau u = 0 \quad \text{on} \quad \partial M, \quad u \to 1 \quad \text{as} \quad r \to \infty. \end{aligned} \tag{29}$$

By $r \to \infty$ we mean as the points move into both the SAF and AC ends. The function ϱ is some smooth non-negative function which vanishes sufficiently fast as we move into the SAF and AC ends, and $\epsilon > 0$ is some small real number. It is easy to check that if we use u as the conformal factor the resulting scalar curvature satisfies $\bar{R} = \left(R + \epsilon\varrho\right) u^{-8}$.

Now, define new vector fields by $\bar{E} = u^{-6} E$ and $\bar{B} = u^{-6} B$. If E and B are divergence free, then so are $\bar{E}$ and $\bar{B}$ as shown by the following computation:

$$\operatorname{div}_{\bar{g}}(\bar{E}) = \frac{1}{\sqrt{|\bar{g}|}} \partial_i(\sqrt{|\bar{g}|}\bar{E}^i) = \frac{u^{-6}}{\sqrt{|g|}} \partial_i(\sqrt{|g|}E^i) = u^{-6} \operatorname{div}_g(E) = 0 \tag{30}$$

and similarly for $\bar{B}$. Furthermore

$$\bar{R} = (R + \epsilon\varrho)u^{-8} \geq 2u^{-8}(|E|_g^2 + |B|_g^2) + u^{-8}\epsilon\varrho > 2(|\bar{E}|_{\bar{g}}^2 + |\bar{B}|_{\bar{g}}^2) \tag{31}$$

and so this deformed data satisfies the strict CDEC. What remains to be verified is that we can find such a conformal factor u which is close to 1 in an appropriate sense, and so that the resulting quantities $\bar{m}$, $\bar{q}$, and $\bar{A}$ are close to the originals.

Since we are looking for a small deformation of 1 we will write $u = 1 + \phi$ for some small function. Then the above elliptic problem becomes

$$\begin{aligned} \Delta_g \phi - \frac{1}{8} R(1+\phi) + \frac{R}{8(1+\phi)^3} + \frac{\epsilon\varrho}{8(1+\phi)^3} = 0 \quad \text{on} \quad M \\ \partial_\tau \phi = 0 \quad \text{on} \quad \partial M, \quad \phi \to 0 \quad \text{as} \quad r \to \infty \end{aligned} \tag{32}$$

and after a bit of rearrangement we can write (32) as

$$\Delta_g \phi - \frac{R}{8}\left(\frac{4\phi + 6\phi^2 + 4\phi^3 + \phi^4}{(1+\phi)^3}\right) + \frac{\epsilon\varrho}{8(1+\phi)^3} = 0. \tag{33}$$

We will find a solution in the appropriate Hölder space by using the implicit function theorem for Banach spaces.

Proposition 7 *The operator*

$$L = \Delta_g - \frac{1}{2}R \ : \ \tilde{C}^{2,\alpha}_{-2/3,\delta} \to C^{0,\alpha}_{-8/3,\delta} \tag{34}$$

is an isomorphism for all $\delta \in \mathbb{R}_+ \setminus D$ where D is some discrete set.

Proof: As we go down each cylindrical end, L approaches the translation invariant operator $L_0 = \partial_t^2 + \Delta_N + \frac{1}{2}R_0$ where Δ_N is the Laplacian on the cross section with respect to the metric h and R_0 is a function independent of t. Using theorem 2.3.12 in [41] we obtain that L is a Fredholm operator for all δ except those in a certain discrete set. We require $\delta > 0$ since we want our functions to decay quickly as we go down the cylindrical ends.

We show that L is injective. Define $M_T = M \setminus \{(t_i, \omega) : t_i > T\}$ (which is M with the AC ends having the t_i-coordinate greater than T in each of the cylindrical ends chopped off). Suppose $\Delta_g\phi - \frac{1}{2}R\phi = 0$ and consider

$$\begin{aligned} 0 = \int_M \phi\left(\Delta_g\phi - \frac{1}{2}R\phi\right)dV &= \lim_{T\to\infty}\int_{M_T}\phi\left(\Delta_g\phi - \frac{1}{2}R\phi\right)dV \\ = \lim_{T\to\infty} - \int_{M_T}\left(|\nabla\phi|_g^2 + \frac{1}{2}R\phi^2\right)dV &+ \int_{\partial M}\phi(\partial_\tau\phi)dS + \int_{\partial M_T\setminus\partial M}\phi(\partial_\tau\phi)dS \\ &= -\int_M\left(|\nabla\phi|_g^2 + \frac{1}{2}R\phi^2\right)dV \end{aligned} \tag{35}$$

where we integrated by parts. We also used the fact that $\partial_\tau\phi = 0$ on ∂M and $|\phi| \to 0$ in the cylindrical ends. Since $R \geq 0$ we conclude $\phi \equiv 0$ and thus L is injective.

To prove surjectivity we define an L_2 inner product on $C^{0,\alpha}_{-8/3,\delta}$ using the formula

$$\langle F, G\rangle = \int_M FG\, dV \tag{36}$$

and the associated norm $\|F\|_{L_2} = \langle F, F\rangle^{1/2}$. Notice this is well defined by the fall-off conditions in the SAF and AC ends. Since L is Fredholm, $\mathrm{Im}(L)$ is closed and L is formally self-adjoint. By Proposition 2.3.16 in [41] (or showing it directly) we have $\mathrm{Im}(L) = \ker(L)^\perp = \{0\}^\perp = C^{0,\alpha}_{-8/3,\delta}$ which proves surjectivity. $\square$

Next consider defining an operator

$$\mathcal{F}(\epsilon, \phi) = \Delta_g\phi - \frac{1}{8}R(1+\phi) + \frac{R}{8(1+\phi)^3} + \frac{\epsilon\varrho}{8(1+\phi)^3} \tag{37}$$

on $\mathbb{R} \times \tilde{C}^{2,\alpha}_{-2/3,\delta}$. The problem is that if $\phi = -1$ in some region then the output of this operator does not lie in any (nice) Banach space. However, if we restrict ourselves to some neighborhood of 0 in $\tilde{C}^{2,\alpha}_{-2/3,\delta}$ then the output will lie in the appropriate Banach space. Specifically, if we let $V = \{\phi \in \tilde{C}^{2,\alpha}_{-2/3,\delta} : \|\phi\|_{\tilde{C}^{2,\alpha}_{-2/3,\delta}} < 1/2\}$ then examining the operator in the form (33) we have $\mathcal{F} : \mathbb{R} \times V \to C^{0,\alpha}_{-8/3,\delta}$. We can now apply the implicit function theorem.

Proposition 8 *The elliptic problem 32 has a solution ε-close to 0 in $\tilde{C}^{2,\alpha}_{-2/3,\delta}$ for all $\delta \in \mathbb{R}_+ \setminus D$, where D is a discrete set.*

Proof: Consider $\mathbb{R} \times V$ and the operator $\mathcal{F} : \mathbb{R} \times V \to C^{0,\alpha}_{-8/3,\delta}$. Notice that $\mathcal{F}(0,0) = 0$ and the linearization of the operator is $d\mathcal{F}(0,0) = \Delta_g - \frac{1}{2}R = L$ is an isomorphism by Proposition 7. The claim now follows by the implicit function theorem for Banach spaces and by choosing sufficiently small ϵ, we can ensure $\|\phi\|_{C^{2,\alpha}_{-2/3}} < \varepsilon$. $\square$

To prove the regularity of our solution we quote the following result.

Proposition 9 *Assume $u \in H^1$ solves*

$$Lu + B(x, u, \nabla u) = f \tag{38}$$

where L is a second order, linear elliptic differential operator and $B(x, u, p)$ is a smooth function of its arguments satisfying

$$|B(x, u, p)| \leq C|p|^2. \tag{39}$$

Given $0 \leq \alpha < 1,\ k + \alpha > 0,\ s > -1,$ if $u \in C^{k,\alpha}$ and $f \in C^s_*$ then $u \in C^{s+2}_*$.

This is Proposition 12B.1 in chapter 14 of [47]. For the definition of C^s_* (the so-called Zygmund space) see Section 8 of Chapter 13 of the same reference. There it is also pointed out that if $k \geq 0$ and $\alpha > 0$ then $C^{k,\alpha} = C^{k+\alpha}_*$.

Proposition 10 *The solution of Proposition 8 is smooth for all sufficiently small ϵ.*

Proof: We rewrite (32) in the form (38) to find

$$B = -\frac{1}{8}R + \frac{R}{8(1+\phi)^3} + \frac{\epsilon\varrho}{8(1+\phi)^3}. \tag{40}$$

However, this does not satisfy (39), since for ϕ close to -1 the terms blows up. We remedy this by the following trick. Take a smooth, real valued, nondecreasing function $\zeta : \mathbb{R} \to \mathbb{R}$ such that $\zeta(x) = x$ for $-1/2 \leq x \leq 1/2$, $\zeta(x) = -3/4$ for $x \leq -3/4$, and $\zeta(x) = 3/4$ for $x \geq 3/4$.

Now, instead of considering the problem (32), we consider

$$\Delta_g\phi - \frac{1}{8}R(1+\phi) + \frac{R}{8(1+\zeta(\phi))^3} + \frac{\epsilon\varrho}{8(1+\zeta(\phi))^3}. \tag{41}$$

For sufficiently small ϵ the solution given by Proposition 8 is a solution of (41), which we can rewrite in the form (38) with $f = 0$ and

$$B = -\frac{1}{8}R + \frac{R}{8(1+\zeta(\phi))^3} + \frac{\epsilon\varrho}{8(1+\zeta(\phi))^3}. \tag{42}$$

which is now smooth and satisfies (39). Applying Proposition 9 and bootstrapping we are done. $\square$

We will also need the following lemma.

Lemma 1 *Suppose that* $\phi \in C^{0,\alpha}_{-2/3}(\Omega)$ *and* $\Delta_g \phi \in C^{0,\alpha}_{-3}(\Omega) \cap L^1(\Omega)$*. Then there is a constant* C *such that*

$$\|\phi\|_{C^{2,\alpha}_{-1}} \leq C\left(\|\Delta_g \phi\|_{C^{0,\alpha}_{-3}\cap L^1} + \|\phi\|_{C^{2,\alpha}_{-2/3}}\right). \tag{43}$$

Proof: This is Lemma 6 of [50]. □

Remark 2 *We point out that the domain in this lemma is* Ω*, the SAF region of our manifold, rather than all of* M*. The purpose of this proposition is to obtain an estimate on the deformed ADM mass, which is a property of the SAF end alone. Therefore, we only need this estimate in* Ω*.*

We can now prove the following:

Proposition 11 *Let* (M, g, E, B) *be an initial data set satisfying the CDEC and the Maxwell constraints without charged matter, having ADM mass* m*, total charge* q^2 *and generalized boundary area* A*. Then for any sufficiently small* $\varepsilon > 0$ *there exists a conformal deformation of this data such that the data set satisfies the same hypotheses, the strict charged dominant energy condition with mass* $\bar{m}$*, total charge* $\bar{q}^2$ *and generalized boundary area* $\bar{A}$ *such that* $q^2 = \bar{q}^2$ *and there exists a constant* $C > 0$ *such that* $|\bar{m} - m|, |\bar{A} - A| < C\varepsilon$*.*

Proof: We consider the solution given by Proposition 8. Define $u = 1 + \phi$, $\bar{g} = u^4 g$, $\bar{E} = u^{-6}E$ and $\bar{B} = u^{-6}B$. As discussed earlier, these vector fields are divergence free and the resulting data set satisfies the strict CDEC. Since $\partial_\tau u = \partial_\tau \phi = 0$ and the boundary is minimal, by the formula for the mean curvature under a conformal change (26) the boundary remains minimal.

Let S be an embedded hypersurface in M. If dS_g is the induced volume form in the g metric, then the induced volume form in $\bar{g}$ is given by $dS_{\bar{g}} = u^4 dS_g$. Since in the AC ends $\phi \sim e^{-\delta t}$ we see these ends remain AC with respect to $\bar{g}$ with the same area. Since $|\phi| < \varepsilon$ we have

$$(1 - 5\varepsilon)|\partial M|_g \leq (1 - \varepsilon)^4 |\partial M|_g \leq |\partial M|_{\bar{g}} \leq (1 + \varepsilon)^4 |\partial M|_g \leq (1 + 5\varepsilon)|\partial M|_g \tag{44}$$

for sufficiently small ε, and so $|\bar{A} - A| \leq 5|\partial M|_g \varepsilon$.

Noticing that if ν is the unit normal to a surface then $\bar{\nu} = u^{-2}\nu$, and using equations (11) we compute

$$\bar{q}_e = \lim_{r\to\infty} \frac{1}{4\pi} \int_{S_r} \bar{E}_j \bar{\nu}^j dS_{\bar{g}} = \lim_{r\to\infty} \frac{1}{4\pi} \int_{S_r} (u^{-2}) E_j (u^{-2}) \nu^j (u^4) dS = \lim_{r\to\infty} \frac{1}{4\pi} \int_{S_r} E_j \nu^j dS = q_e$$

and similarly for $\bar{q}_b$. Hence, the charges remain the same.

Finally, by (33) we have $\Delta_g \phi \in C^{0,\alpha}_{-3} \cap L^1(\Omega) \to 0$ as $\varepsilon \to 0$. Hence by Lemma 1 we have $\phi \in C^{2,\alpha}_{-1}(\Omega) \to 0$ as $\varepsilon \to 0$. Using formula (10) it is then easy to show $|\bar{m} - m| \leq C\varepsilon$ for some constant. □

Notice that we did not make any mention of the generalized boundary being outer-minimizing. This is because this condition is not necessarily preserved by the conformal deformation. However, for any surface S enclosing the generalized boundary it will be true that

$$|S|_{\bar{g}} \geq (1 - 5\varepsilon)|S|_g > (1 - 5\varepsilon)A \tag{45}$$

for all sufficiently small ε which will be enough to establish the charged Penrose inequality.

