Lagrangian correspondences and functors in wrapped Floer theory

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Abstract of the Dissertation

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We study wrapped Floer theory of a product Liouville manifold, and prove that two versions of wrapped Fukaya categories, with respect to different types of Floer data, are both well-defined and equivalent. Based on this, a quilted version of wrapped Floer theory is developed, which allows us to construct functors between suitable enlargement of wrapped Fukaya categories from Lagrangian correspondences that satisfy a properness condition. As applications to the general theory, we present a Knneth formula for wrapped Fukaya categories, and give a discussion on reformulation and extension of the Viterbo restriction functor.

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Contents

1. Introduction

1.1. **Mirror symmetry background.** Prediction from Homological Mirrror Symmetry conjecture [Kon95] roughly says that symplectic and complex manifolds come in mirror pairs with equivalent A-model and B-model categories. While compact Calabi-Yau manifolds are the primary and original objects of concern, the conjecture itself has been widely extended to various context, in particular certain non-compact manifolds which will be of interest to us. The expected mirror of the derived category of coherent sheaves on a non-proper variety is the wrapped Fukaya category, introduced by [AS10]. It admits certain class of non-compact Lagrangians as objects, and is generally not proper. For non-proper categories, the functoriality properties can be somewhat complicated.

The goal of this thesis is to study natural functors between wrapped Fukaya categories which arise from Lagrangian correspondences, with a motivation in investigating the functoriality properties of wrapped Fukaya categories, as well as the relation to the well-established functoriality properties of the devired categories of coherent sheaves via homological mirror symmetry. Eventually, we also hope to fully generalize the theory to other versions of Fukaya categories in order to provide a tool for understanding various Fukaya categories that appear in a mirror symmetry picture, for example in [AAK16].

1.2. **Floer theory on product manifolds.** Consider Liouville manifolds $M = (M, \lambda_M, Z_M)$ and $N = (N, \lambda_N, Z_N)$. Since Lagrangian correspondences are simply Lagrangian submanifolds of the product manifold, it is natural to study first of all wrapped Floer theory of the product Liouville manifold $M \times N$ and relate it to the wrapped Floer theories of both factors. To understand such a relation, it is natural to set up wrapped Floer theory using the split Hamiltonian, which is the sum of admissible Hamiltonians on both factors, i.e. Hamiltonian of the form $\pi_M^* H_M + \pi_N^* H_N$. This together with the product almost complex structure defines a version of wrapped Fukaya category of $M \times N$, which we call the split wrapped Fukaya category and denote by $\mathcal{W}^s(M \times N)$. On the other hand, there are natural choices of cylindrical structures on $M \times N$, which enable us to define the usual wrapped Fukaya category $\mathcal{W}(M \times N)$.

However, as already addressed in [Gao17b], there is a technical issue: the split Hamiltonian is a priori not admissible for wrapped Floer theory, meaning that it might have defined a different category compared to the quadratic Hamiltonian on the product $M \times N$ with respect to a natural cylindrical end. This issue was resolved on the cohomology level there. For the purpose of constructing functors and studying deeper structures related to them, we shall carry out a chain-level discussion, confirming that the wrapped Fukaya category defined with respect to the split Hamiltonians is quasi-isomorphic to the one defined with respect to a quadratic Hamiltonian, for a suitable class of Lagrangian submanifolds of the product manifold which we call admissible, to be discussed in section 4. Thus up to canonical quasi-equivalence, there is no ambiguity in mentioning the wrapped Floer theory for admissible Lagrangian submanifolds of the product $M \times N$. The importance is that it allows us to study quilted version of Floer theory and construct functors between wrapped Fukaya categories later on.

Theorem 1.1. *Let $\mathbb{L}$ be a finite collection of admissible Lagrangian submanifolds of $M \times N$, and $\mathcal{W}^s(\mathbb{L})$ (resp. $\mathcal{W}(\mathbb{L})$) the full subcategory of the wrapped Fukaya* *category $\mathcal{W}^s(M \times N)$ (and resp. $\mathcal{W}(M \times N)$) consisting of objects in $\mathbb{L}$. Then there is a natural quasi-equivalence*

$$R_{\mathbb{L}} : \mathcal{W}^s(\mathbb{L}) \to \mathcal{W}(\mathbb{L}). \tag{1.1}$$

These quasi-equivalences satisfy the property that if $\mathbb{L}'$ is a collection containing $\mathbb{L}$, then $R_{\mathbb{L}'}$ restricted to $\mathcal{W}^s(\mathbb{L}) \to \mathcal{W}(\mathbb{L})$ is homotopic to $R_{\mathbb{L}}$.

As this A_∞-functor is constructed using an action-filtration argument and the linear term is basically the "identity" in each inductive step when restricted to a particular action-filtration window, it will be called the action-restriction functor.

1.3. **Functors associated to Lagrangian correspondences.** Next, we investigate a specific class of Lagrangian correspondences, which are either products or cylindrical with respect to a natural choice of cylindrical end of the product $M^- \times N$. Associated to every such Lagrangian correspondence $\mathcal{L} \subset M^- \times N$, we would like to construct an A_∞-functor from the wrapped Fukaya category of M to that of N. Technically, this is not always possible. To overcome this, we include wider class of objects in the wrapped Fukaya category, which are to be introduced in section 6. Details are to be presented later, but let us first illustrate the main spirit below.

Using a quilted version wrapped Floer cohomology, we first construct an A_∞-functor

$$\mathcal{W}(M^- \times N) \to (\mathcal{W}(M), \mathcal{W}(N))^{bimod}$$

from the wrapped Fukaya category of the product to the A_∞-category of $(\mathcal{W}(M), \mathcal{W}(N))$-bimodules. By purely algebraic considerations, this gives rise to an A_∞-functor

$$\mathcal{W}(M^- \times N) \to func(\mathcal{W}(M), \mathcal{W}(N)^{l-mod})$$

by applying the Yoneda embedding on the second factor $\mathcal{W}(N)$. Concretely, to each admissible Lagrangian correspondence $\mathcal{L} \subset M^- \times N$, we associate an A_∞-bimodule $P_{\mathcal{L}}$ over $(\mathcal{W}(M), \mathcal{W}(N))$ whose value on a pair (L, L') is the quilted wrapped Floer cochain complex $CW^*(L, \mathcal{L}, L')$. Regarding $L' \subset N$ as a testing object, with small amount of homological algebra argument we then immediately get an A_∞-functor

$$\Phi_{\mathcal{L}} : \mathcal{W}(M) \to \mathcal{W}(N)^{l-mod}. \tag{1.2}$$

Then we study the geometric composition of Lagrangian correspondences, which is in general a Lagrangian immersion. In order to be able to include these immersed Lagrangian submanifolds as objects of the wrapped Fukaya category, in section 5 we define the immersed wrapped Fukaya category of M, denoted by $\mathcal{W}_{im}(M)$, whose objects are unobstructed proper exact cylindrical Lagrangian immersions of M (with transverse self-intersections), together with bounding cochains for them. Floer theory for such Lagrangian immersions turns out to work over $\mathbb{Z}$, so that $\mathcal{W}_{im}(M)$ is an A_∞-category over $\mathbb{Z}$ in the usual sense. We also extend this theory to cylindrical Lagrangian immersions with clean self-intersections in section **??**.

The ordinary wrapped Fukaya category $\mathcal{W}(M)$ can be embedded into $\mathcal{W}_{im}(M)$ as a full sub-category. Also, the A_∞-functors $\Phi_{\mathcal{L}}$ and Φ can be extended to the category of modules over the immersed wrapped Fukaya category

$$\Phi_{\mathcal{L}} : \mathcal{W}(M) \to \mathcal{W}_{im}(N)^{l-mod}, \tag{1.3}$$

and

$$\Phi : \mathcal{W}(M^- \times N) \to func(\mathcal{W}(M), \mathcal{W}_{im}(N)^{l-mod}). \tag{1.4}$$

Theorem 1.2. *Let $\mathcal{L} \subset M^- \times N$ be an admissible Lagrangian correspondence between Liouville manifolds M and N, such that the projection $\mathcal{L} \to N$ is proper. Then under some further generic geometric conditions, namely Assumption 6.1, we have:*

(i) For every object $L \in Ob\mathcal{W}(M)$, there is a curved A_∞-algebra associated to the geometric composition $L \circ \mathcal{L}$, defined in terms of wrapped Floer theory for Lagrangian immersions.

(ii) The geometric composition $L \circ \mathcal{L}$ is always unobstructed, with a canonical choice of bounding cochain b for it. Thus $(L \circ \mathcal{L}, b)$ becomes an object of $\mathcal{W}_{im}(N)$. This b is unique such that the next condition is satisfied.

(iii) There is a natural A_∞-functor

$$\Theta_{\mathcal{L}} : \mathcal{W}(M) \to \mathcal{W}_{im}(N), \tag{1.5}$$

which represents $\Phi_{\mathcal{L}}$. On the level of objects, it sends any Lagrangian submanifold $L \in Ob\mathcal{W}(M)$ to the pair $(L \circ \mathcal{L}, b) \in Ob\mathcal{W}_{im}(N)$.

In a more functorial form, the assignment of A_∞-functors to Lagrangian correspondences is functorial in the wrapped Fukaya category of the product manifold $M^- \times N$:

Theorem 1.3. *Let $\mathcal{A}(M^- \times N)$ be the full A_∞-subcategory whose objects are Lagrangian correspondences $\mathcal{L}$ from M to N such that the projection $\mathcal{L} \to N$ is proper, which further satisfies Assumption 6.1. Then there is a canonical A_∞-functor*

$$\Theta : \mathcal{A}(M^- \times N) \to func(\mathcal{W}(M), \mathcal{W}_{im}(N)), \tag{1.6}$$

such that

(i) Θ represents Φ;

(ii) $\Theta(\mathcal{L}) = \Theta_{\mathcal{L}}$ for every $\mathcal{L} \in Ob\mathcal{A}(M^- \times N)$.

1.4. **Some applications.** As a particular application of the construction of functors, we present a well-known and expected (but not fully established) Künneth formula for wrapped Fukaya categories, which relate $\mathcal{W}(M \times N)$ to the A_∞-tensor product $\mathcal{W}(M) \otimes \mathcal{W}(N)$ in an appropriate sense. We also show that under the condition that both $\mathcal{W}(M)$ and $\mathcal{W}(N)$ have finite collections of split-generators, there is a quasi-equivalence between $\mathcal{W}(M \times N)$ and $\mathcal{W}(M) \otimes \mathcal{W}(N)$. These results will be discussed in a more formal way in subsection 6.9.

An especially important instance of Lagrangian correspondence comes from Liouville sub-domains $U_0 \subset M_0$: the graph of the natural inclusion can be completed to a Lagrangian correspondence between U and M. This is called the graph correspondence, denoted by Γ. As an admissible Lagrangian correspondence from M to U, it satisfies the hypothesis of Theorem 1.2. Thus we obtain an A_∞-functor

$$\Theta_\Gamma : \mathcal{W}(M) \to \mathcal{W}_{im}(U). \tag{1.7}$$

There is a full sub-category on which this functor takes a simpler form. First, there is the full sub-category $\mathcal{B}(M)$ consisting of exact cylindrical Lagrangian submanifolds L whose primitive f is locally constant near both ∂M and ∂U (Assumption 7.1). Then, we may further consider a full sub-category $\mathcal{B}_0(M)$ of $\mathcal{B}(M)$ whose objects satisfy an additional geometric condition (Assumption 7.2). Restricted to this sub-category, the functor Θ_Γ induces a functor

$$\Theta_\Gamma : \mathcal{B}_0(M) \to \mathcal{W}(N),$$

whose image lies in the ordinary wrapped Fukaya category consisting of properly embedded exact cylindrical Lagrangian submanifolds, with zero bounding cochains. On the other hand, recall that the Viterbo restriction functor is defined on this sub-category $\mathcal{B}(M)$. More detailed definitions will be given in section 7. We shall see that their linear terms agree, when the Viterbo restriction functor is further restricted to $\mathcal{B}_0(M)$.

Theorem 1.4. *The A_∞-functor Θ_Γ associated to the graph correspondence $\Gamma \subset M^- \times U$ of the Liouville sub-domain restricts to an A_∞-functor on the full sub-category $\mathcal{B}_0(M)$*

$$\Theta_\Gamma : \mathcal{B}_0(M) \to \mathcal{W}(U). \tag{1.8}$$

The linear term Θ_Γ^1 is chain homotopic to the linear term r^1 of the Viterbo restriction functor r.

It is expected that Θ_Γ and r are in fact homotopic as A_∞-functors, when restricted to the full sub-category $\mathcal{B}_0(M)$. Our current method of proof needs improvement in order to prove this more general statement. Moreover, they should be indeed homotopic on the bigger sub-category $\mathcal{B}(M)$ in appropriate sense, if we can manage to show that the geometric composition with the graph correspondence is Floer-theoretically equivalent to the actual restriction of Lagrangian submanifolds. Such points are to be discussed in [Gao].

Going back to the definition of the Viterbo restriction functor, we recall that it is only defined for those Lagrangian submanifolds which satisfy Assumption 7.1. However, the restriction functor Θ_Γ is defined on the whole wrapped Fukaya category, though the image of a Lagrangian submanifold is not necessarily simply its restriction to the sub-domain. However, it is conjectured that the Viterbo restriction functor can be extended to the whole wrapped Fukaya category via some kind of deformation theory, as stated in Conjecture 7.1. It is not completely known but a very interesting question to ask in what cases the extension agrees with the functor Θ_Γ. This is a question for future research.

1.5. **Other remarks.** This thesis is written primarily based on the research work by the author in [Gao17b], [Gao17a], as well as the work in progress [Gao]. Thus there is large text overlap.

The structure of this thesis is as follows. Section 2 provides basic and necessary homological algebra background concerning A_∞-modules, and discusses direct limits in the A_∞-context. Section 3 reviews the basic definition of the wrapped Fukaya category, which provides the geometric setup for constructions and proofs throughout the thesis. Section 4 studies Floer theory on product manifolds and proves the two versions of wrapped Fukaya categories are equivalent. Section 5 generalizes wrapped Floer theory to certain classes of Lagrangian immersions. Section 6 develops the theory of functors between wrapped Fukaya categories associated to Lagrangian correspondences, and presents a Künneth formula at the end. Section 7 concerns sub-domains and discusses how the Viterbo restriction functor is related to our framework. In section 8, we include the analytic arguments used to establish the results in section 4.

2. Homological algebra preliminaries

2.1. **A_∞-modules and bimodules.** In this section, we fix an integral domain R as the ground ring, which is mostly $\mathbb{Z}$ in later sections. We are going to study A_∞-structures over R. Let Ch be the dg-category of chain complexes, regarded as an A_∞-category whose higher order operations $m^k, k \geq 3$ are all zero. Here chain complexes are not necessarily bounded.

By definition, a non-unital left A_∞-module over $\mathcal{A}$ is a non-unital A_∞-functor $\mathcal{A} \to Ch^{op}$, where Ch^{op} is the opposite category of Ch. All left A_∞-modules over $\mathcal{A}$ also form an A_∞-category (in fact a dg-category), $\mathcal{A}^{nu-l-mod} = nu-func(\mathcal{A}, Ch^{op})$.

There are also non-unital right A_∞-modules over $\mathcal{A}$, which are functors from $\mathcal{A}^{op}$ to Ch. These also form a dg-category, $\mathcal{A}^{nu-r-mod} = func(\mathcal{A}^{op}, Ch)$.

When $\mathcal{A}$ comes with strict units, homotopy units or cohomological units, there are also unital versions of left and right A_∞-modules:

$$\mathcal{A}^{l-mod} = func(\mathcal{A}, Ch^{op}),$$

$$\mathcal{A}^{r-mod} = func(\mathcal{A}^{op}, Ch),$$

as unital A_∞-functors.

2.2. **Representable modules.** First, we recall the non-unital version of the Yoneda embedding. The left Yoneda functor is a non-unital A_∞-functor

$$\mathfrak{y}_l : \mathcal{A} \to \mathcal{A}^{nu-mod} \tag{2.1}$$

which sends an object $Y \in Ob\mathcal{A}$ to its left Yoneda module $\mathcal{Y}^l \in \mathcal{A}^{nu-mod}$, which is defined as follows. For objects $X \in Ob\mathcal{A}$,

$$\mathcal{Y}^l(X) = hom_{\mathcal{A}}(Y, X), \tag{2.2}$$

and the module structure is given by the A_∞-structure maps of $\mathcal{A}$:

$$n^d_{\mathcal{Y}^l} : \mathrm{hom}_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \mathrm{hom}_{\mathcal{A}}(X_0, X_1) \otimes \mathcal{Y}^r(X_0) \to \mathcal{Y}^r(X_{d-1}), \tag{2.3}$$

$$n^d_{\mathcal{Y}^l}(a_{d-1}, \cdots, a_1, b) = m^{d+1}_{\mathcal{A}}(a_{d-1}, \cdots, a_1, b). \tag{2.4}$$

On the level of morphisms, $\mathfrak{y}^1_l$ assigns to a morphism $c \in hom_{\mathcal{A}}(Y_0, Y_1)$ a pre-module homomorphism:

$$(\mathfrak{y}^1_l(c))^d : \mathrm{hom}_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \mathrm{hom}_{\mathcal{A}}(X_0, X_1) \otimes \mathcal{Y}^l_1(X_0) \to \mathcal{Y}^l_0(X_{d-1}),$$

$$(\mathfrak{y}^1_l(c))^d(a_{d-1}, \cdots, a_1, b) = m^{d+1}_{\mathcal{A}}(a_{d-1}, \cdots, a_1, b, c). \tag{2.5}$$

Higher order terms $\mathfrak{y}^k_l$ are defined in by analogous formulas:

$$(\mathfrak{y}^k_l(c_k, \cdots, c_1))^d : \mathrm{hom}_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \mathrm{hom}_{\mathcal{A}}(X_0, X_1) \otimes \mathcal{Y}^l_k(X_0) \to \mathcal{Y}^l_0(X_{d-1}),$$

$$(\mathfrak{y}^k_l(c_k, \cdots, c_1))^d(a_{d-1}, \cdots, a_1, b) = m^{d+k}_{\mathcal{A}}(a_{d-1}, \cdots, a_1, b, c_k, \cdots, c_1). \tag{2.6}$$

In case $\mathcal{A}$ is cohomologically unital, the image of the Yoneda functor lies in the A_∞-subcategory of $\mathcal{A}^{nu-l-mod}$ consisting of c-unital right $\mathcal{A}_\infty$-modules over $\mathcal{A}$. We denote this A_∞-subcategory by $\mathcal{A}^{l-mod}$.

An important notion is the representability of an A_∞-functor, in the sense of [Fuk02]. We briefly recall it here.

Definition 2.1. *A left A_∞-module over $\mathcal{A}$, namely an A_∞-functor $\mathcal{M}:\mathcal{A}\to Ch^{op}$ is said to be representable, if it is homotopic (or equivalently quasi-isomorphic) to a left Yoneda module $\mathcal{Y}^l=\mathfrak{y}_l(Y)$ for some object Y of $\mathcal{A}$, as A_∞-functors $\mathcal{A}\to Ch^{op}$.*

There is also a right Yoneda functor

$$\mathfrak{y}_r:\mathcal{A}\to\mathcal{A}^{nu-r-mod}, \tag{2.7}$$

whose c-unital version becomes

$$\mathfrak{y}_r:\mathcal{A}\to\mathcal{A}^{r-mod}. \tag{2.8}$$

Similarly, a right A_∞-module over $\mathcal{A}$ is said to be representable, if it is homotopic to a right Yoneda module $\mathcal{Y}^r=\mathfrak{y}_r(Y)$.

We will see in the following subsection that the notion of representability does not have ambiguity, up to quasi-isomorphism. That is, the representative is unique up to quasi-isomorphism.

2.3. **Yoneda lemma.** The key point related to Yoneda embedding we want to emphasize here is that whenever $\mathcal{A}$ is c-unital, the Yoneda embedding is cohomologically fully faithful, and therefore is an A_∞-homotopy equivalence to its image. This was first proved in [Fuk02] under the assumption that $\mathcal{A}$ is strictly unital, with a new proof given in [Sei08] in case $\mathcal{A}$ is only cohomologically unital. To see this, let $\mathcal{M}$ be any c-unital left A_∞-module over $\mathcal{A}$ and consider the following cochain map

$$\lambda:\mathcal{M}(Y)\to hom_{\mathcal{A}^{mod}}(\mathcal{Y}^l,\mathcal{M}) \tag{2.9}$$

$$\lambda(c)^d(a_{d-1},\cdots,a_1,b)=n_{\mathcal{M}}^{d+1}(a_{d-1},\cdots,a_1,b,c). \tag{2.10}$$

Note in fact this definition also makes sense for general non-unital $\mathcal{A}_\infty$-modules, but we will only emphasize its importance in the c-unital case.

Lemma 2.1. *Then the above cochain map λ is a quasi-isomorphism, for any object Y of $\mathcal{A}$.*

Proof. The mapping cone of the cochain map λ is the following cochain complex:

$$(\mathcal{M}(Y)\oplus \mathrm{hom}_{\mathcal{A}^{l-mod}}(\mathcal{Y}^l,\mathcal{M})[-1],\begin{pmatrix} n^1_{\mathcal{M}} & 0\\ \lambda & -m^1_{\mathcal{A}^{l-mod}}\end{pmatrix})[1]. \tag{2.11}$$

Define a filtration on this cochain complex by first taking the subcomplex $\mathrm{hom}_{\mathcal{A}^{l-mod}}(\mathcal{Y}^l,\mathcal{M})[-1]$, then filtering it by its natural length filtration. Denote $A=H(\mathcal{A}), M=H(\mathcal{M})$. Associated to this filtration there is a spectral sequence whose E_1-page is

$$E_1^{rs}=\begin{cases} M^s(Y), \text{ if } r=0,\\ \prod\limits_{X_0,\cdots,X_{r-1}} Hom^s_R(Hom_A(X_{r-2},X_{r-1})\otimes\cdots\otimes Hom_A(X_0,X_1)\\ \qquad\qquad \otimes Hom_A(Y,X_0), M(X_{r-1})), \text{ if } r>0.\end{cases} \tag{2.12}$$

The differential $d=d_1^{rs}:E_1^{rs}\to E_1^{r+1,s}$ is given by $d(c)(b)=(-1)^{|b|}(c)$ if $r=0$, and

$$\begin{aligned} d(t)(a_r,\cdots,a_1,b)&=(-1)^{|b|+*_r}t(a_r,\cdots,a_1)b+(-1)^{|b|}t(a_r,a_{r-1},\cdots,a_1b)\\ &+\sum_n(-1)^{\Delta}t(a_r,\cdots,a_{n+2}a_{n+1},\cdots,a_1,b),\end{aligned} \tag{2.13}$$

where $*_r = |a_1| + \cdots + |a_r| - r$, and $\Delta = |a_{n+2}| + \cdots + |a_r| + |b| + n + 1 - r$. In the above expression, $a_i a_j$ is the induced composition in the cohomology category A, which is associative, and ab is the induced A-module structure on M from the structure of left A_∞-module on $\mathcal{M}$. This is the standard bar resolution of the cochain complex (2.11), which in the presence of cohomological unit of $\mathcal{A}$, admits a contracting homotopy

$$\kappa : E_1^{r+1,s} \to E_1^{rs} \tag{2.14}$$

$$\kappa(t)(a_{r-1}, \cdots, a_1, b) = t(a_{r-1}, \cdots, a_1, b, e_Y), \tag{2.15}$$

where e_Y is a cochain in $\hom_{\mathcal{A}}(Y, Y)$ representing the identity morphism in $H(\hom_{\mathcal{A}}(Y, Y))$. This shows that the spectral sequence degenerates at the E_1-page, which imples that the cochain complex (2.11) is acyclic, and therefore the cochain map λ is a quasi-isomorphism. □

Corollary 2.1. *If $\mathcal{A}$ is c-unital, then the (c-unital) left Yoneda functor*

$$\mathfrak{y}_l : \mathcal{A} \to \mathcal{A}^{l-mod}$$

is cohomologically fully faithful.

There is a parallel discussion for right Yoneda functors. That is, the right Yoneda functor

$$\mathfrak{y}_r : \mathcal{A} \to \mathcal{A}^{r-mod}$$

is cohomologically fully faithful.

For this reason, we also call the Yoneda functor the Yoneda embedding. And there is no ambiguity for the notion of representability. This allows us to make the following definition.

Definition 2.2. *The A_∞-category $\mathcal{A}^{rep-l-mod}$ (resp. $\mathcal{A}^{rep-r-mod}$) of representable left (resp. right) A_∞-modules over $\mathcal{A}$, is the full A_∞-subcategory of $\mathcal{A}^{l-mod}$ whose objects are representable left (resp. right) A_∞-modules. Equivalently, it is the image of the left (resp. right) Yoneda embedding $\mathfrak{y}_l$ (resp. $\mathfrak{y}_r$) inside $\mathcal{A}^{l-mod}$ (resp. $\mathcal{A}^{r-mod}$).*

2.4. **Bimodules and functors.** Let us relate the story of representable A_∞-modules to that of A_∞-functors. We shall be considering A_∞-functors of the following kind

$$\mathcal{F}_m : \mathcal{A} \to \mathcal{B}^{l-mod}. \tag{2.16}$$

Such an A_∞-functor is called a module-valued functor, which says that it suffices to verify representability on objects.

Definition 2.3. *A module-valued functor*

$$\mathcal{F}_m : \mathcal{A} \to \mathcal{B}^{l-mod}$$

is said to be representable, if there exists an A_∞-functor

$$\mathcal{F} : \mathcal{A} \to \mathcal{B}, \tag{2.17}$$

such that $\mathfrak{y}_l \circ \mathcal{F}$ is homotopic to $\mathcal{F}_m$ as A_∞-functors.

We have the following concrete criterion for representability of a module-valued functor.

Lemma 2.2. *A module-valued functor*

$$\mathcal{F}_m : \mathcal{A} \to \mathcal{B}^{l-mod}$$

is representable, if and only if $\mathcal{F}_m(X)$ is a representable left $\mathcal{B}$-module of any object $X \in Ob\mathcal{A}$.

Proof. The "only if" part is obvious by definition.

Now consider the "if" part. Since for every $X \in Ob\mathcal{A}$, the A_∞-module $\mathcal{F}_m(X)$ over $\mathcal{B}$ is representable, there exists an object $Y = Y(X)$ and an A_∞-module homomorphism

$$T_X : \mathcal{F}_m(X) \to \mathcal{Y}^l, \tag{2.18}$$

which is a quasi-isomorphism of A_∞-modules. We make a choice of a homotopy inverse K_X of T_X for each X.

Let us recall what it means for $\mathcal{F}_m$ to be representable. There should exist an A_∞-functor

$$\mathcal{F} : \mathcal{A} \to \mathcal{B},$$

as well as an A_∞-natural transformation of degree 1

$$T \in \hom_{func(\mathcal{A},\mathcal{B}^{l-mod})}(\mathcal{F}_m, \mathfrak{y}_l \circ \mathcal{F}),$$

such that T is a homotopy between $\mathcal{F}_m$ and $\mathfrak{y}_l \circ \mathcal{F}$.

To define these, we need to pick a homotopy inverse of the Yoneda embedding

$$\mathfrak{y}_l : \mathcal{B} \to \mathcal{B}^{l-mod},$$

when restricted to the image. That is, if we regard the Yoneda embedding as an A_∞-functor

$$\mathfrak{y}_l : \mathcal{B} \to \mathcal{B}^{rep-l-mod},$$

it is a quasi-isomorphism, so that we can choose a homotopy inverse,

$$\lambda_{\mathcal{B}} : \mathcal{B}^{rep-l-mod} \to \mathcal{B}. \tag{2.19}$$

We define $\mathcal{F}$ as follows. On objects, $\mathcal{F}(X) = Y = Y(X)$. For $X_0, X_1 \in Ob\mathcal{A}$, we define

$$\mathcal{F}^1 : \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{B}}(Y_0, Y_1) \tag{2.20}$$

as follows. For each $a \in \hom_{\mathcal{A}}(X_0, X_1)$, the morphism

$$\mathcal{F}_m^1(a) \in \hom_{\mathcal{B}^{l-mod}}(\mathcal{F}_m(X_0), \mathcal{F}_m(X_1))$$

is a A_∞-pre-module homomorphism, which can be composed with K_{X_0} and T_{X_1} in the dg-category $\mathcal{B}^{l-mod}$ by the structure map $m^2_{\mathcal{B}^{l-mod}}$ to get an A_∞-pre-module homomorphism

$$\mathcal{G}_m(a) = m^2_{\mathcal{B}^{l-mod}}(T_{X_1}, m^2_{\mathcal{B}^{l-mod}}(\mathcal{F}_m(a), K_{X_0})) \in \hom_{\mathcal{B}^{l-mod}}(\mathcal{Y}_0^l, \mathcal{Y}_1^l). \tag{2.21}$$

Since $m^2_{\mathcal{B}^{l-mod}}$ is associative, there is no ambiguity of this composition, and therefore this is well-defined in a unique way, once we fix a choice of a homotopy inverse K_X of T_X for every $X \in Ob\mathcal{A}$, as this homotopy inverse is indepedent of a. Note that

$$\mathcal{G}_m(a) \in \hom_{\mathcal{B}^{l-mod}}(\mathcal{Y}_0^l, \mathcal{Y}_1^l)$$

is an A_∞-pre-module homomorphism between left Yoneda modules, which lie in the sub-category $\mathcal{B}^{rep-l-mod}$. Thus we can apply $\lambda_{\mathcal{B}}$ to $\mathcal{G}_m(a)$ to obtain

$$\mathcal{F}^1(a) = \lambda^1_{\mathcal{B}}(\mathcal{G}_m(a)). \tag{2.22}$$

This defines $\mathcal{F}$ on morphisms.

For higher order terms $\mathcal{F}^k$, we again follow the same strategy. That is, we define

$$\mathcal{F}^k : \hom_{\mathcal{A}}(X_{k-1}, X_k) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{B}}(Y_0, Y_d) \tag{2.23}$$

to be the image of the composition of $\mathcal{F}_m^k$ with K_{X_0} and T_{X_k} under $\lambda_{\mathcal{B}}$. That is, for $a_i \in \hom_{\mathcal{A}}(X_{i-1}, X_i)$, we define

$$\mathcal{F}^k(a_k, \cdots, a_1) = \lambda_{\mathcal{B}}^1(m^2_{\mathcal{B}^{l-mod}}(T_{X_k}, m^2_{\mathcal{B}^{l-mod}}(\mathcal{F}_m^k(a_k, \cdots, a_1), K_{X_0}))). \tag{2.24}$$

It is a straightforward computation to check that $\mathcal{F} = \{\mathcal{F}^k\}_{k=1}^{\infty}$ satisfies the A_∞-functor equations.

Then we need to define the homotopy between $\mathfrak{y}_l \circ \mathcal{F}$ and $\mathcal{F}_m$. But this is clear: we simply take

□

A natural source of module-valued functors is given by A_∞-bimodules. Let $\mathcal{P}$ be a left-$\mathcal{B}$ right-$\mathcal{A}$ A_∞-bimodule, or simply called an A_∞-bimodule over $(\mathcal{A}, \mathcal{B})$. This notation is slightly misleading as we write $\mathcal{A}$ on the left and $\mathcal{B}$ on the right, but we shall keep this convention as it is fitted into Floer theory which will be discussed later on. From $\mathcal{P}$ we can define an A_∞-functor

$$\mathcal{F}_{\mathcal{P}} : \mathcal{A} \to \mathcal{B}^{l-mod} \tag{2.25}$$

as follows. For each object $X \in Ob\mathcal{A}$, we set

$$\mathcal{F}_{\mathcal{P}}(X) = \mathcal{P}(X, \cdot).$$

That is, $\mathcal{F}_{\mathcal{P}}(X)$ is the left-$\mathcal{B}$ module that takes value $\mathcal{P}(X, Y)$ for each object $Y \in Ob\mathcal{B}$, and has module structure maps

$$n^k_{\mathcal{F}_{\mathcal{P}}(X)} : \hom_{\mathcal{B}}(Y_{k-1}, Y_k) \otimes \hom_{\mathcal{B}}(Y_0, Y_1) \otimes \mathcal{P}(X, Y_0) \to \mathcal{P}(X, Y_k), \tag{2.26}$$

$$n^k_{\mathcal{F}_{\mathcal{P}}(X)}(b_k, \cdots, b_1, p) = n^{k,0}_{\mathcal{P}}(b_k, \cdots, b_1, p), \tag{2.27}$$

where $n^{k,l}_{\mathcal{P}}$ are the A_∞-bimodule structure maps of $\mathcal{P}$. Next we define the action of $\mathcal{F}_{\mathcal{P}}$ on morphism spaces

$$\mathcal{F}^l_{\mathcal{P}} : \hom_{\mathcal{A}}(X_{l-1}, X_l) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{B}^{l-mod}}(\mathcal{F}_{\mathcal{P}}(X_l), \mathcal{F}_{\mathcal{P}}(X_0)) \tag{2.28}$$

by the formula

$$(\mathcal{F}^l_{\mathcal{P}}(a_l, \cdots, a_1))^k(b_k, \cdots, b_1, p) = n^{k,l}_{\mathcal{P}}(b_k, \cdots, b_1, p, a_l, \cdots, a_1). \tag{2.29}$$

By the A_∞-bimodule equations for $n^{k,l}_{\mathcal{P}}$, it is straightforward to verify:

Lemma 2.3. *The multilinear maps $\{\mathcal{F}^l_{\mathcal{P}}\}_{l=1}^{\infty}$ form an A_∞-functor*

$$\mathcal{F}_{\mathcal{P}} : \mathcal{A} \to \mathcal{B}^{l-mod}.$$

All A_∞-bimodules over $(\mathcal{A}, \mathcal{B})$ form an A_∞-category, denoted by $(\mathcal{A}, \mathcal{B})^{bimod}$. The above construction of a module-valued functor $\mathcal{F}_{\mathcal{P}}$ associated to an A_∞-bimodule $\mathcal{P}$ can be made on the level of the whole category $(\mathcal{A}, \mathcal{B})^{bimod}$.

Proposition 2.1. *There is a canonical A_∞-functor*

$$\mathcal{F} : (\mathcal{A}, \mathcal{B})^{bimod} \to func(\mathcal{A}, \mathcal{B}^{l-mod}), \tag{2.30}$$

such that $\mathcal{F}(\mathcal{P}) = \mathcal{F}_{\mathcal{P}}$ for every object $\mathcal{P} \in Ob(\mathcal{A}, \mathcal{B})^{bimod}$.

In general, it is not possible to expect $\mathcal{F}(X)$ is representable for every $X \in Ob\mathcal{A}$. But we might say that it is representable for some of the objects. This leads to the following definition.

Definition 2.4. *Let $\mathcal{A}_0$ be a full A_∞-subcategory of $\mathcal{A}$. We say that*

$$\mathcal{F} : (\mathcal{A}, \mathcal{B})^{bimod} \to func(\mathcal{A}, \mathcal{B}^{l-mod})$$

is representable on $\mathcal{A}_0$, if $\mathcal{F}(X)$ is a representable left $\mathcal{B}$-module for every $X \in Ob\mathcal{A}_0$.

By the criterion stated in Lemma 2.2, this definition makes sense.

2.5. **Cyclic element and bounding cochain.** In this subsection, we introduce the notion of a cyclic element, which is key to proving the existence and uniqueness of a bounding cochain under certain assumptions. One origianl formulation due to Fukaya (Proposition 3.5 of [Fuk15]), works with filtered A_∞-algebras and filtered A_∞-modules over the Novikov ring. In our case, we consider curved A_∞-algebras and A_∞-modules over $\mathbb{Z}$. There is in general no intrinsic way of making the method work for arbitrary curved A_∞-algebras and A_∞-modules, so additional structures are required, to be explained below.

Let (C, m^k) be a curved A_∞-algebra over $\mathbb{Z}$ ($m^0 \neq 0$), where C is a free $\mathbb{Z}$-module (of finite or infinite rank), and (D, n^k) a left A_∞-module over (C, m^k), where D is also a free $\mathbb{Z}$-module. We need an additional condition, similar to the gappedness condition for filtered A_∞-algebras and filtered A_∞-modules. Although we work with usual A_∞-algebras and A_∞-modules, we still want to have some inductive structures, similar to those for filtered A_∞-algebras and filtered A_∞-modules. For us, the notion we need is a filtration which satisfies certain analogous conditions.

Definition 2.5. *(i) A filtration F_C on (C, m^k) is an $\mathbb{R}$-filtration such that*

$$F_C^{\lambda'} C \subset F_C^{\lambda} C, \text{ if } \lambda < \lambda',$$

and furthermore,

$$m^k(F_C^{\lambda_k} C \otimes \cdots F_C^{\lambda_1} C) \subset F_C^{\sum_{j=1}^k \lambda_j} C. \tag{2.31}$$

(ii) Given a filtration F_C on (C, m^k), a compatible filtration F_D on (D, n^k) is an $\mathbb{R}$-filtration such that

$$F_D^{\lambda'} D \subset F_D^{\lambda} D, \text{ if } \lambda < \lambda',$$

and furthermore,

$$n^k(F_C^{\lambda_k} C \otimes F_C^{\lambda_1} C \otimes F_D^{\lambda'}) \subset F_D^{\lambda' + \sum_{j=1}^k \lambda_j} D. \tag{2.32}$$

Definition 2.6. *A filtration F_C is said to be discrete, if there is a discrete subset Λ_C of $\mathbb{R}$ such that*

$$F_C^{\lambda} C = F_C^{\lambda'} C, \text{ if } [\lambda, \lambda'] \cap \Lambda_C = \varnothing, \tag{2.33}$$

for any $\lambda < \lambda'$. A similar definition applies to a compatible filtration F_D.

A discrete compatible filtration F_D is said to be strictly compatible, if $\Lambda_C = \Lambda_D$.

In order to deform a curved A_∞-algebra to a non-curved A_∞-algebra, we need the notion of a bounding cochain.

Definition 2.7. *A bounding cochain for the curved A_∞-algebra (C, m^k) is an element b, such that the inhomogeneous Maurer-Cartan equation is satisfied*

$$\sum_{k=0}^{\infty} m^k(b, \cdots, b) = 0, \tag{2.34}$$

where the sum stops at a finite stage. That is, there exists K such that for all $k > 0$, $m^k(b, \cdots, b) = 0$ and

$$\sum_{k=0}^{K} m^k(b, \cdots, b) = 0. \tag{2.35}$$

In other words, b is assumed to be nilpotent.

There is a very important class of examples for which nilpotent elements naturally exist. For example, if C as a $\mathbb{Z}$-module has finitely many generators which lie in F_C^0, then any such generator in the strictly positive part of the filtration is nilpotent, because the A_∞-structure maps increase the filtration. For geometric applications, we shall see that wrapped Floer cochain spaces have this property. Thus it does not harm to give a name for such a filtration.

Definition 2.8. *We say that the filtration F_C is bounded above, if Λ_C is bounded above. Equivalently, there exists λ_+ such that*

$$F_C^{\lambda_+} C = 0.$$

Suppose that we have chosen a discrete filtration F_C on (C, m^k) as well as a discrete strictly compatible filtration F_D on (D, n^k). For simplicity, we assume that $0 \in \Lambda_D$. Now we are going to introduce the key notion.

Definition 2.9. *We say that $u \in D$ is a cyclic element, if the following properties are satisfied:*

- *(i) the map $C \to D$ sending x to $n^1(x; u)$ is a filtration-preserving isomorphism of $\mathbb{Z}$-modules, with its inverse also filtration-preserving;*
- *(ii) $u \in F_D^0 D$, and there exists $\lambda_1 > 0$ such that $n^0(u) \in F_D^{\lambda_1} D$.*

Note that $n^0(F_D^\lambda D) \subset F_D^\lambda D$ for every λ. Thus the second condition means when n^0 acts on u, it strictly increases the filtration.

The following result is proved in [Fuk15] in the case of filtered A_∞-algebras and filtered A_∞-modules (Proposition 3.5 of [Fuk15]) over the Novikov ring. We have a similar result in our case, which yields bounding cochains of "finite type".

Lemma 2.4. *Let (C, m^k) be a curved A_∞-algebra, and (D, n^k) a left A_∞-module over (C, m^k). Suppose that the filtrations F_C and F_D are bounded above. Suppose $u \in D$ is a cyclic element such that $u \in F_D^0$. Then there exists a unique nilpotent bounding cochain b of (C, m^k) such that*

$$b \in F_C^{\lambda_1} C$$

for some $\lambda_1 > 0$, and

$$d^b(u) = 0. \tag{2.36}$$

Here $d^b : D \to D$ is defined by

$$d^b(y) = \sum_{k=0}^{\infty} n^k(b, \cdots, b; y).$$

Proof. Because the filtration F_D is strictly compatible with F_C, it is possible to find free generators y_j of D and x_j of C, such that $y_j \in F_D^{\lambda_j} D$ and $x_j \in F_C^{\lambda_j} C$, where $\lambda_j \in \Lambda_D = \Lambda_C$.

We write $u = u_0 y_0 + \cdots + u_l y_l$ in a unique way, where $u_j \in \mathbb{Z}$ and $y_j \in D$ are free generators of the free $\mathbb{Z}$-module D, such that $y_0 \in F_D^0 D$, and $y_j \in F_D^{\lambda_j} D$ but $y_j \notin F_D^{\lambda} D$ if $\lambda > \lambda_j$, where $0 < \lambda_1 < \cdots < \lambda_l$. Because of condition (ii) for a cyclic element, we have that $n^0(y_0) = 0$, so that $n^0(u) = \sum_{j=1}^{l} u_j n^0(y_j) \in F_D^{\lambda_1}$.

Also, b has a unique expression $b = b_1 x_1 + \cdots b_l x_l$, where $b_j \in \mathbb{Z}$ and $x_j \in C$ are free generators of the free $\mathbb{Z}$-module C, such that $x_j \in F_C^{\lambda_j} C$ but $x_j \notin F_C^{\lambda} C$ if $\lambda > \lambda_j$, where $\lambda_1 < \cdots < \lambda_m$.

Let us try to solve the equation (2.36) for such an element b. This breaks down to a system of equations

$$\sum_k \sum_{i_1, \cdots, i_k} \sum_j b_{i_1} \cdots b_{i_k} u_j n^k(x_{i_1}, \cdots, x_{i_k}; y_j) = 0. \tag{2.37}$$

Here the term $n^k(x_{i_1}, \cdots, x_{i_k}; y_j) \in F_D^{\lambda_{i_1} + \cdots + \lambda_{i_k} + \lambda_j} D$, but not in a $F_D^{\lambda'} D$ for any smaller λ'. Because F_D is bounded above, there can be at most finitely many such terms which are non-zero. Thus this system can be separated to finitely many equations, according to the filtration. We order these equations in an increasing order in terms of filtration. Rewrite every equation as the form

$$b_i u_0 n^1(x_i; y_0) + \sum b_{i_1} \cdots b_{i_k} u_j n^k(x_{i_1}, \cdots, x_{i_k}; y_j) = 0 \tag{2.38}$$

where the second sum is taken over all possible indices so that $\lambda_{i_1} + \cdots + \lambda_{i_k} + \lambda_j \leq \lambda_i$. In particular, all $i_j < i$. When $i = 1$, there are no b's in the second sum, and the only possibly nonzero terms are of the form $u_j n^0(u_j)$ such that $\lambda_j \leq \lambda_i + \lambda_0$. Because $n^1(\cdot; u)$ is a filtration-preserving isomorphism of $\mathbb{Z}$-modules, the coefficient of the first term u_0 is invertible: $u_0 = \pm 1$. Thus we can solve for a unique b_1. Then we consider $i = 2$ and argue in the same way to solve for b_2. We repeat this process until we solve for all $b_i, i = 1, \cdots, l$.

It remains to prove that this solution is also a solution to the inhomogeneous Maurer-Cartan equation (2.34). Since $b \in F_C^{\lambda_1} C$ for $\lambda_1 > 0$, and F_C is bounded above, b is automatically nilpotent, and the equation (2.34) can be reduced to (2.35). The strategy is to prove that

$$\sum_k m^k(b, \cdots, b) \in F_C^{\lambda'} C \tag{2.39}$$

for every $\lambda' > 0$, which implies that it must vanish because F_C is bounded above: for λ' large, $F_C^{\lambda'} C = 0$. From the A_∞-equations for n^k and m^k we get the following equation

$$\sum n^{k_1}(b, \cdots, b, n^{k_2}(b, \cdots, b; u)) + \sum n^{k_1}(b, \cdots, m^{k_2}(b, \cdots, b), \cdots, b); u) = 0. \tag{2.40}$$

The first sum vanishes because $\sum n^k(b, \cdots, b; u) = 0$. Following the same kind of argument as before, we rewrite the above equation as the following:

$$\sum b_{i_1} \cdots b_{i_k} u_j n^{k_1}(x_{i_1}, \cdots, x_{i_s}, m^{k_2}(x_{i_{s+1}}, \cdots, x_{i_{s+k_2}}), \cdots, x_{i_k}; y_j) = 0. \tag{2.41}$$

We may further write this equation as a system of equations, ordered by the filtration. Let us try to prove (2.39) by induction on λ, namely we take a discrete sequence $\lambda'_n \to +\infty$ and prove (2.39) for λ'_n. Recall that we have

$$b_i u_0 n^1(x_i; y_0) + \sum b_{i_1} \cdots b_{i_k} u_j n^k(x_{i_1}, \cdots, x_{i_k}; y_j) = 0 \tag{2.42}$$

If we assume (2.39) holds for any λ' smaller than λ'_i, it follows that the terms on the left hand side of (2.41) cancel with each other except that the following terms are left over:

$$b_i u_0 n^1(m^1(x_i); y_0) + \sum b_{i_1} \cdots b_{i_k} u_0 n^1(m^k(x_{i_1}, \cdots, x_{i_k}); y_0) \tag{2.43}$$

where the second sum is taken over all possible indices such that $\lambda_{i_1} + \cdots + \lambda_{i_k} \leq \lambda'_i$. As the map $n^1(\cdot; u)$ is a filtration-preserving isomorphism, and y_0 is the summand of u in lowest filtration, this implies that

$$b_i m^1(x_i) + \sum b_{i_1} \cdots b_{i_k} m^k(x_{i_1}, \cdots, x_{i_k}) = 0.$$

Since m^1 does not decrease the filtration, this implies that (2.39) holds for λ'_i.

□

3. The wrapped Fukaya category: revisited

3.1. **Overview.** There are several ways of defining the wrapped Fukaya category of a Liouville manifold. The basic approaches grow out of the definition of wrapped Floer cohomology, which either uses a cofinal family of Hamiltonians linear at infinity [AS10], or a single Hamiltonian quadratic at infinity [Abo10], or more generally a single Hamiltonian whose growth at infinity is faster than linear. A more functorial approach is via categorical colimits and localization, without having to specify a particular choice of Hamiltonian as introduced in [GPS17]. Of course, all these approaches give quasi-equivalent A_∞-categories.

At first, we set up the wrapped Fukaya category using a single Hamiltonian quadratic at infinity. Although this is not the best definition as it relies on working with a specific choice of Hamiltonian, it is convenient for the construction of functors from Lagrangian correspondences, as they can be constructed in a single step, without having to be defined as a colimit of functors.

Alternatively, we also use linear Hamiltonians to define the wrapped Fukaya category, following [AS10], for the purpose of comparing the two versions of wrapped Fukaya categories of the product manifold.

3.2. **Basic geometric setup.** Consider a Liouville manifold M which is the completion of a Liouville domain M_0 with boundary ∂M, which has a collar neighborhood $\partial M \times (\epsilon, 1]$ so that the Liouville vector field is equal to $\frac{\partial}{\partial r}$ in that neighborhood. We assume that M satisfies $c_1(M) = 0 \in H^2(M; \mathbb{Z})$.

The admissible Lagrangian submanifolds are either closed exact Lagrangian submanifolds in the interior M_0, or exact cylindrical Lagrangian submanifolds of the form $L = L_0 \cup \partial L \times [1, +\infty)$ where $\partial L \subset \partial M$ is a Legendrian submanifold with respect to the contact structure induced from the Liouville one-form. To be more specific, for the latter kind of Lagrangian submanifold L_0 of M_0, there should be a function f on it so that $df = \lambda|_{L_0}$, where λ is the Liouville form. Moreover, we require that f has an extension to a neighborhood of L in M such that it is locally constant near $\partial L \times (\epsilon, +\infty)$. In addition, we shall make the assumption that L has vanishing Maslov class and is spin. We will fix a choice of grading and spin structure for every admissible Lagrangian submanifold. These conditions will ensure that the wrapped Fukaya category of M is defined over $\mathbb{Z}$, and carries $\mathbb{Z}$-gradings.

3.3. **Floer data and consistency.** The moduli space of surfaces controlling the algebraic operations and relations in the Fukaya categories are the moduli spaces of stable marked nodal disks, which was studied in [FOOO09a] and proved to be cellular isomorphic to the moduli spaces of stable metric ribbon trees introduced by Stasheff, which is known to be the operad controlling A_∞-algebras.

Let $\bar{\mathcal{M}}_{k+1}$ be the compactified moduli space of stable $(k+1)$-marked disks. It is proved in [FOOO09a] that $\bar{\mathcal{M}}_{k+1}$ (their notation is $\bar{\mathcal{M}}_{k+1}^{main}$) is a compact smooth manifold with corners and a neighborhood of the stratum $\mathcal{M}_T$ with combinatorial type modeled on a stable ribbon tree T with one root and k-leaves is covered by the image of the gluing map:

$$(-1, 0]^{e(T)} \times \mathcal{M}_T \supset U_T \to \bar{\mathcal{M}}_{k+1}, \tag{3.1}$$

which is smooth and a diffeomorphism onto the image by shrinking U_T if necessary.

To define A_∞-operations on Floer cochain spaces, we need to study moduli spaces of inhomogeneous pseudoholomorphic maps from nodal disks to M with boundary in Lagrangian submanifolds. For this purpose, we also need to include the case where the domain is unstable, or contains unstable components. The unstable curve involved here is the infinite strip Z, whose automorphism is the additive group $\mathbb{R}$. To write down inhomogeneous Cauchy-Riemann equations and achieve transversality of the moduli spaces of solutions, we need several auxilliary data. We briefly recall the notion here.

Definition 3.1. *Given a semistable $(k+1)$-marked nodal disk $S \in \bar{\mathcal{M}}_k$, a Floer datum P_S for S consists of*

- *(i) A collection of positive integers $w_0, \cdots, w_k$.*
- *(ii) A time-shifting function $\rho_S : \partial S \to [1, +\infty)$, which takes the value w_j over the j-th strip-like end ϵ_j.*
- *(iii) A basic one form α_S, whose restriction to every smooth component of S is closed, and whose pullback by ϵ_j agrees with $w_j dt$.*
- *(iv) A Hamiltonian perturbation $H_S : S \to \mathcal{H}(M)$, whose pullback by ϵ_j agrees with $\frac{H \circ \psi^{w_j}}{w_j^2}$.*
- *(v) A domain-dependent perturbation of almost complex structures $J_S : S \to \mathcal{J}(M)$, whose pullback by ϵ_j agrees with $(\psi^{w_j})^* J_t$.*

such that over unstable components of S, i.e. strips, all the three data restricts to translation-invariant data.

In order to ensure that the various operations constructed from moduli spaces of marked inhomogeneous pseudoholomorphic disks satisfy the A_∞-equations, we need to make sure that the Floer data chosen for the underlying semistable marked nodal disks are compatible with respect to gluing maps (3.1). Therefore the following notion is useful: for $k \geq 3$, a universal and conformally consistent choice of Floer data is a choice of Floer data for all $S \in \bar{\mathcal{M}}_{k+1}$ that varies smoothly with respect to S in the compactified moduli space, such that the induced Floer data on for elements in the boundary strata $\partial \bar{\mathcal{M}}_{k+1}$ are conformally equivalent to product Floer data chosen for elements in the moduli spaces of disks with fewer boundary marked points. The notion of universal and conformally consistent choice of Floer data is extended also to the strip Z, as follows: when we glue in a strip at a boundary marked point of a stable marked nodal disk S, we require that the (translation-invariant) Floer datum chosen on Z agree with that on the strip-like end for S near that marked point. The space of choices of Floer data is convex, therefore by induction on the strata of the moduli space of stable marked nodal disks, we can construct universal and conformally consistent choices of Floer data. More detailed explanation is given in [Sei08].

3.4. **Inhomogeneous pseudoholomorphic disks.**

To define the A_∞-operations on the wrapped Fukaya category, we need to study the moduli spaces of inhomogeneous pseudoholomorphic disks with boundary mapped to several Lagrangian submanifolds. Make universal and conformally consistent choices of Floer data P for all semistable marked nodal disks. Denote by S an element in the smooth part of the moduli space $\mathcal{M}_{k+1}$. That is, S is a smooth disk with boundary marked points $(z_0, \cdots, z_k)$ that are cyclically ordered on the boundary. Given admissible Lagrangian submanifolds $L_0, \cdots, L_k$, consider the following inhomogeneous

Cauchy-Riemann equation, for both S and u as variables:

$$\begin{cases}(du - \alpha_S \otimes X_{H_S})^{0,1} = 0; \\ u(z) \in \psi^{\rho_S(z)} L_j, \text{ if } z \text{ lies in between } z_j \text{ and } z_{j+1}; \\ \lim_{s \to -\infty} u \circ \epsilon_0(s, \cdot) = \psi^{w_0} x_0(\cdot) \in \mathcal{X}(\psi^{w_0} L_0, \psi^{w_0} L_k); \\ \lim_{s \to +\infty} u \circ \epsilon_j(s, \cdot) = \psi^{w_j} x_j(\cdot) \in \mathcal{X}(\psi^{w_j} L_{j-1}, \psi^{w_j} L_j), \quad j = 1, \cdots, k.\end{cases} \tag{3.2}$$

The solutions will sometimes also be called Floer's disks.

Suppose for the moment $k \geq 2$. Let $\mathcal{M}_{k+1}(L_0, \cdots, L_k; x_0, \cdots, x_k; P)$ be the moduli space of solutions (S, u) to the above equation with respect to the chosen Floer data P, and let $\bar{\mathcal{M}}_{k+1}(L_0, \cdots, L_k; x_0, \cdots, x_k; P)$ be its stable map compactification. It is proved in [Abo10] that for a generic choice of Floer data P, the zero-dimensional and one-dimensional components of $\bar{\mathcal{M}}_{k+1}(L_0, \cdots, L_k; x_0, \cdots, x_k; P)$ are compact smooth manifolds with corners of dimension

$$\deg(x_0) - \deg(x_1) - \cdots - \deg(x_k) + k - 2.$$

In the unstable case $k = 1$, there is no moduli of S, so we consider the set of solutions u to the above equation. Since the Floer datum P_Z on the strip is chosen to be translation-invariant, we can quotient the parametrized moduli space by this automorphism group. We denote the quotient moduli space by $\mathcal{M}_2(L_0, L_1; x_0, x_1; P_Z)$ as well, and the corresponding Gromov bordification by $\bar{\mathcal{M}}_2(L_0, L_1; x_0, x_1; P_Z)$.

3.5. **Identification of Floer cochain spaces with different weights.** The "count" of rigid elements in the moduli spaces $\bar{\mathcal{M}}_{k+1}(L_0, \cdots, L_k; x_0, \cdots, x_k; P)$ defines operations of the following kind

$$\begin{aligned} &CW^*(\psi^{w_k} L_{k-1}, \psi^{w_k} L_k; \frac{H \circ \psi^{w_k}}{w_k}) \otimes \cdots \otimes CW^*(\psi^{w_1} L_0, \psi^{w_1} L_1; \frac{H \circ \psi^{w_1}}{w_1}) \\ \to &CW^*(\psi^{w_0} L_0, \psi^{w_0} L_k; \frac{H \circ \psi^{w_0}}{w_0}). \end{aligned} \tag{3.3}$$

In order define an actual A_∞-category with morphism spaces being $CW^*(L, L'; H)$ between the two objects L and L' so that the operations are between these morphism spaces, we have to identify $CW^*(\psi^w L, \psi^w L'; \frac{H \circ \psi^w}{w})$ with $CW^*(L, L'; H)$, in a canonical way. Because the Hamiltonian H is quadratic in the radial coordinate of the cylindrical end, such an identification is easily achieved by noting that $\frac{H \circ \psi^w}{w^2}$ behaves the same as H in the cylindrical end where Reeb dynamics occur. Technically, the rescaled Hamiltonian differs from H by a small amount that is supported in the compact part of M, and this can be taken care of by using a compactly supported deformation of Hamiltonian functions, which gives rise to continuation maps that form an A_∞-quasi-isomorphism (of A_∞-bimodule structures on the wrapped Floer cochain spaces).

To summarize, these arguments in previous subsections together imply that the wrapped Fukaya category $\mathcal{W}(M)$ is well-defined, up to quasi-isomorphism.

3.6. **Linear Hamiltonians.** For $i = 1, 2, \cdots$, let $H_i : M \to \mathbb{R}$ be a sequence of Hamiltonians which are positive, depend only on the radial coordinate in the cylindrical end, $H_i(y, r) = h_i(r)$, for $(y, r) \in \partial M \times (0, +\infty)$, and which further satisfies $h_i(r) = k_i r$ for $r \geq 1$, i.e. is linear of slope k_i. Suppose k_i is strictly increasing, $k_i \to \infty$ as $i \to \infty$, and k_i is not equal to the length of any Reeb chord on ∂M between the Legendrian boundaries of any pair of Lagrangian submanifolds in consideration. For simplicity, further suppose that $k_i = ic$ for a constant slope $c > 0$. Since we only work with at most countably many Lagrangians, it is possible to choose such a c.

