Monge-Ampère equation on the complement of a divisor
&
On the Chern-Yamabe flow

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Abstract of the Dissertation

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In this dissertation we discuss two seperate topics. In the first part we consider the complex Monge-Ampère equation on complete Kähler manifolds with cusp singularity along a divisor when the right hand side F has rather weak regularity. We prove a compactness result on the solutions to the ε-perturbed equations of the Monge-Ampère equation when the right hand side F is in some *weighted* W^{1,p_0} space for $p_0 > 2n$ where n is the complex dimension. As an application we show that there exists a classical W^{3,p_0} solution for complex Monge-Ampére equation when F is in the weighted W^{1,p_0}. The key ingredient lies in using the de Giorgi-Nash-Moser theory to derive the uniform estimates of the gradient $\nabla\varphi_\varepsilon$ and the Laplacian $\Delta\varphi_\varepsilon$ in terms of the weighted W^{1,p_0} norm of F.

In the second part we consider the Chern-Yamabe problem of finding constant Chern scalar curvature metrics in the conformal classes. We propose a flow to study the Chern-Yamabe problem and discuss the long time existence of the flow. In the balanced case we show that the Chern-Yamabe problem is the Euler-Lagrange equation of some functional. The monotonicity of the functional along the flow is derived. We also show that the functional is not bounded from below.

To my parents and all my teachers

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Part I

Monge-Ampère equation on the complement of a divisor

1 A brief review of canonical metrics for compact Kähler manifolds

In this section we collect the basic notations of Kähler geometry and briefly review the develop of canonical metrics on the compact Kähler manifolds.

1.1 Kähler manifolds

We begin with Riemannian manifolds. Suppose M is a smooth manifold. A Riemannian metric g on M is a positive definite bilinear form on the tangent bundle TM. Under the local coordinates $(x^1, \cdots, x^n)$, the metric g is locally represented by a smooth matrix valued function $\{g_{ij}\}$, where the matrix with entry $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ is positive definite. The pair (M, g) is called a Riemannian manifold. Recall that the Riemannian manifold (M, g) endows a unique connection which is torsion free and compatible with the Riemannian metric g, namely, the Levi-Civita connection. Let ∇ denote the Levi-Civita connection of g.

Almost complex structure

An almost complex structure on M is an endomorphism $J : TM \to TM$ such that $J^2 = -id$. It is clear that for a Riemannian manifold to endow an almost complex structure, it has to be even dimensional. An almost complex structure is called integrable if there is a set of coordinate charts on M with holomorphic transition functions such that J corresponds to the induced complex multiplication on $TM \otimes \mathbb{C}$. An almost complex structure is not always integrable. An integrable almost complex structure is also called a complex structure. In fact, we have the following theorem due to Newlander-Nirenberg [35].

Theorem 1.1. *An almost complex structure is integrable if and only if the Nijenhuis tensor $N_J : TM \times TM \to TM$*

$$N_J(u, v) := [u, v] + J[Ju, v] + J[u, Jv] - [Ju, Jv] \tag{1.1}$$

vanishes identically.

Given a Riemannian manifold (M, g) with an almost complex structure J, we say that the almost complex structure J is compatible with the Riemannian metric g, if for any tangent vectors $u, v \in TM$

$$g(u, v) = g(Ju, Jv). \tag{1.2}$$

Now we are ready to define the Kähler manifolds.

Definition 1.2. *A Kähler manifold (M, g, J) is a Riemannian manifold (M, g) with a compatible almost complex structure J such that $\nabla J = 0$.*

Notice that $\nabla J = 0$ implies that $N_J = 0$ and thus the almost complex structure J is integrable, hence is a complex structure.

Kähler form

On a Kähler manifold (M, g, J), we can define

$$\omega_g(\cdot, \cdot) = g(J\cdot, \cdot). \tag{1.3}$$

One can derive easily that ω_g is in fact a 2-form on M. We usually call ω_g the Kähler form of g. Since g and J are both parallel with respect to the Levi-Civita connection ∇, it follows that $\nabla\omega_g = 0$, and thus $d\omega_g = 0$. In other words, M admits a symplectic form ω_g such that the almost complex structure J is compatible with ω_g. Conversely, we have the following proposition.

Proposition 1.3. *If (M, g) admits an integrable almost complex structure J which is compatible with the metric g, then $\nabla J = 0$ if and only if $d\omega_g = 0$.*

Proof. The proof of this proposition is purely computational. We refer the interested readers to [38]. □

Curvatures

On a Kähler manifold (M, g, J) with dimension $\dim_{\mathbb{C}} M = n$, it is more convenient to work in local holomorphic coordinates $z^i = x^i + \sqrt{-1}y^i$ for $i = 1, 2, \cdots, n$. Besides the obvious basis $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n}\}$ and $\{dx^1, \cdots, dx^n, dy^1, \cdots, dy^n\}$ of the complexified tangent bundle $TM \otimes \mathbb{C}$ and the complexified cotangent bundle $T^*M \times \mathbb{C}$, we have

$$\frac{\partial}{\partial z^i} = \frac{1}{2}(\frac{\partial}{\partial x^i} - \sqrt{-1}\frac{\partial}{\partial y^i}), \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2}(\frac{\partial}{\partial x^i} + \sqrt{-1}\frac{\partial}{\partial y^i}), \tag{1.4}$$

for $i = 1, 2, \cdots, n$ of $TM \otimes \mathbb{C}$ corresponding to the $\pm\sqrt{-1}$-eigenspaces $T^{1,0}M$ and $T^{0,1}M$ of the complex structure J and similarly

$$dz^i = dx^i + \sqrt{-1}dy^i, \; d\bar{z}^i = dx^i - \sqrt{-1}dy^i, \tag{1.5}$$

for $i = 1, 2, \cdots, n$ of $T^*M \otimes \mathbb{C}$.

We extend the metric g $\mathbb{C}$-linearly to the complexified tangent bundle $TM \otimes \mathbb{C}$. Then we have $g(u, v) = 0$ for $u, v \in T^{1,0}M$ or $u, v \in T^{0,1}M$. In the local holomorphic coordinates, the metric g is therefore written as

$$g = g_{i\bar{j}}(dz^i \otimes d\bar{z}^j + d\bar{z}^i \otimes dz^j). \tag{1.6}$$

where $g_{i\bar{j}} = g(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j})$ and $g_{i\bar{j}} = \overline{g_{j\bar{i}}}$. Thus, the Kähler form ω_g can be written as

$$\omega_g = g_{i\bar{j}} dz^i \wedge d\bar{z}^j. \tag{1.7}$$

The Kähler condition $d\omega_g = 0$ is then equivalent to

$$\frac{\partial g_{i\bar{p}}}{\partial z^j} = \frac{\partial g_{j\bar{p}}}{\partial z^i} \tag{1.8}$$

for any $i, j, p = 1, 2, \cdots, n$.

Further more, we can extend the Levi-Civita connection ∇ $\mathbb{C}$-linearly to $\Gamma(TM \otimes \mathbb{C})$. We write the Christoffel symbols as

$$\nabla \frac{\partial}{\partial z^j} = (\Gamma_{ij}^k dz^i + \Gamma_{\bar{i}j}^k d\bar{z}^i) \otimes \frac{\partial}{\partial z^k} + (\Gamma_{ij}^{\bar{k}} dz^i + \Gamma_{\bar{i}j}^{\bar{k}} d\bar{z}^i) \otimes \frac{\partial}{\partial \bar{z}^k}, \tag{1.9}$$

$$\nabla \frac{\partial}{\partial \bar{z}^j} = (\Gamma_{i\bar{j}}^k dz^i + \Gamma_{\bar{i}\bar{j}}^k d\bar{z}^i) \otimes \frac{\partial}{\partial z^k} + (\Gamma_{i\bar{j}}^{\bar{k}} dz^i + \Gamma_{\bar{i}\bar{j}}^{\bar{k}} d\bar{z}^i) \otimes \frac{\partial}{\partial \bar{z}^k}. \tag{1.10}$$

The Kähler condition $\nabla J = 0$ then implies that all Christoffel symbols vanishes except Γ_{ij}^k and $\Gamma_{\bar{i}\bar{j}}^{\bar{k}} = \overline{\Gamma_{ij}^k}$. In fact, we can compute easily that

$$\frac{\partial}{\partial z^i} g_{j\bar{p}} = g(\nabla_{\frac{\partial}{\partial z^i}} \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^p}) = \Gamma_{ij}^k g_{k\bar{p}}. \tag{1.11}$$

It follows that

$$\Gamma_{ij}^k = g^{k\bar{p}} \frac{\partial g_{j\bar{p}}}{\partial z^i}. \tag{1.12}$$

Given the Levi-Civita connection, the Riemannian curvature tensor $\mathrm{Rm} \in \Gamma(\Lambda^2 T * M \otimes \mathrm{End}(TM))$ is defined as

$$\mathrm{Rm}(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w. \tag{1.13}$$

Similarly, we extent Rm $\mathbb{C}$-linearly to $\Gamma(\Lambda^2 T^*M \otimes \operatorname{End}(TM \otimes \mathbb{C}))$. In the local holomorphic coordinates,

$$\mathrm{Rm} = dz^i \wedge d\bar{z}^j \otimes (R^l_{i\bar{j}k} dz^k \otimes \frac{\partial}{\partial z^l} + R^{\bar{l}}_{i\bar{j}\bar{k}} d\bar{z}^k \otimes \frac{\partial}{\partial \bar{z}^l}) \tag{1.14}$$

where $R^l_{i\bar{j}k} = -\frac{\partial}{\partial \bar{z}^j}\Gamma^j_{ik}$ and $R^{\bar{l}}_{i\bar{j}\bar{k}} = -\overline{R^l_{\bar{j}ik}}$. The Ricci tensor $\mathrm{Ric} \in \Gamma(T^*M \otimes T^*M)$ evaluating on $X, Y \in TM$ is defined as the trace of $\mathrm{Rm}(\cdot, X)Y \in \operatorname{End}(TM)$. Under the local holomorphic coordinates,

$$\mathrm{Ric} = R_{i\bar{j}}(dz^i \otimes d\bar{z}^j + d\bar{z}^i \otimes dz^j) \tag{1.15}$$

where $R_{i\bar{j}} = R^l_{l\bar{j}i}$.

We can define the associated Ricci form

$$(\mathrm{Ric}\,\omega_g)(\cdot,\cdot) = \mathrm{Ric}(J\cdot,\cdot). \tag{1.16}$$

Under the local holomorphic coordinates,

$$\mathrm{Ric}\,\omega_g = R_{i\bar{j}} dz^i \wedge d\bar{z}^j. \tag{1.17}$$

Since

$$R_{i\bar{j}} = -\frac{\partial}{\partial \bar{z}^j}\Gamma^l_{il} = -\frac{\partial}{\partial \bar{z}^j}(g^{l\bar{p}}\frac{\partial g_{l\bar{p}}}{\partial z^i}) = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j}\log\det g, \tag{1.18}$$

there is a global simple formula for the Ricci form

$$\mathrm{Ric}\,\omega_g = -\sqrt{-1}\partial\bar{\partial}\log\det g, \tag{1.19}$$

It is also written as

$$\mathrm{Ric}\,\omega_g = -\sqrt{-1}\partial\bar{\partial}\log\omega_g^n \tag{1.20}$$

since $\omega_g^n = \det(g_{i\bar{j}})\sqrt{-1}dz^i \wedge d\bar{z}^1 \wedge \cdots \wedge \sqrt{-1}dz^n \wedge d\bar{z}^n$. As a consequence, with fixed complex structure J, given another Kähler metric g', the associated Kähler form is given by

$$\mathrm{Ric}\,\omega_{g'} = -\sqrt{-1}\partial\bar{\partial}\log\det g'. \tag{1.21}$$

Hence,

$$\mathrm{Ric}\,\omega_{g'} - \mathrm{Ric}\,\omega_g = -\sqrt{-1}\partial\bar{\partial}\log\frac{\det g'}{\det g} \tag{1.22}$$

where $\log\frac{\det g'}{\det g}$ is in fact a global function on M. Therefore, $\mathrm{Ric}\,\omega_g, \mathrm{Ric}\,\omega_{g'}$ necessarily belongs to the same cohomology class in $H^{1,1}(M,\mathbb{C}) \cap H^2(M,\mathbb{R})$, which is in fact 2π multiple of the first Chern class of (M, J) denoted as $2\pi c_1(M)$.

$\partial\bar{\partial}$-Lemma

A big advantage of Kähler manifolds is that the Christoffel symbols and the Ricci form have very neat formulas. Another big advantage of being Kähler is that we have the following $\partial\bar{\partial}$-lemma.

Lemma 1.4 ($\partial\bar{\partial}$-lemma)**.** *Let (M, g, J) be a closed Kähler manifold. Let $\alpha, \alpha' \in H^{1,1}(M, \mathbb{C})$ are in the same cohomology class. Then there exists a function $F \in C^\infty(M, \mathbb{C})$ such that $\alpha' = \alpha + \sqrt{-1}\partial\bar{\partial}F$.*

Recall that on a complex manifold (M, J), the space of complex valued k-forms on M naturally splits as $\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$, where locally $\Omega^{p,q}(M)$ has basis $dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q}$ for $i_1 < i_2 < \cdots < i_p$ and $j_1 < j_2 < \cdots < j_q$. We have differential operators $\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$ and $\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M)$ defined as the projection of the exterior differential operator d on $\Omega^{p,q+1}(M)$ and $\Omega^{p+1,q}(M)$ components respectively. In fact, $\partial\bar{\partial}$-lemma is also valid for (p, q)-forms with appropriate modifications and the proof requires some ideas from Hodge theory. Since we are only interested in $(1, 1)$-forms on M where the Kähler form lies, in this simple case we provide a quick proof of the $\partial\bar{\partial}$-lemma as below.

Proof. By assumptions there is a 1-form β on M such that $\alpha' - \alpha = d\beta$. Write $\beta = \beta^{1,0} + \beta^{0,1}$. Then

$$d\beta = \partial\beta + \bar{\partial}\beta = \partial\beta^{1,0} + \partial\beta^{0,1} + \bar{\partial}\beta^{1,0} + \bar{\partial}\beta^{0,1}. \tag{1.23}$$

Since $d\beta = \alpha' - \alpha \in H^{1,1}(M, \mathbb{C})$, it follows that $\partial\beta^{1,0} = 0 = \bar{\partial}\beta^{0,1}$. The lemma can be proved if we show that $\partial\beta^{0,1} = \partial\bar{\partial}f$ and $\bar{\partial}\beta^{1,0} = \partial\bar{\partial}g$ for some functions $f, g \in C^\infty(M, \mathbb{C})$. It suffices to show that for any $\bar{\partial}$-closed $(0, 1)$ form, say $\beta^{0,1}$, there exists some function $f \in C^\infty(M, \mathbb{C})$ such that $\beta^{0,1} - \bar{\partial}f$ is ∂-closed. Consider the formal adjoint operator $\bar{\partial}^* : \Omega^{0,1}(M) \to \Omega^0(M)$ of $\bar{\partial}$: for any $\theta = \theta_{\bar{j}} d\bar{z}^j$,

$$\bar{\partial}^*\theta = -g^{i\bar{j}}\theta_{\bar{j},i} \tag{1.24}$$

where $\theta_{\bar{j},i}$ is the $(i, \bar{j})$ entry of $\nabla\theta$. Set $f \in C^\infty(M)$ be the solution to the equation

$$\bar{\partial}^*\bar{\partial}f = \bar{\partial}^*\beta^{0,1}. \tag{1.25}$$

Note that on Kähler manifold we have that $\bar{\partial}^*\bar{\partial} = \frac{1}{2}\Delta_g$ where Δ_g is the usual Laplacian operator with respect to the metric g of M. Moreover, the integral of $\bar{\partial}^*\beta^{0,1}$ over M is zero by integration by parts. Therefore, the existence of the solution f is guaranteed. Let $\eta^{0,1} = \alpha^{0,1} - \bar{\partial}f$. Then we have $\bar{\partial}\eta^{0,1} = 0$ and $\bar{\partial}^*\eta^{0,1} = 0$. It is left to show that $\partial\eta^{0,1} = 0$. Write $\eta^{0,1} = \eta_{\bar{j}} d\bar{z}^j$. We have

$$\begin{aligned} 0 &= \int_M \langle \partial\bar{\partial}^*\eta^{0,1}, \eta^{0,1}\rangle_g d\,\mathrm{Vol}_g = \int_M -g^{i\bar{j}} g^{k\bar{l}} \eta_{\bar{j},ik}\eta_{\bar{l}} d\,\mathrm{Vol}_g \\ &= \int_M g^{i\bar{j}} g^{k\bar{l}} \eta_{\bar{j},ki}\eta_{\bar{l}} d\,\mathrm{Vol}_g = \int_M |\partial\eta^{0,1}| d\,\mathrm{Vol}_g\,. \end{aligned} \tag{1.26}$$

Hence, $\partial\eta^{0,1} = 0$. This finished the proof. □

In particular, if we have $[\alpha] = [\alpha'] \in H^{1,1}(M, \mathbb{C}) \cap H^2(M, \mathbb{R})$, then we have $\alpha' - \alpha = \sqrt{-1}\partial\bar{\partial}F$ for some $F \in C^\infty(M, \mathbb{R})$.

1.2 Calabi conjecture and Kähler-Einstein problem

In the end of last subsection we showed that for every Kähler metric, its Ricci form lies in 2π multiple of the first Chern class $2\pi c_1(M)$. In the 1950's, E. Calabi first raised the question whether each representative in the cohomology class $2\pi c_1(M)$ could be realized as the Ricci form of some Kähler metric. This question is known as the Calabi conjecture. It was a widely open problem for more than two decades until it was solved by S.-T. Yau through PDE theory in 1976.

We now show how we can represent the Calabi conjecture into a problem of solving some complex Monge-Ampère equation. Let (M, g, J) be a closed Kähler manifold and ω is the associated Kähler form. Let $\alpha \in 2\pi c_1(M)$. Calabi's problem is to look for a Kähler metric whose Ricci form is the given α. We might restrict ourselves in the fixed cohomology class $[\omega]$, which consists of all the Kähler metrics cohomologous to ω,

$$\mathcal{H} = \{\omega_\varphi \mid \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0,\ \varphi \in C^\infty(M, \mathbb{R})\}. \tag{1.27}$$

Since α and $\mathrm{Ric}\,\omega$ lies in the same cohomology class $2\pi c_1(M)$, by the $\partial\bar{\partial}$-lemma, there exists some function $F \in C^\infty(M, \mathbb{R})$ such that $\alpha = \mathrm{Ric}\,\omega + \sqrt{-1}\partial\bar{\partial}F$. If α is the Ricci form of some Kähler metric $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$, then the above equation can be written as

$$-\sqrt{-1}\partial\bar{\partial}\log\omega_\varphi^n = -\sqrt{-1}\partial\bar{\partial}\log\omega^n + \sqrt{-1}\partial\bar{\partial}F \tag{1.28}$$

which is equivalent to

$$\log\frac{\omega_\varphi^n}{\omega^n} - F \equiv C \tag{1.29}$$

for some constant C. By taking exponential on both sides, we have that

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = \omega_\varphi^n = e^{F+C}\omega^n. \tag{1.30}$$

By integrating both sides on M, one can determine that the constant

$$C = \log \frac{\int_M e^F \omega^n}{\int_M \omega^n}. \tag{1.31}$$

Without of loss of generality, we can always assume that $F \in C^\infty(M, \mathbb{R})$ satisfying that $\int_M e^F \omega^n = \int_M \omega^n$. Thus, the Calabi conjecture is equivalent to solve the following equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^F \omega^n. \tag{1.32}$$

Under the local holomorphic coordinates, let $\varphi_{i\bar{j}} = \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}$, the above equation is written as

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det(g_{i\bar{j}}) \tag{1.33}$$

which is a complex Monge-Ampère equation.

In 1976, S.-T. Yau solved the Calabi conjecture by solving the complex Monge-Ampère equation using the continuity method. The continuity path he worked on is

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{tF+C_t} \det(g_{i\bar{j}}), \ t \in [0,1] \tag{1.34}$$

where the constant C_t is chosen to make the normalization condition

$$\int_M e^{tF+C_t}\omega^n = \int_M \omega^n \tag{1.35}$$

hold. Set

$$I = \{t \in [0,1] \mid \text{ Equation (1.35) with parameter } t \text{ has a smooth solution } \varphi_t \text{ with } \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t > 0.\} \tag{1.36}$$

It is clear that $0 \in I$ hence I is not empty. The goal is to show that $1 \in I$ by showing that I is both open and closed in $[0,1]$.

The openness is done by Implicit Function Theorem. Suppose $t_0 \in I$ and φ_{t_0} is the solution of (1.35) with $t = t_0$. Set

$$\Phi(t, \varphi) = \log \frac{\det(g_{i\bar{j}} + \varphi_{i\bar{j}})}{\det(g_{i\bar{j}})} - tF - C_t \tag{1.37}$$

as a nonlinear map $\Phi : [0,1] \times \overline{C}^{2,\alpha}(M) \to \overline{C}^{0,\alpha}(M)$ where $\overline{C}^{2,\alpha}(M)$ (*resp.* $\overline{C}^{0,\alpha}(M)$) is the $C^{2,\alpha}(M)$ (*resp.* $C^{0,\alpha}(M)$) space with normalization

$$\overline{C}^{2,\alpha}(M) = \{\varphi \in C^{2,\alpha}(M) \mid \int_M \varphi\omega^n = 0\}. \tag{1.38}$$

It is easy to see that φ is a solution to (1.35) for t if and only if $\Phi(t,\varphi) = 0$. The partial derivative $\mathcal{D}_\varphi\Phi(t_0,\varphi_{t_0}) : \overline{C}^{2,\alpha}(M) \to \overline{C}^{0,\alpha}(M)$ is then given by

$$\mathcal{D}_\varphi\Phi(t_0,\varphi_{t_0})(\phi) = \Delta_{t_0}\phi \tag{1.39}$$

where Δ_{t_0} is the Laplacian operator of the metric $\omega_{\varphi_{t_0}} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_{t_0}$. On closed manifold M, it is an invertible linear map from $\overline{C}^{2,\alpha}(M)$ to $\overline{C}^{0,\alpha}(M)$. By the Implicit Function Theorem it sufficiently implies the openness of I.

The closedness is done by proving a list of *a priori* estimates on φ, among which the C^2 estimate plays a crucial role. In the holomorphic orthonormal frame of the metric g, Yau developed a delicate inequality

$$\begin{aligned}\Delta_\varphi(e^{-C\varphi}(n+\Delta\varphi)) \geq e^{-C\varphi}\Big[&\frac{1}{2}\sum_{i\neq j} R_{i\bar{i}j\bar{j}}\frac{(\varphi_{i\bar{i}}-\varphi_{j\bar{j}})^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} \\ &+ t\Delta F + (n+\Delta\varphi)(-C\Delta_\varphi\varphi)\Big].\end{aligned} \tag{1.40}$$

Taking C such that $C \geq -\inf_M \inf_{i\neq j} R_{i\bar{i}j\bar{j}} + 1$, one can find constants C_1, C_2 and C_3 such that

$$\Delta_\varphi(e^{-C\varphi}(n+\Delta\varphi)) \geq C_1(e^{-C\varphi}(n+\Delta\varphi))^{\frac{n}{n-1}} - C_2(e^{-C\varphi}(n+\Delta\varphi)) - C_3 \tag{1.41}$$

where the constant C_1 depends on $\sup_M|\varphi|$ and C_3 depends on $\sup_M|\Delta F|$. Once C^0 estimate is obtained, the C^2 estimate can be easily deducted from (1.41) by maximum principle. To get the higher order estimates, Yau showed a C^3 estimate in terms of the C^2 estimate, which was first introduced by E. Calabi. Soon after him, it was showed that the $C^{2,\alpha}$ estimate could be derived by the *a priori* interior $C^{2,\alpha}$ estimates of Monge-Ampère equations on domains known as the Evans-Krylov theory [22, 29, 30], which can be used to replace the C^3 estimate and simplify Yau's original proof. All the higher order estimates can be derived via elliptic theory of linear equation by taking differentiation of equation (1.35).