6 The Gluing

We index our n cylindrical ends by i with $1 \leq i \leq n$. Each end has the topology of $[0, \infty) \times S^2$. On each particular AC end we fix a choice of cylindrical coordinates.

Start with the $i = 1$ AC end. Denote by $\tilde{g}_1$ the product metric on $\mathbb{R} \times S^2$ which is the limit of g. Now, using these cylindrical coordinates we consider $M_1 = M \setminus \{(t_1, \omega) : t_1 > T\}$; that is, our manifold with the cylindrical end chopped off beyond $t_1 = T$ in the fixed cylindrical coordinates. We also consider the region $\Sigma_1(T) = \{(t_1, \omega) : T - 3 \leq t_1 \leq T\}$.

Now, we consider a smooth cut-off function $\chi_1(t_1)$ defined in $\Sigma_1(T)$ such that $\chi_1 \equiv 1$ for $T - 3 \leq t_1 \leq T - 2$, decreasing for $T - 2 < t_1 < T - 1$ and $\chi_1 \equiv 0$ for $T - 1 \leq t_1 \leq T$. Next, we define $\chi_2(t_1) = 1 - \chi_1(t_1)$. Finally, we define a new metric g_1 on $\Sigma_1(T)$ by $g_1 = \chi_1 g + \chi_2 \tilde{g}_1$. So, the metric is the product metric for $T - 1 \leq t_1 \leq T$ and our original metric for $T - 3 \leq t \leq T - 2$. We extend g_1 to the rest of M_1 by letting $g_1 = g$ for points in $M_1 \setminus \Sigma_1(T)$. Furthermore, notice that as $T \to \infty$, g_1 approaches $\tilde{g}_1$ uniformly in the region $T - 2 \leq t_1 \leq T - 1$. We denote the cross section $\{t_1 = T\}$ by S_1.

We proceed inductively. Given M_i as obtained above, we perform the same procedure on the $i + 1$-th AC end. For each AC end we use the same parameter T so we don't have to worry about keeping track of n individual parameters representing how far down each cylindrical end we've performed the chopping. Thus, we obtain the manifold M_n with a metric which we denote by g_n, and whose boundary consists of $\partial M \cup (\cup_{i=1}^n S_i)$.

Now, we take two copies of M_n which we denote by M^+ and M^-. We denote the surfaces S_i in each of these sets by S_i^+ and S_i^-, respectively. We also denote the sets $\Sigma_i(T)$ in each $M^\pm$ by $\Sigma_i^\pm(T)$. We glue M^+ and M^- by identifying the points of S_i^+ and S_i^- via the identity map for $1 \leq i \leq n$ to obtain a doubled manifold which we denote by $\hat{M}$.

Definition 9 *Consider the gluing map $G : M^+ \sqcup M^- \to \hat{M}$. For the i-th AC end we define the sets*

$$\Sigma_i(T) = \{(t_i, \omega) : T - 3 \leq t_i \leq T\}, \quad \Gamma_i(\delta) = \{(t_i, \omega) : \delta \leq t_i \leq T\}. \tag{46}$$

Using natural identifications, we can consider these as subsets of $M^\pm$ and denote them by $\Sigma_i(T)^\pm$, $\Gamma_i(\delta)^\pm$. We also define

$$\hat{\Sigma}_i^\pm(T) = G(\Sigma_i^\pm(T)), \quad \hat{\Gamma}_i^\pm(\delta) = G(\Gamma_i^\pm(\delta)) \tag{47}$$

$$\hat{\Sigma}^\pm(T) = \cup_i \hat{\Sigma}_i^\pm(T), \quad \hat{\Gamma}^\pm(\delta) = \cup_i \hat{\Gamma}_i^\pm(\delta) \tag{48}$$

$$\hat{\Sigma}(T) = \hat{\Sigma}^+(T) \cup \hat{\Sigma}^-(T), \quad \hat{\Gamma}(\delta) = \hat{\Gamma}^+(\delta) \cup \hat{\Gamma}^-(\delta). \tag{49}$$

Furthermore, given some general set $P \subset M$ we similarly define corresponding sets $P^\pm \subset M^\pm$, $\hat{P}^\pm = G(P^\pm)$, $\hat{P} = \hat{P}^+ \cup \hat{P}^-$.

We write $\mathcal{F} = \cup_{i=1}^n \hat{S}_i^\pm$ (with $\mathcal{F}$ standing for fixed point set). Since the metrics $g^\pm$ agree on and are cylindrical near each $S_i^\pm$, $\hat{M}$ is a smooth Riemannian manifold with a metric which we denote by $\hat{g}$. It has two SAF ends. Since we can identify both $\hat{M}^\pm$ with the same subset of M, there is a natural correspondence between the points of $\hat{M}^+$ and $\hat{M}^-$. This allows us to define the following map which is clearly an isometry with respect to the $\hat{g}$ metric:

Definition 10 *Consider the sets $\hat{M}^{\pm} \subset \hat{M}$. Given a point $x^{+} \in \hat{M}^{+}$ and the corresponding point $x^{-} \in \hat{M}^{-}$ define the inversion map $I : \hat{M} \to \hat{M}$ by $I(x^{\pm}) = x^{\mp}$.*

It is easy to see that $\mathcal{F}$ is the fixed point set of this isometry. As a result, being totally geodesic, it is a minimal surface. Hence, if we choose one of the SAF ends, which for definiteness we will say is the end contained in $\hat{M}^{+}$, there will be an outermost minimal surface with respect to this end, which we will denote by $\mathcal{S}$. We would now like to apply the KWY theorem to the region in $\hat{M}^{+}$ exterior to this minimal surface. However, currently our region does not necessarily satisfy all of the hypotheses of the theorem. We restore each of these hypotheses in turn.

7 The Divergence Constraint

So far, the manifold $\hat{M}$ does not satisfy the assumptions of the KWY theorem. In fact, as of yet we have not defined any electromagnetic fields on our manifold. Considering the natural identification of $\hat{M}^+$ and $\hat{M}^-$ with subsets of M, we could pushforward E and B to $\hat{M}$. However, these vector fields would not be continuous across $\mathcal{F}$.

However, a bit of thought suggests that we do not need the vector fields defined on the entirety of $\hat{M}$. The point of the doubling is to obtain a minimal surface and a resulting outermost minimal surface with respect to a chosen SAF end to which we can apply the KWY theorem. Therefore, we only need the electomagnetic fields to satisfy the Maxwell constraint without charged matter outside of this minimal surface, and so it is enough to push forward the electromagnetic fields only to some chosen half of $\hat{M}$.

Therefore, consider $\hat{M}^+ \subset \hat{M}$. Using the natural identification of $\hat{M}^+$ with a subset of M we pushforward the fields E and B to vector fields $\hat{E}$ and $\hat{B}$ on one half of $\hat{M}$. The problem is that these vector fields are not necessarily divergence free on $\hat{\Gamma}^+(T-2)$. They are divergence free on $\hat{M}^+ \setminus \hat{\Gamma}^+(T-2)$ since there the $\hat{g}$ and g metrics agree.

The method of proof follows [50]. The idea is to solve Poisson's equation with Dirichlet boundary condition to obtain functions such that $E' = \hat{E} - \nabla\varphi_E$, $B' = \hat{B} - \nabla\varphi_B$ are divergence free with respect to $\hat{g}$. Then we check that the resulting solution is small in an appropriate sense so that the resulting charge is close to the original. The corresponding proof given in [50] has a small gap which we fill appropriately. We begin with the following two propositions.

Proposition 12 *Let $0 < \beta < 1, \nu > 2$, let $h \in C^{0,\alpha}_{-\nu}$ satisfy $h \geq 0$, and let $\phi \in \tilde{C}^{2,\alpha}_{-\beta}$. There is a constant C independent of T such that for all sufficiently large T*

$$||\phi||_{C^{2,\alpha}_{-\beta}} \leq C\left(||\phi||_{C^0_{-\beta}} + ||(\Delta_{\hat{g}} - h)\phi||_{C^{0,\alpha}_{-\beta-2}}\right). \tag{50}$$

Proof: The proof is almost the same as that of Proposition 1 in [50]. The idea is to consider compact K_1, K_2 with $K_1 \subsetneq K_2$ such that $\hat{M}\setminus K_1$ consists of the two SAF ends, Ω_i for $i = 1, 2$. In each of these ends we can use local estimates and the scaling of annuli to obtain estimates

$$||\phi||_{C^{2,\alpha}_{-\beta}(\Omega_i)} \leq C\left(||\phi||_{C^0_{-\beta}(\Omega_i')} + ||(\Delta_{\hat{g}} - h)\phi||_{C^{0,\alpha}_{-\beta-2}(\Omega_i')}\right). \tag{51}$$

where the Ω_i' are the two ends of $\hat{M} \setminus K_2$ (so that $\Omega_i \subset \Omega_i'$).

Now, away from the boundary, the compact set K_2 can be covered by finitely many geodesic balls $B^{\hat{g}}_{q_i}(\rho)$ of radius $\rho > 0$ so that the elliptic constant of $\hat{g}$ when computed in geodesic normal coordinates on a geodesic ball $B^{\hat{g}}_{q_i}(2\rho)$ is uniformly bounded above and below. The number of such balls increases as we increase T, but because the metric in each of the necks converges to the cylindrical metrics, ρ and the bounds can be chosen independent of T. This yields local elliptic estimates

$$||\phi||_{C^{2,\alpha}_{-\beta}(B^{\hat{g}}_{q_i}(\rho))} \leq C\left(||\phi||_{C^0_{-\beta}(B^{\hat{g}}_{q_i}(2\rho))} + ||(\Delta_{\hat{g}} - h)\phi||_{C^{0,\alpha}_{-\beta-2}(B^{\hat{g}}_{q_i}(2\rho))}\right). \tag{52}$$

Near the boundary, we make local elliptic estimates using Lemma 6.29 and the interpolation inequality Lemma 6.35 of [23] (c.f proof of Lemma 6.5 in the same reference) and the fact that $\partial_\tau|_{\partial\hat{M}}\phi = 0$. Putting all of these estimates together yields (50). $\square$

Proposition 13 *Consider $\hat{M}^+$. Let $0 < \beta < 1, \nu > 2$, let $h \in C^{0,\alpha}_{-\nu}$ satisfy $h \geq 0$, and let $\phi \in C^{2,\alpha}_{-\beta}(\hat{M}^+)$ satisfy $\phi|_{\partial\hat{M}^+} = 0$. There is a constant C independent of T such that for all sufficiently large T*

$$||\phi||_{C^{2,\alpha}_{-\beta}} \leq C\left(||\phi||_{C^0_{-\beta}} + ||(\Delta_{\hat{g}} - h)\phi||_{C^{0,\alpha}_{-\beta-2}}\right). \tag{53}$$

Proof: The proof is the same as that of Proposition 12, except that for the estimates near the boundary we use Lemma 6.5 of [23]. □

Proposition 14 *For each T large enough, there exists a unique solution $\varphi \in C^{2,\alpha}_{-1}(\hat{M}^+)$ of the problem:*

$$\Delta_{\hat{g}}\varphi = f, \quad \varphi|_{\partial\hat{M}^+} = 0 \tag{54}$$

on $\hat{M}^+$, where $f = \operatorname{div}_{\hat{g}}\hat{E}$. Furthermore,

$$||\varphi||_{C^{2,\alpha}_{-1}(\hat{\Omega}^+)} \leq \tilde{\varepsilon}(T) \tag{55}$$

where $\lim_{T\to\infty}\tilde{\varepsilon}(T) = 0$.

Remark 3 *The gap in* [50] *occurs here. The authors claim the existence of a $C^{2,\alpha}_{-1}$ solution by referring the reader to* [15] *and then use Proposition 13 to obtain the bound on the solution. However, both* [15] *and Proposition 13 apply in the case $0 < \beta < 1$. We will solve the problem with $\beta = 2/3$ and then apply Lemma 1 to obtain the $C^{2,\alpha}_{-1}$ bound.*

Proof: The existence of a unique solution $\varphi \in C^{2,\alpha}_{-2/3}$ follows from [15]. The smallness of the solution in $C^{2,\alpha}_{-2/3}$ will follow from Proposition 12 once we obtain a $C^0_{-2/3}$ estimate. To do this, first we will establish an unweighted supremum bound by using the maximum principle. We will make use of the existence of a bounded subharmonic function Ψ on $\hat{M}^+$ which satisfies $\Delta_{\hat{g}}\Psi > C > 0$ on $\hat{\Gamma}^+(T-2)$, and which is supported in $\hat{\Sigma}^+(T)$ with C independent of T for all T large enough. The existence of this function is established in Proposition 15.

Notice that $\|f\|_{C^{0,\alpha}_{-8/3}} = \varepsilon(T) \to 0$ as $T \to \infty$. Let $P = \sup_{\hat{M}^+}|\Psi|$ (In fact $P = 1$ as shown in Proposition 15). Then, by the above properties of Ψ the function $\varphi + \varepsilon(T)C^{-1}\Psi$ satisfies $\Delta_{\hat{g}}(\varphi + \varepsilon(T)C^{-1}\Psi) \geq 0$. Since φ vanishes on $\partial\hat{M}^+$ and at ∞, by the maximum principle we have $\varphi + \varepsilon(T)C^{-1}\Psi \leq \varepsilon(T)C^{-1}P$, or $\varphi \leq 2\varepsilon(T)C^{-1}P$. Similarly, by considering the superharmonic function $\varphi - \varepsilon(T)C^{-1}\Psi$ we obtain $\varphi \geq -2\varepsilon(T)C^{-1}P$, so that $\sup_{\hat{M}^+}|\varphi| \leq 2\varepsilon(T)C^{-1}P$.