The wrapped Floer cochain space for a pair (L_0, L_1) is the telescope complex

$$CW^*(L_0, L_1; \{H_i\}) = \bigoplus_i CF^*(L_0, L_1; H_i)[q], \tag{3.4}$$

where $CF^*(L_0, L_1; H_i)$ is the Hamiltonian version of Floer complex for (L_0, L_1) with respect to H_i, and q is a formal variable of degree -1 satisfying $q^2 = 0$. The A_∞-structure maps is made up of a system of maps

$$\mu^{k;F;i_0,\cdots,i_k} : CF^*(L_{k-1}, L_k; H_{i_k})[q] \otimes \cdots \otimes CF^*(L_0, L_1; H_{i_1})[q] \to CF^*(L_0, L_k; H_{i_0})[q], \tag{3.5}$$

with labels indexed by a finite subset $F \subset \{1, \cdots, k\}$, with $i_0 = i_1 + \cdots + i_k + |F|$, which are defined by counting rigid elements in the moduli spaces of popsicle maps [AS10]. The popsicle maps are inhomogeneous pseudoholomorphic disks which satisfy inhomogeneous Cauchy-Riemann equation with a domain-dependent family of Hamiltonians which agree with H_{i_j} near the j-th strip-like end. The structure map on the wrapped Floer cochain space is defined to be the sum of these maps $\mu^{k;F;i_0,\cdots,i_k}$ for all possible $(F, i_0, \cdots, i_k)$.

Finally, we mention that there are two kinds of important subspaces which form filtrations of the wrapped Floer complex thus defined. The first is a subcomplex

$$C_{\geq i} = \bigoplus_{j \geq i} CF^*(L_0, L_1; H_j)[q]. \tag{3.6}$$

These form a decreasing filtration on $CW^*(L_0, L_1; \{H_i\})$. A key property, proved in [AS10], is that each $C_{\geq i}$ is homotopy equivalent to the whole wrapped Floer complex. The second is

$$C_{\leq i} = \bigoplus_{j < i} CF^*(L_0, L_1; H_j)[q] \oplus CF^*(L_0, L_1; H_i), \tag{3.7}$$

where the last summand does not have a q-factor. These form an increasing filtration. While a single $C_{\leq i}$ is too small for it to be homotopy equivalent to the whole wrapped Floer complex, it has the property that the limit of the cohomologies of $C_{\leq i}$ computes the wrapped Floer cohomology:

$$\lim_{i \to \infty} H^*(C_{\leq i}) \cong HW^*(L_0, L_1), \tag{3.8}$$

where the map giving this isomorphism is induced by the natural inclusions $C_{\leq i} \to CW^*(L_0, L_1; \{H_i\})$.

The two approaches of wrapped Fukaya categories yield equivalent A_∞-categories. It seems that this is well-known to experts so we simply describe such an equivalence. However, we cannot find a good reference which defines the A_∞-functor of all orders, so we use it as a working hypothesis.

Assumption 3.1. *Let $\mathcal{W}(M; H)$ denote the wrapped Fukaya category defined with respect to a single quadratic Hamiltonian H, and $\mathcal{W}(M; \{H_i\})$ be the wrapped Fukaya category defined using the sequence of linear Hamiltonians.*

There is an A_∞-homotopy equivalence

$$G : \mathcal{W}(M; \{H_i\}) \to \mathcal{W}(M; H). \tag{3.9}$$

The first order map is very explicit: it is the homotopy direct limit of continuation maps $CF^*(L; H_i) \to CW^*(L; H)$ where we increase each linear Hamiltonian H_i to the quadratic Hamiltonian H. It might happen that for specific choices of Hamiltonians, higher order terms vanish. However, this is far from being true for general Hamiltonians. Writing down higher order terms in a systematic way requires some non-trivial work, which is of its own interest but not the concern of this thesis.

4. Wrapped Floer theory in the product manifold

4.1. **Overview.** In view of Lagrangian correspondences as Lagrangian submanifolds in the product symplectic manifold, we need to study wrapped Floer theory of the product manifold in order to understand the construction of functors between the wrapped Fukaya categories. Then a somewhat technical but essential problem arises: there are two natural models for the wrapped Fukaya category of the product manifold, which are not a priori known to be equivalent. This causes some trouble in understanding wrapped Floer theory of the product manifold.

Denote by $H_{M,N} = \pi_M^* H_M + \pi_N^* H_N$ the split Hamiltonian and $J_{M,N} = J_M \times J_N$ the product almost complex structure. Using $H_{M,N}$ and $J_{M,N}$, we can define a version of the wrapped Fukaya category of the product Liouville manifold $M \times N$, which we call the split wrapped Fukaya category of $M \times N$ and denote it by $\mathcal{W}^s(M \times N)$. On the other hand, there is a natural choice of the cylindrical end $\Sigma \times [1, +\infty)$ for $M \times N$, which allows us to define Hamiltonian functions K that are linear at infinity. Therefore the ordinary wrapped Fukaya category of $M \times N$ can also be defined as usual.

However one important thing is not mentioned explicitly. That is, the classes of Lagrangian submanifolds for which we can define wrapped Floer cohomology and A_∞-algebras are a priori different with respect to the two kinds of Hamiltonians and almost complex structures. For the the split Hamiltonian $H_{M,N}$ and product almost complex structure $J_{M,N}$, the natural class of Lagrangian submanifolds are products of objects in $\mathcal{W}(M)$ and those in $\mathcal{W}(N)$, i.e. products of exact cylindrical Lagrangian submanifolds. For radial Hamiltonians which are linear at infinity and cylindrical almost complex structures, the natural class of Lagrangian submanifolds are cylindrical Lagrangian submanifolds with respect to the cylindrical end $\Sigma \times [1, +\infty)$. If we want to identify these two versions of wrapped Floer theory, we must ensure that both classes of objects can be included in each wrapped Fukaya category. One of our main results in confirms that this is possible if we require a further condition on cylindrical Lagrangian submanifolds, which we will explore in subsection 4.2.

This section is devoted to proving Theorem 1.1, which basically says $\mathcal{W}^s(M \times N)$ is quasi-equivalent to $\mathcal{W}(M \times N)$ when both are considered for the class of admissible Lagrangian submanifolds. Thus the ambiguity in differentiating these two models is removed, which allows us to better understand the functoriality properties of wrapped Fukaya categories from the viewpoint of Lagrangian correspondences.

Remark 4.1. *In this section all the wrapped Fukaya categories in consideration have objects being properly embedded, but the result should continue to hold in the immersed case, if stated in an appropriate way.*

The goal is to construct an A_∞-functor between these two versions of wrapped Fukaya categories which gives the desired equivalence. To perform the construction, we fix a finite collection $\mathbb{L}$ of admissible Lagrangian submanifolds in the product $M \times N$. The strategy consists of two parts:

(i) For products of exact cylindrical Lagrangian submanifolds, which have well-defined Floer theory for the split Hamiltonian and product almost complex structure, we show that they also have well-defined Floer theory for the admissible Hamiltonian and almost complex structure, and build a functor from the split wrapped Fukaya category to the usual wrapped Fukaya category when these objects are concerned.

(ii) For cylindrical Lagrangian submanifolds, which have well-defined Floer theory for the admissible Hamiltonian and almost complex structure with respect to the natural cylindrical structure, we show that they also have well-defined Floer theory for split Hamiltonian and product almost complex structure, and build a functor from the usual wrapped Fukaya category to the split wrapped Fukaya category when these objects are concerned.

The two A_∞-functors as stated above will both be quasi-equivalences, and thus can be inverted in either direction.

In the next subsection we will introduce certain geometric data which we call action-restriction data, which are used to define a A_∞-functor $R_{\mathbb{L}}$ from the split wrapped Fukaya category $\mathcal{W}^s(\mathbb{L})$ consisting of $\mathcal{L}_1, \cdots, \mathcal{L}_d$ as objects, to the usual wrapped Fukaya category $\mathcal{W}(\mathbb{L})$ with the same objects, which acts by identity on the level of objects, and induces quasi-isomorphisms on all morphism spaces. Therefore this A_∞-functor $R_{\mathbb{L}}$ is a quasi-isomorphism $\mathcal{W}^s(\mathbb{L}) \to \mathcal{W}(\mathbb{L})$. Moreover, the A_∞-functor R_d will be cohomologically unital.

Remark 4.2. *Of course, the quasi-isomorphism will depend on choices geometric data involved. But for the purpose of comparing the two versions of wrapped Fukaya categories of the product, this does not matter, since the Fukaya category itself is only well-defined up to quasi-isomorphism, because of the flexibility in the choice of auxilliary data in Floer theory. The choices themselves form a contractible space, so we might work harder to show that different functors constructed in this way are homotopic.*

4.2. **Well-definedness.** The first thing that needs to be discussed is what kinds of Lagrangian submanifolds $\mathcal{L} \subset M \times N$ can be objects of the wrapped Fukaya category with respect to either split Hamiltonians or radial Hamiltonians linear at infinity with respect to the natural cylindrical structure $\Sigma \times [1, +\infty)$.

Definition 4.1. *A Lagrangian submanifold $\mathcal{L} \subset M \times N$ is called admissible, if it is exact and moreover belongs to one of the following kinds:*

(i) a product $L \times L'$, where L is an exact cylindrical Lagrangian submanifold of M, and L' is one of N;

(ii) an exact cylindrical Lagrangian submanifold $\mathcal{L}$ with respect to the natural cylindrical end $\Sigma \times [1, +\infty)$.

For the purpose of identifying two versions of wrapped Floer theory in the product, we shall use linear Hamiltonians and split linear Hamiltonians for technical convenience. This approach does not affect the result up to quasi-equivalence, but gives a way of making the relevant construction work directly.

The main observation is:

Proposition 4.1. *Suppose $\mathcal{L} \subset M \times N$ is an admissible Lagrangian submanifold. Then wrapped Floer A_∞-algebra associated to $\mathcal{L}$ is well-defined, with respect to either split Hamiltonians and product almost complex structures, or radial Hamiltonians and almost complex structures J.*

Sketch of proof. There are two key issues in proving well-definedness: transversality and compactness of the moduli spaces of pseudoholomorphic disks.

Regarding transversality, perturbing the almost complex structure $J_{M,N}$ within the class of product almost complex structures $\mathcal{J}(M) \times \mathcal{J}(N)$ might not ensure enough genericity to make the moduli spaces regular. However, when defining the wrapped Fukaya category, we shall take domain-dependent perturbations of almost complex structures in a suitable way. Namely, if we allow perturbations of $J_{M,N}$ in a neighborhood in $\mathcal{J}(M \times N)$ instead of just product almost complex structures, then transversality can be achieved. On the other hand, if one of the Lagrangian submanifolds involved in m^k is a product Lagrangian $L \times L'$, it is true that transversaliy of the moduli spaces involved can be achieved by perturbations within the class of product almost complex structures. The argument is similar to Wehrheim and Woodward's argument in quilted Floer cohomology (for compact monotone or exact Lagrangian submanifolds), combined with the transversality argument in wrapped Floer theory.

Compactness is the substantial issue that we need to think carefully about. Consider for simplicity the case of a single exact cylindrical Lagrangian submanifolds for which we want to define wrapped Floer cohomology using the split Hamiltonians and product almost complex structures. For a pair of Hamiltonian $\underline{x}_0, \underline{x}_1$ for the split Hamiltonian from $\mathcal{L}_0$ to $\mathcal{L}_1$, the moduli space $\mathcal{M}(\underline{x}_0, \underline{x}_1) = \tilde{\mathcal{M}}(\underline{x}_0, \underline{x}_1)/\mathbb{R}$ of inhomogeneous pseudoholomorphic strips from x to y has a natural Gromov bordification $\bar{\mathcal{M}}(\underline{x}_0, \underline{x}_1)$ by adding broken strips. To ensure that $\bar{\mathcal{M}}(\underline{x}_0, \underline{x}_1)$ is compact, the main ingredient in addition to Gromov compactness is the maximum principle, which prevents strips connecting $\underline{x}_0$ and $\underline{x}_1$ from escaping to infinity. By directly estimating the action of Hamiltonian chords, and using the action-energy equality, we may give an a priori estimate for the energy of inhomogeneous pseudoholomorphic disks.

On the other hand, we need a C^0-bound for inhomogeneous pseudoholomorphic disks. The idea is to compare a split Hamiltonian with one that is linear at infinity. The argument is done when the action-restriction map is constructed and proved to be a cochain map. Let us recall that here. We begin with a sequence of Hamiltonians $\{H_i\}$ on $M \times N$ which are linear at infinity with increasing slope $k_i \to \infty$, with respect to which wrapped Floer theory for a cylindrical Lagrangian submanifold $\mathcal{L} \subset M \times N$ (or a finite collection of such Lagrangian submanifolds) is defined. Then for each action filtration window $(-b, a]$, and each Hamiltonian H_i with slope sufficiently large (depending on b), we find a large compact subset of $M \times N$ of the form $r = r_1 + r_2 \leq A$, where A depends on the slope of H_i, and use a suitable cut-off function depending only on the radial coordinate r, which is 1 when $r = A - \epsilon$ for very small ϵ, and vanishes when r is close to A and $r \geq A$. Using this cut-off function, we interpolate H_i with a suitable constant, and obtained a new Hamiltonian $H_{i,1}$. Then we modify the Hamiltonian $H_{i,1}$ to a new Hamiltonian by adding a split Hamiltonian $H_{M,i} + H_{N,i}$, where $H_{M,i}$ is zero inside $r_1 \leq A$ and is linear of some slope k'_i outside, and $H_{N,i}$ is zero inside $r_2 \leq A$ and is linear of some slope k''_i. Here the slopes k'_i and k''_l are chosen such that they depend on H_i and tend to ∞ as $k_i \to \infty$. The new Hamiltonian agrees with K_i inside of this compact set, and is of split type when $r_1 \geq B$ and $r_2 \geq B$. A corresponding almost complex structure is also constructed, which is a product outside the region $r_1 \leq B, r_2 \leq B$. For a pair of Hamiltonian chords $\underline{x}_0, \underline{x}_1$ for H_i whose action falls in $(-b, a]$, they must be contained in the region $r_1 + r_2 \leq A$, where H_i agrees with $H_{M,j} + H_{N,l}$. Then it is shown that (H_i, J_i)-pseudoholomorphic strips connecting $\underline{x}_0$ and $\underline{x}_1$ are in one-to-one correspondence with $(H_{M,i}+H_{N,i}, J_{M,i}\times J_{N,i})$-pseudoholomorphic strips connecting $\underline{x}_0$ and $\underline{x}_1$. In particular, maximum principle can be applied to give C^0-bound for (H_i, J_i)-pseudoholomorphic strips, and therefore also $(H_{M,i}+H_{N,i}, J_{M,i}\times J_{N,i})$-pseudoholomorphic strips. This prevents inhomogeneous pseudoholomorphic strips with respect to split Floer data from escaping to infinity. Details of such analysis is in section 8.

We have thus shown that the wrapped Floer cohomology for cylindrical Lagrangian submanifolds of $M \times N$ is well-defined with respect to the split Hamiltonians and product almost complex structures. The same kind of argument applies to the definition of higher structure maps m^k.

Now let us consider products of exact cylindrical Lagrangian submanifolds of the form $L \times L'$. These a priori have well-defined wrapped Floer theory with respect to split Hamiltonians and product almost complex structures. Starting with a sequence $\{H_{M,N,i} = H_{M,i} + H_{N,i}\}$ of split Hamiltonians, we produce an sequence of Hamiltonians $\{K_i\}$ which are linear at infinity with respect to the natural cylindrical structure $\Sigma \times [1, +\infty)$, following similar procedure. We first deform the split Hamiltonian $H_{M,i} + H_{N,l}$ in some compact set of the form $r_1 \leq A - \epsilon, r_2 \leq A - \epsilon$, and is equal to a suitable constant outside $r_1 \leq B, r_2 \leq B$. In particular, it is constant in the region $r = r_1 + r_2 \geq B$. Thus we can choose a Hamiltonian linear in r for $r \geq B + \epsilon$ of suitable slope, and add to the modified Hamiltonian to obtain the desired H_i which is linear at infinity. Using similar action filtration argument together with maximum principle applied to split Hamiltonians with product Lagrangian boundary conditions, we can show that $(H_{M,N,i}, J_{M,N,i})$-pseudoholomorphic strips connecting $\underline{x}_0$ and $\underline{x}_1$ are in one-to-one correspondence with (K_i, J_i)-pseudoholomorphic strips connecting $\underline{x}_0$ and $\underline{x}_1$. In particular, these inhomogeneous pseudoholomorphic strips are contained in the region where the two kinds of Hamiltonians agree. This implies that there is a priori C^0-bound for inhomogeneous pseudoholomorphic strips with respect to the radial Hamiltonian H_i and cylindrical almost complex structure J_i. Thus, the moduli space of inhomogeneous pseudoholomorphic strips is compact.

Similar arguments apply to inhomogeneous pseudoholomorphic disks with more punctures with domain-dependent perturbations of Hamiltonians linear at infinity and cylindrical almost complex structures, showing the relevant moduli spaces are all compact.

□

The remaining question is whether we can do Floer theory between cylindrical Lagrangian submanifolds and product Lagrangian submanifolds. This is key to studying functors between wrapped Fukaya categories. Let $\mathcal{L} \subset M \times N$ be an exact cylindrical Lagrangian submanifold, and $L \times L' \subset M \times N$ be a product of exact cylindrical Lagrangian submanifolds. We introduce the following numerical condition on the Lagrangian submanifolds in order to ensure well-definedness of wrapped Floer theory.

Assumption 4.1. *There are discrete subsets $S^1_{\mathcal{L}, L\times L'}, S^2_{\mathcal{L}, L\times L'} \subset \mathbb{R}_+$, which depend only on the Lagrangian submanifolds, such that for every non-degenerate Hamiltonian H on $M \times N$ which is either linear with respect to the cylindrical structure $\Sigma \times [1, +\infty)$, or is of split type away from a compact set, the length of each time-one*

H-chords (x, y) from $\mathcal{L}$ to $L \times L'$, or from $L \times L'$ to $\mathcal{L}$ satisfies:

$$l(x) \in S^1_{\mathcal{L}, L\times L'}, l(y) \in S^2_{\mathcal{L}, L\times L'}. \tag{4.1}$$

Under Assumption 4.1, we can prove the following proposition using the same kind of argument.

Proposition 4.2. *Let $\mathcal{L} \subset M \times N$ be an exact cylindrical Lagrangian submanifold and $L \times L' \subset M \times N$ be a product of exact cylindrical Lagrangian submanifolds. Suppose that Assumption 4.1 holds for $\mathcal{L}$ and $L \times L'$. Then wrapped Floer cohomology is well-defined for $(\mathcal{L}, L \times L')$ and $(L \times L', \mathcal{L})$.*

This can be generalized to the following statement in a straightforward way:

Proposition 4.3. *Let $\mathbb{L}$ be a finite collection of admissible Lagrangian submanifolds in $M \times N$, such that for every exact cylindrical $\mathcal{L} \in \mathbb{L}$ and every product $L \times L' \in \mathbb{L}$, Assumption 4.1 is satisfied. Then one can define both versions of wrapped Fukaya categories whose objects are the Lagrangian submanifolds in the collection $\mathbb{L}$, using*

- *(i) either split Hamiltonians and product almost complex structures;*
- *(ii) or radial Hamiltonians which are linear at infinity and cylindrical almost complex structures.*

The first version is called the split wrapped Fukaya category of the product manifold $M \times N$ (with objects being Lagrangian submanifolds in $\mathbb{L}$), denoted by $\mathcal{W}^s(\mathbb{L})$, and the second version is called the wrapped Fukaya category, denoted by $\mathcal{W}(\mathbb{L})$.

The following proposition describes the way how Theorem 1.1 is proved.

Proposition 4.4. *Concerning wrapped Floer theory on the product manifold $M \times N$, we have:*

- *(i) Let $\mathbb{L}_c$ be a finite collection of exact cylindrical Lagrangian submanifolds of $M \times N$. Then there is a natural quasi-equivalence*

$$R_{\mathbb{L}_c} : \mathcal{W}(\mathbb{L}_c) \to \mathcal{W}^s(\mathbb{L}_c). \tag{4.2}$$

- *(ii) Let $\mathbb{L}_p$ be a finite collection of products of exact cylindrical Lagrangian submanifolds of M and those of N. Then there is a quasi-equivalence*

$$R_{\mathbb{L}_p} : \mathcal{W}^s(\mathbb{L}_p) \to \mathcal{W}(\mathbb{L}_p). \tag{4.3}$$

- *(iii) Let $\mathbb{L}$ be a finite collection of admissible Lagrangian submanifolds, such that Assumption 4.1 is satisfied for any pair in the collection $\mathcal{L}$. Then there is a natural quasi-equivalence*

$$R_{\mathbb{L}} : \mathcal{W}^s(\mathbb{L}) \to \mathcal{W}(\mathbb{L}). \tag{4.4}$$

The formal definitions of these functors will be given in the rest of this section, while analytic details required to give the definition are deferred to section 8.

4.3. **Action-restriction data.** As mentioned at the beginning of section 4 main technical issue in studying Floer theory on the product manifold $M \times N$ and relating that to quilted Floer theory is with regard to the choice of Hamiltonian functions. Having found the cylindrical end $\Sigma \times [1, +\infty)$, one can immediately set up wrapped Floer theory using Hamiltonians that depend only on the radial coordinate r in the cylindrical end are linear for r large with increasing slopes that tend to infinity. On the other hand, what is more directly related to quilted Floer theory is the split Hamiltonian, i.e. the sum of the two Hamiltonians on both factors M and N. However, the split Hamiltonian is not a priori admissible in the usual sense of wrapped Floer theory. Also, there is a similar issue with almost complex structures. Thus it is not immediately clear that the resulting two versions of wrapped Fukaya categories are equivalent.

By the invariance nature of Floer cohomology, it is expected that these two versions should be equivalent. The proof of this is based on an action filtration argument, whose spirit goes back to the work of [Oan06] proving Künneth formula for symplectic cohomology. The basic idea is summarized as follows. First consider the case of exact cylindrical Lagrangian submanifolds of $M \times N$. We start with Hamiltonians H_i that are linear at infinity, and a family of cylindrical almost complex structures, which are used to define the wrapped Floer complex $CW^*(\mathcal{L}_0, \mathcal{L}_1; \{H_i\})$. Continuing the discussion on well-definedness of wrapped Floer theory, for each action filtration window $(-b, a]$, we have produced sequence of Hamiltonians K_i each of which is split when $r_1 \geq A, r_2 \geq A$, and agrees with the given Hamiltonian H_i inside the compact set $r = r_1 + r_2 \leq A$. In the given action filtration window $(-b, a]$, the relevant generators for both Floer complexes within the action filtration window $(-b, a]$ are all contained in the region $r \leq A$, so that the generators of $CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; K_i)$ are automatically generators of $CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; H_i)$. Thus we get a homomorphism of graded modules

$$\bar{R}^1_b : CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; H_i) \to CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; K_i), \tag{4.5}$$

which is just the identity map identifying the Hamiltonian chords for both Hamiltonians contained in the region $r \leq A$. It turns out that it is a cochain map, for a suitable choice of almost complex structure $J_{i,b}$ which is a product almost complex structure outside the compact set $r_1 \leq A, r_2 \leq A$ and is a product of cylindrical almost complex structures when $r_1 \geq A + \epsilon, r_2 \geq A + \epsilon$.

The details on the construction of the map (4.5) and the definitions of the relevant geometric data will be discussed in section 8.

Revisiting the property of the Hamiltonian K_i, we have that $K_i \leq H_i$ everywhere, and $H_i = K_i$ inside $r \leq A - \epsilon$. Moreover, we can construct a decreasing homotopy $H_{i,w}$ from H_i to K_i, parametrized by $w \in [0, 1]$, such that during the homotopy for every $w > 0$, the extra Hamiltonian chords have sufficiently positive action and do not fall in the action filtration window $(-b, a]$.

To extend the above map to the whole wrapped Floer complex and to study its lift to an A_∞-functor, we need to identify each truncated Floer complex $CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; K_i, J_{s,i})$ with one that is defined with respect to Hamiltonians $K_{i,1}$ which are split outside the same compact set independent of i, and almost complex structures $J_{s,i,1}$ which are product outside the same compact set independent of i. It is proved in section 8 that:

Lemma 4.1. *There exists a Hamiltonian $K_{i,1}$ which is split outside $r \leq 1$, and is the sum of two linear Hamiltonians when $r_1 \geq 1, r_2 \geq 1$, and an almost complex structure $J_{s,i,1}$ which is a product outside $r \leq 1$ and is the product of two cylindrical almost complex structures when $r_1 \geq 1, r_2 \geq 1$, such that there is a cochain homotopy equivalence*

$$h_i : CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; K_i, J_i) \to CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; K_{i,1}, J_{i,1}). \tag{4.6}$$

Then we compose the above cochain homotopy equivalence with (4.5) (which we still denote by the same symbol) to obtain a cochain map:

$$R^1_i : CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; H_i) \to CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; K_{i,1}), \tag{4.7}$$

$$R^1_i = h_{i,b} \circ \bar{R}^1_i, \tag{4.8}$$

for Hamiltonians. These maps are shown to commute up to chain homotopy with natural inclusions of sub-complexes under action filtration, so that we are able to take the limit (homotopy direct limit) to obtain the desired cochain quasi-isomorphism on the whole wrapped Floer complexes

$$R^1 : CW^*(\mathcal{L}_0, \mathcal{L}_1; \{H_i\}) \to CW^*(\mathcal{L}_0, \mathcal{L}_1; \{K_{i,1}\}). \tag{4.9}$$

The procedure of taking the limit is as follows. For each b, we consider the homotopy direct limit for $i > i_b$, to obtain maps on the sub-complexes

$$R^1_b : \bigoplus_{i>i_b} CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; H_i)[q] \to \bigoplus_{i>i_b} CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; K_{i,1})[q]. \tag{4.10}$$

Then we compose this with cochain homotopy equivalences

$$\bigoplus_{i>i_b} CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; H_i)[q] \to CW^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; \{H_i\}),$$

and

$$\bigoplus_{i>i_b} CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; K_{i,1})[q] \to CW^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; \{K_{i,1}\}),$$

and their homotopy inverses to obtain

$$R^1_b : CW^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; \{H_i\}) \to CW^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; \{K_{i,1}\}). \tag{4.11}$$

These do not strictly commute with inclusions of sub-complexes under action-filtration, but only up to canonical chain homotopy. Then we take a (homotopy) direct limit of cochain maps by using corrections from the chosen chain homotopies and then obtain the desired map

$$R^1 : CW^*(\mathcal{L}_0, \mathcal{L}_1; \{H_i\}) \to CW^*(\mathcal{L}_0, \mathcal{L}_1; \{K_{i,1}\}). \tag{4.12}$$

Remark 4.3. *The way we define the action-restriction map is not via certain parametrized moduli space, but this map can alternatively be interpreted as the continuation map induced by a monotone homotopy from the split Hamiltonian to the admissible Hamiltonian.*

In the construction of the map R^1_b above, there are several parameters involved in the procedure of modifying the Hamiltonian (e.g. the size of the compact set, the precise behavior of the Hamiltonian K_i, etc.), such that we can obtain the desired estimate on the action of the extra chords. Such estimates are given in section 8. We call a choice of these parameters, or more essentially the collection of geometric data these parameters determine, an action-restriction datum. A precise definition is given below.

Definition 4.2. *An action-restriction datum for the strip $Z = \mathbb{R} \times [0,1]$ and a number $w \in [0,1]$, and the pair $(\mathcal{L}_0, \mathcal{L}_1)$ of exact cylindrical Lagrangian submanifolds of $M \times N$ consists of the following data:*

(i) an action filtratio window $(-b, a]$ for the Floer cochain space $CF^*(\mathcal{L}_0, \mathcal{L}_1; H_i)$ with respect to the action filtration;

(ii) *a Hamiltonian K_i, which agrees with the given Hamiltonian H_i, is split outside the compact set $r \leq A$, and is the sum of two linear Hamiltonians $H_{M,i}$ on M and $H_{N,i}$ on N for $r_1 \geq A, r_2 \geq A$. The number $A = \frac{3k_i}{\delta_i}$, where δ_i is the gap between the slope k_i and*

(iii) *a compatible almost complex structure $J_{s,i}$, which agrees with the given J_i in the compact set $\{r \leq A - \epsilon\}$, and is a product almost complex structure outside the compact set $r_1 \leq A, r_2 \leq A$, and is the product of two cylindrical almost complex structures when $r_1 \geq A, r_2 \geq A$;*

(iv) *a homotopy $H_{i,w}$ between H_i and K_i, which is constant when $r \leq A - \epsilon$, and is a decreasing homotopy outside the compact set $r \leq A - \epsilon$. It is required that extra Hamiltonian chords for $H_{i,w}$ for every $w \in (0, 1]$ have sufficiently positive action for i large and do not fall in the action filtration window $(-b, a]$.*

(v) *a homotopy $J_{i,w}$ from J_i to $J_{s,i}$, parametrized by $w \in [0, 1]$, and constantly equals J_i inside $r \leq A - \epsilon$, and is split outside $r \geq A$ for every $w > 0$.*

The construction of such an action-restriction datum is given in section 8. An action-restriction datum can be thought of as a special kind of Floer datum for continuation maps. The key property of a choice of an action-restriction datum is stated in the following proposition, whose proof is also deferred to section 8.

Proposition 4.5. *For a choice of action-restriction datum for the strip and the pair $(\mathcal{L}_0, \mathcal{L}_1)$, the map (4.5) is a cochain isomorphism. Therefore, the maps (4.9) and (4.7) are cochain homotopy equivalence.*

To define higher order maps similar to the map (4.7), we generalize the notion of action-restriction datum for disks with more punctures.

Definition 4.3. *An action-restriction datum for a disk S with $(k + 1)$-boundary punctures as well as a number $w \in [0, 1]$, and a $(k + 1)$-tuple of Lagrangian submanifolds $(\mathcal{L}_{j_0}, \cdots, \mathcal{L}_{j_k})$ consists of the following data:*

(i) *a finite subset $F \subset \{1, \cdots, k\}$ of labels;*

(ii) *a collection of positive integers $i_0, \cdots, i_k$, called weights, satisfying the condition $i_0 = |F| + i_1 + \cdots + i_k$;*

(iii) *action filtration windows $(-b_i, a_i]$ for $CF^*(\mathcal{L}_{j_{i-1}}, \mathcal{L}_{j_i}; H_i)$, for $i = 1, \cdots, k$, and a truncation $(-b_0, a_0]$ of $CF^*(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; H_i)$, subject to the following condition:*

$$\sum_{i=1}^{k} b_i \leq b_0, \text{ and } \sum_{i=1}^{k} a_i \leq a_0; \tag{4.13}$$

(iv) *a sub-closed one-form α_S on S, which vanishes along the boundary ∂S, and agrees with $i_j dt$ over the j-th strip-like end, such that the differential $d\alpha_S$ also vanishes in a small neighborhood of the boundary;*

(v) *a modified Hamiltonian K_i which agrees with the linear Hamiltonian H_i in the compact set $\{r \leq A_i - \epsilon\}$, and is linear in r outside $\{r \leq A_i\}$. The choice of A_i above should satisfy the additional condition that the extra K_i-chords in the region $\{r \leq A_i\}$ from $\mathcal{L}_{j_{i-1}}$ to $\mathcal{L}_{j_i}$ (respectively from $\mathcal{L}_{j_0}$ to $\mathcal{L}_{j_k}$ all have sufficiently positive action, much bigger than a_i (respectively a_0);*

(vi) an S-parametrized family of Hamiltonians $\tilde{H}_{(S,w);F;i_0,\cdots,i_k;\vec{b}}$*, depending on* w*, such that over the* j*-th strip-like end* ($j = 1, \cdots, k$)*,* $\tilde{H}_{S,w}$ *agrees* $H_{i_j,w}$*.*

(vii) a modified almost complex structure J_i*, which is a cylindrical in* $\{r \leq A_i\}$ *and is generic of contact type outside* $\{r \leq B_i\}$*;*

(viii) an S-parametrized family of admissible almost complex structures $\tilde{J}_{S,w;F;i_0,\cdots,i_k;\vec{b}}$ *depending on* w*, such that over the* j*-th strip-like end* ($j = 1, \cdots k$)*, it agrees with the given homotopy* $J_{i_j,w}$*.*

In particular, when restricting to each strip-like end, the Hamiltonian and the almost complex structure should be independent of the s*-coordinate, at least for sufficiently large* $|s|$*. Here* s *is the coordinate on* $\mathbb{R}_{\pm}$*.*

Remark 4.4. *Although we need only to modify the Hamiltonian near the* 0*-th strip-like end, we include all the relevant information near other strip-like ends because it will be convenient to extend the action-restriction data to nodal disks and describe the gluing process for action-restriction data.*

For products of exact cylindrical Lagrangian submanifolds of M and those of N, we can make similar definitions and perform similar constructions. This time, we start from split linear Hamiltonians $\{H_{M,N,i}\}$ and product almost complex structures $\{J_{M,N,i}\}$, and construct Hamiltonians $K_{i,1}$ which are linear at infinity with respect to the cylindrical structure $\Sigma \times [1, +\infty)$, and almost complex structures $J_{i,1}$ which are cylindrical away from a compact set. We shall not repeat all the sentences, but will give details on the construction of the Hamiltonians later in section 8.

4.4. **Choosing action-restriction data for all curves.** The underlying operad controling A_∞-functors is commonly known as multiplihedra, introduced by J. Stasheff. Before constructing the action-restriction functor, we need a model for multiplihedra whose elements are the underlying domains for various inhomogeneous pseudoholomorphic maps used to define the action-restriction functor which relates the two versions of wrapped Fukaya categories of the product manifold. Since we set up wrapped Floer theory using sequences of linear Hamiltonians, we shall refer to moduli spaces of cascades, introduced in [AS10], as a variant of moduli spaces of disks with time-allocation, introduced in [FOOO09a], [FOOO09b]. On the other hand, since we are working with families of Floer data parametrized by the closed unit interval $[0, 1]$, we use the definition of [AS10] for moduli spaces of cascades, but allowing the parameter ρ to vary in $[0, 1]$, as in the definition of disks with time-allocation in [FOOO09a], [FOOO09b]. The only new input compared to disks with time-allocation is the label F.

We denote by $\mathcal{N}_{k+1}^{F;i_0,\cdots,i_k}$ the smooth locus of the moduli space of cascades, with label F and weights $i_0, \cdots, i_k$. Each such is a copy of $\mathcal{N}_{k+1} = \mathcal{M}_{k+1} \times [0, 1]$. Its compactification $\bar{\mathcal{N}}_{k+1}^{F;i_0,\cdots,i_k}$ can be described as the union of fiber products the same kinds of moduli spaces for admissible cuts of the labeling set F into F_1 and F_2. In other words, a point in the compactification is a tree of elements in $\mathcal{N}_{k+1}^{F;i_0,\cdots,i_k}$ for various $k, F, i_0, \cdots, i_k$, which are joint at boundary marked points, subject to the following condition: if a collection of elements $(S_{k_v+1}^{F_v;\vec{i}_v}, w_v) \in \prod_{v \in T} \mathcal{N}_{k_v+1}^{F_v;\vec{i}_v}$ for some tree T labeling the vertices of the given point in the compactification $\bar{\mathcal{N}}_{k+1}^{F;i_0,\cdot,i_k}$, then we require that for each path in T from the root to any given vertex, the w_v-values decrease along the path. Such a collection of elements in the fiber products of smooth moduli spaces gives a point in the compactification, and is sometimes called a cascade.

Definition 4.4. *A universal choice of action-restriction data over $\mathcal{N}_{k+1}$ is a choice of an action-restriction datum for each $(S, w) \in \mathcal{N}_{k+1}$, which depends smoothly on points in $\mathcal{N}_{k+1}$.*

Fix a collection of Lagrangian labels $(\mathcal{L}_0, \cdots, \mathcal{L}_d)$. A universal choice of action-restriction data over $\mathcal{N}_{k+1}$ for the given Lagrangian labels allows us to define multilinear maps that are similar to (4.5). However, they do not necessarily satisfy any relations. In the next subsection, we shall discuss how the action-restriction data can be extended to the compactification $\bar{\mathcal{N}}_{k+1}$, so that the multilinear maps thus defined satisfy the desired equations for A_∞-functors.

4.5. **Making choices of action-restriction data consistently.** The construction of a cochain homotopy equivalence (4.9) of the two versions of wrapped Floer complexes works well for any pair $(\mathcal{L}_0, \mathcal{L}_1)$ among a fixed finite collection of Lagrangian submanifolds of the product manifold, as that only requires a choice of an action-restriction datum for the strip and that particular pair of Lagrangian submanifolds in consideration. Extending this to an A_∞-functor on the categorical level requires us to choose action-restriction data for many Lagrangian submanifolds.

In section 8, we shall show that for each H_i, the parameter A be can chosen to be the same for any pair of Lagrangian submanifolds in the collection $\mathbb{L}_c$ that are crucial in the definition of an action-restriction datum, and they essentially depend on the initially given split Hamiltonian, product almost complex structure, and Lagrangian submanifolds only. Moreover, we can choose the same parameters for all pairs such that the action-restriction maps are defined, which is possible because of the following three reasons:

(i) the constant $a > 0$ can be chosen to be the largest such that there are no Hamiltonian chords for the given Hamiltonians H_i between any pair of exact cylindrical Lagrangian submanifolds which have action bigger than a
(ii) we have to consider all possible b's satisfying the condition (4.13), thus these are not matters of choices;
(iii) we can choose the same numbers A_i's for every H_i independent of particular pair of Lagrangian submanifolds from the given collection $\mathbb{L}_c$, but only on the collection as a whole. This determines the size of the compact set inside which the linear Hamiltonian H_i agrees with the split one K_i, and the way how H_i is modified to K_i.

Consider the full sub-category $\mathcal{W}(\mathbb{L}_c)$ (respectively $\mathcal{W}^s(\mathbb{L}_c)$) of $\mathcal{W}(M \times N)$ (respectively $\mathcal{W}^{\in}(M \times N)$) whose objects are Lagrangian submanifolds from our collection $\mathbb{L}_c$. We may construct an A_∞-functor

$$R_{\mathbb{L}_c} : \mathcal{W}(\mathbb{L}_c) \to \mathcal{W}^s(\mathbb{L}) \tag{4.14}$$

which is a quasi-equivalence. As mentioned before, to verify the A_∞-equations, we need to arrange the action-restriction data in a consistent way.

To explain the precise meaning of this consistency, we shall first describe how action-restriction data can be glued together when we glue cascades. Suppose a disk S with $k+1$ marked points is glued with several smooth cascades $(S_j, F_j, \vec{i}^j)$. There are no issues with labels and weights, and they are part of the data for gluing cascades in the usual sense. As for the action filtration windows, over the strip-like ends of S_ρ that are obtained from S_j's, we simply take the existing action filtration windows $(-b^l_j, a^l_j]$. But we have not specified an action filtration windows for the 0-th strip-like end of S_ρ which comes from S, as the Floer datum on S does not contain such information. Nonetheless, the previous condition on Hamiltonians give a natural action filtration window over that end. We may simply take $b_0 = \sum_{j=1}^{l} b^j_0$, and take a_0 not less than $\sum_{i=1} a^j_0$ such that the truncated Floer complex $CF^*_{(-b_0,a]}$ is independent of a when $a \geq a_0$.

Because of the above conditions on families of Hamiltonians and almost complex structures over S and S_j's that they agree over the strip-like ends which are to be glued together, we may take the union of the families $H_S, H_{S_j,w_j;F_j;\vec{i}^j;\vec{b}^j}$ and $J_S, J_{S_j,w_j;F_j;\vec{i}^j,\vec{b}^j}$, to obtain families of Hamiltonians and almost complex structures on the glued surface S_ρ with desired properties.

There are more complicated gluings, which happen in higher-codimensional strata. But the corresponding gluing process for action-restriction data is the same.

Now let us formalize all the above ideas in the following definition.

Definition 4.5. *A universal and consistent choice of action-restriction data is a choice of an action-restriction datum for every $k \geq 1$ and for every (representative of) element $\bar{\mathcal{N}}^{F;i_0,\cdots,i_k}_{k+1}$ and every $(k+1)$-tuple of exact cylindrical Lagrangian submanifolds $(\mathcal{L}_{j_0}, \cdots, \mathcal{L}_{j_k})$ of $M \times N$ from the given finite collection $\mathbb{L}_c$, which varies smoothly on $\mathcal{N}_{k+1}$, and satisfies the following conditions:*

(i) For an element $(S, w) \in \mathcal{N}^{F;i_0,\cdots,i_k}_{k+1}$ that is sufficiently close to the boundary strata $\partial\bar{\mathcal{N}}_{k+1}$, then the choice of action-restriction datum for (S, w) is equivalent to the action-restriction datum induced by gluing of action-restriction data and Floer data;

(ii) The following chart

$$U \times \prod_{interior\ vertices\ v} U_v \times \prod_{interior\ edges\ e} (0, a_e), \tag{4.15}$$

for a deleted neighborhood of a boundary stratum $\sigma \subset \partial\bar{\mathcal{N}}_{k+1}$ which has a chart being products of open charts in $\mathcal{M}_{l+1} \times \prod_v \mathcal{N}_{k_v+1}$ cross with open intervals, the restriction of the action-restriction data to the main component $(S, w) \in \mathcal{S}_{l+1}$ induces a family of action-restriction data for (S, w) parametrized by

$$U \times \prod_{e\ adjacent\ to\ the\ root} (0, a_e) \times E.$$

We require this family extends smoothly to

$$U \times \prod_{e\ adjacent\ to\ the\ root} [0, a_e) \times E,$$

and that it agrees on $U \times \prod_{e\ adjacent\ to\ the\ root}\{0\} \times E$ with the family of action-restriction data that was chosen for $\mathcal{N}_{l+1}$, up to a family of conformal rescalings.

In view that $\bar{\mathcal{N}}_{k+1}$ is a generalized smooth space, we may simply say that a universal and consistent choice is one which varies smoothly over the compactification.

Lemma 4.2. *Fix a finite collection $\mathbb{L}_c$ of exact cylindrical Lagrangian submanifolds $\mathcal{L}_1, \cdots, \mathcal{L}_d$ of $M \times N$. Then universal and conformally consistent choices of action-restriction data exist, with Lagrangian labels from this collection.*

Proof. The proof is an inductive argument based upon the inductive structure of the multiplihedra $\bar{\mathcal{N}}^{F;i_0,\cdots,i_k}_{k+1}$.

The initial step starts with $k = 1$ for which we can construct the Hamiltonian K_i and almost complex structure $J_{s,i}$ with desired properties, which depend only on the collection and do not differ for each particular pair of Lagrangian submanifolds. Then we can construct the desired homotopies $H_{i,w}$ and $J_{i,w}$.

For the inductive step, suppose that we have made consistent choices of action-restriction data for elements in $\bar{\mathcal{N}}_{m_i+1}, \bar{\mathcal{N}}_{m+2}$ as well as Floer data for elements in $\bar{\mathcal{M}}_{l+1}, \mathcal{M}_{k-m+1}$, such that the Hamiltonians, almost complex structures, etc. agree over the strip-like ends that are to be glued together. We then use the gluing maps to obtain action-restriction data for elements in a neighborhood of the boundary strata $\partial\bar{\mathcal{N}}_{k+1}$ in $\bar{\mathcal{M}}_{k+1}$, in the way we described before Definitions 4.2 and 4.3. Since $\bar{\mathcal{N}}_{k+1}$ is contractible and the spaces of admissible Hamiltonians and almost complex structures are contractible, we may extend the choices for all elements therein, without changing those for elements in the boundary $\partial\bar{\mathcal{N}}_{k+1}$. This completes the inductive step and therefore finishes the proof.

□

Remark 4.5. *Unlike the case of compact Lagrangian submanifolds, in general $CW^*(\mathcal{L}_0, \mathcal{L}_1)$ is different from $CW^*(\mathcal{L}_1, \mathcal{L}_0)$ even on the level of cohomology, and in general there is no Poincare duality between these two complexes. Thus there is no need to worry about compatibility for the choices of action-restriction datum for $(\mathcal{L}_0, \mathcal{L}_1)$ and that for $(\mathcal{L}_1, \mathcal{L}_0)$ - these can simply be two different, independent choices.*

4.6. **The action-restriction functor: definition.** Suppose we have made a universal and conformally consistent choice of action-restriction data for all $k \geq 1$ and all $\bar{\mathcal{N}}_{k+1}$, with the Lagrangian labels for the boundary components of punctured disks chosen from a fixed finite collection of exact cylindrical Lagrangian submanifolds $\mathbb{L} = \{\mathcal{L}_1, \cdots, \mathcal{L}_d\}$ of $M \times N$. Then we shall construct an A_∞-functor

$$R_{\mathbb{L}} : \mathcal{W}(\mathbb{L}) \to \mathcal{W}^s(\mathbb{L}), \tag{4.16}$$

with the following properties. On the level of objects, it acts as the identity. The first order map R^1_d is the action-restriction map (4.7), for each pair of Lagrangian submanifolds from the collection $\mathbb{L}$.

As regard for higher order structure maps, consider $(k+1)$-tuple of Lagrangian submanifolds $\mathcal{L}_{j_0}, \cdots, \mathcal{L}_{j_k}$, where $j_i \in \{1, \cdots, d\}$. Suppose we have made a universal and conformally consistent choice of action-restriction data, where the Lagrangian labels are limited to the cyclically-ordered tuple $(\mathcal{L}_{j_0}, \cdots, \mathcal{L}_{j_k})$. For any $(S, w) \in \mathcal{N}_{k+1}$, the action-restriction datum from our choice sets up a moduli problem. Varying (S, w) in the moduli space $\mathcal{N}_{k+1}$ gives rise to a family over $\mathcal{N}_{k+1}$, which is a parametrized moduli space $\mathcal{N}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k)$, provided appropriate asymptotic convergence conditions are given:

$$x_l \in CW^*_{(-b_l,a_l]}(\mathcal{L}_{j_{l-1}}, \mathcal{L}_{j_;}; H_{i_l}, J_{i_l}),$$

and

$$\tilde{x}_0 \in CW^*_{(-b_0,a_0]}(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; K_{i_0}, J_{s,i_0}).$$

The moduli space $\mathcal{N}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k)$ has a compactification $\bar{\mathcal{N}}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k)$. The additional elements are either (equivalence classes of) broken inhomogeneous pseudoholomorphic maps from domains being elements in $\bar{\mathcal{N}}_{k+1}$, or broken inhomogeneous pseudoholomorphic maps which break out inhomogeneous pseudoholomorphic strips.

Proposition 4.6. *For generic choice of action-restriction data, the moduli space $\bar{\mathcal{N}}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k)$ satisfies the following properties:*

(i) If the virtual dimension is zero, $\bar{\mathcal{N}}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k)$ is a zero-dimensional compact smooth manifold, hence consists of finitely many points;

(ii) If the virtual dimension is one, $\bar{\mathcal{N}}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k)$ is a one-dimensional compact topological manifold, hence is a disjoint union of finitely many circles and intervals.

By counting rigid solutions in the moduli space $\bar{\mathcal{N}}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k)$, we define the following multilinear maps of degree $1-k$

(4.17)

$$\begin{aligned} \bar{R}^{k;F;i_0,\cdots,i_k}_{\mathbb{L}_c,\vec{b}} :& CF^*_{(-b_k,a_k]}(\mathcal{L}_{j_{k-1}}, \mathcal{L}_{j_k}; H_{i_k}, J_{i_k})[q] \otimes \cdots \otimes CF^*_{(-b_1,a_1]}(\mathcal{L}_{j_0}, \mathcal{L}_{j_1}; H_{i_1}, J_{i_1})[q] \\ &\to CF^*_{(-b_0,a_0]}(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; K_{i_0}, J_{s,i_0})[q]. \end{aligned}$$

Similar to the first-order map, we may compose this with a canonical cochain homotopy equivalence

$$h_{i_0,b_0} : CF^*(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; K_{i_0}, J_{s,i_0}) \to CF^*(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; K_{i_0,1}, J_{s,i_0,1}),$$

to obtain the following map

(4.18)

$$\begin{aligned} R^{k;F;i_0,\cdots,i_k}_{\mathbb{L},\vec{b}} :& CF^*_{(-b_k,a_k]}(\mathcal{L}_{j_{k-1}}, \mathcal{L}_{j_k}; H_{i_k}, J_{s,i_k}) \otimes \cdots \otimes CF^*_{(-b_1,a_1]}(\mathcal{L}_{j_0}, \mathcal{L}_{j_1}; H_{i_1}, J_{s,i_1}) \\ &\to CF^*_{(-b_0,a_0]}(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; K_{i_0,1}, J_{s,i_0,1}). \end{aligned}$$

Remark 4.6. *An alternative interpretation of the action-restriction functor is helpful. Simply put, the action-restriction functor is a continuation functor associated to a particular kind of monotone homotopy from (H_i, J_i) to $(K_{i,1}, J_{s,i,1})$. The reason we perform the construction in a slightly non-standard way as above is that it is not obviously clear that this kind of continuation functor is a homotopy equivalence, as the two kinds of Floer data behave somewhat differently at infinity, and such a homotopy is decreasing and is not of compact support.*

To extend the above maps to the whole wrapped Floer complexes so that we can obtain the desired A_∞-functor (4.16), we are faced with two potential problems. First, we must ensure that these maps are compatible with natural inclusions of wrapped Floer complexes under action filtration. Second, the target wrapped Floer complex is defined with respect to a Hamiltonian K_i and a one-parameter family of almost complex structures J_i which depend on the action filtration windows chosen in the action-restriction data. The solutions to these two problems will have to depend on the precise behavior of the choices of families of Hamiltonians and almost complex structures involved in the action-restriction data. We shall discuss these matters in greater detail in the next subsection.

4.7. **Arranging geometric data in a compatible system.** In Definition (4.3), the condition (4.13) implies that we can define a new k-th order "multiplication" $m^k_{\vec{b}}$ on the truncated wrapped Floer complexes:

(4.19)

$$m^{k;F;i_0,\cdots,i_k}_{\vec{b}} : CF^*_{(-b_k,a_k]}(\mathcal{L}_{j_{k-1}},\mathcal{L}_{j_k};K_{i_k},J_{i_k})[q]\otimes\cdots\otimes CF^*_{(-b_1,a_1]}(\mathcal{L}_{j_0},\mathcal{L}_{j_1};K_{i_1},J_{i_1})[q] \to CF^*_{(-b_0,a_0]}(\mathcal{L}_{j_0},\mathcal{L}_{j_k};K_{i_0},J_{s,i_0})[q]$$

whose definition is given in section 8.8 of section 8. Roughly speaking, this counts rigid inhomogeneous pseudoholomorphic polygons with possibly different asymptotic Floer data (Hamiltonians and almost complex structures) over the strip-like ends. The reason why we introduce these maps is that in general, the maps $\bar{R}^{k;F;i_0,\cdots,i_k}_{\mathbb{L}_c}$ (4.17) satisfy analogues of A_∞-equations with the honest A_∞-structure maps m^k for $\mathcal{W}(M\times N)$ replaced by these new maps $m^k_{\vec{b}}$.

However, the new operations $m^k_{\vec{b}}$ are not the A_∞-operations for the sub-category $\mathcal{W}(\mathbb{L})$ of the wrapped Fukaya category $\mathcal{W}(M\times N)$. This is one of the main reasons that we need to introduce the higher order maps $R^k_{\mathbb{L}}$ to adjust the failure of $R^1_{\mathbb{L}}$ from being an A_∞-functor with respect to the honest A_∞-structures.

Following the construction in section 8.8, we can construct families of Hamiltonians and almost complex structures, which give us conformally consistent choices of action-restriction data for all $k\geq 1$ and for all $\bar{\mathcal{N}}_{k+1}$, with Lagrangian labels from the fixed collection $\mathbb{L}=\{\mathcal{L}_1,\cdots,\mathcal{L}_d\}$. Now we can prove:

Lemma 4.3. *The multilinear maps (4.17) satisfy an analogue of A_∞-functor equations, with respect to the m^k's for $\mathcal{W}^s(M\times N)$ and the modified operations $m^k_{\vec{b}}$:*

(4.20)

$$\begin{aligned}
&m^1_{b_0}\circ\bar{R}^k_{\vec{b}}(x_k,\cdots,x_1)\\
=&\sum_{2\leq l\leq k}\sum_{\substack{s_1\geq 1,\cdots,s_l\geq 1\\ s_1+\cdots+s_l=k}} m^l_{\vec{b}^{l,new}}(\bar{R}^{s_l}_{\vec{b}^{s_l}}(x_{s_1+\cdots+s_l},\cdots,x_{s_1+\cdots+s_{l-1}+1}),\cdots,\bar{R}^{s_1}_{\vec{b}^{s_1}}(x_{s_1},\cdots,x_1))\\
&+\sum_{0\leq s\leq k-1}\sum_i \bar{R}^{s+1}_{\vec{b}^{s+1}}(x_k,\cdots,x_{i+k-s+1},m^{k-s}(x_{i+k-s},\cdots,x_{i+1}),x_i,\cdots,x_1),
\end{aligned}$$

whenever the generators are taken from sub-complexes $C_{>i_{b_l}}$. Here the filtration numbers b's are determined by the following rule. In the first term on the right hand side of (4.20), we have $\vec{b}^{s_i}=(b^{s_i}_0,\cdots,b^{s_i}_{s_i})$, such that $\vec{b}^{l,new}=(b_0,b^{s_1}_0,\cdots,b^{s_l}_0)$. On the left hand side, we have $\vec{b}=(b_0,\cdots,b_k)$. Moreover, we require that when deleting those $b^{s_i}_0$'s for all i and combining the rest together, $(b_0,b^{s_1}_1,\cdots,b^{s_1}_{s_1},\cdots,b^{s_l}_1,\cdots,b^{s_l}_{s_l})$ should agree with $(b_0,b_1,\cdots,b_k)$ from the left hand side. In the second term, we have $\vec{b}^{s+1}=(b_0,\cdots,b_i,b_{i+k-s+1},\cdots,b_k)$.

Proof. Recall that in subsection 8.8, we construct the family of Hamiltonians $\tilde{H}_{S,w,\vec{b}}$ and the family of almost complex structures $\tilde{J}_{S,w,\vec{b}}$ which degenerate to a product of families of similar kinds on domains with fewer punctures, and Floer data on disks with punctures that are used to define A_∞-structure maps in $\mathcal{W}^s(M\times N)$ and the maps $m^k_{\vec{b}}$. Therefore, the compactification $\bar{\mathcal{N}}_{k+1}(\tilde{x}_0;x_1,\cdots,x_k)$ has codimension one boundary strata consisting of the following four kinds of products of moduli spaces:

(i)

$$\mathcal{M}^1_{l+1}(\tilde{x}_0;\tilde{x}_1,\cdots,\tilde{x}_l)$$
$$\times\prod_{i=1}^{l}\mathcal{N}_{s_i+1}(\tilde{x}_i;x_{s_1+\cdots+s_{i-1}+1},\cdots,x_{s_1+\cdots+s_i}), l\geq 2;$$

(ii)

$$\mathcal{N}_{k+1}(\tilde{x}'_0;x_1,\cdots,x_k)\times\mathcal{M}^1_2(\tilde{x}_0;\tilde{x}'_0);$$

(iii)

$$\mathcal{N}_{s+2}(\tilde{x}_0;x_1,\cdots,x_i,x',x_{i+k-s+1},\cdots,x_k)$$
$$\times\mathcal{M}^0_{k-s+1}(x';x_{i+1},\cdots,x_{i+k-s}), s\leq k-2;$$

(iv)

$$\mathcal{M}^0_2(x'_i;x_i)\times\mathcal{N}_{k+1}(\tilde{x}_0;x_1,\cdots,x_{i-1},x'_i,x_{i+1},\cdots,x_k).$$

Here the moduli spaces $\mathcal{M}^1_{l+1}(\cdots)$ with superscript 1 consist of inhomogeneous pseudoholomorphic disks defined with respect to the Floer data $(K_{\vec{b}},J_{\vec{b}})$.

The first two types appear when the domains degenerate, which occur in the compactification of $\mathcal{N}_{k+1}$. The last two types appear when a sequence of pseudoholomorphic maps breaks off pseudoholomorphic strips at one of the strip-like ends. Strata of higher codimensions correspond to further degenerations of the domains, and breaking off more pseudoholomorphic strips. Since for our purpose only codimension one strata need to be considered, we will not spell out the details for strata of higher codimensions.

Considering various operations defined by the moduli spaces appearing in the boundary strata of $\bar{\mathcal{N}}_{k+1}(\tilde{x}_0;x_1,\cdots,x_k)$, we get the desired A_∞-equations (4.20).

We remark that in the above formula (4.20) there are have two kinds of terms, as we have combined the contributions from types (i), (ii) and the types (iii), (iv). □

In order to extend the maps (4.18) over the whole wrapped Floer complexes, we need to check that these maps are compatible with each other (for different values of $\vec{b}=(b_0,\cdots,b_k)$) with respect to the natural inclusions

$$\kappa_l: CF^*_{(-b_l,a_l]}(\mathcal{L}_{j_{l-1}},\mathcal{L}_{j_l};H_{i_l},J_{i_l})\to CF^*_{(-b'_l,a'_i]}(\mathcal{L}_{j_{l-1}},\mathcal{L}_{j_l};H_{i_l},J_{i_l}),$$

and

$$\kappa'_0: CW^*_{(-b_0,a_0]}(\mathcal{L}_{j_0},\mathcal{L}_{j_k};K_{i_0},J_{s,i_0})\to CW^*_{(-b'_0,a'_0]}(\mathcal{L}_{j_0},\mathcal{L}_{j_k};K_{i_0},J_{s,i_0})$$

of the truncated wrapped Floer complexes, whenever $b'_i\geq b_i, a'_i\leq a_i$. In fact, we may fix once-for-all the a's at the beginning for all the truncated wrapped Floer complexes, as changing these numbers do not affect the wrapped Floer complexes as soon as they are chosen so that all the interior chords are included. Thus keeping track of a's is unnecessary.