Calabi also proposed the question of finding "canonical metrics" inside a fixed Kähler class. A particular type of "canonical metrics" is the Kähler-Einstein metric. If the Kähler class is proportional to the first Chern class, it is natural to ask whether we could find a metric ω_φ in the Kähler class such that

$$\operatorname{Ric}\omega_\varphi = \lambda\omega_\varphi \tag{1.42}$$

for some constant λ. This problem is called the Kähler-Einstein problem.

Definition 1.5. *Suppose M is a closed Kähler manifold and $c_1(M)$ is its first Chern class. We say that $c_1(M) > 0$ (*resp. *$c_1(M) < 0$) if there exists a representative $\alpha \in c_1(M)$ such that α is positive definite (*resp. *negative definite).*

By scaling the metric ω_φ one can assume that the constant λ is either $-1, 0$ or 1. In all three cases it requires that the Chern class is definite: $c_1(M) < 0$, $c_1(M) = 0$ and $c_1(M) > 0$, respectively. Note that $\operatorname{Ric}\omega \in 2\pi c_1(M) = \lambda[\omega]$. By the $\partial\bar{\partial}$-lemma, there exists some function $F_\omega \in C^\infty(M, \mathbb{R})$ such that

$$\operatorname{Ric}\omega = \lambda\omega + \sqrt{-1}\partial\bar{\partial}F_\omega. \tag{1.43}$$

It then follows that

$$\operatorname{Ric}\omega_\varphi = \operatorname{Ric}\omega + \sqrt{-1}\partial\bar{\partial}(\lambda\varphi - F_\omega) \tag{1.44}$$

which is equivalent to the following Monge-Ampère equation

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^{F_\omega - \lambda\varphi}\det(g_{i\bar{j}}). \tag{1.45}$$

The case $\lambda = 0$ is already settled by Yau in his resolution of the Calabi conjecture. Such Kähler manifolds with vanishing first Chern class are thus called Calabi-Yau manifolds. The case $\lambda = -1$ is solved independently by Yau [41] and Aubin [2] in the late 1970s. It can be solved in a similar way to Yau's resolution of the Calabi conjecture via the continuity method on a suitable continuity path. In this case Yau's C^2 estimate still holds and moreover the C^0 estimate of the solution can be easily obtained via the maximum principle. The case $\lambda = 1$, i.e., the Kähler-Einstein problem on Fano manifolds, however, turns out to be quite subtle.

In fact, there are many obstructions for the existence of Kähler-Einstein metrics when $c_1(M) > 0$. Let $Aut(M)$ denote the group of biholomorphisms on the complex manifold (M, J). In 1957, Matsushima [34] found that if there exists a Kähler-Einstein metric in the class $2\pi c_1(M) > 0$, then $Aut(M)$ is reductive. Therefore, Kähler manifolds with $c_1(M) > 0$ and non-reductive $Aut(M)$, for instance, $\mathbb{CP}^2$ blowing up with one point, does not possess a Kähler-Einstein metric.

In 1983, Futaki [23] discovered another obstruction known as Futaki invariant. Choose $\omega \in 2\pi c_1(M) > 0$. Let $h_\omega \in C^\infty(M, \mathbb{R})$ such that $\operatorname{Ric}\omega - \omega = \sqrt{-1}\partial\bar{\partial}h_\omega$. The Futaki invariant is defined as $f_M : \eta(M) \to \mathbb{C}$,

$$f_M(X) = \int_M X(h_\omega)\omega^n, \tag{1.46}$$

where $\eta(M)$ is the Lie algebra of $Aut(M)$ that consists of all holomorphic vector fields on M. Futaki showed that f_M is actually independent of the choice of ω. Moreover, if there exists a Kähler-Einstein metric in the class $2\pi c_1(M) > 0$, then $f_M \equiv 0$. In [23], Futaki also constructed an example of 3-dimensional manifold with $c_1(M) > 0$ and $Aut(M)$ reductive but $f_M \neq 0$, hence does not possess a Kähler-Einstein metric.

It is proved by Donaldson-Uhlenbeck-Yau [40, 20] that the existence of Hermitian-Yang-Mills connection is equivalent to the stability of underlying holomorphic line bundle. Inspired by this result, in the late 1980s Yau proposed that the existence of Kähler-Einstein metrics for Fano manifolds should be correspondent to certain stability of the underlying manifold in the geometric invariant theory. This stability condition is later defined more precisely by Tian [37] and Donaldson [21] known as K-stability. This results in the following famous conjecture.

Conjecture 1.6. *(Yau-Tian-Donaldson, [21]) A Fano manifold V admits a Kähler-Einstein metric if and only if (V, K_V^{-1}) is K-stable.*

This conjecture is only fully settled recently by Chen-Donaldson-Sun [13, 14, 15] in 2013.

1.3 Extremal and cscK metrics

It is worth mentioning more general canonical metrics introduced by Calabi besides the Kähler-Einstein metrics. Calabi introduced the L^2-norm of the scalar curvature as a functional on the metrics called the Calabi functional

$$Ca(\varphi) = \int_M R_\varphi^2 \omega_\varphi^n \tag{1.47}$$

where $R_\varphi = \operatorname{tr}_{\omega_\varphi} \operatorname{Ric}(\omega_\varphi)$ is the scalar curvature of ω_φ. Calabi proposed to look for special metrics in the space $\mathcal{H}$ which are the critical points of the Calabi functional. Such metrics are called extremal metrics.

By Calabi's computation, we have the first variation of the Calabi functional

$$\begin{aligned} \delta_\psi Ca(\varphi) &= \int_M 2(\delta_\psi R_\varphi) R_\varphi \omega_\varphi^n + R_\varphi^2 \delta_\psi(\omega_\varphi^n) \\ &= \int_M (g^{\alpha\bar{p}} g^{\beta\bar{q}} R_{\varphi,\bar{p}\bar{q}\alpha\beta}) \psi \omega_\varphi^n. \end{aligned} \tag{1.48}$$

Therefore, the Euler-Lagrange equation for the Calabi functional is

$$g^{\alpha\bar{p}} g^{\beta\bar{q}} R_{\varphi,\bar{p}\bar{q}\alpha\beta} = 0. \tag{1.49}$$

Pairing with R_φ and integrate by parts, we obtain equivalently that φ must satisfy

$$R_{\varphi,\bar{p}\bar{q}} = 0. \tag{1.50}$$

for all $p, q \in \{1, 2, \cdots, n\}$. We define the $(1,0)$-vector field on M given by

$$\nabla^{1,0} R_\varphi = g_\varphi^{i\bar{p}} \frac{\partial R_\varphi}{\partial \bar{z}^p} \frac{\partial}{\partial z^i}.$$

Then (1.50) is equivalent to that $\nabla^{0,1} R_\varphi$ is a holomorphic vector field on M. A metric ω_φ is a extremal metric if and only if the vector field $\nabla^{1,0} R_\varphi$ is holomorphic. In particular, if $\nabla^{1,0} R_\varphi = 0$, we have that $R_\varphi = \underline{R}$ is a constant. Indeed, the constant is a topological invariant

$$\underline{R} = \frac{\int_M R_\varphi \omega_\varphi^n}{\int_M \omega_\varphi^n} = \frac{2\pi c_1(M) \cdot [\omega_0]^{n-1}/(n-1)!}{[\omega_0]^n/n!}. \tag{1.51}$$

Such a metric is then called a constant-scalar-curvature Kähler (cscK) metric.

The existence problem of extremal/cscK metrics can be viewed as a generalization of the Kähler-Einstein problem in the sense that if we work in the cohomology class $2\pi c_1(M) > 0$, then cscK are equivalent to Kähler-Einstein. It is straightforward that Kähler-Einstein metrics are also cscK. To see the inverse, notice that ω is cscK if and only if that $\mathrm{Ric}(\omega)$ is harmonic with respect to ω. Since the harmonic form in $2\pi c_1(M) > 0$ is unique by Hodge theory, it follows that $\mathrm{Ric}(\omega) = \omega$ is Kähler-Einstein. (The form ω is harmonic with respect to itself.)

There are also obstructions for the extremal/cscK metrics similar to the Kähler-Einstein case. In particular, Futaki's invariant can be generalized to the cscK case by setting $f_M : \eta(M) \to \mathbb{R}$,

$$f_M(X) = \int_M X(u_\omega) \omega^n, \tag{1.52}$$

where $u_\omega \in C^\infty(M, \mathbb{R})$ is the solution to the equation $\Delta_\omega u_\omega = R_\omega - \underline{R}$. The definition only depends on the cohomology class $[\omega] \in H^{1,1}(M, \mathbb{R})$, and if there exists a cscK metric then necessarily $f_M \equiv 0$.

The K-stability by Donaldson and Tian is indeed an obstruction for the more general cscK metrics. To this end we have the following more general conjecture.

Conjecture 1.7. *(Yau-Tian-Donaldson, [21]) A smooth polarized manifold (V, L) admits a cscK metric in the class $c_1(L)$ if and only if it is K-stable.*

The space of Kähler metric $\mathcal{H}$ can be endowed with a L^2 Riemannian metric given by

$$\langle \psi_1, \psi_2 \rangle_\varphi = \int_M \psi_1 \psi_2 \omega_\varphi^n \tag{1.53}$$

for any $\psi_1, \psi_2 \in T_\varphi \mathcal{H} = C^\infty(M, \mathbb{R})$. It is proved by X. X. Chen [16] that any Kähler metrics $\varphi_1, \varphi_2 \in \mathcal{H}$ can be joint by a unique $C^{1,1}$ geodesic under the above L^2 metric.

Mabuchi [33] introduced a functional $\mathcal{M}_{\omega_0}$ over $\mathcal{H}$ called K-energy, which has the cscK metrics as its critical point. The K-energy is defined using its derivative: for any $\psi \in T_\varphi \mathcal{H}$,

$$\delta_\psi \mathcal{M}_{\omega_0}(\varphi) = -\int_M \psi (R_\varphi - \underline{R}) \omega_\varphi^n. \tag{1.54}$$

It is proved by Berman & Berndtsson [4] that the K-energy is convex along the geodesics in $\mathcal{H}$. Therefore, the existence of critical points for K-energy would be expected to imply that the properness of K-energy with respect to some geodesic distance. In [8], X. X. Chen made the following conjecture.

Conjecture 1.8. *There exists a cscK metric in a Kähler class $[\omega_0]$ if and only if the K-energy $\mathcal{M}_{\omega_0}$ is proper.*

This conjecture is recently fully affirmatively solved by a series of work. The direction that the existence of cscK implies the properness of K-energy is estabished by Berman-Darvas-Lu [5] very recently. To tackle the existence problem, it is proposed by X. X. Chen to consider the following continuity path

$$t(R_\varphi - \underline{R}) = (1-t)(\operatorname{tr}_\varphi \omega_0 - n). \tag{1.55}$$

It is straightforward that when $t = 0$ then $\varphi = 0$ is a solution. The openness is done by Chen [9] (for $t > 0$) and by Chen-Păun-Zeng [18] (for $t = 0$). To obtain the closedness it is expected to obtain *a priori* estimates for the path (1.55). In this stage one can consider the the more general coupled equations:

$$\det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = e^F \det g_{i\bar{j}}, \tag{1.56}$$

$$\Delta_\varphi F = -f + \operatorname{tr}_\varphi \eta \tag{1.57}$$

where f is a given smooth function and η is a smooth real closed $(1,1)$ form on M. Note that (1.56), (1.57) combined gives that

$$R_\varphi = f + \operatorname{tr}_\varphi(\operatorname{Ric}(\omega_0) - \eta). \tag{1.58}$$

The path (1.55) is equivalent to equation (1.58) with choice

$$f = \underline{R} - \frac{1-t}{t} n, \quad \eta = \operatorname{Ric}(\omega_0) - \frac{1-t}{t}\omega_0.$$

In a series of recent deep work of Chen-Cheng [10, 11, 12], the authors proves the following *a priori* estimates:

Theorem 1.9 (Chen-Cheng)**.** *Let φ be a smooth solution to* (1.56), (1.57) *normalized to be* $\sup_M \varphi = 0$*. Then for any $p < \infty$, there exists a constant C, depending only on the background metric (M, g), $\|f\|_0$, $\max_M |\eta|_{\omega_0}$, p and the upper bound of $\int_M e^F F \omega_0^n$ such that $\|\varphi\|_{W^{4,p}} \leq C$, $\|F\|_{W^{2,p}} \leq C$.*

While f and η has higher regularity, it is easy to obtain higher regularity for φ by bootstrapping. In the same work the authors show that when the K-energy is proper, then the entropy term $\int_M e^F F \omega_0^n = \int_M \log(\frac{\omega_\varphi^n}{\omega_0^n})\omega_\varphi^n$ is indeed bounded from above. This closes the gap of the closedness argument and hence assures the existence of cscK metrics when assuming the properness of the K-energy.

2 Motivation and main results

With the great progress for Kähler geometry for compact Kähler manifolds, there is also a large amount of interest to study the canonical metrics on complete, non-compact Kähler manifolds. We are primarily interested in quasiprojective manifolds. These manifolds are complements of a divisor of a projective manifold.

Let $(\overline{M}, \omega_0)$ be a compact Kähler manifold of complex dimension n. Let D be an effective divisor in $\overline{M}$ with only *simple normal crossings*, namely, $D = \sum_{j=1}^N D_j$ where the irreducible components D_i are smooth and intersect transversely. Let $[D_j]$ be the associated line bundle to D_j, endowed with a smooth hermitian metric $|\cdot|_j$. Let $s_j \in \mathcal{O}([D_j])$ be a holomorphic defining section such that D_j is the zero locus of s_j and let $\rho_j = -\log(|s_j|_j^2)$. Up to scaling $|\cdot|_j$, one can assume that $|s_j|_j^2 \leq e^{-1}$ so that $\rho_j \geq 1$ out of D_j. Let $\rho = \prod_{j=1}^N \rho_j$. Note that $\sqrt{-1}\partial\bar{\partial}\rho_j$ extends to a *smooth* real $(1,1)$-form on the whole $\overline{M}$ which lies in the class $2\pi c_1([D_j])$. For $\lambda > 0$ sufficiently large, set

$$\omega = \lambda\omega_0 + \sqrt{-1}\partial\bar{\partial}\log\rho = \lambda\omega_0 + \sum_{j=1}^N \sqrt{-1}\partial\bar{\partial}\log\rho_j. \tag{2.1}$$

Then ω defines a genuine Kähler form on $M = \overline{M}\backslash D$, with the properties that it is complete, has finite volume and has cusp singularity along D. Indeed, it is asymptotically hyperbolic near the divisor D. Such a metric is usually called metric of *Poincarè type* or *Carlson-Griffiths type.*

It is proved by Tian-Yau [39] and R. Kobayashi [28] in the 1980's that if $K_{\overline{M}} + D$ is ample, one can deform such a Poincaré type metric into a negatively curved complete Kähler-Einstein metric on the complement $\overline{M}\backslash D$. These Kähler-Einstein metrics is also of Poincaré type and have cusp singularities along the divisor D.

It is then natural to consider the same existence problem of cscK metrics over such manifolds. Motivated by Chen-Cheng's work of *a priori* estimates on the cscK metrics over the compact Kähler manifolds, one would hope that such *a priori* estimates can also hold on these manifolds. Note that the cscK problem can be equivalently written as the following coupled equations:

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^F\omega^n, \tag{2.2}$$

$$\Delta_\varphi F = \operatorname{tr}_\varphi \operatorname{Ric} - \underline{R} \tag{2.3}$$

where Ric is the Ricci curvature of ω and $\underline{R}$ is the average of the total scalar curvature which is a topogical constant.

The analogous problem of finding a Kähler metric on M with prescribed volume form which is equivalently to solve the following Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^F\omega^n \quad \text{on } M = \overline{M}\backslash D \tag{2.4}$$

for some suitable function F. was studied by H. Auray [3, Theorem 4] that there exists a solution $\varphi \in C^\infty_{\mathrm{loc}}(M)$ bounded at any order to the Monge-Ampème equation (2.4) on $M = \overline{M}\backslash D$, when $F \in C^\infty_{\mathrm{loc}}(M)$ is of $O(e^{-\nu u})$ at any order for some $\nu > 0$, and $\int_M e^F\omega^n = \int_M \omega^n$. Auvray's result requires that F decays in the exponential order for all its derivatives when approaching to the divisor (the infinity). Moreover, the C^2 estimate essentially uses

Yau's C^2 estimate for compact manifold, which requires F is at least C^2 and the bounded on $\sup_M |\Delta F|$.

In order to control the metric ω_φ through the scalar curvature, it is to natural to study the Monge-Ampére equation (2.4) when the right hand side F has rather weaker regularity less than C^2. Indeed, for compact Kähler manifolds, X. X. Chen and W. Y. He proved that when the right hand side is in W^{1,p_0} for $p_0 > 2n$ where n is the complex dimension, then one can derive *a priori* C^0 bound for $\Delta\varphi$ and W^{3,p_0} bound for φ.

The first part of our dissertation is devoted to a non-compact version of their result on the complement of a divisor. Define

$$\mathcal{I}(F, p_0) := \int_M (|F|^{p_0} + |\nabla F|^{p_0})\rho^{\frac{p_0-2}{2n-2}}\omega^n. \tag{2.5}$$

Our main theorem states as follows.

Theorem 2.1 (Main theorem)**.** *Let $\overline{M}$ be a compact Kähler manifold of complex dimension n and D be an divisor on $\overline{M}$ with only simple normal crossings. Let $M = \overline{M}\backslash D$ endowed with some Poincaré type Kähler metric ω constructed as above. For any function $F \in W^{1,p_0}_{\text{loc}}(M)$ satisfying $\int_M (e^F - 1)\omega^n = 0$ and $\mathcal{I}(F, p_0) < \infty$ for some $p_0 > 2n$, the Monge-Amperé equation* (2.4) *has a classical solution φ in $W^{3,p_0}(M)$.*

We sketch the idea of proof. Following Auvray's proof of the smooth case, we first consider the ε-perturbed equation of (2.4):

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon)^n = e^{F+\varepsilon\varphi_\varepsilon}\omega^n, \tag{2.6}$$

for $\varepsilon \in (0, 1]$. For any fixed $\varepsilon > 0$, by a simple scaling $\tilde{\omega} = \varepsilon\omega, \tilde{\varphi} = \varepsilon\varphi_\varepsilon$, the equation (2.6) can be normalized to

$$(\tilde{\omega} + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi})^n = e^{F+\tilde{\varphi}}\tilde{\omega}^n. \tag{2.7}$$

The equation (2.7) has been well studied by Cheng-Yau [19], R. Kobayashi [28] and Tian-Yau [39] to derive Kähler-Einstein metrics with negative curvature on (M, ω) when $K_{\overline{M}} + D$ is assumed ample. Yet the existence of solution to (2.7) actually does not necessarily need the additional assumption of the ampleness of $K_{\overline{M}} + D$. We thus obtain the existence of solutions φ_ε to (2.6) and *a priori* estimates of φ_ε depending on ε. Then we will show that the family $\{\varphi_\varepsilon\}$ is compact in W^{3,p_0} by securing a uniform W^{3,p_0} estimate for the family. Lastly, we use Arzellà-Ascoli theorem to take a converging subsequence which converges to a W^{3,p_0} solution to (2.4).

In order to get the W^{3,p_0} estimate, we need to first get the C^2 estimate. When the right hand side F has rather weak regularity, the C^2 estimate can not be obtained by a similar Yau's argument. Instead, we follow the strategy of Chen-He [17] to obtain the uniform C^1 and C^2 estimates (the C^1 estimate is needed to derive the C^2 estimate) by integration method. To be specific, we prove the following theorems. While deriving these theorem we could temporarily assume that the right hand side F is smooth with compact support.

Theorem 2.2. *Suppose that $F \in C_c^\infty(M)$ satisfies $\mathcal{I}(F, p_0) < \infty$ for some $p_0 > 2n$, and φ_ε is a solution to the ε-perturbed equation* (2.6)*. Then there exists a constant C independent of ε such that*

$$|\nabla \varphi_\varepsilon| \leq C, \ \forall \varepsilon \in (0, 1]. \tag{2.8}$$

Theorem 2.3. *Suppose that $F \in C_c^\infty(M)$ satisfies $\mathcal{I}(F, p_0) < \infty$ for some $p_0 > 2n$, and φ_ε is a solution to the ε-perturbed equation* (2.6)*. Then there exists a constant C independent of ε such that*

$$|\Delta \varphi_\varepsilon| \leq C, \ \forall \varepsilon \in (0, 1]. \tag{2.9}$$

Different from the compact case considered in [17], there are two main issues when carrying out the integration techniques in our setting. First, we need to deal with the boundary terms when we do integration by parts. Second, the usual Sobolev inequality fails in our context as the injective radius of the Kähler manifold with Poincaré type metric is zero. A generalized Stokes theorem by Gaffeny to complete non-compact manifolds [24] deals with the first issue, which allows us to perform integration by parts the same way as the compact case. For the second issue, we adopt a weighted Sobolev inequality from [3] and show that the similar analysis can be carried out successfully in our context.

The arrangement of this dissertation is as follows: In section 3 we construct the Poincaré type metric on the complement of a divisor as the reference metric. We briefly states the properties of the reference metric and construct the system of quasi-coordinates near the divisor and set up the analytic ingredients for the proof of our main theorems. This part is mainly cited from [3]. The section 4 is devoted to the proof of main theorems. In subsection 5.1 we cite a proof of the C^0 estimate from [3] with little modification. The main estimates are in subsection 4.2 and 4.3 where we prove the C^1 and C^2 estimates following the idea of Chen-He [17], followed by the W^{3,p_0} estimate. The proof of the main result Theorem 2.1 is presented in the end of this section.

3 Kähler manifolds with Poincaré type metrics

Let $\overline{M}$ be a closed Kähler manifold and D be an effective divisor over $\overline{M}$ with only simple normal crossings. In this section we construct a Poincaré type metric on the complement $M = \overline{M} \backslash D$ as the reference metric. We describe the basic properties of the reference metric and construct the system of quasi-coordinates near the divisor. In this section we also develop the analytic ingredients for the proof of our main results. It includes an unweighted Poincaré inequality and a weighted Sobolev inequality. This part is mainly cited from [3]. Throughout this section and the following sections, we denote by $d\mu$ the volume form of ω.

3.1 Poincaré type metrics

Let $(\overline{M}, \omega_0)$ be a compact Kähler manifold with $\dim_{\mathbb{C}} \overline{M} = n$. Let D be an effective divisor with simple normal crossings, namely, $D = \sum_{j=1}^{N} D_j$ decomposes into smooth irreducible components. For each j, let s_j be a holomorphic defining section of D_j. Let $\rho_j = -\log(|s_j|^2)$. We can assume that $\rho_j \geq 1$ out of D_j by scaling. Note that $\sqrt{-1}\partial\bar{\partial}\rho_j$ extends to a smooth real $(1,1)$-form on the whole $\overline{M}$, whose class is $2\pi c_1([D_j])$. Let $\rho = \prod_{j=1}^{N} \rho_j$. Set

$$\omega = \lambda\omega_0 - \sqrt{-1}\partial\bar{\partial}\log\rho = \lambda\omega_0 - \sum_{j=1}^{N} \sqrt{-1}\partial\bar{\partial}\log(-\log|s_j|^2) \tag{3.1}$$

for some positive constant λ.