To obtain the wighted estimate, we take the SAF region $\hat{\Omega}^+$ and solve the problem

$$\Delta_{\hat{g}}v = 0, \quad v = 1 \text{ on } \partial\hat{\Omega}^+, \quad v \to 0 \text{ at } \infty. \tag{56}$$

There is some constant C_2 such that $0 < v < C_2\sigma^{-1}$. Then the functions $\pm\varphi + 2\varepsilon(T)C^{-1}Pv$ are harmonic in $\hat{\Omega}^+$, tend to 0 at infinity, and are non-negative on $\partial\Omega$. By the maximum principle, $\pm\varphi + 2\varepsilon(T)C^{-1}Pv \geq 0$ on Ω, and so $|\varphi| \leq 2\varepsilon(T)C^{-1}Pv \leq 2\varepsilon(T)C^{-1}PC_2\sigma^{-1}$ which yields $\sigma|\varphi| \leq C_3\varepsilon(T)$. This implies $\|\varphi\|_{C^0_{-2/3}} \leq C_3\varepsilon(T)$.

By Proposition 13 we have

$$\|\varphi\|_{C^{2,\alpha}_{-2/3}} \leq C\left(C_3\varepsilon(T) + \|f\|_{C^{0,\alpha}_{-8/3}}\right) \leq C_4\varepsilon(T) \tag{57}$$

and by Lemma 1

$$\|\varphi\|_{C^{2,\alpha}_{-1}}(\hat{\Omega}^+) \leq CC_4\varepsilon(T) = \tilde{\varepsilon}(T). \tag{58}$$

□

Remark 4 *By elliptic regularity, the solution of Proposition 14 is smooth on $\hat{M}^+$.*

We now establish the existence of the subharmonic function Ψ used in the previous proposition.

Proposition 15 *For all T large enough there exists a bounded subharmonic function Ψ on $\hat{M}^+$ and supported in $\hat{\Sigma}^+(T)$ such that $\Delta_{\hat{g}}\Psi > C > 0$ on $\hat{\Gamma}^+(T-2)$ and $\sup_{\hat{M}^+}|\Psi| = 1$ with C independent of T.*

Proof: Pick one of the AC ends (for convenience we will omit the index i from the notation). Consider the region of $\hat{M}^+$ which corresponds to $T-4 \leq t \leq T$. Define the function $s(t)$ (defined for $t \in \mathbb{R}$) by

$$s(t) = \begin{cases} 0, & t \leq 0 \\ e^{-1/t}, & t > 0 \end{cases}.$$

Next, define the smooth function

$$b(t) = \frac{s(t-(T-3))}{s(t-(T-3)) + s((T-2)-t)} \tag{59}$$

which is 0 for $T-4 \leq t \leq T-3$, increasing on $T-3 < t < T-2$ and identically 1 for $T-2 \leq t \leq T$. Now, consider the function $k(t) = e^{\gamma(t-T)}$. Finally, consider $b(t)k(t)$ for $T-4 \leq t \leq T$. This function is identically 0 for $T-4 \leq t \leq T-3$ and so we can extend it to be 0 on the rest of $\hat{M}^+$. We claim that for sufficiently large γ and T this is a subharmonic function.

Recall that in local coordinates the Laplace-Beltrami operator takes the form

$$\Delta_{\hat{g}} f = \frac{1}{\sqrt{|\hat{g}|}}\partial_i\left(\sqrt{|\hat{g}|}\hat{g}^{ij}\partial_j f\right) \tag{60}$$

and if our function is only a function of t then

$$\Delta_{\hat{g}} f = \frac{1}{\sqrt{|\hat{g}|}}\partial_i\left(\sqrt{|\hat{g}|}\hat{g}^{it}\partial_t f\right) = g^{tt}\partial_t^2 f + \partial_i(g^{it})\partial_t f + \frac{\partial_i(\sqrt{|\hat{g}|})}{\sqrt{|\hat{g}|}}g^{it}\partial_t f. \tag{61}$$

Also, we have by the well known property of the Laplace-Beltrami operator:

$$\Delta_{\hat{g}}(bk) = b\Delta_{\hat{g}}k + k\Delta_{\hat{g}}b + 2\hat{g}(\nabla b, \nabla k). \tag{62}$$

Since both k and b are nondecreasing functions of t, the term $\hat{g}(\nabla b, \nabla k)$ is nonnegative. Next, consider

$$\Delta_{\hat{g}}k = \left(g^{tt}\gamma^2 + \partial_i(g^{it})\gamma + \frac{\partial_i(\sqrt{|\hat{g}|})}{\sqrt{|\hat{g}|}}g^{it}\gamma\right)e^{\gamma(t-T)}. \tag{63}$$

Due to the cylindrical geometry, we have $g^{tt} \to 1$, $g^{it} \to 0$ for $i \neq t$, and $\partial_i g^{it}, \partial_t(\sqrt{|\hat{g}|})/\sqrt{|\hat{g}|} \to 0$ as $T \to \infty$. Hence, there will be some T_0 and γ_0 such that for all $T > T_0$ and all $\gamma > \gamma_0$, $\Delta_{\hat{g}} k > 0$ for $T - 4 \leq t \leq T$. Finally, let us look at the term

$$\Delta_{\hat{g}} b = g^{tt}\partial_t^2 b + \partial_i(g^{it})\partial_t b + \frac{\partial_i(\sqrt{|\hat{g}|})}{\sqrt{|\hat{g}|}} g^{it}\partial_t b. \tag{64}$$

Let us examine how this term behaves as we take $t \to T - 3^+$. We see that for the leading terms we have

$$\partial_t^2 b \approx C_2 \frac{e^{-1/(t-T-3)}}{(t-(T-3))^4} \tag{65}$$

and

$$\partial_t b \approx C_3 \frac{e^{-1/(t-(T-3))}}{(t-(T-3)^2} \tag{66}$$

where $C_2, C_3 > 0$. And so, near $t = T - 3$ we will have $\Delta_{\hat{g}} b \geq 0$. More specifically, there will be some ϵ (independent of T) such that $\Delta_{\hat{g}} b \geq 0$ on $[T-3, T-3+\epsilon]$. On $[T-3+\epsilon, T]$ we can write

$$b\Delta_{\hat{g}} k + k\Delta_{\hat{g}} b = \left(bg^{tt}\gamma^2 + \partial_i(g^{it})(b\gamma + \partial_t b) + \frac{\partial_i(\sqrt{|\hat{g}|})}{\sqrt{|\hat{g}|}} g^{it}(b\gamma + \partial_t b) + g^{tt}\partial_t^2 b \right) e^{\gamma(t-T)} \tag{67}$$

Thus, since $b(t) \geq b(T-3+\epsilon) > 0$ on $[T-3+\epsilon, T]$, we see that for sufficiently large γ, $b\Delta_{\hat{g}} k + k\Delta_{\hat{g}} b > 0$, and so there will be some constant C such that $b\Delta_{\hat{g}} k + k\Delta_{\hat{g}} b > C$ on $\hat{\Gamma}^+(T-2)$. Notice that for sufficiently large T we can choose a fixed sufficiently large γ so that the constant C becomes independent of T. We construct such functions in each of the cylindrical ends, and then take their sum, defining it to be Ψ, which is thus the required bounded subharmonic function. Notice $|\Psi| \leq 1$ and the constant C can be chosen independently of T for all sufficiently large T. $\square$

Define $E' = E - \nabla_{\hat{g}}\varphi$, which is now a smooth, divergence free vector field on $\hat{M}^+$. Using formula 11 we compute the accompanying charge

$$|q_{e,T} - q_e| = \left| \lim_{r\to\infty} \frac{1}{4\pi} \int_{S_r} (\nabla_{\hat{g}}\varphi)_j \nu^j dS \right| \leq C\tilde{\varepsilon}(T) \tag{68}$$

for some constant C independent of T. We similarly obtain the vector field B' and conclude that $|q_{b,T} - q_b| \leq C\tilde{\varepsilon}(T)$. Therefore, if we denote the total squared charge by q_T^2, we see that $\lim_{T\to\infty} q_T^2 = q^2$.

8 The CDEC Constraint

We look for a conformal deformation $\bar{g} = u^4 \hat{g}$ on $\hat{M}$ such that the scalar curvature $\bar{R}$ of this metric satisfies the charged dominant energy condition. However, recall the vector fields E', B' are only defined on $\hat{M}^+$. The simplest way around this is the following. Recall that by Proposition 11, we can assume that our original data set satisfies the strict charged dominant energy condition. Given the vector fields E', B' we define $\bar{E} = u^{-6}E', \bar{B} = u^{-6}B'$ which are defined and divergence free on $\hat{M}^+$. In order to restore the charged dominant energy condition on $\hat{M}^+$ we want

$$\bar{R} \geq 2(|\bar{E}|^2_{\bar{g}} + |\bar{B}|^2_{\bar{g}}) = 2u^{-8}\left(|E'|^2_{\hat{g}} + |B'|^2_{\hat{g}}\right). \tag{69}$$

We begin with the following Lemma, whose importance will soon become apparent. Recall the inversion map defined previously by $I(x^{\pm}) = x^{\mp}$.

Lemma 2 *For each T there exists a function $\zeta \geq 0$ on $\hat{M}$ which satisfies*

(i) $\hat{R} + \zeta \geq 2\left(|E'|^2_{\hat{g}} + |B'|^2_{\hat{g}}\right)$ *on* $\hat{M}^+$

(ii) ζ *is smooth and* $\zeta(x) = \zeta(I(x))$

(iii) $4\hat{R} + 3\zeta > 0$

(iv) $\|\zeta\|_{C^{0,\alpha}_{-8/3}} \to 0$ *as* $T \to \infty$.

Proof: We can write (i) as

$$\hat{R} + \zeta \geq 2\left(|E'|^2_{\hat{g}} + |B'|^2_{\hat{g}}\right) = 2\left(|\hat{E}|^2_{\hat{g}} + |\hat{B}|^2_{\hat{g}}\right) + 4\langle\hat{E}, \nabla\varphi_E\rangle + 4\langle\hat{B}, \nabla\varphi_B\rangle + |\nabla\varphi_E|^2 + |\nabla\varphi_B|^2$$

or

$$\left[\hat{R} - 2\left(|\hat{E}|^2_{\hat{g}} + |\hat{B}|^2_{\hat{g}}\right)\right] + \zeta - \left(4\langle\hat{E}, \nabla\varphi_E\rangle + 4\langle\hat{B}, \nabla\varphi_B\rangle + |\nabla\varphi_E|^2 + |\nabla\varphi_B|^2\right) \geq 0$$

.

On $\hat{M}^+$ outside of $\hat{\Gamma}^+(T-2)$ the first term is positive by the strict CDEC. In $\hat{\Gamma}^+(T-2)$ it satisfies $\hat{R} - 2\left(|\hat{E}|^2_{\hat{g}} + |\hat{B}|^2_{\hat{g}}\right) \geq -\varepsilon(T)$ for some $\varepsilon(T) > 0$ such that $\varepsilon(T) \to 0$ as $T \to \infty$. Therefore, we look for a smooth function such that

$$\zeta \geq |4\langle\hat{E}, \nabla\varphi_E\rangle + 4\langle\hat{B}, \varphi_B\rangle + |\nabla\varphi_E|^2 + |\nabla\varphi_B|^2| \tag{70}$$

everywhere on $\hat{M}^+$ and

$$\zeta \geq 2\varepsilon(T) + |4\langle\hat{E}, \nabla\varphi_E\rangle + 4\langle\hat{B}, \varphi_B\rangle + |\nabla\varphi_E|^2 + |\nabla\varphi_B|^2| \tag{71}$$

on $\hat{\Gamma}^+(T-2)$. We require $2\varepsilon(T)$ in order for (iii) to hold. Obviously we can find such a function satisfying (ii) and (iv) follows since $\|\varphi_E\|_{C^{2,\alpha}_{-1}}, \|\varphi_B\|_{C^{2,\alpha}_{-1}} \to 0$ as $T \to \infty$ so we can choose ζ to fall-off rapidly at infinity. In fact, since E, B are $O(1/|x|^2)$ we can obtain $\|\zeta\|_{C^{0,\alpha}_{-4}} \to 0$ as $T \to \infty$. $\square$

Remark 5 *The purpose of section 2.3 was to achieve condition (iii) in the above lemma. It plays an important role in the next proposition.*

With ζ in hand we seek to solve the problem $\bar{R} = (\hat{R} + \zeta)u^{-8}$, $\partial_\tau u = 0$ which by using (25) can be written as

$$\begin{gathered} \Delta_{\hat{g}} u - \frac{1}{8}\left(\hat{R}u - \frac{\hat{R}+\zeta}{u^3}\right) = 0 \\ \partial_\tau u = 0 \quad \text{on} \quad \partial\hat{M}. \end{gathered} \tag{72}$$

Proposition 16 *Let*

$$h = \frac{1}{2}\hat{R} + \frac{3}{8}\zeta. \tag{73}$$

Then there is a constant C independent of T, for T large enough, such that if $\phi \in \tilde{C}^{2,\alpha}_{-2/3}$ then

$$||\phi||_{C^{2,\alpha}_{-2/3}} \leq C||(\Delta_{\hat{g}} - h)\phi||_{C^{0,\alpha}_{-8/3}}. \tag{74}$$

Remark 6 *This is a slight generalization of Proposition 3 in* [50] *and follows the same proof.*

Proof: In order not to encumber notation we will treat the case of a single AC end. The generalization to finitely many AC ends is then clear.