Let us describe in more detail the compatibility conditions. Consider the two composed maps $R^{k;F;i_0,\cdots,i_k}_{d,\vec{b}'}\circ(\kappa_k\otimes\cdots\otimes\kappa_1)$ and $\kappa_0\circ R^k_{d,\vec{b}}$. If they strictly agreed, then the maps $R^k_{d,\vec{b}}$ would be the restriction of the single map (4.18) to the truncated wrapped Floer complexes. These two compositions in general differ from each other. The compatibility condition we require should therefore be phrased that the sequence of maps $R^{k;F;i_0,\cdots,i_k}_{d,\vec{b}'}\circ(\kappa_k\otimes\cdots\otimes\kappa_1)$ is homotopic to $\kappa_0\circ R^{k;F;i_0,\cdots,i_k}_{d,\vec{b}}$, for $b'_i \geq b_i$. The meaning of such kind of homotopy is stated in the following proposition.

Proposition 4.7. *Consider the maps $R^{k;F;i_0,\cdots,i_k}_{d,\vec{b}}$ defined in (4.18). Then there are multilinear maps $T^{k;F;i_0,\cdots,i_k}_{\vec{b},\vec{b}'}$, which after summing over F and weights $i_0,\cdots,i_k$, form a homotopy between the two sequences of maps*

$$R^k_{d,\vec{b}'} \circ (\kappa_k \otimes \cdots \otimes \kappa_1)$$

and

$$\kappa_0 \circ R^k_{d,\vec{b}},$$

in the sense of the obvious analogue of a homotopy between A_∞-functors. That is, these homotopies satisfy the following analogue of A_∞-equations:

(4.21)

$$\begin{aligned}
&R^k_{d,\vec{b}'} \circ (\kappa_k \otimes \cdots \otimes \kappa_1)(x_k, \cdots, x_1) - \kappa_0 \circ R^k_{d,\vec{b}}(x_k, \cdots, x_1)\\
=&\sum_{r,i} \sum_{s_1,\cdots,s_r} (-1)^* m^r(\kappa_0 \circ R^{s_r}_{d,\vec{b}}(x_k, \cdots, x_{k-s_r+1}), \cdots,\\
&\kappa_0 \circ R^{s_{i+1}}_{d,\vec{b}}(x_{s_1+\cdots+s_{i+1}}, \cdots, x_{s_1+\cdots+s_i+1}), T^{s_i}_{\vec{b},\vec{b}'}(x_{s_1+\cdots+s_i}, \cdots, x_{s_1+\cdots+s_{i-1}+1}),\\
&R^{s_i}_{d,\vec{b}'} \circ (\kappa_{s_1+\cdots+s_{i-1}} \otimes \cdots \otimes \kappa_{s_1+\cdots+s_{i-2}+1})(x_{s_1+\cdots+s_{i-1}}, \cdots, x_{s_1+\cdots+s_{i-2}+1}),\\
&\cdots, R^{s_1}_{d,\vec{b}'} \circ (\kappa_{s_1} \otimes \cdots \otimes \kappa_1))\\
+&\sum_{m,l} (-1)^{**} T^{k-m+1}_{\vec{b},\vec{b}'}(x_k, \cdots, x_{m+l+1}, \mu^m(x_{m+l}, \cdots, x_{l+1}), x_l, \cdots, x_1).
\end{aligned}$$

Here the symbols μ^k (temporarily) denote the A_∞-structure maps in the wrapped Fukaya category defined with respect to $(H_{M,N}, J_{M,N})$, and m^k denote those in the wrapped Fukaya category defined with respect to (K, J). The signs are

$$* = s_1 + \cdots + s_{i-1} - \deg(x_1) - \cdots - \deg(x_{s_1+\cdots s_{i-1}}),$$

and

$$** = \deg(x_1) + \cdots + \deg(x_l) - l - 1.$$

Sketch of proof. This follows from the commutative diagram. The key reasoning is that the spaces of admissible Hamiltonians and of almost complex structures are contractible. □

This kind of compatibility condition stated in Proposition 4.7 as above then implies that the homotopy direct limit of $R^k_{d,\vec{b}}$ exists as $b_i \to +\infty$ for all i.

Corollary 4.1. *There exist multilinear maps*

$$\begin{aligned}
R^k_d : &CW^*(\mathcal{L}_{j_{k-1}}, \mathcal{L}_{j_k}; \{H_i\}, \{J_i\}) \otimes \cdots \otimes CW^*(\mathcal{L}_{j_0}, \mathcal{L}_{j_1}; \{H_i\}, \{J_i\})\\
&\to CW^*(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; \{K_{i,1}\}, \{J_{i,1}\}),
\end{aligned} \tag{4.22}$$

such that when restricted to any truncated wrapped Floer complex, it is homotopic to the maps (4.17). Moreover, they satisfy the equations for A_∞-functors from $\mathcal{W}^s_d$ to $\mathcal{W}_d$.

Proof. We can modify the maps $R^k_{d,\vec{b}}$ in (4.17) by composing them with self-homotopy equivalences on the truncated wrapped Floer complexes

$$CW^*_{(-b_i,a_i]}(\mathcal{L}_{j_{i-1}}, \mathcal{L}_{j_i}; H_{M,N}, J_{M,N})$$

and also

$$CW^*_{(-b_0,a_0]}(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; K, J),$$

so that $R^k_{d,\vec{b}'} \circ (\kappa_k \otimes \cdots \otimes \kappa_1)$ and $\kappa_0 \circ R^k_{d,\vec{b}}$ strictly agree after such modification. Thus the direct limit of the modified maps exists. The homotopy direct limit

$$R^k_d = \varinjlim_{b_i \to +\infty} R^k_{d,\vec{b}} \tag{4.23}$$

is defined to be the direct limit of the modified maps. The more detailed definition is an analogue of R^1_d, by considering sub-complexes $C_{>i_{b_i}}$ for various b_i and then transfer the structures to the whole wrapped Floer complexes using homotopy inverses of the inclusions.

The A_∞-functor equations for the sequence of multilinear maps $\{R^k_d\}_{k=1}^{\infty}$ follow from the universal and conformally consistent choice of action-restriction data. Recall that we have analogues of A_∞-functor equations for the maps (4.17). By taking homotopy direct limit as above, we obtain the desired A_∞-equations for the maps R^k_d.

□

We have thus completed the construction of the A_∞-quasi-isomorphism (4.16), in the case of finitely many exact cylindrical Lagrangian submanifolds of $M \times N$.

For products of exact cylindrical Lagrangian submanifolds of M and those of N, we can perform similar constructions, by exchanging the conditions on the input Hamiltonians and almost complex structures and the output ones. The construction from split Hamiltonians to ones linear in the cylindrical end $\Sigma \times [1, +\infty)$ is given in section 8.

4.8. **The action-restriction functor: invariance.** To finish the proof of Theorem 1.1, it remains to verify that $R_{\mathbb{L}'}$ is homotopic to $R_{\mathbb{L}}$ when restricted to $\mathcal{W}^s(\mathbb{L})$.

Let $\mathbb{L}'$ be a finite collection containing $\mathbb{L}$, which can be written as

$$\mathbb{L}' = \{\mathcal{L}_1, \cdots, \mathcal{L}_d, \mathcal{L}_{d+1}, \cdots, \mathcal{L}_{d'}\},$$

for some $d' > d$. For simplicity, denote $\mathcal{W}^s(\mathbb{L}) = \mathcal{W}^s_d$, $\mathcal{W}^s(\mathbb{L}') = \mathcal{W}^s_{d'}$, $\mathcal{W}(\mathbb{L}) = \mathcal{W}_d$, and $\mathcal{W}(\mathbb{L}') = \mathcal{W}_{d'}$. Let

$$i_{d,d'} : \mathcal{W}^s_d \to \mathcal{W}^s_{d'}$$

and

$$j_{d,d'} : \mathcal{W}_d \to \mathcal{W}_{d'}$$

denote the natural inclusion functors, which are strict A_∞-functors having vanishing higher order terms. Also denote the functors $R_{\mathbb{L}} = R_d$ and $R_{\mathbb{L}'} = R_{d'}$.

Since there are fixed d' Lagrangian submanifolds $\mathcal{L}_1, \cdots, \mathcal{L}_{d'}$ to consider at each time, the A_∞-functors R_d and $R_{d'}$ are quite concrete: they are determined by our choices of action-restriction data. There are two families of such choices - one for the collection of Lagrangians $\mathbb{L}$, denoted by D_0, the other for $\mathbb{L}'$, denoted by D_1 (meaning choices for all elements in $\bar{\mathcal{N}}_{k+1}$ in a consistent way. We observe that the conditions for action-restriction data D_1 for $\mathbb{L}'$ will also be satisfied for those for $\mathbb{L}$. This is because the way the estimates in section 8 work essentially depends on the choice of A, which in turn depends on the gaps between slopes and length spectra of Reeb chords. For the bigger collection $\mathbb{L}'$, the common gaps will be smaller than those for $\mathbb{L}$. Thus the constants A are bigger when constructing $R_{d'}$ such that the modified Hamiltonians agree with the given ones on bigger compact sets. Such Hamiltonians will also make the estimates work for Lagrangian submanifolds in $\mathbb{L}$, and therefore make the action-restriction functor well-defined. Thus we may also restrict D_1 to the collection $mathbbL$ to obtain consistent choices of action-restriction data for $\mathbb{L}$, still denoted by D_1. The idea of proving that R_d and $j_{d,d'}^{-1} \circ R_{d'} \circ i_{d,d'}$ are homotopic as A_∞-functors from $\mathcal{W}_d^s$ to $\mathcal{W}_d$ is to choose a particular one-parameter family D_t of action-restriction data interpolating these two (for the sub-categories $\mathcal{W}_d^s$ and $\mathcal{W}_d$), and then use the resulting parametrized moduli spaces to construct the desired homotopy between the two action-restriction functors determined by D_0 and D_1 respectively. The choice is very natural: since the Hamiltonian chosen for the collection $\mathbb{L}'$ is bigger than the corresponding one chosen for the collection $\mathbb{L}$ inside compact sets $r \le A$, we may simply take a linear homotopy decreasing the bigger Hamiltonian to the smaller one, without bringing in extra chords with action in the given action filtration windows. Thus we may find the desired homotopy D_t.

To construct a homotopy between the functors associated to these two data D_0, D_1, we need a one-dimensional higher analogue of the multiplihedra, which we call the homotopehedra and denote by $\bar{\mathcal{P}}_{k+1}$. Define $\mathcal{P}_{k+1} = \mathcal{N}_{k+1} \times [0,1]$. The compactification $\bar{\mathcal{P}}_{k+1}$ has boundary strata made of products of copies of $\mathcal{M}_{i+1}, \mathcal{N}_{j+1}$ as well as $\mathcal{P}_{l+1}$.

Now we consider moduli space $\mathcal{P}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k)$ of elements (S, w, t, u) where $(S, w, t) \in \mathcal{P}_{k+1}$, and $u : (S, w) \to M$ is a continuation disk satisfying the Cauchy-Riemann equation with respect to the Hamiltonian and almost complex structure from the action-restriction datum D_t, which converges to some K_{i_0}-chord $\tilde{x}_0$ over the 0-th strip-like end, and to some H_{i_l}-chord x_l over the l-th strip-like end.

There is a natural bordification $\bar{\mathcal{P}}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k)$, whose codimension one boundary strata consist of a union of product moduli spaces of the following form:

(4.24)
$$\begin{aligned}
&\partial\bar{\mathcal{P}}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k) \\
&= \coprod_{m,n} \mathcal{P}_{k-m}(\tilde{x}_0; x_1, \cdots, x_n, x', x_{n+m+1}, \cdots, x_k) \times \mathcal{M}_{m+1}(x'; x_{n+1}, \cdots, x_{n+m}) \\
&\cup \coprod_{\substack{r,s \\ i_1+\cdots i_r = k}} \coprod_{\tilde{x}'_1, \cdots, \tilde{x}'_r} \mathcal{M}_{r+1}(\tilde{x}_0; \tilde{x}'_1, \cdots \tilde{x}'_r) \times (\mathcal{N}_{i_1+1}(\tilde{x}'_1; x_1, \cdots, x_{i_1}; D_0) \\
&\times \cdots \times \mathcal{N}_{i_{s-1}+1}(\tilde{x}'_{s-1}; x_{i_1+\cdots+i_{s-2}+1}, \cdots, x_{i_1+\cdots+i_{s-1}}; D_0) \\
&\times \mathcal{P}_{i_s+1}(\tilde{x}'_s; x_{i_1+\cdots+i_{s-1}+1}, \cdots, x_{i_1+\cdots+i_s}; D_t) \\
&\times \mathcal{N}_{i_{s+1}+1}(\tilde{x}'_{s+1}; x_{i_1+\cdots+i_s+1}, \cdots, x_{i_1+\cdots+i_{s+1}}; D_1) \\
&\times \cdots \times \mathcal{N}_{i_r+1}(\tilde{x}'_r; x_{i_1+\cdots+i_{r-1}+1}, \cdots, x_k; D_1)) \\
&\cup \mathcal{N}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k; D_1) \\
&\cup \mathcal{N}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k; D_0).
\end{aligned}$$

Here the x's without tilde denote $H_{M,N}$-chords, while the $\tilde{x}$'s denote K-chords.

By counting rigid elements in the moduli spaces $\bar{\mathcal{P}}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k)$, we construct from this one-parameter family of action-restriction data D_t a multilinear map

(4.25)

$$T^k : CW^*(\mathcal{L}_{j_{k-1}}, \mathcal{L}_{j_k}; \{H_i\}) \otimes \cdots \otimes CW^*(\mathcal{L}_{j_0}, \mathcal{L}_{j_1}; \{H_i\}) \to CW^*(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; \{K_{i,1}\})$$

of degree $-k$, whose $(\tilde{x}_0; x_1, \cdots, x_k)$ component is the count of rigid elements in the moduli space $\bar{\mathcal{P}}_{k+1}(\tilde{x}_0; x_1, \cdots, x_k)$.

Setting $T^0 = 0$, we claim that $T = \{T^k\}_{k=0}^{\infty}$ is a homotopy between the functors R_d and $j_{d,d'}^{-1} \circ R_{d'} \circ i_{d,d'}$, where the former is defined by the action-restriction datum D_0, and the latter by D_1. Verifying the relation $m^1_{\mathcal{Q}}(T) = j_{d,d'}^{-1} \circ R_{d'} \circ i_{d,d'} - R_d$ amounts to looking at the boundary strata of the bordification $\bar{\mathcal{P}}_{k+1}(x_0'; x_1, \cdots, x_k)$ as above (4.24). This relation is precisely the condition for R_d and $j_{d,d'}^{-1} \circ R_{d'} \circ i_{d,d'}$ to be homotopic as A_∞-functors. Thus the proof of Theorem 1.1 is complete.

Remark 4.7. *We have check compatibility conditions on every sub-category consisting of finitely many objects. While this in principle should yield an A_∞-functor between the whole wrapped Fukaya categories (at least consisting of countably many objects), the notion of direct limit of A_∞-categories and A_∞-functors is quite delicate and beyond our current scope. However, we can prove that for every K, we may get an A_K-functor between the wrapped Fukaya categories. But since this is not important for applications, we shall avoid that discussion.*

5. Wrapped Floer theory for Lagrangian immersions

5.1. **Overview of immersed Lagrangian Floer theory.** In this section, we extend wrapped Floer theory to certain classes of Lagrangian immersions. The main purpose of such an extension is to prove representability of functors associated to Lagrangian correspondences in general (to be discussed in section 6), though in many concrete and interesting cases, it is sufficient to study embedded Lagrangian submanifolds.

In order for the Lagrangian immersions in consideration to have well-behaved Floer theory, we must impose some conditions: they should satisfy a condition similar to being exact, be embedded in the cylindrical end of M, and possibly have transverse self-intersections in the interior part of M. Without loss of generality, we assume these self-intersections are at most double points. In general, there will be pseudoholomorphic disks bounded by the image of such a Lagrangian immersion, and these disks will interact with inhomogeneous pseudoholomorphic disks (solutions to Floer's equation). Therefore, we should pick a good model for the compactifications of the relevant moduli spaces of disks. Fortunately, this can be done fairly directly, as the Lagrangian immersions we are going to consider still satisfy an "exactness" condition, which will be introduced in the next subsection.

Wrapped Floer theory should contain information about Reeb dynamics on the boundary contact manifold, in addition to the cohomological generators of the Lagrangian submanifolds. The construction of the A_∞-structure maps involves both inhomogeneous pseudoholomorphic disks and homogeneous pseudoholomorphic disks. The entire picture would be an analogue of the setup of holomorphic curves in relative symplectic field theory (see [BEH+03]). Fortunately, there is a purely Floer-theoretic formulation, where we can construct moduli spaces of maps which satisfy certain Floer's equation, and the virtual techniques used in proving transversality does not go beyond the theory of Kuranishi structures, because the pseudoholomorphic curves that we are dealing with are all of genus zero with connected boundary, considered by Fukaya-Oh-Ohta-Ono [FOOO09a], [FOOO09b].

Wrapped Floer theory assigns to such a Lagrangian immersion $\iota : L \to M$ a curved A_∞-algebra $(CW^*(L, \iota; H), m^k)$ over $\mathbb{Z}$ (compare to the case of general compact Lagrangian immersions studied by Akaho and Joyce [AJ10]). To define the immersed wrapped Fukaya category, we shall consider unobstructed Lagrangian immersions, i.e. those for which the curved A_∞-algebra $CW^*(L, \iota; H)$ has a bounding cochain.

In the next several subsections, we shall give a particular chain model for the wrapped Floer cochain spaces, and describe the moduli spaces that are used to define the A_∞-structures. The technical part in defining the A_∞-structures would be to use Kuranishi structures to construct virtual fundamental chains on the relevant moduli spaces. Since there will be nothing new about the theory of Kuranishi structures, the details of constructing the virtual fundamental chains are omitted, which we shall refer to [FO99], [FOOO09a], [FOOO09b], [FOOO12], [FOOO13]. See also [Gao17a] for a more detailed discussion.

5.2. **Gradings and spin structures.** For Floer theory to carry an absolute $\mathbb{Z}$-grading as well as to have coefficients in $\mathbb{Z}$, we need to introduce the notions of gradings and spin structures.

Definition 5.1. *Say that the Lagrangian immersion $\iota : L \to M$ is graded, if the square phase function $\alpha_L : L \to S^1$ has a lift $\tilde{\alpha}_L$ to $\mathbb{R}$. Here the square phase function is defined by sending $x \in L$ to $(d\iota)_x(T_xL)$, an element in the Lagrangian Grassmannian $\mathcal{LAG}(TM)$, then mapping that to S^1 by pairing any orthonormal basis for the Lagrangian plane $(d\iota)_x(T_xL)$ with the quadratic volume form, which is independent of the choice of an orthonormal basis. Such a lift is called a grading for this Lagrangian immersion $\iota : L \to M$.*

From now on we shall make the following assumption.

Assumption 5.1. *The manifold L is spin with a chosen spin structure v. Also, the Lagrangian immersion is graded in the sense of Definition 5.1 with a chosen grading $\tilde{\alpha}_L$.*

A grading for the Lagrangian immersion $\iota : L \to M$ defines an absolute Maslov index for each generator c (note that if c is a critical point of f together with a capping half-disk w, the disk Maslov index of w agrees with the Morse index of f at p), which endows with the wrapped Floer cochain space a $\mathbb{Z}$-grading. A spin structure v determines orientations on the moduli spaces we are going to introduce in subsection 5.6.

Remark 5.1. *The condition that the immersion $\iota : L \to M$ be graded implies that the Maslov class of $\iota : L \to M$ is zero. However, it does not prohibit the existence of holomorphic disks with boundary on $\iota(L)$ of non-zero Maslov indices.*

5.3. **Exact cylindrical Lagrangian immersions and orientation local systems.** Let $\iota : L \to M$ be a Lagrangian immersion. To develop wrapped Floer theory for it, we shall make some geometric assumptions.

Definition 5.2. *A Lagrangian immersion $\iota : L \to M$ is said to have clean self-intersections, if the following conditions are satisfied:*

- *(i) The fiber product $L \times_\iota L = \{(p, q) \in L \times L | \iota(p) = \iota(q)\}$ is a smooth submanifold of $L \times L$;*
- *(ii) At each point $(p, q) \in L \times_\iota L$, we have that*

$$T_{(p,q)}(L \times_\iota L) = \{(V, W) \in T_pL \times T_qL | d_p\iota(V) = d_q\iota(V)\} \tag{5.1}$$

Definition 5.3. *Let $\iota : L \to M$ be a Lagrangian immersion with clean self-intersections. We say that it is exact, if there exists $f : L \to \mathbb{R}$ such that $df = \iota^*\lambda_M$.*

We say that $\iota : L \to M$ is cylindrical, if there exists an embedded closed Legendrian submanifold l of ∂M such that the geometric image $\iota(L)$ satisfies

$$\iota(L) \cap (\partial M \times [1, +\infty)) = l \times [1, +\infty), \tag{5.2}$$

and moreover, the preimage $\iota^{-1}(l \times [1, +\infty)$ is a union of copies of $l \times [1, +\infty)$, so that the restriction of ι is a trivial discrete covering map.

For an exact cylindrical Lagrangian immersion $\iota : L \to M$ with clean self-intersections, we have a finer description of the decomposition of the fiber product $L \times_\iota L$ into its connected components. That is, we shall specify those connected components which are mapped to the cylindrical end $l \times [1, +\infty)$ of the image. We denote

$$L \times_\iota L = \coprod L_a \cup \coprod L_b, \tag{5.3}$$

where the L_b's are the connected components part of which are mapped to the cylindrical end $l \times [1, +\infty)$, including the diagonal component $\Delta_L \cong L$, and the L_a's are the ones which are not, and correspond to those self-intersections only contained in the interior part of M.

Here is a more refined description of the components L_b. Suppose that $\iota^{-1}(l \times [1, +\infty)$ is a union of copies of $l \times [1, +\infty)$, labeled by $i \in I$ for some index set I, which can be infinite. Then the labels b correspond to pairs (i, j) for $i, j \in I$, distinguishing the copies L_b in the fiber product.

For wrapped Floer theory to be well-defined and to give desired A_∞-structures, we shall from now on assume that the immersion $\iota : L \to M$ be proper. In particular, the covering of the cylindrical end $l \times [1, +\infty)$ has at most finite sheets, say d-fold.

Similar to the setup of Morse homology of a Morse-Bott function, we need to take into account certain local systems on the components L_a and L_b of the fiber product $L \times_\iota L$, in order to obtain canonical orientations of the moduli spaces of pseudoholomorphic disks considered in wrapped Floer theory. Most of the definitions follow from Chapter 8 of [FOOO09b], so we just list the essential definitions that we need to fix notations, provide part of the proofs while leaving the full details to the reference.

Let L_a be any connected component different from the diagonal component $\Delta_L \cong L$. Since ι is a Lagrangian immersion, for each $x \in L_a$ we get two Lagrangian subspaces

$$\lambda_{a,x,l} = d\iota_x(T_{pr_1(x)}L), \tag{5.4}$$

$$\lambda_{a,x,r} = d\iota_x(T_{pr_2(x)}L) \tag{5.5}$$

of $T_{\iota(x)}M$, where $pr_1, pr_2 : L_a \subset L \times_\iota L \to L$ are induced by the projections to the two factors.

Let $\mathcal{P}_{a,x}$ be the set of all smooth maps $\lambda_{a,x} : [0, 1] \to \mathcal{LAG}(T_{\iota(x)}M)$, such that $\lambda_{a,x}(0) = \lambda_{a,x,l}, \lambda_{a,x}(1) = \lambda_{a,x,r}$. Associated to each $\lambda_{a,x} \in \mathcal{P}_{a,x}$, there are two Cauchy-Riemann operators

$$\bar{\partial}^-_{\lambda_{a,x}} : L^p_{1,\delta}(Z_-; T_{\iota(x)}M, \lambda_{a,x}) \to L^p_\delta(Z_-; T_{\iota(x)}M) \tag{5.6}$$

and

$$\bar{\partial}^+_{\lambda_{a,x}} : L^p_{1,\delta}(Z_+; T_{\iota(x)}M, \lambda_{a,x}) \to L^p_\delta(Z_+; T_{\iota(x)}M) \tag{5.7}$$

on the weighted Sobolev spaces. Here

$$Z_- = \{z \in \mathbb{C} | Re(z) \leq 0, |z| \leq 1\} \cup \{z \in \mathbb{C} | Re(z) \geq 0, -1 \leq Im(z) \leq 1\}, \tag{5.8}$$

and

$$Z_+ = \{z \in \mathbb{C} | Re(z) \geq 0, |z| \leq 1\} \cup \{z \in \mathbb{C} | Re(z) \leq 0, -1 \leq Im(z) \leq 1\}. \tag{5.9}$$

The weighted Sobolev space $L^p_{1,\delta}(Z_-; T_{\iota(x)}M, \lambda_{a,x})$ is the $L^p_{1,\delta}$-completion of the space of smooth maps $u : Z_- \to T_{\iota(x)}M$ satisfying the following conditions:

(i) $u(s + \sqrt{-1}) \in \lambda_{a,x,l}$, for all $s \geq 0$;
(ii) $u(s - \sqrt{-1}) \in \lambda_{a,x,r}$, for all $s \geq 0$;
(iii) $u(e^{\sqrt{-1}(\frac{\pi}{2} + \pi t)}) \in \lambda_{a,x}(t)$, for $t \in [0, 1]$;
(iv) $\int_{Z_-} e^{\delta |Re(z)|} ||\nabla u||^p dz d\bar{z} < \infty$.

The other weighted Sobolev spaces are defined in similar fashion.

These operators $\bar{\partial}^-_{\lambda_{a,x}}$ and $\bar{\partial}^+_{\lambda_{a,x}}$ are Fredholm. Consider their determinant lines:

$$\Theta^{\pm}_{\lambda_{a,x}} = \det(\bar{\partial}^{\pm}_{\lambda_{a,x}}). \tag{5.10}$$

We wish to move x as well as $\lambda_{a,x}$, so that $\bar{\partial}^{\pm}_{\lambda_{a,x}}$ form a family index bundle $Ind(D^{\pm})$, and the associated determinant line bundle $\underline{\det}Ind(D^{\pm})$ has fiber being (5.10). To make sense of this discussion, we shall first define the space over which the family index bundle is defined.

Definition 5.4. *Define a fiber bundle $\mathcal{I}_{a,x} \to \mathcal{P}_{a,x}$ in the following five steps:*

(i) *First define $(I_{a,x})_{\lambda_{a,x}}$ to be the space of all smooth maps $\sigma_{a,x} : [0,1] \times \mathbb{R}^n \to TM$ such that for each $t \in [0,1]$, the map $\sigma_{a,x}(t,\cdot)$ is a linear isometry between $\mathbb{R}^n$ and $\lambda_{a,x}(t)$. That is, $(I_{a,x})_{\lambda_{a,x}}$ is the space of trivializations along the path $\lambda_{a,x}$ of Lagrangian subspaces in $T_{\iota(x)}M$.*

(ii) *Let $P_{SO}(L)$ be the principal $SO(n)$-bundle associated to the tangent bundle of L. For the given points $p_{\pm}$, let $P_{Spin}(L)_{p_{\pm}}$ be the double cover of the fiber $P_{SO}(L)_{p_{\pm}}$ of $P_{SO}(L)$ at $p_{\pm}$. If $x = (p_-, p_+) \in L_a$, we set*

$$P_x = (P_{Spin}(L)_{p_-} \times P_{Spin}(L)_{p_+})/\{\pm 1\}. \tag{5.11}$$

(iii) *Define a map $(I_{a,x})_{\lambda_{a,x}} \to P_{SO}(L)_{p_-} \times P_{SO}(L)_{p_+}$ as follows. For each $\sigma_{a,x}$, consider its restriction to the endpoints $t = 0, 1$. By definition, $\sigma(0,\cdot)$ is an isometry between $\mathbb{R}^n$ and $d\iota_x(T_{p_-}L)$, hence is canonically identified as an element in the fiber $P_{SO}(L)_{p_-}$. A parallel argument applies to $t = 1$.*

(iv) *Then define $(\mathcal{I}_{a,x})_{\lambda_{a,x}}$ to be the fiber product*

$$(\mathcal{I}_{a,x})_{\lambda_{a,x}} = (I_{a,x})_{\lambda_{a,x}} \times_{P_{SO}(L)_{p_-} \times P_{SO}(L)_{p_+}} P_x. \tag{5.12}$$

(v) *Finally, we consider the union over all paths $\lambda_{a,x}$:*

$$\mathcal{I}_{a,x} = \cup_{\lambda_{a,x} \in \mathcal{P}_{a,x}} (\mathcal{I}_{a,x})_{\lambda_{a,x}}. \tag{5.13}$$

This is the desired fiber bundle $\mathcal{I}_{a,x} \to \mathcal{P}_{a,x}$.

Lemma 5.1. *Suppose L is spin with a chosen spin structure. Then the union $\cup_{x \in L_a} \mathcal{I}_{a,x}$ restricts to a fiber bundle $\mathcal{I}_a$ over the 3-skeleton of L_a.*

Proof. Since L is spin, there is a globally defined fiberwise double cover $P_{Spin}(L)$ of $P_{SO}(L)$ over the 3-skeleton $(L_a)_{[3]}$ of L_a, determined by the spin structure. Thus when defining $\mathcal{I}_{a,x}$, the definition of P_x as in (5.11) can be made globally over $(L_a)_{[3]}$. □

From now on we shall always assume L to be spin, with a chosen spin structure v. Returning to the concern about family index bundles, by moving x and $\lambda_{a,x}$, the operators $\bar{\partial}^{\pm}_{\lambda_{a,x}}$ form a family index bundle $Ind(D^{\pm})$ over $\mathcal{I}_a$, whose determinant line bundle $\underline{\det}Ind(D^{\pm})$ is a real line bundle with fiber $\Theta^{\pm}_{\lambda_{a,x}}$.

First observe that:

Lemma 5.2. *On each fiber $\mathcal{I}_{a,x}$ of the fiber bundle $\mathcal{I}_a \to (L_a)_{[3]}$, the pullback of the determinant line bundle $\underline{\det}Ind(D^{\pm})$ is trivial.*

Proof. Fix a reference point $[\lambda_{a,x}, \sigma_{a,x}, s_1, s_2] \in \mathcal{I}_{a,x}$. Consider the family of operators $D'^- = \{\bar{\partial}_{\lambda'_{a,x}, Z_-}\}$ parametrized by $[\lambda'_{a,x}, \sigma'_{a,x}, s'_1, s'_2] \in \mathcal{I}_{a,x}$. By gluing D'^- with $\bar{\partial}_{\lambda_{a,x}, Z_+}$, where the latter is a single operator regarded as a constant family, we obtain a family of Dolbeault operators with domain D^2, with boundary conditions given by the family of real sub-bundles parametrized by $S^1 = \partial D^2$, specified by the union of the paths $\lambda_{a,x}$ and $\lambda'_{a,x}$.

Since $[\lambda_{a,x}, \sigma_{a,x}, s_1, s_2]$ and $[\lambda'_{a,x}, \sigma'_{a,x}, s'_1, s'_2]$ determine the spin structures on the family of real sub-bundles consistently, the determinant line bundle of the family of Dolbeault operators is trivial. By definition, this family of Dolbeault operators is obtained from gluing D'^- with a constant family of operators, so the determinant line bundle of the family D'^- is also trivial, which completes the proof.

□

Lemma 5.3. *There exist local systems $\Theta_a^{\pm}$ on L_a, such that their pullbacks to $\mathcal{I}_a$ are isomorphic to $Ind(D^{\pm})$. Moreover, there is an isomorphism*

$$\Theta_a^- \otimes \Theta_a^+ \cong \det TL_a. \tag{5.14}$$

Proof. Recall that $\mathcal{P}_{a,x}$ is the space of smooth paths in $\mathcal{LAG}(T_{\iota(x)}M)$ connecting the Lagrangian subspaces $\lambda_{a,x,l}$ and $\lambda_{a,x,r}$ in (5.4). Hence it is homotopy equivalent to the based loop-space of the Lagrangian Grassmannian $\mathcal{LAG}(n)$ of linear Lagrangian subspaces of $\mathbb{R}^{2n}$. Set $I_{a,x} = \cup_{\lambda_{a,x}}(I_{a,x})_{\lambda_{a,x}}$. The fiber bundle $I_{a,x} \to \mathcal{P}_{a,x}$ is homotopy equivalent to the free loop-space of the Lagrangian Grassmannian $\mathcal{LAG}(n)$. Therefore, $\pi_0(I_{a,x}) = \mathbb{Z}$ so that $I_{a,x}$ has $\mathbb{Z}$-worth of connected components, labeled by $I_{a,x;k}$.

Also recall that $\mathcal{I}_{a,x}$ is a double cover of $I_{a,x}$, which is non-trivial. Let $\mathcal{I}_{a,x;k}$ be the pullback of $I_{a,x;k}$ to $\mathcal{I}_{a,x}$, which is therefore connected as $I_{a,x;k}$ is connected. Let $\mathcal{I}_{a;k}$ be the union of $\mathcal{I}_{a,x;k}$ over $x \in (L_a)_{[3]}$. From the fibration

$$\mathcal{I}_{a,x;k} \hookrightarrow \mathcal{I}_{a;k} \to (L_a)_{[3]},$$

we obtain a long exact sequence of homotopy groups:

$$\pi_1(\mathcal{I}_{a,x;k}) \to \pi_1(\mathcal{I}_{a;k}) \to \pi_1((L_a)_{[3]}) \to \{1\}. \tag{5.15}$$

This implies that $\pi_1(\mathcal{I}_{a;k}) \to \pi_1((L_a)_{[3]})$ is surjective. Now we have a real line bundle $\Theta_a^{\pm}$ on $(L_a)_{[3]}$, which is classified by a homomorphism $\pi_1((L_a)_{[3]}) \to \mathbb{Z}/2$, whose pullback to the homomorphism $\pi_1(\mathcal{I}_{a;k}) \to \mathbb{Z}/2$ defined by the real line bundle $\underline{\det} Ind(D^{\pm})$. The homomorphism $\pi_1(\mathcal{I}_{a;k}) \to \mathbb{Z}/2$ is well-defined because of Lemma 5.2.

Since any line bundle on the 3-skeleton has a unique extension to the whole space, we obtain the desired $\Theta_a^{\pm}$ on L_a. □

Definition 5.5. *The local system Θ_a^- is called the orientation local system for the cleanly self-intersecting component L_a.*

The next lemma explains how the local systems change under different choices of spin structures.

Lemma 5.4. *Let v_1, v_2 be two spin structures on L. Let $\Theta_a^-(v_1), \Theta_a^-(v_2)$ be the orientation local systems defined by v_1 and v_2, respectively. Then the local system*

$$\Theta_a^-(v_1) \otimes \Theta_a^-(v_2)$$

is classified by the $\mathbb{Z}/2$-cohomology class

$$pr_1^*(v_1 - v_2) - pr_2^*(v_1 - v_2) \in H^1(L_a, \mathbb{Z}/2). \tag{5.16}$$

The proof of this lemma also follows from an argument by gluing families of elliptic operators. As we shall not quite use it, we refer the reader to Chapter 8 of [FOOO09b] for the detailed proof.

5.4. **The wrapped Floer cochain space for a cylindrical Lagrangian immersion.** In order to choose a suitable chain model for the wrapped Floer cochain space for an exact cylindrical Lagrangian immersion, we consider the following geometric setup. We take a smooth Hamiltonian $H : M \to \mathbb{R}$ which is zero in the interior part M_0 of the Liouville manifold M. This in particular implies that all the Hamiltonian chords from $\iota(L)$ to itself which are contained in the interior part of M are constant. Over the cylindrical end $\partial M \times [1, +\infty)$, more precisely on a smaller subset $\partial M \times [1+\epsilon, +\infty)$ for some small $\epsilon > 0$, the Hamiltonian H is quadratic, i.e. of the form r^2.

Let $\mathcal{P}(M; \iota(L))$ be the space of paths in M from $\iota(L)$ to itself. Fix a choice of a basepoint x_* in every connected component of $\mathcal{P}(M; \iota(L))$. Let x be a time-one H-chord from $\iota(L)$ to itself (either constant or non-constant), which lies in a connected component of $\mathcal{P}(M; \iota(L))$ where x_* is located. A capping half-disk of w with reference to x_* is a map

$$w : [0,1] \times [0,1] \to M, \tag{5.17}$$

such that $w(s,i) \in \iota(L)$ for $i = 0, 1$, and $w(0,t) = x_*(t), w(1,t) = x(t)$. Now if x is a non-constant time-one H-chord contained in the cylindrical end $\partial M \times [1, +\infty)$, there is a unique homotopy class of capping half-disk for x, since ι is a trivial covering map there and is exact in the usual sense. For such a Hamiltonian chord, it is not necessary to specify the homotopy class of capping half-disks, just as in the case of an embedded exact Lagrangian submanifold.

Because the Hamiltonian H is constantly zero in the interior part, it is natural to introduce a Morse-Bott setup for wrapped Floer theory. For convenience, we shall choose the Morse complex for computing the cohomology of M and that of L, so that there are finitely many generators involved. To define the wrapped Floer cochain space $CW^*(L, \iota; H)$, we choose an auxiliary Morse function f_a on each connected component L_a of the self fiber product $L \times_\iota L$. Let $p_{a,j}$ be the critical points of f_a. The corresponding Morse complex computes the cohomology of the fiber product $L \times_\iota L$. Let $\mathcal{X}_+(\iota(L); H)$ be the set of all non-constant time-one H-chords from $\iota(L)$ (the geometric image) to itself which are contained in the cylindrical end $M \times [1, +\infty)$. These non-constant time-one H-chords naturally correspond to Reeb chords of all times from l to itself on the contact boundary ∂M.

Definition 5.6. *We define the wrapped Floer cochain space $CW^*(L, \iota; H)$ as to be the graded free $\mathbb{Z}$-module generated by the following two kinds of generators*

- *(i) $(p, w) \otimes \theta_p$, where $p \in Crit(f_a)$, w is a Γ-equivalence class of capping half-disks for p, and $\theta_p \in (\Theta_a^-)_p$;*
- *(ii) (x, b), where x is a non-constant time-one H-chord from $\iota(L)$ to itself which is contained in the cylindrical end $\partial M \times [1, +\infty)$, and b is a lifting index, corresponding to a pair (i, j) where i, j label the copies of the preimage of the covering ι when restricted to the cylindrical end.*

That is,

$$CW^*(L,\iota;H) = (\bigoplus_a CM^*(L_a, f_a; \Theta_a^-)) \oplus \bigoplus_{b=(i,j)=(1,1)}^{(d,d)} \mathbb{Z}\mathcal{X}_+(\iota(L);H). \tag{5.18}$$

Here $\bigoplus_b \mathbb{Z}\mathcal{X}_+(\iota(L);H)$ *means the direct sum of several copies of* $\mathbb{Z}\mathcal{X}_+(\iota(L);H)$*, one for each index* $b=(i,j)$*.*

In the definition of the wrapped Floer cochain space, we include several copies of the free module generated by non-constant H-chords, in order to keep track of which component of the preimage the boundary map is lifted to.

In particular, when $\iota : L \to M$ is a proper embedding, we see that this cochain space is isomorphic to the usual Morse-Bott wrapped Floer cochain space.

5.5. **Pearly trees.** The moduli spaces we use to set up wrapped Floer theory for a cylindrical Lagrangian immersion $\iota : L \to M$ combine two geometric configurations: both pseudoholomorphic disks and gradient flow trees. Moreover, there are two different kinds of disks - both inhomogeneous pseudoholomorphic disks and homogeneous pseudoholomorphic disks. To describe elements in the moduli spaces, we introduce the following objects as the underlying domains of "pseudoholomorphic maps" to M.

Definition 5.7. *A colored rooted tree with k-leaves* ($k \geq 0$) *consists of the following data:*

(i) a planar oriented metric ribbon tree (T,V,E,r) *with* $k+1$ *ends, where* V *is the set of vertices,* E *is the set of edges, and* $r : E \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ *is a length function;*

(ii) a decomposition of the set E *of edges into the set of exterior edges* E_{ext} *and the set of interior edges* E_{int}*, such that* E_{ext} *consists of* $k+1$ *semi-infinite edges corresponding to the* $k+1$ *ends: one is called the root* e_0 *and the other* k *are called the leaves* $e_1, \cdots, e_k$*, while the interior edges are finite;*

(iii) a coloring $c : V \to \{0,1\}$ *and a coloring* $d : E_{ext} \to \{0,1\}$*, which satisfy the property that if* $d(e) = 1$ *for an enterior edge* e*, then its endpoint must have color* 1*.*

Note that we should also allow one exceptional case: T has no vertices and only one edge e which is infinite in both directions, with color $d(e) = 0$. This does not quite fit into the definition of a colored rooted tree, but we shall still call it one. This infinite edge e should be thought of as joining the root and one leaf together, so that it also comes with a preferred orientation.

The orientation on T induces an orientation on every interior edge $e \in E_{int}$, so the two endpoints of each edge can be naturally distinguished - one is called the source, denoted by $s(e)$, the other is called the target, denoted by $t(e)$. On the other hand, exterior edge are semi-infinite, and the root e_0 has only the target $t(e_0)$ as its endpoint, while each e_i has only the source $s(e_i)$.

Remark 5.2. *Note that in our definition, we do not require the valency* $val(v)$ *of a vertex* v *to be greater than or equal to* 2*. In fact, the presence of vertices* v *with* $val(v) = 1$ *will be important in the story.*

Definition 5.8. *A colored rooted tree* T *is called admissible, if it is obtained from a colored rooted tree* T_0 *whose vertices all have color* 1 *by attachment of colored* *rooted trees T_j whose vertices all have color 0. These T_j's are attached to T_0 by edges e_j (not the leaves) whose color are 0. Moreover, T_0 and T_j's are subtrees of T.*

For our purpose of setting up wrapped Floer theory, we shall consider only admissible colored rooted trees, and simply call them colored rooted trees by abuse of name.

Given a colored rooted tree with k-leaves as above, we can construct from it a topological space S_T canonically in the following way.

To every vertex v, we assign a punctured disk $S_v = D \setminus \{z_{v,0}, \cdots, z_{v,val(v)-1}\}$, where each puncture $z_{v,j}$ corresponds to an edge adjacent to v.

To every interior edge e, we assign a finite interval I_e of length $r(e)$, joining the two disks (possibly with punctures) associated to $s(e), t(e)$ at the punctures on $S_{s(e)}$ and $S_{t(e)}$ which correspond to e. The length $r(e)$ is allowed to be zero, in which case I_e topologically becomes a point, but we still think of e as an edge combinatorially.

To every exterior edge e with $d(e) = 0$, we assign a semi-infinite interval $I_e = (-\infty, 0]$ if e is the root e_0, or $I_e = [0, +\infty)$ if e is any of the leaves $e_i, i = 1, \cdots, k$. The semi-infinite interval I_e is attached at $\{0\}$ to the corresponding puncture on $S_{s(e)}$ or $S_{t(e)}$.

Finally, to every exterior edge e with $d(e) = 1$, we assign a semi-infinite strip $Z_e = (-\infty, 0] \times [0,1]$ if e is the root e_0, or $Z_e = [0, +\infty) \times [0,1]$ if e is any of the leaves e_i. This semi-infinite strip should be identified with a strip-like end near the corresponding puncture on $S_{s(e)}$ or $S_{t(e)}$.

The topological space S_T is the union of all the above configurations, which are glued together according to the combinatorial data of the tree T.

One special case is when T has only one vertex v one root e_0 and one leaf e_1 with colors $d(e_0) = d(e_1) = 1$. In this case, S_T should be a disk with two boundary punctures, which is identified with $Z = \mathbb{R} \times [0,1]$.

In the exceptional case, i.e. when T has no vertices and only one edge e which is infinite in both directions, we assign $S_T = I_e = \mathbb{R}$.

In order for such geometric objects S_T to have a reasonable moduli problem, we should then equip S_T with an additional structure - a complex structure j_v on each disk component D_v. We briefly denote that by j. We call (S_T, j) or simply S_T a (rooted) pearly tree with k-leaves. However, these pearly trees do not have a moduli space because we do not impose stability conditions at the moment.

Pearly trees will be the underlying domains of the pseudoholomorphic maps in wrapped Floer theory of cylindrical Lagrangian immersions. However, they are not enough, as positive-dimensional families of pseudoholomorphic maps can degenerate to broken pseudoholomorphic maps. To describe those, we introduce broken colored rooted trees as well as broken pearly trees.

We need some terminology when talking about degeneration of colored rooted trees. When the length of an interior edge e of a colored rooted tree T tends to infinity, we obtain a pair of colored rooted trees T_0, T_1, so that e breaks into a new leaf $e_{0,new}$ of T_0 and the root $e_{1,0}$ of T_1. In such a picture, we say that the pair of root and leaf $e_{1,0}, e_{0,new}$ is connected at infinity.

Now we formalize the definition of a broken colored rooted tree.

Definition 5.9. *A broken colored rooted tree is a tuple $(T_0, \cdots, T_m)$, where each T_i is an admissible colored rooted tree, such that it satisfies the following conditions:*

(i) *(rooting) The root* $e_{0,0}$ *of* T_0 *is not connected at infinity to any leaf of any* $T_i, i \neq 0$.
(ii) *(ordering) For each* $j \neq 0$*, there is a unique* $l(j)$ *(which can be* 0*) such that the root* $e_{j,0}$ *of* T_j *is connected at infinity to some (unique) leaf* $e_{l(j),p(j)}$ *of* $T_{l(j)}$.
(iii) *(compatible coloring) The root* $e_{j,0}$ *of* T_j *and the leaf* $e_{l(j),p(j)}$ *of* $T_{l(j)}$ *which are connected at infinity should have the same coloring, i.e.* $d(e_{j,0}) = d(e_{l(j),p(j)})$.

Given a broken colored rooted tree $(T_0, \cdots, T_m)$ as above, as well as m positive real numbers $\rho_1, \cdots, \rho_m$, we may perform a gluing construction as follows. For each $T_j, j \neq 0$ and the corresponding $l(j)$, recall that we have identifications

$$\begin{aligned} e_{j,0} &\cong (-\infty, 0], \\ e_{l(j),p(j)} &\cong [0, +\infty), \end{aligned} \tag{5.19}$$

cut off $(-\infty, -\rho_j/2]$ from $e_{j,0}$ and $[\rho_j/2, +\infty)$ from $e_{l(j),p(j)}$, and glue the remaining intervals at the endpoints $\{-\rho_j/2\} \sim \{\rho_j/2\}$. We may suitably reparametrize the interval so it has a nicer form, but that is not important; the only important information is that the resulting edge has length ρ_j. After doing this process for all $j = 1, \cdots, m$, we obtain a colored rooted tree

$$T = \sharp_{\rho_1, \cdots, \rho_m}(T_0, \cdots, T_m). \tag{5.20}$$

If the resulting colored rooted tree T is admissible, we call this an admissible gluing, and call $(T_0, \cdots, T_m)$ an admissible broken colored rooted tree. From now on we shall only consider admissible broken colored rooted trees, and call them broken colored rooted trees for simplicity. Partial gluings are also allowed, which again give us broken colored rooted trees. It can be defined in a similar way, but the gluing process is only done for a sub-collection of edges connected at infinity. Let $J \subset \{1, \cdots, m\}$ index such a sub-collection, and we denote the result of partial gluing by

$$\sharp_{\rho_j : j \in J}(T_0, \cdots, T_m). \tag{5.21}$$

Similar to the case of a colored rooted tree, we can assign to a broken colored rooted tree a topological space as follows.

Definition 5.10. *A broken pearly tree* $(S_{T_0}, \cdots, S_{T_m})$ *associated to a broken colored rooted tree* $(T_0, \cdots, T_m)$ *is simply the union of pearly trees* S_{T_i} *associated to each component.*

As the underlying broken colored rooted tree $(T_0, \cdots, T_m)$ can be glued root-to-leaf in an admissible way, we can also glue the associated broken pearly tree to get a pearly tree. There are two cases. If $d(e_{j,0}) = d(e_{l(j),p(j)}) = 0$, the 0-th end $\epsilon_{j,0}$ of S_{T_j} is the negative half-ray $(-\infty, 0]$ and the $p(j)$-th end $\epsilon_{l(j),p(j)}$-th is the positive half-ray $[0, +\infty)$. In this case perform the gluing in the same way as we have done for the underlying trees. If $d(e_{j,0}) = d(e_{l(j),p(j)}) = 1$, the 0-th end $\epsilon_{j,0}$ of S_{T_j} is the negative infinite half-strip $(-\infty, 0] \times [0, 1]$ and the $p(j)$-th end $\epsilon_{l(j),p(j)}$-th is the positive infinite half-strip $[0, +\infty) \times [0, 1]$. We cut off $(-\infty, \rho_j/2] \times [0, 1]$ from $\epsilon_{j,0}$ and $[\rho_j/2, +\infty) \times [0, 1]$ from $\epsilon_{l(j),p(j)}$, then glue the resulting finite strips along the boundary intervals $\{-\rho_j/2\} \times [0, 1] \sim \{\rho_j/2\} \times [0, 1]$. Doing this process for all j, we obtain a pearly tree denoted by

$$S_T = \sharp_{\rho_1, \cdots, \rho_m}(S_{T_0}, \cdots, S_{T_m}). \tag{5.22}$$

Also, we can perform partial gluing in a similar way as we have done for broken colored rooted trees.

5.6. **Moduli spaces of stable pearly trees.** The moduli spaces involved in wrapped Floer theory for the Lagrangian immersion $\iota : L \to M$ are analogues and modifications of those used by [BO16] to set up linearized contact homology in Hamiltonian formulation, without circle action and symmetry in our case. In addition, the inhomogeneous pseudoholomorphic disks in the interior part which have asymptotic limits being the constant chords are also in consideration.

Two issues bring up complication in the construction of the moduli spaces. Unlike the case of an embedded exact Lagrangian submanifold, the image $\iota(L)$ in general bounds J-holomorphic disks, and limits of inhomogeneous pseudoholomorphic disks might bubble off homogeneous pseudoholomorphic disks with boundary on $\iota(L)$. These should be suitably packaged into the moduli spaces. On the other hand, the A_∞-structure maps are typically defined by appropriate counts of inhomogeneous pseudoholomorphic disks; in particular, we expect the zeroth order map m^0 of the curved A_∞-algebra structure to be defined by inhomogeneous pseudoholomorphic disks with one puncture. This causes some potential problems, in particular in the verification of A_∞-relations, as the elements in some boundary strata do not satisfy Floer's equations because they arise from disk bubbling.

In order to treat homogeneous and inhomogeneous pseudoholomorphic disks with one puncture in a uniform way, we have chosen our Hamiltonian H to be constantly zero in the interior part of M, so that inhomogeneous pseudoholomorphic disks which are contained in the interior part automatically satisfy the homogeneous Cauchy-Riemann equation. Because of the chain model we pick for the wrapped Floer cochain space, we shall construct a version of Morse-Bott moduli spaces combining pseudoholomorphic disks and gradient flow trees.

The issues can be resolved using the exactness condition for the Lagrangian immersion $\iota : L \to M$. The key properties about the behavior of pseudoholomorphic disks are given by the following lemmas.

Lemma 5.5. *For any compatible almost complex structure J of contact type near the boundary ∂M, all J-holomorphic disks with boundary on $\iota(L)$ are contained in the interior part of M, and have to pass through a self-intersection point of $\iota : L \to M$.*

For any self-intersection point (p_-, p_+) ($p_- \neq p_+$) such that $\iota(p_-) = \iota(p_+) = p$, there are finitely many relative homology classes β of J-holomorphic disks bounded by $\iota(L)$, with one boundary marked point mapped to the point p.

Proof. The first statement follows from the assumption that ι is an embedding over the cylindrical end $\partial M \times [1, +\infty)$ of an exact Lagrangian submanifold of the form $l \times [1, +\infty)$, using a standard argument by the maximum principle.

The second statement follows from the exactness condition. For any such a J-holomorphic disk u, the exactness condition $df = \iota^* \lambda_M$ implies that its energy is fixed:

$$E(u) = f(p_-) - f(p_+),$$

by integration by parts. On the other hand, the energy is also equal to $\omega(\beta)$, which implies that there can only be finitely many such homology classes β.

□

The next three lemmas all follow from maximum principle.

Lemma 5.6. *Let $u : D \to M$ be a J-holomorphic disk with boundary on $\iota(L)$. Then $u(D)$ cannot be entirely contained in the cylindrical end $\partial M \times [1, +\infty)$.*

Lemma 5.7. *Let $u : S \to M$ be a J-holomorphic curve with boundary on $\iota(L)$, for any smooth Riemann surface S with boundary $\partial S \neq \varnothing$. If $u(\partial D)$ is contained in a compact subset of the interior part of M away from ∂M, then the image of the entire disk $u(D)$ must be contained in the interior part of M away from ∂M.*

Lemma 5.8. *Let $u : S \to M$ be a smooth map from a one-punctured disk to M with boundary on $\iota(L)$, which satisfies the Floer's equation*

$$(du - \gamma \otimes X_H)^{0,1} = 0. \tag{5.23}$$

If u asymptotically converges to either a constant H-chord, or a self-intersection point of ι at the puncture, then $u(S)$ must be entirely contained in the interior part of M, where the Hamiltonian H vanishes. In particular, u has a natural smooth extension to the closed disk D, and satisfies the J-holomorphic curve equation

$$(du)^{0,1} = 0 \tag{5.24}$$

in the interior of D.

Now let us proceed to describe elements in the moduli spaces. To write down the inhomogeneous Cauchy-Riemann equations, we shall need certain geometric data, e.g. Morse functions, Hamiltonians, and almost complex structures. Fix the original Hamiltonian H, and let $J = J_{L,\iota}$ be a compatible almost complex structure of contact type. The various geometric data needed for writing down the relevant equations are packaged in the following way.

Definition 5.11. *A Floer datum on a pearly tree S_T consists of the following data:*

- *(i) A time-shifting function $\rho_{S_T} : \partial S_T \to [1, +\infty)$, where ∂S_T denotes the union of boundary components of the disk components of S_T, as well as the intervals. ρ_{S_T} should be equal to a constant over the strip-like end near each puncture.*
- *(ii) For each vertex v with $c(v) = 0$, a constant family of almost complex structures $J_{S_v} = J$.*
- *(iii) For each vertex v with $c(v) = 1$ and $val(v) = 1$, a constant family of Hamiltonians $H_{S_v} = H$ and a constant family of almost complex structures $J_{S_v} = J$.*
- *(iv) For each vertex v with $c(v) = 1$ and $val(v) = 2$, a time-dependent family of almost complex structures $J_{v,t}$, rescaled by weight $w_{v,0} = w_{v,1}$. This is a time-dependent perturbation of J in the class of almost complex structures of contact type. Moreover, we require that the choices $J_{v,t} = J_t$ be the same for all such vertices v, but possibly rescaled by differente weights according to the values of ρ_{S_T}.*
- *(v) For each vertex v with $c(v) = 1$ and $val(v) \geq 3$, a domain-dependent family of Hamiltonians H_{S_v} and a domain-dependent family of almost complex structures J_{S_v}, such that H_{S_v} and J_{S_v} asymptotically agrees with H and respectively J_t rescaled by weight $w_{v,j}$, over the strip-like end near each puncture $z_{v,j}$. Moreover, we require that the family of Hamiltonians H_{S_v} be a compactly-supported domain-dependent perturbation of H, i.e. $H_{S_v} = H$ in a neighborhood of the boundary ∂S_v.*

(vi) For each interior edge e, an s-dependent family of Morse functions $f_{e,s}$ on I_e.

(vii) For each exterior edge e with $d(e) = 0$, a family of Morse functions $f_{e,s}$ on L parametrized by $s \in I_e$, which agrees with f for $|s| \gg 0$.

(viii) For each exterior edge e with $d(e) = 1$, a family of time-dependent Hamiltonians $H_{e,s,t} = H_{e,t}$, and a family of time-dependent almost complex structures $J_{e,s,t} = J_{e,t}$ which agree with H and respectively J for $|s| \gg 0$. Moreover, as the semi-infinite strips are glued to the punctured disk $S_{s(e)}$ or $S_{t(e)}$, we require that these data extend smoothly over the glued domain.

In the exceptional case where T does not have vertices and has a single infinite edge e of color $d(e) = 0$, so that $I_e = \mathbb{R}$, we require $f_{e,s} = f$ for all s.

In order to define the A_∞-structures, a necessary condition is to make sure that the Floer data chosen for various on various pearly trees satisfy certain consistency conditions.

The simplest consistency condition to state is when two colored rooted trees T_1, T_2 are glued together root-to-leaf in a way that the resulting colored rooted tree $T = T_1 \sharp_{0,i,\rho} T_2$ is still admissible. The consistency condition means that the Floer datum on S_T is obtained from gluing the Floer data on S_{T_1} and S_{T_2}. There are higher consistency conditions, which we refer the reader to [Sei08] for a detailed description.