Lemma 3.1. *For $\lambda > 0$ sufficiently large, ω defines a Kähler metric on $M = \overline{M} \backslash D$.*

Proof. By a simple computation we have

$$-\sqrt{-1}\partial\bar{\partial}\log\rho_j = \frac{\sqrt{-1}\partial\rho_j \wedge \bar{\partial}\rho_j}{\rho_j^2} - \frac{\sqrt{-1}\partial\bar{\partial}\rho_j}{\rho_j}. \tag{3.2}$$

The first term in the right hand side is a positive $(1,1)$-form. Note that $\sqrt{-1}\partial\bar{\partial}\rho_j$ extends to a smooth real $(1,1)$-form on the whole $\overline{M}$ which lies in $2\pi c_1([D_j])$. For each j, there is some positive $\lambda_j > 0$ such that $\sqrt{-1}\partial\bar{\partial}\rho_j \leq \lambda_j\omega_0$ on $\overline{M}$. Note that $\rho_j \geq 1$. Hence, $\lambda_j\omega_0 + \sqrt{-1}\partial\bar{\partial}\log\rho_j > 0$ on $M\backslash D_j$. Let $\lambda = \sum_j \lambda_j$, then

$$\omega = \lambda\omega_0 - \sqrt{-1}\partial\bar{\partial}\log\rho = \sum_j(\lambda_j\omega_0 - \sqrt{-1}\partial\bar{\partial}\log\rho_j) > 0 \tag{3.3}$$

on $M = \overline{M}\backslash D$. $\square$

Let Δ_r be the disc in $\mathbb{C}$ of radius r and let $\Delta_r^* = \Delta_r - \{0\}$. A simple model for the Poincaré type of metrics is the punctured disc Δ_κ^* with the Poincaré metric

$$-\sqrt{-1}\partial\bar{\partial}\log(-\log|z|^2) = \frac{\sqrt{-1}dz \wedge d\bar{z}}{|z|^2\log^2|z|^2} \tag{3.4}$$

for some small positive $\kappa < e^{-1}$. For higher dimensions, our local model is given by the punctured polydisc $(\Delta_\kappa^*)^k \times \Delta_1^{n-k}$ with the model metric

$$\omega_{mdl} = \sum_{j=1}^{k} \frac{\sqrt{-1}dz^j \wedge d\bar{z}^j}{|z^j|\log^2|z^j|^2} + \sum_{j=k+1}^{n} \sqrt{-1}dz^j \wedge d\bar{z}^j. \tag{3.5}$$

The model metric is simply the product metric of the Poincarè metric on $(\Delta_\kappa^*)^k$ and the Euclidean metric on Δ_1^{n-k}.

Indeed, the asymptotics of the reference metric near D can be compared with the local model. Let $x \in D_1 \cap \cdots \cap D_k - D_{k+1} \cup \cdots \cup D_N$. The simple normal crossing assumption allows us to take a coordinate polydisc $U = \Delta_\kappa^k \times \Delta_1^{n-k}$ centered at x such that

$$\begin{aligned} &U \cap D_j = \{z \in U : z^j = 0\} \quad (1 \le j \le k), \\ &U\backslash D = (\Delta_\kappa^*)^k \times \Delta_1^{n-k}. \end{aligned} \tag{3.6}$$

Lemma 3.2. *In the coordinates $(z^1, \ldots, z^k, z^{k+1}, \ldots, z^n)$, we have*

$$\omega = \sum_{j=1}^{k} \frac{\sqrt{-1}dz^j \wedge d\bar{z}^j}{|z^j|^2\log^2|z^j|^2} + \big(\lambda\omega_0 - \sum_{j=k+1}^{N} \sqrt{-1}\partial\bar{\partial}\log\rho_j\big) + O(\rho_1^{-1} + \cdots + \rho_k^{-1}). \tag{3.7}$$

In particular, ω is quasi-isometric to ω_{mdl} on $U\backslash D$, i.e., there exists some constant $C > 0$ such that

$$C^{-1}\omega_{mdl} \le \omega \le C\omega_{mdl}. \tag{3.8}$$

Proof. For any index j in $\{1, \cdots, k\}$, there is some smooth function f though D such that $|s_j|^2 = e^f|z^j|^2$. Hence, $\rho_j = -\log|z^j|^2 - f \sim -\log|z^j|^2$. A simple computation shows that

$$\begin{aligned}
-\sqrt{-1}\partial\bar{\partial}\log\rho_j &= \frac{\sqrt{-1}dz^j\wedge d\bar{z}^j}{|z^j|^2\rho_j^2} - \frac{\sqrt{-1}\partial\bar{\partial}f}{\rho_j} \\
&+ \frac{\sqrt{-1}(z^j dz^j\wedge\bar{\partial}f + \bar{z}^j\partial f\wedge d\bar{z}^j + |z^j|^2\partial f\wedge\bar{\partial}f)}{|z^j|^2\rho_j^2} \\
&= \frac{\sqrt{-1}dz^j\wedge d\bar{z}^j}{|z^j|^2\log^2|z^j|^2} + O(\rho_j^{-1}).
\end{aligned} \tag{3.9}$$

Sum up the above equality for $j = 1, \cdots, k$, we obtain (3.7). Note that $\lambda\omega_0 - \sum_{j=k+1}^{N}\sqrt{-1}\partial\bar{\partial}\log\rho_j$ is smooth though D. It is quasi-isometric to the Euclidean metric when restricted to Δ_1^{n-k}, while it is dominated by the Poincarè metric on $(\Delta_\kappa^*)^k$ when restricted on $(\Delta_\kappa^*)^k$. Hence, ω is quasi-isometric to the model metric ω_{mdl}. □

Lemma 3.3. *Let ω be the Poincarè type metric constructed as above. Then*

(1) The Kähler manifold (M, ω) is complete, it has finite volume and its injectivity radius goes to 0 as the points approach to the divisor.

(2) There is some constant $B > 0$ such that

$$\inf_M \inf_{i\neq j} R_{i\bar{i}j\bar{j}} \geq -B, \ \ \sup_M |R| \leq B, \ \ \textit{and} \ \sup_M \rho^{-1}|\nabla\rho| \leq B$$

where $R_{i\bar{i}j\bar{j}}$ and R are the holomorphic sectional curvature and scalar curvature, respectively.

Proof. The assertion (1) is clear from the properties of the local model. We only need to consider the assertion (2) near the divisor D. Since D can be covered by finitely many local coordinate chart, it suffices to show the inequalities hold in each coordinate chart. Let $U = \Delta_\kappa^k \times \Delta_1^{n-k}$ with the properties (3.6). The metric ω has the asymptotics in $U\backslash D$ which is quasi-isometric to the model metric ω_{mdl} on $(\Delta_\kappa^*)^k \times \Delta_1^{n-k}$. Note that the Poincarè metric on $(\Delta_\kappa^*)^k$ has constant holomorphic sectional curvature -1, while the Euclidean metric on Δ_1^{n-k} has constant sectional curvature 0. Hence, the holomorphic sectional curvature of ω on $U\backslash D$ is bounded from below and the scalar curvature bounded on $U\backslash D$.

To see the last inequality, let us assume without loss of the generality that $(U\backslash D, \omega)$ is the local model $((\Delta_\kappa^*)^k \times \Delta_1^{n-k}, \omega_{mdl})$. Note that

$$\rho^{-1}|\nabla\rho| = |\nabla\log\rho| \leq \sum_{j=1}^{k}|\nabla\log\rho_j| + \sum_{j=k+1}^{N}|\nabla\log\rho_j|. \tag{3.10}$$

For each j in $\{k+1, \cdots, N\}$, $|\nabla \log \rho_j|$ is bounded because ρ_j is smooth across D. For each j in $\{1, \cdots, k\}$, $\rho_j = -\log |z^j|^2 + f \sim -\log |z^j|^2$ for some function f smooth across D. We have

$$|\nabla \log \rho_j| \leq \rho_j^{-1}(|\nabla \log |z^j|^2| + |\nabla f|) = \rho_j^{-1}(-\log |z^j|^2 + |\nabla f|) = \mathcal{O}(1) \tag{3.11}$$

as $z^j \to 0$. Hence, $|\nabla \log \rho_j|$ is also bounded when $j \in \{1, \cdots, k\}$.

Since we can cover the divisor D by finitely many local coordinate charts, the constant B in assertion (2) can be taken independent of the choice of local coordinate charts. □

3.2 Quasi-coordinates

The usual coordinate system is not convenient to use near the divisor because that the injectivity radius goes to zero when points become closer and closer to the divisor. To overcome this issue, it was first by Cheng-Yau [19], then followed by Tian-Yau [39] and R. Kobayashi [28], to use the so-called quasi-coordinates to study the complete Kähler manifolds with Poincaré metrics. Given a point $x \in D$, we can take some open neighborhood $U \subset \overline{M}$ of x, such that $U \backslash D$ is biholomorphic to $(\Delta_\kappa^*)^k \times \Delta_1^{n-k}$ for some k. Hence, $U \backslash D$ has a branched covering which is a smooth open manifold. The idea of quasi-coordinates is to use the local coordinates on the branched covering instead of the local coordinates of $U \backslash D$. A good reference of the quasi-coordinates for Kähler manifolds with Poincaré metrics can be found in [28, Section 2].

Definition 3.4. *Let V be an open set in $\mathbb{C}^n$ and $(z_1, \cdots, z_n)$ be the Euclidean coordinates on V. A holomorphic map Ψ from V into a complex manifold M of dimension n is called a* quasi-coordinate map *iff it is of maximal rank everywhere in V. The pair $(V; z_1, \cdots, z_n; \Psi)$ is called a local quasi-coordinate of M.*

We now state how to construct the quasi-coordinates near the divisor explicitly. We begin with the punctured disc Δ_κ^* with the model Poincaré metric

$$\omega_{mdl} = \frac{\sqrt{-1} dz \wedge d\bar{z}}{|z|^2 \log^2 |z|^2}. \tag{3.12}$$

The map

$$\exp : \mathbb{C} \to \Delta_1^*, \; w \mapsto \exp(w)$$

is the universal covering map of the punctured disc. Yet to cover the image, we only need to restrict the map on the banded region $\{w \in \mathbb{C} : -\pi <$

$\operatorname{Im} w \leq \pi\}$. Now we define a family of holomorphic maps $\{\tilde{\psi}_\delta\}$ parametrized by a real parameter $\delta \in (0,1)$, which is given by

$$\tilde{\psi}_\delta(w) : \Delta_{3/4} \to \mathbb{C},\ w \mapsto \delta^{-1}(w+1)/(w-1). \tag{3.13}$$

which maps the disc $\Delta_{3/4}$ onto the ball of radius $\frac{24}{7\delta}$ centered at $(-\frac{25}{7\delta}, 0)$. Each map is a biholomorphism from the disc $\Delta_{3/4}$ to its image. The union of the images $\bigcup_{\delta\in(0,1)} \tilde{\psi}_\delta(\Delta_{3/4})$ covers the banded region $\{w \in \mathbb{C} : -\pi < \operatorname{Im} w \leq \pi,\ \operatorname{Re} w < -K\}$ for some $K > 0$ sufficient large. Under the exponential map, the image of the above banded region covers a small punctured disc Δ^*_κ with $\kappa = \exp(-K) > 0$ a small constant.

Let $\psi_\delta = \exp \circ \tilde{\psi}_\delta$. Then ψ_δ is a holomorphic map which has maximal rank everywhere on $\Delta_{3/4}$. By the discussion above, there exists some constant $\kappa > 0$ sufficiently small, such that

$$\Delta^*_\kappa \subset \bigcup_{\delta\in(0,1)} \psi_\delta(\Delta_{3/4}). \tag{3.14}$$

It is easy to check the following properties of the map ψ_δ.

- The pullback of the model metric ω_{mdl} is

$$\psi^*_\delta \omega_{mdl} = \frac{\sqrt{-1}dw \wedge d\bar{w}}{(1-|w|^2)^2} \tag{3.15}$$

  which is independent of δ and C^∞-quasi-isometric to the Euclidean metric on the disc $\Delta_{3/4}$.

- The pullback of the function $-\log|z|^2$ is

$$\psi^*_\delta(-\log|z|^2) = 2\delta^{-1} \operatorname{Re}\left(\frac{1+\omega}{1-\omega}\right) = \mathcal{O}(\delta^{-1}) \text{ as } \delta \to 0. \tag{3.16}$$

Now let us come back to our manifold. Given a point $x \in D$, we can take some open neighborhood U of x such that under the local coordinates $(z_1, \cdots, z_n)$, $U \cap D_j = \{z \in U : z^j = 0\}$ for $1 \leq j \leq k$ and $U \backslash D = (\Delta^*_\kappa)^k \times \Delta_1^{n-k}$. Let $\delta = (\delta^1, \ldots, \delta^k) \in (0,1)^k$ be a multi-index. Define

$$\Pi_\delta = \prod_{j=1}^k \delta^j. \tag{3.17}$$

Define the following map

$$\begin{aligned}\Psi_\delta : \Delta^k_{3/4} \times \Delta^{n-k}_1 &\to (\Delta^*_\kappa)^k \times \Delta^{n-k}_1, \\ (w_1, \ldots, w_n) &\mapsto \big(\psi_{\delta^1}(w_1), \ldots, \psi_{\delta^k}(w_k), w_{k+1}, \ldots, w_m\big)\end{aligned} \tag{3.18}$$

where each ψ_{δ_j} is defined as previous. For each multi-index $\delta \in (0,1)^k$, the map Ψ_δ is a holomorphic map from $\Delta^k_{3/4} \times \Delta^{n-k}_1$ to $U \backslash D$ which is of maximal rank everywhere in $\Delta^k_{3/4} \times \Delta^{n-k}_1$. Thus, the triple

$$(\Delta^k_{3/4} \times \Delta^{n-k}_1; w_1, \cdots, w_n; \Psi_\delta)$$

is a local quasi-coordinate for $U \backslash D$.

Similar to the punctured disc case, we have the following properties of the map Ψ_δ.

- $U \backslash D$ is covered by $\bigcup_{\delta \in (0,1)^k} \Psi_\delta(\Delta^k_{3/4} \times \Delta^{n-k}_1)$.
- The pullback of the model metric ω_{mdl} on $(\Delta^*_\kappa)^k \times \Delta^{n-k}_1$ is

$$\Psi^*_\delta \omega_{mdl} = \sum_{j=1}^k \frac{\sqrt{-1} dw^j \wedge d\bar{w}^j}{(1 - |w^j|^2)^2} + \sum_{j=k+1}^n \sqrt{-1} dw^j \wedge d\bar{w}^j \tag{3.19}$$

  which is independent of the multi-index δ and C^∞-quasi-isometric to the Euclidean metric on $\Delta^k_{3/4} \times \Delta^{n-k}_1$. Since the Poincarè type metric ω on $U \backslash D$ is quasi-isometric to the model metric ω_{mdl}, the pullback $\Psi^*_\delta \omega$ is quasi-isometric to the Euclidean metric on $\Delta^k_{3/4} \times \Delta^{n-k}_1$.
- For $j \in \{1, \cdots, k\}$, $\rho_j \sim -\log |z^j|^2$ under the local coordinates, hence, $\Psi^*_\delta \rho_j = \mathcal{O}((\delta^j)^{-1})$. For $j \in \{k+1, \cdots, N\}$, ρ_j is smooth across D, hence, $\Psi^*_\delta \rho_j$ is bounded. It follows that

$$\Psi^*_\delta \rho = \prod_{j=1}^N \Psi^*_\delta \rho_j = \mathcal{O}(\prod_{j=1}^k (\delta^j)^{-1}) = \mathcal{O}(\Pi^{-1}_\delta) \text{ as } \Pi_\delta \to 0. \tag{3.20}$$

We can cover an open neighborhood of the divisor D by the local quasi-coordinate charts constructed above, and cover the complement of the neighborhood by a finite number of unit balls in $\mathbb{C}^n$. Note that the latter are automatically local quasi-coordinate charts. Thus, this gives a local quasi-coordinate system to our manifold (M, ω).

The quasi-coordinate system can be used to define the Hölder norms and Hölder spaces which are useful for the Schauder estimate on (M, ω).

Definition 3.5. *Given a local quasi-coordinate system* $\{(V_\alpha, \Psi_\alpha) : \alpha \in \mathcal{A}\}$ *for* (M, ω)*, for any non-negative integer* k *and real number* $\lambda \in (0,1)$*, define*

$$\|u\|_{k,\lambda} := \sup_{\alpha\in\mathcal{A}} \|u \circ \Psi_\alpha\|_{C^{k,\lambda}(V_\alpha)} \tag{3.21}$$

where $\|\cdot\|_{C^{k,\lambda}(V_\alpha)}$ *is the usual Hölder norm on* $V_\alpha \subset \mathbb{C}^n$*. The Hölder space* $C^{k,\lambda}(M)$ *is defined as*

$$C^{k,\lambda}(M) := \{u \in C^k_{\text{loc}}(M) : \|u\|_{k,\lambda} < \infty\}.$$

The Hödler space $C^{k,\lambda}(M)$ is a Banach space with the norm $\|\cdot\|_{k,\lambda}$. The definition of the norm $\|\cdot\|_{k,\lambda}$ depends on the choice of the local quasi-coordinate system. However, given two local quasi-coordinate systems and the associated $C^{k,\lambda}$ norms, one can show that they are indeed equivalent.

3.3 Poincaré inequality

Lemma 3.6 (Auvray, [3, Lemma 1.10])**.** *There exists a constant* $C_P > 0$ *such that for all* $u \in H^1(M, \omega)$*, we have*

$$\int_M |u - \bar{u}|^2 d\mu \le C_P \int_M |du|^2 d\mu \tag{3.22}$$

where $\bar{u} = \frac{1}{Vol(M)} \int_M u d\mu$*.*

Proof. Start, for simplicity, by the case where D is smooth. We cover it in $\overline{M}$ with open sets of coordinates U_j, $j = 1, \ldots, s$, of the form $\Delta_{3/4} \times \Delta_1^{n-1}$, so that $D \cap U_j = \{|z| = 0\}$. Consider a neighbourhood U of D such that $U \subset \bigcup_{j=1}^M U_j$. Let $v \in C_c^\infty(\overline{U}\backslash D)$ such that $v|_{\partial U} \equiv 0$. We are first seeing that there exists $c > 0$ such that for all j,

$$\int_{U_j\backslash D} |v|^2 d\mu \le c \int_{U_j\backslash D} |dv|^2 d\mu. \tag{3.23}$$

We can assume, up to modifying c, that ω restricted to $U_j\backslash D$ writes as $\omega = \frac{\sqrt{-1}dz\wedge d\bar{z}}{|z|^2\log^2|z|^2} + ds^2$ where ds^2 the Euclidean metric on Δ_1^{n-1}. Now change the coordinates by setting $t = \log(\log^2|z|^2) \in (A, \infty)$ and $\theta = \arg z \in S^1$. Then the metric ω becomes $dt^2 + e^{-2t}d\theta^2 + ds^2$, and the volume form is $e^{-t}dtd\theta ds$. Thus,

$$\int_{U_j\backslash D} |v|^2 d\mu = \int_{S^1\times\Delta_1^{n-1}} d\theta ds \int_A^{+\infty} |v|^2 e^{-t} dt \tag{3.24}$$

and

$$\int_{U_j\backslash D}|dv|^2d\mu=\int_{S^1\times\Delta_1^{n-1}}d\theta ds\int_A^{+\infty}|dv|^2e^{-t}dt \tag{3.25}$$

Note that

$$|dv|^2=(\partial_t v)^2+e^{2t}(\partial_\theta v)^2+|d_{\Delta_1^{m-1}}v|^2_{ds^2}\geq(\partial_t v)^2. \tag{3.26}$$

To obtain (3.23), it suffices to show that

$$\int_A^{+\infty}v^2e^{-t}dt\leq c\int_A^{+\infty}(\partial_t v)^2e^{-t}dt \text{ for all } (\theta,s). \tag{3.27}$$

Set $w(t)=e^{-t}$. Let $'$ stands for ∂_t, then we have $(v^2w)'=2vv'w+v^2w'=2vv'w-v^2w$, hence by integrating with fixed θ and s, $0=2\int_A^{+\infty}vv'e^{-t}dt-\int_A^{+\infty}v^2e^{-t}dt$ because $v\equiv 0$ on $\{t=A\}$ and for t big enough. We rewrite this as:

$$\int_A^{+\infty}v^2e^{-t}dt=2\int_A^{+\infty}vv'e^{-t}dt\leq 2\Big(\int_A^{+\infty}v^2e^{-t}dt\Big)^{\frac{1}{2}}\Big(\int_A^{+\infty}v'^2e^{-t}dt\Big)^{\frac{1}{2}} \tag{3.28}$$

by Cauchy-Schwartz, hence

$$\int_A^{+\infty}v^2e^{-t}dt\leq 4\int_A^{+\infty}v'^2e^{-t}dt. \tag{3.29}$$

This ends the first point of demonstration. We then have that for any $v\in C_c^\infty\big(\overline{U}\backslash D\big)$

$$\int_{U\backslash D}|v|^2d\mu\leq\sum_{j=1}^{s}\int_{U_j\backslash D}|v|^2d\mu\leq c\sum_{j=1}^{s}\int_{U_j\backslash D}|dv|^2d\mu\leq cs\int_{U\backslash D}|dv|^2d\mu. \tag{3.30}$$

Now seek a contradiction, and take a sequence of functions $f_j\in C_c^\infty(M)$ violating the theorem; we thus can consider that

- for all j, $\int_M f_jd\mu=0$ and $\int_M f_j^2d\mu=1$;
- $\lim_{j\to\infty}\int_M|df_j|^2d\mu=0$.

Observe that (f_j) is bounded in $H^1(M,\omega)$, hence up to an extraction converges weakly in $H^1(M,\omega)$ to a function $f\in H^1(M,\omega)$. In particular, $\|df\|_{L^2}=0$, that is to say f is constant, since the df_j tend to 0 in L^2. Now finally, by weak L^2 convergence, $\int_M fd\mu=\lim_{j\to\infty}\int_M f_jd\mu=0$, hence $f\equiv 0$.

Take $\varepsilon > 0$ small, such that $3\varepsilon^2 < (cs)^{-1}$ say, and a domain $V \subset\subset M$ wide enough so that $U^c \subset\subset V$ and there exists a smooth cut-off function χ equal to 1 on U^c, 0 on V^c, and such that $0 \leq \chi \leq 1$ and $|d\chi| \leq \varepsilon$. For all j set $u_j = (1-\chi)f_j$ and $v_j = \chi f_j$ so that $u_j \in C_c^\infty\big(\overline{U}\backslash D\big)$, $(u_j)_{|\partial U} \equiv 0$, $v_j \in C_c^\infty(V)$ and $f_j = u_j + v_j$. Thus for all j,

$$\int_M f_j^2 d\mu \leq 2\Big(\int_M u_j^2 d\mu + \int_M v_j^2 d\mu\Big) = 2\Big(\int_{U\backslash D} u_j^2 d\mu + \int_V v_j^2 d\mu\Big).$$

Now on the one hand, (v_j) converges weakly to 0 in $H^1\big(\overline{V}, g\big)$ — just see that for all test function φ (resp. test 1-form α) on V, $\chi\varphi$ is again a test function (resp. $\chi\alpha$ a test 1-form and $(d\chi, \alpha)_g$ a test function) — and since $\overline{V}$ is compact with boundary, we can assume (forgetting another extraction) that (v_j) strongly converges to 0 in L^2, necessarily to 0.

On the other hand, according to the beginning of this demonstration, for all j we have

$$\begin{aligned}\int_{U\backslash D} u_j^2 d\mu &\leq cs \int_{U\backslash D} |du_j|^2 d\mu \\ &= cs\left(\int_{U\backslash D} \chi^2 |df_j|^2 d\mu + \int_{U\backslash D} f_j^2 |d\chi|^2 d\mu + 2\int_{U\backslash D} f_j \chi (df_j, d\chi)_\omega d\mu\right).\end{aligned}$$

In the latter line, the first integral is bounded above by $\int_M |df_j|^2 d\mu$ which tends to 0; the second one by $\varepsilon^2 \int_M f_j^2 d\mu = \varepsilon^2$, and the third by the square root of the first two. It thus follows that $\int_M f_j^2 d\mu \leq 2cs\varepsilon^2 < 1$ when j is big enough, a contradiction, hence the theorem for $C_c^\infty(M)$ functions, and then for $H^1(M, \omega)$ functions by density.