Suppose to the contrary. Then there is a sequence $T_j \to \infty$ and $\phi_j \in \tilde{C}^{2,\alpha}_{-2/3}$ satisfying

$$||\phi_j||_{C^{2,\alpha}_{-2/3}} = 1 \quad \forall j, \quad ||(\Delta_{\hat{g}} - h)\phi||_{C^{0,\alpha}_{-8/3}} \to 0 \quad \text{as} \quad j \to \infty. \tag{75}$$

By Proposition 12 we have

$$||\phi_j||_{C^{2,\alpha}_{-2/3}} \leq C\left(||\phi_j||_{C^0_{-2/3}} + ||(\Delta_{\hat{g}} - h)\phi_j||_{C^{0,\alpha}_{-8/3}}\right) \tag{76}$$

with C independent of j. In view of (75) we must have that there exists some $\varepsilon > 0$ such that

$$\varepsilon \leq ||\phi_j||_{C^0_{-2/3}} \leq 1 \tag{77}$$

for all j. We now consider two cases:
(i) There is a $\tau > 0$ such that for any $\delta > 0$ we have

$$\limsup_j ||\phi_j||_{C^0(\hat{\Gamma}(\delta))} \geq \tau \tag{78}$$

(ii) For every $\tau > 0$ there exists a $\delta > 0$ such that

$$\limsup_j ||\phi_j||_{C^0(\hat{\Gamma}(\delta))} < \tau \tag{79}$$

Note that $\hat{\Gamma}(\delta)$ is the union of the two necks cut at δ.

Case (i): We take the AC coordinates on our end. For each positive integer j large enough take $\delta_j = j$. Then there is a i_j large enough such that $T_{i_j} > j$ and there exists a $p_{i_j} \in \hat{\Gamma}(j)$

with $|\phi_{i_j}(p_{i_j})| \geq \tau/2$. Without loss of generality, we can assume that $p_{i_j} \in \hat{\Gamma}^+(j)$. Passing to a subsequence we can assume that $i_j = j$. In the obvious way for an end doubled at T we can consider the t-coordinate on $\hat{\Gamma}(0)$ as lying in $[0, 2T]$. That is, $t \in [0, T]$ corresponds to $\hat{\Gamma}^+(0)$ and $[T, 2T]$ corresponds to $\hat{\Gamma}^-(0)$.

Let $s_j = t(p_j)$ be the t-coordinate of the point p_j. Notice that for fixed k and $j > k$ we have $\{s_j - k/2 \leq t \leq s_j - k/2\} \subset \{j/2 \leq t \leq 2T_j - j/2\}$. So in each of these regions the coefficients of the operator

$$\mathcal{T} = \Delta_{\hat{g}} - h \tag{80}$$

converge uniformly to those of

$$\mathcal{T}_0 = \Delta_0 - \frac{1}{2}R_0 \tag{81}$$

where Δ_0 is the Laplace-Beltrami operator and R_0 the scalar curvature with respect to the cylindrical metric. Notice that $R_0 \geq 0$, R_0 is independent of t, and $R_0(\omega) > 0$ for some $\omega \in S^2$.

Consider the set $\Lambda_k = \{-k/2 \leq t \leq k/2\} \times S^2$. For each j we can consider the functions ϕ_j as a functions on Λ_k translated appropriately so that the point p_j now has 0 coordinate. For each k the sequence ϕ_j is a bounded sequence on Λ_k. Since the embedding $C^{2,\alpha'}(\Lambda_k) \hookrightarrow C^{2,\alpha}(\Lambda_k)$ is compact for $0 \leq \alpha' < \alpha$ there is a function ϕ_0 on $\mathbb{R} \times S^2$ and a subsequence, which we continue to denote by ϕ_j, converging to ϕ_0 on each Λ_k. Furthermore, there is a point p_0 in the $t = 0$ cross section of Λ so that $|\phi_0(p_0)| \geq \tau/2$, so ϕ_0 is not identically zero.

By our previous comments, we can consider $(\Delta_{\hat{g}} - h)\phi_j$ as functions on Λ_k, translated in the same fashion as each corresponding ϕ_j. We can similarly translate the operator $\mathcal{T}$ so that its coefficients converge uniformly to the coefficients of $\mathcal{T}_0$ for each fixed k. By a slight abuse of notation we write $\mathcal{T}\phi_j \to \mathcal{T}_0\phi_0$ in $C^0(\Lambda_k)$. Since we have that $\mathcal{T}\phi_j \to 0$ in $C^{0,\alpha}(\Lambda_k)$ by (75) we conclude that ϕ_0 is a non-trivial solution of the linear equation $\mathcal{T}_0\phi_0 = 0$. But then ϕ_0 must exhibit exponential growth (either for $t \to \infty$ or $t \to -\infty$), contradicting (77).

Case (ii): First, we take the region Ω with its AF coordinates. Hence there is some r_0 such that for $s \geq r_0 > 0$ we have the region $\mathcal{E}(s) = \{r > s\}$. Define $\mathcal{B}(s) = M \setminus \mathcal{E}(s)$, and also define $D(\delta) = \{(t, \omega) : t > \delta\}$. Notice with this definition $D(\delta_1) \supset D(\delta_2)$ if $\delta_1 < \delta_2$. Finally, define $A_\delta = \mathcal{B}(r_0) \setminus D(\delta + 1)$. We now have the following lemma:

Lemma 3 *Suppose ϕ_j satisfies* (75) *and (ii). Then for each $\delta > 0$ there holds $||\phi_j||_{C^{1,\alpha}(A_\delta)} \to 0$ as $j \to \infty$.*

Proof: To be precise, $A_\delta \subset M$, while the functions ϕ_j are functions on $\hat{M}$. However, by the natural identification, we can consider the functions as defined on $A_\delta \subset \hat{M}^+$ (at least for sufficiently large j).

Suppose the lemma is false and let j_k be a subsequence which converges to ϕ_0 in $C^{1,\alpha'}(A_\delta)$ for $\alpha' < \alpha$. Then ϕ_0 is not identically equal to 0 and since $h > 0$ on A_δ we have

$$\lim_k \int_{A_\delta} h\phi_{j_k}^2 = \int_{A_\delta} h\phi_0^2 \geq \int_{A_\delta} \frac{1}{2}R\phi_0^2 > 0. \tag{82}$$

This was why we needed to ensure that our data satisfied the strict CDEC.

We will now show that

$$\limsup_j \int_{A_\delta} h\phi_j^2 = 0 \tag{83}$$

yielding a contradiction. Without loss of generality, by passing to a subsequence we may assume that

$$\int_{A_\delta} h\phi_j^2 \to \limsup_j \int_{A_\delta} h\phi_j^2. \tag{84}$$

If χ is any smooth cut-off function with compact support in $\hat{M}^+$, with $0 \le \chi \le 1$ and $\chi = 1$ in A_δ then

$$\int_{A_\delta} h\phi_j^2 \le \int_{\hat{M}^+} \chi^2(|\nabla\phi_j|_{\hat{g}}^2 + h\phi_j^2) = -\int_{\hat{M}^+} \chi^2\phi_j(\Delta_{\hat{g}} - h)\phi_j - 2\int_{\hat{M}^+} \chi\phi_j\hat{g}(\nabla\chi, \nabla\phi_j). \tag{85}$$

We will choose a sequence of cutoffs χ_k such that the right hand side tends to 0 yielding a contradiction. By (ii), for each integer k we can choose $\delta_k > \delta$ such that $|\phi_j| < 1/k$ on $\hat{\Gamma}(\delta_k)$ for all j sufficiently large. Now we can choose χ_k supported on $\mathcal{B}(k) \setminus D(\delta_k + 2)$ with :

$$\mathrm{supp}\nabla\chi_k \subset [\mathcal{B}(k) \setminus \mathcal{B}(r_0)] \cup [D(\delta_k + 1) \setminus D(\delta_k + 2)] \tag{86}$$

satisfying

$$|\nabla\chi_k|_{\hat{g}} \le C/k \quad \text{on } \mathcal{B}(k) \setminus \mathcal{B}(r_0) \tag{87}$$

$$|\nabla\chi_k|_{\hat{g}} \le C \quad \text{on } D(\delta_k + 1) \setminus D(\delta_k + 2) \tag{88}$$

By (75) we can choose $j_k > j_{k_1}$ so that

$$\left(\int_{\mathbb{R}^3\setminus\mathcal{B}(r_0)} \sigma^{-10/3}\right) \|(\Delta_{\hat{g}} - h)\phi_{j_k}\|_{C^{0,\alpha}_{-8/3}} \le \frac{1}{k}. \tag{89}$$

It then follows that

$$-\int_{M^+} \chi_k^2\phi_{j_k}(\Delta_{\hat{g}} - h)\phi_{j_k} < \frac{1}{k} \tag{90}$$

and

$$\begin{aligned} -\int_{M^+} \chi_k\phi_{j_k}\hat{g}(\nabla\chi_k, \nabla\phi_{j_k}) &\le C\int_{D(\delta_k+1)\setminus D(\delta_k+2)} |\phi_{j_k}| + Ck^{-1}\int_{\mathcal{B}(k)\setminus\mathcal{B}(r_0)} \sigma^{-7/3} &(91)\\ &\le Ck^{-1} + Ck^{-1}\int_{r_0}^k \frac{dr}{r^{1/3}} &(92)\\ &\le C(k^{-1} + k^{-1/3}). &(93)\end{aligned}$$

This completes the proof of the lemma. □

Now choose $\delta > 0$ such that

$$\limsup_j \|\phi_j\|_{C^0(\hat{\Gamma}(\delta))} < \varepsilon \tag{94}$$

where ε is defined by (77). Chop off the cylindrical end for $t > \delta+1$ obtaining an S^2 boundary component. Define a new manifold (M_*, g_*) by gluing in a solid ball at this boundary and smoothly extending the metric $\hat{g}$ to obtain g_*. The glued in region will be denoted by $\bar{D}(\delta+1) \subset M_*$. Then extend the potential function h smoothly to $\bar{D}(\delta+1)$ so that $h_* \geq 0$ (again, this can be done because $h > 0$). Let χ be a smooth cut off function with $0 \leq \chi \leq 1$ with $\chi = 1$ outside $D(\delta)$ and $\chi = 0$ on $\bar{D}(\delta+1)$. Taking the values of ϕ_j from M^+ we can consider $\chi\phi_j$ as a function on M^*, and find

$$(\Delta_{g_*} - h_*)\chi\phi_j = (\Delta_{\hat{g}} - h)\chi\phi_j = \chi(\Delta_{\hat{g}} - h)\phi_j + 2\hat{g}(\nabla\chi, \nabla\phi_j) + \phi_j\Delta_{\hat{g}}\chi. \tag{95}$$

Therefore we can estimate:

$$\|(\Delta_{g_*} - h_*)\chi\phi_j\|_{C^{0,\alpha}_{-8/3,\delta}(M^*)} \leq C\left(\|(\Delta_{\hat{g}} - h)\phi_j\|_{C^{0,\alpha}_{-8/3}} + \|\phi_j\|_{C^{1,\alpha}(A_\delta)}\right) \to 0 \tag{96}$$

by (75) and Lemma 3. It follows by Theorem 2 part (b) in [50] applied to (M_*, g_*) that

$$\|\chi\phi_j\|_{C^{2,\alpha}_{-2/3}(M^*)} \to 0. \tag{97}$$

Hence we obtain

$$\|\phi_j\|_{C^0_{-2/3}(M^+\setminus\Gamma^+(\delta))} \to 0 \tag{98}$$

and similarly

$$\|\phi_j\|_{C^0_{-2/3}(M^-\setminus\Gamma^-(\delta))} \to 0 \tag{99}$$

so it follows that

$$\|\phi_j\|_{C^0_{-2/3}(\hat{M}\setminus\hat{\Gamma}(\delta))} \to 0. \tag{100}$$

Combining this with (94) we conclude that

$$\limsup_j \|\phi_j\|_{C^0_{-2/3}(\hat{M})} < \varepsilon \tag{101}$$

in contradiction to (77). This completes the proof of Proposition 16. □

Proposition 17 *There exists a unique solution $u = 1 + \psi$ to (72) such that $\|\psi\|_{\tilde{C}^{2,\alpha}_{-2/3}} \to 0$ as $T \to \infty$.*

Proposition 16 will be key here. We follow the procedure and notation outlined in [50]. We will look for solutions in the spaces $\tilde{C}^{k,\alpha}_{-\beta}(\hat{M})$, defined earlier. We look for a conformal perturbation $u = 1 + \psi$, in which case (72) takes the form

$$\Delta_{\hat{g}}\psi - \frac{1}{8}\left(\hat{R}\psi - \frac{\hat{R}+\zeta}{(1+\psi)^3}\right) - \frac{1}{8}\hat{R} = 0. \tag{102}$$

Therefore we define an operator $\mathcal{N} : \tilde{C}^{2,\alpha}_{-2/3} \to C^{0,\alpha}_{-8/3}$ by

$$\mathcal{N}(\psi) = \Delta_{\hat{g}}\psi - \frac{1}{8}\left(\hat{R}\psi - \frac{\hat{R}+\zeta}{(1+\psi)^3}\right) - \frac{1}{8}\hat{R} \tag{103}$$

Computing the Frechet derivative at $\psi = 0$ we obtain

$$d\mathcal{N} = \Delta_{\hat{g}} - \frac{1}{8}\left(4\hat{R} + 3\zeta\right) = \Delta_{\hat{g}} - h : \ \tilde{C}^{2,\alpha}_{-2/3} \to C^{0,\alpha}_{-8/3}. \tag{104}$$

By our choice of ζ, $4\hat{R} + 3\zeta > 0$, and so by the Fredholm alternative, this operator is invertible, that is there exists $d\mathcal{N}^{-1} : C^{0,\alpha}_{-8/3} \to \tilde{C}^{2,\alpha}_{-2/3}$. By Proposition 16 this operator is uniformly bounded, that is, there exists a constant C independent of T such that

$$||d\mathcal{N}^{-1}x||_{\tilde{C}^{2,\alpha}_{-2/3}} \leq C||x||_{C^{0,\alpha}_{-8/3}}. \tag{105}$$

We define the quadratic part of $\mathcal{N}$ by

$$\mathcal{Q}(\psi) \equiv \mathcal{N}(\psi) - \mathcal{N}(0) - d\mathcal{N}(\psi) \tag{106}$$

and a subsequent calculation shows

$$\mathcal{Q}(\psi) = \frac{\hat{R} + \zeta}{8}\left(\frac{6 + 8\psi + 3\psi^2}{(1+\psi)^3}\right)\psi^2. \tag{107}$$

Hence, there exists some η_0 such that for $||\psi||_{\tilde{C}^{2,\alpha}_{-2/3}} < \eta < \eta_0$

$$||\mathcal{Q}(\psi)||_{C^{0,\alpha}_{-8/3}} \leq C\eta^2 \tag{108}$$

$$||\mathcal{Q}(\psi_1) - \mathcal{Q}(\psi_2)||_{C^{0,\alpha}_{-8/3}} \leq 2C\eta||\psi_1 - \psi_2||_{\tilde{C}^{2,\alpha}_{-2/3}} \tag{109}$$

where $C > 0$ is some constant. The key to notice is that for all ζ with $\|\zeta\|_{C^{0,\alpha}_{-8/3}} < C_1$ we can choose C to be independent of ζ.