Remark 5.3. *Such consistency conditions make sense because the Floer datum is designed so that perturbations are not put on the punctured disks S_v for vertices v with $c(v) = 0$, that is, the family of almost complex structure J_{S_v} on such a component is constant equal to J.*

Furthermore, a conformally consistent choice of Floer data is not for the purpose of achieving transversality, even the choices are "generic" for all the components that allow non-constant families of perturbations of Hamiltonians and almost complex structures. Such a conformally consistent choice is made mainly to ensure that the moduli spaces of pearly trees have good compactifications so that the boundary strata of the compactifications are products of moduli spaces of the same type, as we shall see below.

Having made a conformally consistent choice of Floer data as above, we now describe elements in the moduli spaces.

Definition 5.12. *Let $I \subset \{0, \cdots, k\}$ be a subset. Let $\alpha : I \to S(L, \iota)$ be a map, labeling those marked points which are mapped to some self-intersection point of ι. A stable pearly tree map is a triple (S_T, u, l) satisfying the following conditions:*

(i) S_T is a pearly tree modeled on a colored rooted tree T with k-leaves.

(ii) $u : S_T \to M$ is a continuous map.

(iii) For a vertex v with $c(v) = 0$, let u_v be the restriction of u to the punctured disk S_v associated to v. Then u_v satisfies the homogeneous Cauchy-Riemann equation

$$(du_v)^{0,1} = 0, \tag{5.25}$$

with respect to the family J_{S_v} of almost complex structures. In case $val(v) = 2$, $J_{S_v} = J_t$.

(iv) *For a vertex v with $c(v) = 1$, let u_v be the restriction of u to the punctured disk S_v associated to v. Then u_v satisfies the inhomogeneous Cauchy-Riemann equation*

$$(du_v - \gamma_v \otimes X_{H_{S_v}})^{0,1} = 0. \tag{5.26}$$

(v) *For an interior edge e, let u_e be the restriction of u to the interval I_e associated to e. Then u_e comes with a preferred lift $\tilde{u}_e : I_e \to L$ and satisfies the gradient flow equation:*

$$\frac{d\tilde{u}_e}{ds} + \nabla f_{e,s}(\tilde{u}_e) = 0. \tag{5.27}$$

In the exceptional case where $S_T = \mathbb{R}$, this gradient flow equation takes the form

$$\frac{d\tilde{u}_e}{ds} + \nabla f(\tilde{u}_e) = 0, \tag{5.28}$$

by our choice of Floer datum.

(vi) *$u(z) \in \psi_M^{\rho_{S_T}(z)} \iota(L)$, for $z \in \partial S_T$.*

(vii) *$l : \partial S_T \to L$ is a continuous map, which specifies the boundary lifting condition, i.e. $\iota \circ l = u|_{\partial S_T}$.*

(viii) *If e is an exterior edge such that I_e is either half-infinite or $\mathbb{R}$, then $\lim_{s\to\pm\infty} \tilde{u}_e(s, \cdot) = p$ for some critical point p of f.*

(ix) *If e_i is the i-th exterior edge with color $d(e_i) = 1$, $\lim_{s\to\pm\infty} u_{v(e_i)} \circ \epsilon_i(s, \cdot) = \psi_M^{w_i} x_i(\cdot)$, where $v(e_i)$ is the endpoint of e, and ϵ_i is the strip-like end associated to e_i. Here x_i is some non-constant time-one H-chord from $\iota(L)$ to itself.*

(x) *$(\lim_{\theta\uparrow 0} l(e^{\sqrt{-1}\theta}\zeta_i), \lim_{\theta\downarrow 0} l(e^{\sqrt{-1}\theta}\zeta_i)) = \alpha(i) \in S(L, \iota)$ for $i \in I$. In addition, $\iota(\alpha(i)) = \tilde{u}_{e_i}(z_{v(e_i),j(v(e_i))})$. Here $\tilde{u}_{e_i}(z_{v(e_i),j(v(e_i))})$ means the asymptotic value of $\tilde{u}_{e_i}$ at the end of the half-infinite ray associated to the i-th exterior edge e_i, where $z_{v(e_i),j(v(e_i))}$ is the puncture on the punctured disk $S_{v(e_i)}$ that corresponds to the exterior edge e_i.*

(xi) *If $d(e) = 0$, then the preferred lift $\tilde{u}_e$ of u_e should be compatible with the restriction of l to I_e, such that the above condition hold. That is, if $\tilde{u}_e$ is a constant map to a component of $S(L, \iota)$, i.e. a discrete point, then the restriction of l to I_e should satisfy the same limiting condition in (x).*

(xii) *The homology class of u together with its asymptotic non-constant Hamiltonian chords oppositely oriented, is $\beta \in H_2(M, \iota(L); \mathbb{Z})$.*

The last condition in the definition of a stable pearly tree map needs some explanation. Since some of the asymptotic convergence conditions are non-constant Hamiltonian chords, which might be non-contractible relative $\iota(L)$, the map u does not define a homology class in $H_2(M, \iota(L); \mathbb{Z})$. However, if we compactify S_T at infinity by adding $\{\pm\infty\} \times [0, 1]$ to all exterior strip-like ends, and glue the oppositely oriented Hamiltonian chords to the map u, then the resulting map defines a homology class in $H_2(M, \iota(L); \mathbb{Z})$.

There is an obvious notion of two triples (S_T, u, l) and $(S'_{T'}, u', l')$ being isomorphic. First of all, there should be an isomorphism of rooted colored trees $\bar{\phi} : T \to T'$, an isomorphism $\phi : S_T \to S'_{T'}$ compatible with $\bar{\phi}$, such that $u' \circ \phi = u$, and the pullback of the Floer datum chosen for $S'_{T'}$ by ϕ agrees with that for S_T.

Denote by c either a pair (p, w), where p is a critical point of f and w is a homotopy class of capping half-disk, or a pair (x, w), where x is a non-constant H-chord, and w is a homotopy class of capping half-disk with reference to some chosen based point in the connected component of x in the space $\mathcal{P}(M; \iota(L))$ of paths in M from $\iota(L)$ to itself. Let $\mathcal{M}_{k+1}(\alpha, \beta; J, H; c_0, \cdots, c_k)$ be the moduli space of the above triples (S_T, u, l) which satisfy the convergence conditions to $C_0, \cdots, c_k$ at the root e_0 and leaves $e_1, \cdots, e_k$. Clearly, there is a decomposition according to the combinatorial type of T:

$$\mathcal{M}_{k+1}(\alpha, \beta; J, H; c_0, \cdots, c_k) = \coprod_T \mathcal{M}_T(\alpha, \beta; J, H; c_0, \cdots, c_k). \tag{5.29}$$

This description is only for the purpose of visualizing elements in the moduli space, but not for the purpose of proving transversality results by induction on combinatorial types.

5.7. **Compactification: stable broken pearly trees.** The moduli space of stable pearly trees $\mathcal{M}_{k+1}(\alpha, \beta; J, H; c_0, \cdots, c_k)$ is generally non-compact, from which one cannot expect to extract invariants. In order to define the desired curved A_∞-algebra whose structure constants are given by appropriate counts of elements in the moduli space, we need to compactify it.

Lemma 5.9. *The moduli space $\mathcal{M}_{k+1}(\alpha, \beta; J, H; c_0, \cdots, c_k)$ has a natural compactification $\bar{\mathcal{M}}_{k+1}(\alpha, \beta; J, H; c_0, \cdots, c_k)$, which has a natural topology being compact and Hausdorff.*

Lemma 5.10. *The compactification $\bar{\mathcal{M}}_{k+1}(\alpha, \beta; J, H; c_0, \cdots, c_k)$ consists of products of moduli spaces $\bar{\mathcal{M}}_{k_i+1}(\alpha_i, \beta_i; J, H; c'_{i,0}, \cdots, c'_{i,k_i})$ of the same type. More precisely, the codimension-m strata*

$$S_m \bar{\mathcal{M}}_{k+1}(\alpha, \beta; J, H; c_0, \cdots, c_k),$$

i.e. those points of codimension exactly m consist of the union of fiber products of the form

(5.30)

$$\bar{\mathcal{M}}_{k_0+1}(\alpha_0, \beta_0; J, H; c_{0,0}, \cdots, c_{0,k_0}) \times \cdots \times \bar{\mathcal{M}}_{k_m+1}(\alpha_m, \beta_m; J, H; c_{m,0}, \cdots, c_{m,k_m}),$$

where $c_j = c_{i_j,k_j}$ for some i_j, k_j, for every $j = 0, \cdots, k$. Here fiber products are taken over a discrete set of generators, so they are written as products. In fact, by the inductive construction, for any element in the codimension-m strata, the underlying domain is a broken pearly tree $(S_{T_0}, \cdots, S_{T_m})$ consisting of exactly $m+1$ pearly trees. Denote by u_j the restriction of the stable map on S_{T_j}. For each $j \neq 0$, there is a unique $l(j)$ for which u_j converges over the 0-th end $\epsilon_{j,0}$ (corresponding to the root) of S_{T_j} to the same generator as $u_{l(j)}$ converges over the end $\epsilon_{l(j),p(j)}$ (corresponding to some leaf) of $S_{T_{l(j)}}$. That is to say, $c_{j,0} = c_{l(j),p(j)}$ for some $p(j)$.

Sketch of proof of Lemma 5.9 and Lemma 5.10. The definition of the compactification is given in the statement of Lemma 5.10, which is valid because of the Gromov compactness theorem. This inductive fiber product structure makes sense, because the various factors satisfy an induction condition with respect to (E, k) in lexicographic order, where E the energy of the stable maps from broken pearly trees, and k is the number of leaves. The fact that $\bar{\mathcal{M}}_{k+1}(\alpha, \beta; J, H; c_0, \cdots, c_k)$ is compact follows from the standard maximum principle and Gromov compactness theorem, as explained below.

The proof of the rest is basically standard, so we shall simply explain the crucial reason. Fixing the homology class β provides a priori energy bound for all elements in the moduli space $\bar{\mathcal{M}}_{k+1}(\alpha,\beta;J,H;c_0,\cdots,c_k)$, so that there are only finitely many types of product moduli spaces that appear in the boundary strata of the compactfication. Thus by the maximum principle, there is a compact subset C of M depending on k,α,β,J,H and the asymptotic convergence conditions $c_0,\cdots,c_k$ but not on individual maps in the moduli space, so that any element in $\bar{\mathcal{M}}_{k+1}(\alpha,\beta;J,H;c_0,\cdots,c_k)$ is represented by a stable map whose image is contained in C. We may take C to be some sub-level set $\{r \leq A\}$ by possibly enlarging the subset. Then Gromov compactness theorem applies.

Hausdorff-ness follows from the stability condition. For a detailed proof of Hausdorff-ness, we refer the reader to [FOOO09b].

□

Note that $\bar{\mathcal{M}}_{k+1}(\alpha,\beta;J,H;c_0,\cdots,c_k)$ naturally compactifies the Gromov bordification of the moduli space of inhomogeneous pseudoholomorphic disks with boundary on $\iota(L)$, encoding additional data that specify the boundary lifts to the preimage of the immersion, as introduced in [AJ10]. Here by the Gromov bordification, we mean the moduli space of broken stable inhomogeneous pseudoholomorphic disks as those in [Abo10] (see also [Sei08] in the unwrapped setting), which in our case is not compact because the limit of a sequence of a broken stable inhomogeneous pseudoholomorphic disk can bubble off homogeneous pseudoholomorphic disks with one marked points (such disks necessarily pass through some self-intersection point of $\iota : L \to M$. In words, the above compactification is obtained by adding all such disks.

Using the techniques from [FOOO09a], [FOOO09b], [FOOO12], we may construct Kuranishi structures on these compactified moduli spaces $\bar{\mathcal{M}}_{k+1}(\alpha,\beta;J,H;c_0,\cdots,c_k)$ in a coherent way. An important geometric assumption for us is that the ambient symplectic manifold M is exact, which allows us to take single-valued multisections on the Kuranishi spaces $\bar{\mathcal{M}}_{k+1}(\alpha,\beta;J,H;c_0,\cdots,c_k)$, because in this case the isotropy group of every Kuranishi chart is trivial. This is the simplest instance of the more general result of [FOOO13]. We may choose these single-valued multisections $s_{\alpha,\beta;J,H;c_0,\cdots,c_k}$ in a coherent way, which are compatible with at the boundary strata (5.30) with fiber product multisections.

5.8. **The curved A_∞-algebra associated to an exact cylindrical Lagrangian immersion.** Based on the discussion presented above, we may extract from the moduli spaces $\bar{\mathcal{M}}_{k+1}(\alpha,\beta;J,H;c_0,\cdots,c_k)$ a structure of a curve A_∞-algebra on the underlying wrapped Floer cochain space $CW^*(L,\iota;H)$.

Proposition 5.1. *A coherent choice of single-valued multisections on $\bar{\mathcal{M}}_{k+1}(\alpha,\beta;J,H;c_0,\cdots,c_k)$ defines a structure of a curved A_∞-algebra on $CW^*(L,\iota;H)$. Moreover, this curved A_∞-algebra is independent of the choice up to homotopy.*

Sketch of proof. We sketch the main steps of the proof while referring the reader to techniques developed in [FOOO09a], [FOOO09b]. The discussion in the previous subsections provides coherent system of Kuranishi structures on all the moduli spaces $\bar{\mathcal{M}}_{k+1}(\alpha,\beta;J,H;c_0,\cdots,c_k)$. Make a coherent choice of single-valued multisections on these Kuranishi spaces, which are compatible at the boundary strata with fiber product multisections.

To define the structure maps of the curved A_∞-algebra, we shall consider only the rigid cases, which means that we only consider generators c_i of appropriate degrees such that the virtual dimension of the moduli space $\bar{\mathcal{M}}_{k+1}(\alpha, \beta; J, H; c_0, \cdots, c_k)$ is zero. The zero-sets of the chosen single-valued multisections define an integral virtual fundamental chain, which therefore gives us an integer number

$$a_{\alpha,\beta;J,H;c_0,\cdots,c_k} \in \mathbb{Z}.$$

Then we set

$$m^k(c_k, \cdots, c_1) = \sum_{\substack{\alpha,\beta,c_0 \\ \deg(c_0)-\deg(c_1)-\cdots-\deg(c_k)=2-k}} a_{\alpha,\beta;J,H;c_0,\cdots,c_k} c_0. \tag{5.31}$$

To make sense of the formula (5.31), we must show that this sum is in fact finite. If $k = 0$, the zeroth order map

$$m^0(1) = \sum_{\substack{\alpha,beta,c_0 \\ \deg(c)=2}} a_{\alpha,\beta;J,H;c_0} c_0$$

"counts" inhomogeneous pseudoholomorphic disks with one marked point, which necessarily pass through some self-intersection point. That is, c_0 is of the form (p, w). Then, Lemma 5.5 implies that there can only be finitely many such homology classes β appearing so that the moduli space $\bar{\mathcal{M}}_1(\alpha, \beta; J, H; c_0)$ is non-empty. For $k \geq 1$, we can again use the action-energy identity, which holds because the Lagrangian immersions $\iota : L \to M$ is exact, to show that if the inputs $c_1, \cdots, c_k$ are fixed, there are only finitely many possible c_0 and finitely many possible homology classes β, for which the moduli space $\bar{\mathcal{M}}_{k+1}(\alpha, \beta; J, H; c_0, \cdots, c_k)$ is non-empty. This implies that the sum (5.31) is a finite sum. Thus, it gives rise to a well-defined multilinear map

$$m^k : CW^*(L, \iota; H)^{\otimes k} \to CW^*(L, \iota; H).$$

To verify that these maps satisfy the A_∞-equations, we need to study boundary strata of moduli spaces $\bar{\mathcal{M}}_{k+1}(\alpha, \beta; J, H; c_0, \cdots, c_k)$ of virtual dimension one. Recall that the multisection $s_{\alpha,\beta;J,H;c_0,\cdots,c_k}$ is isomorphic to the fiber product multisections (**??**) on (5.30). In particular, if the virtual dimension of $\bar{\mathcal{M}}_{k+1}(\alpha, \beta; J, H; c_0, \cdots, c_k)$ is one, the only boundary strata that we have to consider are of codimension one, and have the form

$$\bar{\mathcal{M}}_{k_0+1}(\alpha_0, \beta_0; J, H; c_{0,0}, \cdots, c_{0,k_0}) \times \bar{\mathcal{M}}_{k_1+1}(\alpha_1, \beta_1; J, H; c_{1,0}, \cdots, c_{1,k_1}). \tag{5.32}$$

The numbering of these generators is as follows. The root $e_{1,0}$ of a stable pearly tree map in $\bar{\mathcal{M}}_{k_1+1}(\alpha_1, \beta_1; J, H; c_{1,0}, \cdots, c_{1,k_1})$ is connected to some leaf $e_{0,l(1)}$ of a stable pearly tree map in $\bar{\mathcal{M}}_{k_0+1}(\alpha_0, \beta_0; J, H; c_{0,0}, \cdots, c_{0,k_0})$. Thus $c_{1,0} = c_{0,l(1)} = c_{new}$, while the other generators agree with the original ones:

$$c_{0,i} = c_i, i = 0, \cdots, l(1), \tag{5.33}$$

$$c_{1,j} = c_{j+l(1)-1}, j = 1, \cdots, k_1, \tag{5.34}$$

$$c_{0,i} = c_{i+k_1-1}, i = l(1)+1, \cdots, k_0. \tag{5.35}$$

So we rewrite the fiber product as

$$\begin{aligned} &\bar{\mathcal{M}}_{k_0+1}(\alpha_0, \beta_0; J, H; c_0, \cdots, c_{l(1)-1}, c_{new}, c_{l(1)+k_1}, \cdots, c_{k_0+k_1-1}) \\ &\times \bar{\mathcal{M}}_{k_1+1}(\alpha_1, \beta_1; J, H; c_{new}, c_{l(1)}, \cdots, c_{l(1)+k_1-1}). \end{aligned} \tag{5.36}$$

Thus we have isomorphisms of multisections:

(5.37)

$$\begin{aligned} & s_{\alpha,\beta;J,H;c_0,\cdots,c_k} \\ & \cong s_{\alpha_0,\beta_0;J,H;c_0,\cdots,c_{l(1)-1},c_{new},c_{l(1)+k_1},\cdots,c_{k_0+k_1-1}} \times s_{\alpha_1,\beta_1;J,H;c_{new},c_{l(1)},\cdots,c_{l(1)+k_1-1}}, \end{aligned}$$

when restricting the multisection $s_{\alpha,\beta;J,H;c_0,\cdots,c_k}$ to the boundary strata. This implies that the operations m^k defined in (5.31) satisfy the A_∞-equations.

The independence of choices of multisections up to homotopy is a consequence of general theory of Kuranishi structures. To prove that the curved A_∞-algebra is independent of choice of almost complex structure J up to homotopy, we may introduce parametrized moduli spaces of pearly tree maps with time-allocation, associated to such a homotopy $\{J_t\}_{t\in[0,1]}$, and construct Kuranishi structures on these moduli spaces in a well-arranged way so that they define cobordisms between the Kuranishi structures on the moduli spaces of pearly tree maps $\bar{\mathcal{M}}_{k+1}(\alpha,\beta;J_i,H;c_0,\cdots,c_k), i = 0,1$ with respect to different almost complex structures. Here recall that the generators $c_0,\cdots,c_k$ themselves do not depend on the almost complex structures used to define pearly tree maps, so both kinds of moduli spaces make sense.

□

The way of obtaining a cohomology group from the curved A_∞-algebra $(CW^*(L,\iota;H),m^k)$ is to deform the operations m^k algebraically to kill the curvature. This can be achieved by finding solutions to the inhomogeneous Maurer-Cartan equation:

$$\sum_{k=0}^{\infty} m^k(b,\cdots,b) = 0. \tag{5.38}$$

As we are dealing with ordinary (curved) A_∞-algebras over $\mathbb{Z}$, in order for this equation to make sense, we shall further impose the condition that b is nilpotent, so that the sum stops at a final stage: there exists K such that $m^k(b,\cdots,b) = 0$ for all $k > K$ and

$$\sum_{k=0}^{K} m^k(b,\cdots,b) = 0.$$

Because of the presence of the inhomogeneous term $m^0(1)$, this equation might not have any solution in general. But if it does have one solution $b \in CW^*(L,\iota;H)$ which is nilpotent, we call b a bounding cochain for (L,ι), and say (L,ι) is unobstructed in the sense of wrapped Floer theory. In this case, we can deform the curved A_∞-algebra to an A_∞-algebra with vanishing curvature:

$$m^{k;b}(c_k,\cdots,c_1) = \sum_{\substack{i\geq 0 \\ i_0+\cdots+i_k=i}} m^{k+i}(\underbrace{b,\cdots,b}_{i_k \text{ times}},c_k,\underbrace{b,\cdots,b}_{i_{k-1} \text{ times}},\cdots,c_1,\underbrace{b,\cdots,b}_{i_0 \text{ times}}). \tag{5.39}$$

In particular, $m^{1;b}$ squares to zero, and we can define a cohomology group of $CW^*(L,\iota;H)$ with respect to the differential $m^{1;b}$, which we denote by $HW^*(L,\iota,b;H)$. We call this the wrapped Floer cohomology group of the Lagrangian immersion $\iota : L \to M$ with respect to the bounding cochain b.

5.9. **Wrapped Floer cochain space of a pair of Lagrangian immersions.** Now consider a pair $(\iota_0 : L_0 \to M, \iota_1 : L_1 \to M)$ of exact cylindrical proper Lagrangian immersions with clean self-intersections.

Definition 5.13. *The pair* $(\iota_0 : L_0 \to M, \iota_1 : L_1 \to M)$ *is said to have clean intersections, if the following conditions are satisfied:*

(i) the fiber product

$$L_0 \times_{\iota_0, \iota_1} L_1$$

is a smooth manifold, possibly disconnected with different components having different dimensions,

$$L_0 \times_{\iota_0, \iota_1} L_1 = \coprod_a C_a. \tag{5.40}$$

(ii) the tangent space of the fiber product at each point is given by

$$T_{(p_0,p_1)}(L_0 \times_{\iota_0,\iota_1} L_1) = \{(V_0, V_1) \in T_{p_0}L_0 \times T_{p_1}L_1 : d_{p_0}\iota_0(V_0) = d_{p_1}\iota_1(V_1)\}. \tag{5.41}$$

In wrapped Floer theory, it is important to keep track of the geometry of the cylindrical ends. Recall that for $i = 0, 1$, $\iota_i : L_i \to M$ is assumed to be a discrete trivial covering of a cylindrical end $l_i \times [1, +\infty)$ over the cylindrical end $\partial M \times [1, +\infty)$ modeled on some Legendrian submanifold l_i of ∂M. And by the properness assumption, the covering is finitely-sheeted, say d_i-fold. We shall make the following assumption on how this pair intersects, distinguishing from the case of a single Lagrangian immersion.

Assumption 5.2. $\iota_0 : L_0 \to M$ *and* $\iota_1 : L_1 \to M$ *do not intersect in the cylindrical end* $\partial M \times [1, +\infty)$*. That is, the Legendrian boundaries* l_0, l_1 *are disjoint in the contact boundary* ∂M*.*

This is a generic assumption on Legendrian submanifolds, in case $l_0 \neq l_1$. The case $l_0 = l_1$ but $\iota_0 \neq \iota_1$ is different and slightly more involved, but can be studied in a similar way.

To set up wrapped Floer theory for such a pair of Lagrangian immersions, we need to choose a chain model for the wrapped Floer cochain space. We shall for each component C_a an auxiliary Morse function $f_a : C_a \to \mathbb{R}$, which is C^2-small and satisfies the Morse-Smale condition. Let $Crit(f_a)$ be the set of critical points of f_a. The specific choice will be made below.

Lemma 5.11. *There exists a* C^2*-small generic perturbation* K *of* H *so that all the time-one* K*-chords that are contained in the interior part of* M *are non-degenerate and constant. Moreover, these* K*-chords correspond bijectively to the critical points of the lift of* K *to* C_a*.*

Thus it is natural to choose f_a to be the lift of K to C_a.

As in the case of a single cylindrical Lagrangian immersion with clean self-intersections, we have rank-one $\mathbb{Z}/2$-local systems $\Theta_a^{\pm}$ on C_a. They satisfy $\Theta_a^- \otimes \Theta_a^+ \cong TC_a$. We call Θ_a^- the orientation local system on C_a.

Let $\mathcal{X}_+(\iota_0(L_0), \iota_1(L_1); H)$ be the set of non-constant time-one H-chords from $\iota_0(L_0)$ to $\iota_1(L_1)$. These H-chords are contained in the cylindrical end $\partial M \times [1, +\infty)$ and naturally correspond to Reeb chords on the contact manifold ∂M from l_0 to l_1 of all lengths.

Definition 5.14. *The wrapped Floer cochain space* $CW^*((L_0, \iota_0), (L_1, \iota_1); H)$ *for a pair of exact cylindrical Lagrangian immersions with clean intersections is the free* $\mathbb{Z}$*-module generated by the following two kinds of generators:*

(1) $(p, w) \otimes \theta_p$, *where* $p \in Crit(f_a)$, w *is a* Γ*-equivalence class of capping half-disks for* p, *and* $\theta_p \in (\Theta_a^-)_p$;

(2) (x, b), *where* x *is a non-constant time-one* H*-chord from* $\iota_0(L_0)$ *to* $\iota_1(L_1)$ *which is contained in the cylindrical end* $\partial M \times [1, +\infty)$, *and* b *is a lifting index, corresponding to a pair* (i, j) *where* i *labels the copy of the preimage of the covering* ι_0, *and* j *labels the copy of the preimage of the cover* ι_1, *when restricted to the cylindrical end.*

That is,

(5.42)

$$CW^*((L_0, \iota_0), (L_1, \iota_1); H) = \bigoplus_a CM^*(C_a, f_a; \Theta_a^-) \oplus \bigoplus_{b=(i,j)=(1,1)}^{(d_0,d_1)} \mathbb{Z}\mathcal{X}_+(\iota_0(L_0), \iota_1(L_1); H).$$

5.10. **Moduli spaces of Floer trajectories.** To study wrapped Floer theory for a pair, we consider moduli spaces of stable broken Floer trajectories. In our situation, because the Lagrangian submanifolds are immersed, we also need to add some refinement of the data to the moduli spaces, which are similar to those for a single Lagrangian immersion.

Choose for each Lagrangian immersion $\iota_i : L_i \to M$ an admissible almost complex structure $J_i, i = 0, 1$ of contact type, using which the curved A_∞-algebras $(CW^*(L_i, \iota_i; H), m^k)$ are defined. Also, choose a path of admissible almost complex structures J_t of contact type connecting J_0 and J_1. We shall consider the moduli spaces of the following kinds of maps.

Let $I_0 \subset \{1, \cdots, k\}$ and $I_1 \subset \{1, \cdots, l\}$, and $\alpha_i : I_i \to L_i \otimes_{\iota_i} L_i \setminus \Delta_{L_i}$ be maps, labeling those marked points which are mapped to some switching components of ι_0 and ι_1 respectively. Also, let $\beta \in H_2(M, \iota_0(L_0) \cup \iota_1(L_1))$ be a relative homology class.

Definition 5.15. *A* (k, l)*-marked Floer trajectory is a quadruple* $(\Sigma, \vec{s}, u, \vec{l})$ *satisfying the following conditions:*

(i) $\Sigma = \mathbb{R} \times [0, 1]$ *is the infinite strip.*

(ii) $\vec{s} = (\vec{s}^0, \vec{s}^1)$ *with* $\vec{s}^0 = (s_1^0, \cdots, s_k^0)$ *and* $\vec{s}^1 = (s_1^1, \cdots, s_l^1)$ *are collections of real numbers, such that*

$$s_j^0 > s_{j+1}^0, j = 1, \cdots, k-1,$$

$$s_j^1 < s_{j+1}^1, j = 1, \cdots, l-1.$$

(iii) $u : \Sigma \to M$ *is a continuous map.*

(iv) u *satisfies the inhomogeneous Cauchy-Riemann equation (Floer's equation)*

$$\frac{\partial u}{\partial s} + J_t(\frac{\partial u}{\partial t} - X_H(u)) = 0.$$

(v) $u(s, 0) \in \iota_0(L_0), u(s, 1) \in \iota_1(L_1)$, *for all* $s \in \mathbb{R}$.

(vi) u *asymptotically converges to time-one* H*-chords* x_-, x_+ *from* $\iota_0(L_0)$ *to* $\iota_1(L_1)$ *as* $s \to -\infty$ *and respectively* $+\infty$, *where these* H*-chords might be constant or non-constant.*

(vii) The homology class of u *is* $\beta \in H_2(M, \iota_0(L_0) \cup \iota_1(L_1))$.

(viii) $\vec{l} = (l_0, l_1)$ *is a pair of smooth maps* $l_i : \mathbb{R} \times \{i\} \to L_i \times_{\iota_i} L_i$, *such that* $u(s, i) = \iota_i \circ l_i(s)$, *for* $s \in \mathbb{R} \times \{i\} \setminus \{s_j^i | j \in I_i\}$. $\vec{l}$ *is the boundary lifting condition of* u.

(ix) $(\lim_{s\uparrow s^i_j} l_i(s), \lim_{s\downarrow s^i_j} l_i(s)) = \alpha_i(j)$*, for every* $j \in I_i$.

(x) $(\Sigma, \vec{s}, u, l)$ *is stable.*

There is an obvious $\mathbb{R}$-action by translations on the set of all marked Floer trajectories $(\Sigma, \vec{s}, u, l)$. We denote by

$$\mathcal{N}_{k,l}((L_0, \iota_0), (L_1, \iota_1); \vec{\alpha}, \beta; \{J_t\}_t, H; c_-, c_+)$$

the set of equivalence classes of such $(\Sigma, \vec{s}, u, l)$. For convenience, we sometimes omit the notations $(L_0, \iota_0), (L_1, \iota_1)$ in case no confusion can occur.

The above moduli space has a natural compactification, called the moduli space of stable broken Floer trajectories, to be described in two steps: first, we need to add trees of disk bubbles to each boundary of a Floer trajectory; second, we need to add broken Floer trajectories.

First consider stable unbroken Floer trajectories, which are marked Floer trajectories with stable pearly trees attached to the two boundary components of the strip $\mathbb{R} \times [0, 1]$.

In order to describe these elements in detail, we add more data to marked Floer trajectories. Let $(\Sigma, \vec{s}, u, l)$ be a marked Floer trajectory.

Definition 5.16. *A decoration for* $(\Sigma, \vec{s}, u, l)$ *is an assignment of coloring* $c : \vec{s} \to \{0, 1\}$.

For each s^i_j with color $c(s^i_j) = 0$, we attach a half-infinite ray $I_{s^i_j} = [0, +\infty)$ to Σ at s^i_j. For each s^i_j with color $c(s^i_j) = 1$, we remove s^i_j and add a half-infinite strip $Z^+_{s^i_j} = [0, +\infty) \times [0, 1]$ as a strip-like end near the puncture s^i_j.

Now we extend the map u to the decorated domain $\tilde{\Sigma}$. On each newly-added half-infinite ray $I_{s^i_j}$, we extend u by a map $u_{s^i_j} : I_{s^i_j} \to L_i$, which comes with a preferred lift $\tilde{u_{s^i_j}} : I_{s^i_j} \to L_i \times_{\iota_i} L_i$, which satisfies the gradient flow equation:

$$\frac{du_{s^i_j}}{dt} + \nabla f_{i,s}(u_{s^i_j}) = 0, \tag{5.43}$$

and asymptotically converges to a critical point of f_i, where $f_{i,s}$ a family of perturbations of f_i parametrized by $s \in I_{s^i_j}$, so that $f_{i,s} = f_i$ for $s \gg 0$. On each newly-added half-infinite strip $Z^+_{s^i_j}$, we extend u by a map $u_{s^i_j} : Z^+_{s^i_j} \to M$, such that $u_{s^i_j}$ maps the boundary to $\iota_i(L_i)$, and satisfies the inhomogeneous Cauchy-Riemann equation:

$$\partial_s u_{s^i_j} + J_i(\partial_t u_{s^i_j} - X_H(u_{s^i_j})) = 0, \tag{5.44}$$

and asymptotically converges to some time-one H-chord x.

These critical points of f_i or non-constant H-chords from $\iota_i(L_i)$ to itself should come equipped with choices of capping half-disks, which make them into generators c^j_i of the wrapped Floer cochain space $CW^*((L_0, \iota_0), (L_1, \iota_1); H)$.

We also extend the maps $\vec{l} = (l_0, l_1)$ to these newly-added components, so that the extended map $\tilde{l}_i$ satisfies $\iota_i \circ \tilde{l}_i = \tilde{u}$. In particular, the extension to a half-infinite ray $I_{s^i_j}$ is precisely the preferred lift $\tilde{u}_{s^i_j}$.

Definition 5.17. *We call the extended map* $(\tilde{\Sigma}, \vec{s}, \tilde{u}, \vec{\tilde{l}})$ *a decorated Floer trajectory. Denote by*

$$\mathcal{N}^{dec}_{k,l}(\vec{\alpha}, \beta; \{J_t\}_t, H; c_-, c_+; c^0_1, \cdots, c^0_k, c^1_1, \cdots, c^1_l) \tag{5.45}$$

the moduli space of decorated Floer trajectories.

For convenience, denote by $k = k_0$ and $l = k_1$. Let n_0, n_1, m_0, m_1 and $m_{0,1}, \cdots, m_{0,n_0}$, and $m_{1,1}, \cdots, m_{1,n_1}$ be non-negative integers such that

$$k_i = m_i + \sum_{a=1}^{n_i} m_{i,a}.$$

Let $A_i \subset \{1, \cdots, m_i + n_i\}$ be a subset of n_i elements and put

$$A_i = \{\sigma_i(1), \cdots, \sigma_i(n_i)\}, \tag{5.46}$$

where $\sigma_i : \{1, \cdots, n_i\} \to \{1, \cdots, m_i + n_i\}$ is an injective map satisfying $\sigma_i(a) < \sigma_i(a+1)$. Let $\pi_{i,1} : (Crit(f_i) \coprod \mathcal{X}_+(\iota_i(L_i), H))^{k_i+n_i} \to (Crit(f_i) \coprod \mathcal{X}_+(\iota_i(L_i), H))^{n_i}$ be the projection

$$\pi_{i,1}(x_1, \cdots, x_{k_i+n_i}) = (x_{\sigma_i(1)}, \cdots, x_{\sigma_i(n_i)}).$$

Let $\pi_{i,2} : (Crit(f_i) \coprod \mathcal{X}_+(\iota_i(L_i), H))^{k_i+n_i} \to (Crit(f_i) \coprod \mathcal{X}_+(\iota_i(L_i), H))^{k_i}$ be the projection to the other factors.

For simplicity, denote by A the collection of data $n_0, n_1, m_0, m_1, m_{0,1}, \cdots, m_{0,n_0}$, and $m_{1,1}, \cdots, m_{1,n_1}$ as well as A_0, A_1. We put

(5.47)

$$\begin{aligned}
&\mathcal{N}^A_{k_0,k_1}(\vec{\alpha}, \beta; \{J_t\}_t, H; c_-, c_+; \{c^0_a, c^0_{a,1}, \cdots, c^0_{a,m_{0,a}}\}_{a=1}^{n_0}, \{c^1_b, c^1_{b,1}, \cdots, c^1_{b,m_{1,b}}\}_{b=1}^{n_1}) \\
&= \bigcup_{\beta' \sharp \sum \beta_{0,a} \sharp \sum \beta_{1,b} = \beta} \coprod_{\substack{\alpha'_0 \cup \bigcup_a \alpha_{0,a} = \alpha_0 \\ \alpha'_1 \cup \bigcup_b \alpha_{1,b} = \alpha_1}} \mathcal{N}^{dec}_{n_0+m_0, n_1+m_1}(\vec{\alpha}', \beta'; \{J_t\}_t, H; x_-, x_+; \\
&c^0_1, \cdots, c^0_{n_0+m_0}, c^1_1, \cdots, c^1_{n_1+m_1}) \\
&\times_{(\pi_{0,1}, \pi_{1,1}) \circ ev, \vec{ev}_0} \Big(\prod_{a=1}^{n_0} \bar{\mathcal{M}}_{m_{0,a}+1}(\alpha_{0,a}, \beta_{0,a}; J_0, H; c^0_a, c^0_{a,1}, \cdots, c^0_{a,m_{0,a}}) \\
&\times \prod_{b=1}^{n_1} \bar{\mathcal{M}}_{m_{1,b}+1}(\alpha_{1,b}, \beta_{1,b}; J_1, H; c^1_b, c^1_{b,1}, \cdots, c^1_{b,m_{1,b}})\Big),
\end{aligned}$$

where $\vec{ev}^0 = (ev^0, \cdots, ev^0)$ are the evaluation maps at the 0-th marked point, and the moduli spaces

$$\bar{\mathcal{M}}_{m_{0,a}+1}(\alpha_{0,a}, \beta_{0,a}; J_0, H; c^0_a, c^0_{a,1}, \cdots, c^0_{a,m_{0,a}})$$

and

$$\bar{\mathcal{M}}_{m_{1,b}+1}(\alpha_{1,b}, \beta_{1,b}; J_1, H; c^1_b, c^1_{b,1}, \cdots, c^1_{b,m_{1,b}})$$

are moduli spaces of stable pearly tree maps.

To simplify the notations, we denote by $\vec{c}^0_A$ the collection $\{c^0_a, c^0_{a,1}, \cdots, c^0_{a,m_{0,a}}\}_{a=1}^{n_0}$ of generators and similarly $\vec{c}^1_A$ for the other collection. Also, denote by $\vec{c}^0, \vec{c}^1$ for any possible collections among c^0_A, c^1_A for all type A.

Definition 5.18. *We put*

$$\mathcal{N}^{unbr}_{k_0,k_1}(\alpha_0,\alpha_1;\beta;c_-,c_+;\vec{c}^0,\vec{c}^1) = \bigcup_A \mathcal{N}^A_{k_0,k_1}(\alpha_0,\alpha_1;\beta;c_-,c_+;\vec{c}^0_A,\vec{c}^1_A), \tag{5.48}$$

and call it the moduli space of stable unbroken Floer trajectories.

In words, the elements in this moduli space are Floer trajectories with trees of disks attached to each boundary of $\mathbb{R}\times[0,1]$. We do not have to include sphere bubbles because the ambient symplectic manifold M is exact.

5.11. **Compactification: stable broken Floer trajectories.** Next, as the limit of a sequence of stable unbroken Floer trajectories can break into several Floer trajectories, we introduce the moduli space of stable broken Floer trajectories.

Definition 5.19. *A stable broken Floer trajectory is a tuple*

$$((\tilde{\Sigma}(1),\vec{s}(1),\tilde{u}(1),\vec{\tilde{l}}(1)),\cdots,(\tilde{\Sigma}(K),\vec{s}(K),\tilde{u}(K),\vec{\tilde{l}}(K)),$$

where each $(\tilde{\Sigma}(a),\vec{s}(a),\tilde{u}(a),\vec{\tilde{l}}(a))$ *is a stable unbroken Floer trajectory, such that the asymptotic convergence conditions match for successive stable unbroken Floer trajectories:*

$$c_+(a) = c_-(a+1). \tag{5.49}$$

The moduli space of stable broken Floer trajectories is denoted by

$$\begin{aligned}
&\bar{\mathcal{N}}_{k_0,k_1}(\vec{\alpha},\beta;\{J_t\}_t,H;c_-,c_+;\vec{c}^0,\vec{c}^1)\\
&=\coprod_K \coprod_{\substack{\alpha_0(1),\cdots,\alpha_0(K)\\ \alpha_0(1)\cup\cdots\cup\alpha_0(K)=\alpha_0}} \coprod_{\substack{\alpha_1(1),\cdots,\alpha_1(K)\\ \alpha_1(1)\cup\cdots\cup\alpha_1(K)=\alpha_0}} \coprod_{\beta(1)\sharp\cdots\sharp\beta(K)=\beta}\\
&\coprod_{\substack{k_{0,1},\cdots,k_{0,K}\\ k_{0,1}+\cdots+k_{0,K}=k_0}} \coprod_{\substack{k_{1,1},\cdots,k_{1,K}\\ k_{1,1}+\cdots+k_{1,K}=k_1}} \prod_{a=1}^{K-1}\\
&\mathcal{N}^{unbr}_{k_{0,a},k_{1,a}}(\alpha_0(a),\alpha_1(a),\beta(a);\{J_t\}_t,H;c_-(a),c_+(a);\vec{c}^0(a),\vec{c}^1(a)),
\end{aligned} \tag{5.50}$$

where $c(1)=c_-, c(K)=c_+$. *Here the disjoint union is taken over all* $K,\vec{\alpha},\vec{c},k_{0,a},k_{1,a}$.

Now it is standard to prove:

Proposition 5.2. *The moduli space of stable broken Floer trajectories*

$$\bar{\mathcal{N}}_{k_0,k_1}(\vec{\alpha},\beta;\{J_t\}_t,H;c_-,c_+;\vec{c}^0,\vec{c}^1)$$

is compact. Its codimension-one boundary strata consist of

(5.51)
$$\begin{aligned}
&\partial\bar{\mathcal{N}}_{k_0,k_1}(\vec{\alpha},\beta;\{J_t\}_t,H;c_-,c_+;\vec{c}^0,\vec{c}^1)\\
&\cong \coprod_{\substack{k_0'+k_0''=k_0\\ 1\le i\le k_0'\\ \alpha_0}} \coprod_{\substack{(\vec{c}'^0,\vec{c}'''^0,\vec{c}''^0)=\vec{c}^0\\ \vec{c}'^0 \text{ is an } i\text{-tuple}}} \bar{\mathcal{N}}_{k_0'+1,k_1}(\alpha_0',\alpha_1,\beta';\{J_t\}_t,H;c_-,c_+;(\vec{c}'^0,\vec{c}''^0),\vec{c}^1)\\
&\times_{ev_0^i,ev^0}\bar{\mathcal{M}}_{k_0''+1}(\alpha_0'',\beta_0;J_0,H;\vec{c}'''^0)\\
&\cup \coprod_{\substack{k_1'+k_1''=k_1\\ 1\le i\le k_1'\\ \alpha_1}} \coprod_{\substack{(\vec{c}'^1,\vec{c}'''^1,\vec{c}''^1)=\vec{c}^1\\ \vec{c}'^1 \text{ is an } i\text{-tuple}}} \bar{\mathcal{N}}_{k_0,k_1'+1}(\alpha_0,\alpha_1',\beta;\{J_t\}_t,H;c_-,c_+;\vec{c}^0,\vec{c}'^1,\vec{c}''^1)\\
&\times_{ev_1^i,ev^0}\bar{\mathcal{M}}_{k_1''+1}(\alpha_1'',\beta_1;J_1,H;\vec{c}'''^1)\\
&\cup \coprod_{c_+'=c_-'} \bar{\mathcal{N}}_{k_0',k_1'}(\alpha_0',\alpha_1',\beta';\{J_t\}_t,H;c_-,c_+';\vec{c}'^0,\vec{c}'^1)\\
&\times \bar{\mathcal{N}}_{k_0'',k_1''}(\alpha_0'',\alpha_1'',\beta'';\{J_t\}_t,H;c_-',c_+;\vec{c}''^0,\vec{c}''^1)
\end{aligned}$$

Then we may construct Kuranishi structures on these moduli spaces of stable broken Floer trajectories

$$\bar{\mathcal{N}}_{k_0,k_1}(\vec{\alpha},\beta;\{J_t\}_t,H;c_-,c_+;\vec{c}^0,\vec{c}^1),$$

such that they are compatible with fiber product Kuranishi structures on moduli spaces of stable broken Floer trajectories as well as moduli spaces of stable pearly tree maps at the boundary (5.51). Single-valued multisections can also be chosen in a coherent way.

5.12. **The curved A_∞-bimodule associated to a pair of exact cylindrical Lagrangian immersions with transverse self-intersections.** For a pair of exact cylindrical Lagrangian immersions as above, we shall construct a curved A_∞-bimodule structure on the wrapped Floer cochain space $CW^*((L_0,\iota_0),(L_1,\iota_1);H)$ over the curved A_∞-algebras $(CW^*(L_0,\iota_0;H),m_0^k),(CW^*(L_1,\iota_1;H),m_1^k)$.

Proposition 5.3. *There is a natural curved A_∞-bimodule structure on the wrapped Floer cochain space $CW^*((L_0,\iota_0),(L_1,\iota_1);H)$ over the curved A_∞-algebras for $\iota_i : L_i \to M$,*

$$(CW^*(L_0,\iota_0;H),CW^*(L_1,\iota_1;H)).$$

The structure maps $n^{k,l}$ are defined by appropriate counts of broken stable Floer trajectories.

Suppose both $\iota_i : L_i \to M$ are unobstructed with choices of bounding cochains $b_i \in CW^*(L_i,\iota_i;H)$. Then the (b_0,b_1)-deformation $n^{k,l;b_0,b_1}$ defines a non-curved A_∞-bimodule over the deformed A_∞-algebras

$$((CW^*((L_0,\iota_0;H),m^{k;b_0}),(CW^*((L_1,\iota_1;H),m^{k;b_1})).$$

Proof. Consider moduli spaces of stable broken Floer trajectories which are of virtual dimension zero. Then the virtual fundamental chains associated to the chosen single-valued multisections $s_{k_0,k_1;\vec{\alpha},\beta;c_-,c_+;\{J_t\}_t,H;\vec{c}^0,\vec{c}^1}$ gives rise to an integer number

$$a_{k,l;\vec{\alpha},\beta;c_-,c_+;\{J_t\}_t,H;\vec{c}^0,\vec{c}^1} = (s_{k,l;\vec{\alpha},\beta;c_-,c_+;\{J_t\}_t,H;\vec{c}^0,\vec{c}^1})^{-1}(0) \in \mathbb{Z}. \tag{5.52}$$

We define multilinear maps

(5.53)

$$n^{k,l}: CW^*(L_0,\iota_0)^{\otimes k}\otimes CW^*((L_0,\iota_0),(L_1,\iota_1))\otimes CW^*(L_1,\iota_1)^{\otimes l}\to CW^*((L_0,\iota_0),(L_1,\iota_1)),$$

for $k=k_0, l=k_1\geq 0$, by the formula:

$$\begin{aligned}&n^{k,l}(c^0_k,\cdots,c^0_1,c_+,c^1_1,\cdots,c^1_l)\\ &=\sum_{\substack{\alpha,\beta,c_-\\ \dim \bar{\mathcal{N}}_{k_0,k_1}(\vec{\alpha},\beta;c_-,c_+;\{J_t\}_t,H;\vec{c}^0,\vec{c}^1)=0}} a_{k,l;\vec{\alpha},\beta;c_-,c_+;\{J_t\}_t,H;\vec{c}^0,\vec{c}^1}c_-.\end{aligned}\tag{5.54}$$

Again, this is a finite sum by the same argument as that for (5.31), so that the multilinear map $n^{k,l}$ is well-defined.

These maps satisfy the A_∞-equations for a curved A_∞-bimodule, by Proposition (**??**).

□

In general, $n^{0,0}$ does not square to zero because of the contribution of m^0 from each (L_i,ι_i). To obtain a differential, we need to assume that both $\iota_0: L_0\to M$ and $\iota_1: L_1\to M$ are unobstructed, in which case we can deform the curved A_∞-bimodule structure by the chosen bounding cochains for $\iota_0: L_0\to M$ and $\iota_1: L_1\to M$ respectively:

$$\begin{aligned}&n^{k,l;b_0,b_1}(c^0_k,\cdots,c^0_1,c_+,c^1_1,\cdots,c^1_l)\\ &=\sum_{\substack{i\geq 0,j\geq 0\\ i_0+\cdots+i_k=i\\ j_0+\cdots+j_l=j}} n^{k+i,l+j}(\underbrace{b_0,\cdots,b_0}_{i_k\text{ times}},c^0_k,\underbrace{b_0,\cdots,b_0}_{i_{k-1}\text{ times}},\cdots,c^0_1,\underbrace{b_0,\cdots,b_0}_{i_0\text{ times}},\\ &c_+,\underbrace{b_1,\cdots,b_1}_{j_0\text{ times}},c^1_1,\underbrace{b_1,\cdots,b_1}_{j_1\text{ times}},\cdots,c^1_l,\underbrace{b_1,\cdots,b_1}_{j_l\text{ times}}).\end{aligned}\tag{5.55}$$

Because of the Maurer-Cartan equations that the bounding cochains satisfy, $n^{0,0;b_0,b_1}$ squares to zero, and thus defines a differential on $CW^*((L_0,\iota_0),(L_1,\iota_1);H)$. We call the resulting cohomology group the wrapped Floer cohomology group of the pair of the exact cylindrical Lagrangian immersions $\iota_0: L_0\to M$ and $\iota_1: L_1\to M$, with respect to the bounding cochains b_0 and b_1, and denote it by $HW^*((L_0,\iota_0,b_0),(L_1,\iota_1,b_1);H)$.

6. A_∞-functors associated to Lagrangian correspondences

6.1. **Extension of quilted wrapped Floer cohomology to Lagrangian immersions.** This section provides chain-level refinements of the construction of cohomological functors in [Gao17b]. That is, we are going to prove that admissible Lagrangian correspondences give rise to functors between appropriate versions of wrapped Fukaya categories. For simplicity, we let the source of the functors be the wrapped Fukaya category of M consisting of embedded exact cylindrical Lagrangian submanifolds. All A_∞-functors are to be understood as cohomologically unital A_∞-functors.

As an introductory part of the main construction, we first give a naive attempt in extending quilted wrapped Floer cohomology to exact cylindrical Lagrangian immersions with transverse or clean self-intersections. For our purpose of constructing functors from Lagrangian correspondences, we shall only consider the case where $L \subset M$ and $\mathcal{L} \subset M^- \times N$ are properly embedded, while $L' \subset N$ is replaced by an exact cylindrical Lagrangian immersion $\iota : L' \to N$.

One short-cut definition for the quilted wrapped Floer cochain space is given as follows. As the underlying $\mathbb{Z}$-module, the quilted wrapped Floer cochain space

$$CW^*(L, \mathcal{L}, (L', \iota))$$

is defined as the wrapped Floer cochain space

$$CW^*(\mathcal{L}, L \times (L', \iota)),$$

for the pair of exact cylindrical Lagrangian immersions in $M^- \times N$, where $L \times (L', \iota)$ is the obvious product Lagrangian immersion with clean self-intersections. This pair has clean intersections, whose wrapped Floer cochain space is defined in section **??**. The quilted Floer "differential" n^0 is defined as the zeroth-order curved A_∞-structure map on the above wrapped Floer cochain space $CW^*(\mathcal{L}, L \times (L', \iota))$. Here we put the quotation mark because n^0 might not square to zero in general. Alternatively, there is another straightforward definition, using moduli spaces of inhomogeneous pseudoholomorphic quilted strips, following the standard setup of quilted wrapped Floer theory. In fact, these inhomogeneous pseudoholomorphic quilted strips are in natural bijection to inhomogeneous pseudoholomorphic strips in the product manifold, and we can choose the same perturbations (by multisections) for both moduli spaces. Thus the second definition is equivalent to the first one.

The second definition is more suitable for discussing the A_∞-bimodule structure on the quilted wrapped Floer cochain space $CW^*(L, \mathcal{L}, (L', \iota))$. The details of the construction will be discussed in subsection 6.2.

6.2. **The module-valued functors associated to Lagrangian correspondences.** The guiding principle for constructing A_∞-functors from Lagrangian correspondences is to use moduli spaces of inhomogeneous pseudoholomorphic quilted maps. For these A_∞-functors to be defined over $\mathbb{Z}$, we must ensure that these moduli spaces carry coherent orientations. The reader is referred to the Appendix of [Gao17b] for the discussion on orientations on the relevant moduli spaces of quilted inhomogeneous pseudoholomorphic maps, where the discussion focused on one particular kind of moduli space but can be easily generalized to all the other ones which we actually use here. For a more general discussion on orientations of moduli spaces of pseudoholomorphic quilts, we refer the reader to [WW15a], but remark that our approach is independent because some of the quilted surfaces we use are not included there.

The starting point is to to relate the wrapped Fukaya category $\mathcal{W}(M^- \times N)$ of the product manifold, to the dg-category of A_∞-bimodules over $(\mathcal{W}(M), \mathcal{W}(N))$.

Proposition 6.1. *There is a canonical A_∞-functor*

$$\Phi : \mathcal{W}(M^- \times N) \to (\mathcal{W}(M), \mathcal{W}(N))^{bimod}, \tag{6.1}$$

satisfying the following properties:

(i) Φ is non-trivial for any non-trivial $\mathcal{W}(M^- \times N)$;
(ii) If either M or N is a point, Φ is the Yoneda functor for N or M;
(iii) If $\mathcal{L} = L \times L'$ is a product Lagrangian correspondence, then the A_∞-bimodule $\Phi(\mathcal{L})$ splits. That is, there is an isomorphism of A_∞-bimodules

$$\Phi(\mathcal{L}) \cong \mathfrak{y}_r(L) \otimes \mathfrak{y}_l(L'), \tag{6.2}$$

where $\mathfrak{y}_r$ and $\mathfrak{y}_l$ are the right and left Yoneda functors.

The idea of proof is to develop a quilted version of wrapped Floer theory, extending that in [Gao17b]. The natural construction should yield an A_∞-functor from the split the wrapped Fukaya category $\mathcal{W}^s(M^- \times N)$. However, based on the results of section 4, that is equivalent to the ordinary wrapped Fukaya category $\mathcal{W}(M^- \times N)$. Thus it does not matter which version of wrapped Fukaya category of the product manifold we use as far as algebraic structures are concerned, at least up to quasi-equivalence.

Definition 6.1. *A Lagrangian correspondence $\mathcal{L} \subset M^- \times N$ from M to N is said to be admissible, if it is admissible for wrapped Floer theory in the product manifold $M^- \times N$ in the sense of Definition 4.1, i.e. is an object of the wrapped Fukaya category $\mathcal{W}(M^- \times N)$.*

As a result, by evaluating the above A_∞-functor at each given object $\mathcal{L}$ of $\mathcal{W}(M^- \times N)$, i.e. an admissible Lagrangian correspondence, we then get an A_∞-bimodule over $(\mathcal{W}(M), \mathcal{W}(N))$. By purely algebraic consideration involving the Yoneda embedding, we have the following A_∞-functor associated to $\mathcal{L}$:

Corollary 6.1. *For any admissible Lagrangian correspondence $\mathcal{L} \subset M^- \times N$, there is an associated A_∞-functor:*

$$\Phi_{\mathcal{L}} : \mathcal{W}(M) \to \mathcal{W}(N)^{l-mod}, \tag{6.3}$$

to the dg-category of left A_∞-modules over $\mathcal{W}(N)$.

In particular, we remark that such an A_∞-functor is defined for any admissible Lagrangian correspondence $\mathcal{L} \subset M^- \times N$, without any properness assumption. However, as we shall see in the next subsection, a suitable properness assumption is needed in order to prove that this module-valued functor is representable, thus can be improved to a filtered A_∞-functor to the immersed wrapped Fukaya category $\mathcal{W}_{im}(N)$.

Now let us discuss in detail the construction of the bimodule-valued functor (6.1). On the level of objects, the functor Φ should assign an A_∞-bimodule $\Phi(\mathcal{L})$ over $(\mathcal{W}(M), \mathcal{W}(N))$ to an admissible Lagrangian correspondence $\mathcal{L} \subset M^- \times N$. The first order bimodule structure map of $\Phi(\mathcal{L})$ has already been constructed in [Gao17b]. We now give an extension of that, defining the A_∞-bimodule structure maps of all orders in a uniform treatment.

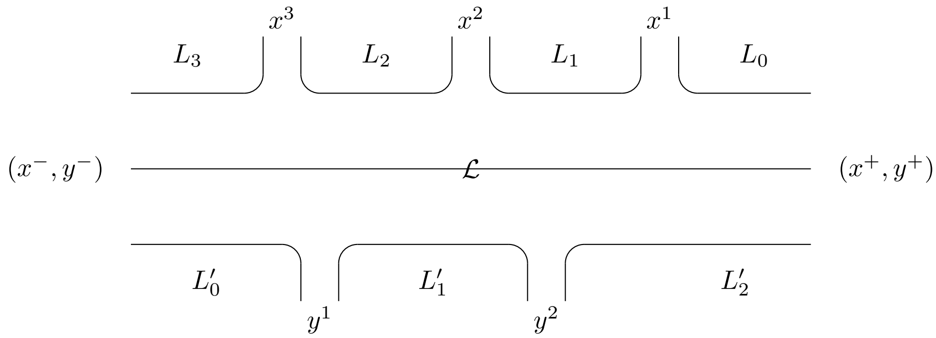

FIGURE 1. The quilted map defining the bimodule

Consider the quilted surface $\underline{S}^{k,l}$ consisting of two patches S_0^k, S_1^l, where S_0^k is a disk with $(k+2)$ boundary punctures $z_0^-, z_0^1, \cdots, z_0^k, z_0^+$, and S_1^l is a disk with $(l+2)$ boundary punctures $z_1^-, z_1^1, \cdots, z_1^l, z_1^+$. Let $I_0^{\pm}$ be the boundary component of S_0^k between z_0^+ and z_0^-, and $I_1^{\pm}$ the boundary component of S_1^l between z_1^+ and z_1^-. The quilted surface is obtained by seaming the two patches along these two boundary components. After seaming the two patches, the quilted surface $\underline{S}^{k,l}$ has $(k+l)$ positive strip-like ends $\epsilon_0^1, \cdots, \epsilon_0^k$ and $\epsilon_1^1, \cdots, \epsilon_1^l$ as well as two quilted ends (one positive and one negative), each of which consists of two strip-like ends. See the picture below.