Now let us consider the case where D admits crossings. If we have an inequality for smooth functions with a compact support near D like (3.30), the end of the argument will apply unchanged. To get this inequality though, cover D with polydiscs of coordinates $\kappa\mathcal{P}_k = \Delta_\kappa^k \times \Delta^{n-k}$ such that D is given in those by $\{z^1 \cdots z^k = 0\}$. One point is that to get the desired inequality with U an open set relatively compact in the union of our polydiscs, it is enough to show such an inequality for functions $v \in C_c^\infty\big(\overline{\kappa\mathcal{P}_k}\backslash D\big)$ with $v \equiv 0$ on $\{|z^1| = a_k\} \cap \cdots \cap \{|z^k| = a_k\}$. But this we can do assuming ω is the product metric $\sum_{j=1}^k \frac{\sqrt{-1}dz^j \wedge d\bar{z}^j}{|z^j|^2 \log^2 |z^j|^2} + ds^2$, i.e. $dt_1^2 + \cdots + dt_k^2 + e^{-2t_1} d\theta_1^2 + \cdots + e^{-2t_k} d\theta_k^2 + ds^2$ where $t^j = \log(\log^2 |z^j|^2) \in (A_k, \infty)$, $\theta_j = \arg z^j \in S^1$, $j = 1, \ldots, k$. Finally, express $(t_1, \ldots, t_k)$ in polar coordinates $(r, \varphi_1, \ldots, \varphi_{k-1})$, φ_1, ..., $\varphi_{k-1} \in (0, \pi/2)$, $r \in (r(\varphi_1, \ldots, \varphi_{k-1}), \infty)$, and do the same integration by parts as above with $'$ standing for ∂_r in order to conclude. □

3.4 Weighted Sobolev inequality

The usual Sobolev inequality fails on (M,ω) because the volume form is degenerate near the infinity (the divisor D). In this section we introduce a *weighted* Sobolev inequality from [3]. But ahead of that let us first present a useful lemma which connects the integration on M with integration on the quasi-coordinate charts. Let $r\mathcal{P}_k = \Delta_r^k \times \Delta_1^{n-k}$ and let $r\mathcal{P}_k^* = (\Delta_r^*)^k \times \Delta_1^{n-k}$. Let $d\mu_{mdl}$ be the volume form of the model metric ω_{mdl} on $\kappa\mathcal{P}_k^* = (\Delta_\kappa^*)^k \times \Delta_1^{n-k}$. Let $d\mu_0$ be the standard Euclidean volume on $\mathbb{C}^n$.

Lemma 3.7. *There exists a sequence of multi-indices* $\{\delta_l = (\delta_l^1, \cdots, \delta_l^k) : l = 1, 2, \cdots\}$ *and a constant* $c > 0$*, such that for any* $f \in L^1_{\mathrm{loc}}(\kappa\mathcal{P}_k^*, d\mu_{mdl})$ *we have*

$$c^{-1}\sum_{l=1}^{\infty}\Pi_{\delta_l}\int_{\frac{3}{4}\mathcal{P}_k}|\Psi_{\delta_l}^* f|d\mu_0 \;\le\; \int_{\kappa\mathcal{P}_k^*}|f|d\mu_{mdl} \;\le\; c\sum_{l=1}^{\infty}\Pi_{\delta_l}\int_{\frac{1}{2}\mathcal{P}_k}|\Psi_{\delta_l}^* f|d\mu_0 \tag{3.31}$$

where Π_{δ_l} *is defined as* (3.17) *and* Ψ_{δ_l} *is the quasi-coordinate map constructed as* (3.18) *for each multi-index* δ_l.

Proof. The proof is technical. We shall begin with the simplest case $k = 1$, namely, $\kappa\mathcal{P}_k^* = \Delta_\kappa^* \times \Delta_1^{n-1}$. Then $\Pi_{\delta_l} = \delta_l^{-1}$. Let

$$\mathcal{B} = \{z \in \mathbb{C} \mid -\pi < \operatorname{Im} z < \pi,\ \ -\infty < \operatorname{Re} z < \log\kappa\}.$$

The exponential map $\exp : w \to \exp(w)$ is a biholomorphism from $\mathcal{B}$ to Δ_κ^* minus the positive real line. We can thus pullback the integral over $\kappa\mathcal{P}_1^*$ to $\mathcal{B} \times \Delta_1^{n-1} \subset \mathbb{C}^n$ by change of variables. Note that

$$\exp^*\omega_{mdl} = \frac{\sqrt{-1}dw_1 \wedge d\bar{w}_1}{(2\operatorname{Re} w_1)^2} + \sum_{j=1}^{n}\sqrt{-1}dw_j \wedge d\bar{w}_j. \tag{3.32}$$

Let $\hat{f}(w_1, w_2, \cdots, w_n) = f(\exp(w_1), w_2, \cdots, w_n)$. Then

$$\int_{\kappa\mathcal{P}_1^*}|f|d\mu_{mdl} = \int_{\mathcal{B}\times\Delta_1^{n-1}}\frac{|\hat{f}|}{(2\operatorname{Re} w_1)^2}d\mu_0. \tag{3.33}$$

On the other hand, note that $\Phi_\delta^* f = \tilde{\psi}_\delta^*\hat{f}$ where $\tilde{\psi}_\delta$ is holomorphic map $\tilde{\psi}_\delta(w) = \delta^{-1}\frac{1+w}{1-w}$ from Δ_r to $\mathbb{C}^n$. Let $\sim$ denotes that two quantities are mutually bounded each other by constant factors independent of δ. By change of variables, it follows that

$$\int_{r\mathcal{P}_1} |\Phi_\delta^* f| d\mu_0 \sim \int_{\tilde{\psi}_\delta(\Delta_r)\times\Delta_1^{n-1}} \frac{|\hat{f}|}{\delta^2} d\mu_0. \tag{3.34}$$

Let $\eta_r = (1+r^2)/(1-r^2)$ and $\zeta_r = 2r/(1-r^2)$. Note that $\psi_\delta(\Delta_r)$ is the ball centered at $(-\delta^{-1}\eta_r, 0) \in \mathbb{R}^2$ with radius $\delta^{-1}\zeta_r$. Let $B_{r,\delta}$ be the open square centered at $(-\delta^{-1}\eta_r, 0)$ with side length $2\delta^{-1}\zeta_r$ and $B'_{r,\delta}$ be the open square centered at $(-\delta^{-1}\eta_r, 0)$ with side length $\sqrt{2}\delta^{-1}\zeta_r$. Let

$$\mathcal{B}_{\delta,r} = \{z \in B_{r,\delta} \mid -\pi < \operatorname{Im} z < \pi\},\ \mathcal{B}'_{\delta,r} = \{z \in B'_{r,\delta} \mid -\pi < \operatorname{Im} z < \pi\}.$$

Then we have $B'_{r,\delta} \subset \tilde{\psi}_\delta(\Delta_r) \subset B_{r,\delta}$. Over the square $B_{r,\delta}$, $\operatorname{Re} w \sim \delta^{-1}$ and

$$\int_{B_{\delta,r}\times\Delta_1^{n-1}} |\hat{f}| d\mu_0 \sim \delta^{-1} \int_{\mathcal{B}_{\delta,r}\times\Delta_1^{n-1}} |\hat{f}| d\mu_0, \tag{3.35}$$

$$\int_{B'_{\delta,r}\times\Delta_1^{n-1}} |\hat{f}| d\mu_0 \sim \delta^{-1} \int_{\mathcal{B}'_{\delta,r}\times\Delta_1^{n-1}} |\hat{f}| d\mu_0. \tag{3.36}$$

It then follows that there exist constant C_r and C'_r depending only on r such that

$$\int_{\mathcal{B}_{\delta,r}\times\Delta_1^{n-1}} \frac{|\hat{f}|}{(2\operatorname{Re} w_1)^2} d\mu_0 \geq C_r \delta^{-1} \int_{\Delta_r\times\Delta_1^{n-1}} |\Phi_\delta^* f| d\mu_0. \tag{3.37}$$

$$\int_{\mathcal{B}'_{\delta,r}\times\Delta_1^{n-1}} \frac{|\hat{f}|}{(2\operatorname{Re} w_1)^2} d\mu_0 \leq C'_r \delta^{-1} \int_{\Delta_r\times\Delta_1^{n-1}} |\Phi_\delta^* f| d\mu_0. \tag{3.38}$$

Now pick a sequence $\{\delta_l, l = 1, 2, \cdots\}$ such that $\delta_1 = -\log\kappa, \delta_{l+1} = 2\delta_l$. For $r \geq 1/2$, it is easy to check that $\mathcal{B} \subseteq \bigcup_{l=1}^\infty \mathcal{B}'_{r,\delta_l} \subseteq \bigcup_{l=1}^\infty \mathcal{B}_{r,\delta_l}$. Moreover, each $\mathcal{B}_{r,\delta_l}$ intersects with other squares in the sequence $\{\mathcal{B}_{r,\delta_l} \mid l = 1, 2, \cdots\}$ at most a fixed number of times N_r depending on r only. Thus, we have

$$\begin{aligned}\int_{\mathcal{B}\times\Delta_1^{n-1}} \frac{|\hat{f}|}{(2\operatorname{Re} w_1)^2} d\mu_0 &\leq \sum_{l=1}^\infty \int_{\mathcal{B}_{r,\delta_l}\times\Delta_1^{n-1}} \frac{|\hat{f}|}{(2\operatorname{Re} w_1)^2} d\mu_0 \\ &\leq C'_r \sum_{l=1}^\infty \delta_l^{-1} \int_{\Delta_r\times\Delta_1^{n-1}} |\Phi_{\delta_l}^* f| d\mu_0\end{aligned} \tag{3.39}$$

and

$$N_r \int_{\mathcal{B}\times\Delta_1^{n-1}} \frac{|\hat{f}|}{(2\operatorname{Re} w_1)^2} d\mu_0 \geq \sum_{l=1}^{\infty} \int_{\mathcal{B}_{r,\delta_l}\times\Delta_1^{n-1}} \frac{|\hat{f}|}{(2\operatorname{Re} w_1)^2} d\mu_0 \geq C_r \sum_{l=1}^{\infty} \delta_l^{-1} \int_{\Delta_r\times\Delta_1^{n-1}} |\Phi_{\delta_l}^* f| d\mu_0. \tag{3.40}$$

It follows that

$$N_r^{-1} C_r \sum_{l=1}^{\infty} \delta_l^{-1} \int_{r\mathcal{P}_1} |\Phi_{\delta_l}^* f| d\mu_0 \leq \int_{\kappa\mathcal{P}_1^*} |f| d\mu_{mdl} \leq C_r' \sum_{l=1}^{\infty} \delta_l^{-1} \int_{r\mathcal{P}_1} |\Phi_{\delta_l}^* f| d\mu_0. \tag{3.41}$$

Taking $r = 3/4$ for the left inequality and $r = 1/2$ for the right inequality, we then have (3.31) for some constant c. This proves the case $k = 1$. For $k \geq 2$, it can be done by induction on k. □

Lemma 3.8. *Suppose $x \in D$ is on a normal crossing of codimension k. Take a polydisc U centered at x such that $U \backslash D = \kappa\mathcal{P}_k^*$. There exist a sequence of multi-indices $\{\delta_l = (\delta_l^1, \cdots, \delta_l^k) : l = 1, 2, \cdots\}$ and a constant $c > 0$, such that for any $u \in W^{m,p}_{\text{loc}}(U\backslash D, d\mu)$ we have*

$$c^{-1} \sum_{l=1}^{\infty} \|\Psi_{\delta_l}^* u\|^p_{W^{m,p}(\frac{3}{4}\mathcal{P}_k)} \leq \int_{U\backslash D} \sum_{j=0}^{m} |\nabla^j u|^p \rho d\mu \leq c \sum_{l=1}^{\infty} \|\Psi_{\delta_l}^* u\|^p_{W^{m,p}(\frac{1}{2}\mathcal{P}_k)}. \tag{3.42}$$

Proof. By Lemma 3.7, there exists a sequence of multi-indices $\{\delta_l = (\delta_l^1, \cdots, \delta_l^k) : l = 1, 2, \cdots\}$ and a constant $c' > 0$ such that $\int_{U\backslash D} \sum_{j=0}^{m} |\nabla^j u|^p \rho d\mu$ is mutually bounded with

$$(c')^{-1} \sum_{l=1}^{\infty} \Pi_{\delta_l} \int_{\frac{3}{4}\mathcal{P}_k} \sum_{j=0}^{m} |\Psi_{\delta_l}^*(\nabla^j u)|^p (\Psi_{\delta_l}^* \rho) d\mu_0 \leq \int_{U\backslash D} \sum_{j=0}^{m} |\nabla^j u|^p \rho d\mu \leq c' \sum_{l=1}^{\infty} \Pi_{\delta_l} \int_{\frac{1}{2}\mathcal{P}_k} \sum_{j=0}^{m} |\Psi_{\delta_l}^*(\nabla^j u)|^p (\Psi_{\delta_l}^* \rho) d\mu_0. \tag{3.43}$$

Note that $\Psi_{\delta_l}^* \rho$ is mutually bounded with $\Pi_{\delta_l}^{-1}$ uniformly. Moreover, the metric ω is quasi-isometric to the model metric ω_{mdl} on $U\backslash D = \kappa\mathcal{P}_k^*$. The pullback of ω_{mdl} under Ψ_{δ_l} is invariant of the multi-index δ_l and is C^∞-quasi-isometric to the Euclidean metric on $\mathbb{C}^n$. Thus, $\sum_{j=0}^{m} |\Psi_{\delta_l}^*(\nabla^j u)|^p$ is mutually bounded with $\sum_{j=0}^{m} |\nabla_{g_0}(\Psi_{\delta_l}^* u)|_{g_0}^p$ where g_0 is the Euclidean metric on $\mathbb{C}^n$ and ∇_{g_0} is the covariant derivative with respect to g_0. Thus, we obtain the inequality (3.42). □

Now we prove the following weighted Sobolev inequality on (M, ω).

Lemma 3.9 (Auvray, [3, Lemma 4.4])**.** *For any function $u \in W_{\mathrm{loc}}^{1,p}(M)$, there exists a positive constant $C_p = C(p, M, \omega)$ such that for any $q \geq p$ with $1/p \leq 1/(2n) + 1/q$ we have*

$$\left(\int_M |u|^q \rho d\mu \right)^{1/q} \leq C_p \left(\int_M (|v|^p + |\nabla v|^p) \rho d\mu \right)^{1/p}. \qquad (3.44)$$

Proof. We can cover an open neighborhood of D by a finite number of polydiscs $\{U_1, \cdots, U_s\}$ and cover the complement of the neighborhood by a finite number of unit balls $\{\mathbb{B}_1, \cdots, \mathbb{B}_t\}$ in $\mathbb{C}^n$.

On each $\mathbb{B}_j$, since it is away from the divisor, ρ is bounded on $\mathbb{B}_j$, hence, by the usual Sobolev inequality in bounded domain of $\mathbb{C}^n$ there exists a constant C'_p depending on p such that for any $p > 0$ with $1/p \leq 1/(2n) + 1/q$

$$\left(\int_{\mathbb{B}_j} |u|^q \rho d\mu \right)^{1/q} \leq C'_p \left(\int_{\mathbb{B}_j} (|v|^p + |\nabla v|^p) \rho d\mu \right)^{1/p}, \ j = 1, 2, \cdots, s.$$

On each $U_j \backslash D$, we can cover it by quasi-coordinate charts $\bigcup_\delta \Psi_\delta(\Delta_{3/4}^{k_j} \times \Delta_1^{n-k_j})$. For the each pullback $\Psi_\delta^* u$ on the polydisc $\Delta_{3/4}^{k_j} \times \Delta_1^{n-k_j}$ we have the standard Sobolev inequality

$$\|\Psi_\delta^* u\|_{L^q(\frac{3}{4}\mathcal{P}_{k_j})} \leq C'_p \|\Psi_\delta^* u\|_{W^{1,p}(\frac{3}{4}\mathcal{P}_{k_j})}, \ \forall \delta, \ j = 1, 2, \cdots, t$$

for any $q > 0$ with $1/p \leq 1/(2n) + 1/q$. For each j, by Lemma 3.8, there exist a sequence of multi-indices $\{\delta_l : l = 1, 2, \cdots\}$ and a positive constant c_j, so that

$$\begin{aligned}
\int_{U_j \backslash D} |u|^q \rho d\mu &\leq c_j \sum_{l=1}^{\infty} \|\Psi_{\delta_l}^* u\|_{L^q(\frac{3}{4}\mathcal{P}_k)}^q \leq c_j C_p'^q \sum_{l=1}^{\infty} \|\Psi_{\delta_l}^* u\|_{W^{1,p}(\frac{3}{4}\mathcal{P}_k)}^q \\
&\leq c_j C_p'^q \left(\sum_{l=1}^{\infty} \|\Psi_{\delta_l}^* u\|_{W^{1,p}(\frac{3}{4}\mathcal{P}_k)}^p \right)^{q/p} \quad (\text{since } q \geq p) \qquad (3.45) \\
&\leq c_j^{1+q/p} C_p'^q \left(\int_{U_j \backslash D} (|u|^p + |\nabla u|^p) \rho d\mu \right)^{q/p}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\left(\int_{U_j\setminus D}|u|^q\rho d\mu\right)^{1/q} &\le c_j^{1/p+1/q}C_p'\left(\int_{U_j\setminus D}(|v|^p+|\nabla v|^p)\rho d\mu\right)^{1/p}\\ &\le c_j^{2/p}C_p'\left(\int_{U_j\setminus D}(|v|^p+|\nabla v|^p)\rho dV\right)^{1/p}.\end{aligned} \tag{3.46}$$

It follows that

$$\begin{aligned}&\Big(\int_M|u|^q\rho d\mu\Big)^{1/q}\le\sum_{j=1}^s\Big(\int_{\mathbb{B}_j}|u|^q\rho d\mu\Big)^{1/q}+\sum_{j=1}^t\Big(\int_{U_j\setminus D}|u|^q\rho d\mu\Big)^{1/q}\\ &\le\sum_{j=1}^s C_p'\Big(\int_{\mathbb{B}_j}(|v|^p+|\nabla v|^p)\rho d\mu\Big)^{1/p}+\sum_{j=1}^t c_j^{2/p}C_p'\Big(\int_{U_j\setminus D}(|v|^p+|\nabla v|^p)\rho d\mu\Big)^{1/p}\\ &\le(s+\sum_{j=1}^t c_j^{2/p})C_p'\Big(\int_M(|v|^p+|\nabla v|^p)\rho d\mu\Big)^{1/p}.\end{aligned} \tag{3.47}$$

The proof is finished by taking $C_p=(s+\sum_{j=1}^t c_j^{2/p})C_p'$. □

As a corollary of Lemma 3.9, we show that the C^0 norm of F is controlled as long as $\mathcal{I}(F,p_0)<\infty$ for some $p_0>2n$.

Corollary 3.10. *Suppose $F\in W^{1,p_0}_{\mathrm{loc}}(M)$ satisfies $\mathcal{I}(F,p_0)<\infty$ for some $p_0>2n$. Then these exists some positive constants $C=C(\mathcal{I}(F,p_0),p_0,n,\omega)$ and such that $\|F\|_{C^0}\le C$.*

Proof. Note that

$$\int_M(|F|^{p_0}+|\nabla F|^{p_0})\rho d\mu\le\mathcal{I}(F,p_0)$$

since $p_0>2n$ and $\rho\ge1$. Moreover, $1/p_0\le 1/q+1/2n$ for any $q\ge p_0$. It follows from Lemma 3.9 that

$$\Big(\int_M|F|^q\rho d\mu\Big)^{1/q}\le C_{p_0}\Big(\int_M(|F|^{p_0}+|\nabla F|^{p_0})\rho d\mu\Big)^{1/p_0}\le C_{p_0}\mathcal{I}(F,p_0)^{1/p_0}.$$

The lemma follows by letting $q\to\infty$. □

3.5 The ε-perturbed equations

The ε-perturbed equation (2.6) can be normalized into the Monge-Ampère equation (2.7). The latter has been well studied by Tian-Yau [39] and K. Kobayashi [28]. The existence and uniqueness of the solution of (2.7) can be done by the continuity method in the quasi-coordinates. To summarize, the theorem of Tian-Yau and Kobayashi is the following:

Theorem 3.11 (Tian-Yau, Kobayashi)**.** *Let $\overline{M}$ be a compact Kähler manifold and D be an effective divisor on $\overline{M}$ with only simple normal crossings. Let $M = \overline{M} - D$ and ω be of Kähler metric on M of Poincaré type. For any $F \in C^{k,\lambda}(M)$ for some $k \geq 3$, there exists a solution $\varphi \in C^{k+2,\lambda}(M)$ to the following equation*

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{F+\varphi}\omega^n.$$

In particular, if $K_{\overline{M}} + [D]$ is ample, then there exists a (unique) Kähler-Einstein metric of curvature -1 equivalent to ω.

We temporarily assume that $F \in C_c^\infty(M)$, as we will show that F can be approximated by smooth functions with compact support if F lies in the weighted Sobolev space with $\mathcal{I}(F, p_0) < \infty$ for some $p_0 > 2n$. The existence of the solution to the ε-perturbed equation (2.6) can be derived from the theorem of Tian-Yau and Kobayashi. We summarize it in the following lemma.

Lemma 3.12. *Let $\overline{M}$ be a compact Kähler manifold and D be an effective divisor on $\overline{M}$ with only simple normal crossings. Let $M = \overline{M} - D$ and ω be of Kähler metric on M of Poincaré type. For any $F \in C_c^\infty(M)$, there exists a solution $\varphi_\varepsilon \in \bigcap_{k,\lambda} C^{k,\lambda}(M)$ to the equation*

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon)^n = e^{F+\varepsilon\varphi_\varepsilon}\omega^n$$

for any $\varepsilon \in (0, 1]$.

Our goal is to show that $\{\varphi_\varepsilon\}$ is compact in the usual $W^{3,p_0}(M)$. It amounts to show some uniform estimates on the gradient and Laplacian of φ_ε in terms of the integral bound $\mathcal{I}(F, p_0)$. In particular, this precludes the use of maximum principle in the Yau's classical Laplacian estimate, as it requires to use the C^2 norm of F. We will take the Chen-He's integration method to obtain the estimate. In order to deal with the boundary terms in the integration by parts, we need the following Gaffney-Stokes theorem.

Lemma 3.13 (Gaffney-Stokes, [24])**.** *Let (M, g) be an orientable complete Riemannnian manifold whose Riemannian tensor is of class C^2. Let η be an $(n-1)$-form of class C^1 such that both η and $d\eta$ are in L^1. Then $\int_M d\eta = 0$.*

Lemma 3.13 states a Stokes theorem for complete manifolds under suitable conditions.

4 Proof of the main theorem

In this section we present the proof of the main theorem 2.1.

4.1 Uniform C^0 estimate

The first step of deriving the uniform W^{3,p_0} estimate is to derive the uniform C^0 estimate. This part has been done by Auvray in [3]. For readers' convenience we cite the proof here. In what follows let $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon$ and ∇', Δ' and $d\mu'$ be the covariant derivative, Laplacian and volume form of the Kähler metric ω'. The constant C may vary from line to line, but always only depends on $\mathcal{I}(F, p_0)$, p_0, ω and n.