Next, notice

$$||\mathcal{N}(0)||_{C^{0,\alpha}_{-8/3}} = \left|\left|\frac{\zeta}{8}\right|\right|_{C^{0,\alpha}_{-8/3}} \leq \varepsilon(T) \tag{110}$$

where $\varepsilon(T) \to 0$ as $T \to \infty$.

Now choose $0 < \lambda < 1$ and $\eta > 0$ such that

$$\eta < \frac{\lambda}{2C^2}$$

and T large enough so that

$$\varepsilon(T) < C\eta^2.$$

Notice that the larger we take T, the smaller $\varepsilon(T)$ becomes, and hence the smaller we can choose η.

Now, consider the operator

$$F(\psi) \equiv -d\mathcal{N}^{-1}\left(\mathcal{N}(0) + \mathcal{Q}(\psi)\right) \tag{111}$$

and notice

$$\begin{aligned}||F(\psi)||_{\tilde{C}^{2,\alpha}_{-2/3}} &\leq C\left(||\mathcal{N}(0)||_{C^{0,\alpha}_{-8/3}} + ||\mathcal{Q}(\psi)||_{C^{0,\alpha}_{-8/3}}\right)\\ &\leq C(\varepsilon(T) + C\eta^2) \leq 2C^2\eta^2 < \frac{2C^2\lambda}{2C^2}\eta < \eta\end{aligned}$$

while

$$\begin{aligned}||F(\psi_1) - F(\psi_2)||_{\tilde{C}^{2,\alpha}_{-2/3}} &\leq C||\mathcal{Q}(\psi_1) - \mathcal{Q}(\psi_2)||_{C^{0,\alpha}_{-8/3}}\\ &\leq 2C^2\eta||\psi_1 - \psi_2||_{\tilde{C}^{2,\alpha}_{-2/3}}\\ &< \lambda||\psi_1 - \psi_2||_{\tilde{C}^{2,\alpha}_{-2/3}}.\end{aligned}$$

Therefore, F is a contraction mapping of the ball of radius η to itself, and hence possesses a unique fixed point, which we continue to denote by ψ. Since

$$F(\psi) = \psi = -d\mathcal{N}^{-1}(\mathcal{N}(0) + \mathcal{Q}(\psi)) \tag{112}$$

then operating by $d\mathcal{N}$ and rearranging

$$d\mathcal{N}(\psi) + \mathcal{N}(0) + \mathcal{Q}(\psi) = \mathcal{N}(\psi) = 0 \tag{113}$$

and so $u = 1 + \psi$ is precisely the conformal factor we were looking for. Furthermore $||\psi||_{\tilde{C}^{2,\alpha}_{-2/3}} < \eta \to 0$ as $T \to \infty$. $\square$

Corollary 1 *The conformal factor u satisfies $u(x) = u(I(x))$.*

Proof: Define $\tilde{u}(x) = u(I(x))$. Since $\hat{R}(x) = \hat{R}(I(x))$ and $\zeta(x) = \zeta(I(x))$, $\tilde{u}$ satisfies (72). Since the solution u was obtained using the contraction mapping principle, it is unique, and so $\tilde{u} = u$. $\square$

Proposition 18 *The conformal factor u is smooth.*

Proof: The proof is the same as that of Proposition 10. $\square$

Corollary 2 *The surface $\mathcal{F} \subset \hat{M}$ is a minimal surface of $(\hat{M}, \bar{g})$ where $\bar{g} = u^4\hat{g}$.*

Proof: The map I is an isometry of $(\hat{M}, \bar{g})$ and $\mathcal{F}$ is its fixed point set. $\square$

To finish our proof, we need to obtain a bound on $\|\phi\|_{C^{2,\alpha}_{-1}}$ in order to have $\bar{m}$ close to m. As remarked earlier, we only need this in the SAF region Ω. Notice

$$\Delta_{\hat{g}}\psi = \frac{1}{8}\hat{R}\psi - \frac{\hat{R} + \zeta}{8(1+\psi)^3} + \frac{1}{8}\hat{R}. \tag{114}$$

Consider the right hand side as a function on Ω. We first show that the right hand side tends to 0 in $C^{0,\alpha}_{-3}(\Omega) \cap L^1(\Omega)$ as $T \to \infty$. We rewrite the right hand side as

$$\frac{1}{8}\hat{R}\psi - \frac{\zeta}{8(1+\psi)^3} + \frac{\hat{R}}{8}\frac{(1+\psi)^3 - 1}{(1+\psi)^3} = \frac{1}{8}\hat{R}\psi - \frac{\zeta}{8(1+\psi)^3} + \frac{\hat{R}}{8}\frac{3\psi + 3\psi^2 + \psi^3}{(1+\psi)^3}. \tag{115}$$

We use the fact that if $f_i \in C^{0,\alpha}_{-\beta_i}, i = 1, 2$ and $\beta_1 + \beta_2 > 3$, then

$$\|f_1 f_2\|_{C^{0,\alpha}_{-3} \cap L^1} = \|f_1 f_2\|_{C^{0,\alpha}_{-3}} + \|f_1 f_2\|_{L^1} \leq C \|f_1\|_{C^{0,\alpha}_{-\beta_1}} \|f_2\|_{C^{0,\alpha}_{-\beta_2}} \tag{116}$$

and so if one of the terms on the right-hand side is bounded and the other tends to 0, the left-hand side tends to 0 as well. By our previous remarks, we have that $\zeta \in C^{0,\alpha}_{-4}$ with $\|\zeta\|_{C^{0,\alpha}_{-4}} \to 0$ as $T \to \infty$, while the other two terms are of the form $f\psi$ with $f \in C^{0,\alpha}_{-3}$ bounded, and $\|\psi\|_{C^{0,\alpha}_{-2/3}} \to 0$. We conclude that

$$\|\Delta_{\hat{g}} \psi\|_{C^{0,\alpha}_{-3} \cap L^1} \to 0. \tag{117}$$

Therefore, we apply Lemma 1 to conclude

$$\|\psi\|_{C^{2,\alpha}_{-1}} \to 0 \quad \text{as} \quad T \to \infty. \tag{118}$$

9 Proof of Theorem 1: Part 1

Consider the manifold $(\hat{M}, \bar{g})$ and notice that $\mathcal{F} \cup \partial \hat{M}^+$ is a minimal surface. Therefore, with respect to our chosen asymptotically flat end (that is, the end on which $\bar{E}$ and $\bar{B}$ are defined) there is some outermost minimal surface which we denote by $\mathcal{S}_T$, to show the dependence on T. Now, we consider the areas of this surface measured in the $\bar{g}$, $\hat{g}$ and g metrics. We will also write m_T and $|q_T|$ to show the dependence of the mass and charge on T.

Of course, as we increase T and move down the cylindrical ends, the surface $\mathcal{S}_T$ changes. However, as we do so, $\hat{g} \to g$, and $\bar{g} \to \hat{g}$ since $\psi \to 0$. Hence, we obtain an inequality of the form

$$(1+5\varepsilon(T))|\mathcal{S}_T|_g \geq |\mathcal{S}_T|_{\bar{g}} = 4\pi\rho_T^2 \geq (1-5\varepsilon(T))|\mathcal{S}_T|_g > (1-5\varepsilon(T))A = 4\pi(1-5\varepsilon(T))\rho^2. \tag{119}$$

where $\varepsilon(T) \to \infty$ as $T \to \infty$.

On the other hand, the mass and charge of our chosen end satisfy

$$|m_T - m| < \varepsilon(T) \tag{120}$$

and

$$||q_T| - |q|| < \varepsilon(T) \tag{121}$$

by (10) and (118).

If we denote $\lim_{T\to\infty} \rho_T = \bar{\rho}$ (which exists, after potentially passing to a subsequence) we have by Corollary 1.2 of [37] that

$$\rho_T \leq m_T + \sqrt{m_T^2 - q_T^2} \tag{122}$$

and now, taking the limit and noting $\rho \leq \bar{\rho}$ we obtain

$$\rho \leq \bar{\rho} \leq m + \sqrt{m^2 - q^2} \tag{123}$$

proving the upper bound in (3).

As for (2), suppose $\rho \geq |q|$. Take any $0 < \lambda < 1$ and instead of taking our initial data set (M, g, E, B) consider $(M, g, \lambda E, \lambda B)$ which has charge $\lambda|q|$. Therefore $\rho > \lambda|q|$ (if $|q| = 0$ this is trivially tue). In that case, eventually the ρ_T satisfy $\rho_T > \lambda|q_T|$ and thus by Theorem 1.1 of [37] the solutions satisfy

$$m_T \geq \frac{1}{2}\left(\rho_T + \frac{\lambda^2 q_T^2}{\rho_T}\right). \tag{124}$$

Taking the limit, we obtain

$$m \geq \frac{1}{2}\left(\bar{\rho} + \frac{\lambda^2 q^2}{\bar{\rho}}\right) \geq \frac{1}{2}\left(\rho + \frac{\lambda^2 q^2}{\rho}\right) \tag{125}$$

where the last inequality follows from $\bar{\rho} \geq \rho > \lambda|q|$. Since this holds for arbitrary λ, taking the limit $\lambda \to 1$ we are done.

Therefore we have proven our theorem in the case that our initial data satisfies the strict charged dominant energy condition and the outer-minimizing generalized boundary condition. For the case where the data merely satisfies the charged dominant energy condition, we apply Proposition 11 to obtain a data set satisfying the strict charged dominant energy condition with mass m_ε and the same charge $|q|$, with $m_\varepsilon \to m$ as $\varepsilon \to 0$. By (45) we have that any surface S enclosing the generalized boundary satisfies $|S| > 4\pi\rho_\varepsilon$ where $\rho_\varepsilon \to \rho$ as $\varepsilon \to 0$. Therefore the data satisfies

$$\rho_\varepsilon \leq m_\varepsilon + \sqrt{m_\varepsilon^2 - q^2} \tag{126}$$

for all ε and taking the limit $\varepsilon \to 0$ we have obtained

$$\rho \leq m + \sqrt{m^2 - q^2}. \tag{127}$$

For the full Penrose inequality assume $\rho \geq |q|$ and choose any λ with $0 < \lambda < 1$. Then for all sufficiently small ε we have $\rho_\varepsilon \geq \lambda|q|$ and so

$$m_\varepsilon \geq \frac{1}{2}\left(\rho_\varepsilon + \frac{\lambda^2 q^2}{\rho_\varepsilon}\right) \tag{128}$$

and taking the limit $\varepsilon \to 0$ we obtain

$$m \geq \frac{1}{2}\left(\rho + \frac{\lambda^2 q^2}{\rho}\right) \tag{129}$$

and finally letting $\lambda \to 1$ we obtain the full charged Penrose inequality. $\square$

The case of equality is treated in the next section.

10 Proof of Theorem 1: The Case of Equality

First we remark that equality in the upper bound in (3) is equivalent to equality in (2). Next we apply the argument of Section 7 of [37] to conclude that if equality occurs, our initial data must have either a single boundary component or a single asymptotically cylindrical end. The case of a single boundary component is treated in [18]. Thus, we are left with the case of a single asymptotically cylindrical end. We will use the same strategy as in [18], which depends on the existence of a solution to the IMCF.

Proposition 19 *Suppose our initial data set has a single AC end. Then there exists a solution of the IMCF on all of M.*

Proof: Let $(t, \omega) \in [0, \infty) \times S^2$ be coordinates on the AC end (recall that by our convention $t \to \infty$ corresponds to moving down the AC end). Take any cross section S_n with $n \in \mathbb{N}$. By Theorem 3.1 in [28], there exists a solution u_n on $\Omega_n = M \setminus E_n$ which satisfies the gradient estimate

$$|\nabla u_n(x)| \leq \sup_{\partial E_n \cap B_r(x)} H_+ + \frac{C}{r} \quad \text{a.e.} \quad x \in \Omega_n \tag{130}$$

for each $0 < r \leq \sigma(x)$. Here, $H_+ = \max(0, H)$, $E_n = \{(t, \omega) : t > n\}$, $\sigma(x)$ is a positive, continuous function of x, and C is a constant depending only on the dimension of M.