Choosing a Floer datum for $\underline{S}^{k,l}$ allows us to define inhomogeneous pseudoholomorphic quilted maps from $\underline{S}^{k,l}$ to the pair (M, N) with appropriate moving Lagrangian boundary conditions and asymptotic convergence conditions over the various ends. We shall choose Floer data for all (representatives of) such quilted surfaces in the moduli spaces, and extend the choices by automorphism-invariant Floer data on unstable components, i.e. quilted strips, of semistable quilted surfaces, such that they are compatible under gluing maps with the universal and conformally consistent choices we made for disks.

Let $\mathcal{R}^{k,l}((x^-, y^-); (x^+, y^+), \vec{x}, \vec{y})$ be the moduli space, namely the set of isomorphism classes of inhomogeneous pseudoholomorphic quilted maps $(\underline{S}^{k,l}, \underline{u})$ as pictured in Figure 1. There is a natural Gromov bordification

$$\bar{\mathcal{R}}^{k,l}((x^-, y^-); (x^+, y^+), \vec{x}, \vec{y}), \tag{6.4}$$

which compactifies $\mathcal{R}^{k,l}((x^-, y^-); (x^+, y^+), \vec{x}, \vec{y})$. The codimension one boundary strata of $\bar{\mathcal{R}}^{k,l}((x^-, y^-); (x^+, y^+), \vec{x}, \vec{y})$ are covered by a union of products of moduli spaces of the following form:

$$
\begin{aligned}
&\coprod_{\substack{1\le i\le k\\(\vec{x}',\vec{x}'')=\vec{x}}}\coprod_{x_{new}}\bar{\mathcal{R}}^{k-i+1,l}((x^-,y^-);(x^+,y^+),\vec{x}',x_{new},\vec{y})\times\bar{\mathcal{M}}_{i+1}(x_{new},\vec{x}'')\\
\cup&\coprod_{\substack{1\le j\le l\\(\vec{y}',\vec{y}'')=\vec{y}}}\coprod_{y_{new}}\bar{\mathcal{R}}^{k,l-j+1}((x^-,y^-);(x^+,y^+),\vec{x},\vec{y}',y_{new})\times\bar{\mathcal{M}}_{j+1}(y_{new},\vec{y}'')\\
\cup&\coprod_{\substack{k'+k''=k,l'+l''=l\\(x_1^+,y_1^+)}}\bar{\mathcal{R}}^{k',l'}((x^-,y^-);(x_1^+,y_1^+),\vec{x}',\vec{y}')\\
&\qquad\times\bar{\mathcal{R}}^{k'',l''}((x_1^+,y_1^+);(x^+,y^+),\vec{x}'',\vec{y}'')
\end{aligned}
\tag{6.5}
$$

Here by the notation $\bar{\mathcal{R}}^{k-i+1,l}((x^-,y^-);(x^+,y^+),\vec{x}',x_{new},\vec{y})$ we mean to insert the new Hamiltonian chord x_{new} in every possible place that splits the tuple $\vec{x}$ of Hamiltonian chords to the two tuples $\vec{x}'$ and $\vec{x}''$, as long as the cyclic order is preserved. Similar remarks apply to the y's.

In [WW12], it is demonstrated that this kind of moduli space is locally modeled on a Fredholm complex. Since there are no disk bubbles, we can use the standard transversality argument to prove that for generic universal and conformally consistent choices of Floer data, the Gromov bordification $\bar{\mathcal{R}}^{k,l}((x^-,y^-);(x^+,y^+),\vec{x},\vec{y})$ is a compact smooth manifold with corners of expected dimension

$$k-2+l-2+\deg((x^-,y^-))-\deg((x^+,y^+))-\sum\deg(x^i)-\sum\deg(y^j). \tag{6.6}$$

And moreover, we can arrange the perturbations so that every stratum is regular.

A finiteness result, which says that given inputs $(x^+,y^+),\vec{x},\vec{y}$, the moduli spaces $\bar{\mathcal{R}}^{k,l}((x^-,y^-);(x^+,y^+),\vec{x},\vec{y})$ are empty for all but finitely many outputs (x^-,y^-), can be deduced from the action-energy equality, which plays a crucial role in the well-definedness of various maps in wrapped Floer theory ([AS10], [Abo10], also see [Gao17b] in which we used the quilted version in special cases $k\le 1, l\le 1$). This ensures that the count of rigid elements of all moduli spaces $\bar{\mathcal{R}}^{k,l}((x^-,y^-);(x^+,y^+),\vec{x},\vec{y})$ for fixed inputs $(x^+,y^+),\vec{x},\vec{y}$ is finite, which gives rise to a map

$$
\begin{aligned}
n^{k|0|l}:{}&CW^*(L_{k-1},L_k)\otimes\cdots\otimes CW^*(L_0,L_1)\otimes CW^*(L_k,\mathcal{L},L_0')\\
&\otimes CW^*(L_{l-1}',L_l')\otimes\cdots\otimes CW^*(L_0',L_1')\to CW^*(L_0,\mathcal{L},L_l').
\end{aligned}
\tag{6.7}
$$

By analyzing the boundary strata of one dimensional moduli spaces as described in (6.5), we find that the operation $n^{k|0|l}$ satisfies the following equation

$$
\begin{aligned}
&m^1\circ n^{k|0|l}([\vec{x}],[x^+,y^+],[\vec{y}])\\
&=\sum n^{k-i+1|0|l}([\vec{x}'],m^i([\vec{x}'']),[x^+,y^+],[\vec{y}])\\
&+\sum n^{k|0|l-j+1}([\vec{x}],[x^+,y^+],[\vec{y}'],m^j([\vec{y}'']))\\
&+\sum_{\substack{k'+k''=k,l'+l''=l\\(\vec{x}',\vec{x}'')=\vec{x}}} n^{k'|0|l'}([\vec{x}'],n^{k''|0|l''}([\vec{x}''],[x^+,y^+],[\vec{y}'']),[\vec{y}']).
\end{aligned}
\tag{6.8}
$$

This precisely means that the operations $n^{k|0|l}$ define an A_∞-bimodule structure on $\Phi(\mathcal{L})$ over $(\mathcal{W}(M),\mathcal{W}(N))$.

Next we want to study what the A_∞-functor (6.1) does to morphisms, and how this is related with the A_∞-structure maps of $\mathcal{W}(M)$ and $\mathcal{W}(N)$. A Floer cochain $[\gamma] \in CW^*(\mathcal{L}_0, \mathcal{L}_1)$ should give rise to an A_∞-bimodule homomorphism

$$\Phi_\gamma : \Phi_{\mathcal{L}_0} \to \Phi_{\mathcal{L}_1}. \tag{6.9}$$

Moreover, this should be functorial in the wrapped Fukaya category of the product manifold $M^- \times N$ as stated in Proposition 6.1. More specifically, it means that there are multilinear maps

$$\begin{aligned} n^{k|1|l} : CW^*(\mathcal{L}_0, \mathcal{L}_1) \to \hom(&CW^*(L_{k-1}, L_k) \otimes \cdots \otimes CW^*(L_0, L_1) \\ &\otimes CW^*(L_0, \mathcal{L}_0, L'_l) \otimes CW^*(L'_{l-1}, L'_l) \\ &\otimes \cdots \otimes CW^*(L'_0, L'_1), CW^*(L_k, \mathcal{L}_1, L'_0)), \end{aligned} \tag{6.10}$$

such that when evaluated on $[\gamma] \in CW^*(\mathcal{L}_0, \mathcal{L}_1)$, the resulting maps form the desired A_∞-bimodule homomorphism (6.9).

To define the maps (6.10), we shall study moduli spaces of inhomogeneous pseudoholomorphic maps from another kind of quilted surface $\underline{S}^{1,k,l}$, which we describe as follows. It has two patches $S_0^{1,k}, S_1^{1,l}$, where $S_0^{1,k}$ is a disk with $(k+3)$ boundary punctures $z_0^+, z_0^-, z_0^p, z_0^1, \cdots, z_0^k$, and $S_1^{1,l}$ is a disk with $(l+3)$ boundary punctures $z_1^+, z_1^-, z_1^p, z_1^1, \cdots, z_1^k$. We denote by $I_{0,+}$ the boundary component of $S_0^{1,k}$ between z_0^+ and z_0^p, and by $I_{0,-}$ that between z_0^p and z_0^-. We use similar notations for $S^{1,l}$. The quilted surface is obtained by seaming the two patches along the two pairs of boundary components $(I_{0,+}, I_{1,+})$ and $(I_{0,-}, I_{1,-})$.

Choose a Floer datum for $\underline{S}^{1,k,l}$ so that we can write down the inhomogeneous Cauchy-Riemann equation for quilted maps $\underline{u} : \underline{S}^{1,k,l} \to (M, N)$:
(6.11)

$$\begin{cases} (du_0 - \alpha_{S_0^{1,k}} \otimes X_{H_{S_0^{1,k}}})^{0,1} = 0 \\ (du_1 - \alpha_{S_1^{1,l}} \otimes X_{H_{S_1^{1,l}}})^{0,1} = 0 \\ u_0(z) \in \psi_M^{\rho_{S_0^{1,k}}(z)} L_i, \text{ if } z \in \partial S_0^{1,k} \text{ lies between } z_0^i \text{ and } z_0^{i+1} \\ u_0(z) \in \psi_M^{\rho_{S_0^{1,k}}(z)} L_0, \text{ if } z \in \partial S_0^{1,k} \text{ lies between } z_0^+ \text{ and } z_0^1 \\ u_0(z) \in \psi_M^{\rho_{S_0^{1,k}}(z)} L_k, \text{ if } z \in \partial S_0^{1,l} \text{ lies between } z_0^k \text{ and } z_0^- \\ u_1(z) \in \psi_N^{\rho_{S_1^{1,l}}(z)} L'_j, \text{ if } z \in \partial S_1^{1,l} \text{ lies between } z_1^j \text{ and } z_1^{j+1} \\ u_1(z) \in \psi_N^{\rho_{S_1^{1,l}}(z)} L'_0, \text{ if } z \in \partial S_1^{1,l} \text{ lies between } z_1^+ \text{ and } z_1^1 \\ u_1(z) \in \psi_N^{\rho_{S_1^{1,l}}(z)} L'_l, \text{ if } z \in \partial S_1^{1,l} \text{ lies between } z_1^l \text{ and } z_1^- \\ (u_0(z), u_1(z)) \in (\psi_M^{\rho_{S_0^{1,k}}(z)} \times \psi_N^{\rho_{S_1^{1,l}}(z)})\mathcal{L}_1, \text{ if } z \in \partial S_0^{1,k} \text{ lies between } z_0^- \text{ and } z_0^p \\ (u_0(z), u_1(z)) \in (\psi_M^{\rho_{S_0^{1,k}}(z)} \times \psi_N^{\rho_{S_1^{1,l}}(z)})\mathcal{L}_0, \text{ if } z \in \partial S_0^{1,k} \text{ lies between } z_0^p \text{ and } z_0^+ \\ \lim_{s\to-\infty} (u_0 \circ \epsilon_0^-(s,\cdot), u_1 \circ \epsilon_1^-(s,\cdot)) = (x^-(\cdot), y^-(\cdot)) \\ \lim_{s\to+\infty} (u_0 \circ \epsilon_0^+(s,\cdot), u_1 \circ \epsilon_1^+(s,\cdot)) = (x^+(\cdot), y^+(\cdot)) \\ \lim_{s\to+\infty} (u_0 \circ \epsilon_0^p(s,\cdot), u_1 \circ \epsilon_1^p(s,\cdot)) = \gamma(\cdot) \\ \lim_{s\to+\infty} u_0 \circ \epsilon_0^i(s,\cdot) = x^i(\cdot), i = 1, \cdots, k \\ \lim_{s\to+\infty} u_1 \circ \epsilon_1^j(s,\cdot) = y^j(\cdot), j = 1, \cdots, l \end{cases}$$

Here $[x^i] \in CW^*(L_{i-1}, L_i), [y^i] \in CW^*(L'_{j-1}, L'_j)$ are Hamiltonian chords in M and N respectively, $[\gamma] \in CW^*(\mathcal{L}_0, \mathcal{L}_1)$ is a Hamiltonian chord in $M^- \times N$ with respect to the split Hamiltonian, and $[(x^-, y^-)] \in CW^*(L_k, \mathcal{L}_1, L'_0)$, $[(x^+, y^-)] \in CW^*(L_0, \mathcal{L}_0, L'_l)$ are generalized chords for the corresponding Lagrangian boundary and seaming conditions. We omit suitable rescalings of the asymptotic Hamiltonian chords by the Liouville flow for the purpose of simplifying notation, but shall keep in mind that these can be chosen and have been chosen in a consistent way. The Lagrangian boundary conditions are ordered as follows: on the boundary of the first patch, $L_k, \cdots, L_0$ are in order from the negative quilted puncture to the positive quilted puncture; on the boundary of the second patch, $L'_0, \cdots, L'_l$ are in order from the negative quilted puncture to the positive quilted puncture; on the seam, $\mathcal{L}_d, \cdots, \mathcal{L}_0$ are in order from the negative quilted puncture to the positive quilted puncture.

Let $\mathcal{R}^{1,k,l}((x^-, y^-); \vec{x}, \gamma, (x^+, y^+), \vec{y})$ be the moduli space of solutions $(\underline{S}^{1,k,l}, \underline{u})$ to the above equation. Here by $\underline{S}^{1,k,l}$ in the triple we mean a complex structure on $\underline{S}^{1,k,l}$ up to isomorphism. The Gromov bordification $\bar{\mathcal{R}}^{1,k,l}((x^-, y^-); \vec{x}, \gamma, (x^+, y^+), \vec{y})$

is in fact a compactification, with its codimension one stratum covered by the following union of fiber products of moduli spaces

(6.12)

$$\begin{aligned}
&\coprod_{\substack{1\le i\le k\\ (\vec{x}',\vec{x}'')=\vec{x}}}\coprod_{x_{new}}\bar{\mathcal{R}}^{1,k-i+1,l}((x^-,y^-);\vec{x}',x_{new},\gamma,(x^+,y^+),\vec{y})\times\bar{\mathcal{M}}_{i+1}(x_{new},\vec{x}'')\\
&\cup\coprod_{\substack{1\le j\le l\\ (\vec{y}',\vec{y}'')=\vec{y}}}\coprod_{y_{new}}\bar{\mathcal{R}}^{1,l,l-j+1}((x^-,y^-);\vec{x},\gamma,(x^+,y^+),\vec{y}')\times\bar{\mathcal{M}}_{j+1}(y_{new},\vec{y}'')\\
&\cup\coprod_{\gamma_1}\bar{\mathcal{R}}^{1,k,l}((x^-,y^-);\vec{x},\gamma_1,(x^+,y^+),\vec{y})\times\bar{\mathcal{M}}(\gamma_1,\gamma)\\
&\cup\coprod_{\substack{k'+k''=k,l'+l''=l\\ (\vec{x}',\vec{x}'')=\vec{x},(\vec{y}',\vec{y}'')=\vec{y}}}\coprod_{(x_1^+,y_1^+)}\bar{\mathcal{R}}^{1,k',l'}((x^-,y^-);\vec{x}',\gamma,(x_1^+,y_1^+),\vec{y}')\\
&\times\bar{\mathcal{R}}^{0,k'',l''}((x_1^+,y_1^+);\vec{x}'',(x^+,y^+),\vec{y}'')\\
&\cup\coprod_{\substack{k'+k''=k,l'+l''=l\\ (\vec{x}',\vec{x}'')=\vec{x},(\vec{y}',\vec{y}'')=\vec{y}}}\coprod_{(x_1^+,y_1^+)}\bar{\mathcal{R}}^{0,k',l'}((x^-,y^-);\vec{x}',(x_1^+,y_1^+),\vec{y}')\\
&\times\bar{\mathcal{R}}^{1,k'',l''}((x_1^+,y_1^+);\vec{x}'',\gamma,(x^+,y^+),\vec{y}'').
\end{aligned}$$

In this situation, the underlying quilted surface is not obtained by gluing patches tangentially, thus the limit of a sequence of inhomogeneous pseudoholomorphic quilted maps does not create a figure-eight bubble ([WW15b]). Therefore, the usual Sard-Smale theorem can be used to prove transversality. The upshot is that for generic choices of Floer data compatible with the choices made for puncture disks involved in the definition of wrapped Fukaya categories, these moduli spaces $\bar{\mathcal{R}}^{1,k,l}((x^-,y^-);\vec{x},\gamma,(x^+,y^+),\vec{y})$ are compact smooth manifolds with corners of expected dimension

(6.13)

$$k-1+l-1+\deg((x^-,y^-))-\deg((x^+,y^+))-\deg(\alpha)-\sum\deg(x^i)-\sum\deg(y^j),$$

and moreover each stratum is regular. Counting rigid elements in the zero dimensional moduli space $\bar{\mathcal{R}}^{1,k,l}((x^-,y^-);\vec{x},\gamma,(x^+,y^+),\vec{y})$ gives rise to the desired map (6.10).

We then extend the construction to higher orders. For this purpose, we consider the quilted surface $\underline{S}^{d,k,l}$ which consists of two patches $S_0^{d,k}, S_1^{d,l}$, where $S_0^{d,k}$ is a disk with $(k+d+2)$ boundary punctures $z_0^+, z_0^-, z_0^{p_1}, \cdots, z_0^{p_d}, z_0^1, \cdots, z_0^k$, and $S_1^{d,l}$ is a disk with $(l+d+2)$ boundary punctures $z_1^+, z_1^-, z_1^{p_1}, \cdots, z_1^{p_d}, z_1^1, \cdots, z_1^k$. After seaming these two patches together, the strip-like end near $z_0^{p_i}$ and the one near $z_1^{p_i}$ together form a quilted cylindrical end.

Consider the moduli spaces $\mathcal{R}^{d,k,l}((x^-,y^-);\vec{x},\gamma^d,\cdots,\gamma^1,(x^+,y^+),\vec{y})$ of inhomogeneous pseudoholomorphic quilted maps with appropriate boundary conditions and asymptotic convergence conditions. These are similar to that in Figure 1, but now there are also punctures on the seam which have appropriate asymptotic convergence conditions to generalized chords γ's.

By a standard argument using Gromov compactness theorem and the maximum principle, we may prove that the Gromov bordification

$$\bar{\mathcal{R}}^{d,k,l}((x^-,y^-);\vec{x},\gamma^d,\cdots,\gamma^1,(x^+,y^+),\vec{y}) \tag{6.14}$$

is compact. Thus it is possible to count rigid elements therein, which gives rise to multilinear maps

$$\begin{aligned} n^{k|d|l} &: CW^*(\mathcal{L}_{d-1},\mathcal{L}_d)\otimes\cdots\otimes CW^*(\mathcal{L}_0,\mathcal{L}_1)\to \\ &\hom(CW^*(L_{k-1},L_k)\otimes\cdots\otimes CW^*(L_0,L_1)\otimes CW^*(L_0,\mathcal{L}_0,L'_l) \\ &\otimes CW^*(L'_{l-1},L'_l)\otimes\cdots CW^*(L'_0,L'_1), CW^*(L_k,\mathcal{L}_d,L'_0)) \end{aligned} \tag{6.15}$$

Lemma 6.1. *The multilinear maps $\{n^{k|d|l}\}$ satisfy the A_∞-functor equations for the A_∞-functor (6.1). More concretely, for varying k,l and testing objects L_i and L'_j, the multilinear maps $n^{k|d|l}$ define for each d-tuple of composable Floer cochains in $\mathcal{W}(M^-\times N)$ a pre-bimodule homomorphism $\Phi_{\mathcal{L}_0}\to\Phi_{\mathcal{L}_d}$; moreover, the assignments of pre-bimodule homomorphisms for d-tuples of composable Floer cochains satisfy the A_∞-functor equations.*

Proof. To verify that the multilinear maps $n^{k|d|l}$ satisfy the desired A_∞-equations, we look at the codimension one boundary strata of the moduli space (6.14). It is covered by a union of the following products of moduli spaces:

(6.16)

$$\begin{aligned} &\partial\bar{\mathcal{R}}^{d,k,l}((x^-,y^-);\vec{x},\gamma^d,\cdots,\gamma^1,(x^+,y^+),\vec{y}) \\ \cong &\coprod_{\substack{1\le i\le k\\(\vec{x}',\vec{x}''',\vec{x}'')=\vec{x}}}\coprod_{x_{new}}\bar{\mathcal{M}}_{i+1}(x_{new},\vec{x}''') \\ &\times\bar{\mathcal{R}}^{d,k-i+1,l}((x^-,y^-);\vec{x}',x_{new},\vec{x}''\gamma^d,\cdots,\gamma^1,(x^+,y^+),\vec{y}) \\ \cup &\coprod_{\substack{1\le j\le l\\(\vec{y}',\vec{y}''',\vec{y}'')=\vec{y}}}\coprod_{y_{new}}\bar{\mathcal{M}}_{j+1}(y_{new},\vec{y}''') \\ &\times\bar{\mathcal{R}}^{d,k,l-j+1}((x^-,y^-);\vec{x},\gamma^d,\cdots,\gamma^1,(x^+,y^+),\vec{y}',y_{new},\vec{y}'') \\ \cup &\coprod_{0\le d_1\le d}\coprod_{\substack{k'+k''=k,l'+l''=l\\(\vec{x}',\vec{x}'')=\vec{x},(\vec{y}',\vec{y}'')=\vec{y}}}\bar{\mathcal{R}}^{d_2,k'',l''}((x^-,y^-);\vec{x}'',\gamma^d,\cdots,\gamma^{d_1+1},(x_1^+,y_1^+),\vec{y}'') \\ &\times\bar{\mathcal{R}}^{d_1,k',l'}((x_1^+,y_1^+);\vec{x}',\gamma^{d_1},\cdots,\gamma^1,(x^+,y^+),\vec{y}') \\ \cup &\coprod_{\substack{d_1+d_2=d+1\\0\le s\le d_1}}\coprod_{\gamma_{new}}\bar{\mathcal{M}}_{d_2+1}(\gamma^{s+d_2},\cdots,\gamma^{s+1},\gamma_{new}) \\ &\times\bar{\mathcal{R}}^{d_1,k,l}((x^-,y^-);\vec{x},\gamma^d,\cdots,\gamma^{s+d_2+1},\gamma_{new},\gamma^s,\cdots,\gamma^1,(x^+,y^+)). \end{aligned}$$

The above description of the codimension-one boundary strata of

$$\bar{\mathcal{R}}^{d,k,l}((x^-,y^-);\vec{x},\gamma^d,\cdots,\gamma^1,(x^+,y^+),\vec{y}),$$

similar to that in (6.16), implies the following series of identities which the operations $n^{k|d|l}$ satisfy:

$$
\begin{aligned}
& m^1 \circ n^{k|d|l}([\vec{x}], [\gamma^d], \cdots, [\gamma^1], [x^+, y^+], [\vec{y}]) \\
& = \sum n^{k-i+1|d|l}([\vec{x}'], m^i([\vec{x}''']), [\vec{x}''], [\gamma^d], \cdots, [\gamma^1], [x^+, y^+], [\vec{y}]) \\
& + \sum n^{k|d|l-j+1}([\vec{x}], [\gamma^d], \cdots, [\gamma^1], [x^+, y^+], [\vec{y}'], m^j([\vec{y}''']), [\vec{y}'']) \\
& + \sum n^{k''|d_2|l''}([\vec{x}''], [\gamma^d], \cdots, [\gamma_{d_1+1}], \\
& n^{k'|d_1|l'}([\vec{x}'], [\gamma^{d_1}], \cdots, [\gamma^1], [x^+, y^+], [\vec{y}']), [\vec{y}'']) \\
& + \sum n^{k|d_1|l}([\vec{x}, [\gamma^d], \cdots, [\gamma^{s+d_2+1}], \\
& m^{d_2}([\gamma^{s+d_2}], \cdots, [\gamma^{s+1}]), [\gamma^s], \cdots, [\gamma^1], [x^+, y^+])
\end{aligned}
\tag{6.17}
$$

Note in particular that the term $n^{k''|d_2|l''}(\cdots, n^{k'|d_1|l'}(\cdots), \cdots)$ accounts for the second order structure map m^2 in the dg-category $(\mathcal{W}(M), \mathcal{W}(N))^{bimod}$ of A_∞-bimodules over $(\mathcal{W}(M), \mathcal{W}(N))$. Rewriting the above identity in a suitable way, combining the quilted Floer differentials and Floer differentials in both $\mathcal{W}(M)$ and $\mathcal{W}(N)$ into the first order structure map in $(\mathcal{W}(M), \mathcal{W}(N))^{bimod}$, we obtain the desired A_∞-functor equations for the assignment $\mathcal{L} \mapsto \Phi_{\mathcal{L}}$, from $\mathcal{W}(M^- \times N)$ to the dg-category of A_∞-bimodules over $(\mathcal{W}(M), \mathcal{W}(N))$.

□

We have thus completed the construction of the A_∞-functor (6.1). As mentioned before, by evaluation and the Yoneda embedding we obtain the module-valued functor (6.3) for each admissible Lagrangian correspondence $\mathcal{L} \subset M^- \times N$.

Remark 6.1. *Note that in our construction, a Lagrangian correspondence $\mathcal{L} \subset M^- \times N$ gives rise to a bimodule over $(\mathcal{W}(M), \mathcal{W}(N))$, rather than $(\mathcal{W}(M^-), \mathcal{W}(N))$. The sign is important and is due to the fact that the quilted inhomogeneous pseudoholomorphic maps are defined with respect to the almost complex structure with the correct sign, forcing the boundary conditions to be ordered in the desired way demanded by the structure of a bimodule over $(\mathcal{W}(M), \mathcal{W}(N))$.*

6.3. **The quilted Floer bimodule for Lagrangian immersions.** Now we would like to extend the A_∞-bimodule $\Phi(\mathcal{L})$ over $(\mathcal{W}(M), \mathcal{W}(N))$ to an A_∞-bimodule over $(\mathcal{W}(M), \mathcal{W}_{im}(N))$. The construction can be viewed as a generalization of that in subsection 6.1.

Proposition 6.2. *The A_∞-bimodule $\Phi(\mathcal{L})$ over $(\mathcal{W}(M), \mathcal{W}(N))$ extends to an A_∞-bimodule over $(\mathcal{W}(M), \mathcal{W}_{im}(N))$. That is, there is a A_∞-bimodule over $(\mathcal{W}(M), \mathcal{W}_{im}(N))$, which composed with the pullback*

$$j^* : \mathcal{W}_{im}(N)^{l-mod} \to \mathcal{W}(N)^{l-mod}$$

agrees with $\Phi(\mathcal{L})$, up to homotopy equivalence of bimodules. Here $j : \mathcal{W}(N) \to \mathcal{W}_{im}(N)$ is the quasi-embedding (**??**).

The construction of this A_∞-bimodule structure involves moduli spaces similar to $\mathcal{R}^{k,l}((x^-, y^-); (x^+, y^+), \vec{x}, \vec{y})$ as (6.4), where now some of the conditions for the inhomogeneous pseudoholomorphic quilted maps are slightly modified. First, the Lagrangian submanifolds L'_j as boundary conditions are replaced by the images of Lagrangian immersions $\iota_j : L'_j \to N$. Second, we need to include some additional information: the switching labels α_j for the Lagrangian immersions $\iota_j : L'_j \to N$, and the relative homotopy class β of the map. Third, the Hamiltonian chords y_j from L'_{j-1} to L'_j are now replaced by appropriate generators for the wrapped Floer cochain space $CW^*((L'_{j-1}, \iota_{j-1}), (L'_j, \iota_j); H)$, which are either critical points of an auxiliary Morse function on the intersection components, or non-constant time-one Hamiltonian chords contained in the cylindrical end of N. Denoting these generators by the same letters $\vec{y} = (y_1, \cdots, y_l)$, we write the corresponding moduli space by

$$\mathcal{R}^{k,l}(\vec{\alpha}, \beta; (x^-, y^-); (x^+, y^+), \vec{x}, \vec{y}),$$

with the additional information included.

There is a natural compactification

$$\bar{\mathcal{R}}^{k,l}(\vec{\alpha}, \beta; (x^-, y^-); (x^+, y^+), \vec{x}, \vec{y})$$

which is obtained from $\mathcal{R}^{k,l}(\vec{\alpha}, \beta; (x^-, y^-); (x^+, y^+), \vec{x}, \vec{y})$ by adding stable broken inhomogeneous pseudoholomorphic disks in M and stable broken pearly tree maps in N with boundary on the image of one of the Lagrangian immersions $\iota_j : L'_j \to N$, as well as broken quilted maps. As an analogue to (6.5), the codimension-one boundary strata can thus be described as follows:

(6.18)

$$\begin{aligned}
&\coprod_{\substack{1 \le i \le k \\ (\vec{x}', \vec{x}'') = \vec{x}}} \coprod_{x_{new}} \bar{\mathcal{R}}^{1,k-i+1,l}((x^-, y^-); \vec{x}', x_{new}, \gamma, (x^+, y^+), \vec{y}) \times \bar{\mathcal{M}}_{i+1}(x_{new}, \vec{x}'') \\
\cup &\coprod_{\substack{1 \le j \le l \\ (\vec{y}', \vec{y}'') = \vec{y}}} \coprod_{y_{new}} \bar{\mathcal{R}}^{1,l,l-j+1}((x^-, y^-); \vec{x}, \gamma, (x^+, y^+), \vec{y}') \times \bar{\mathcal{M}}_{j+1}(y_{new}, \vec{y}'') \\
\cup &\coprod_{\gamma_1} \bar{\mathcal{R}}^{1,k,l}((x^-, y^-); \vec{x}, \gamma_1, (x^+, y^+), \vec{y}) \times \bar{\mathcal{M}}(\gamma_1, \gamma) \\
\cup &\coprod_{\substack{k'+k''=k, l'+l''=l \\ (\vec{x}', \vec{x}'') = \vec{x}, (\vec{y}', \vec{y}'') = \vec{y}}} \coprod_{(x_1^+, y_1^+)} \bar{\mathcal{R}}^{1,k',l'}((x^-, y^-); \vec{x}', \gamma, (x_1^+, y_1^+), \vec{y}') \\
\times &\bar{\mathcal{R}}^{0,k'',l''}((x_1^+, y_1^+); \vec{x}'', (x^+, y^+), \vec{y}'') \\
\cup &\coprod_{\substack{k'+k''=k, l'+l''=l \\ (\vec{x}', \vec{x}'') = \vec{x}, (\vec{y}', \vec{y}'') = \vec{y}}} \coprod_{(x_1^+, y_1^+)} \bar{\mathcal{R}}^{0,k',l'}((x^-, y^-); \vec{x}', (x_1^+, y_1^+), \vec{y}') \\
\times &\bar{\mathcal{R}}^{1,k'',l''}((x_1^+, y_1^+); \vec{x}'', \gamma, (x^+, y^+), \vec{y}'').
\end{aligned}$$

Following the same lines as in the construction of Kuranishi structures on the moduli space of inhomogeneous pseudoholomorphic disks that are used to constructe curved A_∞-structures for the immersed wrapped Fukaya category introduced in section 5, we can also construct Kuranishi structures on these moduli spaces. This compactification is obtained from $\mathcal{R}^{k,l}(\vec{\alpha}, \beta; (x^-, y^-); (x^+, y^+), \vec{x}, \vec{y})$ by adding broken inhomogeneous pseudoholomorphic punctured disks in both M and N, as well as broken quilted maps. In particular, those broken inhomogeneous pseudoholomorphic disks in N with boundary conditions given by the Lagrangian immersions $\iota_j : L'_j \to N$ form moduli spaces which carry Kuranishi structures as discussed before. Thus, by the inductive nature of the construction of Kuranishi structures, it remains to build Kuranishi charts on codimension zero strata of various moduli spaces $\mathcal{R}^{k',l'}(\vec{\alpha}', \beta'; (x^-, y^-); (x^+, y^+), \vec{x}', \vec{y}')$ for $k' \leq k, l' \leq l$. That is, we need to build Kuranishi charts over the locus in the moduli spaces whose elements have smooth domains.

Suppose we are given an inhomogeneous pseudoholomorphic quilted map $\sigma = (\underline{S}, (u, v))$ in $\mathcal{R}^{k,l}(\vec{\alpha}, \beta; (x^-, y^-); (x^+, y^+), \vec{x}, \vec{y})$, where $\underline{S}$ is the moduli parameter of the underlying quilted surface, and (u, v) is map to the manifold pair (M, N). The linearized operator $D_\sigma \bar{\partial}$ of the inhomogeneous Cauchy-Riemann equations for σ is Fredholm by non-degeneracy assumption. Thus the Fredholm complex, defined with respect to appropriate Sobolev norm $W^{1,p}$, has finite-dimensional reductions. We choose an obstruction space E_σ, which is a finite-dimensional subspace of $\Omega^{0,1}(\underline{S}; u^*TM \times v^*TN)$ such that for each $V \in E_\sigma$, the support of V is contained in a closed subset of the domain $\underline{S}$ away from the boundary components and the seam. Then, following the lines in section **??**, we can build a Kuranishi chart $(U_\sigma, E_\sigma, s_\sigma, \psi_\sigma, \Gamma_\sigma = \{1\})$ at this point σ.

In order to modify these charts for all elements in the moduli space so that they together define a Kuranishi structure, we need to make sure that these charts glue well with Kuranishi charts for inhomogeneous pseudoholomorphic disks in M and N. To formulate this, consider the following moduli spaces

(i) $\mathcal{R}^{k',l'}(\vec{\alpha}', \beta'; (x^-, y^-); (x^+, y^+), \vec{x}', \vec{y}')$,
(ii) $\mathcal{M}_{k''+1}(x_{new}, \vec{x}'')$,
(iii) $\mathcal{M}_{l''+1}(\alpha'', \beta''; y_{new}, \vec{y}'')$,

where $\vec{x} = (x_1, \cdots, x_k)$, and $\vec{x}' = (x_1, \cdots, x_i, x_{new}, x_{i+k''+1}, \cdots, x_k)$, and $\vec{x}'' = (x_{i+1}, \cdots, x_{i+k''})$; similarly for the y's. The union of the product moduli spaces

$$\coprod_{\substack{i \\ x_{new}}} \coprod_{\substack{j \\ y_{new}}} \mathcal{R}^{k',l'}(\vec{\alpha}', \beta'; (x^-, y^-); (x^+, y^+), \vec{x}', \vec{y}') \times \mathcal{M}_{k''+1}(x_{new}, \vec{x}'') \times \mathcal{M}_{l''+1}(y_{new}, \vec{y}'') \tag{6.19}$$

is a boundary stratum of the compactification $\bar{\mathcal{R}}^{k,l}(\vec{\alpha}, \beta; (x^-, y^-); (x^+, y^+), \vec{x}, \vec{y})$, and under the gluing maps, it can be thickened to a neighborhood of the boundary in the compactification. The gluing happens near the ends for the quilted surface and respectively the punctured disks, over which the quilted map and respectively the inhomogeneous pseudoholomorphic map converge to x_{new}; similarly for the ends with convergence condition y_{new}. Since the gluing construction is local, the process is the same as gluing inhomogeneous pseudoholomorphic disks along strip-like ends. Since the various obstruction spaces are chosen such that the vectors have compact support away from the boundary of the disks and quilted surfaces, the obstruction spaces also glue well under the gluing map. Thus we may apply the process in section **??** to modify the Kuranishi charts so that they form a Kuranishi structure on the moduli space $\bar{\mathcal{R}}^{k,l}(\vec{\alpha}, \beta; (x^-, y^-); (x^+, y^+), \vec{x}, \vec{y})$.

Moreover, such construction extends to boundary strata of higher codimension, by an inductive argument. This implies that we can construct fiber product Kuranishi structures on (6.18), which are are compatible with the Kuranishi structure on $\bar{\mathcal{R}}^{k,l}(\vec{\alpha}, \beta; (x^-, y^-); (x^+, y^+), \vec{x}, \vec{y})$. This proves:

Proposition 6.3. *There exists an oriented Kuranishi structure on every such moduli space* $\bar{\mathcal{R}}^{k,l}(\vec{\alpha}, \beta; (x^-, y^-); (x^+, y^+), \vec{x}, \vec{y})$*, which is compatible with the fiber product Kuranishi structures on* (6.18)*. That is, the restriction of the Kuranishi structure on* $\bar{\mathcal{R}}^{k,l}(\vec{\alpha}, \beta; (x^-, y^-); (x^+, y^+), \vec{x}, \vec{y})$ *to* (6.18) *agrees with the fiber product Kuranishi structure.*

Choosing single-valued multisections for these Kuranishi structures gives rise to virtual fundamental chains on these moduli spaces, using which the desired A_∞-bimodule structure maps are defined.

Corollary 6.2. *The* A_∞*-bimodule* $\Phi(\mathcal{L})$ *over* $(\mathcal{W}(M), \mathcal{W}(N))$ *extends to an* A_∞*-bimodule over* $(\mathcal{W}(M), \mathcal{W}_{im}(N))$*. That is, there is an* A_∞*-bimodule over* $(\mathcal{W}(M), \mathcal{W}_{im}(N))$*, which composed with the pullback*

$$j^* : \mathcal{W}_{im}(N)^{l-mod} \to \mathcal{W}(N)^{l-mod}$$

agrees with $\Phi(\mathcal{L})$*, up to homotopy of bimodules. Here* $j : \mathcal{W}(N) \to \mathcal{W}_{im}(N)$ *is the quasi-embedding* (**??**)*.*

Proof. Such extension is presented above. Thus the only thing that we need to prove is that the pullback by j^* agrees with $\Phi(\mathcal{L})$ up to homotopy. One way to prove this is that we apply virtual techniques to construction virtual fundamental chains on the moduli space of inhomogeneous pseudoholomorphic quilted maps (6.4), regarding all Lagrangian submanifolds as Lagrangian immersions. If we stick with classical transversality methods for those moduli spaces, the other way is a straightforward analogue of the proof of Proposition **??**. As there is nothing essentially new, we leave the details to the interested reader.

□

By a parallel argument, we can also extend the the module-valued functor (6.3) to one with values in category of modules over the immersed wrapped Fukaya category:

Proposition 6.4. *There is a canonical extension of the* A_∞*-functor* (6.3) *to an* A_∞*-functor*

$$\Phi_{\mathcal{L}} : \mathcal{W}(M) \to \mathcal{W}_{im}(N)^{l-mod} \tag{6.20}$$

to the A_∞*-category of left* A_∞*-modules over* $\mathcal{W}_{im}(N)$*. That is, the composition by the pullback*

$$j^* : \mathcal{W}_{im}(N)^{l-mod} \to \mathcal{W}(N)^{l-mod}$$

agrees with (6.3)*, up to homotopy.*

Of course, this also follows from the process of converting a bimodule to a module-valued functor.

6.4. **Geometric composition of Lagrangian correspondences.** In order to obtain a functor that takes value in the actual immersed wrapped Fukaya category $\mathcal{W}_{im}(N)$ instead of the category of modules over it, we need to prove that the A_∞-functor (6.20) is representable, in the sense of [Fuk02]. It has been long noted that the geometric compositions of Lagrangian correspondences are natural candidates for the objects representing the modules defined above. A good reference for the definition and basic properties of geometric compositions is [WW12], in the case where the Hamiltonian perturbation is not present.

There are two issues in proving representability of (6.20). First, representability does not always hold if we only consider embedded Lagrangian submanifolds, as geometric compositions of Lagrangian correspondences are generally Lagrangian immersions. This is why we must go to the immersed wrapped Fukaya category and consider the A_∞-functor (6.20) instead of (6.3). Second, the geometric composition of Lagrangian correspondences might not have good geometric properties, thus not a priori admissible for wrapped Floer theory on the nose. For this, we must imposed further conditions so that they are admissible in the immersed wrapped Fukaya category.

Let us first recall the definition of geometric composition of Lagrangian correspondences. Given an admissible Lagrangian submanifold $L \subset M$ and an admissible Lagrangian correspondence $\mathcal{L} \subset M^- \times N$, in a generic situation the fiber product over the graph $\Gamma(\phi_{H_M}) \subset M^- \times M$ over the Hamiltonian symplectomorphism ϕ_{H_M}

$$L' = L \times_{\Gamma(\phi_{H_M})} \mathcal{L}$$

is a smooth submanifold of $M \times M \times N$. The composition of the embedding $L' \subset M \times M \times N$ with the projection $M \times M \times N \to N$ is a Lagrangian immersion:

$$\iota : L \times_{\Gamma(\phi_{H_M})} \mathcal{L} \to N.$$

We call this Lagrangian immersion the geometric composition of L with $\mathcal{L}$, under the perturbation by the Hamiltonian flow of H_M. By abuse of name, we sometimes also call the image of ι the geometric composition for simplicity, and often denote it by $L \circ_{H_M} \mathcal{L}$.

If the geometric composition happens to be properly embedded, it comes with a natural choice of a primitive, which makes it an exact Lagrangian submanifold of N. Because of the presence of the Hamiltonian perturbation, this primitive is slightly different from the naive sum, but instead takes the following form:

$$g = f + F \circ (\phi_{H_M} \times id_N) + i_{X_{H_M}} \lambda_M \tag{6.21}$$

where X_{H_M} is the Hamiltonian vector field of H_M, thought of as a Hamiltonian pulled back to $M^- \times N$. This formula is calculated in [Gao17b], which follows directly from the following formula for the change of the primitive for an exact Lagrangian submanifold under a Hamiltonian isotopy.

Lemma 6.2. *Let f be a primitive for L. Then the following function*

$$f + \iota_X \lambda \tag{6.22}$$

is a primitive for $\phi_H(L)$. Here X is the Hamiltonian vector field of H.

Proof. The proof is a straightforward calculation, based on the well-known fact that a Hamiltonian symplectomorphism is exact and adds to the primitive of the symplectic form the following:

$$d \int_0^1 \iota_{X_t} \lambda dt. \tag{6.23}$$

Now since our Hamiltonian is time-independent, this is simply equal to

$$d\iota_X \lambda. \tag{6.24}$$

Thus a primitive for $\phi_H(L)$ can be taken to be

$$f + \iota_X \lambda \tag{6.25}$$

□

Depending on the geometry of L and $\mathcal{L}$, the geometric composition might be or not be cylindrical, even if it is embedded. Therefore a proof of its admissibility in wrapped Floer theory is completely necessary. This is done in [Gao17b] in the case where the geometric composition is properly embedded, with the above choice of the primitive. Technically, there we only proved that the wrapped Floer differential converges (and is in fact finite), but the same argument can be utilized to prove that higher order structure maps are also finite.

In general, the geometric composition $\iota : L \circ_{H_M} \mathcal{L} \to N$ is not an embedding, but we still expect it to have some favorable properties. This primitive g still makes ι an "exact" Lagrangian immersion, in the sense that $\iota^* \lambda_N = dg$.

There is also the notion of geometric composition in the usual sense. Instead of taking the fiber product over the graph $\Gamma(\phi_{H_M})$, we take the fiber product over the diagonal Δ_M. The geometric composition $L \circ \mathcal{L}$ is the map

$$\iota : L \times_{\Delta_M} \mathcal{L} \to N.$$

This is a Lagrangian immersion if the fiber product is generically taken, and is also exact in the generalized sense, with given primitive

$$h = f + F.$$

In [Gao17b], we proved well-definedness of wrapped Floer cohomology in the case where the geometric composition is properly embedded. The argument can be generalized to well-definedness of A_∞-structure maps of all orders. However, we shall take a slightly different (but Floer-theoretically equivalent) point of view, in order to make our construction more functorial and canonical. In [Gao17b], we considered the geometric composition under the large perturbation $L \circ_{H_M} \mathcal{L}$, and proved an isomorphism of Floer cohomology groups. While that definition is natural as one can find a natural one-to-one correspondence between the generators, we find it better to work with the geometric composition in the usual sense, when the whole categorical structure is in concern. Of course, the left-module structures associated to the geometric composition under large perturbation and the geometric composition in the usual sense are homotopy equivalent, so the essential difference is minor.

For the geometric composition of Lagrangian correspondences $L \circ \mathcal{L}$ to have well-defined wrapped Floer theory in general, we must have good control of the behavior of inhomogeneous pseudoholomorphic disks(modeled as stable pearly tree maps) bounded by the image of the geometric composition. For this purpose, we need to make sure that the geometry of $L \circ \mathcal{L}$ at infinity does not behave too badly. Thus it is natural to introduce the following assumption.

Assumption 6.1. *For the Lagrangian submanifold L in consideration, the geometric composition $L \circ \mathcal{L}$ is a proper Lagrangian immersion with transverse or clean self-intersections, which is cylindrical in the generalized sense for a Lagrangian immersion.*

When defining the wrapped Fukaya category $\mathcal{W}(M)$, we have to specify a class of Lagrangian submanifolds as objects. Since Assumption 6.1 is generic, it is possible for us to choose a countable collection of Lagrangian submanifolds as objects of the wrapped Fukaya category $\mathcal{W}(M)$, such that for every L in this collection, Assumption 6.1 is satisfied. Then it follows almost from the definition that:

Proposition 6.5. *Under Assumption 6.1, there is a well-defined curved A_∞-algebra for the geometric composition $\iota : L \times_{\Delta_M} \mathcal{L} \to N$, in the sense of immersed wrapped Floer theory discussed in sections 5, for every L from the collection of objects of $\mathcal{W}(M)$.*

Compared to the cohomological result in [Gao17b], which works with the geometric composition under large perturbation, the assumption of Proposition 6.5 is in fact simpler as we assume the geometric composition $L \circ \mathcal{L}$ to be cylindrical. As discussed before, it is automatically exact, so the results of section 5.

6.5. **Unobstructedness of the geometric composition.** We have shown that if $\mathcal{L} \to N$ is proper and if Assumption 6.1 is satisfied, there is a curved A_∞-algebra structure on the wrapped Floer cochain space $CW^*(L \circ \mathcal{L}; H_N)$ for the geometric composition $L \circ \mathcal{L}$. To make it into an object of the immersed wrapped Fukaya category, we must also prove that this curved A_∞-algebra is unobstructed. The main result of this subsection says that the geometric composition $L \circ \mathcal{L}$ is unobstructed with a canonical and unique choice of bounding cochain b satisfying a distinguished property.

Theorem 6.1. *Suppose L is a properly embedded exact cylindrical Lagrangian submanifold of M, and $\mathcal{L} \subset M^- \times N$ is a properly embedded exact cylindrical Lagrangian correspondence between M and N, such that the projection $\mathcal{L} \to N$ is proper. Let $L \circ \mathcal{L}$ be their geometric composition. Then $L \circ \mathcal{L}$ is unobstructed in the sense of wrapped Floer theory, with a canonical and unique choice of bounding cochain b determined by L and $\mathcal{L}$, with the property that b gives rise to non-curved deformations for both the quilted Floer module $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$ and the curved A_∞-algebra $CW^*(L \circ \mathcal{L})$.*

In particular, $(L \circ \mathcal{L}, b)$ becomes an object in the immersed wrapped Fukaya category $\mathcal{W}_{im}(N)$.

The idea to prove Theorem 6.1 is to use Lemma 2.4 to provide an algebraic argument for the existence and uniqueness of such a bounding cochain. For that purpose, we shall first equip the quilted wrapped Floer cochain space $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$ with a curved A_∞-module structure.

Lemma 6.3. *There is a natural curved left A_∞-module structure on $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$ over the curved A_∞-algebra $(CW^*(L \circ \mathcal{L}), m^k)$.*

The construction involves moduli spaces of inhomogeneous pseudoholomorphic quilted maps of the following kind.

Consider the quilted surface $\underline{S}^{mod}$ consisting of two patches, S_0^{mod}, S_1^{mod}, each of which is a disk with one negative puncture z_i^- and one positive puncture z_i^+. Given generalized chords $(x^0, y^0), (x^1, y^1)$ for $(L, \mathcal{L}, L \circ \mathcal{L})$, consider quilted maps

$\underline{u} : \underline{S}^{mod} \to (M, N)$ satisfying the following conditions

$$
(6.26) \qquad \begin{cases} \frac{\partial u_0}{\partial s} + J_M \circ (\frac{\partial u_0}{\partial t} - X_{H_M}(u)) = 0 \\ \frac{\partial u_0}{\partial s} + J_M \circ (\frac{\partial u_0}{\partial t} - X_{H_M}(u)) = 0 \\ u_0(s,0) \in L \\ u_1(s,1) \in L \circ \mathcal{L} \\ (u_0(s,1), u_1(s,0)) \in \mathcal{L} \\ \lim_{s\to-\infty} u_0(s,\cdot) = x^0(\cdot) \\ \lim_{s\to+\infty} u_0(s,\cdot) = x^1(\cdot) \\ \lim_{s\to-\infty} u_1(s,\cdot) = y^0(\cdot) \\ \lim_{s\to+\infty} u_1(s,\cdot) = y^1(\cdot) \end{cases}
$$

We also need to specify some additional data such as the switching conditions, the lifting conditions and the homology classes that are introduced in section 5.6. Quotienting by translations, we obtain moduli spaces of solutions to the above equation with these additional requirements.

To define a curved A_∞-module structure, we also need to study inhomogeneous pseudoholomorphic quilted maps of the same kind, but with more punctures. Break the boundary of S_1^{mod} that gets mapped to $L \circ \mathcal{L}$ into several pieces, namely replace S_1^{mod} by $S_1^{mod,k}$, a disk with $(k+2)$-boundary punctures, $z_1^-, z_1^1, \cdots, z_1^k, z_1^+$. We denote this new quilted surface by $\underline{S}^{mod,k}$.

Since the map u_1 on the second patch of the quilted surface has boundary condition being an immersed Lagrangian submanifold, it is necessary to include additional data α, β, l, etc. that indicate switching conditions of the boundary lifting, homology classes of the maps and so on. These data are analogous to those discussed in sections 5 and **??**, so we omit the details here.

There is a natural compactification of the moduli space of these quilted maps, which we denote by

$$
\bar{\mathcal{Q}}_k(\alpha, \beta; (x^-, y^-); y^1, \cdots, y^k, (x^+, y^+)).
$$

A typical element in the compactified moduli space is a broken inhomogeneous pseudoholomorphic quilted map $\{(u_i, v_i)\}$ with trees of inhomogeneous pseudoholomorphic disks in N attached to the lower boundary components of the second patches v_i's of the broken quilted map. In general, we have the following description of the codimension-one boundary strata of the above moduli space:

$$
(6.27) \qquad \begin{aligned} &\partial\bar{\mathcal{Q}}_k(\alpha, \beta; (x^-, y^-); y^1, \cdots, y^k, (x^+, y^+)) \\ \cong &\coprod_{0\le i\le k} \coprod_{\substack{\alpha_1\cup\alpha_2=\alpha \\ \beta_1\sharp\beta_2=\beta}} \coprod_{(x_1^+, y_1^+)} \bar{\mathcal{Q}}_i(\alpha_1, \beta_1; (x^-, y^-); y^1, \cdots, y^i, (x_1^+, y_1^+)) \\ &\times \bar{\mathcal{Q}}_{k-i}(\alpha_2, \beta_2; (x_1^+, y_1^+); y^{i+1}, \cdots, y^k, (x^+, y^+)) \\ \cup &\coprod_{\substack{k_1+k_2=k+1 \\ 1\le i\le k_1}} \coprod_{\substack{\alpha_1\cup\alpha_2=\alpha \\ \beta_1\sharp\beta_2=\beta}} \coprod_{y_{new}} \bar{\mathcal{Q}}_{k_1}(\alpha_1, \beta_1; (x^-, y^-); y^1, \cdots, y^{k_1}, \\ &y_{new}, y^{i+k_2+1}, \cdots, y^k, (x^+, y^+)) \\ &\times \bar{\mathcal{M}}_{i+1}(\alpha_2, \beta_2; y_{new}, y^{k_1+1}, \cdots, y^{i+k_2}). \end{aligned}
$$

Since L and $\mathcal{L}$ are exact, there are no pseudoholomorphic disks bubbling off L or $\mathcal{L}$. Thus the above fiber products (6.27) cover all the boundary strata of the compactification.

As usual, we can construct Kuranishi structures on these moduli spaces compatibly and use single-valued multisections to define virtual fundamental chains.

Proposition 6.6. *There exists an oriented Kuranishi structure on the moduli space*

$$\bar{\mathcal{Q}}_k(\alpha, \beta; (x^-, y^-); y^1, \cdots, y^k, (x^+, y^+)),$$

such that the induced Kuranishi structures on the boundary strata are isomorphic to the fiber product Kuranishi structures on (6.27).

In addition, we may choose single-valued multisections on these moduli spaces, such that the multisection on $\bar{\mathcal{Q}}_k(\alpha, \beta; (x^-, y^-); y^1, \cdots, y^k, (x^+, y^+))$ *is compatible with the fiber product multisections at the boundary strata* (6.27).

The virtual fundamental chains on these moduli spaces associated to a coherent choice of multisections then define a curved A_∞-module structure on the quilted wrapped Floer cochain space $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$ over the curved A_∞-algebra $CW^*(L \circ \mathcal{L})$.

To apply the result in section 2.5, we also need to find natural filtrations for the curved A_∞-algebra $CW^*(L \circ \mathcal{L})$ and the curved A_∞-module $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$ over it. These filtrations are given by the symplectic action functional.

Lemma 6.4. *The action filtration on* $CW^*(L \circ \mathcal{L})$ *defines a discrete filtration for the curved* A_∞*-algebra* $(CW^*(L \circ \mathcal{L}), m^k)$.

The action filtration on $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$ *defines a discrete filtration for the curved* A_∞*-module* $(CW^*(L, \mathcal{L}, L \circ \mathcal{L}), n^k)$ *is compatible with the action filtration for* $(CW^*(L \circ \mathcal{L}), m^k)$.

Moreover, these filtrations are bounded above.

Proof. The proof of the fact that action filtrations are compatible with the curved A_∞-algebra structure and the curved A_∞-module structure follows immediately from the action-energy relation.

To prove that the action filtration on $CW^*(L \circ \mathcal{L})$ is discrete, we recall that the generators of a wrapped Floer cochain space consist of two kinds: first, critical points of auxiliary Morse functions on components of the self fiber product of the preimage of the immersion; second, non-constant Hamiltonian chords in the cylindrical ends together with lifting indices. There are finitely many critical points, and we can arrange the primitive and choose the auxiliary Morse functions carefully so that their actions are different. On the other hand, ecause the Hamiltonian is non-degenerate in the cylindrical end, these non-constant Hamiltonian chords are non-degenerate and have a discrete action spectrum. A similar argument applies to show that the action filtration on $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$ is discrete.

Compatibility follows from the action-energy relation applied to inhomogeneous pseudoholomorphic quilted maps in the moduli spaces $\bar{\mathcal{Q}}_k(\alpha, \beta; (x^-, y^-); y^1, \cdots, y^k, (x^+, y^+))$, which are used to define this curved A_∞-module structure.

The fact that these filtrations are bounded above follows from the definition of the wrapped Floer cochain space: there are only finitely many free generators which have positive action, as those infinitely many generators, the non-constant Hamiltonian chords in the cylindrical end, all have negative action.

□

To finish the proof of Theorem 6.1, we need to find a cyclic element for the curved A_∞-module $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$ over the curved A_∞-algebra $CW^*(L \circ \mathcal{L})$. Recall from section 5.4 that in the wrapped Floer cochain space $CW^*(L \circ \mathcal{L})$, we have a distinguished generator - the minimum of chosen Morse function f on the diagonal component $\Delta_{L \times_{\Delta_M} \mathcal{L}}$ of the self fiber product $(L \times_{\Delta_M} \mathcal{L}) \times_\iota (L \times_{\Delta_M} \mathcal{L})$. This corresponds to the fundamental chain of the manifold $L \times_{\Delta_M} \mathcal{L}$ in the singular chain model. This generator is the homotopy unit for the curved A_∞-algebra $(CW^*(L \circ \mathcal{L}), m^k)$.

As graded $\mathbb{Z}$-modules, we have that $CW^*(L, \mathcal{L}, L \circ \mathcal{L}) \cong CW^*(L \circ \mathcal{L})$, given by a natural bijective correspondence between the sets of generators. This follows directly from the definition of the geometric composition. Under this correspondence, we get a distinguished element $e_{L \circ \mathcal{L}} \in CW^*(L, \mathcal{L}, L \circ \mathcal{L})$, corresponding to the homotopy unit of $CW^*(L \circ \mathcal{L})$.

Lemma 6.5. *The element $e_{L \circ \mathcal{L}} \in CW^*(L, \mathcal{L}, L \circ \mathcal{L})$ defined above is a cyclic element.*

Proof. Recall that a cylic element has to satisfy two conditions. First, the map

$$CW^*(L \circ \mathcal{L}) \to CW^*(L, \mathcal{L}, L \circ \mathcal{L})$$

defined by

$$x \mapsto n^1(x; e_{L \circ \mathcal{L}})$$

is an isomorphism of $\mathbb{Z}$-modules. Second, $e_{L \circ \mathcal{L}}$ lies in F^0, and applying n^0 to $e_{L \circ \mathcal{L}}$ should strictly increase the action filtration.

The first condition follows from the fact that $e_{L \circ \mathcal{L}}$ corresponds to the homotopy unit of $CW^*(L \circ \mathcal{L})$. Multiplication with the homotopy unit yields a self map on $CW^*(L \circ \mathcal{L})$, which can be written as an upper-triangular matrix with respect to a basis for $CW^*(L \circ \mathcal{L})$ ordered in increasing action. Moreover, the diagonal entries of this upper-triangular matrix are all equal to the identity. Now we consider the basis for $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$ which corresponds to the chosen basis for $CW^*(L \circ \mathcal{L})$ under the natural one-to-one correspondence between generators: each generalized chord for $(L, \mathcal{L}, L \circ \mathcal{L})$ corresponds to a unique Hamiltonian chord from $L \circ \mathcal{L}$ to itself (the same applies to critical points). This basis is also ordered in increasing action, so that the map $x \mapsto n^1(x; e_{L \circ \mathcal{L}})$ can be written as an upper-triangular matrix whose diagonal entries are all equal to the "identity", where this "identity" means the natural one-to-one correspondence between generators.