Proposition 4.1. *Let φ_ε be the solution for the ε-perturbed equation* (2.6)*. There exists a constant $C = C(\|F\|_{C^0}, n, \omega)$ independent of ε such that $\|\varphi_\varepsilon\|_{L^2} \leq C$.*

Proof. Let $T_\varepsilon = (\omega')^{n-1} + (\omega')^{n-2} \wedge \omega + \cdots + \omega^{n-1}$. Then $\omega^n - (\omega')^n = -\sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon \wedge T_\varepsilon$. It follows that

$$\int_M \varphi_\varepsilon(1 - e^{F+\varepsilon\varphi_\varepsilon})d\mu = \int_M \varphi_\varepsilon(\omega^n - (\omega')^n) = -\int_M \varphi_\varepsilon\sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon \wedge T_\varepsilon. \quad (4.1)$$

Note that

$$\varphi_\varepsilon\sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon \wedge T_\varepsilon = \frac{1}{2}\sqrt{-1}\partial\bar{\partial}(\varphi_\varepsilon^2 T_\varepsilon) - \sqrt{-1}\partial\varphi_\varepsilon \wedge \bar{\partial}\varphi_\varepsilon \wedge T_\varepsilon \quad (4.2)$$

and by Gaffney-Stokes (Lemma 3.13) we have $\int_M \sqrt{-1}\partial\bar{\partial}(\varphi_\varepsilon^2 T_\varepsilon) = 0$. It follows that

$$\int_M \varphi_\varepsilon(1 - e^{F+\varepsilon\varphi_\varepsilon})d\mu = \int_M \sqrt{-1}\partial\varphi_\varepsilon \wedge \bar{\partial}\varphi_\varepsilon \wedge T_\varepsilon. \quad (4.3)$$

Note that $\sqrt{-1}\partial\varphi_\varepsilon \wedge \bar{\partial}\varphi_\varepsilon \wedge T_\varepsilon \geq \sqrt{-1}\partial\varphi_\varepsilon \wedge \bar{\partial}\varphi_\varepsilon \wedge \omega^{n-1}$. Moreover,

$$\varphi_\varepsilon(1 - e^{F+\varepsilon\varphi_\varepsilon}) = \varphi_\varepsilon(1 - e^F) + e^F\varphi_\varepsilon(1 - e^{\varepsilon\varphi_\varepsilon}) \leq \varphi_\varepsilon(1 - e^F)$$

since $1 - e^{\varepsilon\varphi_\varepsilon}$ has the opposite sign of φ_ε. It follows that

$$\int_M |\nabla\varphi_\varepsilon|^2 d\mu = \int_M \sqrt{-1}\partial\varphi_\varepsilon \wedge \bar{\partial}\varphi_\varepsilon \wedge \omega^{n-1} \leq \int_M \varphi_\varepsilon(1 - e^F)d\mu. \tag{4.4}$$

Let $\psi_\varepsilon = \varphi_\varepsilon - \bar{\varphi}_\varepsilon$ where $\bar{\varphi}_\varepsilon = \frac{1}{\mathrm{Vol}(M)}\int_M \varphi_\varepsilon d\mu$ is the average of φ_ε over M. Note that $\int_M (e^F - 1)d\mu = 0$. It follows that

$$\int_M |\nabla\psi_\varepsilon|^2 d\mu = \int_M \psi_\varepsilon(1 - e^F)d\mu. \tag{4.5}$$

By the unweighted Poincaré inequality (Lemma 3.6) and Cauchy-Schwartz inequality, it then follows that

$$\|\psi_\varepsilon\|_{L^2} \leq C_P\|1 - e^F\|_{L^2} \leq C. \tag{4.6}$$

Now we estimate the average $\bar{\varphi}_\varepsilon$. To get an upper bound, first notice that

$$\int_M e^{F+\varepsilon\varphi_\varepsilon} d\mu = \int_M (\omega + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon)^n = \mathrm{Vol}(M). \tag{4.7}$$

By Jensen's inequality, it follows that $\int_M \varepsilon\varphi_\varepsilon e^F d\mu \leq 0$. Hence,

$$0 \geq \int_M \varphi_\varepsilon e^F d\mu = \int_M \varphi_\varepsilon(e^F - 1)d\mu + \mathrm{Vol}(M)\bar{\varphi}_\varepsilon. \tag{4.8}$$

It follows that

$$\begin{aligned}\bar{\varphi}_\varepsilon &\leq \frac{1}{\mathrm{Vol}(M)}\int_M \varphi_\varepsilon(1 - e^F)d\mu \\ &= \frac{1}{\mathrm{Vol}(M)}\int_M \psi_\varepsilon(1 - e^F)d\mu \leq \frac{C_p\|1 - e^F\|_{L^2}^2}{\mathrm{Vol}(M)} \leq C.\end{aligned} \tag{4.9}$$

On the other hand, to get a lower bound, notice that

$$\mathrm{Vol}(M) = \int_M e^F d\mu = \int_M e^{-\varepsilon\varphi_\varepsilon} d\mu'. \tag{4.10}$$

By Jensen's inequality it implies that $\int_M \varphi_\varepsilon d\mu' \geq 0$. It follows that

$$\begin{aligned}\mathrm{Vol}(M)\bar{\varphi}_\varepsilon &\geq \int_M \varphi_\varepsilon(d\mu - d\mu') = \int_M \psi_\varepsilon(1 - e^{F+\epsilon\varphi_\varepsilon})d\mu \\ &\geq -\|1 - e^{F+\varepsilon\varphi_\varepsilon}\|_{L^2}\|\psi_\varepsilon\|_{L^2}.\end{aligned} \tag{4.11}$$

For each $\varepsilon \in (0,1]$, by applying the maximum principle to the perturbed equation $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon)^n = e^{F+\varepsilon\varphi_\varepsilon}\omega^n$, it follows that $\|\epsilon\varphi_\varepsilon\|_{C^0} \leq \|F\|_{C^0}$. Hence,

$$\|1 - e^{F+\varepsilon\varphi_\varepsilon}\|_{L^2} \leq (1 + e^{2\|F\|_{C^0}}) \operatorname{Vol}(M)^{1/2}. \tag{4.12}$$

It then follows that

$$\bar{\varphi}_\varepsilon \geq -\operatorname{Vol}(M)^{-1/2}(1 + e^{2\|F\|_{C^0}})\|\psi_\varepsilon\|_{L^2} \geq -C. \tag{4.13}$$

It then follows from (4.6), (4.9) and (4.13) to obtain that $\|\varphi_\varepsilon\|_{L^2} \leq C$ where the constant C depends only on $\|F\|_{C^0}$, n and ω. □

Proposition 4.2. *Suppose $F \in C_c^\infty(M)$ with $\mathcal{I}(F, p_0) < \infty$ for some $p_0 > 2n$. Let φ_ε be the solution for the ε-perturbed equation (2.6). There exists a constant $C = C(\mathcal{I}(F, p_0), p_0, n, \omega)$ independent of ε such that $\|\varphi_\varepsilon\|_{C^0} \leq C$.*

Proof. Step 1. For $p \geq 2$, by a direct computation we have

$$\begin{aligned}
&\int_M |\varphi_\varepsilon|^{p-2}\varphi_\varepsilon(1 - e^{F+\varepsilon\varphi_\varepsilon})d\mu = \int_M |\varphi_\varepsilon|^{p-2}\varphi_\varepsilon(\omega^n - (\omega')^n) \\
&= \int_M |\varphi_\varepsilon|^{p-2}\varphi_\varepsilon\sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon \wedge T_\varepsilon \\
&= \frac{1}{p}\int_M \sqrt{-1}\partial\bar{\partial}(|\varphi_\varepsilon|^{p-1}\varphi_\varepsilon T_\varepsilon) - (p-1)\int_M |\varphi_\varepsilon|^{p-2}\sqrt{-1}\partial\varphi_\varepsilon \wedge \bar{\partial}\varphi_\varepsilon \wedge T_\varepsilon.
\end{aligned} \tag{4.14}$$

By Gaffney-Stokes (Lemma 3.13), we have $\int_M \sqrt{-1}\partial\bar{\partial}(|\varphi_\varepsilon|^{p-1}\varphi_\varepsilon T_\varepsilon) = 0$. Moreover, $\varphi_\varepsilon(1 - e^{F+\varepsilon\varphi_\varepsilon}) \leq \varphi_\varepsilon(1 - e^F)$ and

$$\begin{aligned}
&\int_M |\varphi_\varepsilon|^{p-2}\sqrt{-1}\partial\varphi_\varepsilon \wedge \bar{\partial}\varphi_\varepsilon \wedge T_\varepsilon = \frac{4}{p^2}\int_M \sqrt{-1}\partial(|\varphi_\varepsilon|^{p/2}) \wedge \bar{\partial}(|\varphi_\varepsilon|^{p/2}) \wedge T_\varepsilon \\
&\geq \frac{4}{p^2}\int_M \sqrt{-1}\partial(|\varphi_\varepsilon|^{p/2}) \wedge \bar{\partial}(|\varphi_\varepsilon|^{p/2}) \wedge \omega^{n-1} = \frac{4}{p^2}\int_M |\nabla|\varphi_\varepsilon|^{p/2}|^2 d\mu.
\end{aligned} \tag{4.15}$$

It follows that

$$\int_M \left|\nabla|\varphi_\varepsilon|^{p/2}\right|^2 d\mu \leq \frac{p^2}{4(p-1)}\int_M |\varphi_\varepsilon|^{p-2}\varphi_\varepsilon(1 - e^F)d\mu. \tag{4.16}$$

Step 2. Let $\gamma = 2n/(2n-1)$. Note that $1 = 1/(2n) + 1/\gamma$. By the weighted Sobolev inequality (Lemma 3.9),

$$\Big(\int_M \big(|\varphi_\varepsilon|^p\rho^{-1}\big)^\gamma \rho d\mu\Big)^{\frac{1}{\gamma}} \leq C\Big(\int_M \big|\nabla(|\varphi_\varepsilon|^p\rho^{-1})\big|\rho d\mu + \int_M |\varphi_\varepsilon|^p d\mu\Big). \tag{4.17}$$

Note that $\rho^{-1}|\nabla\rho|$ is bounded on M. An easy computation yields

$$\begin{aligned}
&\int_M \big|\nabla(|\varphi_\varepsilon|^p\rho^{-1})\big|\rho d\mu \le \int_M \big|\nabla|\varphi_\varepsilon|^p\big|d\mu + \int_M |\varphi_\varepsilon|^p(\rho^{-1}|\nabla\rho|)d\mu \\
&= \int_M 2|\varphi_\varepsilon|^{p/2}|\nabla|\varphi_\varepsilon|^{p/2}|d\mu + \int_M |\varphi_\varepsilon|^p(\rho^{-1}|\nabla\rho|)d\mu \\
&\le \int_M |\nabla|\varphi_\varepsilon|^{\frac{p}{2}}|^2 d\mu + C\int_M |\varphi_\varepsilon|^p d\mu.
\end{aligned} \tag{4.18}$$

By the unweighted Poincaré inequality Lemma 3.6 (with a mean term) to $|\varphi_\varepsilon|^{p/2}$, we have

$$\int_M |\varphi_\varepsilon|^p d\mu \le C_P\int_M \big|\nabla|\varphi_\varepsilon|^{p/2}\big|^2 d\mu + \mathrm{Vol}(M)^{-1}\big(\int_M |\varphi_\varepsilon|^{p/2}d\mu\big)^2 \tag{4.19}$$

Combining (4.16), (4.17), (4.18) and (4.19) we get

$$\big(\int_M |\varphi_\varepsilon|^{\gamma p}\rho^{-\frac{1}{2n-1}}d\mu\big)^{1/\gamma} \le Cp\int_M |\varphi_\varepsilon|^{p-1}|e^F-1|d\mu + C\big(\int_M |\varphi_\varepsilon|^{p/2}d\mu\big)^2. \tag{4.20}$$

Let $d\tilde{\mu}$ denote the measure $\rho^{-1/(2n-1)}d\mu$. Note that $|e^F-1| \le C|F|$ for constant C depending on $\|F\|_{C^0}$. We have

$$\big(\int_M |\varphi_\varepsilon|^{\gamma p}d\tilde{\mu}\big)^{1/\gamma} \le Cp\int_M |\varphi_\varepsilon|^{p-1}|F|\rho^{\frac{1}{2n-1}}d\tilde{\mu} + C\big(\int_M |\varphi_\varepsilon|^{p/2}\rho^{\frac{1}{2n-1}}d\tilde{\mu}\big)^2. \tag{4.21}$$

Let $q_0 > 0$ such that $1/p_0 + 1/q_0 = 1$. By Hölder inequality,

$$\begin{aligned}
\int_M |\varphi_\varepsilon|^{p-1}|F|\rho^{\frac{1}{2n-1}}d\tilde{\mu} &\le \big(\int_M |F|^{p_0}\rho^{\frac{p_0}{2n-1}}d\tilde{\mu}\big)^{\frac{1}{p_0}}\big(\int_M |\varphi_\varepsilon|^{(p-1)q_0}d\tilde{\mu}\big)^{\frac{1}{q_0}} \\
&\le (\mathcal{I}(F,p_0))^{1/p_0}\big(\int_M |\varphi_\varepsilon|^{(p-1)q_0}d\tilde{\mu}\big)^{1/q_0} \\
&\le C\|\varphi_\varepsilon\|^p_{L^{pq_0}(d\tilde{\mu})}.
\end{aligned} \tag{4.22}$$

Let $1/p_1 + 1/(2q_1) = 1$ and $n/(2n-1) < q_1 < 2n/(2n-1)$, then $p_1 < 2n$. By Hölder inequality,

$$\big(\int_M |\varphi_\varepsilon|^{p/2}\rho^{\frac{1}{2n-1}}d\mu\big)^2 \le \big(\int_M \rho^{\frac{p_1}{2n-1}}d\mu\big)^{1/p_1}\big(\int_M |\varphi_\varepsilon|^{pq_1}d\mu\big)^{1/q_1} \tag{4.23}$$

Since $p_1 < 2n$, we have

$$\int_M \rho^{\frac{p_1}{2n-1}}d\tilde{\mu} = \int_M \rho^{\frac{p_1-1}{2n-1}}d\mu < \infty.$$

Hence,

$$\Big(\int_M |\varphi_\varepsilon|^{p/2}\rho^{\frac{1}{2n-1}}d\mu\Big)^2 \le C\|\varphi_\varepsilon\|^p_{L^{pq_1}(d\tilde{\mu})}. \tag{4.24}$$

It then follows that

$$\|\varphi_\varepsilon\|^p_{L^{\gamma p}(d\tilde{\mu})} \le Cp\|\varphi_\varepsilon\|^p_{L^{pq_0}(d\tilde{\mu})} + C\|\varphi_\varepsilon\|^p_{L^{pq_1}(d\tilde{\mu})} \tag{4.25}$$

with $q_0, q_1 < 2n/(2n-1)$. Take $q_2 = \max(q_0, q_1)$. Then $q_2 < \gamma$ and

$$\|\varphi_\varepsilon\|_{L^{\gamma p}(d\tilde{\mu})} \le C^{1/p}p^{1/p}\|\varphi_\varepsilon\|_{L^{pp_2}(d\tilde{\mu})}. \tag{4.26}$$

Hence, by standard iteration process we have

$$\|\varphi_\varepsilon\|_{C^0} \le C\|\varphi_\varepsilon\|_{L^2(d\tilde{\mu})} \le C\|\varphi_\varepsilon\|_{L^2(d\mu)} \le C. \tag{4.27}$$

□

4.2 Uniform C^1 estimate

In this section we prove the C^1 estimate Theorem 2.2. In what follows let $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon$ and ∇', Δ' and $d\mu'$ be the covariant derivative, Laplacian and volume form of the Kähler metric ω'. The constant C may vary from line to line, but always only depends on $\mathcal{I}(F, p_0)$, p_0, ω and n.

Proof. Step 1. Let $A(t)$ be a one-variable smooth real function which will be determined later. Following a similar computation in [17, equation (3.11)], we have the following inequality

$$\begin{aligned} &\Delta'\Big(e^{-A(\varphi_\varepsilon)}|\nabla\varphi_\varepsilon|^2\Big) \\ &\ge e^{-A(\varphi_\varepsilon)}|\nabla\varphi_\varepsilon|^2\Big(-A''|\nabla'\varphi_\varepsilon|^2_{\omega'} + (A' - \inf_{i\ne j} R_{i\bar{i}j\bar{j}})\operatorname{tr}_{\omega'}\omega\Big) \\ &+ (2A' - B)e^{-A(\varphi_\varepsilon)}|\nabla'\varphi_\varepsilon|^2_{\omega'} - \big((n+2)A' + 2\varepsilon\big)\, e^{-A(\varphi_\varepsilon)}|\nabla\varphi_\varepsilon|^2 \\ &+ e^{-A(\varphi_\varepsilon)}(\Delta\varphi_\varepsilon - n + \operatorname{tr}_{\omega'}\omega) - 2e^{A(\varphi_\varepsilon)}|\nabla F||\nabla\varphi_\varepsilon|. \end{aligned} \tag{4.28}$$

Let $B > 0$ be a positive constant such that $\inf_{i\ne j} R_{i\bar{i}j\bar{j}} \ge -B$. Let $C_0 = 1 + \|\varphi_\varepsilon\|_{C^0}$. Choose

$$A(t) = (B+2)t - \frac{t^2}{2C_0}. \tag{4.29}$$

Then

$$B + 1 \le A'(\varphi_\varepsilon) = B + 2 - \frac{\varphi_\varepsilon}{C_0} \le B + 3,\ \ A''(\varphi_\varepsilon) = -\frac{1}{C_0}.$$

It is easy to see that

$$\operatorname{tr}_{\omega'} \omega \geq (\exp(F + \varepsilon\varphi_\varepsilon) \operatorname{tr}_\omega \omega')^{1/(n-1)}. \tag{4.30}$$

By (4.30), we compute

$$\begin{aligned}
&- A''|\nabla'\varphi_\varepsilon|^2_{\omega'} + (A' - \inf_{i\neq j} R_{i\bar{i}j\bar{j}}) \operatorname{tr}_{\omega'} \omega \\
&\geq \frac{1}{C_0}|\nabla'\varphi_\varepsilon|^2_{\omega'} + (\exp(F + \varepsilon\varphi_\varepsilon) \operatorname{tr}_\omega \omega')^{1/(n-1)} \\
&\geq n(n-1)^{-\frac{n-1}{n}} C_0^{-\frac{1}{n}} \left(\exp(F + \varepsilon\varphi_\varepsilon)(\operatorname{tr}_\omega \omega')|\nabla'\varphi_\varepsilon|^2_{\omega'}\right)^{\frac{1}{n}} \\
&\geq C|\nabla\varphi_\varepsilon|^{2/n}
\end{aligned} \tag{4.31}$$

for some C depending on $||F||_{C^0}, ||\varphi_\varepsilon||_{C^0}$ and n. Take (4.31) into (4.28) and drop the the nonnegative terms $(2A' - B)e^{-A(\varphi_\varepsilon)}|\nabla'\varphi_\varepsilon|^2_{\omega'}$ and $e^{-A(\varphi_\varepsilon)}tr_{g'}g$, and take the equality $n + \Delta\varphi_\varepsilon = tr_\omega\omega'$ into account, we have

$$\begin{aligned}
\Delta'\left(e^{-A(\varphi_\varepsilon)}|\nabla\varphi_\varepsilon|^2\right) \geq\ & Ce^{-A(\varphi_\varepsilon)}|\nabla\varphi_\varepsilon|^{2+\frac{2}{n}} - ((n+2)A' + 2)e^{-A(\varphi_\varepsilon)}|\nabla\varphi_\varepsilon|^2 \\
& + e^{-A(\varphi_\varepsilon)}(tr_\omega\omega' - 2n) - 2e^{-A(\varphi_\varepsilon)}|\nabla F||\nabla\varphi_\varepsilon|.
\end{aligned} \tag{4.32}$$

We can interpolate $|\nabla\varphi_\varepsilon|^2$ by $|\nabla\varphi_\varepsilon|^{2+2/n}$ and constants, i.e., there exists some sufficiently small positive constant ε and sufficiently large constant $C(\varepsilon)$ depending on ε such that

$$|\nabla\varphi_\varepsilon|^2 \leq \varepsilon|\nabla\varphi_\varepsilon|^{2+\frac{2}{n}} + C(\varepsilon). \tag{4.33}$$

Let $u = \exp(-A(\varphi_\varepsilon))|\nabla\varphi_\varepsilon|^2$. Note that $\exp(-A(\varphi_\varepsilon))$ is uniformly bounded. By (4.32) and (4.33) we have

$$\Delta' u \geq Cu^{1+\frac{1}{n}} - C + C \operatorname{tr}_\omega \omega' - C|\nabla F|u^{1/2}. \tag{4.34}$$

Step 2. We will do an integration scheme to obtain the uniform C^1 bound from (4.34). We multiplying (4.34) by u^p for $p > 0$ and take integral with respect to the volume form $d\mu'$ to obtain

$$\begin{aligned}
&- \int_M pu^{p-1}|\nabla' u|^2_{\omega'} d\mu' + \int_M \nabla'(u^p\nabla' u)d\mu' = \int_M u^p \Delta' u d\mu' \\
&\geq \int_M u^p(Cu^{1+\frac{1}{n}} - C)d\mu' + C\int_M u^p(\operatorname{tr}_\omega \omega')d\mu' - C\int_M |\nabla F|u^{p+\frac{1}{2}} d\mu'
\end{aligned} \tag{4.35}$$

By Lemma 3.13 (Gaffney-Stokes),

$$\int_M \nabla'(u^p\nabla' u)d\mu' = 0.$$

Hence,

$$\begin{aligned}&\int_M \Big(pu^{p-1}|\nabla' u|^2_{\omega'} + Cu^p(\operatorname{tr}_\omega \omega')\Big)d\mu' \\ &\leq C\int_M |\nabla F|u^{p+\frac{1}{2}}d\mu' - \int_M u^p(Cu^{1+\frac{1}{n}} - C)d\mu'\end{aligned} \tag{4.36}$$

Note that we have the following pointwise inequality

$$|\nabla' u|^2_{\omega'} + (\operatorname{tr}_\omega \omega') \geq 2\sqrt{(\operatorname{tr}_\omega \omega')|\nabla' u|^2_{\omega'}} \geq 2|\nabla \varphi_\varepsilon|. \tag{4.37}$$

In addition,

$$d\mu' = \exp(F + \varepsilon\varphi_\varepsilon)d\mu$$

is equivalent to $d\mu$. Hence, we have

$$C\sqrt{p}\int_M u^{p-\frac{1}{2}}|\nabla u|d\mu \leq C\int_M |\nabla F|u^{p+\frac{1}{2}}d\mu - \int_M u^p(Cu^{1+\frac{1}{n}} - C)d\mu. \tag{4.38}$$

It follows from (4.38) that

$$\int_M |\nabla u^{p+\frac{1}{2}}|d\mu \leq C\sqrt{p}\int_M |\nabla F|u^{p+\frac{1}{2}}d\mu - C\sqrt{p}\int_M u^p(u^{1+\frac{1}{n}} - C)d\mu. \tag{4.39}$$

Step 3. Let us rewrite (4.39) as follows by taking a shift on p: for $p > 1/2$ we have

$$\int_M |\nabla u^p|d\mu \leq C\sqrt{p}\int_M |\nabla F|u^p d\mu - C\sqrt{p}\int_M u^{p-\frac{1}{2}}(u^{1+\frac{1}{n}} - C)d\mu \tag{4.40}$$

Note that

$$\begin{aligned}\int_M |\nabla(u^p\rho^{-1})|\rho d\mu &= \int_M |\nabla u^p|d\mu + \int_M \rho^{-1}|\nabla\rho||u|^p d\mu \\ &\leq \int_M |\nabla u^p|d\mu + C\int_M |u|^p d\mu.\end{aligned} \tag{4.41}$$

as $\rho^{-1}|\nabla\rho|$ is bounded on M by Lemma 3.3. Let $\gamma = 2n/(2n-1)$. By the weighted Sobolev inequality (Lemma 3.9), we have

$$\Big(\int_M |u^p\rho^{-1}|^\gamma \rho d\mu\Big)^{1/\gamma} \leq C\Big(\int_M |\nabla(u^p\rho^{-1})|\rho d\mu + \int_M |u^p\rho^{-1}|\rho d\mu\Big). \tag{4.42}$$