Consider this sequence of solutions. We have $u_n = 0$ for $x \in E_n = \{(t, \omega) : t > n\}$ and so in order to obtain convergence, we need to normalize the functions. This is easy to do. Pick any point $x_0 \in M$. Now, define new functions $\tilde{u}_n(x) = u_n(x) - u_n(x_0)$ so that all of these functions agree at the point x_0 and $\nabla u_n = \nabla \tilde{u}_n$. For simplicity we will subsequently drop the tilde and refer to these normalized functions as u_n.

As mentioned before, $\sigma(x)$ is a continuous function (for its full definition we refer the reader to [28]), so around every x we can find an open neighborhood B_x diffeomorphic to a ball such that for all $y \in B_x$ we have $\sigma(y) \geq \frac{1}{2}\sigma(x)$. Notice that as we take cross sections further down the cylindrical end, H_+ tends to 0. Thus, for every B_x we can find a constant K_x such that $|\nabla u_n(y)| \leq K_x$ for a.e $y \in B_x$, so the family $\{u_n\}$ is equicontinuous in B_x.

Now, take any point x and take a finite length smooth curve with trace I of length L joining it to x_0. For each $z \in I$ we have open neighborhoods with constants K_z as above. By compactness, we can take a finite subcover, and then take a maximum of the constants to obtain that $|u_n(x)| \leq KL$ for all n. Then, similarly we can obtain $|u_n(y)| \leq D_x$ for all $y \in B_x$ where D_x is some constant depending on B_x.

By the Arzela-Ascoli theorem, the sequence $\{u_n\}$ has a subsequence which converges uniformly on compact subsets to a function u. By Theorem 2.1 in [28], u is a solution of the weak IMCF. $\square$

Denote $S_{i,\tau} = \{x : u_i(x) = \tau\}$ and $S_\tau = \{x : u(x) = \tau\}$. By the remarks following Theorem 2.1 in [28] (which themselves depend on the Regularity Theorem 1.3.ii in this same paper) we have that

$$S_{i,\tau} \to S_\tau \quad \text{locally in } C^{1,\alpha}. \tag{131}$$

Furthermore, the flow has no jump discontinuities, since those only occur when the flow surfaces touch a minimal surface. We could have then added this surface to the boundary, or if it enclosed the cylindrical end, we could have used it as the boundary. Either way, equality would be impossible.

The key property of weak IMCF is the monotonicity of the charged Hawking mass. The charged Hawking mass is defined by

$$M_{CH}(S) = \sqrt{\frac{|S|}{16\pi}}\left(1 + \frac{4\pi q^2}{|S|} - \frac{1}{16\pi}\int_S H^2\right). \tag{132}$$

If we denote the surfaces of the IMCF by S_τ then the charged Hawking mass satisfies the well known property

$$\begin{aligned}\frac{d}{d\tau}M_{CH}(S_\tau) &= -\frac{1}{2}\sqrt{\frac{\pi}{|S_\tau|}}q^2 + \sqrt{\frac{|S_\tau|}{16\pi}}\left(\frac{1}{2} - \frac{1}{4}\chi(S_\tau)\right)\\ &+\frac{1}{16\pi}\sqrt{\frac{|S_\tau|}{16\pi}}\int_{S_\tau}\left(2\frac{|\nabla_{S_\tau}H|^2}{H^2} + |II|^2 - \frac{1}{2}H^2 + R\right)\end{aligned} \tag{133}$$

which holds in the appropriate weak sense. Here II denotes the (weak) second fundamental form [28], [18].

Proof of Theorem 1: The Case of Equality Assume equality holds in 2. We will use our IMCF to obtain smooth coordinates which show that our initial data set is diffeomorphic to extreme Reissner-Nordström. Take the u solving the weak IMCF which we constructed earlier. We point out a few facts about the resulting flow. First, by the outer-minimizing condition and (131) we have $|S_\tau| > A$. It is also clear we have $|S_\tau| \to A$ as we decrease τ.

Furthermore, our solution u only takes values $\tau \in (c, \infty)$ where c is some finite real number. Notice that this is to be expected. As we move down the AC end, the mean curvature of the flow surfaces S_τ decreases, which means that the speed with which they move increases, and so we foliate the infinite AC region in a finite amount of time. This can be seen as follows. It is well [28] known that the surfaces satisfy the exponential growth condition

$$|S_\tau| = |S_{\tau_0}|e^{\tau-\tau_0} \tag{134}$$

and so if we take some surface with fixed τ_0 we have that

$$|S_{\tau_0}|e^{\tau-\tau_0} > A \tag{135}$$

and so $\tau > \ln(A/|S_{\tau_0}|) + \tau_0$. Therefore by normalizing we can assume $\tau \in (0, \infty)$.

As mentioned earlier, the surfaces of the flow are locally $C^{1,\alpha}$ for all τ and so the monotonicity formula holds. After normalizing, we have $\lim_{\tau\to 0^+}|S_\tau| = A$. We also need to verify that $\lim_{\tau\to 0^+}\int_{S_\tau}H^2 \to 0$. To do so, consider the surface S_τ with $|S_\tau|$ for fixed τ. Also consider the surfaces $S_{n,\tau}$. These are the surfaces of weak IMCF which start from the $t = n$ cross-section. Let τ_n be the parameter for which $S_{n,\tau_n} = \{t = n\}$ (this parameter is not 0 since we normalized our functions as above). We see that $\tau_n > 0$. In Section 5 of [28] we find the estimate

$$\frac{d}{d\tau}\int_{S_\tau}H^2 \le 8\pi \tag{136}$$

and therefore we have

$$\int_{S_{n,\tau}} H^2 \leq 8\pi(\tau - \tau_n) + \int_{S_{n,\tau_n}} H^2 \leq 8\pi\tau + \int_{S_{n,\tau_n}} H^2. \tag{137}$$

Taking the limit $n \to \infty$ we find that

$$\int_{S_\tau} H^2 \leq 8\pi\tau \tag{138}$$

and finally taking the limit $\tau \to 0^+$ we obtain the desired conclusion.

Next we argue that the flow is smooth for all τ. The argument is almost a verbatim repetition of the argument in Section 8 of [28]. We repeat it merely for completeness. By the charged dominant energy condition, (133) is greater than or equal to 0. To see this, notice that by Hölder's inequality and the charged dominant energy condition

$$\begin{aligned} 16\pi^2 q^2 &= \left(\int_{S_\tau} E_j\nu^j\right)^2 + \left(\int_{S_\tau} B_j\nu^j\right)^2 \leq \left(\int_{S_\tau} |E_j\nu^j|\right)^2 + \left(\int_{S_\tau} |B_j\nu^j|\right)^2 \\ &\leq \left(\int_{S_\tau} |E|\right)^2 + \left(\int_{S_\tau} |B|\right)^2 \leq |S_\tau| \int_{S_\tau} (|E|^2 + |B|^2) \leq \frac{|S_\tau|}{2} \int_{S_\tau} R \end{aligned} \tag{139}$$

which implies that

$$-\frac{1}{2}\sqrt{\frac{\pi}{|S_\tau|}}q^2 + \frac{1}{16\pi}\sqrt{\frac{|S_\tau|}{16\pi}} \int_{S_\tau} R \geq 0 \tag{140}$$

while $|II|^2 - (1/2)H^2$ is nonnegative (in the weak sense) [28].

Hence in order for equality to occur, (133) must equal 0 for almost every τ. By Lemma 5.1 in [28] we have $H > 0$ a.e. on on S_τ for a.e. τ, and so $\int_{S_\tau} |\nabla_{S_\tau} H|^2 = 0$ for a.e. τ and therefore by (1.10) in [28] and lower semicontinuity

$$\int_{S_\tau} |\nabla_{S_\tau} H|^2 = 0 \quad \text{for all} \quad \tau \tag{141}$$

Therefore

$$H_{S_\tau}(x) = H(\tau) \quad \text{for a.e. } x \in S_\tau, \quad \text{for all } \tau \geq 0 \tag{142}$$

so that each S_τ has constant mean curvature. Since H is locally bounded and S_τ has locally uniform C^1 estimates, it follows by elliptic theory that each S_τ is smooth. Furthermore, the flow does not jump at any τ since that would contradict the assumption that our initial data set does not contain any compact minimal surfaces. Hence, $H > 0$ for $\tau > 0$ which implies $H(\tau)$ is locally uniformly positive for $\tau > 0$.

By Lemma 2.4 of [28], for each $s > 0$ there is some maximal T such that the flow $(S_\tau)_{s\leq\tau\leq T}$ is smooth. By the above regularity, S_τ has uniform space and time derivatives as $\tau \nearrow T$ and so the evolution can be smoothly continued past $\tau = T$. Hence $T = \infty$ and the flow is smooth everywhere.

The next part follows from the arguments in [18] (see also [1,37]). Since the flow is smooth and (133) vanishes we must have that equality holds everywhere in (139). In particular

$$R = 2(|E|^2 + |B|^2), \quad E_j\nu^j = constant, \quad B_j\nu^j = constant, \quad E = e(\tau)\nu, \quad B = b(\tau)\nu \tag{143}$$

on each S_τ. Furthermore, since $|II|^2 - \frac{1}{2}H^2 = (\lambda_1 - \lambda_2)^2$, we must have that $\lambda_1 = \lambda_2$ at each point in S_τ (where λ_i are the principal curvatures). This then shows that $|II|^2 = \frac{1}{2}H^2$ is constant on each S_τ.

By equation (1.3) in [28]

$$\partial_\tau H = -\Delta_{S_\tau}(H^{-1}) - (|II|^2 + \mathrm{Ric}(\nu,\nu))H^{-1} \tag{144}$$

which (since $H > 0$ and constant on each S_τ) means

$$\partial_\tau H = (|II|^2 + \mathrm{Ric}(\nu,\nu))H^{-1} \tag{145}$$

which then (since H and $|II|$ only depend on τ) implies $\mathrm{Ric}(\nu,\nu) = \textit{constant}$ on each S_τ. Taking two traces of the Gauss equation and solving for Gaussian curvature K of S_τ we obtain (c.f 5.5 in [18])

$$K = \frac{1}{2}R - \mathrm{Ric}(\nu,\nu) + \frac{1}{2}H^2 - \frac{1}{2}|II|^2 \tag{146}$$

which shows that the Gaussian curvature is constant on each S_τ. Therefore each S_τ is isometric to a round sphere with metric $r^2(\tau)d\sigma^2$ for the function $r(\tau)$ defined by the relation $4\pi r^2(\tau) = A(S_\tau) = Ae^\tau$. By noting the fact that $d\tau = 2r^{-1}dr$ and using the Gauss lemma, the metric can be written in the form

$$g = H^{-2}d\tau^2 + g|_{S_\tau} = \frac{4H^{-2}}{r^2}dr^2 + r^2 d\sigma^2. \tag{147}$$

Since $M_{CH}(S_\tau) = m$ for all τ we can solve for H^2 to find

$$H^2 = \frac{4}{r^2}\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) \tag{148}$$

and combining with (147) we obtain the metric is Reissner-Nordström. Since it has a cylindrical end, it must be extreme, with $m = |q|$.

A simple calculation then shows that

$$E = \frac{q_e}{r^2}\nu_r \quad \text{and} \quad B = \frac{q_b}{r^2}\nu_r \tag{149}$$

where ν_r is the outward unit normal to the coordinate spheres, completing the proof. □

11 Proof of Theorem 2

The idea is to use the same methods as those in the proof of Theorem 1. However, we no longer need to assume the electromagnetic fields are divergence free or that the generalized boundary is outer-minimizing. Begin with our initial data set (M, g, E, B) and conformaly deform it by the same procedure as in section 5 to obtain an initial data set satisfying the strict charged dominant energy condition. We denote this deformed data set by $(M_\varepsilon, g_\varepsilon, E_\varepsilon, B_\varepsilon)$ which has ADM mass m_ε satisfying $|m_\varepsilon - m| \leq C\varepsilon$ and $|q_\varepsilon| = |q|$.

Since we don't need the electro-magnetic fields to be divergence free, we don't need to apply any of the techniques found in section 7. Instead, we perform the gluing as in section 6 and restore the CDEC by solving the elliptic problem of section 8 (in fact, the function ζ can now be chosen to be supported in $\Sigma(T)$).

The set $\mathcal{F}$ is a minimal surface, and so we can take the outermost minimal surface with respect to the end $\hat{M}^+$. We then apply the ordinary positive mass theorem with charge [22, 34] to obtain

$$m_{\varepsilon,T} \geq |q| \tag{150}$$

and taking the limit $T \to \infty$ we obtain

$$m_\varepsilon \geq |q|. \tag{151}$$

Finally, taking the limit $\varepsilon \to 0$ we obtain

$$m \geq |q|. \tag{152}$$

□

This gives an alternate proof to the same result which was obtained in [1] using spinorial methods. However, we remark that our techniques should readily generalize to the non-time symmetric case.