Now let us check the second condition. First, the element $e_{L \circ \mathcal{L}}$ itself is a free generator of the quilted wrapped Floer cochain space $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$, as it corresponds to the homotopy unit of $CW^*(L \circ \mathcal{L})$ under the natural one-to-one correspondence between generators. Moreover, we can choose the primitive carefully such that $e_{L \circ \mathcal{L}}$ has zero action. To prove that n^0 applied $e_{L \circ \mathcal{L}}$ strictly increases the action, it suffices to prove that there are no constant inhomogeneous pseudoholomorphic quilted strips with input being $e_{L \circ \mathcal{L}}$. Such a constant quilted strip, if existed, would correspond to a constant inhomogeneous pseudoholomorphic strip with boundary on the image of $L \circ \mathcal{L}$ with input being the homotopy unit of $CW^*(L \circ \mathcal{L})$. But there are no such constant strips, because the homotopy unit is the minimum of the chosen Morse function on the diagonal component $\Delta_{L \times_{\Delta_M} \mathcal{L}}$ of the self fiber product. As a consequence, $n^0(e_{L \circ \mathcal{L}})$ can be written as a linear combination of some generators of $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$, none of which is any critical point on the diagonal component of the self fiber product, or any non-constant Hamiltonian chord in the cylindrical end. Therefore $n^0(e_{L\circ\mathcal{L}})$ can be written as a linear combination of generators, which correspond to critical points on the switching components of the self fiber product of the Lagrangian immersion $L \circ \mathcal{L}$ under the natural one-to-one correspondence between the generators of $CW^*(L \circ \mathcal{L})$ and those of $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$. In particular, applying n^0 to $e_{L\circ\mathcal{L}}$ must strictly increase the action, because it is defined by counting non-constant inhomogeneous pseudoholomorphic quilted maps, which have strictly positive energy.

□

By a purely algebraic argument using Lemma 2.4, the existence of a cyclic element implies the unobstructedness of the geometric composition, with a unique bounding cochain b satisfying the following property:

Corollary 6.3. *There exists a unique (nilpotent) bounding chain $b \in CW^*(L \circ \mathcal{L})$ such that $b \in F^\epsilon$ for some $\epsilon > 0$, and*

$$d^b(e_{L\circ\mathcal{L}}) = 0,$$

where $d^b(\cdot) = \sum_{k=0}^{\infty} n^k(b, \cdots, b; \cdot)$.

Thus the proof of Theorem 6.1 is complete.

Remark 6.2. *Note that $CW^*(L \circ \mathcal{L})$ with respect to the undeformed structure maps is generally not a curved A_∞-module over itself, because of the non-vanishing of the curvature term m^0. However, the b-deformed structure maps $m^{k;b}$ make $CW^*(L \circ \mathcal{L}, b)$ an A_∞-module over itself, as the b-deformed A_∞-algebra is non-curved.*

Nonetheless, the curved A_∞-module structure on the quilted wrapped Floer cochain space $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$ is essentially different from that on $CW^*(L \circ \mathcal{L}, b)$ as a A_∞-module over itself, although the underlying $\mathbb{Z}$-modules are isomorphic. The quilted Floer-theoretic setup is essential for this curved A_∞-module structure to exist.

Concerning the wrapped Fukaya category, we have the following vanishing result of the bounding cochain b for the geometric composition, in the case where it is in fact a proper exact Lagrangian embedding.

Proposition 6.7. *If the geometric composition $\iota : L \circ \mathcal{L} \to N$ is a proper exact cylindrical Lagrangian embedding, whose primitive (coming from the primitive for L and that for $\mathcal{L}$) extends to a function on N which is locally constant in the cylindrical end of N, then the bounding cochain b from Theorem 6.1 vanishes.*

Sketch of proof. Recall that the bounding cochain b is the unique solution to the equation

$$n^{0;b}(e_{L\circ\mathcal{L}}) = \sum_k n^k(b, \cdots, b; e_{L\circ\mathcal{L}}) = 0, \tag{6.28}$$

with the property that $b \in F^{>0}$. This implies that for this choice of bounding cochain b, the map

$$gc^1 : CW^*(L, \mathcal{L}, (L \circ \mathcal{L}, b)) \to CW^*(L \circ \mathcal{L}, b) \tag{6.29}$$

is a cochain map with respect to the deformed differentials on both sides. Recall that the deformed differential on the quilted wrapped Floer cochain space is

$$n^{0;b}(x) = \sum_{k \geq 0} n^k(\underbrace{b, \cdots, b}_{k \text{ times}}; x),$$

while the deformed differential on the wrapped Floer cochain space $CW^*(L \circ \mathcal{L}, b)$ is

$$m^{1;b}(x) = \sum_{k_0, k_1 \geq 0} m^k(\underbrace{b, \cdots, b}_{k_0 \text{ times}}, x, \underbrace{b, \cdots, b}_{k_1 \text{ times}}).$$

Because of the assumption that ι is a proper embedding, 0 is a bounding cochain for $L \circ \mathcal{L}$. We want to prove that, if we choose 0 as the bounding cochain for $L \circ \mathcal{L}$, the cyclic element is closed under the undeformed differential on $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$. The proof will be separated in the following three lemmas.

□

The first lemma is a general statement about the bounding cochain b, where we do not assume the geometric composition $L \circ \mathcal{L}$ is an embedding.

Lemma 6.6. *The bounding cochain b from Theorem 6.1 is supported only at the critical points, or non-constant Hamiltonian chords in the cylindrical end which have positive action.*

Proof. The statement follows immediately from the condition that $b \in F^{>0}$.

□

The second lemma lists some equivalent conditions for 0 to be the desired bounding cochain under the assumption that the geometric composition is an embedding.

Lemma 6.7. *Suppose that the geometric composition $\iota : L \circ \mathcal{L} \to N$ is a proper exact cylindrical Lagrangian embedding. Consider the following conditions:*

- *(i) The cyclic element $e_{L\circ\mathcal{L}}$ is closed under the undeformed differential on $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$.*
- *(ii) There does not exist an inhomogeneous pseudoholomorphic quilted strip with boundary condition $(L, \mathcal{L}, L \circ \mathcal{L})$, which converges to $e_{L\circ\mathcal{L}}$ over the positive quilted end (as input).*
- *(iii) There is no figure eight bubble which asymptotically converges to a generalized chord for $(L, \mathcal{L}, \mathcal{L}, L)$ over the negative quilted end.*

Then (iii) $\implies$ (ii) $\implies$ (i).

Proof. This lemma is proved in [Gao17b]. Let us briefly recall it here.

(ii) implies (i) by the definition of the quilted Floer module structure map on $CW^*(L, \mathcal{L}, L \circ \mathcal{L})$.

(iii) implies (ii) by a strip-shrinking argument. Suppose that there is no figure eight bubble as in (iii). If there is an inhomogeneous pseudoholomorphic quilted strip converging to $e_{L\circ\mathcal{L}}$ over the positive quilted end, then by shrinking it we get an inhomogeneous pseudoholomorphic strip in N with boundary on $L \circ \mathcal{L}$, which asymptotically converges to the unit of $CW^*(L \circ \mathcal{L})$ over the positive end. This is certainly not possible since $L \circ \mathcal{L}$ is a proper exact Lagrangian embedding.

□

Thus, to prove that the cyclic element $e_{L\circ\mathcal{L}}$ is closed under the undeformed quilted wrapped Floer differential, it suffices to prove condition (iii). That uses the third lemma stated as below, which adds the assumption that the primitive is locally constant.

Lemma 6.8. *Suppose that $L \circ \mathcal{L}$ is a proper exact cylindrical Lagrangian embedding and that the primitive for $L \circ \mathcal{L}$ extends to a locally constant function in the* *cylindrical end of N. Then any Hamiltonian chord from $L \circ \mathcal{L}$ which lies outside the compact domain N_0 has negative action.*

Proof. Since the primitive is locally constant in the cylindrical end, the action of any Hamiltonian chord y from $L \circ \mathcal{L}$ to itself contained in the cylindrical end is

$$\mathcal{A}(y) = -\int y^*\lambda_N + H_N(y(t))dt = -r^2 < 0,$$

if y lies on the level set $\partial N \times \{r\}$. □

Suppose condition (iii) of Lemma (6.7) does not hold under the assumption of Proposition 6.7. Note that there is a uniform positive lower bound for the energy of any figure eight bubble. Thus if there were such a figure eight bubble, it would converge to some generalized chord for $(L, \mathcal{L}, \mathcal{L}, L)$ of positive action $\geq \epsilon > 0$, for some uniform constant $\epsilon > 0$ that is independent of individual pseudoholomorphic quilted maps but depends only on the background geometry - Liouville structures, Lagrangian submanifolds, Hamiltonians and almost complex structures. Then this generalized chord corresponds to some generator y of $CW^*(L \circ \mathcal{L})$ which has positive action. By the lemma above, this generator cannot be any non-constant Hamiltonian chord contained in the cylindrical end.

Thus, it suffices to prove that this generator y cannot be a critical point either. The strategy is to consider Lagrangian Floer theory without wrapping. To carry out this idea, we consider the sub-complex $CW^*_0(L \circ \mathcal{L})$ of $CW^*(L \circ \mathcal{L})$, generated by only critical points. Up to A_∞-homotopy equivalence, this is just the Morse complex of $H'|_{L\circ\mathcal{L}}$ with its Morse A_∞-structure defined by counting gradient flow trees. There is a similar subspace $CW^*_0(L, \mathcal{L}, L \circ \mathcal{L})$ of the quilted wrapped Floer cochain space, generated by generalized chords of low action (in absolute value), i.e. those generalized chords which correspond to Hamiltonian chords that are not contained in the cylindrical end of the product $M^- \times N$. Since $L \circ \mathcal{L}$ is assumed to be a proper exact cylindrical Lagrangian embedding, $CW^*_0(L, \mathcal{L}, L \circ \mathcal{L})$ is indeed a cochain complex equipped with the quilted Floer differential, which can be alternatively defined with respect to a pair of Hamiltonians on (M, N) that are C^2-small in a compact set and linear at infinity of small slope less than the minimal length of a Reeb chord. Then the map (6.41) restricted to $CW^*_0(L, \mathcal{L}, L \circ \mathcal{L})$ has image contained in the sub-complex $CW^*_0(L \circ \mathcal{L})$. Since these complexes can be identified with Floer complexes without Hamiltonian perturbations (or with small Hamiltonian perturbations if transversality is demanded) up to chain homotopy equivalences, this map (6.30) in fact becomes a cochain map

$$gc : CW^*_0(L, \mathcal{L}, L \circ \mathcal{L}) \to CW^*_0(L \circ \mathcal{L}). \tag{6.30}$$

after restriction without correction by any bounding cochain, by the argument of [LL13]. Moreover, it induces an isomorphism on cohomology groups. Note that the cyclic element $e_{L\circ\mathcal{L}}$ in fact lies in the subspace $CW^*_0(L, \mathcal{L}, L \circ \mathcal{L})$. In particular, it follows that $e_{L\circ\mathcal{L}}$ is closed under the undeformed quilted Floer differential on $CW^*_0(L, \mathcal{L}, L \circ \mathcal{L})$. Thus we can argue by an analogue of Lemma (6.7) in the setup of quilted Floer theory without wrapping, and conclude that the generator y in question cannot be any nonzero element in $CW^*_0(L \circ \mathcal{L})$. Therefore, $y = 0$ and there cannot be a figure eight bubble, which contradicts our assumption. The proof of Proposition 6.7 is now complete.

6.6. **Representability.** The previously constructed functor $\Phi_{\mathcal{L}}$ (6.20) is not good enough for understanding the functoriality properties of wrapped Fukaya categories, as modules over a non-proper A_∞-category can be very complicated. Thus we must find a more geometric replacement. In the case of compact monotone Lagrangian submanifolds in compact monotone symplectic manifolds, there are results from [WW12], [LL13] on the level of cohomology, which establish an isomorphism between the quilted Floer cohomology group and the Floer cohomology group of the geometric composition:

$$HF^*(L, \mathcal{L}, L') \cong HF^*(L \circ \mathcal{L}, L').$$

Now we would like to generalize this statement on the categorical level, aiming to prove that the Yoneda module associated to the geometric composition is homotopy equivalent to the module $\Phi_{\mathcal{L}}(L)$ defined in terms of quilted wrapped Floer theory, and moreover that such homotopy equivalences are functorial in the wrapped Fukaya category of M. Such a result can be improved to the statement that the module-valued functor $\Phi_{\mathcal{L}}$ (6.20) is representable.

In the previous subsection 6.5, we have shown that if the natural map $\mathcal{L} \to N$ is proper and if Assumption 6.1 holds, the geometric composition $L \circ \mathcal{L}$ is always unobstructed, and that there is a canonical choice of a bounding cochain b for it, determined by L and $\mathcal{L}$. Then, via the (left) Yoneda embedding

$$\mathfrak{y}_l : \mathcal{W}_{im}(N) \to \mathcal{W}_{im}(N)^{l-mod},$$

the distinguished object $(L \circ \mathcal{L}, b)$ defines a left A_∞-module over $\mathcal{W}_{im}(N)$. The main result of this subsection claims that this A_∞-module is homotopy to the module $\Phi_{\mathcal{L}}(L)$, which therefore yields the representability of the functor $\Phi_{\mathcal{L}}$ (6.20).

Theorem 6.2. *Suppose that $\mathcal{L} \subset M^- \times N$ is an admissible Lagrangian correspondence such that the map $\mathcal{L} \to N$ is proper, and Assumption 6.1 is satisfied. Then the A_∞-functor $\Phi_{\mathcal{L}}$ (6.20) is representable. That is, there exists a canonical A_∞-functor*

$$\Psi_{\mathcal{L}} : \mathcal{W}(M) \to \mathcal{W}_{im}(N)^{rep-l-mod} \tag{6.31}$$

such that $i \circ \Psi_{\mathcal{L}}$ is homotopic to $\Phi_{\mathcal{L}}$ as A_∞-functors, where

$$i : \mathcal{W}_{im}(N)^{rep-l-mod} \to \mathcal{W}_{im}(N)^{l-mod}$$

is the obvious inclusion of the sub-category of representable modules to the category of all modules.

The proof of this theorem will occupy the rest of this subsection. An immediate consequence of this theorem is that we get a functor to the immersed wrapped Fukaya category $\mathcal{W}_{im}(N)$:

Corollary 6.4. *There is an A_∞-functor*

$$\Theta_{\mathcal{L}} : \mathcal{W}(M) \to \mathcal{W}_{im}(N), \tag{6.32}$$

which represents the module-valued functor (6.20), in the sense that $\mathfrak{y}_l \circ \Theta_{\mathcal{L}}$ is homotopic to $\Phi_{\mathcal{L}}$.

Proof. The Yoneda lemma says that the left Yoneda functor is a homotopy equivalence onto its image, i.e.,

$$\mathfrak{y}_l : \mathcal{W}_{im}(N) \to \mathcal{W}_{im}(N)^{rep-l-mod}$$

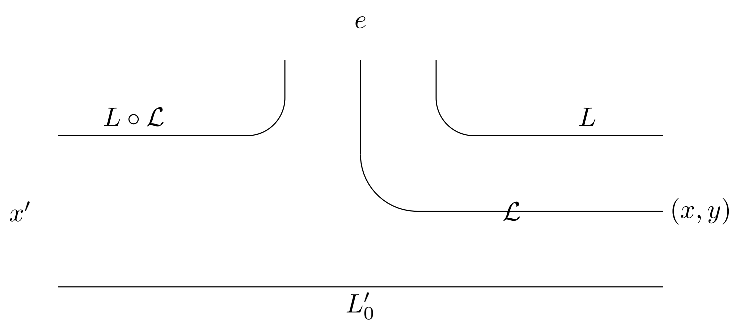

FIGURE 2. the quilted map defining the map gc

is a homotopy equivalence. Thus we may choose a homotopy inverse

$$\lambda_l : \mathcal{W}_{im}(N)^{rep-l-mod} \to \mathcal{W}_{im}(N),$$

and compose the functor $\Psi_{\mathcal{L}}$ with λ_l to obtain the desired functor $\Theta_{\mathcal{L}}$ (6.32). □

Our proof of Theorem 6.2 is a generalization of the proof of isomorphism of Floer cohomology groups under the geometric composition, as discussed in [Gao17b]. For any properly embedded exact cylindrical Lagrangian submanifold $L'_0 \subset N$, there is a canonical isomorphism

$$gc : CW^*(L, \mathcal{L}, L'_0) \to CW^*(L \circ \mathcal{L}, L'_0) \tag{6.33}$$

of $\mathbb{Z}$-modules, which become a chain quasi-isomorphism if we equip the latter cochain space $CW^*(L \circ \mathcal{L}, L'_0)$ with the differential deformed by the bounding cochain b for the geometric composition $L \circ \mathcal{L}$ provided by Theorem 6.1. We call this map the geometric composition quasi-isomorphism. Recall that in [Gao17b] we already defined this map in a slightly different form:

$$gc' : CW^*(L, \mathcal{L}, L'_0) \to CW^*(L \circ_{H_M} \mathcal{L}, L'_0)$$

in case the geometric composisition is properly embedded, and proved that it is a quasi-isomorphism. This map is related to (6.33) via the homotopy equivalence of left modules associated to $L \circ \mathcal{L}$ and $L \circ_{H_M} \mathcal{L}$ - these are modules over the curved A_∞-category $\mathcal{W}_{ob,im}(M)$ whose objects are proper exact cylindrical Lagrangian immersions (possibly without bounding cochains). This curved A_∞-category has been introduced in a somewhat implicit way when we defined the immersed wrapped Fukaya category, before the contributions of the bounding cochains to the structure maps are included. In fact, the same proof applies to the current setup: the map (6.33) is defined using moduli spaces

$$\mathcal{C}((x, y); x'; e) \tag{6.34}$$

of appropriate inhomogeneous pseudoholomorphic quilted maps, as pictured in Figure 2.

The asymptotic condition at the top quilted end (in Figure 2) is that the quilted map asymptotically converges to the generalized Hamiltonian chord e for the triple $(L, \mathcal{L}, L \circ \mathcal{L})$ representing the cyclic element $e = e_{L \circ \mathcal{L}} \in CW^*(L, \mathcal{L}, L \circ \mathcal{L})$, which in turn corresponds to the homotopy unit of $CW^*(L \circ \mathcal{L})$ under the $\mathbb{Z}$-module isomorphism (6.33). This cyclic element is discussed in the previous subsection, 6.5.

In order to see the contributions from the bounding cochain b for $L \circ \mathcal{L}$, we must modify these moduli spaces appropriately. Instead of looking at a single moduli space like (6.34), we consider a sequence of moduli spaces

$$\mathcal{C}_k((x, y); \underbrace{b, \cdots, b}_{k \text{ times}}; x'; e), \tag{6.35}$$

where we add k punctures to the boundary component of a quilted map as in Figure 2 which is mapped to $L \circ \mathcal{L}$, and impose the asymptotic convergence conditions at these punctures to be given by the bounding cochain b for $L \circ \mathcal{L}$. By counting elements in these moduli spaces of virtual dimension zero, we get a map

$$gc : CW^*(L, \mathcal{L}, L_0') \to CW^*((L \circ \mathcal{L}, b), L_0'). \tag{6.36}$$

The "count" requires careful treatment. As the geometric composition is in general no longer an embedding, we cannot use domain-dependent perturbations of Hamiltonians and almost complex structures to achieve transversality of the moduli spaces. The count is instead given by virtual fundamental chains associated to a coherent choice of single-valued multisections for Kuranishi structures on the moduli spaces. Such constructions have been discussed several times and should be routine by now, so we leave the details to the interested reader.

Lemma 6.9. *The map (6.36) is a cochain map, where the differential on $CW^*((L \circ \mathcal{L}, b), L_0')$ is given by the b-deformed structure map. That is,*

$$gc \circ n^0 = m^{1;b} \circ gc,$$

where n^0 denotes the quilted Floer differential on $CW^*(L, \mathcal{L}, L_0')$, and $m^{1;b}$ is the b-deformed Floer differential on $CW^*(L \circ \mathcal{L}, L_0')$.

Proof. By looking at the codimension-one boundary strata of the moduli spaces $\mathcal{C}_k((x, y); \underbrace{b, \cdots, b}_{k \text{ times}}; x'; e)$ (6.35), we find that the map gc as in (6.36) satisfies the following equation:

$$gc \circ n^0((x, y)) = m^{1;b} \circ gc((x, y)) + d^b(e). \tag{6.37}$$

Because the cyclic element e satisfies the condition that $d^b(e) = 0$, the last term vanishes, so the map (6.36) is a cochain map.

With the capping half-disks taken into account, there is a well-defined single-valued action of the generators, so that we can use an action-filtration argument to prove that (6.36) is a cochain isomorphism, as follows. If we truncate the Floer complex using the action filtration, then this map can be written as an upper-triangular matrix with all diagonal entries equal to the "identity", as counting non-trivial inhomogeneous pseudoholomorphic quilted maps as above necessarily increases the action. Here by "identity", we mean the natural one-to-one correspondence between the set of generators for $CW^*(L, \mathcal{L}, L_0')$ and that for $CW^*(L \circ \mathcal{L}, L_0')$. □

In the general case where L_0' is an exact cylindrical Lagrangian immersion, the proof of the map (6.36) being a cochain homotopy equivalence is in fact quite similar. The only difference is that the homotopy unit may no longer be closed under the previously-mentioned quilted Floer differential, because there are pseudoholomorphic disks with one marked point in N with boundary on the image of L'_0. However, L'_0 itself comes with a bounding cochain b'_0 for it to be an object of the immersed wrapped Fukaya category, and as long as we use the bounding cochain b'_0 on L'_0 to cancel the contribution of those disks, the homotopy unit of L'_0 becomes closed under the deformed quilted Floer differential. In this case, the map (6.36) takes the following form

$$gc : CW^*(L, \mathcal{L}, (L'_0, b'_0)) \to CW^*((L \circ \mathcal{L}, b), (L'_0, b'_0)). \tag{6.38}$$

Next, we shall construct a A_∞-pre-module homomorphism extending the map (6.38) as its first order term. We set $gc^0 = 0$. For $d \geq 2$, we define multilinear maps gc^d, for all possible $(d-1)$-tuple of testing objects $(L'_0, b'_0), \cdots, (L'_{d-1}, b'_{d-1})$ of $\mathcal{W}_{im}(N)$, as follows:

$$\begin{aligned} gc^d : &CW^*(L, \mathcal{L}, (L'_{d-1}, b'_{d-1})) \otimes CW^*((L'_{d-2}, b'_{d-2}), (L'_{d-1}, b'_{d-1})) \\ &\otimes \cdots \otimes CW^*((L'_0, b'_0), (L'_1, b'_1)) \to CW^*((L \circ \mathcal{L}, b), (L'_0, b'_0)) \end{aligned} \tag{6.39}$$

defined by appropriate "count" of elements in the moduli spaces

$$\bar{\mathcal{C}}_{k,d,l_0,\cdots,l_{d-1}}((x,y); \underbrace{b'_0, \cdots, b'_0}_{l_0 \text{ times}}, x'_1, \underbrace{b'_1, \cdots, b'_1}_{l_1 \text{ times}} \cdots, x'_{d-1}, \underbrace{b'_{d-1}, \cdots, b'_{d-1}}_{l_{d-1} \text{ times}}; \underbrace{b, \cdots, b}_{k \text{ times}}; x'; e) \tag{6.40}$$

of quilted maps, defined in a way similar to (6.35), but with multiple punctures and Lagrangian labels on the boundary components of the second patch of the quilted surface. The corresponding asymptotic convergence conditions are given by generators $x'_i \in CW^*((L'_{i-1}, b'_{i-1}), (L'_i, b'_i))$, and also those given by bounding cochains $b'_i \in CW^*((L'_i, b'_i)), i = 0, \cdots, d-1$.

Lemma 6.10. *The maps $\{gc^d\}$ form an A_∞-pre-module homomorphism*

$$gc : \Phi_{\mathcal{L}}(L) \to \mathfrak{y}_l((L \circ \mathcal{L}, b)) \tag{6.41}$$

from the left A_∞-module $\Phi_{\mathcal{L}}(L)$ over $\mathcal{W}_{im}(N)$, to the left Yoneda module $\mathfrak{y}_l((L \circ \mathcal{L}, b))$ over $\mathcal{W}_{im}(N)$. Moreover, this A_∞-pre-module homomorphism is in fact an A_∞-module homomorphism.

Proof. The verification of the A_∞-equations for pre-module homomorphisms can be done by looking at the boundary of the above-mentioned moduli spaces

$$\bar{\mathcal{C}}_{k,d,l_0,\cdots,l_{d-1}}((x,y); \underbrace{b'_0, \cdots, b'_0}_{l_0 \text{ times}}, x'_1, \underbrace{b'_1, \cdots, b'_1}_{l_1 \text{ times}} \cdots, x'_{d-1}, \underbrace{b'_{d-1}, \cdots, b'_{d-1}}_{l_{d-1} \text{ times}}; \underbrace{b, \cdots, b}_{k \text{ times}}; x'; e).$$

For this, simply recall that the A_∞-module structure on $CW^*(L, \mathcal{L}, \cdot)$ over $\mathcal{W}_{im}(N)$ is defined via suitable moduli spaces of inhomogeneous pseudoholomorphic quilted maps, while that on $CW^*((L \circ \mathcal{L}, b), \cdot)$ is defined via suitable moduli spaces of stable broken Floer trajectories in N, with decorations by the bounding cochain b. These are compatible with the compactification

$$\bar{\mathcal{C}}_{k,d,l_0,\cdots,l_{d-1}}((x,y); \underbrace{b'_0, \cdots, b'_0}_{l_0 \text{ times}}, x'_1, \underbrace{b'_1, \cdots, b'_1}_{l_1 \text{ times}} \cdots, x'_{d-1}, \underbrace{b'_{d-1}, \cdots, b'_{d-1}}_{l_{d-1} \text{ times}}; \underbrace{b, \cdots, b}_{k \text{ times}}; x'; e),$$

meaning that the two kinds of moduli spaces arise in the boundary strata of this moduli space. This fact can be used to show that the maps defined above satisfy the A_∞-equations for pre-module homomorphisms.

Now if we deform the structure maps on $CW^*(L \circ \mathcal{L}, \cdot)$ by the bounding cochain b, the cyclic element $e_{L \circ \mathcal{L}} \in CW^*(L, \mathcal{L}, (L \circ \mathcal{L}, b))$ becomes closed under the b-deformed Floer differential, which is the module differential for the left Yoneda module $\mathfrak{y}_l((L \circ \mathcal{L}, b))$. Thus the above A_∞-pre-module homomorphism is a cocycle in the functor category, i.e. an A_∞-module homomorphism.

□

Corollary 6.5. *The A_∞-module homomorphism gc (6.41) is a quasi-isomorphism of A_∞-modules.*

Proof. This follows from the fact that the first-order map gc^1 is an isomorphism of cochain complexes.

□

This module homomorphism gc is more than just a quasi-isomorphism, but indeed a homotopy equivalence. This statement is rather important because we are working over the integers. The proof is in fact very simple, based on our construction of gc.

Proposition 6.8. *The A_∞-module homomorphism*

$$gc : \Phi_{\mathcal{L}}(L) \to \mathfrak{y}_l((L \circ \mathcal{L}, b))$$

is a homotopy equivalence of A_∞-modules.

Proof. A homotopy inverse can be constructed using moduli spaces similar to (6.40), but we interchange the inputs and the outputs. That is, we regard x' as the input and (x, y) as the output, and construct a sequence of multilinear maps of the form

(6.42)
$$\begin{aligned} op^d : & CW^*((L \circ \mathcal{L}, b), (L'_{d-1}, b'_{d-1})) \otimes CW^*((L'_{d-2}, b'_{d-2}), (L'_{d-1}, b'_{d-1})) \otimes \cdots \\ & \otimes CW^*((L'_0, b'_0), (L'_1, b'_1)) \to CW^*(L, \mathcal{L}, (L'_0, b'_0)). \end{aligned}$$

These form an A_∞-module homomorphism

$$op : \mathfrak{y}_l((L \circ \mathcal{L}, b)) \to \Phi_{\mathcal{L}}(L).$$

Standard gluing argument in Floer theory implies that $gc^1 \circ op^1$ and $op^1 \circ gc^1$ are both chain homotopic to the identity, which implies that op is the a homotopy inverse of gc.

□

Thus, we set

$$\Psi_{\mathcal{L}}(L) = \mathfrak{y}_l((L \circ \mathcal{L}, b)), \tag{6.43}$$

the left Yoneda module of $(L \circ \mathcal{L}, b) \in Ob\mathcal{W}_{im}(N)$, for every object $L \in Ob\mathcal{W}(M)$. The next step is to prove that such an A_∞-module homotopy equivalence of A_∞-modules is functorial in $\mathcal{W}(M)$. For this purpose, we shall define multilinear maps

(6.44)
$$T^d : CW^*(L_{d-1}, L_d) \otimes \cdots \otimes CW^*(L_0, L_1) \to \hom_{\mathcal{W}_{im}(N)^{l-mod}}(\Phi_{\mathcal{L}}(L_0), \Psi_{\mathcal{L}}(L_d))[-d]$$

of degree $-d$, which satisfy the equations for A_∞-pre-natural transformations. In more concrete terms, we shall define a multilinear map for all possible Floer cochains

$x_1 \in CW^*(L_0, L_1), \cdots, x_d \in CW^*(L_{d-1}, L_d)$ as well as cylindrical Lagrangian immersions in N equipped with bounding cochains $(L'_0, b'_0), \cdots, (L'_{k-1}, b'_{k-1})$:

$$\begin{aligned}
&(T^d(x_d, \cdots, x_1))^k : CW^*((L_0 \circ \mathcal{L}, b_0), (L'_{k-1}, b'_{k-1})) \\
&\otimes CW^*((L'_{k-2}, b'_{k-1}), (L'_{k-1}, b'_{k-1})) \otimes \cdots \otimes CW^*((L'_0, b'_0), (L'_1, b'_1)) \\
&\to CW^*((L_d \circ \mathcal{L}, b_d), (L'_0, b'_0)),
\end{aligned} \tag{6.45}$$

which is linear with respect to each x_i, and satisfies the following equation:

$$\begin{aligned}
&m^1_{\mathcal{W}_{im}(N)^{l-mod}}(\Psi^d_{\mathcal{L}}(x_d, \cdots, x_1)) \\
&+ \sum_s m^2_{\mathcal{W}_{im}(N)^{l-mod}}(\Psi^{d-s}_{\mathcal{L}}(x_d, \cdots, x_{s+1}), \Psi^s_{\mathcal{L}}(x_s, \cdots, x_1)) \\
&= \sum_{n,k} (-1)^* \Psi^{d-k+1}_{\mathcal{L}}(x_d, \cdots, x_{n+k+1}, m^k_{\mathcal{W}(M)}(x_{n+k}, \cdots, x_{n+1}), x_n, \cdots, x_1),
\end{aligned} \tag{6.46}$$

where (only here) $* = |x_1| + \cdots + |x_n| - n$.

We consider the following quilted surfaces $\underline{S}^{nf}$ consisting of two patches S^{nf}_0, S^{nf}_1 where S^{nf}_0 is a disk with $(d+1)$ positive boundary punctures $z^{+,1}_0, z^1_0, \cdots, z^d_0$, and one negative puncture $z^{-,2}_0$, and S^{nf}_0 is a disk with $(k+1)$ positive boundary punctures $z^{+,1}_1, z^1_1, \cdots, z^{k-1}_1$, and one negative boundary puncture $z^-_1, z^{-,2}_1$. We denote by I^+_0 the boundary component of S^{nf}_0 between $z^{+,1}_0$ and $z^{+,2}_0$, and by $I^{\pm}_1$ the boundary component of S^{nf}_1 between z^-_1 and z^+_1. $\underline{S}^{nf}$ is obtained by seaming together the two patches along the pair $(I^+_0, I^{\pm}_1)$ of boundary components. We need to consider semi-stable nodal quilted surfaces arising as domains of limits of stable maps from such quilted surfaces, and we denote them by the same symbol.

Suppose strip-like ends and quilted ends for all semistable nodal quilted surfaces $\underline{S}^{nf}$ have been chosen. Make conformally consistent choices of Floer data for all such $\underline{S}^{nf}$, requiring the choices to automorphism-invariant Floer data for semistable nodal quilted surfaces that are domains of stable maps to M with Lagrangian boundary conditions that are to be specified below. Let $e_d = e_{L_d \circ \mathcal{L}}$ be the generator of $CW^*(L_d, \mathcal{L}, (L_d \circ \mathcal{L}, b_d))$ corresponding to the fundamental chain of $L_d \circ \mathcal{L}$. Consider the moduli space

$$\mathcal{T}^{nf}_{d,k-1}(\alpha, \beta; y^-; x^1, \cdots, x^d; y^1, \cdots, y^{k-1}; (x^+, y^+); e_d) \tag{6.47}$$

of triples $(\underline{S}^{nf}, \underline{u}, l_1)$, satisfying the following conditions:

(i) $\underline{u} : \underline{S}^{nf} \to (M, N)$ is a quilted map with marked points or punctures $\vec{\underline{z}}$ satisfying the following equations:

$$
\begin{cases}
(du_0 - \alpha_{S_0^{nf}} \otimes X_{H_{S_0^{nf}}})^{0,1} = 0 \\
(du_1 - \alpha_{S_1^{nf}} \otimes X_{H_{S_1^{nf}}})^{0,1} = 0 \\
u_0(z) \in \psi_M^{\rho_{S_0^{nf}}(z)} L_0, \text{ if } z \text{ lies between } z_0^{+,1} \text{ and } z_0^1 \\
u_0(z) \in \psi_M^{\rho_{S_0^{nf}}(z)} L_i, \text{ if } z \text{ lies between } z_0^i \text{ and } z_0^{i+1} \\
u_0(z) \in \psi_M^{\rho_{S_0^{nf}}(z)} L_d, \text{ if } z \text{ lies between } z_0^d \text{ and } z_0^{+,2} \\
u_1(z) \in \psi_N^{\rho_{S_1^{nf}}(z)} \iota_0(L_0'), \text{ if } z \text{ lies between } z_1^{+,1} \text{ and } z_1^1 \\
u_1(z) \in \psi_N^{\rho_{S_1^{nf}}(z)} \iota_j(L_j'), \text{ if } z \text{ lies between } z_1^j \text{ and } z_1^{j+1} \\
u_1(z) \in \psi_N^{\rho_{S_1^{nf}}(z)} \iota_{k-1}(L_{k-1}'), \text{ if } z \text{ lies between } z_1^{k-1} \text{ and } z_1^- \\
u_1(z) \in \psi_N^{\rho_{S_1^{nf}}(z)} L_d \circ_{H_M} \mathcal{L}, \text{ if } z \text{ lies between } z_1^- \text{ and } z_1^{+,2} \\
(u_0(z), u_1(z)) \in (\psi_M^{\rho_{S_0^{nf}}(z)} \times \psi_N^{\rho_{S_1^{nf}}(z)})\mathcal{L}, \text{ if } z \text{ lies on the seam} \\
\lim_{s \to -\infty} u_1 \circ \epsilon_1^-(s, \cdot) = \psi_N^{w_1^-} y^-(\cdot) \\
\lim_{s \to +\infty} (u_0 \circ \epsilon_0^{+,1}(s, \cdot), u_1 \circ \epsilon_1^{+,1}(s, \cdot)) = \psi_{M \times N}^{w^+}(x^+(\cdot), y^+(\cdot)) \\
\lim_{s \to +\infty} u_0 \circ \epsilon_0^i(s, \cdot) = \psi_M^{w_0^i} x^i(\cdot) \\
\lim_{s \to +\infty} (u_0 \circ \epsilon_0^{-,2}(s, \cdot), u_1 \circ \epsilon_1^{-,2}(s, \cdot)) = \psi_{M \times N}^{w^e} e_d(\cdot) \\
\lim_{s \to +\infty} u_1 \circ \epsilon_1^j(s, \cdot) = \psi_N^{w_1^j} y^j(\cdot)
\end{cases}
\tag{6.48}
$$

(ii) $l_1 : \partial S_1^{nf} \setminus \{z_1^i : i \in I\} \to (L_0' \times_{\iota_0} L_0') \cup \cdots (L_{k-1}' \times_{\iota_{k-1}} L_{k-1}') \cup ((L_d \circ \mathcal{L}) \times_\iota (L_d \circ \mathcal{L}))$ is a smooth map.

(iii) $\psi_N^{\rho_{S_1^{nf}}(l_1(z))} \iota_i = u_1 \circ l_1(z)$, when $z \in \partial S_1^{nf}$ lies between z_1^i and z_1^{i+1}, for every $1 \leq i \leq k-2$. If z lies between $z_1^{+,1}$ and z_1, the corresponding Lagrangian immersion should be replaced by $\iota_0 : L_0' \to N$. If z lies between $z_1^{+,2}$, the corresponding Lagrangian immersion should be replaced by $\iota_{k-1} : L_{k-1}' \to N$. If z lies between $z_1^{+,2}$ and $z_1^{+,1}$, the corresponding Lagrangian immersion should be replaced by $\iota : L_d \circ \mathcal{L} \to N$.

(iv) the relative homology class of $\underline{u}$ is β.

(v) the triple $(\underline{S}^{nf}, \underline{u}, l_1)$ is stable, meaning that it has finite automorphism group.

The above conditions are analogous to those in the case of a single Lagrangian immersion for which we defined the moduli space of inhomogeneous pseudoholomorphic disks. Recall the relevant notations in section 5.6.

There is a natural compactification $\bar{\mathcal{T}}_{d,k-1}^{nf}(y^-; x^1, \cdots, x^d; y^1, \cdots, y^{k-1}; (x^+, y^+))$ of this moduli space $\mathcal{T}_{d,k-1}^{nf}(y^-; x^1, \cdots, x^d; y^1, \cdots, y^{k-1}; (x^+, y^+))$, which consists of broken quilted maps of the same type. In particular, the codimension-one boundary

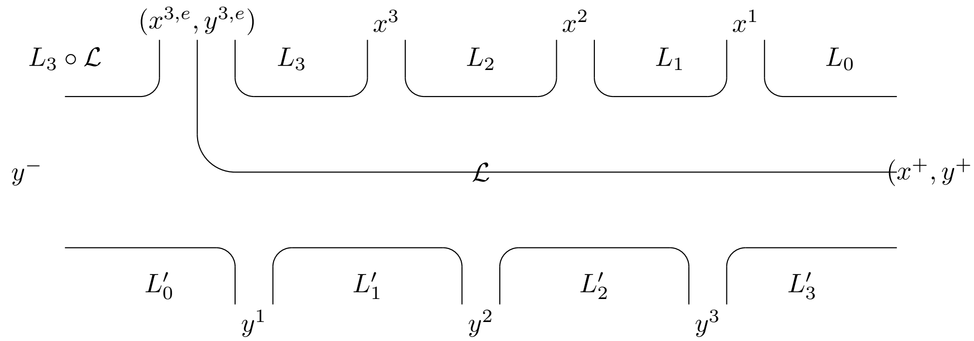

FIGURE 3. the quilted map defining the homotopy between two module-valued functors

strata consist of union of fiber products

$$
\begin{aligned}
&\partial\bar{\mathcal{T}}^{nf}_{d,k-1}(y^-; x^1, \cdots, x^d; y^1, \cdots, y^{k-1}; (x^+, y^+)) \\
&\cong \coprod \bar{\mathcal{T}}^{nf}_{d_1,k_1-1}(y^-; x^1, \cdots, x^i, x^{new}, x^{i+k_1+1}, \cdots, x^d; \\
&y^1, \cdots, y^j, y^{new}, y^{j+k_2+1}, \cdots, y^{k-1}; (x^+, y^+)) \\
&\times \bar{\mathcal{M}}_{d_2+1}(x^{new}, x^{i+1}, \cdots, x^{i+d_2}) \times \bar{\mathcal{M}}_{k_2+1}(y^{new}, y^{j+1}, \cdots, y^{j+k_2}) \\
&\cup \coprod \bar{\mathcal{T}}_{d_1,k_1-1}((x^-, y^-); x^1, \cdots, x^{d_1}; y^1, \cdots, y^{k_1-1}; (x_1^+, y_1^+)) \\
&\times \bar{\mathcal{T}}_{d_2,k_2-1}((x_1^+, y_1^+); x^{d_1+1}, \cdots, x^d; y^{k_1}, \cdots, y^{k-1}; (x^+, y^+))
\end{aligned}
\tag{6.49}
$$

where the compactified moduli space $\bar{\mathcal{T}}^{nf}_{d',k'-1}(\cdots)$ for $d' < d, k' < k$ are built inductively in this way. The stability condition ensures that the moduli space $\bar{\mathcal{T}}^{nf}_{d,k-1}(y^-; x^1, \cdots, x^d; y^1, \cdots, y^{k-1}; (x^+, y^+))$ is compact and Hausdorff.

To include the contribution from the bounding cochains b'_j for $\iota_j : L'_j \to N$, we modify elements in the moduli space $\bar{\mathcal{T}}^{nf}_{d,k-1}(y^-; x^1, \cdots, x^d; y^1, \cdots, y^{k-1}; (x^+, y^+))$ by adding more punctures on each boundary component of S_1^{nf} that is mapped to the image of one of the Lagrangian immersions $\iota_j : L'_j \to N$ and $L_d \circ \mathcal{L}$, and imposing the asymptotic convergence conditions at these additional puctures given by the bounding cochains b'_j. The resulting moduli space is denoted by

$$
\bar{\mathcal{T}}^{nf}_{d,k-1,s_0,\cdots,s_{k-1},s}(y^-; x^1, \cdots, x^d; y^1, \cdots, y^{k-1}; b'_0, \cdots, b'_0; \cdots; b'_{k-1}, \cdots, b'_{k-1}; (x^+, y^+)).
\tag{6.50}
$$

The picture of such a quilted map is shown in Figure 3, where we have omitted the bounding cochains, but shall remember that they are also included as suitable asymptotic convergence conditions on the boundary components of the second patch, with the prescribed number of punctures added.

Following the argument in section ??, we can construct Kuranishi structures on these moduli spaces, which are compatible with the fiber product Kuranishi structures at the boundary. By making a coherent choice of multisections on these Kuranishi spaces, we obtain the virtual fundamental chains, which give rise to the desired homotopy equivalence between the A_∞-modules $i \circ \Psi_{\mathcal{L}}$ and $\Phi_{\mathcal{L}}$.

It is also possible to extend $\Phi_{\mathcal{L}}, \Psi_{\mathcal{L}}$ as well as $\Theta_{\mathcal{L}}$ to the immersed wrapped Fukaya category $\mathcal{W}_{im}(M)$, which we omit as it is not needed for our current purposes.

6.7. **Categorification of the functors.** So far we have discussed the representability for the A_∞-functor (6.3) associated to a single Lagrangian correspondence $\mathcal{L} \subset M^- \times N$. It is straightforward to generalize this functorially in the wrapped Fukaya category of the product manifold $M^- \times N$. That is, we ask whether (6.1) is representable. The answer is affirmative, stated in Theorem 1.3.

Proof of Theorem 1.3. The A_∞-functor (6.1) defines in a natural way an A_∞-functor

$$\Phi : \mathcal{W}(M^- \times N) \to func(\mathcal{W}(M), \mathcal{W}(N)^{l-mod}). \tag{6.51}$$

This statement is proved in a purely algebraic way in section 2.4. Summarizing the argument, we compose the A_∞-functor (6.1) with the algebraically-defined A_∞-functor

$$(\mathcal{W}(M), \mathcal{W}(N))^{bimod} \to func(\mathcal{W}(M), \mathcal{W}(N)^{l-mod}) \tag{6.52}$$

to obtain the desired functor.

In the previous two subsections, we have proved that if the projection $\mathcal{L} \to N$ is proper and if Assumption 6.1 holds for every $L \in Ob\mathcal{W}(M)$, then the filtered module-valued functor $\Phi_{\mathcal{L}}$ is representable. Therefore, Φ is representable over the full subcategory $\mathcal{A}(M^- \times N) \subset \mathcal{W}(M^- \times N)$, in the sense of Definition 2.4. Thus we may rewrite the above A_∞-functor as

$$\Psi : \mathcal{A}(M^- \times N) \to func(\mathcal{W}(M), \mathcal{W}_{im}(N)^{rep-l-mod}). \tag{6.53}$$

Composing this with a homotopy inverse

$$\lambda_l : \mathcal{W}_{im}(N)^{rep-l-mod} \to \mathcal{W}_{im}(N)$$

of the left Yoneda functor

$$\mathfrak{y}_l : \mathcal{W}_{im}(N) \to \mathcal{W}_{im}(N)^{rep-l-mod},$$

we obtain the desired A_∞-functor Θ (1.6). Technically speaking, we shall require that Assumption 6.1 hold for every Lagrangian correspondence $\mathcal{L}$ in the subcategory $\mathcal{A}(M^- \times N)$, which again is a generic condition on the class of objects of the wrapped Fukaya category of the product manifold $M^- \times N$.

□

6.8. **A geometric realization of the cochain map for the correspondence functor.** In practice, it is helpful to have a more direct and geometric construction of the functor (6.32), without referring to the implicit construction with the help of the Yoneda lemma. At this time there are still some technical issues in fully realizing this, but it is possible to construct a cochain map, which is homotopic to the first order map of (6.32). This construction is useful in some applications, for example when studying the relation to the Viterbo restriction functor.

Fix an admissible Lagrangian correspondence $\mathcal{L} \subset M^- \times N$ such that the projection $\mathcal{L} \to N$ is proper. Suppose $L \subset M$ is an admissible Lagrangian submanifold, which can be made as an object of $\mathcal{W}(M)$. Recall that the geometric composition

$\iota : L \circ \mathcal{L} \to N$ comes with a canonical and unique bounding cochain b. Given a pair (L_0, L_1), we define a map

$$\Pi_{\mathcal{L}} : CW^*(L_0, L_1; H_M) \to CW^*((L_0 \circ \mathcal{L}, b_0), (L_1 \circ \mathcal{L}, b_1); H_U) \tag{6.54}$$

in the following way. Consider the moduli spaces

$$\mathcal{U}_{l_0,l_1}(\alpha, \beta; x; y; y_{0,1}, \cdots, y_{0,l_0}; y_{1,1}, \cdots, y_{1,l_1}; e_0, e_1)$$

of quilted inhomogeneous pseudoholomorphic maps $(\underline{S}, (u, v))$ in (M, N), with the following properties:

(i) The quilted surface $\underline{S} = (S_0, S_1$ has two patches. S_0 is a disk with 3 punctures $z_0^1, z_0^{1,-}, z_0^{2,-}$, where $z_0^{1,-}, z_0^{2,-}$ are special punctures. S_1^k is a disk with $3 + l_0 + l_1$ punctures $z_1^{1,-}, z_1^{0,1}, \cdots, z_1^{0,l_0}, z_1^0, z_1^{1,1}, \cdots, z_1^{1,l_1}, z_1^{2,-}$, which are ordered in a counterclockwise order on the boundary, where $z_1^{1,-}, z_1^{2,-}$ are special punctures. The quilted surface is obtained by seaming the two patches along the boundary component I_0^- of S_0 between $z_0^{1,-}, z_0^{2,-}$ and the boundary component I_1^- of S_1 between $z_1^{1,-}, z_1^{2,-}$. Here, we regard the punctures z_0^1 and z_1^0 as being fixed, while $z_1^{0,1}, \cdots, z_1^{0,l_0}$ and $z_1^{1,1}, \cdots, z_1^{1,l_1}$ are allowed to move.

(ii) $u : S_0 \to M$ is inhomogeneous pseudoholomorphic with respect to (H_{S_0}, J_{S_0}), for a family of Hamiltonians H_{S_0} on M parametrized by S_0, which agrees with H_M near z_0^1, and a family of almost complex structures J_{S_0} parametrized by S_0, which agrees with J_M near z_0^1.

(iii) $v : S_1 \to N$ is inhomogeneous pseudoholomorphic with respect to (H_{S_1}, J_{S_1}), for a family of Hamiltonians H_{S_1}, which agrees with H_U near each of the punctures $z_1^{0,1}, \cdots, z_1^{0,l_0}, z_1^0, z_1^{1,1}, \cdots, z_1^{1,l_1}$, and a family of almost complex structures J_{S_1}, which agrees with J_U each of the punctures

$$z_1^{0,1}, \cdots, z_1^{0,l_0}, z_1^0, z_1^{1,1}, \cdots, z_1^{1,l_1}.$$

(iv) u maps the boundary component of S_0 between $z_0^{1,-}$ and z_0^1 to L_0, the boundary component between z_0^j and z_0^{j+1} to L_j (for $j = 1, \cdots, k-1$), and the boundary component between z_0^k and $z_0^{2,-}$ to L_k.

(v) v maps the boundary component of S_1 between $z_1^{1,-}$ and $z_1^{0,1}$, the boundary component between $z_1^{0,j}$ and $z_1^{0,j+1}$ (for $j = 1, \cdots, l_0 - 1$) as well as the boundary component between z_1^{0,l_0} and z_1^0 to the image of the geometric composition $L_0 \circ_{H_M} \mathcal{L}$. v maps the boundary component between z_1^0 and $z_1^{1,1}$, the boundary component between $z_1^{1,j}$ and $z_1^{1,j+1}$ (for $j = 1, \cdots, l_1 - 1$) as well as the boundary component between z_1^{1,l_1} and $z_1^{2,-}$ to the image of the geometric composition $L_1 \circ_{H_M} \mathcal{L}$.

(vi) On the seam, the matching condition for (u, v) is given by the Lagrangian correspondence $\mathcal{L}$.

(vii) u asymptotically converges to some time-one H_M-chord x at z_0^1.

(viii) v asymptotically converges to some generator y for $CW^*(L_0 \circ \mathcal{L}, L_1 \circ \mathcal{L})$ at z_1^0. In the case where y is a time-one H_N-chord from the image of $L_0 \circ \mathcal{L}$ to that of $L_1 \circ \mathcal{L}$, this condition is the same as those for u. In the case where y is a critical point (this happens only when $L_0 = L_1$), the domain S_1 and the map v have to be slightly modified, to be described later on.

(ix) v asymptotically converges to some generator $y_{0,j}$ of $CW^*(L_0 \circ \mathcal{L}; H_N)$ at $z_1^{0,j}$ for $j = 1, \cdots, l_0$, and to some generator $y_{1,j}$ of $CW^*(L_1 \circ \mathcal{L}; H_N)$ at $z_1^{1,j}$ for $j = 1, \cdots, l_1$.

(x) Over the first quilted end, the quilted map (u, v) asymptotically converges to the cyclic element e_0 for $(L_0, \mathcal{L}, L_0 \circ \mathcal{L})$. Over the second quilted end, the quilted map (u, v) asymptotically converges to the cyclic element e_1 for $(L_1, \mathcal{L}, L_1 \circ \mathcal{L})$.

Now let us describe the necessary modification when $L_0 = L_1$ and y is a critical point of the chosen Morse function on the self fiber product of $L_0 \circ \mathcal{L}$. In this case, we require that the family of Hamiltonians H_{S_1} is chosen so that it vanishes near the strip-like end of z_1^0, so that the map v converges to a point on the image of $L_1 \circ \mathcal{L}$. The domain S_1 should also be further modified, by attaching an infinite half ray $(-\infty, 0]$ to it at the negative puncture (which now becomes a marked point as the Hamiltonian vanishes near there). Then we require that the lift of the map on the infinite half ray, which is a gradient flow, converges to y at $-\infty$. To make the statement concise and unified, in both cases we shall briefly say that the map v asymptotically converges to y.

There is a natural stable map compactification of this moduli space, denoted by

$$\bar{\mathcal{U}}_{l_0,l_1}(\alpha, \beta; x; y; y_{0,1}, \cdots, y_{0,l_0}; y_{1,1}, \cdots, y_{1,l_1}; e_0, e_1), \tag{6.55}$$

which is constructed in an inductive nature. This compactification is obtained by adding all possible broken inhomogeneous pseudoholomorphic quilted maps. These broken quilted maps arise when energy escapes over the strip-like ends near the punctures (this phenomenon is often called strip breaking), or when the domains degenerate. There are several cases:

(i) Inhomogeneous pseudoholomorphic disks bubbling off the boundary of the image of $L_0 \circ \mathcal{L}$. The resulting broken quilted map has a main component which is similar to such a quilted map $(\underline{S}, (u, v))$, with possibly less punctures $l_0' \le l_0$, and some other components consisting of trees of inhomogeneous pseudoholomorphic disks with boundary on the image of $L_0 \circ \mathcal{L}$.

(ii) Inhomogeneous pseudoholomorphic disks bubbling off the boundary of the image of $L_1 \circ \mathcal{L}$. The resulting broken quilted map has a main component which is similar to such a quilted map $(\underline{S}, (u, v))$, with possibly less punctures $l_1' \le l_1$, and some other components consisting of trees of inhomogeneous pseudoholomorphic disks with boundary on the image of $L_1 \circ \mathcal{L}$.

(iii) Inhomogeneous pseudoholomorphic strips breaking out at the strip-like end near z_0^1.

(iv) Inhomogeneous pseudoholomorphic strips breaking out at the strip-like end near z_1^0.

(v) Inhomogeneous pseudoholomorphic quilted strips breaking out at the quilted ends.

Thus, there is an isomorphism of the codimension-one boundary strata of the compactified moduli space:

(6.56)

$$\begin{aligned}
&\partial\bar{\mathcal{U}}_{l_0,l_1}(\alpha,\beta;x;y;y_{0,1},\cdots,y_{0,l_0};y_{1,1},\cdots,y_{1,l_1};e_0,e_1)\\
\cong &\coprod_{\substack{x_1\\ \deg(x_1)=\deg(x)+1}} \bar{\mathcal{M}}(x_1,x)\\
&\times \bar{\mathcal{U}}_{l_0,l_1}(\alpha,\beta;x_1;y;y_{0,1},\cdots,y_{0,l_0};y_{1,1},\cdots,y_{1,l_1};e_0,e_1)\\
&\cup \coprod_{\substack{\alpha'\sharp\alpha''=\alpha\\ \beta''\sharp\beta''=\beta}} \coprod_{\substack{y_1\\ \deg(y_1)=\deg(y)-1}} \bar{\mathcal{M}}(\alpha'',\beta'';y,y_1)\\
&\times \bar{\mathcal{U}}_{l_0,l_1}(\alpha',\beta';x;y_1;y_{0,1},\cdots,y_{0,l_0};y_{1,1},\cdots,y_{1,l_1};e_0,e_1)\\
&\cup \coprod_{\substack{1\le l_0'\le l_0\\ l_0'+l_0''=l_0+1}} \coprod_{0\le i_0\le l_0'} \coprod_{\substack{\alpha'\sharp\alpha''=\alpha\\ \beta''\sharp\beta''=\beta}} \coprod_{\substack{y_{0,new}\\ \deg(y_{0,new})=\deg(y_{0,i_0+1})+\cdots+\deg(y_{0,i_0+l_0''})+2-l_0''}}\\
&\bar{\mathcal{U}}_{l_0',l_1}(\alpha',\beta';x;y;y_{0,1},\cdots,y_{0,i_0},y_{0,new},y_{0,i_0+l_0''+1},\cdots,y_{0,l_0};y_{1,1},\cdots,y_{1,l_1};e_0,e_1)\\
&\times \bar{\mathcal{M}}_{l_0''+1}(\alpha'',\beta'';y_{0,new};y_{0,i_0+1},\cdots,y_{0,i_0+l_0''})\\
&\cup \coprod_{\substack{1\le l_1'\le l_1\\ l_1'+l_1''=l_1+1}} \coprod_{1\le i_1\le l_1'} \coprod_{\substack{\alpha'\sharp\alpha''=\alpha\\ \beta''\sharp\beta''=\beta}} \coprod_{\substack{y_{1,new}\\ \deg(y_{1,new})=\deg(y_{1,i_1+1})+\cdots+\deg(y_{1,i_1+l_1''})+2-l_1''}}\\
&\bar{\mathcal{U}}_{l_0,l_1'}(\alpha',\beta';x;y;y_{0,1},\cdots,y_{0,l_0};y_{1,1},\cdots,y_{1,i_1},y_{1,new},y_{1,i_1+l_1''+1},\cdots,y_{1,l_1})\\
&\times \bar{\mathcal{M}}_{l_1''+1}(\alpha'',\beta'';y_{1,new};y_{1,i_1+1},\cdots,y_{1,i_1+l_1''})\\
&\cup \coprod_{\substack{0\le l_0'\le l_0\\ l_0'+l_0''=l_0}} \coprod_{\substack{\alpha'\sharp\alpha''=\alpha\\ \beta''\sharp\beta''=\beta}} \coprod_{\substack{(x_{0,new},y_{0,new})\\ \deg((x_{0,new},y_{0,new}))=\deg(e_0)+1}}\\
&\bar{\mathcal{N}}_{l_0''}(\alpha'',\beta'';y_{0,1},\cdots,y_{0,l_0''};(x_{0,new},y_{0,new}),e_0)\\
&\times \bar{\mathcal{U}}_{l_0',l_1}(\alpha',\beta';x;y;y_{0,l_0''+1},\cdots,y_{0,l_0};y_{1,1},\cdots,y_{1,l_1};(x_{0,new},y_{0,new}),e_1)\\
&\cup \coprod_{\substack{0\le l_1'\le l_1\\ l_1'+l_1''=l_1}} \coprod_{\substack{\alpha'\sharp\alpha''=\alpha\\ \beta''\sharp\beta''=\beta}} \coprod_{\substack{(x_{1,new},y_{1,new})\\ \deg((x_{1,new},y_{1,new}))=\deg(e_1)+1}}\\
&\bar{\mathcal{U}}_{l_0,l_1'}(\alpha',\beta';x;y;y_{0,1},\cdots,y_{0,l_0};y_{1,1},\cdots,y_{1,l_1'};e_0,(x_{1,new},y_{1,new}))\\
&\times \bar{\mathcal{N}}_{l_1''}(\alpha'',\beta'';y_{1,l_1'+1},\cdots,y_{1,l_1};(x_{1,new},y_{1,new}),e_1).
\end{aligned}$$

Some notations need to be explained. Here

$$\bar{\mathcal{U}}_{l_0',l_1}(\alpha',\beta';x;y;y_{0,l_0''+1},\cdots,y_{0,l_0};y_{1,1},\cdots,y_{1,l_1};(x_{0,new},y_{0,new}),e_1)$$

is the moduli space of quilted maps of the same kind, except that the asymptotic convergence condition at the quilted end $(z_0^{1,-},z_1^{1,-})$ is replaced by a new generalized chord $(x_{0,new},y_{0,new})$ for $(L_0,\mathcal{L},L_0\circ_{H_M}\mathcal{L})$; similarly for

$$\bar{\mathcal{U}}_{l_0,l_1'}(\alpha',\beta';x;y;y_{0,1},\cdots,y_{0,l_0};y_{1,1},\cdots,y_{1,l_1'};e_0,(x_{1,new},y_{1,new})).$$

And

$$\bar{\mathcal{N}}_{l_0''}(\alpha'',\beta'';y_{0,1},\cdots,y_{0,l_0''};(x_{0,new},y_{0,new}),e_0)$$

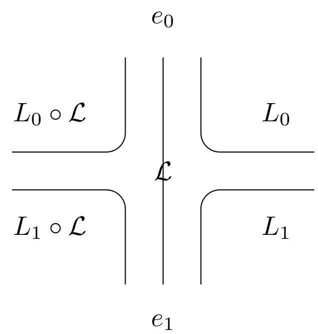

FIGURE 4. the quilted map defining the cochain map

is the moduli space of broken decorated inhomogeneous pseudoholomorphic quilted strips connecting $(x_{0,new}, y_{0,new})$ and e_0, with punctures on the boundary of the first patch; similarly for

$$\bar{\mathcal{N}}_{l_1''}(\alpha'', \beta''; y_{1,l_1'+1}, \cdots, y_{1,l_1}; (x_{1,new}, y_{1,new}), e_1).$$

When the virtual dimension is zero, the virtual fundamental chains of the moduli spaces $\bar{\mathcal{U}}_{l_0,l_1}(\alpha, \beta; x; y; y_{0,1}, \cdots, y_{0,l_0}; y_{1,1}, \cdots, y_{1,l_1}; e_0, e_1)$ give rise to multilinear maps

$$\begin{aligned} a_{l_0,l_1} :& CW^*(L_0 \circ \mathcal{L}; H_N)^{\otimes l_0} \otimes CW^*(L_0, L_1; H_M) \otimes CW^*(L_1 \circ \mathcal{L}; H_N)^{\otimes l_1} \\ &\to CW^*(L_0 \circ \mathcal{L}, L_1 \circ \mathcal{L}; H_N)). \end{aligned} \tag{6.57}$$

By specializing $y_{0,j} = b_0$ and $y_{1,j} = b_1$, we define

$$\Pi_{\mathcal{L}}(x) = \sum_{l_0,l_1=0}^{\infty} a_{l_0,l_1}(\underbrace{b_0, \cdots, b_0}_{l_0 \text{ times}}; x; \underbrace{b_1, \cdots, b_1}_{l_1 \text{ times}}). \tag{6.58}$$

This is the definition of the map $\Pi_{\mathcal{L}}$.