By (4.40), (4.41) and (4.42), we have

$$\begin{aligned}&\Big(\int_M |u^p\rho^{-1}|^\gamma \rho d\mu\Big)^{1/\gamma}\\ &\le C\sqrt{p}\int_M u^p|\nabla F|d\mu - C\sqrt{p}\int_M u^{p-\frac{1}{2}}(u^{1+\frac{1}{n}} - C\sqrt{u} - C)d\mu.\end{aligned} \tag{4.43}$$

Notice that for $p > 1/2$ the function $f(t) = t^{p-\frac{1}{2}}(t^{1+\frac{1}{n}} - C\sqrt{t} - C)$ is bounded from below on $\mathbb{R}$. Therefore,

$$\Big(\int_M |u^p\rho^{-1}|^\gamma \rho d\mu\Big)^{1/\gamma} \le C\sqrt{p}\Big(\int_M u^p|\nabla F|d\mu + 1\Big). \tag{4.44}$$

Let $d\tilde{\mu} = \rho^{-1/(2n-1)}d\mu$. Then (4.44) can be written as

$$\Big(\int_M |u|^{p\gamma}d\tilde{\mu}\Big)^{1/\gamma} \le C\sqrt{p}\Big(\int_M |u|^p|\nabla F|\rho^{\frac{1}{2n-1}}d\tilde{\mu} + 1\Big). \tag{4.45}$$

Let q_0 be that $1/p_0 + 1/q_0 = 1$. From Hölder inequality we have

$$\int_M |u|^p|\nabla F|\rho^{\frac{1}{2n-1}}d\tilde{\mu} \le \Big(\int_M |u|^{pq_0}d\tilde{\mu}\Big)^{1/q_0}\Big(\int_M |\nabla F|^{p_0}\rho^{\frac{p_0}{2n-1}}d\tilde{\mu}\Big)^{1/p_0}. \tag{4.46}$$

Note that

$$\int_M |\nabla F|^{p_0}\rho^{\frac{p_0}{2n-1}}d\tilde{\mu} = \int_M |\nabla F|^{p_0}\rho^{\frac{p_0-1}{2n-1}}d\mu = \mathcal{I}(F, p_0). \tag{4.47}$$

From (4.45), (4.46) and (4.47), we get

$$\|u\|^p_{L^{\gamma p}(d\tilde{\mu})} \le C\sqrt{p}\big(\|u\|^p_{L^{q_0 p}(d\tilde{\mu})} + 1\big). \tag{4.48}$$

Let $\beta = q_0^{-1}\gamma$, then $\beta > 1$. Take a sequence $\{p_\ell\}$ with $p_\ell = q_0^{-1}\beta^{\ell-1}$ for $\ell \ge 1$. Then by (4.48) we get

$$\|u\|^{p_\ell}_{L^{q_0 p_{\ell+1}}(d\tilde{\mu})} \le C\sqrt{p_\ell}\big(\|u\|^{p_\ell}_{L^{q_0 p_\ell}(d\tilde{\mu})} + 1\big). \tag{4.49}$$

We may assume that $C\sqrt{p_\ell} \ge 1$. Let

$$\mathcal{B}_\ell = \max\big(\|u\|_{L^{p_0 p_\ell}(d\tilde{\mu})}, 1\big).$$

Then

$$\mathcal{B}_{\ell+1} \le (2C)^{1/p_\ell}p_\ell^{2/p_\ell}\mathcal{B}_\ell. \tag{4.50}$$

By iteration we have

$$\mathcal{B}_\ell \le \Big(\prod_{j=1}^{\ell-1} (2C)^{1/p_j} p_j^{2/p_j} \Big) \mathcal{B}_1. \tag{4.51}$$

It is easy to check that

$$\log \Big(\prod_{j=1}^{\infty} (2C)^{1/p_j} p_j^{2/p_j} \Big) \le \sum_{j=0}^{\infty} p_0 \beta^{-j} \left(2j \log \beta - 2 \log p_0 + \log(2C) \right) < \infty.$$

Let $k \to \infty$, it follows that

$$\|u\|_{C^0} \le C\mathcal{B}_1 = C \cdot \max \big(\|u\|_{L^1(d\tilde{\mu})}, 1 \big). \tag{4.52}$$

To get a bound of $\|u\|_{L^1(d\tilde{\mu})}$, notice that $d\tilde{\mu} = \rho^{-1/(2n-1)} d\mu \le d\mu$, hence, we have

$$\begin{aligned} \|u\|_{L^1(d\tilde{\mu})} &= \int_M \exp(-A(\varphi_\varepsilon)) |\nabla \varphi_\varepsilon|^2 d\tilde{\mu} \le C \int_M |\nabla \varphi_\varepsilon|^2 d\mu \\ &\le -C \int_M \varphi_\varepsilon (\Delta \varphi_\varepsilon + n - n) d\mu \\ &\le C ||\varphi_\varepsilon||_{C^0} \int_M (\Delta \varphi_\varepsilon + n) d\mu - Cn \int_M \varphi_\varepsilon d\mu \\ &\le 2nC \|\varphi_\varepsilon\|_{C^0}. \end{aligned} \tag{4.53}$$

From (4.52) and (4.53) and that $\|\varphi_\varepsilon\|$ is uniformly bounded, we have

$$\|\nabla \varphi_\varepsilon\|_{C^0} \le C$$

for some constant C depends only on $\mathcal{I}(F, p_0)$, p_0, ω and n. □

4.3 Uniform C^2 estimate

In this section we prove the C^2 estimate Theorem 2.3. In what follows let $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon$ and ∇', Δ' and $d\mu'$ be the covariant derivative, Laplacian and volume form of the Kähler metric ω'. The constant C may vary from line to line, but always only depends on $\mathcal{I}(F, p_0)$, p_0, ω and n.

Proof. Step 1. By the same computations as in [41], we obtain in the orthonormal frame of background metric ω,

$$\begin{aligned} \Delta'(e^{-\beta\varphi_\varepsilon}(n + \Delta\varphi_\varepsilon)) \ge e^{-C\varphi_\varepsilon} \Big[\frac{1}{2} \sum_{i \ne j} R_{i\bar{i}j\bar{j}}(\omega) \frac{\big((\varphi_\varepsilon)_{i\bar{i}} - (\varphi_\varepsilon)_{j\bar{j}} \big)^2}{(1 + (\varphi_\varepsilon)_{i\bar{i}})(1 + (\varphi_\varepsilon)_{j\bar{j}})} \\ + \Delta F + \varepsilon \Delta \varphi_\varepsilon + (n + \Delta\varphi_\varepsilon)(-\beta \Delta' \varphi_\varepsilon) \Big]. \end{aligned} \tag{4.54}$$

Choose

$$\beta \geq -\inf_M \inf_{i \neq \ell} R_{i\bar{i}j\bar{j}} + 1.$$

It follows that

$$\Delta'(e^{-\beta\varphi_\varepsilon}(n+\Delta\varphi_\varepsilon)) \geq e^{-\beta\varphi_\varepsilon}\Big[(n+\Delta\varphi)\operatorname{tr}_{\omega'}\omega + (\varepsilon - 2n\beta)(n+\Delta\varphi_\varepsilon) + \Delta F - n\varepsilon\Big]. \tag{4.55}$$

Let

$$w = e^{-\beta\varphi_\varepsilon}(n + \Delta\varphi_\varepsilon).$$

Note that

$$\operatorname{tr}_{\omega'}\omega \geq (\exp(F + \varepsilon\varphi_\varepsilon)\operatorname{tr}_\omega \omega')^{1/(n-1)}$$

and $\operatorname{tr}_\omega \omega' = n + \Delta\varphi_\varepsilon$. It follows that

$$\Delta' w \geq C w^{\frac{n}{n-1}} - C + e^{-\beta\varphi_\varepsilon}\Delta F. \tag{4.56}$$

Step 2. We compute, for $p > 0$,

$$\begin{aligned} \int_M |\nabla w^p|^2 dV &\leq \int_M (tr_\omega \omega')|\nabla' w^p|^2_{\omega'} e^{-(F+\varepsilon\varphi_\varepsilon)} d\mu' \\ &= \int_M e^{(\beta-\varepsilon)\varphi_\varepsilon - F} w |\nabla' w^p|^2_{\omega'} d\mu' \\ &\leq C \int_M w|\nabla' w^p|^2_{\omega'} d\mu'. \end{aligned} \tag{4.57}$$

Take integration by parts

$$\int_M w|\nabla' w^p|^2_{\omega'} d\mu' = -\frac{p}{2}\int_M w^{2p}\Delta' w d\mu' + \frac{p}{2}\int_M \nabla'(w^{2p}\nabla' w) d\mu'. \tag{4.58}$$

By Lemma 3.13 (Gaffney-Stokes), the integral

$$\int_M \nabla'(w^{2p}\nabla' w) d\mu' = 0. \tag{4.59}$$

Combining (4.57), (4.58) and (4.59), we have

$$\begin{aligned} \int_M |\nabla w^p|^2 d\mu &\leq \frac{C}{2} p \int w^{2p}(-\Delta' w) d\mu' \\ &= \frac{C}{2} p \int_M w^{2p}(-\Delta' w) e^{F+\varepsilon\varphi_\varepsilon} d\mu \\ &\leq Cp \int_M w^{2p}(-\Delta' w) d\mu \end{aligned} \tag{4.60}$$

By (4.53), we have

$$\begin{aligned}&\int_M |\nabla w^p|^2 d\mu \le -Cp\int_M w^{2p}(Cw^{\frac{n}{n-1}} - C)d\mu - Cp\int_M w^{2p}e^{-\beta\varphi_\varepsilon}\Delta F d\mu\\&\le -Cp\int_M w^{2p}(Cw^{\frac{n}{n-1}} - C)d\mu + Cp\int_M 2e^{-\beta\varphi_\varepsilon}w^p|\nabla w^p||\nabla F|d\mu\\&+ Cp\int_M \beta e^{-\beta\varphi_\varepsilon}w^{2p}|\nabla\varphi_\varepsilon||\nabla F|d\mu\\&\le -Cp\int_M w^{2p}(w^{\frac{n}{n-1}} - C)d\mu + Cp\int_M w^p|\nabla w^p||\nabla F|d\mu\\&+ Cp\int_M w^{2p}|\nabla F|d\mu\end{aligned} \tag{4.61}$$

By Hölder inequality,

$$Cp\int_M |\nabla w^p|w^p|\nabla F|d\mu \le \frac{1}{2}\int_M |\nabla w^p|^2 d\mu + \frac{C^2p^2}{2}\int_M w^{2p}|\nabla F|^2 d\mu \tag{4.62}$$

and

$$\int_M w^{2p}|\nabla F|d\mu \le \frac{1}{2}\int_M w^{2p}(|\nabla F|^2 + 1)d\mu. \tag{4.63}$$

Combine (4.61), (4.62) and (4.63), we have

$$\int_M |\nabla w^p|^2 d\mu \le -Cp\int_M w^{2p}(w^{\frac{n}{n-1}} - C)d\mu + Cp^2\int_M w^{2p}|\nabla F|^2 d\mu. \tag{4.64}$$

Step 3. As $\rho^{-1}|\nabla\rho|$ is bounded on M, we have

$$\begin{aligned}\int_M |\nabla(w^p\rho^{-\frac{1}{2}})|^2\rho d\mu &\le \int_M \big(2|\nabla w^p|^2 + \frac{1}{2}(\rho^{-1}|\nabla\rho|)^2 w^{2p}\big)d\mu\\&\le 2\int_M |\nabla w^p|d\mu + C\int_M w^{2p}d\mu.\end{aligned} \tag{4.65}$$

Let $\gamma' = n/(n-1)$. Note that $1/2 = 1/(2n) + 1/(2\tau)$. By the weighted Sobolev inequality (Lemma 3.9), we have

$$\Big(\int_M |w^p\rho^{-\frac{1}{2}}|^{2\gamma'}\rho d\mu\Big)^{\frac{1}{2\gamma'}} \le C\Big(\int_M |\nabla(w^p\rho^{-\frac{1}{2}})|^2\rho d\mu + \int_M w^{2p}d\mu\Big)^{\frac{1}{2}}. \tag{4.66}$$

Combining (4.64), (4.65) and (4.66), we get

$$\Big(\int_M |w^p\rho^{-\frac{1}{2}}|^{2\gamma'}\rho d\mu\Big)^{\frac{1}{\gamma'}} \le -Cp\int_M w^{2p}(w^{\frac{n}{n-1}} - C)d\mu + Cp^2\int_M w^{2p}|\nabla F|^2 d\mu. \tag{4.67}$$

Note that for $p > 0$ the function $f(t) = t^{2p}(t^{n/(n-1)} - C)$ is bounded from below. Hence,

$$\int_M w^{2p}(w^{\frac{n}{n-1}} - C)d\nu \geq -C. \tag{4.68}$$

Let $d\nu = \rho^{-1/(n-1)}d\mu$, we can rewrite

$$\int_M |w^p \rho^{-\frac{1}{2}}|^{2\gamma'} \rho d\mu = \int_M |w|^{2\gamma' p} d\nu. \tag{4.69}$$

It follows that

$$\Big(\int_M |w|^{2\gamma' p} d\nu\Big)^{\frac{1}{\gamma'}} \leq Cp + Cp^2 \int_M w^{2p} |\nabla F|^2 \rho^{\frac{1}{n-1}} d\nu \tag{4.70}$$

Let q_0' be that $1/q_0 + 2/p_0 = 1$. By Hölder inequality we have

$$\int_M w^{2p} |\nabla F| \rho^{\frac{1}{n-1}} d\nu = \Big(\int_M w^{2pq_0} d\nu\Big)^{\frac{1}{q_0}} \Big(\int_M |\nabla F|^{p_0} \rho^{\frac{p_0}{2n-2}} d\nu\Big)^{\frac{2}{p_0}}. \tag{4.71}$$

Notice that

$$\int_M |\nabla F|^{p_0} \rho^{\frac{p_0}{2n-2}} d\nu = \int_M |\nabla F|^{p_0} \rho^{\frac{p_0-2}{2n-2}} d\mu = \mathcal{I}(F, p_0). \tag{4.72}$$

Hence, by (4.70), (4.71) and (4.72), it follows that

$$\|w\|^p_{L^{2\gamma' p}(d\nu)} \leq Cp^2 (\|w\|^p_{L^{2q_0' p}(d\nu)} + 1) \tag{4.73}$$

By a similar iteration argument as in the end of Section 4.2, it follows that

$$\|w\|_{C^0} \leq C\|w\|_{L^1(d\nu)}. \tag{4.74}$$

To obtain the bound for $\|w\|_{L^1(d\nu)}$, notice that $d\nu = \rho^{-1/(n-1)}d\mu$, it follows that

$$\|w\|_{L^1(d\nu)} \leq \int_M e^{-\beta\varphi_\varepsilon}(n + \Delta\varphi_\varepsilon)d\mu \leq C\int_M (n + \Delta\varphi_\varepsilon)d\mu \leq C. \tag{4.75}$$

Therefore, we have

$$\|\Delta\varphi_\varepsilon\|_{C^0} \leq C$$

for some constant C depending only on $\mathcal{I}(F, q_0)$, q_0, ω and n. □

4.4 Hölder estimate of the second order

In Yau's original resolution of the Calabi conjecture, the C^3 estimate is needed to obtain higher order regularity from Schauder theory. For Monge-Amperè equations, such C^3 estimates date back to Calabi's seminal third order estimates [7]. Later on Evans [22] and Krylov [29, 30] proved that Hölder estimates of second order hold for fully nonlinear concave uniform elliptic operators. All the results are originally stated for the right hand side F with second derivatives or higher. The Hölder estimates of second order derivatives has also been studied for uniform elliptic operators when the right hand side has weaker regularity. These estimates can be localized. In particular, for Monge-Ampére equation, Błocki proved that the Hölder esitmates hold when F is Lipschitz and the Laplacian of the Kähler potential is bounded [6]. Chen-He extended Blocki's result to the case when F is only in W^{1,p_0} for some $p_0 > 2n$ [17]. We cite Chen-He's result in the following lemma. The readers are referred to their paper for the detail of the proof.

Lemma 4.3 (Chen-He, [17, Lemma 4.1]). *Let v be a C^4-psh function in an open $\Omega \subset \mathbb{C}^n$ such that*

$$\det(v_{i\bar{j}}) = F.$$

Assume that there are some positive constants Λ and K such that

$$0 < \Lambda^{-1} \leq \Delta v \leq \Lambda, \ \|v\|_{L^\infty} \leq K \text{ and } \|v\|_{W^{1,p_0}(\Omega)} \leq K.$$

Then for any $\Omega' \subset\subset \Omega$, there exists some $\lambda = \lambda(\Omega, \Omega', \Lambda, K)$ with $0 < \lambda < 1$ such that

$$\|v\|_{C^{2,\lambda}(\Omega')} \leq C(\Omega, \Omega', \Lambda, K).$$

We can cover (M, ω) by a quasi-coordinate system and apply the Hölder estimate in each quasi-coordinate chart. In particular, we have

Proposition 4.4. *Let $F \in C_c^\infty(M, g)$ such that $\mathcal{I}(F, p_0) < \infty$ for some $p_0 > 2n$. Let φ_ε be the solution of perturbed equation* (2.6). *Then there exists some $\lambda = \lambda(\mathcal{I}(F, p_0), p_0, \omega, n)$ such that*

$$\|\varphi_\varepsilon\|_{2,\lambda} \leq C(\mathcal{I}(F, p_0), p_0, \omega, n) \tag{4.76}$$

where $\|\cdot\|_{2,\lambda}$ is the $(2, \lambda)$*-Hölder norm defined in* (3.21) *by the quasi-coordinates.*

Proof. We can cover a neighborhood of the divisor D by a finite number of polydiscs U's and cover the complement of the neighborhood by a finite number of open sets V's. For each V we can take some $V' \subset\subset V$ such that

$$\overline{M} \backslash \bigcup U \subset \bigcup V'.$$

On each V with some local coordinates, we can find a local potential Q_0 such that $g_{i\bar{j}} = (Q_0)_{i\bar{j}}$. Then the perturbed equation (2.6) can be written as

$$\det(\vartheta_{i\bar{j}}) = f,$$

with $\vartheta = Q_0 + \varphi_\varepsilon$ and $f = \exp(F + \varepsilon\varphi_\varepsilon)\det(g_{i\bar{j}})$. The uniform estimates on $\|\nabla\varphi_\varepsilon\|_{C^0}$ and $\|\Delta\varphi_\varepsilon\|_{C^0}$ and the boundedness of $\mathcal{I}(F, p_0)$ imply that

$$0 \leq \Lambda^{-1} \leq \Delta\vartheta \leq \Lambda,\ \|\vartheta\|_{L^\infty(V)} < K \text{ and } \|f\|_{W^{1,p_0}(V)} \leq K$$

for some positive constants Λ and K. Then by Lemma 4.3, these exists some constant λ' and C' depending on V such that

$$\|\vartheta\|_{C^{2,\lambda'}(V')} \leq C'. \tag{4.77}$$

For each U, we can assume that $U\backslash D = (\Delta^*_\kappa)^k \times \Delta_1^{n-k}$ for some $k < n$ and $\kappa > 0$ sufficient small. The set $U\backslash D$ can be covered by a family of quasi-coordinate charts

$$U\backslash D \subset \bigcup_{\delta\in(0,1)^k} \Psi_\delta(\frac{1}{2}\mathcal{P}_k) \subset \bigcup_{\delta\in(0,1)^k} \Psi_\delta(\frac{3}{4}\mathcal{P}_k)$$

where $r\mathcal{P}_k = \Delta_r^k \times \Delta_1^{n-k}$ and $\Psi_\delta : \frac{3}{4}\mathcal{P}_k \to U\backslash D$ is the quasi-coordinate map constructed in Section 3.2, for each multi-index δ. Under the quasi-coordinates, suppose the metric is written as $\tilde{g}_{i\bar{j}}$, the potential $\tilde{\varphi}_\varepsilon$ and the right hand side $\tilde{F}$ on the polydisc $\frac{3}{4}\mathcal{P}_k \subset \mathbb{C}^n$. Note that in the quasi-coordinates the pullback metric $\tilde{g}$ is quasi-isometric to the Euclidean metric g_0 on $\mathbb{C}^n$:

$$0 < C^{-1}g_0 \leq \tilde{g} \leq Cg_0$$

for some positive constant C independent of the multi-index δ. We can find some local potential $\tilde{Q}_0$ on $\frac{3}{4}\mathcal{P}_k$ such that $\tilde{g}_{i\bar{j}} = (\tilde{Q}_0)_{i\bar{j}}$. Then in the quasi-coordinates the perturbed equation (2.6) can be rewritten as

$$\det(\tilde{\vartheta}_{i\bar{j}}) = \tilde{f} \tag{4.78}$$

where $\tilde{\vartheta} = \tilde{Q}_0 + \tilde{\varphi}_\varepsilon$ and $\tilde{f} = \exp(\tilde{F} + \varepsilon\tilde{\varphi}_\varepsilon)\det(\tilde{g}_{i\bar{j}})$. The uniform estimates on $\|\nabla\varphi_\varepsilon\|_{C^0}$ and $\|\Delta\varphi_\varepsilon\|_{C^0}$ readily imply that there exist constants Λ and K such that

$$0 < \Lambda^{-1} \leq \Delta\tilde{\vartheta} \leq \Lambda,\ \|\tilde{\vartheta}\|_{L^\infty(\frac{3}{4}\mathcal{P}_k)} \leq K \text{ and } \|f\|_{L^\infty(\frac{3}{4}\mathcal{P}_k)} \leq K.$$

We now show that $f \in W^{1,p_0}(\frac{3}{4}\mathcal{P}_k)$ and $\|f\|_{W^{1,p_0}(\frac{3}{4}\mathcal{P}_k)} \le K$. It suffices to show that $\|\nabla_{g_0}\tilde{F}\|_{L^p(\frac{3}{4}\mathcal{P}_k)} \le K$. By Lemma 3.7, we have

$$\begin{aligned}\int_{\frac{3}{4}\mathcal{P}_k} |\nabla_{g_0}\tilde{F}|_{g_0}^{p_0} d\mu_0 &\le C\int_{\frac{3}{4}\mathcal{P}_k} (\Psi_\delta^*\rho)^{-1}\Psi_\delta^*(|\nabla F|^{p_0}\rho)d\mu_0 \\ &\le C\int_{U\setminus D} |\nabla F|^{p_0}\rho d\mu \le \mathcal{I}(F,p_0).\end{aligned} \tag{4.79}$$

Hence,

$$\|f\|_{W^{1,p_0}(\frac{3}{4}\mathcal{P}_k)} \le K.$$

By Lemma 4.3, there exists λ'' and constant C'' depending on U such that

$$\|\tilde{\vartheta}\|_{C^{2,\lambda''}(\frac{1}{2}\mathcal{P}_k)} \le C''. \tag{4.80}$$

Note that the constants λ'' and C'' do not depend on the multi-index δ. Since there are only a finite number of U's and V's, we can take some common λ and C such that in either local coordinates for V's, or local quasi-coordinates for U's, the estimates of (4.77) and (4.80) both holds. Therefore, by taking supreme over all the quasi-coordinate charts, we get

$$\|\varphi_\varepsilon\|_{2,\lambda} \le C. \tag{4.81}$$

□

To obtain W^{3,p_0} estimate, we localize the estimate in the quasi-coordinate charts as what we did in the proof of Proposition 4.4. Under the quasi-coordinates, suppose the metric is written as $\tilde{g}_{i\bar{j}}$, the potential $\tilde{\varphi}_\varepsilon$ and the right hand side $\tilde{F}$ on the polydisc $\frac{3}{4}\mathcal{P}_k \subset \mathbb{C}^n$. The perturbed equation (2.6) is written under the quasi-coordinates as

$$\det(\tilde{g}_{i\bar{j}} + (\tilde{\varphi}_\varepsilon)_{i\bar{j}}) = e^{F+\varepsilon\tilde{\varphi}_\varepsilon}\det(\tilde{g}_{i\bar{j}}). \tag{4.82}$$

Let ∂ be an arbitrary first order differential operator on the quasi-coordinate chart $\frac{3}{4}\mathcal{P}_k$. Once the Hölder estimate of second order is proved, we compute in the quasi-coordinate chart

$$\Delta_{\tilde{g}}\partial\tilde{\varphi}_\varepsilon = \partial\tilde{F} + \varepsilon\partial\tilde{\varphi}_\varepsilon + (\tilde{g}^{i\bar{j}} - \tilde{g}^{i\bar{j}}_{\varphi_\varepsilon})\partial\tilde{g}_{i\bar{j}} \tag{4.83}$$

where the $\tilde{g}_{\varphi_\varepsilon}$ is the metric of $(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon)$ in the quasi-coordinates. Note that we already have $\|\tilde{\varphi}_\varepsilon\|_{C^{2,\alpha}(\frac{3}{4}\mathcal{P}_k)}$ and $\|\tilde{F}\|_{W^{1,p_0}(\frac{3}{4}\mathcal{P}_k)}$ bounded, hence the

L^{p_0} norm of right hand side bounded. It then follows from L^p theory, for example see [27] Chapter 9, that

$$\|\partial\tilde{\varphi}_\varepsilon\|_{W^{2,p_0}(\frac{1}{2}\mathcal{P}_k)} \le C. \tag{4.84}$$

It follows that

$$\|\Psi^*_\delta\varphi_\varepsilon\|_{W^{3,p_0}(\frac{1}{2}\mathcal{P}_k)} \le C,\ \forall\delta \tag{4.85}$$

By Lemma 3.7, we have

$$\begin{aligned} ||\varphi_\varepsilon||^{p_0}_{W^{3,p_0}(U\backslash D)} &\le c\sum_\ell \Pi_{\delta_\ell}\|\Psi^*_{\delta_\ell}\varphi_\varepsilon\|^{p_0}_{W^{3,p_0}(\frac{1}{2}\mathcal{P}_k)} \\ &\le C^{p_0}c\sum_\ell \Pi_{\delta_\ell}\int_{\frac{1}{2}\mathcal{P}_k} 1d\mu_0 \\ &\le C^{p_0}c^2\int_{U\backslash D} 1d\mu \le C^{p_0}c^2\,\mathrm{Vol}(M). \end{aligned} \tag{4.86}$$

We can a neighborhood of D by a finite number of such U's and cover the complement of the neighborhood by a finite number of unit balls. Collect the inequalities on each of them, we get the following proposition.