12 Proof of Theorem 3: Bekenstein Bounds

Consider an initial data set (M, g, k, E, B) satisfying the charged dominant energy condition $|\mu_{EM}| \geq |J_{EM}|$. A body Ω will be described as a connected open subset of M having compact closure and smooth boundary $\partial\Omega$. The total charge within the body is then given by

$$q^2 = \left(\frac{1}{4\pi}\int_\Omega \operatorname{div} E\right)^2 + \left(\frac{1}{4\pi}\int_\Omega \operatorname{div} B\right)^2, \tag{153}$$

and it will always be presumed that there is no charged matter outside Ω. In order to characterize the angular momentum of the body, the initial data will be assumed to be axisymmetric. That is, there is a $U(1)$ subgroup within the group of isometries of the Riemannian manifold (M, g), and all relevant quantities are invariant under the $U(1)$ action. Without axisymmetry it is problematic to define quasi-local angular momentum [46]. Moreover, with this hypothesis all angular momentum is contained within the matter fields, as gravitational waves carry no angular momentum. Let η be the generator of the $U(1)$ symmetry, then the angular momentum of the body is

$$\mathcal{J} = \int_\Omega J_i \eta^i. \tag{154}$$

The basic strategy to obtain Bekenstein type bounds (8) is to use monotonicity of the Hawking mass along inverse mean curvature flow. This worked well in [2, 3, 17] because of the maximal assumption $Tr_g k = 0$. More precisely, monotonicity of the Hawking mass relies on nonnegativity of the scalar curvature, and this is achieved with the dominant energy condition if the data are maximal. Here we do not assume that the data are maximal, and thus this method breaks down. However, we may follow an approach similar to that in the proof of the positive mass theorem [44], where the initial data are deformed by $(M, g, k) \to (M, \bar{g})$ with $\bar{g}_{ij} = g_{ij} + u^2 f_i f_j$ for some functions $u > 0$ and f. In [44] the function $u = 1$ and f is chosen to solve the so called Jang equation, which is designed to impart positivity properties to the scalar curvature $\overline{R}$ of $\bar{g}$. In the present setting it is more appropriate to utilize an embellished version of the Jang equation

$$\left(g^{ij} - \frac{u^2 f^i f^j}{1 + u^2|\nabla f|^2}\right)\left(\frac{u\nabla_{ij} f + u_i f_j + u_j f_i}{\sqrt{1 + u^2|\nabla f|^2}} - k_{ij}\right) = 0, \tag{155}$$

where ∇_{ij} are second covariant derivatives with respect to g and $f^i = g^{ij} f_j$. This equation also yields desirable features for the scalar curvature which now takes the form

$$\overline{R} = 16\pi(\mu - J(w)) + |h - k|^2_{\bar{g}} + 2|Q|^2_{\bar{g}} - 2u^{-1}\operatorname{div}_{\bar{g}}(uQ), \tag{156}$$

where

$$h_{ij} = \frac{u\nabla_{ij} f + u_i f_j + u_j f_i}{\sqrt{1 + u^2|\nabla f|^2}}, \qquad w_i = \frac{u f_i}{\sqrt{1 + u^2|\nabla f|^2}}, \qquad Q_i = \frac{u f^j}{\sqrt{1 + u^2|\nabla f|^2}}(h_{ij} - k_{ij}). \tag{157}$$

These formulas along with their geometric interpretations are explained in [13, 14]. Observe that the first term on the right-hand side of (156) is nonnegative if the dominant energy condition is satisfied, since $|w| \leq 1$. Furthermore, all other terms are manifestly nonnegative except possibly the divergence term. The deformed scalar curvature may then be described as 'weakly' nonnegative, since integrating it against u produces a nonnegative quantity modulo boundary terms.

In order to optimize the positivity of $\overline{R}$ with regards to IMCF we choose u as follows. Let $\{\overline{S}_t\}_{t=t_0}^{\infty}$ be an IMCF in the deformed data $(M, \bar{g})$, where $t_0 = 0$ or $-\infty$ depending on whether the flow starts at a surface or a point. A weak version of the flow always exists [28] in the asymptotically flat setting, although for the purposes of exposition we may assume that the flow is smooth. Then set $u = \sqrt{|\overline{S}_t|/16\pi}\,\overline{H}$ to be the product of the square root of area and mean curvature for the flow surfaces. Consider now the Hawking mass of the flow surfaces within the deformed data

$$M_H(\overline{S}_t) = \sqrt{\frac{|\overline{S}_t|}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\overline{S}_t} \overline{H}^2\right). \tag{158}$$

By the monotonicity formula for the ordinary Hawking mass [28], if $t_2 > t_1$ then

$$M_H(\overline{S}_{t_2}) - M_H(\overline{S}_{t_1}) \geq \frac{1}{16\pi}\int_{t_1}^{t_2} \sqrt{\frac{|\overline{S}_t|}{16\pi}} \int_{\overline{S}_t} \overline{R}. \tag{159}$$

The first two terms on the right-hand side in the expression (156) will provide lower bounds for (159) involving the charge and angular momentum, while the divergence expression will contribute to the Hawking energies.

Consider now the case when the flow starts from a point x_0 within the body Ω on the axis of rotation, so that the starting time is $t_0 = -\infty$. Observe that in (159) with $t_1 = -\infty$ and $t_2 = \infty$ several simplifications occur. Namely, since the Hawking mass of a point is zero and the limit of Hawking masses as $t \to \infty$ is no larger than the ADM mass, the left-hand side of (159) may be replaced with the ADM mass m. Note that this total mass is a priori with respect to the deformed metric $\bar{g}$. However by placing the natural boundary conditions at infinity for solutions of the Jang equation, namely $f \to 0$ in the asymptotic end, the total mass of g and $\bar{g}$ are equivalent [44]. Furthermore if the charged dominant energy condition holds then $\mu - J(w) \geq \frac{1}{8\pi}(|E|^2 + |B|^2)$, as it may be assumed without loss of generality in axisymmetry that the electric and magnetic fields have no component in the Killing direction so that $E \times B(w) = 0$. In [18] a deformation of the electromagnetic field $(E, B) \to (\overline{E}, \overline{B})$, tailored to the Jang metric $\bar{g}$, was given which preserves total charge as well as zero charge density and has less energy density than the original field $|E| \geq |\overline{E}|$, $|B| \geq |\overline{B}|$. From this a lower bound for the right-hand side of (159) is obtained in terms of the energy density of $(\overline{E}, \overline{B})$, and since the surface integrals are computed with respect to $\bar{g}$ a relation with total charge is produced as in [18]. In particular

$$\int_{-\infty}^{\infty} \sqrt{\frac{|\overline{S}_t|}{16\pi}} \int_{\overline{S}_t} (\mu - J(w)) \geq \frac{1}{8\pi}\int_{t_*}^{\infty} \sqrt{\frac{|\overline{S}_t|}{16\pi}} \int_{\overline{S}_t} (|\overline{E}|^2 + |\overline{B}|^2) \geq \frac{q^2}{2\overline{\mathcal{R}}_{t_*}}, \tag{160}$$

where $\overline{\mathcal{R}}_{t_*} = \sqrt{|\overline{S}_{t_*}|/4\pi}$ is the area radius of $\overline{S}_{t_*}$. The time t_* may be chosen arbitrarily, however in order to obtain the optimal inequality for the body, t_* will denote the first (smallest) time such that the flow surface $\overline{S}_{t_*}$ completely encloses Ω. Moreover since the flow will change depending on the choice of its starting point x_0, optimization requires that we choose the x_0 for which the area radius at t_* is smallest. Such a starting point exists within the body since Ω is compact. The radius $\overline{\mathcal{R}}$ of Ω will then be defined as in [17] to be this optimal area radius, and in (160) the radius $\overline{\mathcal{R}}_{t_*}$ may be replaced with $\overline{\mathcal{R}}$.

Within the scalar curvature formula (156) the second term on the right-hand side encodes a contribution from angular momentum. In order to extract this contribution we first make some observations. The metric $\bar{g}$ arises as the induced metric on the graph of the function f [14], and the surfaces $\overline{S}_t$ may be interpreted as a flow within the graph. There is then a natural projection of $\overline{S}_t$ into (M,g) which will be denoted S_t. Since the flow starts from a point on the symmetry axis, each of the surfaces $\overline{S}_t$, S_t is axisymmetric. As is shown in the appendix under mild hypotheses, it then follows that $h(\eta,\nu)=0$ on S_t, where ν is the unit normal to S_t. Therefore assuming that angular momentum density vanishes outside the body and using Hölder's inequality produces

$$\begin{aligned}(8\pi)^2\mathcal{J}^2=\left(\int_{S_t}k(\eta,\nu)\right)^2&=\left(\int_{S_t}[k(\eta,\nu)-h(\eta,\nu)]\right)^2\\&\leq\left(\int_{S_t}|k-h|_g|\eta|\right)^2\leq\int_{\overline{S}_t}|k-h|^2_{\bar{g}}\int_{\overline{S}_t}|\eta|^2,\end{aligned}\tag{161}$$

where we have also used the fact that $\bar{g}$ measures areas to be at least as large as does g. This estimate is suited to give a lower bound for the ADM mass which may be expressed properly with the 'circumference' radius

$$\overline{\mathcal{R}}_c^{-2}=\sqrt{|\overline{S}_{t_*}|}\int_{t_*}^{\infty}\frac{\sqrt{|\overline{S}_t|}}{\int_{\overline{S}_t}|\eta|^2}.\tag{162}$$

The radius $\overline{\mathcal{R}}_c$ was used and studied in [2, 3], where it was shown that if the flow has reasonably nice properties then this radius may be related to more traditional measures of size for the body. In particular if the flow remains convex outside of Ω, as it is known to be for large times $|t|>>0$ or in spherical symmetry, then $\overline{\mathcal{R}}_c\leq\sqrt{5/2}\max_{S_{t_*}}|\eta|$ which is proportional to the circumference of the largest orbit within S_{t_*}. Because it provides an upper bound for $\overline{\mathcal{R}}_c$, when the flow is convex the circumference may be used in place of the this radius in

$$\frac{1}{16\pi}\int_0^{\infty}\sqrt{\frac{|\overline{S}_t|}{16\pi}}\int_{\overline{S}_t}|h-k|^2_{\bar{g}}\geq\frac{1}{2}\frac{\mathcal{J}^2}{\overline{\mathcal{R}}\,\overline{\mathcal{R}}_c^2}.\tag{163}$$

It is now possible to combine (159), (160), and (163) to obtain the Bekenstein-type bound of Theorem 3. □

Note that the proof above relies on the existence of a solution to the Jang equation coupled to IMCF through the choice of the function u. Due to the fact that solutions to the Jang equation tend to blow-up at apparent horizons [24], it will be assumed that the initial data are devoid of these surfaces. Under this hypothesis, the desired solutions to the Jang/IMCF system have been shown to always exist in spherical symmetry [13], and it is reasonable to expect that existence continues to hold at least in a weak sense in axisymmetry.

This theorem generalizes the results of [3, 17] to the non-maximal setting. Although it is in the spirit of the Bekenstein bound (8), these two inequalities are distinct in that one does not directly imply the other. Nevertheless, as will be shown in the next section inequality (9) does indirectly imply a lower bound for m^2 which has the same structure as (8).

13 Proof of Theorem 4: Penrose-like Inequalities

In this section we will adapt the techniques discussed in the previous section to establish a version of the Penrose inequality with angular momentum and charge (10). This will then yield an alternate version of the Bekenstein bound (9). Recall that an apparent horizon is a surface $S \subset M$ which has zero null expansion, that is, a shell of light emitted from the surface is (infinitesimally) neither growing nor shrinking in area as it leaves the surface. These surfaces indicate the presence of a strong gravitational field, and may be interpreted as quasi-local versions of black hole event horizons from the initial data point of view. Mathematically they are expressed by one of the two equations $\theta_{\pm} := H \pm Tr_S k = 0$, where the signs $+/-$ indicate a future/past horizon. An apparent horizon is called outermost within an initial data set if it is not enclosed by any other apparent horizon.

In contrast to the previous section, here we will work with an IMCF starting at a closed axisymmetric surface S so that $t_0 = 0$ is the starting time of the flow, and S will either be an outermost apparent horizon or the boundary of a body $\partial\Omega$. First consider the case in which $S = \partial\Omega$, and assume that the boundary of the body is completely untrapped $H > |Tr_S k|$. This allows for the prescription of a Neumann type boundary condition for solutions of the Jang equation (155)

$$\frac{u\partial_\nu f}{\sqrt{1+u^2|\nabla f|^2}} = H^{-1}Tr_S k. \tag{164}$$

It was shown in [13,31], in the context of spherical symmetry, that solutions of the Jang/IMCF system exist satisfying this boundary condition. Moreover it was also shown that with (164) the boundary integrals arising from the divergence expression associated with $\overline{R}$ in (159), combine with the Hawking mass on the left-hand side of (159), to yield

$$m - M_{SH}(S) \geq \int_0^\infty \sqrt{\frac{|\overline{S}_t|}{16\pi}} \int_{\overline{S}_t} \left((\mu - J(w)) + \frac{1}{16\pi}|h-k|_{\bar{g}}^2 \right) \tag{165}$$

where the spacetime Hawking mass is given by

$$M_{SH}(S) = \sqrt{\frac{|S|}{16\pi}} \left(1 - \frac{1}{16\pi}\int_S \theta_+\theta_- \right). \tag{166}$$

It should be pointed out that (165) depends on appropriate behavior of the IMCF. For instance in the weak formulation of Huisken/Ilmanen [28], the flow may instantaneously jump from the desired starting surface S to another surface $\tilde{S}$ enclosing it with less area. If this occurs, then in inequality (165) the role of S should be replaced by $\tilde{S}$. Such 'jumping' behavior can be prevented by requiring that S be outer area minimizing in $(M, \bar{g})$, in that any surface which encloses S should have greater area. In order to achieve this property with respect to the deformed data metric $\bar{g}$, further geometric hypotheses on S with respect to the original initial data may be required. For the purposes of the present work, which does not seek to fully examine the analytical problem of solving the Jang/IMCF system in generality, we will simply refer to solutions with these suitable properties as *proper solutions*. As pointed out, it is known that proper solutions always exist under the hypothesis of spherical symmetry and small perturbations thereof.

As in the previous section, the two terms on the right-hand side of (165) yield contributions of angular momentum and charge. More precisely, applying (160) and (163) produces

$$m \geq M_{SH}(S) + \frac{q^2}{2\overline{\mathcal{R}}_0} + \frac{1}{2}\frac{\mathcal{J}^2}{\overline{\mathcal{R}}_0\overline{\mathcal{R}}_c^2} \tag{167}$$

where $\overline{\mathcal{R}}_0$ is the area radius of $S_0 = S$, and $\mathcal{J}$, q denote the angular momentum and charge contained within S. This inequality will lead to a Bekenstein bound for bodies in the presence of a sufficiently strong gravitation field, as well as a version of the Penrose inequality.