The main observation is that the map $\Pi_{\mathcal{L}}$ is homotopic to the cochain map $\Theta_{\mathcal{L}}^1$.

Proposition 6.9. *The map $\Pi_{\mathcal{L}}$ is a cochain map with respect to the usual Floer differential on $CW^*(L_0, L_1; H_M)$ and the (b_0, b_1)-deformed differential on $CW^*(L_0 \circ \mathcal{L}, L_1 \circ \mathcal{L}; H_N)$.*

Moreover, this map is chain homotopic to the first order map $\Theta_{\mathcal{L}}^1$ of the A_∞-functor $\Theta_{\mathcal{L}}$.

Proof. The first statement follows from the facts that the b_0-deformed and b_1-deformed A_∞-structures have vanishing zeroth order terms, as well as the fact that the cyclic elements e_0, e_1 are closed under the b_0-deformed and respectively the b_1-deformed quilted Floer differentials. The underlying geometric idea is as described below. In order to get the correct "count" of elements, we shall impose the restricted asymptotic convergence conditions $y_{0,j} = b_0$ at $z_1^{0,j}$ and $y_{1,j} = b_1$ at $z_1^{1,j}$. The meaning of this is as follows. If b_0 (resp. b_1) is a formal linear combination of generators of $CW^*(L_0 \circ \mathcal{L})$ (resp. $CW^*(L_1 \circ \mathcal{L})$), then we consider these moduli spaces whose asymptotic convergence conditions are given by all such generators, and take the corresponding formal linear combination of the virtual fundamental chains with the same coefficients as those for b_0 and b_1. The

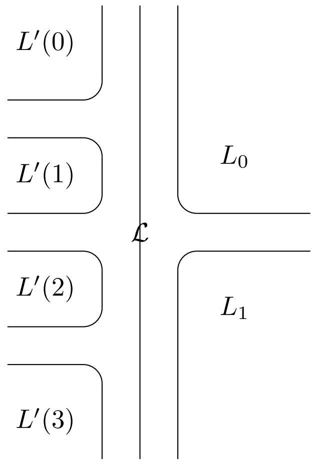

FIGURE 5. the quilted map defining the linear term of the module-valued functor

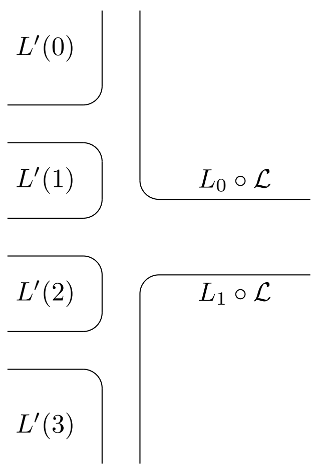

FIGURE 6. inhomogeneous pseudoholomorphic disk defining the module structure on the Yoneda module

numerical effect is that when inserting the bounding cochains b_0 and b_1 for the corresponding asymptotic convergence conditions, the total contributions from any kind of the following moduli spaces $\bar{\mathcal{M}}_{l_0''+1}(\alpha'', \beta''; y_{0,new}; y_{0,i_0+1}, \cdots, y_{0,i_0+l_0''})$, or $\bar{\mathcal{M}}_{l_1''+1}(\alpha'', \beta''; y_{1,new}; y_{1,i_1+1}, \cdots, y_{1,i_1+l_1''})$, or $\bar{\mathcal{N}}_{l_0''}(\alpha'', \beta''; y_{0,1}, \cdots, y_{0,l_0''}; (x_{0,new}, y_{0,new}), e_0)$, or $\bar{\mathcal{N}}_{l_1''}(\alpha'', \beta''; y_{1,l_1'+1}, \cdots, y_{1,l_1}; (x_{1,new}, y_{1,new}), e_1)$ are all zero. In other words, no quilted strip breaking can occur numerically. Now, by looking at (6.56), we note that the only non-trivial contributions are from strip breaking at the strip-like ends near z_0^1 or z_1^0, and therefore conclude that $\Pi_{\mathcal{L}}$ is a cochain map.

The proof of the second statement is based upon the proof of representability of the module-valued functor $\Phi_{\mathcal{L}}$. We shall compose this map $\Pi_{\mathcal{L}}$ with the Yoneda functor and compare the resulting map to the linear term of module-valued functor

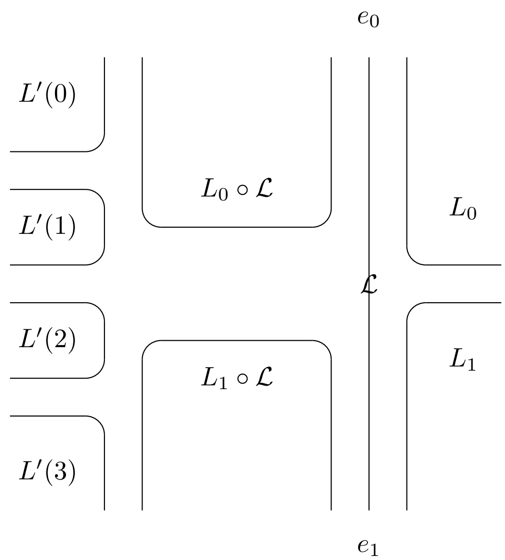

FIGURE 7. the quilted map for the composition of the cochain map with the Yoneda functor

$\Phi_{\mathcal{L}}$. This will be done by analyzing how the relevant inhomogeneous pseudoholomorphic quilted maps can be related. Recall that the module-valued functor $\Phi_{\mathcal{L}}$ defines for each object L of $\mathcal{W}(M)$ an A_∞-module $\Phi_{\mathcal{L}}(L)$ over $\mathcal{W}_{im}(N)$, with the module structure maps defined by "counting" elements moduli spaces of quilted inhomogeneous pseudoholomorphic maps (u^0, v^0), whose first component u^0 in M is a strip with one boundary component mapped to L, and second component v^0 is a disk with several punctures, and the boundary components are mapped to objects of $\mathcal{W}_{im}(N)$, which are testing objects for the A_∞-module $\Phi_{\mathcal{L}}$. Floer cochains in $CW^*(L_0, L_1; H_M)$ give rise to pre-module homomorphisms from $\Phi_{\mathcal{L}}(L_0)$ to $\Phi_{\mathcal{L}}(L_1)$, which are defined by using moduli spaces of quilted inhomogeneous pseudoholomorphic maps (u^1, v^1), as shown in Figure 5. These quilted maps are similar to the previous ones (u^0, v^0), but satisfying somewhat different conditions:

(i) One boundary component of the domain of u^1 is mapped to L_0 and the other to L_1, with the prescribed Floer cochain in $CW^*(L_0, L_1; H_M)$ as the asymptotic convergence condition at the puncture in between the two boundary components.

(ii) The boundary components of the domain of v^1 are mapped to testing objects of $\mathcal{W}_{im}(N)$. For instance, when defining the d-th order map

(6.59)
$$\begin{aligned}(\Phi^1_{\mathcal{L}})^d :&CW^*(L_0, L_1; H_M)\\ &\to \hom(CW^*(L_0, \mathcal{L}, (L'(0), b(0))) \otimes CW^*((L'(0), b(0)), (L'(1), b(1))) \otimes \cdots\\ &\otimes CW^*((L'(d-2), b(d-2)), (L'(d-1), b(d-1))),\\ &CW^*(L_1, \mathcal{L}, (L'(d-1), b(d-1))),\end{aligned}$$

the boundary conditions for v^1 are given by $(L'(0), b(0)), \cdots, (L'(d-1), b(d-1))$. With bounding cochains included, these are understood as follows.

There are additional several punctures on these boundary components, whose asymptotic convergence conditions are given by the bounding cochains $b(j)$. Here $(L'(j), b(j))$ are general testing objects, which do not have to be the geometric compositions of L_0 or L_1 with $\mathcal{L}$.

(iii) The asymptotic convergence conditions over the two quilted ends are the input and output for the modules $\Phi_{\mathcal{L}}(L_0)$ and $\Phi_{\mathcal{L}}(L_1)$.

On the other hand, recall that the proof of the representability of the module-valued functor $\Phi_{\mathcal{L}}$ uses the moduli spaces $\bar{\mathcal{C}}(x'; (x,y); e)$ (6.34) and $\bar{\mathcal{C}}_d(x'; x'_1, \cdots, x'_{d-1}; (x,y); e)$ (6.40), whose count yields the homotopy equivalence (6.41). Such a quilted map in $\bar{\mathcal{C}}_d(x'; x'_1, \cdots, x'_{d-1}; (x,y); e)$ converges to the cyclic element e of the corresponding triple $(L, \mathcal{L}, L \circ_{H_M} \mathcal{L})$ over one quilted end. Algebraically, the virtual fundamental chains of these moduli spaces give rise to a homotopy equivalence from the module $\Phi_{\mathcal{L}}(L)$ to the Yoneda module $\mathfrak{y}_l((L \circ \mathcal{L}, b))$ for every L. Now glue the other quilted end of two copies of such quilted maps (u_0, v_0) and (u_1, v_1) in the moduli spaces $\bar{\mathcal{C}}(x'; (x,y); e)$ (one for the triple $(L_0, \mathcal{L}, L_0 \circ \mathcal{L})$ and the other for $(L_1, \mathcal{L}, L_1 \circ_{H_M} \mathcal{L})$) to one quilted inhomogeneous pseudoholomorphic map (u^1, v^1), along the quilted ends over which (u_i, v_i) have a common asymptotic convergence condition with (u^1, v^1). The new quilted map $(\tilde{u}, \tilde{v})$ has two quilted ends, over which the asymptotic convergence conditions are given by the cyclic elements e_i for the triple $(L_i, \mathcal{L}, L_i \circ \mathcal{L})$. Also, there are two new punctures on the domain of $\tilde{v}$, and correspondingly two new boundary components which are mapped to the image of the geometric composition $L_0 \circ \mathcal{L}$ and $L_1 \circ \mathcal{L}$ respectively. Over the strip-like ends of these two new punctures, the map $\tilde{v}$ asymptotically converges to some generator of $CW^*((L_0 \circ \mathcal{L}, b_0), (L'(0), b(0)))$ and one of $CW^*((L_1 \circ \mathcal{L}, b_1), (L'(d-1), b(d-1)))$., respectively.

Now let us look at the composition of $\Pi_{\mathcal{L}}$ with the linear term of Yoneda functor

$$\begin{aligned} \mathfrak{y}_l \circ \Pi_{\mathcal{L}} : & CW^*(L_0, L_1; H_M) \\ & \to CW^*((L_0 \circ \mathcal{L}, b_0), (L_1 \circ \mathcal{L}, b_1); H_N) \\ & \to \hom(\mathfrak{y}_l((L_0 \circ \mathcal{L}, b_0)), \mathfrak{y}_l((L_1 \circ \mathcal{L}, b_1))), \end{aligned} \tag{6.60}$$

which concretely consists of multilinear maps

$$\begin{aligned} \mathfrak{y}_l \circ \Pi_{\mathcal{L}} : & CW^*(L_0, L_1; H_M) \\ & \to CW^*((L_0 \circ \mathcal{L}, b_0), (L_1 \circ \mathcal{L}, b_1); H_N) \\ & \to \hom(CW^*(L_0, \mathcal{L}, (L'(0), b(0))) \otimes CW^*((L'(0), b(0)), (L'(1), b(1))) \otimes \cdots \\ & \otimes CW^*((L'(d-2), b(d-2)), (L'(d-1), b(d-1))), \\ & CW^*(L_1, \mathcal{L}, (L'(d-1), b(d-1))), \end{aligned} \tag{6.61}$$

Such maps are defined using moduli spaces of inhomogeneous pseudoholomorphic quilted maps which are obtained from gluing those quilted maps (u, v) in the moduli spaces

$$\bar{\mathcal{U}}_{l_0, l_1}(\alpha, \beta; x; y; y_{0,1}, \cdots, y_{0,l_0}; y_{1,1}, \cdots, y_{1,l_1}; e_0, e_1),$$

with those inhomogeneous pseudoholomorphic maps which are used to define pre-module homomorphisms between the Yoneda modules. The inhomogeneous pseudoholomorphic maps of the latter kind are just ordinary A_∞-disks used to define the A_∞-structure on $\mathcal{W}_{im}(N)$, because the module structure maps and the pre-module homomoprhisms for Yoneda modules are precisely given by the original

A_∞-structure maps for the corresponding objects in $\mathcal{W}_{im}(N)$, as shown in Figure 6. After gluing, we get a quilted map as shown in Figure 7.

It is then an immediate observation that such a quilted map is of the same type as the previously constructed one $(\tilde{u}, \tilde{v})$. Recall that the quilted maps (u_0, v_0) and (u_1, v_1) are shown in Figure 2. Gluing the two quilted maps (u_0, v_0) and (u_1, v_1) to (u^1, v^1) would replace the two quilted ends of (u^1, v^1) by two new quilted ends, over which the new map $(\tilde{u}, \tilde{v})$ asymptotically converges to the cyclic elements e_0 and e_1 respectively, and moreover create two new punctures on the component $\tilde{v}$, over which $\tilde{v}$ asymptotically converges to generators of $CW^*((L_0 \circ \mathcal{L}, b_0), (L'(0), b(0)))$ and $CW^*((L_1 \circ \mathcal{L}, b_1), (L'(d-1), b(d-1)))$. Comparing this to Figure 7, we see that they are of the same type.

Such an argument can be generalized to the compactified moduli spaces in a straightforward way, by repeating the same process for broken quilted maps. It is clear that this process respects the structure of the boundary strata as described in (6.56). Translating the result algebraically, we conclude that the map $\mathfrak{y}_l \circ \Pi_{\mathcal{L}}$ is chain homotopic to $\Phi^1_{\mathcal{L}}$. Since $\Theta_{\mathcal{L}}$ represents the module-valued functor $\Phi_{\mathcal{L}}$, we conclude that the map $\Pi_{\mathcal{L}}$ is chain homotopic to $\Theta^1_{\mathcal{L}}$. Therefore the proof of the second statement is complete.

□

6.9. **A Künneth formula for the wrapped Fukaya category.** As in classical Floer cohomology theory, it is natural to expect that the wrapped Fukaya category of the product manifold $\mathcal{W}(M \times N)$ can be expressed as a tensor product of $\mathcal{W}(M)$ and $\mathcal{W}(N)$. This is some kind of Künneth formula in wrapped Floer theory. As an application of the construction of functors discussed before, in particular the bimodule-valued functors, we can phrase this in a precise way, where we take the statement in [GPS17].

There are some issues regarding the statement of the Künneth formula. First, $\mathcal{W}(M \times N)$ generally has more objects than product Lagrangian submanifolds $L \times L'$, so in general we cannot expect this to be equivalent to the tensor product. However, passing to the split-closure gives some hope of establishing this kind of equivalence, provided that $\mathcal{W}(M \times N)$ is split-generated by product Lagrangian submanifolds.

Second, the definition of A_∞-tensor product of A_∞-categories is delicate. Rather than giving a definition by certain "universal property", we shall take a particular model of the A_∞-tensor product $\mathcal{W}(M) \otimes \mathcal{W}(N)$ so that the construction of bimodule-valued functors associated to Lagrangian correspondences can be untilized in the proof. For the definition of the A_∞-tensor product, we follow the construction of [SU04], in which a diagonal for the Stasheff associahedra is constructed. In particular, in this model of A_∞-tensor product $\mathcal{W}(M) \otimes \mathcal{W}(N)$, the objects are pairs (L, L') of Lagrangian submanifolds of M and N respectively (or formal products $L \times L'$, the underlying morphism spaces are the usual tensor products of wrapped Floer complexes

$$\hom((L_0, L'_0), (L_1, L'_1)) = CW^*(L_0, L_1) \otimes CW^*(L'_0, L'_1), \tag{6.62}$$

and the first order structure map is the tensor product wrapped Floer differential:

$$m^1 = m^1_{L_0, L_1} \otimes id + id \otimes m^1_{L'_0, L'_1}. \tag{6.63}$$

However, we remark that all the different constructions are quasi-isomorphic (see [MS06]).

Proposition 6.10. *Assume both M and N are non-degenerate, i.e. their wrapped Fukaya categories satisfy Abouzaid's generation criterion for finite collections of Lagrangian submanifolds. Then there is a canonical quasi-equivalence:*

$$\mathcal{W}(M \times N)^{perf} \to (\mathcal{W}(M) \otimes \mathcal{W}(N))^{perf}. \tag{6.64}$$

The outline of the proof goes as follows. First, non-degeneracy implies that there are finite collections of Lagrangian submanifolds $L_1, \cdots, L_k$ of M and $L'_1, \cdots, L'_l$ of N which split-generate their wrapped Fukaya categories, and moreover, the products $L_i \times L'_j$ split-generate $\mathcal{W}(M \times N)$. Thus it will be enough to consider the full A_∞-subcategory $\mathcal{P}$ of $\mathcal{W}(M \times N)$ whose objects are products $L_i \times L'_j$. Second, the framework of Lagrangian correspondence gives us a bimodule-valued functor

$$\mathcal{P} \to (\mathcal{W}(M^-), \mathcal{W}(N))^{bimod}, \tag{6.65}$$

which is defined at the beginning of this section. Third, there is a canonical algebraically defined Yoneda-type A_∞-functor

$$\mathcal{W}(M) \otimes \mathcal{W}(N) \to (\mathcal{W}(M^-), \mathcal{W}(N))^{bimod}, \tag{6.66}$$

to the dg-category of A_∞-bimodules over $(\mathcal{W}(M^-), \mathcal{W}(N))$. An appropriate version of Yoneda lemma says that this is cohomologically full and faithful, thus is a quasi-isomorphism onto the image. Fourth, note that the image of $\mathcal{P}$ under the A_∞-functor (6.65) lands in the image of $\mathcal{W}(M) \otimes \mathcal{W}(N)$ under the A_∞-functor (6.66) in the dg-category of bimodules. Inverting the functor (6.66) on the image gives us an A_∞-functor

$$\mathcal{P} \to \mathcal{W}(M) \otimes \mathcal{W}(N). \tag{6.67}$$

Finally, using a direct argument by analyzing the moduli space of inhomogeneous pseudoholomorphic quilted strips, we can easily prove the Künneth formula on the level of cohomology. This isomorphism on cohomology can be rephrased as the statement that the above A_∞-functor is a quasi-isomorphism, which allows us to compare the images of $\mathcal{P}$ and $\mathcal{W}(M) \otimes \mathcal{W}(N)$ in the dg-category of bimodules $(\mathcal{W}(M^-), \mathcal{W}(N))^{bimod}$, and show that they are quasi-isomorphic. Passing to the split-closure, we get a quasi-equivalence:

$$\mathcal{W}(M \times N)^{perf} \to (\mathcal{W}(M) \otimes \mathcal{W}(N))^{perf}. \tag{6.68}$$

While this is an outline, a complete detailed proof only requires careful writing out the formulas for the algebraic structures involving A_∞-tensor products that we use here. The reader is referred to [SU04] and [MS06] for detailed account of A_∞-tensor products and related algebraic results.

Remark 6.3. *In fact, a chain-level A_∞-quasi-equivalence between (the split-closure of) the split wrapped Fukaya category $\mathcal{W}_s(M \times N)$ and (the split-closure of) A_∞-tensor product of (suitable dg-replacements of) $\mathcal{W}(M)$ and $\mathcal{W}(N)$ was already established in [Gan13]. That is, Proposition 6.10 was proved there for the split wrapped Fukaya category $\mathcal{W}^s(M \times N)$ of the product manifold, with objects being products of exact cylindrical Lagrangian submanifolds of individual factors.*

Without assuming non-degeneracy of M or N, there is a general version of Künneth formula for the wrapped Fukaya category.

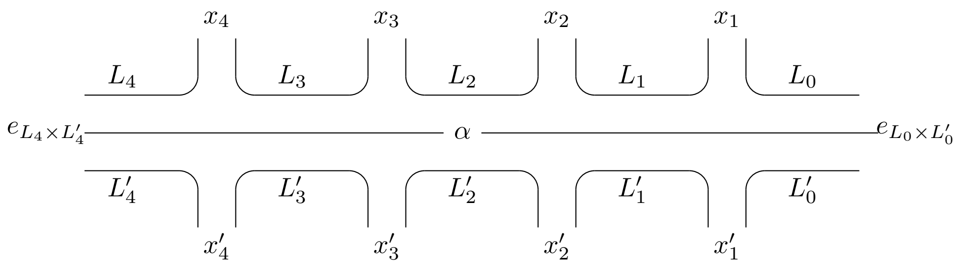

FIGURE 8. the quilted map defining the Künneth bifunctor

Proposition 6.11. *There is a natural cohomologically fully faithful A_∞-bifunctor (or say A_∞-bilinear functor)*

$$\mathfrak{Kun} : \mathcal{W}(M) \times \mathcal{W}(N) \to \mathcal{W}(M \times N). \tag{6.69}$$

This determines a unique cohomologically fully faithful A_∞-functor

$$\mathfrak{Kun}' : \mathcal{W}(M) \otimes \mathcal{W}(N) \to \mathcal{W}(M \times N), \tag{6.70}$$

up to homotopy.

Proof. Define $\mathcal{W}(M \times N)$ using the split wrapped Fukaya category $\mathcal{W}^s(M \times N)$. Then the A_∞-bifunctor (6.69) can be defined in a straightforward way. On the level of objects, the bifunctor simply takes a pair (L, L') to their direct product,

$$(L, L') \mapsto L \times L'.$$

On the level of morphism spaces, the bifunctor

$$CW^*(L_0, L_1) \times CW^*(L_0', L_1') \to CW^*(L_0 \times L_0', L_1 \times L_1')$$

is defined as follows. For any basic Floer cochains $x \in CW^*(L_0, L_1)$ and $x' \in CW^*(L_0', L_1')$ (here basic means that the Floer cochain is represented by a single Hamiltonian chord), the product $x \times x'$ is naturally a Floer cochain in $CW^*(L_0 \times L_0', L_1 \times L_1')$ since the latter is defined with respect to the split Hamiltonian. Higher-order terms are of the form

$$\begin{aligned}\mathfrak{Kun}^{k,l} :&(CW^*(L_{k-1}, L_k) \otimes \cdots \otimes CW^*(L_0, L_1))\\ &\times (CW^*(L_{l-1}', L_l') \otimes \cdots \otimes CW^*(L_0', L_1')) \to CW^*(L_0 \times L_0', L_k \times L_l'),\end{aligned} \tag{6.71}$$

which are defined as follows. Let $x_i \in CW^*(L_{i-1}, L_i)$ and $x_j' \in CW^*(L_{j-1}', L_j')$ be basic Floer cochains. The image of $(x_k \otimes \cdots \otimes x_1) \times (x_l' \otimes \cdots \otimes x_1')$ under the map (6.71) is defined by counting inhomogeneous pseudoholomorphic (generalized) quilted maps of the following kind:

Here the output is $\alpha \in CW^*(L_0 \times L_0', L_k \times L_l')$, and the asymptotic conditions near the quilted ends are given by the elements

$$e_{L_0 \times L_0'} \in CW^*(L_0, L_0 \times L_0', L_0'),$$

and

$$e_{L_k \times L_l'} \in CW^*(L_k, L_k \times L_l', L_l')$$

which correspond to the homotopy unit

$$1_{L_0 \times L_0'} \in CW^*(L_0 \times L_0'),$$

and respectively

$$1_{L_k \times L'_l} \in CW^*(L_k \times L'_l).$$

This construction includes $k = l = 1$ as a special case, as such a quilted map is necessarily constant when $k = l = 1$.

It remains to prove that $\mathfrak{Kun}$ (6.69) as defined above is cohomologically fully faithful. This follows immediately from the definition of $\mathfrak{Kun}$, as the first order map is the "identity": sending (x, x') to the product $x \times x'$, which induces an isomorphism on wrapped Floer cohomology groups, proven in [Gao17b] as a special case of the results in section 4.

The statement regarding $\mathfrak{Kun}'$ (6.70) follows from the universal property of the A_∞-tensor product.

□

Now let us go back to the situation where both M and N are non-degenerate and compare the statement of Proposition 6.10 and that of Proposition 6.11. Non-degeneracy implies that the Kunneth A_∞-functor (6.70) is invertible after passing to the idempotent completion, and the inverse is given by the quasi-equivalence (6.64).

7.1. **Sub-domains and restrictions of exact Lagrangian submanifolds.** Let M_0 be a Liouville domain, and M its completion. A Liouville sub-domain U_0 is a codimension zero exact symplectic submanifold of M_0 with smooth boundary, such that the Liouville vector field on M_0 points outward the boundary ∂U_0 transversely. U_0 is itself a Liouville domain with the induced Liouville structure. We may attach to U_0 an infinite half-cylinder $\partial U_0 \times [1, +\infty)$ to the boundary to get a complete Liouville manifold U.

Denote by M^- the Liouville manifold with opposite symplectic form $-\omega$, with the Liouville form also being the opposite $-\lambda_M$, similarly for U^-. There is a natural Lagrangian correspondence between M and U, defined as follows. Because U_0 is a Liouville sub-domain, the graph of the embedding $U_0 \subset M_0$ is naturally an exact Lagrangian submanifold (with corners) of $U_0^- \times M_0$ with respect to the product exact one-form $-\lambda_{U_0} \times \lambda_{M_0}$. The natural completion of that in $U^- \times M$, is an exact cylindrical Lagrangian submanifold of $U^- \times M$, with respect to the contact hypersurface Σ. We denote this Lagrangian correspondence by Γ. Alternatively, consider the natural embedding

$$i : U \to M \tag{7.1}$$

induced by the inclusion $U_0 \subset M_0$, whose is definition is given by the following formula:

$$i(x) = \begin{cases} x, & \text{if } x \in U_0, \\ \psi_M^r(i(y)), & \text{if } x = (y, r) \in \partial U \times [1, +\infty). \end{cases} \tag{7.2}$$

Then Γ is the graph of i.

Equipped with the opposite primitive, it can also be regarded as an exact Lagrangian submanifold of $M^- \times U$. To make Γ into an admissible Lagrangian correspondence in Floer-theoretic sense, one needs a spin structure on it. A natural spin structure exists because we assume both M and U have vanishing first Chern classes. We shall give this Lagrangian correspondence a special name.

Definition 7.1. *The Lagrangian submanifold $\Gamma \subset M^- \times U$ equipped with the natural spin structure is called the graph correspondence.*

Given an exact Lagrangian submanifold L_0 of M_0, possibly with non-empty boundary ∂L_0, the intersection $L_0' = L_0 \cap U_0$ is naturally an exact Lagrangian submanifold of U_0, possibly with non-empty boundary $\partial L_0'$, even if L_0 has empty boundary.

If L_0 is either a closed exact Lagrangian submanifold, or an exact cylindrical Lagrangian submanifold, it can be naturally completed to an object L of $\mathcal{W}(M)$. However, in general this is not true for L_0', as the boundary $\partial L_0'$ might not behave nicely. In the next subsection we shall seek geometric assumptions such that L_0' can be completed into an object of $\mathcal{W}(U)$, so that we may attempt to define a restriction functor.

7.2. **Restriction and the associated functor.** First, note that the graph correspondence Γ is admissible for wrapped Floer theory in the product $M^- \times U$. This implies that the module-valued functor Φ_Γ is well-defined. More importantly, note that the projection $\Gamma \to U$ is proper, thus the module-valued functor Φ_Γ is representable, and represented by the A_∞-functor

$$\Theta_\Gamma : \mathcal{W}(M) \to \mathcal{W}_{im}(U). \tag{7.3}$$

In this and next subsections, we shall study this functor in a more specific way, compare it with the Viterbo restriction functor defined in [AS10], and prove Theorem 1.4.

Remark 7.1. *The graph correspondence Γ can also be viewed as a Lagrangian correspondence from U to M. If we ask whether it defines a functor from $\mathcal{W}(U)$ to $\mathcal{W}_{im}(M)$, the answer is not always yes. This is because the projection $\Gamma \to M$ is not always proper, contrary to $\Gamma \to U$. In that case, the module-valued functor is not necessarily representable in our sense.*

Consider an object L in $\mathcal{W}(M)$. Suppose it is the completion of an exact cylindrical Lagrangian submanifold L_0 of M_0 which intersects U_0 nicely in the following sense.

Assumption 7.1. *$\partial L' = L_0 \cap \partial U_0$ is a Legendrian submanifold of ∂U_0; and moreover, the primitive of L_0 is locally constant near both ∂M_0 and ∂U_0.*

If L_0 satisfies Assumption 7.1, then $L'_0 = L_0 \cap U_0$ is an exact cylindrical Lagrangian submanifold of U_0 and can therefore be completed to an exact cylindrical Lagrangian submanifold $L' \subset U$. Because of the geometric nature of the A_∞-functor

$$\Theta_\Gamma : \mathcal{W}(M) \to \mathcal{W}_{im}(U),$$

we expect that it behaves like a restriction functor, and takes L to L' on the level of objects.

However, the true story is a bit more complicated, and in fact the above expectation does not always hold true. In order for the geometric composition $L \circ_{H_M} \Gamma$ to be related to L', we further need the following condition:

Assumption 7.2. *L is invariant under the Liouville flow in the intermediate region $M_0 \setminus int(U_0)$.*

This assumption means that, the Lagrangian L is the Liouville completion of the restriction $L'_0 = L \cap U_0$, in the sense that L is the union of L'_0 and all trajectories of the Liouville flow on M starting from points on $\partial L'$.

Example 7.1. *Let M_0 and U_0 be (critical) Weinstein domains. If $M_0 \setminus int(U_0)$ is a Weinstein cobordism, and L is the union of some cylinders over Legendrians and the stable or unstable manifolds of those critical points of a Lyapunov function which are contained in the cobordism $M_0 \setminus int(U_0)$, then L satisfies Assumption 7.2.*

Under Assumption 7.2, it is easy to see that the geometric composition $L \circ \Gamma$ is precisely L', the completion of the restriction L'_0 of L to U_0. In fact, the geometric composition in the usual sense (without Hamiltonian perturbation) is the map

$$L \circ \Gamma = \{(p_1, p_2, q) \in M \times M \times U : p_1 \in L, p_1 = p_2, i(q) = p_2\} \to U$$

which sends (p_1, p_2, q) to $q \in U$. In this case this is an exact Lagrangian embedding, whose image is

$$\{q \in U : \exists p \in M, \text{ such that } p \in L, i(q) = p\}.$$

By the definition of the map $i : U \to M$ as in (7.2), this is the set of all points $q \in U$ such that either $q \in L \cap U_0$, or $q = (y, r) \in \partial U \times [1, +\infty)$ such that $\psi_M^r(y) = p$. By Assumption 7.2, this is exactly L'.

Remark 7.2. *In general, every exact cylindrical Lagrangian L which intersects ∂U to a Legendrian submanifold is invariant under the Liouville flow in a neighborhood of ∂U and one of ∂M, after suitable Hamiltonian perturbation.*

On the other hand, under assumption Assumption 7.1, we can show that the bounding cochain in fact vanishes by Proposition 6.7, so that the image of the functor Θ_Γ lands in the completed wrapped Fukaya category $\mathcal{W}(U)$ whose objects are embedded Lagrangian submanifolds. This is the first half of the statement of Theorem 1.4.

The full sub-category of $\mathcal{W}(M)$ consisting of Lagrangian submanifolds which satisfy Assumption 7.1 and Assumption 7.2 will be denoted by $\mathcal{B}_0(M)$. This is a full sub-category of $\mathcal{B}(M)$. The restriction of the functor Θ_Γ to this sub-category is

$$\Theta_\Gamma : \mathcal{B}_0(M) \to \mathcal{W}(U), \tag{7.4}$$

such that the image of any $L \in Ob\mathcal{B}_0(M)$ is $L' \in Ob\mathcal{W}(U)$. This finishes the proof of the first half of Theorem 1.4.

7.3. **The Viterbo restriction functor revisited.** To prove the second half of Theorem 1.4, we shall briefly review the definition of the Viterbo restriction functor first, originally due to [AS10]. As the setup of the wrapped Fukaya category is slightly different (but equivalent), we shall consider inhomogeneous pseudoholomorphic maps which behave somewhat differently than cascade maps. However, they should yield an equivalent definition of the Viterbo restriction functor, although we will not attempt to include a proof.

The relevant inhomogeneous pseudoholomorphic maps that are used to define the Viterbo restriction functor will be called "climbing disks". Climbing disks are in fact very similar to cascades introduced in [AS10]. Roughly speaking, these are inhomogeneous pseudoholomorphic disks in M, for which the Floer data consist of a family of Hamiltonians and a family of almost complex structures interpolating the Floer data on M and those on U, and the boundary conditions are given by a family of Lagrangian submanifolds interpolating L and L'. In order to visualize Floer theory of U in M, we shall use a rescaling trick. More details are presented below.

To define climbing disks, we introduce the following geometric data. Conceptually, we want to consider geometric objects which reflect the way how the Liouville manifolds M and U can be interpolated in a suitable sense. The geometric data on M and those on U should be related in a one-parameter family. While in [AS10] they think of the size of the collar neighborhood of ∂U inside U_0 as the parameter (shrinking the sub-domain by a conformal factor $\rho \in (0, 1]$ which is taken to be the parameter), we fix the size of the collar neighborhood (as in [Vit99]), but rather change the height of the Hamiltonian and regard that as a parameter.

Note that the Liouville structure on U_0 induced by that on M_0 provides a symplectic embedding of a collar neighborhood $\partial U \times [1-\epsilon, 1]$ to U_0 such that $\partial U \times \{1\}$ is mapped to the boundary ∂U. Suppose we have chosen the quadratic Hamiltonian H_M on M and H_U on U, such that they agree on $U_0 \subset M_0$. Define a family of piecewise smooth (and lower semi-continuous) Hamiltonians K_A depending on a parameter $A \in [0, +\infty)$,

$$K_A = \begin{cases} 0, & \text{on } U_0 \setminus (\partial U \times (1-\epsilon, 1), \\ A(r-1+\epsilon)^2, & \text{if } p = (y, r) \in \partial U \times (1-\epsilon, 1), \\ A, & \text{on } M_0 \setminus U_0, \\ A + h(r), & \text{if } x = (y, r) \in \partial M \times [1, +\infty), \end{cases} \tag{7.5}$$

where $h(r)$ is the same radial function as that for H_M, i.e. $h(r) = r^2$ if r is slightly bigger than 1. Note that when $A = 0$, this is simply an admissible Hamiltonian on M which vanishes in the interior part M_0 and is quadratic in the cylindrical end $\partial M \times [1, +\infty)$. Let H_A be a C^2-small perturbation of K_A, which is smooth and non-degenerate, such that H_A agrees with K_A in the collar neighborhood $\partial U \times [1-\epsilon, 1]$ as well as in the cylindrical end $\partial M \times [1, +\infty)$.

We also need a suitable class of almost complex structures in order to write down the inhomogeneous Cauchy-Riemann equations for climbing disks. Suppose we have chosen almost complex structures $J_M = J_M$ on M and $J_U = J_U$ on U. What is needed is a family of almost complex structures interpolating these two families, defined below.

Definition 7.2. *An interpolating family of almost complex structures is a family J_A of compatible almost complex structures on M parametrized by $A \in [0, +\infty)$ such that the following properties are satisfied:*

- *(i)* $J_0 = J_M$*;*
- *(ii) For each A, $J_A = J_M$ outside U_0;*
- *(iii) For each A, the restriction of J_A to $M \setminus U_0$ is of contact type in the cylindrical end $\partial M \times [1, +\infty)$;*
- *(iv) There exists $A_0 > 0$ such that for $A > A_0$, the restriction of J_A to $U_0 \setminus (\partial U \times [1-\epsilon, 1])$ agrees with $(\psi_U^A)_* J_U$, and the restriction of J_A to a collar neighborhood of ∂U in M is of contact type on some neighborhood of $\partial U \times \{1-\epsilon\} \subset \partial U \times [1-\epsilon, 1]$.*

It is not difficult to show that such interpolating families exist, and in fact that they form a contractible space.

Given an exact cylindrical Lagrangian submanifold $L \subset M$ satisfying Assumption 7.1, the completion L' of its restriction $L'_0 = L \cap U_0$ is exact cylindrical with a locally constant primitive near ∂U. Call that constant c_L. We define a slightly shrunk Lagrangian submanifold

$$L^{1-\epsilon} = \begin{cases} \psi_U^{1-\epsilon} L'_0, & \text{on } U_0 \setminus (\partial U \times [1-\epsilon, 1]), \\ \partial L' \times [1-\epsilon, 1], & \text{on } \partial U \times [1-\epsilon, 1], \\ L \end{cases} \tag{7.6}$$

This is well-defined, because L is invariant under the Liouville flow in the collar neighborhood $\partial U \times [1-\epsilon, 1]$.

In Floer theory, when defining inhomogeneous pseudoholomorphic maps, we also need to perturb Hamiltonians and almost complex structures. Our strategy is to perturb almost complex structures, in a way such that they depend on domains of the inhomogeneous pseudoholomorphic maps. If the domain is a strip, then an interpolating family should also depend on an additional parameter t. That is, an interpolating family for the strip is a family of almost complex structures $J_{A,t}$, such that for every fixed t, $J_{A,t}$ is an interpolating family in the sense of Definition 7.2.

Given H_A and $J_{A,t}$ defined as above, we now introduce a Floer datum on the strip. Such a Floer datum, roughly speaking, is a homotopy between $(H_M, J_{M,t})$ and $(H_A, J_{A,t})$, parametrized by $s \in \mathbb{R}$.

Definition 7.3. *A Floer datum on the strip $Z = \mathbb{R} \times [0,1]$ consists of the following data:*

(i) A family of Hamiltonians H_s depending on $s \in \mathbb{R}$, thought of as a family on the strip Z independent of t, such that for $s \gg 0$, H_s agrees with H_M, and for $s \ll 0$, H_s agrees with H_A.

(ii) A family of almost complex structures $J_{(s,t)}$ parametrized by $(s,t) \in Z$, such that for $s \gg 0$, $J_{(s,t)}$ agrees with $J_M = J_{M,t}$, and for $s \ll 0$, $J_{(s,t)}$ agrees with $J_{A,t}$.

To define a climbing strip, we also need appropriate Lagrangian boundary conditions.

Definition 7.4. *A moving Lagrangian labeling for the strip $Z = \mathbb{R} \times [0,1]$ is a pair $(L_{0,s}, L_{1,s})$ where each $L_{j,s}$ is an exact Lagrangian isotopy (through exact cylindrical Lagrangian submanifolds) parametrized by $s \in \mathbb{R}$, such that for $s \gg 0$, $L_{j,s} = L_j$, and for $s \ll 0$, $L_{j,s} = L_j^{1-\epsilon}$ as defined in (7.6).*

Note that there is a natural choice of moving Lagrangian labeling as follows. Let $\lambda : \mathbb{R} \to [1-\epsilon, 1]$ be a smooth non-decreasing function which is $1-\epsilon$ for $s \ll 0$ and is 1 for $s \gg 0$. Then we can define $L_{j,s} = L_j^{\lambda(s)}$ in a similar way to that of (7.6) (replacing $1-\epsilon$ by $\lambda(s)$). Without loss of generality, we may assume that $\lambda(s) = 1$ for $s \geq 0$.

Definition 7.5. *A climbing strip is a smooth map*

$$w : \mathbb{R} \times [0,1] \to M,$$

with the following properties:

(i) w satisfies the inhomogeneous Cauchy-Riemann equation:

$$\partial_s w + J_{(s,t)}(\partial_t w - X_{H_s}(w)) = 0. \tag{7.7}$$

(ii) The boundary conditions for w are:

$$w(s,0) \in L_{0,s}, w(s,1) \in L_{1,s}. \tag{7.8}$$

(iii) The asymptotic convergence conditions of w are:

$$\lim_{s \to -\infty} w(s, \cdot) = x_A(\cdot),$$

for a time-one H_A-chord x_A from $L_{0,A}$ to $L_{1,A}$, and

$$\lim_{s \to +\infty} w(s, \cdot) = x(\cdot),$$

for a time-one H_M-chord x from L_0 to L_1. We require that the H_A-chord x_A satisfy a further condition that

$$-A\epsilon^2 \leq \mathcal{A}(x_A) \leq \delta, \tag{7.9}$$

for a universal small constant δ, whose meaning is to be explained later.

There are two conditions listed above which need further explanation. First, the asymptotic convergence conditions should be, as usual, Hamiltonian chords for the Hamiltonian $H_{A,s}$, as $s \to \pm\infty$. When $s \to +\infty$, $H_{A,s} = H_M$, and the boundary conditions $(L_{0,s}, L_{1,s})$ agree with (L_0, L_1) when $s \gg 0$, so the asymptotic convergence condition as $s \to +\infty$ is given by a time-one H_M-chord x from L_0 to L_1. When $s \to -\infty$, $H_{A,s} = H_A$, and the boundary conditions $(L_{0,s}, L_{1,s})$ agree with $(L_0^{1-\epsilon}, L_1^{1-\epsilon})$ when $s \ll 0$, so the asymptotic convergence condition as $s \to +\infty$ is given by a time-one H_A-chord x_A from $L_0^{1-\epsilon}$ to $L_1^{1-\epsilon}$. Now we shall see how such a time-one H_A-chord is related to a time-one H_U-chord x' from L'_0 to L'_1. Note that inside $U_0 \setminus (\partial U \times [1-\epsilon, 1])$, the Hamiltonian K_A vanishes, so that H_A is C^2-small, which can be taken to agree with $\frac{1}{1-\epsilon} H_U \circ \psi_U^{1-\epsilon}$ (as H_U is also small, the rescaled Hamiltonian is close to H_U); in the collar neighborhood $\partial U \times [1-\epsilon, 1]$, the Hamiltonian H_A is the rescaling of a quadratic Hamiltonian on U by a factor A, i.e. $\frac{1}{A} H_U \circ \psi_U^A$; and outside U_0, H_A agrees with $H_M + A\epsilon^2$. Regarding the Lagrangian boundary conditions, inside $U_0 \setminus (\partial U \times [1-\epsilon, 1])$ the Lagrangian submanifold $L_j^{1-\epsilon}$ agrees with $\psi_U^{1-\epsilon} L'_{j,0}$; and outside U_0, $L_j^{1-\epsilon}$ agrees with L_j. Thus, time-one H_A-chords from $L_0^{1-\epsilon}$ to $L_1^{1-\epsilon}$ inside $U_0 \setminus (\partial U \times [1-\epsilon, 1])$ naturally correspond to time-one H_U-chords from L'_0 to L'_1 inside U_0, and time-one Hamiltonian chords of H_A which are contained in the collar neighborhood $\partial U \times [1-\epsilon, 1]$ are in one-to-one correspondence with time-one Hamiltonian chords for H_U from L'_0 to L'_1 restricted to a finite part of the cylindrical end, $\partial U \times [1, A\epsilon + 1]$, on which H_U is quadratic with leading coefficient 1.

The second point that needs explanation is the action constraint (7.9). The reason that we impose the constraint on the action of x' is that for the Hamiltonian K_A, the Hamiltonian chords are either constants in $U_0 \setminus (\partial U \times [1-\epsilon, 1])$ or $M_0 \setminus U_0$), and non-constant chords in the collar neighborhood $\partial U \times [1-\epsilon, 1]$ or in the cylindrical end $\partial M \times [1, +\infty)$. For the constant chords, the action is uniformly bounded by a small constant, denoted by δ', which depends only on the primitives of the Lagrangian submanifolds and can be chosen very small. Thus if we make a small perturbation of K_A to H_A, the Hamiltonian chords which are contained inside $U_0 \setminus (\partial U \times [1-\epsilon, 1])$ also have action bounded by a small constant δ. For the non-constant chords, we observe that the restriction of the Hamiltonian vector field X_{K_A} to some level $\partial U \times \{r\} \subset \partial U \times [1-\epsilon, 1]$ is $2A(r-1+\epsilon)Y_{\partial U}$, where $Y_{\partial U}$ is the Reeb vector field on the contact boundary ∂U of U_0. Thus a Hamiltonian chord on level $r \in [1-\epsilon, 1]$ has action $-A(r-1+\epsilon)^2$ (there is no extra contribution as the primitive is locally constant there), which is at least $-A\epsilon^2$. Thus the action constraint (7.9) for the Hamiltonian chord x_A for H_A is reasonable. By imposing this condition, we mean that for each A, we only consider climbing strips w whose asymptotic Hamiltonian chord at $-\infty$ satisfies such constraint on its action.

For each given A, the moduli space of such climbing strips is denoted by $\mathcal{P}^A(x_A, x)$, i.e.

(7.10)

$$\begin{aligned}\mathcal{P}^A(x_A, x) = \{w :& w \text{ is a climbing strip with respect to the Floer datum } (H_{A,s}, J_{A,(s,t)}),\\ & \text{with Lagrangian boundary conditions given by } (L_{0,s}, L_{1,s}),\\ & \text{with asymptotic convergence conditions } x_A \text{ at } -\infty \text{ and } x \text{ at } +\infty\},\end{aligned}$$

Note that the equation (7.7) is not translation invariant, and we are not varying the parameter A in this case, so the virtual dimension of the moduli space $\mathcal{P}^A(x_A, x)$ is

$$v-\dim \mathcal{P}^A(x_A, x) = \deg(x_A) - \deg(x).$$

If the virtual dimension is zero, the moduli space $\mathcal{P}^A(x_A, x)$ is a compact smooth manifold of dimension zero, for generic choices of Floer data. Thus, by counting rigid elements in this moduli space, we may define a map

$$\tilde{r}^1_A : CW^*(L_0, L_1; H_M) \to CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_A), \tag{7.11}$$

by

$$\tilde{r}^1_A(x) = \sum_{x_A:\deg(x_A)=\deg(x)} \sum_{w\in\mathcal{P}^A(x_A,x)} o_w, \tag{7.12}$$

where $o_w : o_x \to o_{x_A}$ is the canonical isomorphism of orientation lines induced by w. Here $CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_A)$ is a sub-complex of the wrapped Floer complex $CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_A)$, generated by time-one H_A-chords from $L_0^{1-\epsilon}$ to $L_1^{1-\epsilon}$ which are contained in U_0. Because of the natural correspondence between time-one H_A-chords from $L_0^{1-\epsilon}$ to $L_1^{1-\epsilon}$ which are contained in U_0 and time-one H_U-chords from L'_0 to L'_1 inside a finite part of U, this space $CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_A)$ is related to the wrapped Floer complex $CW^*(L'_0, L'_1; H_U)$ by the following map

$$\tau_A : CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_A) \to CW^*_{(-A\epsilon^2,\delta)}(L'_0, L'_1; H_U), \tag{7.13}$$

for some fixed small $\delta > 0$ such that all time-one H_U-chords have action less than δ (this can be achieved by choosing the Hamiltonian and the primitives appropriately). This map is a chain-level isomorphism, and in fact sends any time-one H_A-chord x_A to its corresponding time-one H_U-chord x'. By composing the map $\tilde{r}^1_A$ with τ_A, we get a map

$$r^1_A = \tau_A \circ \tilde{r}^1_A : CW^*(L_0, L_1; H_M) \to CW^*_{(-A\epsilon^2,\delta)}(L'_0, L'_1; H_U). \tag{7.14}$$

To prove that r^1_A is a cochain map, we need to study the compactification of the moduli space $\mathcal{P}^A(x_A, x)$. Because of the asymptotic behavior of the elements w, it is natural to introduce a compactification

$$\bar{\mathcal{P}}^A(x_A, x)$$

of this moduli space, by adding broken climbing strips. In the codimension one boundary strata, broken climbing strips are of the following two types:

(i) A pair (w, u), where w is a climbing strip, and u is an inhomogeneous pseudoholomorphic strip in M. w and u have a common asymptotic convergence condition x_0 at the positive end of w and the negative end of u. This occurs because as $s \to +\infty$, the family $H_{A,s}$ agrees with H_M (and similarly for the almost complex structures and Lagrangian boundary conditions), so when the energy of a sequence of climbing strips escapes from $+\infty$, a (H_M, J_M)-pseudoholomorphic strip breaks out.

(ii) A pair (u', w), where u' is pseudoholomorphic with respect to the Floer datum (H_A, J_A), and w is a climbing strip. u' and w have a common asymptotic convergence condition $x_{A,0}$ at the positive end of u' and the negative end of w. This occurs because as $s \to -\infty$, the family $H_{A,s}$ agrees with H_A (and similarly for the almost complex structures and Lagrangian boundary conditions), so when the energy of a sequence of climbing strips escapes from $+\infty$, a (H_A, J_A)-pseudoholomorphic strip breaks out. Moreover, for such an inhomogeneous pseudoholomorphic strip, if the input $x_{A,0}$ is in the sub-complex $CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_A)$, the output must also be in this sub-complex.

This implies that we have an isomorphism of the codimension one boundary strata:

(7.15)
$$\partial\bar{\mathcal{P}}^A(x_A, x) \cong \coprod_{\substack{x_0 \\ \deg(x_0)=\deg(x)}} \mathcal{P}^A(x_A, x_0)\times\mathcal{M}(x_0, x) \cup \coprod_{\substack{x_{A,0} \\ \deg(x_{A,0})=\deg(x_A)}} \mathcal{M}(x_A, x_{A,0})\times\mathcal{P}^A(x_{A,0}, x).$$

Here $\mathcal{M}(x_A, x_{A,0})$ is the moduli space of (H_A, J_A)-pseudoholomorphic strips with asymptotic convergence conditions $x_A, x_{A,0}$, which can be identified with the moduli space $\mathcal{M}(x', x'_0)$ of (H_U, J_U)-pseudoholomorphic strips in U, by a similar rescaling argument.

In order for the above isomorphism to hold, we must assume from now on that we have make conformally consistent choice of Floer data for all kinds of strips. As the Lagrangian submanifolds involved are all exact cylindrical, standard transversality methods allow us to prove that, in the case of virtual dimension being one, the compactified moduli space $\bar{\mathcal{P}}^A(x_A, x)$ of virtual dimension one is a compact topological manifold of dimension one, if the Floer data are chosen generically. Thus, by a standard gluing argument, based on on the structure of the codimension-one boundary strata of $\bar{\mathcal{P}}^A(x_A, x)$ as shown in (7.15), we can show:

Lemma 7.1. *The map*
$$r^1_A : CW^*(L_0, L_1; H_M) \to CW^*_{(-A\epsilon^2,\delta)}(L'_0, L'_1; H_U)$$
is a cochain map.

Proof. In addition to the standard gluing argument, one must ensure that no inhomogeneous pseudoholomorphic strip in U_0 connecting two Hamiltonian chords for H_A inside U_0 escapes from U_0. This is because that the hypersurface $\partial U \times \{1\}$ is pseudo-convex with respect to the chosen almost complex structure, so we can apply the maximum principle.

□

However, the problem is that this depends on an extra parameter A. To remove this ambiguity, we introduce the following trick. Note that the action of any H_A-chord in the collar neighborhood $\partial \times [1-\epsilon, 1]$ satisfies:
$$\mathcal{A}_{H_A}(x_A) = -A(r - 1 + \epsilon)^2,$$
if x_A lies on $\partial U \times \{r\}$. There is no contribution from the primitive f, because f is locally constant there, so that $f_A(x_A(1)) = f_A(x_A(0))$. For the corresponding H_U-chord x', the same estimate is satisfied, thus we see that x' is a time-one H_U-chord contained in $\partial U \times [1, A\epsilon + 1]$. Thus, there exists $\delta > 0$ sufficiently small, such that the map r^1_A in fact descends to a sub-complex

$$r^1_A : CW^*(L_0, L_1; H_M) \to CW^*_{(-A\epsilon^2,\delta)}(L'_0, L'_1; H_U). \tag{7.16}$$

To get rid of the dependence of A, the idea is to take the direct limit as $A \to +\infty$. In order for such direct limit to exist, we must ensure that these maps are compatible with the natural inclusions
$$i_{A,A'} : CW^*_{(-A\epsilon^2,\delta)}(L'_0, L'_1; H_U) \to CW^*_{(-A'\epsilon^2,\delta)}(L'_0, L'_1; H_U), \text{ for } A < A'.$$

Lemma 7.2. *There is a way of choosing families of Hamiltonians when defining the various maps involved, such that for every $A < A'$, the following diagram commutes up to chain homotopy*

(7.17)

$$\begin{array}{ccccc} CW^*(L_0, L_1; H_M) & \xrightarrow{\tilde{r}^1_A} & CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_A) & \xrightarrow{\tau_A} & CW^*_{(-A\epsilon^2,\delta)}(L_0', L_1'; H_U) \\ & \searrow^{\tilde{r}^1_{A'}} & \downarrow c_{A,A'} & & \downarrow i_{A,A'} \\ & & CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_{A'}) & \xrightarrow{\tau_{A'}} & CW^*_{(-A'\epsilon^2,\delta)}(L_0', L_1'; H_U) \end{array}$$

Here

$$c_{A,A'} : CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_A) \to CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_{A'})$$

is the continuation map induced by a monotone homotopy between H_A and $H_{A'}$, and

$$i_{A,A'} : CW^*_{(-A\epsilon^2,\delta)}(L_0', L_1'; H_U) \to CW^*_{(-A'\epsilon^2,\delta)}(L_0', L_1'; H_U)$$

is the natural inclusion map.

Proof. When defining the continuation map $c_{A,A'}$, we can choose the family of Hamiltonians $H_{A,A',s}$, such that the composition of $H_{A,s}$ and $H_{A,A',s}$ agrees with the family $H_{A',s}$, after identifying the glued strip with $\mathbb{R} \times [0, 1]$ by a suitable reparametrization. Thus, by a standard gluing argument, the left triangle commutes up to chain homotopy, which is unique up to higher homotopies.