Proposition 4.5. *Let $F \in C^\infty_c(M, g)$ such that $\mathcal{I}(F, p_0) < \infty$. Let φ_ε be the solution of perturbed equation* (2.6)*. Then we have*

$$\|\varphi_\varepsilon\|_{W^{3,p_0}(M)} \le C \tag{4.87}$$

for some constant C depends only on $\mathcal{I}(F, p_0)$, p_0, ω and n.

4.5 Proof of Theorem 2.1

All the estimates are proved with the temporary assumption that F is in $C^\infty_c(M)$. When F is only in the weighted Sobolev space with $\mathcal{I}(F, p_0) < \infty$, we show that F can be approximated by C^∞_c functions in the weighted Sobolev spaces.

Lemma 4.6. *Suppose $F \in W^{1,p_0}_{\mathrm{loc}}(M)$ satisfies $I(F, p_0) < \infty$ for some $p_0 > 2n$. Then there is a sequence of $F_k \in C^\infty_c$ such that $\mathcal{I}(F - F_k, p_0) \to 0$ as $k \to 0$. In particular, $F_k \to F$ in $W^{1,p_0}(M, g)$.*

Proof. We can assume that F is smooth. The Ricci curvature of (M, g) is bounded from below. Let $r = r(x)$ denote the distance function to some fixed point. By a theorem of Yau ([36, Theorem 4.2]), there is a proper

$C^\infty(M)$ function d such that $d \geq Cr$ and $|\nabla d| \leq C$, for some constant C. Let $\chi : [0,\infty) \to \mathbb{R}$ be a cut-off function such that: (i) $\chi(t) \equiv 1$ for $t \leq 1$, $\chi(t) \equiv 0$ for $t \geq 2$ and $0 \leq \chi \leq 1$; (ii) $|\chi'(t)| < 2$. Let $F_k(x) = \chi\,(d(x)/k)\,F(x)$. Then $F_k \in C_c^\infty(M,g)$. It remains to show $I(F - F_k, p_0) \to 0$. To see this, note that $\nabla F_k = \chi\,(d/k)\,\nabla F + F\chi'\,(d/k)\,\nabla d/k$. Hence,

$$\begin{aligned}
&\int (|F - F_k|^{p_0} + |\nabla F - \nabla F_k|^{p_0})\rho^{\frac{p_0-2}{2n-2}} dV \\
\leq &\int (1 - \chi\,(d/k))\,|F|^{p_0} + \big((1-\chi(d/k))|\nabla F| + Ck^{-1}|F|\big)^{p_0}\,\rho^{\frac{p_0-2}{2n-2}} dV \\
\leq &\, C\int_{\{d\geq k\}} (|F|^{p_0} + |\nabla F|^{p_0})\rho^{\frac{p_0-2}{2n-2}} dV + Ck^{-1}\int |F|^{p_0}\rho^{\frac{p_0-2}{2n-2}} dV
\end{aligned} \tag{4.88}$$

The RHS goes to 0 as $k \to \infty$. □

Finally, we proof the main theorem.

Proof of theorem 2.1. Let F_k be a sequence of smooth functions with compact support such that $\mathcal{I}(F_k - F, p_0) \to 0$. In particular, we can assume $\mathcal{I}(F_k, p_0) \leq \mathcal{I}(F, p_0) + 1$ for any k. For each ε and k, there is a smooth solution $\varphi_{\varepsilon,k}$ which solves the perturbed equation

$$(\omega + \sqrt{-1}\partial\bar\partial\varphi_{\varepsilon,k})^n = e^{F_k + \varepsilon\varphi_{\varepsilon,k}}\omega^n \tag{4.89}$$

such that $(\omega + \sqrt{-1}\partial\bar\partial\varphi_{\varepsilon,k}) > 0$. By Proposition 4.5 we have

$$\|\varphi_{\varepsilon,k}\|_{W^{3,p_0}} \leq C(\mathcal{I}(F,p_0), p_0, \omega, n). \tag{4.90}$$

There is a subsequence of $(\varphi_{\varepsilon,k})$ that converges to some $\varphi \in W^{3,p_0}(M,g)$ such that $\omega + \sqrt{-1}\partial\bar\partial\varphi > 0$ defines a W^{1,p_0} Kähler metric. □

Part II

On the Chern-Yamabe problem

[1]

5 Introducation to Chern-Yamabe problem

5.1 Chern scalar curvature

Let (X, ω) be a compact complex manifold of complex dimension $n \geq 2$ endowed with a Hermitian metric ω. Besides the usual Riemannian scalar curvature which is scalar curvature with respect to the the Levi-Civita connection, one can define the *Chern scalar curvature* of (X, ω) to be the scalar curvature with respect to the Chern connection associated to ω. The Chern scalar curvature can be succinctly expressed as

$$\mathrm{S}^{\mathrm{C}}(\omega) = -\operatorname{tr}_\omega i\partial\bar{\partial}\log\omega^n, \tag{5.1}$$

where ω^n denotes the volume form. It is easy to see that under conformal transformation the Chern scalar curvature changes as

$$\mathrm{S}^{\mathrm{C}}\left(\exp(2f/n)\omega\right) = \exp(-2f/n)\left(\mathrm{S}^{\mathrm{C}}(\omega) - \Delta^{\mathrm{C}}_\omega f\right), \tag{5.2}$$

where $\Delta^{\mathrm{C}}_\omega$ is the Chern Laplacian operator[2] with respect to ω, which is defined as

$$\Delta^{\mathrm{C}}_\omega f := 2\operatorname{tr}_\omega i\partial\bar{\partial}f. \tag{5.3}$$

5.2 Chern-Yamabe problem

Inspired by the Yamabe problem for Riemannian manifolds, Angella-Calamai-Spotti [1] proposed the *Chern-Yamabe* problem of finding metrics of constant Chern scalar curvature in the given conformal classes. By the conformal transformation formula of Chern scalar curvature, the Chern-Yamabe problem is equivalent to find a pair $(f, \lambda) \in C^\infty(X; \mathbb{R}) \times \mathbb{R}$ solving

$$-\Delta^{\mathrm{C}}_\omega f + \mathrm{S}^{\mathrm{C}}(\omega) = \lambda\exp(2f/n). \tag{5.4}$$

In this way, the conformal metric $\exp(2f/n)\omega$ then has the constant Chern scalar curvature equal to λ.

[1] This is a joint work with Simone Calamai.
[2] We use the analysts' convention of Laplacian operator.

5.3 Normalization

Given a hermitian metric ω, the torsion 1-form of ω is defined as some 1-form θ_ω such that $d\omega^{n-1} = \theta_\omega \wedge \omega^{n-1}$. The hermitian metric ω is said to be *balanced* if $\theta_\omega = 0$, equivalently, $d\omega^{n-1} = 0$. The hermitian metric ω is said to be *Gauduchon* if $d^*\theta_\omega = 0$. From the definition one can easily see that a balanced metric is automatically a Gauduchon metric. The reserve is not necessarily true.

In [25], P. Gauduchon proved the following fundamental theorem.

Theorem 5.1 ([25, Théoremè 1]). *Fix a compact complex manifold X with $\dim_{\mathbb{C}} X \geq 2$. For every conformal class of hermitian metrics there exists a unique Gauduchon metric ω such that $\int_X \omega^n/n! = 1$.*

By a computation in [26, pages 502-503], the Chern Laplacian can be decomposed as

$$\Delta^{\mathrm{C}}_\omega f = \Delta_d f + \langle df, \theta_\omega \rangle_\omega \tag{5.5}$$

where Δ_d is the Hodge's Laplacian and θ_ω is the torsion 1-form of ω. The Gauduchon metrics have the advantage that if ω is Gauduchon, then

$$\int_M \Delta^{\mathrm{C}}_\omega f dV_\omega = \int_M \Delta_d f + f(d^*\theta_\omega) dV_\omega = 0. \tag{5.6}$$

From this point, we would like to take the unique Gauduchon metric with unit volume in each conformal class as the background metric. This is also guaranteed by Gauduchon's theorem that the Gauduchon metric always exist in each conformal class. From now on we assume the background metric ω in the Chern-Yamabe problem is Gauduchon with unit volume and let $d\mu$ denote its volume form.

We can also normalize f so that

$$\int_X \exp(2f/n) d\mu = 1. \tag{5.7}$$

Then the constant λ is exactly the total Chern scalar curvature of the background metric

$$\lambda = \int_X \mathrm{S}^{\mathrm{C}}(\omega)\, d\mu = \int_X 2\pi c_1(X) \wedge [\omega]^{n-1}/(n-1)!. \tag{5.8}$$

where $c_1(X)$ is the first Chern class of the complex manifold (X, J). (Note that ω has unit volume.) The constant λ is uniquely determined by the conformal class (more precisely the unique Gauduchon metric in that conformal class with unit volume). It is called the Gauduchon degree of the conformal class.

5.4 Dissertation work

When the Gauduchon degree $\lambda \leq 0$, it has been proved in [1] that there exists unique solution to (5.4) with normalization (5.7) via direct PDE method. This result is reproved in [31] through a flow method. The positive case $\lambda > 0$ is still an open problem.

In [1, 31] two different flows were defined to approach the study of Hermitian metrics with constant Chern scalar curvature. Here we define a different flow, in Section 6, which has the advantage of preserving some quantities and being monotone when the problem is known to be variational (when the background is moreover balanced). Our first result is

Proposition 5.2. *The Chern-Yamabe flow exists as long as the maximum of Chern scalar curvature stays bounded.*

This result is not satisfactory as we need to assume the upper bound of the Chern scalar curvature in order to obtain the long time existence. We conjecture that

Conjecture 5.3. *The Chern scalar curvature under the Chern-Yamabe flow does not blow up in finite time.*

If the background metric ω is even balanced, then $\Delta_\omega^{\mathrm{C}} = \Delta_d$ is sysmetric. In this special case, the Chern Yamabe problem is variational. There exists a functional $\mathcal{F}$ on space of smooth functions with the normalization condition (5.7) such that the Chern-Yamabe equation is the Euler-Lagrange equation of the functional. We show that the functional is decreasing along the flow. Our second result regards with the boundedness of this functional.

Proposition 5.4. *Suppose the background Gauduchon metric ω is also balanced. There is a functional $\mathcal{F}$ whose critical points are the conformal metrics with constant Chern scalar curvature. When the Gauduchon degree $\lambda \leq 0$, this functional $\mathcal{F}$ is bounded from below. When $\lambda > 0$, $\mathcal{F}$ is unbounded from below.*

Second variation of the functional is computed. We show that in some examples the functional can possess saddle points.

Some additional property of the flow is presented under additional assumptions in the end of this part.

6 Chern-Yamabe Flow

Let $f(x;t)$ be a family of C^∞ functions on X parametrized by a real parameter t. Let $S(x;t) = \mathrm{S}^{\mathrm{C}}(\exp(2f(x;t)/n)\omega)$. The *Chern-Yamabe flow* is the flow $f(x;t)$ defined by the following flow equation:

$$\frac{\partial f}{\partial t} = \frac{n}{2}\big(\lambda - S\big) = \frac{n}{2}\exp(-2f/n)\left(\Delta^{\mathrm{C}}_\omega f - \mathrm{S}^{\mathrm{C}}(\omega) + \lambda\exp(2f/n)\right) \tag{6.1}$$

with some initial value f_0 satisfying the normalization constraint

$$\int_X \exp(2f_0/n)d\mu = 1. \tag{6.2}$$

Under the flow some quantities are preserved.

Lemma 6.1. *Along the flow we have*

1.
$$\int_X \exp(2f/n)d\mu \equiv 1.$$

2.
$$\int_X S\exp(2f/n)d\mu \equiv \lambda.$$

Proof. 1. Let
$$\phi(t) = \int_X \exp(2f/n)d\mu.$$
By the initial data (6.2) and the flow equation (6.1), we have $\phi(0) = 1$ and

$$\begin{aligned}\phi'(t) &= \frac{2}{n}\int_X \exp(2f/n)\frac{\partial f}{\partial t}d\mu \\ &= \int_X \left(\Delta^{\mathrm{C}}_\omega f - \mathrm{S}^{\mathrm{C}}(\omega) + \lambda\exp(2f/n)\right)d\mu \\ &= \lambda\big(\phi(t) - 1\big).\end{aligned}$$

It is straightforward to show that $\phi(t) \equiv 1$.

2. It follows that

$$\int_X S\exp(2f/n)d\mu = \int_X (\mathrm{S}^{\mathrm{C}}(\omega) - \Delta^{\mathrm{C}}_\omega f)d\mu \equiv \lambda.$$

□

6.1 Evolution of the Chern scalar curvature

Under the Chern-Yamabe flow the Chern scalar curvature $S(x;t) = \mathrm{S}^{\mathrm{C}}(\exp(2f/n)\omega)$ evolves according to the following equation

$$\frac{\partial S}{\partial t} = \frac{n}{2}\exp(-2f/n)\,\Delta^{\mathrm{C}}_{\omega} S + S(S-\lambda) \tag{6.3}$$

with initial value $S(x;0) = \mathrm{S}^{\mathrm{C}}(\exp(2f_0/n)\omega)$.

The following lemma gives a uniform lower bound of the Chern scalar curvature.

Lemma 6.2. *Let* $(S_0)_{min} = \min_{x\in X} S(x;0)$. *We have*

$$S(x;t) \geq \min\{(S_0)_{min}, 0\},\ \forall x \in X.$$

Proof. Let $S_{min}(t) = \min_{x\in X} S(x;t)$. Applying maximum principle to (6.3) we obtain

$${S_{min}}'(t) \geq S_{min}(S_{min} - \lambda) \geq -\lambda S_{min}.$$

Hence,

$$S(x;t) \geq S_{min}(t) \geq (S_0)_{min}\exp(-\lambda t), \forall x \in X.$$

If $(S_0)_{min} \geq 0$, then $S(x;t) \geq 0$; otherwise $S(x;t) \geq (S_0)_{min}$. Hence,

$$S(x;t) \geq \min\{(S_0)_{min}, 0\}.$$

□

Remark 6.3. *For Lemma 6.2,* $S(x,t) \geq (S_0)_{min}\exp(-\lambda t)$ *as long as the flow exists. In particular, if the initial Chern scalar curvature is strictly positive, then the positiveness is preserved along the flow.*

We can always take a special initial f_0 *so that the initial Chern scalar curvature is strictly positive. Let* $h \in C^{\infty}(X;\mathbb{R})$ *such that*

$$\Delta^{\mathrm{C}}_{\omega} h = \mathrm{S}^{\mathrm{C}}(\omega) - \lambda \ \textit{with} \ \int_X \exp(2h/n)d\mu = 1.$$

We have $\mathrm{S}^{\mathrm{C}}(\exp(2h/n)\omega) = \lambda\exp(-2h/n) > 0$. *Hence, the Chern-Yamabe flow with this specific initial* $f(x;0) = h(x)$ *has the positive Chern scalar curvature as long as the flow exists.*

6.2 Long time existence

In this section we show that the Chern-Yamabe flow exists as long as the maximum of Chern scalar curvature stays bounded. The short time existence of the flow is straightforward as the principal symbol of the second-order operator of the right-hand side of the Chern-Yamabe flow is strictly positive definite. To obtain the long time existence, one needs to show the *a priori* C^k estimate

$$\max_{0\le t<T} \|f(x;t)\|_{C^k(X)} \le C_k(T) < \infty$$

for any $T < \infty$ and any positive integer k. We use $C(T)$ to denote a constant depending on T. The constants $C(T)$ may vary from line to line. We begin with a C^0 estimate on the flow $f(x;t)$.

Lemma 6.4 (C^0 estimate)**.** *Suppose that the Chern-Yamabe flow exists on $\Omega_T = X \times [0,T)$ for some $T > 0$. Then these exists some constant $C_0(T)$ depending only on (X,ω) and initial data f_0 such that*

$$\sup_{0\le t<T} \|f(x;t)\|_{C^0(X)} \le C_0(T).$$

Proof. Let $h \in C^\infty(X;\mathbb{R})$ such that

$$\Delta^{\mathrm{C}}_\omega h = \mathrm{S}^{\mathrm{C}}(\omega) - \lambda \quad \text{with} \quad \int_X \exp(2h/n)d\mu = 1.$$

Such a function h exists because of (5.8). Similarly, by Lemma 6.1 there exists some $v(t) \in C^\infty(X \times [0,T);\mathbb{R})$ such that

$$\Delta^{\mathrm{C}}_\omega v = \exp(2f/n) - 1. \tag{6.4}$$

Differentiating (6.4) with respect to t, by the flow equation (6.1) we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(\Delta^{\mathrm{C}}_\omega v\right) &= \Delta^{\mathrm{C}}_\omega f - \mathrm{S}^{\mathrm{C}}(\omega) + \lambda\exp(2f/n)\\ &= \Delta^{\mathrm{C}}_\omega f - (\mathrm{S}^{\mathrm{C}}(\omega) - \lambda) + \lambda(\exp(2f/n) - 1)\\ &= \Delta^{\mathrm{C}}_\omega f - \Delta^{\mathrm{C}}_\omega h + \lambda\,\Delta^{\mathrm{C}}_\omega v.\end{aligned}$$

Hence,

$$\Delta^{\mathrm{C}}_\omega \left(\frac{\partial v}{\partial t} - f + h - \lambda v\right) = 0.$$

We can normalize $v(x;t)$ (by adding some function depending only on t if necessary) so that

$$\frac{\partial v}{\partial t} - f + h - \lambda v = 0 \tag{6.5}$$

with initial value $v(x;0) = v_0(x)$ for some v_0 satisfying

$$\Delta_\omega^{\mathrm{C}} v_0 = \exp(2f_0/n) - 1 \quad \text{and} \quad \int_X v_0 d\mu = 0.$$

Let $w(x;t) = \partial v/\partial t$. Differentiating (6.5) with respect to t, we have

$$\begin{cases} \dfrac{\partial w}{\partial t} = \dfrac{\partial f}{\partial t} + \lambda w = \dfrac{n}{2}\exp(-2f/n)\,\Delta_\omega^{\mathrm{C}} w + \lambda w, \\ w(x;0) = f_0(x) - h(x) + \lambda v_0(x). \end{cases} \tag{6.6}$$

Let $w_{max}(t) = \max_{x\in X} w(x;t)$ and $w_{min}(t) = \min_{x\in X} w(x;t)$. By maximum principle, we have

$$\frac{d}{dt}w_{max} \le \lambda w_{max} \quad \text{and} \quad \frac{d}{dt}w_{min} \ge \lambda w_{min}.$$

It follows that

$$w_{min}(t) \ge w_{min}(0)\exp(\lambda t) \quad \text{and} \quad w_{max}(t) \le w_{max}(0)\exp(\lambda t).$$

Hence, we have

$$\|w(x;t)\|_{C^0(X)} \le K\exp(\lambda t) \quad \text{with} \quad K = \max\big(|w_{min}(0)|, |w_{max}(0)|\big).$$

It then follows that

$$\begin{aligned} |v(x;t)| &= \left| v_0(x) + \int_0^t w(x;t)dt \right| \\ &\le \|v_0\|_{C^0(X)} + \int_0^t \|w(x;t)\|_{C^0(X)}dt \le \|v_0\|_{C^0(X)} + \frac{K}{\lambda}\exp(\lambda t). \end{aligned}$$

By (6.5) we have $f(x;t) = w(x;t) + h(x) - \lambda v(x;t)$. Hence,

$$\begin{aligned} \|f(x;t)\|_{C^0(X)} &\le \|w(x;t)\|_{C^0(X)} + \|h\|_{C^0(X)} + \lambda\|v(x;t)\|_{C^0(X)} \\ &\le K\exp(\lambda t) + \|h\|_{C^0(X)} + \lambda\left(\|v_0\|_{C^0(X)} + \frac{K}{\lambda}\exp(\lambda t)\right) \\ &\le \|h\|_{C^0(X)} + \lambda\|v_0\|_{C^0(X)} + 2K\exp(\lambda t) := C_0(T). \end{aligned}$$

Since the functions h, v_0 and w_0 are uniquely determined by (X,ω) and f_0, the constant $C_0(T)$ only depends on (X,ω) and f_0. □

Lemma 6.5. *Suppose the Chern-Yamabe flow exists on $\Omega_T = X \times [0, T)$ for some $T > 0$. Moreover, suppose that*

$$\sup_{0 \le t < T} \|S\|_{C^0(X)} \le C(T) < \infty.$$

Then for any $k \in \mathbb{N}$ there exists constant $C_k(T)$ such that

$$\sup_{0 \le t < T} \|f(x;t)\|_{C^k(X)} \le C_k(T).$$

Proof. We first get the parabolic Hölder norm bound[3] for f. For any $p \ge 1$ and $0 \le t < T$,

$$\begin{aligned}
&\|f(x;t)\|_{W^{2,p}(X)} \le C_p\Big(\|f(x;t)\|_{L^p(X)} + \|\Delta^{\mathrm{C}}_{\omega} f(x;t)\|_{L^p(X)}\Big) \\
&\le C\Big(\sup_{0\le t<T}\|f(x;t)\|_{C^0(X)} + \sup_{0\le t<T}\|S(x;t)\|_{C^0(X)} + \|\mathrm{S}^{\mathrm{C}}(\omega)\|_{C^0(X)}\Big) \\
&\le C(T).
\end{aligned}$$

By Sobolev embedding, there exists some α with $0 < \alpha < 1$,

$$\sup_{0 \le t < T} \|f(x;t)\|_{C^\alpha(X)} \le C(T).$$

Moreover, note that $|\partial f/\partial t| = (n/2)|\lambda - S| \le C(T)$. Hence, we have

$$\|f\|_{C^\alpha(X\times[0,T))} \le C(T).$$

Let $\mathcal{L}$ be any differential operator in x and t. A simple calculation shows that

$$\frac{\partial}{\partial t}(\mathcal{L}f) - \frac{n}{2}\exp(-2f/n)\,\Delta^{\mathrm{C}}_{\omega}(\mathcal{L}f) + S(\mathcal{L}f) = -\frac{n}{2}\exp(2f/n)(\mathcal{L}\,\mathrm{S}^{\mathrm{C}}(\omega)).$$

By the interior Schauder estimate for parabolic equations (Theorem 4.9 in [32]), for any τ, τ' with $0 \le \tau < \tau' < T$, we have

$$\|\mathcal{L}f\|_{C^{2+\alpha}(X\times(\tau',T))} \le C_{Sch}(\|\mathcal{L}f\|_{C^0(X\times(\tau,T))} + \|\mathcal{L}\,\mathrm{S}^{\mathrm{C}}(\omega)\|_{C^\alpha(X\times(\tau,T))})$$

where the constant C_{Sch} depends on τ, τ', $\|S\|_{C^0(X\times(\tau,T))}$ and $\|f\|_{C^\alpha(X\times(\tau,T))}$. It then follows by the standard bootstrapping argument to obtain that for any $\tau > 0$, $k \in \mathbb{N}$ and $0 < \alpha < 1$, there exists constant $C(k, \alpha, \tau, T)$ such that

$$\|f\|_{C^{k+\alpha}(X\times(\tau,T))} \le C(k, \alpha, \tau, T).$$

[3] See the definition in Chapter IV, [32].