The arguments above seem to rely on the assumption that S is untrapped, as otherwise the boundary condition (164) would imply that $u\partial_\nu f = \pm\infty$. However for the Jang equation, blow-up solutions are natural as first observed in the proof of the positive mass theorem [44]. Blow-up occurs at apparent horizons, and can be prescribed at outermost apparent horizons as well [24]. Therefore in place of the boundary condition (164), at an outermost apparent horizon S we will prescribe blow-up as the boundary condition. In this situation the graph of the solution to Jang's equation asymptotes to a cylinder over S, and the area of this surface in the deformed metric and the original coincide $|\overline{S}| = |S|$. Moreover, at an apparent horizon $\theta_+\theta_- = 0$ so that $M_{SH}(S) = \sqrt{|S|/16\pi}$.

Upon multiplying (167) by the first two terms on the right-hand side we find

$$m^2 \geq \left(M_{SH}(S) + \frac{q^2}{2\overline{\mathcal{R}}_0}\right) m \geq \left(M_{SH}(S) + \frac{q^2}{2\overline{\mathcal{R}}_0}\right)^2 + M_{SH}(S)\frac{1}{2}\frac{\mathcal{J}^2}{\overline{\mathcal{R}}_0\overline{\mathcal{R}}_c^2}. \tag{168}$$

From this the desired inequality in Theorem 4 arises from the arguments above. □

This result is similar to the conjectured Penrose inequality (10) with the primary difference arising in the angular momentum term. Instead of area, this term involves the squared radius defined in the previous section.

Furthermore (168) may be used to yield a Bekenstein bound. Suppose that $S = \partial\Omega$ is the boundary of a body immersed in a strong gravitational field. By this we mean that $\lambda := 1 - (|S|/16\pi)\sup_S \theta_+\theta_- > 0$, or rather that $\theta_+\theta_-$ has sufficiently small positive part. In particular surfaces S which are close to being an apparent horizon satisfy this property, as do trapped surfaces. For surfaces S which satisfy this property, the spacetime Hawking energy is bounded below by the product of λ and the area radius up to a universal constant. Let $\lambda_0 > 0$ be fixed and consider the class of bodies with boundaries experiencing a appropriately strong gravitational field so that $\lambda \geq \lambda_0$. Then for bodies of this type a Bekenstein bound follows immediately from (168)

$$m^2 \geq \frac{q^4}{4\overline{\mathcal{R}}_0^2} + \lambda_0\frac{\mathcal{J}^2}{4\overline{\mathcal{R}}_c^2}. \tag{169}$$

This inequality has the same structure as the Bekenstein inequality (8), although the radius associated with the angular momentum term is more complicated than the area radius.

14 Proof of Theorem 5: An Inequality For Extended Bodies

Consider two concentric bodies $\Omega_1 \subset \Omega_2$, each having the topology of a 3-dimensional ball, inside an axisymmetric asymptotically flat initial data set (M, g, k, E, B). The model astronomical body in this context is a typical star, where there is a highly dense core and interior (represented by Ω_1) compared to the outermost layer or corona (represented by $\Omega_2 \setminus \Omega_1$) with very little matter density. For simplicity of the model we will assume that the charge density and momentum density in the Killing direction vanish in the annular region $\Omega_2 \setminus \Omega_1$, so that $\operatorname{div} E = \operatorname{div} B = J(\eta) = 0$. If there are no apparent horizons in the initial data, then as discussed in Section **??** we may take a solution of the Jang/IMCF system of equations with the flow emanating from a point $x_0 \in \Omega_1$ on the axis of rotation. Let t_1 and t_2 be the first times for which the flow completely encloses the boundaries $\partial\Omega_1$ and $\partial\Omega_2$, respectively. From the arguments used to obtain (165), together with the fact that the Hawking energy of a point is zero, we find that

$$M_{SH}(S_{t_2}) \geq \int_{-\infty}^{t_2} \sqrt{\frac{|\bar{S}_t|}{16\pi}} \int_{\bar{S}_t} \left((\mu - J(w)) + \frac{1}{16\pi} |h - k|_{\bar{g}}^2 \right) \tag{170}$$

if the Jang solution f is prescribed to be zero (or more generally constant) on S_{t_2}. Note that this boundary condition differs from (164) which is used to obtain (165). This is due to the fact that the boundary integrals that arise from the divergence term in $\overline{R}$ have different signs on the inner and outer boundaries [31]. In fact the boundary terms at the outer boundary have an advantageous sign, and it is likely that this Dirichlet boundary condition used for (170) is not needed.

Proceeding as in the previous sections, lower bounds for the right-hand side of (170) may be extracted in terms of the total charge and angular momentum of Ω_1. In addition, a contribution from the nonelectromagnetic matter fields will also occur. To see this observe that as in (160)

$$\begin{aligned} \int_{-\infty}^{t_2} \sqrt{\frac{|\bar{S}_t|}{16\pi}} \int_{\bar{S}_t} (\mu - J(w)) \geq & \int_{-\infty}^{t_1} \sqrt{\frac{|\bar{S}_t|}{16\pi}} \int_{\bar{S}_t} (\mu_{EM} - J_{EM}(w)) + \frac{1}{8\pi} \int_{t_1}^{t_2} \sqrt{\frac{|\bar{S}_t|}{16\pi}} \int_{\bar{S}_t} \left(|\overline{E}|^2 + |\overline{B}|^2 \right) \\ \geq & \frac{4\pi}{3} \overline{\mathcal{R}}_1^3 \min_{\tilde{\Omega}_1} (\mu_{EM} - |J_{EM}|) + \frac{q^2}{2\overline{\mathcal{R}}_1} \left(1 - \sqrt{\frac{\overline{\mathcal{R}}_1}{\overline{\mathcal{R}}_2}} \right), \end{aligned} \tag{171}$$

where $\tilde{\Omega}_1$ is the domain enclosed by $\overline{S}_{t_1}$ and $\overline{\mathcal{R}}_1$, $\overline{\mathcal{R}}_2$ are the area radii of $\overline{S}_{t_1}$, $\overline{S}_{t_2}$. Notice that if the charged dominant energy condition is valid then the first term on the right in (171) is nonnegative, and the second term also has this property since areas are nondecreasing in an IMCF. Similarly, applying the arguments of (163) to the current setting produces

$$\frac{1}{16\pi} \int_{-\infty}^{t_2} \sqrt{\frac{|\bar{S}_t|}{16\pi}} \int_{\bar{S}_t} |h - k|_{\bar{g}}^2 \geq \frac{1}{2} \frac{\mathcal{J}^2}{\overline{\mathcal{R}}_1 \overline{\mathcal{R}}_{ac}^2}, \tag{172}$$

where the circumference radius is with respect to the annular domain

$$\overline{\mathcal{R}}_{ac}^{-2} = \sqrt{|\bar{S}_{t_1}|}\int_{t_1}^{t_2} \frac{\sqrt{|\bar{S}_t|}}{\int_{\bar{S}_t}|\eta|^2}. \tag{173}$$

Furthermore assuming that the outer surface S_{t_2} is untrapped, so that $H > |Tr_{S_{t_2}}k|$, implies that the Hawking mass may be estimated above by the area radius $M_{SH}(S_{t_2}) \leq \frac{1}{2}\overline{\mathcal{R}}_2$. Therefore combining (170), (171), and (172) yields

$$\frac{1}{2}\overline{\mathcal{R}}_2 \geq \frac{4\pi}{3}\overline{\mathcal{R}}_1^3 \min_{\tilde{\Omega}_1}(\mu_{EM} - |J_{EM}|) + \frac{q^2}{2\overline{\mathcal{R}}_1}\left(1 - \sqrt{\frac{\overline{\mathcal{R}}_1}{\overline{\mathcal{R}}_2}}\right) + \frac{1}{2}\frac{\mathcal{J}^2}{\overline{\mathcal{R}}_1\overline{\mathcal{R}}_{ac}^2}. \tag{174}$$

□

The geometric inequality (174) relates the size of the body $\Omega_2 \supset \Omega_1$ to its core nonelectromagnetic matter content, total charge, and total angular momentum. It may be interpreted as stating that a material body of fixed size can only contain a certain fixed amount of matter energy, charge, and angular momentum. The primary hypotheses which were used to derive this inequality consist of the assumption that the outer region is untrapped, the annular region $\Omega_2 \setminus \Omega_1$ has no charge and momentum density in the Killing direction, and most importantly that the initial data are void of apparent horizons. This latter assumption is used to obtain regular solutions of the Jang equation, and following [45] we may turn this around to obtain a black hole existence result.

It is a basic folklore belief that if enough matter/energy is concentrated in a sufficiently small region, then gravitational collapse must ensue. This is typically referred to as the hoop conjecture or trapped surface conjecture [43, 48], and is quite difficult to formulate precisely, see the references in [38]. One of the most general results in this direction is due to Schoen and Yau [45], who exploited the techniques developed in their proof of the positive mass theorem [44] to prove the existence of apparent horizons whenever matter density is highly concentrated. Their strategy was to show that the concentration hypothesis forces solutions of the Jang equation to blow-up, and since blow-up can only occur at an apparent horizon the existence of such a surface in the initial data is established.

In our case, if a body with the hypotheses above, minus any assumption on apparent horizons, satisfies

$$\frac{1}{2}\overline{\mathcal{R}}_2 < \frac{4\pi}{3}\overline{\mathcal{R}}_1^3 \min_{\tilde{\Omega}_1}(\mu_{EM} - |J_{EM}|) + \frac{q^2}{2\overline{\mathcal{R}}_1}\left(1 - \sqrt{\frac{\overline{\mathcal{R}}_1}{\overline{\mathcal{R}}_2}}\right) + \frac{1}{2}\frac{\mathcal{J}^2}{\overline{\mathcal{R}}_1\overline{\mathcal{R}}_{ac}^2} \tag{175}$$

then an apparent horizon must be present within the initial data. The reasoning is that if there were no apparent horizons, then we may apply the arguments above to conclude that (174) holds, a contradiction.

This relies on the analysis of the Jang/IMCF system of equations, which has been established rigorously in the case of spherical symmetry [13]. This conclusion concerning the existence of an apparent horizon implies that the spacetime arising from the initial data contains a singularity, or more accurately is null geodesically incomplete by the Hawking-Penrose singularity theorems [25], and assuming cosmic censorship it must therefore possess a black hole. This result may be interpreted as stating that if a body of fixed size contains sufficient amounts of nonelectromagnetic matter energy, charge, or angular momentum, then it must collapse to form a black hole.

In addition, we would prefer to have a result that only depends on the initial data, without having to solve the generalized Jang equation. Indeed, if the initial data is time symmetric (175) simplifies to

$$\frac{1}{2}\mathcal{R}_2 < \frac{4\pi}{3}\mathcal{R}_1^3 \min_{\Omega_1}(\mu_{EM}) + \frac{q^2}{2\mathcal{R}_1}\left(1 - \sqrt{\frac{\mathcal{R}_1}{\mathcal{R}_2}}\right) \tag{176}$$

and so any data satisfying this inequality must contain a minimal surface.

A Vanishing Extrinsic Curvature

Consider an axisymmetric closed surface S within an axisymmetric Riemannian 3-manifold (M, g). If η is the generator for the axisymmetry and ν is the unit normal to S, then under mild hypotheses $h(\eta, \nu) = 0$ along S, where h is the tensor associated with the solution f of Jang's equation and is given in (157). Geometrically the tensor h represents the extrinsic curvature of the graph of f in a static spacetime constructed from the metric $\bar{g}$ and function u [14]. The vanishing of this particular component of h is used throughout the main body of the paper in order to allow for angular momentum contributions to the various inequalities. Here we will confirm this property of h.

In [16] it was shown that if M is asymptotically flat and simply connected then a global cylindrical coordinate system exists, denoted by (ρ, z, ϕ) and referred to as Brill coordinates, such that the metric takes the following form

$$g = e^{-2U+2\alpha}(d\rho^2 + dz^2) + \rho^2 e^{-2U}(d\phi + Ad\rho + Bdz)^2 \tag{177}$$

for some functions U, α, A, and B all depending only on (ρ, z). The Killing field is given by $\eta = \partial_\phi$, and if $U = \alpha = A = B = 0$ then g reduces to the typical expression of the flat metric on Euclidean 3-space in cylindrical coordinates. For simplicity it will be assumed that $A = B = 0$ so that η is perpendicular to the orbit space or ρz-half plane. Observe that since u and f are axisymmetric, that is $\partial_\phi f = \partial_\phi u = 0$, it follows that

$$h(\eta, \nu) = \frac{u\nabla_{\phi\nu} f}{\sqrt{1 + u^2|\nabla f|^2}} = -\frac{u\left(\Gamma^\rho_{\phi\nu}\partial_\rho f + \Gamma^z_{\phi\nu}\partial_z f\right)}{\sqrt{1 + u^2|\nabla f|^2}} \tag{178}$$

where the Γ^l_{ij} are Christoffel symbols. Since the surface is axisymmetric ∂_ϕ is tangent to S, and thus $g(\eta, \nu) = 0$. A straightforward calculation then yields

$$\Gamma^\rho_{\phi\nu} = \frac{1}{2} g^{\rho i}\partial_\nu g_{\phi i} = 0, \qquad \Gamma^z_{\phi\nu} = \frac{1}{2} g^{zi}\partial_\nu g_{\phi i} = 0, \tag{179}$$

and the desired conclusion follows.

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