Together with the short time existence near $t = 0$, we have

$$\sup_{0 \le t < T} \|f(x;t)\|_{C^k(X)} \le C_k(T) < \infty.$$

□

With Lemma 6.2 in hand, we only need $S(x;t)$ being upper bounded from infinity to obtain the C^k estimate of the flow. Therefore, we have the following long time existence result.

Proposition 6.6. *The Chern-Yamabe flow exists as long as the maximum of Chern scalar curvature stays bounded.*

We therefore put forward the following conjecture to fully resolve the long time existence of the flow.

Conjecture 6.7. *Suppose the Chern-Yamabe flow exists on $\Omega_T = X \times [0, T)$ form some $T > 0$. Then there exists some constant $C(T)$ depending on T such that*

$$S(x;t) \le C(T), \ \forall (x,t) \in \Omega_T.$$

7 Balanced Case

7.1 The variational functional

When the background metric is balanced, we have that $\Delta^{\mathrm{C}}_\omega = \Delta_d$ is symmetric. The partial differential equation (5.4) with normalization (5.7) is the Euler-Lagrange equation for the following functional

$$\mathcal{F}(f) := \frac{1}{2} \int_X |df|^2_\omega d\mu + \int_X \mathrm{S}^{\mathrm{C}}(\omega)\, f d\mu \tag{7.1}$$

with constraint

$$\int_X \exp(2f/n) d\mu = 1. \tag{7.2}$$

To solve the partial differential equation (5.4) is then equivalent to find a critical point of the functional (7.1) with constraint (7.2).

7.2 Monotonicity along the Chern-Yamabe flow

Let $\mathcal{F}(t) = \mathcal{F}(f(\cdot;t))$. We have the following lemma showing the monotonicity of the $\mathcal{F}$ functional along the flow.

Lemma 7.1.
$$\frac{d}{dt}\mathcal{F}(t) = -\int_X (S-\lambda)^2 \exp(2f/n) d\mu.$$

Proof. First, by Lemma 6.1, we have
$$\int_X \frac{\partial f}{\partial t} \exp(2f/n) d\mu = 0.$$

Hence,
$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t) &= \int_X \frac{\partial f}{\partial t}(-\Delta_d f + \mathrm{S}^{\mathrm{C}}(\omega)) d\mu \\ &= \int_X \frac{\partial f}{\partial t}(-\Delta_d f + \mathrm{S}^{\mathrm{C}}(\omega) - \lambda \exp(2f/n)) d\mu \\ &= -\int_X (S-\lambda)^2 \exp(2f/n) d\mu \leq 0.\end{aligned}$$

The proof is finished. □

7.3 Regarding the lower bound of the functional

As the functional is decreasing along the flow, it would be nice if the functional could be bounded from below. This is true when *lambda* ≤ 0, but not the case when $\lambda > 0$ and the complex dimension $n \geq 2$.

Proposition 7.2. *Suppose the Gauduchon degree $\lambda \leq 0$, then there exists some constants $0 < c < 1/2$ and C such that*
$$\mathcal{F}(f) \geq c \int_X |df|_\omega^2 d\mu - C \tag{7.3}$$
for any f with the normalization $\int_M \exp(2f/n) d\mu = 1$.

Proof. Let h be the solution to $\Delta h = \mathrm{S}^{\mathrm{C}} - \lambda$. Note that $\int_M f d\mu \leq 0$ by Jensen inequality. Hence,
$$\begin{aligned}\mathcal{F}(f) &= \frac{1}{2}\int_X |df|_\omega^2 d\mu + \int_X (\mathrm{S}^{\mathrm{C}} - \lambda) f d\mu + \lambda \int_X f d\mu \\ &\geq \frac{1}{2}\int_X |df|_\omega^2 d\mu - \int_X |df|_\omega |dh|_\omega d\mu \qquad (7.4) \\ &\geq c \int_X |df|_\omega^2 d\mu - C.\end{aligned}$$

□

Proposition 7.3. *For (X, ω) with complex dimension $n \geq 2$. Suppose the Gauduchon degree $\lambda > 0$, we have*

$$\inf \left\{ \mathcal{F}(f) : f \in C^\infty(X) \text{ with } \int_X \exp(2f/n) d\mu = 1 \right\} = -\infty.$$

Proof. We will construct a family of Lipschitz functions $\{f_r\}$ parameterized by a positive real number r, each of which satisfies the constraint (7.2), yet $\lim_{r\to 0} \mathcal{F}(f_r) = -\infty$. Choose an arbitrary point $p \in X$ as the center. The function $f_r(x)$ is defined as constants both inside the geodesic ball $B_r(p)$ and outside the larger ball $B_{2r}(p)$, while interpolated linearly on the annulus $B_{2r}(p)/B_r(p)$, namely,

$$f_r(x) = \begin{cases} c_r, & |x| \leq r \\ (\log r - c_r)\big(|x|/r - 1\big) + c_r, & r \leq |x| \leq 2r \\ \log r, & |x| \geq 2r \end{cases}$$

where $|x|$ denotes the distance to the center of the geodesic ball and c_r is a constant depending on r. Choose the radius r sufficiently small, then the geodesic ball $B_r(0)$ is close to a Euclidean ball and $\log r < 0$. The constant c_r is determined so that

$$\int_X \exp(2f_r/n) d\mu = 1.$$

We claim

$$c_r \leq -n^2 \log r - \frac{n}{2} \log C$$

for some dimensional constant $C = C(n)$. To see this,

$$1 = \int_X \exp(2f_r/n) d\mu \geq \int_{B_r(p)} \exp(2c_r/n) d\mu = \exp(2c_r/n) \operatorname{Vol}(B_r(p)).$$

Hence,

$$c_r \leq -\frac{n}{2} \log \operatorname{Vol}(B_r(p)) = -\frac{n}{2} \log(Cr^{2n}) = -n^2 \log r - \frac{n}{2} \log C.$$

Now we show that $\lim_{r\to 0} \mathcal{F}[f_r] = -\infty$. First of all, we have

$$\begin{aligned} \mathcal{F}(f_r) &= \int_X |df_r|^2_\omega d\mu + \int_X \mathrm{S}^{\mathrm{C}}(\omega)\, f_r d\mu \\ &= \int_{B_{2r}(p)\backslash B_r(p)} |df_r|^2_\omega d\mu + \int_{B_{2r}(p)} \mathrm{S}^{\mathrm{C}}(\omega)\, f_r d\mu \\ &\quad + \int_{X\backslash B_{2r}(p)} \mathrm{S}^{\mathrm{C}}(\omega)\, f_r d\mu_\omega. \end{aligned} \tag{7.5}$$

By continuity there exists some $r_0 > 0$ such that

$$\int_{B_{2r}(p)} \mathrm{S}^{\mathrm{C}}(\omega)\, d\mu \leq \frac{\lambda}{2}, \forall r \leq r_0.$$

Note that $\lambda = \int_X \mathrm{S}^{\mathrm{C}}(\omega)\, d\mu$, hence,

$$\int_{X \backslash B_{2r}(p)} \mathrm{S}^{\mathrm{C}}(\omega)\, d\mu \geq \frac{\lambda}{2}, \forall r \leq r_0.$$

Take r sufficiently small so that $\log r < 0$. Then $c_r > 0$ since

$$\int_X \exp(2f_r/n) d\mu = 1.$$

It follows that

$$\begin{aligned} \mathcal{F}(f_r) &\leq \frac{(c_r - \log r)^2}{r^2} \mathrm{Vol}\left(B_{2r}(p) \backslash B_r(p)\right) \\ &\quad + \|\mathrm{S}^{\mathrm{C}}(\omega)\|_{C^0(X)} c_r \mathrm{Vol}(B_{2r}(p)) + \frac{\lambda}{2} \log r \\ &\leq C(c_r - \log r)^2 r^{2n-2} + C r^{2n} c_r + \frac{\lambda}{2} \log r \\ &= \frac{\lambda}{2} \log r + O(r^{2n-2} \log r). \end{aligned} \tag{7.6}$$

When $n \geq 2$, we have $\lim_{r \to 0} r^{2n-2} \log r = 0$. The leading term for $\mathcal{F}(f_r)$ is $\frac{\lambda}{2} \log r$. Therefore, $\lim_{r \to 0} \mathcal{F}(f_r) = -\infty$. This finishes the proof. □

7.4 Second variation

Lemma 7.4. *The second variation of $\mathcal{F}$ functional is given by*

$$\delta^2 \mathcal{F}(u, v)\,|_f = \int_X \left((du, dv)_\omega - \frac{2\lambda}{n} \exp(2f/n) uv \right) d\mu \tag{7.7}$$

for any u and v in the tangent space of f, namely

$$\int_X \exp(2f/n) u d\mu = 0 \quad \textit{and} \quad \int_X \exp(2f/n) v d\mu = 0.$$

Proof. Note that the unconstrained functional is

$$\tilde{\mathcal{F}}(f) = \frac{1}{2} \int_X |df|^2_\omega d\mu + \int_X \mathrm{S}^{\mathrm{C}}(\omega)\, f d\mu - \frac{n\lambda}{2} \left(\int_X \exp(2f/n) d\mu - 1 \right).$$

The second variation follows by simple calculation. □

Given some specific direction v, we have the second variation at v as

$$\delta^2\mathcal{F}(v) = \int_X \left(|dv|^2 - \frac{2\lambda}{n}\exp(2f/n)v^2 \right) d\mu.$$

It's interesting that the positivity of the second variation may have some relation with the Rayleigh quotient, or the first principal eigenvalue of the Laplacian operator λ_1. In the special case when the background Gauduchon metric is itself a constant Chern-Scalar curvature metric, we have $f = 0$ is a critical point.

If $\lambda_1 \geq 2\lambda/n$, then

$$\delta^2\mathcal{F}(v) \geq (\lambda_1 - 2\lambda/n)\int_X v^2 d\mu \geq 0, \quad \forall v \;\text{ with }\; \int_X v d\mu = 0$$

shows that $f = 0$ is a local minimum.

If $\lambda_1 < 2\lambda/n$, then we can take some non-zero eigenvector v_0 with $\int_X v_0 d\mu = 0$ and

$$\delta^2\mathcal{F}(v_0) \leq (\lambda_1 - 2\lambda/n)\int_X v_0^2 d\mu < 0.$$

Hence, $f = 0$ is a saddle point and unstable.

To construct concrete example for the above argument, one can consider $\mathbb{P}^1 \times \theta\mathbb{P}^1$ with $\mathbb{P}^1$ and $\mathbb{P}^1$ both endowed with the standard Fubini-Study metrics. For such family of complex manifolds, the background Fubini Study metrics ω_θ are constant Chern scalar curvature metrics; so we write down the functional $\mathcal{F}$ with respect to the reference metric ω_θ, and $f = 0$ represents a constant scalar Chern curvature metric with $\mathcal{F}(0) = 0$. By adjusting the scaling parameter θ, it is not hard to adjust λ_1 and the total Chern scalar curvature λ such that $-\frac{\lambda_1}{2} + \lambda < 0$; this makes possible to find a sequence of conformal factors f_k that are arbitrarily close to $f = 0$, and with $\mathcal{F}(f_k) < 0$. Since the flow decreases the functional $\mathcal{F}$, then the flow starting at f_k will not converge to $f = 0$. The conclusion we can draw is that saddle points are possible and we should not expect only local minima in general. Together with the fact, proved in Lemma 7.1, that the $\mathcal{F}$ functional always is not bounded from below, the techniques for only minima is not enough.

7.5 Under additional assumptions

We have already shown in Lemma 7.3 that the functional $\mathcal{F}$ is unbounded from below. So it is impossible to find a global minimum. Yet it is still possible that the functional is bounded from below along the flow for some specific initial value. In particular, if the flow finally converges to a solution, one of the necessary conditions is that the functional is bounded under the flow.

In this section we assume the flow exists on $[0, \infty)$ and

$$\lim_{t\to\infty} \mathcal{F}(t) \geq C > -\infty. \tag{7.8}$$

What can we say about the flow?

Since the functional is decreasing and bounded from below, we can find a sequence of time slices $\{t_k\}$, so that $\frac{d}{dt}\mathcal{F}(t_k) \to 0$. Let $f_k = f(t_k)$ and $S_k = S\big(\exp(2f_k/n)\omega\big)$. Note that by Lemma 7.1,

$$\frac{d}{dt}\mathcal{F}(t) = -\int_X (S-\lambda)^2 \exp(2f/n) d\mu = \lambda^2 - \int_X S^2 \exp(2f/n) d\mu.$$

On the other hand, by Lemma 6.2, we have $S(x;t) > -C$. Hence, we have

$$\begin{cases} \displaystyle\int_X \exp(2f_k/n) d\mu = 1, \\ \displaystyle\int_X S_k^2 \exp(2f_k/n) d\mu \to \lambda^2 \text{ and } S_k > -C, \\ |\mathcal{F}(f_k)| \leq C. \end{cases} \tag{7.9}$$

In this section we assume that there exists uniform upper bound for the sequence $\{f_k\}$ in (7.9). We show that there exists a smooth solution to the Chern-Yamabe equation 5.4. In what follows the constant C may vary from line to line.

Lemma 7.5. *Suppose there is a sequence $\{f_k\}$ satisfying (7.9). Suppose additionally there exists some constant C_0 such that $\max_{x\in X} f_k(x) \leq C_0, \forall k$. Then $\|f_k\|_{H^2} \leq C$.*

Proof. First of all, we have

$$\int_X (S_k \exp(2f_k/n))^2 d\mu \leq \exp(2C_0/n) \int_X S_k^2 \exp(2f_k/n) d\mu \leq C.$$

Note that $S_k = \exp(-2f_k/n)(\mathrm{S}^{\mathrm{C}}(\omega) - \Delta f_k)$, namely, $\Delta f_k = \mathrm{S}^{\mathrm{C}}(\omega) - S_k \exp(2f_k/n)$. Hence, we have $\|\Delta f_k\|_{L^2(X)} \leq C$.

Claim. *Let $\bar{f}_k = \int_X f_k d\mu$. There exists some constant $C > 0$ such that $-C \leq \bar{f}_k \leq 0$.*

Proof of the Claim. First of all, since $\mathrm{Vol}(X) = 1$, we have

$$\exp(2\bar{f}_k/n) = \exp\Big(\int_X (2f_k/n)d\mu\Big) \le \int_X \exp(2f_k/n)d\mu = 1.$$

Hence, $\bar{f}_k \le 0$.

For the other side, first note that

$$\begin{aligned}
&\int_X S_k \exp(2f_k/n) f_k d\mu \\
&= \int_X (-\Delta f_k + \mathrm{S^C}(\omega)) f_k d\mu = 2\mathcal{F}(f_k) - \int_X (\mathrm{S^C}(\omega) - \lambda) f_k - \lambda \bar{f}_k \\
&= 2\mathcal{F}(f_k) - \int_X \Delta h f_k - \lambda \bar{f}_k = 2\mathcal{F}(f_k) - \int_X h \Delta f_k - \lambda \bar{f}_k \\
&\ge 2\mathcal{F}(f_k) - \|h\|_{L^2(X)} \|\Delta f_k\|_{L^2(X)} - \lambda \bar{f}_k \\
&\ge C - \lambda \bar{f}_k.
\end{aligned} \tag{7.10}$$

On the other hand, since $S_k > -C$, we have

$$\begin{aligned}
&\int_X S_k \exp(2f_k/n) f_k d\mu \\
&= \int_X (S_k + C) \exp(2f_k/n) f_k d\mu - C \int_X \exp(2f_k/n) f_k d\mu \\
&\le C_0 \exp(C_0/n) \int_X S_k \exp(f_k/n) d\mu + C_0 C + C \cdot \frac{n}{2e} \\
&\le C_0 \exp(C_0/n) \Big(\int_X S_k^2 \exp(2f_k/n) d\mu\Big)^{1/2} + C \le C.
\end{aligned} \tag{7.11}$$

By (7.10) and (7.11), we obtain that $\bar{f}_k \ge -C$. This finishes the proof of the Claim. $\square$

We continue our proof for the Lemma. By Poincare inequality, there exists some constant C_p so that

$$\int_X (f_k - \bar{f}_k)^2 d\mu \le C_p \int_X |\nabla f_k|^2 d\mu.$$

On the other hand,

$$\int_X |\nabla f_k|^2 d\mu = \int_X (-f_k \Delta f_k) d\mu \le \frac{1}{2C_p} \int_X f_k^2 d\mu + \frac{C_p}{2} \int_X (\Delta f_k)^2 d\mu.$$

Hence,

$$\int_X f_k^2 d\mu - \bar{f_k}^2 \le \frac{1}{2}\int_X f_k^2 d\mu + \frac{C_p^2}{2}\int_X (\Delta f_k)^2 d\mu.$$

Hence,

$$\int_X f_k^2 d\mu \le 2\bar{f_k}^2 + C_p^2 \int_X (\Delta f_k)^2 d\mu. \tag{7.12}$$

It then follows by Sobolev estimate that

$$\|f_k\|_{H^2(X)} \le C\Big(\|f_k\|_{L^2(X)} + \|\Delta f_k\|_{L^2(X)}\Big) \le C\Big(|\bar{f_k}| + \|\Delta f_k\|_{L^2(X)}\Big) \le C.$$

This finishes the proof. □

Proposition 7.6. *Suppose there is a sequence $\{f_k\}$ satisfying* (7.9). *Suppose additionally there exists some constant C_0 such that*

$$\max_{x\in X} f_k(x) \le C_0, \forall k.$$

Then there exists a function $f_\infty \in C^\infty(X)$ which solves the differential equation (5.4).

Proof. By Lemma 7.5, we have $\|f_k\|_{H^2(X)} \le C$. Hence, by passing to a subsequence if necessary, we have $f_k \rightharpoonup f_\infty$ weakly in $H^2(X)$ for some f_∞. It follows that $f_k \to f_\infty$ strongly in $L^2(X)$ and $\Delta f_k \rightharpoonup \Delta f_\infty$ weakly in $L^2(X)$. As a result of the strong convergence in $L^2(X)$, by passing to a subsequence if necessary, we have $f_k \to f_\infty$ $d\mu$-a.e.. Then by Egonov's theorem, for any $\delta > 0$, there exists a subset $\Omega_\delta \subset X$ with $\mathrm{Vol}(X\backslash\Omega_\delta) < \delta$, such that $f_k \to f_\infty$ uniformly on Ω_δ. We have

$$\begin{aligned}
&\int_{\Omega_\delta} (\Delta f_k - \mathrm{S^C}(\omega))^2 \exp(-2f_k/n) d\mu \\
&= \int_{\Omega_\delta} (\Delta f_k - \mathrm{S^C}(\omega))^2 \exp(-2f_\infty/n) d\mu + \int_{\Omega_\delta} (\Delta f_k - \mathrm{S^C}(\omega))^2 \big(e^{-2f_k/n} - e^{-2f_\infty/n}\big) d\mu \\
&\ge \int_{\Omega_\delta} (\Delta f_k - \mathrm{S^C}(\omega))^2 \exp(-2f_\infty/n) d\mu - C\|e^{-2f_k/n} - e^{-2f_\infty/n}\|_{L^\infty(\Omega_\delta)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\liminf_{k\to\infty} \int_{\Omega_\delta} (\Delta f_k - \mathrm{S^C}(\omega))^2 \exp(-2f_k/n) d\mu \\
&\ge \liminf_{k\to\infty} \int_{\Omega_\delta} (\Delta f_k - \mathrm{S^C}(\omega))^2 \exp(-2f_\infty/n) d\mu \\
&\ge \int_{\Omega_\delta} (\Delta f_\infty - \mathrm{S^C}(\omega))^2 \exp(-2f_\infty/n) d\mu.
\end{aligned}$$

Notice that

$$\int_X (\Delta f_k - \mathrm{S}^{\mathrm{C}}(\omega))^2 \exp(-2f_k/n) d\mu = \int_X S_k^2 \exp(2f_k/n) d\mu \to \lambda^2, \quad \text{as } k \to \infty.$$

Hence,

$$\int_{\Omega_\delta} (\Delta f_\infty - \mathrm{S}^{\mathrm{C}}(\omega))^2 \exp(-2f_\infty/n) d\mu \leq \lambda^2.$$

Let $\delta \to 0$, we obtain that

$$\int_X (\Delta f_\infty - \mathrm{S}^{\mathrm{C}}(\omega))^2 \exp(-2f_\infty/n) d\mu \leq \lambda^2. \tag{7.13}$$

Note that $f_k \to f_\infty$ $d\mu$-a.e., and $f_k \leq C_0$ by assumption, we have $f_\infty \leq C_0$ $d\mu$-a.e.. Then by Dominance Convergence Theorem, we have

$$\int_X \exp(2f_\infty/n) d\mu = \lim_{k\to\infty} \int_X \exp(2f_k/n) d\mu = 1. \tag{7.14}$$

By (7.13) and (7.14), we have

$$\int_X \Big(\Delta f_\infty - \mathrm{S}^{\mathrm{C}}(\omega) + \lambda \exp(2f_\infty/n)\Big)^2 \exp(-2f_\infty/n) d\mu \leq 0.$$

It follows that the equality holds and

$$\Delta f_\infty - \mathrm{S}^{\mathrm{C}}(\omega) + \lambda \exp(2f_\infty/n) = 0, \; d\mu - \text{a.e.}. \tag{7.15}$$

Since $f_\infty \leq C_0$ $d\mu$-a.e., we have $\Delta f_\infty = \mathrm{S}^{\mathrm{C}}(\omega) - \lambda \exp(2f_\infty/n) \in L^\infty(X)$. Hence, $f_\infty \in W^{2,p}(X)$ for any $p > 1$. By Sobolev embedding theorem, this implies that $f_\infty \in C^{1,\alpha}(X)$. Then $\Delta f_\infty \in C^{1,\alpha}(X)$. By the standard bootstrapping argument, we eventually have $f_\infty \in C^\infty(X)$. This finishes the proof. $\square$

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