To prove the homotopy commutativity of the right square, we need to study the continuation map $c_{A,A'}$ in more details. For this, we first write down a specific choice of the homotopy $H_{A,A',s}$, such that in the collar neighborhood $\partial U \times [1-\epsilon, 1]$, the homotopy takes the form

$$H_{A,A',s}(y, r) = ((1 - \lambda(s))A + \lambda(s)A')(r - 1 + \epsilon)^2, (y, r) \in \partial U \times [1 - \epsilon, 1], \tag{7.18}$$

where $\lambda : \mathbb{R} \to [0, 1]$ is a smooth non-increasing function, such that $\lambda(s) = 1$ for $s \ll 0$, and $\lambda(s) = 0$ for $s \gg 0$. Let x_A be a time-one H_A-chord, which corresponds to a time-one H_U-chord x' under τ_A. We shall first prove that, if A' is sufficiently close to A, then the image of x_A under $c_{A,A'}$ is the unique time-one $H_{A'}$-chord which corresponds to the same time-one H_U-chord x' under $\tau_{A'}$. There are two cases to consider:

(i) x_A is a small perturbation of a constant Hamiltonian chord for K_A. Such a Hamiltonian chord is contained in $U_0 \setminus (\partial U \times [1 - \epsilon, 1])$, where K_A and $K_{A'}$ are both zero. Thus we can take their small perturbations H_A and $H_{A'}$ such that they agree inside $U_0 \setminus (\partial U \times [1 - \epsilon, 1])$. Since for a constant homotopy of Hamiltonians the continuation map is necessarily the identity, we conclude that $c_{A,A'}(x_A)$ must be the same Hamiltonian chord as x_A, now regarded as a Hamiltonian chord for $H_{A'}$.

(ii) x_A lies in the collar neighborhood $\partial U \times [1 - \epsilon, 1]$, and corresponds to a non-constant time-one H_U-chord x' in the cylindrical end. It is a general property shared by continuation maps that the count of rigid continuation strips does not jump except for a discrete set of A's. Thus for A' sufficiently close to A, we have that $c_{A,A'}(x_A) = x_{A'}$, as $c_{A,A} = id$ is the identity map (on the chain level), as it is the continuation map with respect to the constant homotopy of Hamiltonians. In particular, for our specific choice of homotopy of Hamiltonians (7.18), there is an explicit formula for such a continuation strip:

$$u_{A,A'}(s,t) = x_{(1-\lambda(s))A+\lambda(s)A'}(t). \tag{7.19}$$

In other words, if A' is sufficiently close to A, then the unique continuation strip which asymptotically converges to x_A at $+\infty$ is the trace of all Hamiltonian chords of $H_{A''}$ for varying $A'' \in [A, A']$, starting from the given time-one H_A-chord x_A.

This proves that if A' is sufficiently close to A, the right square strictly commutes on the chain level.

For general $A' > A$, commutativity might not hold on strictly on the chain level, as there might be extra off-diagonal terms appearing in the continuation map. To prove homotopy commutativity we use the following trick. For each $B \in [A, A']$, there is a small $\delta_B > 0$ such that for $B' \in (B - \delta_B, B + \delta_B)$, we have that there is a choice of almost complex structure $J_{B'}$ in a small neighborhood of J_B homotopic to J_B, for which the relevant moduli spaces of inhomogeneous pseudoholomorphic strips are regular, and the chain-level continuation map

$$c_{B',B} : CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_{B'}, J_{B'}) \to CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_B, J_B),$$

if $B' \leq B$, or

$$c_{B,B'} : CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_B, J_B) \to CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_{B'}, J_{B'})$$

if $B \leq B'$, are diagonal matrices with respect to the natural basis. The intervals $(B - \delta_B, B + \delta_B)$ form an open cover of $[A, A']$, from which we choose a finite subcover, $(B_0 - \delta_{B_0}, B_0 + \delta_{B_0}), \cdots, (B_N - \delta_{B_N}, B_N + \delta_{B_N})$, where $A < B_0 < \cdots < B_N < A'$. For each B_j and B_{j+1}, $(B_j - \delta_{B_j}, B_j + \delta_{B_j}) \cap (B_{j+1} - \delta_{B_{j+1}}, B_{j+1} + \delta_{B_{j+1}})$ is non-empty, so we can pick a number B'_j in this intersection. Then by a standard homotopy argument, the continuation map $c_{B_j,B_{j+1}}$ is chain homotopic to the composition $c_{B'_j,B_{j+1}} \circ c_{B'_j,B_{j+1}}$. With respect to the natural basis (ordered according to action) for these Floer cochain complexes, both c_{B_j,B'_j} and $c_{B'_j,B_{j+1}}$ can be written as matrices of the form $(I, 0)$, where I is the identity matrix (the number of columns of c_{B_j,B'_j} should be equal to the number of rows of $c_{B'_j,B_{j+1}}$). Now the continuation map $c_{A,A'}$ is chain homotopic to the composition $c_{B_{N-1},B_N} \circ \cdots \circ c_{B_0,B_1}$, and is therefore chain homotopic to a map $\kappa_{A,A'}$ which can be written as a matrix of the form $(I, 0)$. Now it is easy to see that the map $\kappa_{A,A'}$, when replacing $c_{A,A'}$, makes the right square strictly commutes, because we have $\kappa_{A,A'}(x_A) = x_{A'}$ by definition, for the time-one H_A-chord x_A and the time-one $H_{A'}$-chord $x_{A'}$ corresponding to the same time-one H_U-chord x'. Thus, for the continuation map $c_{A,A'}$ itself, the right square commutes up to chain homotopy.

□

As an immediate consequence of this lemma, we have the following:

Corollary 7.1. *The homotopy direct limit map*

$$r^1 = \lim_{A \to +\infty} r^1_A : CW^*(L_0, L_1; H_M) \to CW^*(L'_0, L'_1; H_U) \tag{7.20}$$

is well-defined. It is a cochain map.

Proof. Simply note that the direct limit $\lim_{A \to +\infty} CW^*_{(-A\epsilon^2,\delta)}(L'_0, L'_1; H_U)$ is homotopy equivalent to the whole wrapped Floer cochain complex $CW^*(L_0, L_1; H_U)$.

The second item follows immediately from the fact that r^1_A is a cochain map for every A.

□

The map (7.20) is called the linear Viterbo restriction homomorphism.

7.4. **Non-linear terms of the Viterbo restriction functor.** The higher order terms of the Viterbo restriction functor are defined by counting inhomogeneous pseudoholomorphic maps of similar kind, whose domains are disks with several punctures. These are defined in a slightly different manner, as we shall also include A as a parameter. Let $k \geq 2$ be a positive integer and let $(Z^{k+1}, A) \in \mathcal{S}^{k+1} = \mathcal{R}^{k+1} \times (0, +\infty)$ be an element of (the smooth part of) the multiplihedra, where Z^{k+1} is a disk with $k+1$ boundary punctures. The punctures are labeled by $z_0, \cdots, z_k$ in a counterclockwise order, where z_0 is negative, while other punctures are positive. These punctures should come with chosen strip-like ends:

$$\epsilon_0 : (-\infty, 0] \times [0,1] \to Z^{k+1},$$

and

$$\epsilon_j : [0, +\infty) \times [0,1] \to Z^{k+1},$$

for $j = 1, \cdots, k$. The boundary component between z_0 and z_1 is denoted by I_0^-, the boundary component between z_0 and z_k is denoted by I_k^-, and for $j = 1, \cdots, k-1$, the boundary component between z_j and z_{j+1} is denoted by I_j^+.

The boundary conditions for (Z^{k+1}, A) is specified by the following definition.

Definition 7.6. *A moving Lagrangian label for (Z^{k+1}, A) is a collection of families of Lagrangian submanifolds, one for each boundary component of Z^{k+1}. The assignment is as follows:*

- *(i) Assigned to I_j^+, the family of Lagrangian submanifolds is the constant family L_j, for $j = 1, \cdots, k-1$;*
- *(ii) Assigned to I_0^-, the family of Lagrangian submanifolds is the family $L_{0,A,s}$;*
- *(iii) Assigned to I_k^-, the family of Lagrangian submanifolds is the family $L_{k,A,s}$.*

Here $L_{0,A,s}, L_{k,A,s}$ are the exact Lagrangian isotopies introduced in Definition **??**.

To write down inhomogeneous Cauchy-Riemann equations adapted to our setup, we need to introduce an appropriate class of Floer data on these domains.

Definition 7.7. *A Floer datum on (Z^{k+1}, A) consists of the following data:*

- *(i) A collection of weights $\lambda_0, \cdots, \lambda_k$.*
- *(ii) A basic one-form $\beta_{Z^{k+1},A}$ on Z^{k+1}, such that over the j-th strip-like ends it agrees with $\lambda_j dt$. Here by a basic one-form we mean a sub-closed one-form which vanishes along the boundary of Z^{k+1}, and whose differential vanishes in a neighborhood of the boundary of Z^{k+1}.*
- *(iii) A family of Hamiltonians $H_{Z^{k+1},A}$ depending on (Z^{k+1}, A), such that near the j-th strip-like end, it agrees with $\lambda_j H_A$ up to some conformal rescaling factor ρ_j;*
- *(iv) A family of almost complex structures $J_{Z^{k+1},A}$ depending on (Z^{k+1}, A), such that near the j-th strip-like end, it agrees with J_A up to the same conformal rescaling factor ρ_j.*
- *(v) A shifting function $\rho_{(Z^{k+1},A)} : \partial Z^{k+1} \to (0, +\infty)$ which takes the value ρ_j on the boundary of the j-th strip-like end.*

Definition 7.8. *Suppose we have chosen a moving Lagrangian labeling for* (Z^{k+1}, A), *as well as a Floer datum on* (Z^{k+1}, A). *A climbing disk (with* $k+1$ *punctures) is a triple* (Z^{k+1}, A, w^{k+1}) *where* $(Z^{k+1}, A) \in \mathcal{S}^{k+1}$, *and*

$$w^{k+1} : Z^{k+1} \to M$$

is a smooth map with the following properties:

(i) w^{k+1} *satisfies the inhomogeneous Cauchy-Riemann equation:*

$$(dw - \beta_{Z^{k+1},A} \otimes X_{H_{Z^{k+1},A}}(w)) + J_{Z^{k+1},A} \circ (dw - \beta_{Z^{k+1},A} \otimes X_{H_{Z^{k+1},A}}(w)) \circ j_{Z^{k+1},A} = 0. \tag{7.21}$$

(ii) The boundary conditions for w *are given by the chosen moving Lagrangian labeling* $(L_{0,A}, L_1, \cdots, L_{k-1}, L_{k,A})$.

(iii) The asymptotic convergence conditions of w^{k+1} *are:*

$$\lim_{s \to -\infty} w^{k+1} \circ \epsilon_0(s, \cdot) = x_{0,A}(\cdot)$$

for a time-one H_A*-chord* $x_{0,A}$ *from* $L_{0,A}$ *to* $L_{k,A}$, *and*

$$\lim_{s \to +\infty} w^{k+1} \circ \epsilon_j(s, \cdot) = x_j(\cdot)$$

for a time-one H_M*-chord* x_j *from* L_{j-1} *to* L_j, *where* $j = 1, \cdots, k$. *Similar to the case of climbing strips, we require that the* H_A*-chord* $x_{A,0}$ *satisfy the following condition*

$$-A\epsilon^2 \leq \mathcal{A}(x_{0,A}) \leq \delta.$$

The definition of a Floer datum easily generalizes to broken disks in the multiplihedra $\bar{\mathcal{S}}^{k+1}$. There is also a notion of universal and conformally consistent choices of Floer data, similar to that in the case for ordinary A_∞-disks and also that in the case of the action-restriction functor. Now let us assume from now on that universal and conformally consistent choices of Floer data have been made for all elements in $\bar{\mathcal{S}}^{k+1}$ and for all k.

Note that for each climbing disk (Z^{k+1}, A, w^{k+1}), the asymptotic convergence condition $x_{0,A}$ at the negative strip-like end depends on the moduli parameter A. In order to define a moduli space of such climbing disks, such that the Hamiltonian chord at the negative strip-like end can be fixed as the output, we must find a way of identifying these $x_{0,A}$ for different A. This issue can be resolved using the observation that time-one H_A-chords inside U_0 can be identified with certain time-one H_U-chords. Let x'_0 be the time-one H_U-chord corresponding to $x_{0,A}$, under the natural map

$$\tau_A : CW^*_-(L_{0,A}, L_{k,A}; H_A) \to CW^*_{(-A\epsilon^2, \delta)}(L'_0, L'_k; H_U).$$

As discussed in the proof of Lemma 7.2, the image of $x_{0,A}$ under the continuation map $c_{A,A'}$ is a time-one $H_{A'}$-chord $x_{0,A'}$ corresponding to the same H_U-chord x'_0 under the isomorphism $\tau_{A'}$. Thus, we may write that asymptotic convergence condition as

$$\lim_{s \to -\infty} w^{k+1}(s, \cdot) = \tau_A(x'_0)(\cdot), \tag{7.22}$$

for a fixed time-one H_U-chord x'_0 from L'_0 to L'_k. Then, the moduli space of such climbing disks, with the given asymptotic convergence conditions x'_0 at the negative strip-like end, and $x_1, \cdots, x_k$ at the positive strip-like ends, is denoted by

$\mathcal{P}_{k+1}(x_0'; x_1, \cdots, x_k)$. The virtual dimension of this moduli space is

$$v-\dim \mathcal{P}_{k+1}(x_0'; x_1, \cdots, x_k) = \deg(x_0') - \deg(x_1) - \cdots - \deg(x_k) + k - 1,$$

where $k-1$ is the dimension of the moduli space $\mathcal{S}^{k+1}$ of the underlying domains (Z^{k+1}, A).

There is a natural compactification of the moduli space of climbing disks with $k+1$ punctures, denoted by

$$\bar{\mathcal{P}}_{k+1}(x_0'; x_1, \cdots, x_k).$$

The elements in this compactified moduli space are broken climbing disks, generalizing the broken climbing strips. There are four types of such broken configurations:

(i) A pair $((Z^{k+1}, A, w^{k+1}), u)$, where (Z^{k+1}, A, w^{k+1}) is a climbing disk with $k+1$ punctures, and u is an inhomogeneous pseudoholomorphic strip in M. w^{k+1} and u have a common asymptotic convergence condition x_{new} at some positive end of w^{k+1} and the negative end of u, for some time-one H_M-chord x_{new}. This occurs as the limit of a sequence of climbing disks where the energy escapes through some positive strip-like end.

(ii) A pair $(u', (Z^{k+1}, A, w^{k+1}))$, where u' is an inhomogeneous pseudoholomorphic strip in U with respect to the Floer datum (H_A, J_A), and (Z^{k+1}, A, w^{k+1}) is a climbing disk with $k+1$ punctures. u' and w^{k+1} have a common asymptotic convergence condition x'_{new} at the positive end of u' and the negative end of w^{k+1} (up to a conformal rescaling), for some time-one H_U-chord x'_{new}. This occurs as the limit of a sequence of climbing disks as the energy escapes through the negative strip-like end.

(iii) A pair $((Z^{k-m+2}, A, w^{k-m+2}), u)$, where $(Z^{k-m+2}, A, w^{k-m+2})$ is a climbing disk with $k-m+2$ punctures (for some $m > 1$) and u is an inhomogeneous pseudoholomorphic $(m+1)$-punctured disk in M. w^{k-m+2} and u have a common asymptotic convergence condition x_{new} at some positive end of w^{k-m+2}, and the negative end of u, for some time-one H_M-chord x_{new}. This occurs as the limit of a sequence of climbing disks for which the parameter A tends to 0 or remains finite.

(iv) A tuple $(u', (Z^{s_1+1}, A_1, w^{s_1+1}), \cdots, (Z^{s_l+1}, A_l, w^{s_l+1}))$, where $(Z^{s_j+1}, A_j, w^{s_j+1})$ is a climbing disk with s_j+1 punctures, and u' is an inhomogeneous pseudoholomorphic $(l+1)$-punctured disk with respect to the Floer datum (H_A, J_A) (for some $l > 1$). u' and w^{s_j+1} have a common asymptotic convergence condition $x'_{new,j}$ at the j-th positive end of u' (up to conformal rescaling), and the negative end of w^{s_j+1}, for some time-one H_U-chord $x'_{new,j}$. This occurs as the limit of a sequence of climbing disks for which the parameter A tends to $+\infty$.

Note that the broken configurations of type (i) and type (iii) can be written in a uniform way, as (i) is the special case $m=1$. And the broken configurations of type (ii) and type (iv) can be written in a uniform way, as (ii) is the special case $l=1$. In case (ii), by an (H_A, J_A)-pseudoholomorphic disk u' with $l+1$ punctures, we mean it satisfies the inhomogeneous Cauchy-Riemann equation with respect to a domain-dependent family of Hamiltonians which agrees with H_A (up to rescaling) over the strip-like ends, and a domain-dependent family of almost complex structures which agrees with J_A (up to rescaling) over the strip-like ends. Such an inhomogeneous pseudoholomorphic disk then corresponds to an (H_U, J_U)-pseudoholomorphic disk under the conformal rescaling by Liouville flow. In case (iv), such an (H_A, J_A)-pseudoholomorphic strip u' corresponds to an (H_U, J_U)-pseudoholomorphic strip under conformal rescaling. Thus, we have an isomorphism of the codimension one boundary strata:

$$
\begin{aligned}
&\partial\bar{\mathcal{P}}_{k+1}(x_0'; x_1, \cdots, x_k) \\
\cong &\coprod_i \coprod_{\substack{x_{new} \\ \deg(x_{new}) = \deg(x_{i+1}) + \cdots + \deg(x_{i+m}) + 2 - k}} \mathcal{P}_{k-m+2}(x_0'; x_1, \cdots, x_i, x_{new}, x_{i+m+1}, \cdots, x_k) \\
&\times \mathcal{M}_{m+1}(x_{new}, x_{i+1}, \cdots, x_{i+m}) \\
\cup &\coprod_{\substack{s_1, \cdots, s_l \\ s_1 + \cdots + s_l = k}} \coprod_{\substack{x'_{new,1}, \cdots, x'_{new,l} \\ \deg(x'_{new,j}) = \deg(x_{s_1 + \cdots + s_{j-1} + 1}) + \cdots + \deg(x_{s_1 + \cdots + s_j}) + 1 - s_j}} \mathcal{M}_{l+1}(x_0', x'_{new,1}, \cdots, x'_{new,l}) \\
&\times \mathcal{P}_{s_1+1}(x'_{new,1}; x_1, \cdots, x_{s_1}) \times \cdots \times \mathcal{P}_{s_l+1}(x'_{new,l}; x_{s_1+\cdots+s_{l-1}+1}, \cdots, x_k).
\end{aligned}
\tag{7.23}
$$

Note that the components $\mathcal{M}_{l+1}(x_0', x'_{new,1}, \cdots, x'_{new,l})$ are moduli spaces of (H_U, J_U)-pseudoholomorphic disks in U, which enter this picture as the identification of the moduli spaces $\mathcal{M}_{l+1}(x_{0,A}; x_{new,1,A}, \cdots, x_{new,l,A})$ of (H_A, J_A)-pseudoholomorphic disks, provided that a rigid climbing disk exists for that A. As before, this isomorphism holds if the Floer data are chosen in a consistent way.

If we choose Floer data generically, the compactified moduli space $\bar{\mathcal{P}}_{k+1}(x_0'; x_1, \cdots, x_k)$, when the virtual dimension is zero or one, is a compact smooth/topological manifold of dimension zero/one. In particular, $\mathcal{P}_{k+1}(x_0'; x_1, \cdots, x_k)$ is compact if the dimension is zero, and therefore consists of finitely many points with natural orientations. Then by counting rigid elements in the moduli space $\mathcal{P}_{k+1}(x_0'; x_1, \cdots, x_k)$, we can define multilinear maps

$$r^k : CW^*(L_{k-1}, L_k; H_M) \otimes \cdots \otimes CW^*(L_0, L_1; H_M) \to CW^*(L_0', L_k'; H_U). \tag{7.24}$$

That is, for each rigid climbing disk (Z^{k+1}, A, w^{k+1}), we get a canonical isomorphism of orientation lines

$$o_{(Z^{k+1}, A, w^{k+1})} : o_{x_1} \otimes \cdots \otimes o_{x_k} \to o_{x_{0,A}}.$$

We then compose it with the map τ_A to get an isomorphism

$$o_{x_1} \otimes \cdots \otimes o_{x_k} \to o_{x_0'},$$

and sum over all such (Z^{k+1}, A, w^{k+1}) and all x_0' to get the desired map r^k.

Now consider the situation where the virtual dimension of $\bar{\mathcal{P}}(x', x)$ is one. Then the structure of the codimension one boundary strata of the moduli spaces $\bar{\mathcal{P}}(x', x)$ (7.15) and the moduli spaces $\bar{\mathcal{P}}_{k+1}(x_0'; x_1, \cdots, x_k)$ (7.23) immediately implies:

Lemma 7.3. *The multilinear maps*

$$r^k : CW^*(L_{k-1}, L_k; H_M) \otimes \cdots \otimes CW^*(L_0, L_1; H_M) \to CW^*(L_0', L_k'; H_U)$$

satisfy the equations for A_∞-functors.

Proof. The non-trivial part involves taking the direct limit $\lim\limits_{A \to +\infty}$ for various Floer cochain spaces and various maps, but that can be dealt with in a similar way as Lemma □

Varying the objects L_j's in the full sub-category $\mathcal{B}(M)$, we get an A_∞-functor

$$r : \mathcal{B}(M) \to \mathcal{W}(U). \tag{7.25}$$

To summarize, the spirit of our construction is as follows. Implicit, we have defined a version of homotopy direct limit

$$\lim_{A\to+\infty} \mathcal{W}_-(M; H_A)$$

where $\mathcal{W}_-(M; H_A)$ is an A_∞-category whose morphism spaces are the truncated wrapped Floer complexes $CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_A)$, and whose A_∞-structure maps are defined by counting inhomogeneous pseudoholomorphic disks with respect to Floer data (H_A, J_A). Equivalently, this category is quasi-isomorphic to a truncated version of the wrapped Fukaya category of U. There are natural functors

$$c_A : \mathcal{W}(U) \to \mathcal{W}_-(M; H_A),$$

which are induced by the continuation functors defined in terms of Floer theory in U, given a homotopy of Hamiltonians from H_U, which is quadratic with leading coefficient 1, to a Hamiltonian which is quadratic with leading coefficient A. These functors c_A are compatible with natural continuation functors

$$c_{A,A'} : \mathcal{W}_-(M; H_A) \to \mathcal{W}_-(M; H_{A'}),$$

so we can take the direct limit

$$\lim_{A\to+\infty} c_A : \mathcal{W}(U) \to \lim_{A\to+\infty} \mathcal{W}_-(M; H_A).$$

By an action filtration argument, this functor is a quasi-isomorphism. Thus we are able to find a homotopy inverse

$$k : \lim_{A\to+\infty} \mathcal{W}_-(M; H_A) \to \mathcal{W}(U).$$

From this point of view, the main part of our construction is to use the moduli spaces $\mathcal{P}_{k+1}(x_0'; x_1, \cdots, x_k)$ to define an A_∞-functor

$$\lim_{A\to+\infty} r_A : \mathcal{B}(M) \to \lim_{A\to+\infty} \mathcal{W}_-(M; H_A).$$

By composing this with the functor k, we get

$$r = k \circ \lim_{A\to+\infty} r_A : \mathcal{B}(M) \to \mathcal{W}(U).$$

Definition 7.9. *The A_∞-functor*

$$r : \mathcal{B}(M) \to \mathcal{W}(U)$$

is called the Viterbo restriction functor.

7.5. **Comparison between the linear terms.** In this subsection, we shall prove the second half of Theorem 1.4. Formulated in a more precise way, what we need to prove is that the linear term of the functor Θ_Γ

$$\Theta_\Gamma^1 : CW^*(L_0, L_1; H_M) \to CW^*(L_0', L_1'; H_U)$$

is chain homotopic to the linear Viterbo restriction map

$$r^1 : CW^*(L_0, L_1; H_M) \to CW^*(L_0', L_1'; H_U).$$

We shall from now on restrict the Viterbo restriction functor to the full sub-category $\mathcal{B}_0(M)$, and obtain an A_∞-functor

$$r : \mathcal{B}_0(M) \to \mathcal{W}(U).$$

That is to say, we shall consider only Lagrangian submanifolds L in the full subcategory $\mathcal{B}_0(M)$, i.e. they satisfy Assumption 7.1 and Assumption 7.2.

When comparing Θ^1_Γ to the linear term r^1 of the Viterbo restriction functor, we shall instead work with the cochain map Π_Γ as defined in (6.54). The reason why we study this map is that its construction is straightforward and geometric, without explicitly referring to representability.

Recall that the map Π_Γ is defined using the moduli spaces $\bar{\mathcal{U}}(x; x'; e_0, e_1)$ of inhomogeneous pseudoholomorphic quilted maps, which are variants of the moduli spaces

$$\bar{\mathcal{U}}_{l_0,l_1}(\alpha, \beta; x; y; y_{0,1}, \cdots, y_{0,l_0}; y_{1,1}, \cdots, y_{1,l_1}; e_0, e_1),$$

where now there are no those $l_0 + l_1$ punctures on the boundary components of the second patch, as the Lagrangian submanifolds are embedded and the bounding cochains b_0 and b_1 vanish. On the other hand, the linear Viterbo restriction map r^1 is the homotopy direct limit of maps r^1_A, constructed by appropriate count of elements in the moduli spaces of climbing strips $\bar{\mathcal{P}}^A(x_A, x)$. The idea of proving that these two maps coincide up to chain homotopy is to compare these two kinds of relevant moduli spaces, so as to establish a natural bijective correspondence between them. For this purpose, we shall investigate the geometric conditions for inhomogeneous pseudoholomorphic quilted maps in the moduli spaces $\bar{\mathcal{U}}(x; x'; e_0, e_1)$, and prove that any inhomogeneous psuedoholomorphic quilted map can be converted to a climbing strip in a canonical way.

Let us first recall the picture of these inhomogeneous pseudoholomorphic quilted maps in $\bar{\mathcal{U}}(x; x'; e_0, e_1)$, or rather, the smooth locus $\mathcal{U}(x; x'; e_0, e_1)$. Let (u, v) be an inhomogeneous pseudoholomorphic quilted map in $\mathcal{U}(x; x'; e_0, e_1)$. Then it satisfies the following conditions:

(i) The quilted surface has two patches $\underline{S} = (S_0, S_1)$. S_0 is a punctured disk with a positive puncture z^1_0 and two special punctures $z_0^{-,1}, z_0^{-,2}$, with chosen strip-like ends near the punctures. S_1 is a punctured disk with a negative puncture z^0_1 and two special punctures $z_1^{-,1}, z_1^{-,2}$. The quilted surface is obtained by seaming the two patches along the boundary component I_0^- of S_0 between $z_0^{-,1}, z_0^{-,2}$ and the boundary component I_1^- of S_1 between $z_1^{-,1}, z_1^{-,2}$. The strip-like ends near special punctures form quilted strip-like ends for $(z_0^{-,1}, z_1^{-,1}$ and $(z_0^{-,2}, z_1^{-,2})$.

(ii) $u : S_0 \to M$ is pseudoholomorphic with respect to the Floer datum (H_{S_0}, J_{S_0}), which maps the two (non-seam) boundary components to L_0 and L_1 respectively, which asymptotically converges to a time-one H_M-chord x from L_0 to L_1 at the puncture z^1_0.

(iii) $v : S_1 \to U$ is pseudoholomorphic with respect to the Floer datum (H_{S_1}, J_{S_1}), which maps the two (non-seam) boundary components to L', which asymptotically converges to an H_U-chord x' from L'_0 to L'_1 at z^0_1.

(iv) Over the seam (I_0^-, I_1^-), the matching condition for the pair of maps (u, v) is given by the Lagrangian correspondence Γ, i.e., $(u(z), v(z)) \in \Gamma$ for z on the seam.

(v) At the two quilted strip-like ends, (u, v) asymptotically converges to the unique generalized chord representing the cyclic $e_j \in CW^*(L_j, \Gamma, L'_j)$. In fact, e_j comes from the fundamental chain of L'_j, under the natural identification between the generalized intersections of (L_j, Γ, L'_j) and the self-intersections of L'_j.

Given these conditions, we find that the isomorphism class of the underlying quilted surface $\underline{S}$ is unique, i.e. the moduli space of such quilted surfaces is a singleton. Thus we often denote such a quilted map simply by (u, v).

The main task of this subsection is to prove the following:

Proposition 7.1. *Suppose the virtual dimension of $\mathcal{U}(x; x'; e_0, e_1)$ is zero or one. Let $A > 0$ be any positive number such that*

$$\mathcal{A}(x') \geq -A\epsilon^2. \tag{7.26}$$

Among generic choices of Floer data which make $\mathcal{U}(x; x'; e_0, e_1)$ regular, we can make a specific choice of Floer data (without losing genericity), for which the moduli space $\mathcal{U}(x; x'; e_0, e_1)$ is isomorphic to a moduli space $\tilde{\mathcal{P}}^A(x_A, x)$ of inhomogeneous pseudoholomorphic maps, which is orientedly cobordant to the moduli space of climbing strips $\mathcal{P}^A(x_A, x)$.

Remark 7.3. *For technical reasons, our construction does not immediately give rise to a climbing strip from a given inhomogeneous pseudoholomorphic quilted map, but rather yields a map satisfying the inhomogeneous Cauchy-Riemann equation with respect to the same Floer datum, but different boundary conditions. The moduli space of such inhomogeneous pseudoholomorphic maps are cobordant to the moduli space of climbing strips in a natural way by means of exact Lagrangian isotopies of the boundary conditions, which will be explained in the proof.*

In order to turn such a quilted map $(\underline{S}, (u, v))$ into an inhomogeneous pseudoholomorphic map in M, we shall consider the Morse-Bott setup of both wrapped Floer cochain spaces $CW^*(L_j; H_M)$ and $CW^*(L'_j; H_U)$, for $j = 0, 1$. This process can be done without repeating the details by regarding L' as a Lagrangian immersion. Recall that the asymptotic conditions over the quilted ends are specified by the cyclic element $e_j \in CW^*(L_j, \Gamma, L'_j)$, which in turn corresponds to the homotopy unit of the A_∞-algebra $CW^*(L'_j; H_U)$, under the map (6.33). This homotopy unit is represented by the minimum of a Morse function on L'_j, whose image in U is inside the interior part U_0 where the Hamiltonian vanishes. Thus we may in fact demand that the families of Hamiltonians (H_{S_0}, H_{S_1}) be zero near the quilted ends: this condition is consistent with the Morse-Bott setup of the wrapped Floer theory for L_j and L'_j.

A naive trial is to simply "glue" the two components u and v together. Since Γ is the completion of the graph of the natural inclusion $U_0 \to M_0$, the condition that a point $(p, q) \in M \times U$ lies on Γ is equivalent to the condition that p is the image of q under the natural map $i : U \to M$. Thus the matching condition on Γ in fact allows us to "fold" the quilted map to obtain a map $\bar{w}$ in M, after composing the second component v with the embedding $i : U \to M$. That is, glue the two patches of the quilted surface together and define a map $\tilde{w}$ which is u on S_0 and which is $i \circ v$ on S_1. From the conditions for $(\underline{S}, (u, v))$ one can immediately see that $\tilde{w}$ satisfies the following properties:

(i) The domain of the map $\bar{w}$ is a 4-punctured disk. One is a negative puncture ξ^0, which corresponds to the negative puncture of S_1. One is a positive puncture ξ^1, which corresponds to the positive puncture of S_0. The other two are special punctures $\xi^{1,-}, \xi^{2,-}$, which come from gluing the quilted punctures.

(ii) $\bar{w}$ is smooth, as the matching condition on Γ means that the maps u and $i \circ v$ agree on the seam along which the two patches are glued together.

(iii) $\bar{w}$ has removable singularities at the special punctures $\xi^{1,-}, \xi^{2,-}$. This is because the families of Hamiltonians (H_{S_0}, H_{S_1}) are chosen such that they vanish over the two quilted ends, and the quilted map (u, v) asymptotically converges to e'_0 and e'_k, which in the Morse-Bott chain model correspond to the fundamental chains of L'_0 and L'_k.

We have thus obtained a map $\bar{w}$ to M from the inhomogeneous pseudoholomorphic quilted map (u, v). However, the domain of the map $\bar{w}$ is of an unfamiliar form, and it does not satisfy the desired inhomogeneous Cauchy-Riemann equation for a climbing strip. It is therefore necessary to perform suitable modification on the map $\bar{w}$. By condition (iii), we may take conformal transformations mapping S_0 to the positive half-strip $Z_+ = [0, +\infty) \times [0, 1]$, sending the positive puncture to $+\infty$, and the two quilted punctures to the corner points $(0, 0), (0, 1)$, and mapping S_1 to the negative half-strip $Z_- = (-\infty, 0] \times [0, 1]$, sending the negative puncture to $-\infty$, and the two quilted punctures to the corner points $(0, 0), (0, 1)$. Glue these two half-strips together along the common boundary $\{0\} \times [0, 1]$, which correspond to the seam of the quilted surface. The result of gluing is a strip with two marked points $z_{1,+} = (0, 0), z_{2,+} = (0, 1)$. This strip is the domain for our new map. In addition, condition (iii) also implies that the incidence conditions for these two marked points are given by the fundamental chains of L'_0 and L'_1, thus these incidence conditions are in fact free conditions.

Now we shall construct Floer datum on this new domain from the original Floer datum for the quilted surface $\underline{S}$. The guiding principle is to use the Liouville structure to rescale the original Floer datum to obtain the new one. Without loss of generality, we may assume that $A > 1$, as when taking the direct limit $\lim\limits_{A \to +\infty}$ we only have to consider sufficiently large A. Fix a choice of a smooth increasing function

$$\rho_A : (-\infty, 0] \to [1, A]$$

such that for $s \ll 0$, $\rho_A(s) = A$ and $\rho_A(s) = 1$ for s close to 0. We can also extend this function to the whole of $\mathbb{R}$ by setting $\rho_A(s) = 1$ for all $s \geq 0$.

First let us discuss how to obtain a family of Hamiltonians from the given families of Hamiltonians (H_{S_0}, H_{S_1}) for the quilted surface. We define a family of Hamiltonians $H_{A,s}$ on M parametrized by Z, which depends only on the coordinate $s \in \mathbb{R}$, by setting

$$H_{A,s} = \begin{cases} \frac{1}{\rho_A(s)} H_{Z_-,s} \circ \psi_U^{\rho_A(s)} \circ i^{-1}, & \text{on } Z_-, \\ H_{Z_+,s}, & \text{on } Z_+, \end{cases} \tag{7.27}$$

where H_{Z_-} is the family H_{S_1} composed with the conformal transformation from Z_- to S_1, and H_{Z_+} is the family H_{S_0} composed with the conformal transformation from Z_+ to S_0. Here ϵ is the small constant that we have fixed for which the collar neighborhood $\partial U \times [1-\epsilon, 1]$ embeds into U_0. And $\frac{1}{\rho_A(s)} H_{Z_-,s} \circ \psi_U^{\rho_A(s)} i^{-1}$ means a Hamiltonian on M which takes value $\frac{1}{\rho_A(s)} H_{Z_-,s} \circ \psi_U^{\rho_A(s)}(v(s,t))$ at the point $i \circ v(s,t) \in M$, and is constant (equal to $A\epsilon^2$) elsewhere. In fact, we do not have to specify what values the Hamiltonian take for points outside $i(U)$, as the image of the point on the domain of the inhomogeneous pseudoholomorphic map at which the domain-dependent family of Hamiltonians is evaluated is inside $i(U)$. A priori, the resulting family of Hamiltonians also depends on the t-coordinate, but we may choose (H_{S_0}, H_{S_1}) appropriately such that the resulting families H_{Z_+} and H_{Z_-} is independent of t (though this is indeed irrelevant for our purpose). This is possible, because of the following reasons: over the positive strip-like end S_0, H_{S_0} agrees with H_M, and over the negative strip-like end of S_1, H_{S_1} agrees with H_U, while over the quilted ends, (H_{S_0}, H_{S_1}) vanish - all of these are independent of t.

For the almost complex structure, given the families of almost complex structures (J_{S_0}, J_{S_1}) for the quilted surface, we define a family of almost complex structures $J_{A,(s,t)}$ on M parametrized by $(s,t) \in Z$, by setting

$$J_{A,(s,t)} = \begin{cases} (\psi_U^{\rho_A(s)})_* J_{Z_-,(s,t)} \circ i^{-1}, & \text{on } Z_-, \\ J_{Z_+,(s,t)}, & \text{on } Z_+. \end{cases} \tag{7.28}$$

We must explain why (7.27) and (7.28) are well-defined. There are several cases to discuss, depending on the positions of the Hamiltonian chords x' and x to which the quilted map $(\underline{S}, (u, v))$ converges:

(i) x' is a small perturbation of a constant Hamiltonian chord in U_0, and x is a small perturbation of a constant Hamiltonian chord in M_0. In this case, by a maximum principle argument, it is necessary that, for the quilted map (u, v), the image of u is contained in M_0 and the image of v is contained in U_0. Thus, the matching condition on Γ simply implies that $u(0,t) = v(0,t)$. If furthermore x is contained in U_0, then in fact $x' = x$ and the quilted map is constant. This is essentially the only case that needs some discussion. In this case, we may assume our choice of H_M agrees with the rescaling of H_U inside U_0, so that the above formula (7.27) is consistent. Now if x is in $M_0 \setminus U_0$, then such a quilted map (u, v) is non-trivial. In this case we may choose (H_{S_0}, H_{S_1}) such that $H_{Z_-,0} = 0$ and $H_{Z_+,0} = 0$. Thus it is automatic that the two formulas in (7.27) agree when $s = 0$.

(ii) x' is a small perturbation of a constant Hamiltonian chord in U_0, while x is a non-constant H_M-chord in the cylindrical end $\partial M \times [1, +\infty)$. This is similar to the previous case, since such a quilted map is necessarily non-constant, so that we have the freedom to choose the families (H_{S_0}, H_{S_1}) so that they are zero along the seam.

(iii) x' is a non-constant H_U-chord in the cylindrical end $\partial U \times [1, +\infty)$, while x is a small perturbation of a constant Hamiltonian chord in M_0. This x' corresponds to a time-one chord x_A for the rescaled Hamiltonian. Then there is in fact no such quilted map or climbing strip, as the action of x' (or x_A) is very negative, and the action of x is positive, which is of course greater than the action of x' (or x_A). Thus it is not necessary to consider this case.

(iv) x' is a non-constant H_U-chord in the cylindrical end $\partial U \times [1, +\infty)$, and x is a non-constant H_M-chord in the cylindrical end $\partial M \times [1, +\infty)$. Now x' corresponds to a unique time-one H_A-chord x_A, and the quilted map is non-constant. The discussion is similar to those in cases (i) and (ii).

Thus we have shown that (7.27) is well-defined. For the family of almost complex structures, a parallel argument implies that (7.28) is well-defined.

To construct a climbing strip from the given quilted map (u, v), and $A > 0$, we first define a map

$$w_0(s,t) = \begin{cases} i \circ \psi_U^{\rho_A(s)} \circ v(s,t), & \text{if } s < 0, \\ u(s,t), & \text{if } s \geq 0. \end{cases} \tag{7.29}$$

This is well-defined and smooth, since when $s = 0$, we have $\rho_A(0) = 1$ so that the first formula reads $i \circ v(0,t)$, which is equal to $u(0,t)$ by the matching condition on Γ. By definition, the boundary conditions for the map w_0 are as follows:

$$w_0(s,j) \in \begin{cases} i(\psi_U^{\rho_A(s)} L'_j), & \text{if } s < 0, \\ L_j, & \text{if } s \geq 0, \end{cases} \text{where } j = 0, 1. \tag{7.30}$$

Note that when $s \leq 0$ is close to zero, we have $\rho_A(s) = 1$ so that $i(\psi_U^{\rho_A(s)} L'_j) = i(L'_j) \subset L_j$, because of Assumption 7.2. Let us rewrite the boundary conditions as

$$w_0(s,j) \in L_{j,A,s},$$

for

$$L_{j,A,s} := \begin{cases} i(\psi_U^{\rho_A(s)} L'_j), & \text{if } s < 0, \\ L_j, & \text{if } s \geq 0. \end{cases} \tag{7.31}$$

When $s \geq 0$, the boundary condition can also be written as $L_{j,A,s} = \psi_M^{\rho_A(s)} L_j$, as we have extended ρ_A such that $\rho_A(s) = 1$ for all $s \geq 0$. Since i is the map induced by the Liouville strucutre, it maps L'_j into L_j, and $i(\psi_U^{\rho_A(s)} L'_j)$ is mapped into a unique exact cylindrical Lagrangian submanifold of M whose cylindrical end is contained in that of L_j. In fact, the completion of $i(\psi_U^{\rho_A(s)} L'_j)$ is $\psi_M^A L_j$, because L_j is assumed to be invariant under the Liouville flow in the Liouville cobordism $M_0 \setminus int(U_0)$. Thus $L_{j,A,s}$ can be regarded as a family of Lagrangian submanifolds of M parametrized by $s \in \mathbb{R}$, if we understand $i(\psi_U^{\rho_A(s)} L'_j)$ as its completion. Moreover, this family is an exact Lagrangian isotopy. Such boundary conditions are slightly different from those for a climbing strip. The way to fix this is to use Lagrangian isotopy to move one kind of boundary conditions to the other - such an argument will be presented later.

Note that w_0 satisfies the inhomogeneous Cauchy-Riemann equation on the positive half-strip Z_+, with respect to the Floer datum $(H_{A,s}, J_{A,(s,t)})$ defined as above, as on that region the Floer datum is simply given by the original one (H_{S_0}, J_{S_0}) for the quilted surface. On the negative half-strip Z_-, w_0 might not be pseudoholomorphic, because an extra term appears when differentiating $i \circ \psi_U^{\rho_A(s)} \circ v(s,t)$ with respect to s, which is related to the differential of $\psi_U^{\rho_A(s)}$. By a straightforward calculation, we have

$$\partial_s(i \circ \psi_U^{\rho_A(s)} \circ v(s,t)) = di_{\psi_U^{\rho_A(s)} \circ v(s,t)} \circ (d\psi_U^{\rho_A(s)})_{v(s,t)} \rho'_A(s)(\partial_s v(s,t)), \tag{7.32}$$

where $\rho'_A(s)$ is the derivative of the function ρ_A with respect to s. On the other hand, in the inhomogeneous term contributed by the Hamiltonian vector field, there is not such a factor, as we have for $s < 0$,

$$X_{H_{A,s}} = \rho_A(s) di \circ d\psi_U^{\rho_A(s)} X_{H_{Z_-}}, \tag{7.33}$$

as calculating the Hamiltonian vector field from the Hamiltonian does not involve differentiation with respect to the variable s. In order to obtain a map which satisfies the inhomogeneous Cauchy-Riemann equation, we must therefore perturb the map w_0. In fact, we may choose the function ρ_A suitably such that its derivative is small, say

$$|\rho'_A(s)| < C \cdot inj(M, g),$$

where $inj(M,g)$ is the injectivity radius of a family of metrics $g = g(s,t)$ on M for which the Lagrangian submanifolds $L_{0,A,s}, L_{1,A,s}$ are totally geodesic, and C is an appropriate constant to be determined in the proof of the following perturbation lemma.

Lemma 7.4. *For any given $A > 0$ sufficiently large, and for each w_0 defined as above, there is a unique map $\tilde{w} : Z \to M$ closest to w_0, which satisfies the inhomogeneous Cauchy-Riemann equation:*

$$\partial_s w + J_{A,(s,t)}(\partial_t w - X_{H_{A,s}}(w)) = 0, \tag{7.34}$$

and the same boundary conditions $L_{0,A,s}, L_{1,A,s}$ as those for w_0.

Proof. For the point $w_0 \in \mathcal{B}$, there is the exponential map

$$Exp_{w_0} : O \subset T_{w_0}\mathcal{B} \to \mathcal{B},$$

defined with respect to a family of metrics $g = g(s,t)$ on M parametrized by $(s,t) \in \mathbb{R} \times [0,1]$, for which $L_{0,A,s}$ and $L_{1,A,s}$ are totally geodesic. Here $\mathcal{B}$ is the Banach manifold (with respect to some Sobolev norm $W^{m,p}$) of maps satisfying the same conditions for a climbing strip, except the inhomogeneous Cauchy-Riemann equation. We require that O is an open neighborhood of zero such that the exponential map Exp_{w_0} is an isomorphism onto the image: the size of O is at least the injectivity radius of the family of metrics g:

$$inj(M,g) = \min_{(s,t)\in\mathbb{R}\times[0,1]} inj(M, g(s,t)).$$

In fact, $g = g(s,t)$ is determined by the symplectic form and the chosen family of almost complex structures $J_{A,(s,t)}$. As a result, when $s \ll 0$ or $s \gg 0$, $g(s,t)$ agrees with a family independent of s. Therefore, the minimum over $(s,t) \in \mathbb{R} \times [0,1]$ is indeed taken over a compact set, hence well-defined. The tangent space $T_{w_0}\mathcal{B}$ is

$$T_{w_0}\mathcal{B} = \{V \in W^{m,p}(Z; w_0^*TM; w_0^*TL_{0,A,s}, w_0^*TL_{1,A,s}\}.$$

Here the boundary conditions mean that $V(s,0) \in T_{w_0(s,0)}L_{0,A,s}, V(s,1) \in T_{w_1(s,1)}L_{1,A,s}$.

For any $V \in O$, consider the inhomogeneous Cauchy-Riemann equation

$$\partial_s Exp_{w_0}(V) + J_{A,(s,t)}(\partial_t Exp_{w_0}(V) - X_{H_{A,s}}(Exp_{w_0}(V))) = 0. \tag{7.35}$$

We wish to find a solution $V \in O$ to this equation. We denote the inhomogeneous Cauchy-Riemann operator with respect to the Floer datum $(H_{A,s}, J_{A,(s,t)})$ by

$$\bar{\partial}_A(\cdot) = \partial_s(\cdot) + J_{A,(s,t)}(\partial_t(\cdot) - X_{H_{A,s}}(\cdot)).$$

Although w_0 does not satisfy the equation $\bar{\partial}_A w_0 = 0$, we have chosen a connection on the bundle w_0^*TM relative to $(w_0^*TL_{0,A,s}, w_0^*TL_{1,A,s})$ when defining the exponential map Exp, so that the linearized operator at w_0

$$D_{w_0}\bar{\partial}_A : T_{w_0}\mathcal{B} \to \mathcal{E}_{w_0} \tag{7.36}$$

is well-defined, where $\mathcal{E}_{w_0}$ is the space of $(0,1)$-forms with values in w_0^*TM, namely $\mathcal{E}_{w_0} = W^{k,p}(Z; w_0^*TM \otimes \Lambda_Z^{0,1})$. The regularity assumption for the moduli space $\mathcal{P}^A(x', x)$ implies that the linearized operator $D_w\bar{\partial}_A$ is surjective for any w which satisfies the equation $\bar{\partial}_A w = 0$. However, this is an open condition - for those maps close to an actual solution, the linearized operator is also surjective. In particular, as w_0 is close to an actual solution, $D_{w_0}\bar{\partial}_A$ is surjective. It follows that there is a bounded right inverse $Q_{w_0,A}$ of $D_{w_0}\bar{\partial}_A$. Now if we set

$$C = \frac{1}{||Q_{w_0,A}||},$$

then the vector

$$V = Q_{w_0}(-\bar{\partial}_A w_0) \tag{7.37}$$

is contained in the neighborhood O. This is the desired solution. The meaning of this formula is as follows. As w_0 does not satisfy the inhomogeneous Cauchy-Riemann equation, $\bar{\partial}_A w_0$ is non-zero vector in $\mathcal{E}_{w_0}$. Note that the exponential map *Exp* provides a linear structure on a neighborhood of w_0 in $\mathcal{B}$ by identifying that with $O \subset T_{w_0}\mathcal{B}$. Thus, we may "add" the inverse image of the negative of this non-zero vector $\bar{\partial}_A w_0$ to the original map w_0 to kill the deviation from being zero, so that the resulting map satisfies the inhomogeneous Cauchy-Riemann equation.

On the other hand, it is not hard to see such a solution V is unique.

□

Let us now denote by $\tilde{\mathcal{P}}^A(x_A, x)$ the moduli space of maps $\tilde{w}$ as above satisfying the inhomogeneous Cauchy-Riemann equation with respect to the Floer datum $(H_{A,s}, J_{A,(s,t)}$ as well as the Lagrangian boundary conditions $(L_{0,A,s}, L_{1,A,s})$. This moduli space behaves very similarly to the moduli space of climbing strips $\mathcal{P}^A(x_A, x)$, and has a natural compactification $\bar{\tilde{\mathcal{P}}}^A(x_A, x)$, by adding broken maps whose new components are inhomogeneous pseudoholomorphic strips in M with respect to the Floer datum (H_M, J_M) and Lagrangian boundary conditions (L_0, L_1), or those with respect to the Floer datum $(H_A, J_{A,t})$ and Lagrangian boundary conditions given by the completions of $(i(\psi_U^A L_0'), i(\psi_U^A L_1')$, whose asymptotic Hamiltonian chords are contained inside U_0. The latter kind of inhomogeneous pseudoholomorphic strips are also in one-to-one correspondence with inhomogeneous pseudoholomorphic strips in U with respect to the Floer datum (H_U, J_U) and Lagrangian boundary conditions (L_0', L_1').

Thus, the above lemma provides a natural bijection between the moduli spaces

$$\mathcal{U}(x; x'; e_0, e_1) \cong \tilde{\mathcal{P}}^A(x_A, x),$$

when the virtual dimension is zero, for generic choices of Floer data. Then it remains to extend this bijection to the level of compactified moduli spaces.

Lemma 7.5. *Consider the situation where the virtual dimension of the moduli space $\bar{\mathcal{U}}(x, x'; e_0, e_1)$ is zero or one. Suppose we have chosen Floer data generically such that both the compactified moduli space $\bar{\mathcal{U}}(x, x'; e_0, e_1)$ and the compactified moduli space $\bar{\tilde{\mathcal{P}}}^A(x_A, x)$ are regular. This means that these are compact smooth/topological manifolds of dimension zero/one. Then, among the generic choices of Floer data, there is a specific kind of choices for which there is a natural bijection from $\bar{\mathcal{U}}(x, x'; e_0, e_1)$ to $\bar{\tilde{\mathcal{P}}}^A(x_A, x)$.*

Moreover, when the virtual dimension is one, this bijection is an isomorphism of moduli spaces, in the sense that it comes with a natural virtual isomorphism of Fredholm complexes, and commutes with the gluing maps.

Proof. The assignment $(u, v) \mapsto w$ gives the natural bijection between elements of the uncompactified moduli spaces:

$$\mathcal{U}(x, x'; e_0, e_1) \cong \tilde{\mathcal{P}}^A(x_A, x).$$

whenever the virtual dimension is zero or one. If the virtual dimension is zero, these moduli spaces are compact, because we have chosen Floer data generically so that all such moduli spaces as well as moduli spaces of inhomogeneous pseudoholomorphic strips are regular. Thus there is nothing more to prove.

Now consider the case where the virtual dimension is one. Note that the two compactifications are both obtained by adding the same kinds of inhomogeneous pseudoholomorphic strips in M or inhomogeneous pseudoholomorphic strips in U. That is to say, there are isomorphisms:

$$\begin{aligned} &\partial\bar{\mathcal{U}}(x, x'; e_0, e_1) \\ \cong &\coprod \mathcal{M}(x', x'_1) \times \mathcal{U}(x, x'_1; e_0, e_1) \\ &\cup \coprod \mathcal{U}(x_1, x'; e_0, e_1) \times \mathcal{M}(x_1, x). \end{aligned}$$

and

$$\begin{aligned} &\partial\bar{\tilde{\mathcal{P}}}^A(x_A, x) \\ \cong &\coprod \tilde{\mathcal{M}}(x_A, x_{1,A}) \times \tilde{\mathcal{P}}^A(x_{1,A}, x) \\ &\cup \coprod \tilde{\mathcal{P}}^A(x_A, x_1) \times \mathcal{M}(x_1, x). \end{aligned}$$

Here $\tilde{\mathcal{M}}(x_A, x_{1,A})$ is the moduli space of inhomogeneous pseudoholomorphic strips f_A with respect to the Floer datum $(H_A, J_{A,t})$ and Lagrangian boundary conditions given by the completions of $(i(\psi_U^A L'_0), i(\psi_U^A L'_1)$, whose asymptotic Hamiltonian chords are contained inside U_0. This is naturally isomorphic to $\mathcal{M}(x', x'_1)$, the moduli space of inhomogeneous pseudoholomorphic strips f' in U with respect to the Floer datum (H_U, J_U) and Lagrangian boundary conditions (L'_0, L'_1), when the time-one H_U-chords x' and x'_1 correspond to the time-one H_A-chords x_A and $x_{1,A}$ respectively. Thus, for any kind of broken quilted map $(f', (u, v))$ or $((u, v), f)$, there is a unique broken map $(f_A, \tilde{w})$ or $(\tilde{w}, f)$ associated to it, where f_A and f' correspond to each other under the above-mentioned isomorphism between $\tilde{\mathcal{M}}(x_A, x_{1,A})$ and $\mathcal{M}(x', x'_1)$. In this way, the bijection extends over the compactifications.

This bijective correspondence naturally commutes with the gluing maps, because gluing happens near the usual strip-like ends, not the quilted ends.

□

As a corollary, this implies that when choosing Floer data generically in such special class, the counts of elements in these moduli spaces are equal. The algebraic consequence of this can be stated as follows. We have the wrapped Floer cochain space $CW^*(\psi_M^A L_0, \psi_M^A L_1; H_A)$, on which the differential is defined by counting rigid elements in the moduli spaces $\tilde{\mathcal{M}}(x_A, x_{1,A})$ of inhomogeneous pseudoholomorphic strips. And we also have a sub-complex $CW_-^*(\psi_M^A L_0, \psi_M^A L_1; H_A)$, generated by "interior" Hamiltonian chords. In fact, for this sub-complex, one can equivalently write it as $CW_-^*(i(\psi_U^A L'_0), i(\psi_U^A L'_1); H_A)$, because any inhomogeneous pseudoholomorphic strip with asymptotic convergence conditions given by those Hamiltonian chords will be contained in the image of i. In a similar way to the definition of the map

$$\tilde{r}^1_A : CW^*(L_0, L_1; H_M) \to CW^*_-(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_A),$$

we may define a map

$$t^1_A : CW^*(L_0, L_1; H_M) \to CW^*_-(i(\psi^A_U L'_0), i(\psi^A_U L'_1); H_A) \tag{7.38}$$

by counting rigid elements in the moduli space $\bar{\tilde{\mathcal{P}}}^A(x_A, x)$. Then Lemma 7.5 implies that $\Pi_\Gamma = t^1_A$, when the former map is restricted to the sub-complex generated by those generators whose images under Π_Γ fall within the action filtration window $(-A\epsilon^2, \delta)$.

The remaining task is to compare the map t^1_A with r^1_A. For that purpose, the underlying geometric idea is to relate the moduli space $\tilde{\mathcal{P}}^A(x_A, x)$ to the moduli space $\mathcal{P}^A(x_A, x)$ of climbing strips. As mentioned before, the difference between a map $\tilde{w}$ obtained from a quilted map (u, v) and an actual climbing strip w is that they have different boundary conditions. However, as we shall see, their moduli spaces are naturally cobordant to each other.

Lemma 7.6. *The moduli space $\tilde{\mathcal{P}}^A(x_A, x)$ is orientedly cobordant to the moduli space $\mathcal{P}^A(x_A, x)$ of climbing strips. Moreover, the same holds for compactified moduli spaces $\bar{\tilde{\mathcal{P}}}^A(x_A, x)$ and $\bar{\mathcal{P}}^A(x_A, x)$, namely they are also cobordant.*

Proof. For each $s \in \mathbb{R}$, the exact cylindrical Lagrangian submanifold $L_{j,A,s}$ as defined in (7.31) is exact Lagrangian isotopic to $L_{j,s} = L_j^{\lambda(s)}$. Let $L_{j,A,s,\sigma}$ be such an exact Lagrangian isotopy, parametrized by $\sigma \in [0, 1]$. It is possible to find such isotopies such that the two-dimensional family $L_{j,A,s,\sigma}$ parametrized by $(s, \sigma) \in \mathbb{R} \times [0, 1]$ is smooth. Furthermore, $L_{j,A,s,\sigma}$ is constant (i.e. independent of both s and σ) for $s \gg 0$, where $L_{j,s} = L_j$. The reason is as follows. Since L_j is assumed to be invariant under the Liouville flow in the Liouville cobordism $M_0 \setminus int(U_0)$, the completion of $i(\psi^A_U L'_j)$ is exact Lagrangian isotopic to either L_j or $L_j^{1-\epsilon}$. Recall that the completion of $i(\psi^A_U L'_j)$ is precisely $\psi^A_M L_j$. This is isotopic to L_j via the exact Lagrangian isotopy $\psi^{\rho_A(s)}_M L_j$, where $\rho_A : \mathbb{R} \to [1, A]$ is the previously used function which is A for $s \ll 0$ and 1 for $s \geq 0$. On the other hand, $L_j^{1-\epsilon}$ is isotopic to L_j via the exact Lagrangian isotopy $L_j^{\lambda(s)}$. Thus we may compose these two isotopies to obtain an isotopy from $\psi^A_M L_j$ to $L_j^{1-\epsilon}$. For each s, we reparametrize these isotopies by $\sigma \in [0, 1]$ to obtain an isotopy from $L_{j,A,s}$ to $L_{j,s} = L_j^{\lambda(s)}$, namely,

$$L_{j,A,s,\sigma} = \begin{cases} \psi^{\rho_A((1-2\sigma)s)}_M L_j, & \text{if } \sigma \in [0, \frac{1}{2}], \\ L_j^{\lambda_{1-2\sigma}(s)}, & \text{if } \sigma \in [\frac{1}{2}, 1], \end{cases} \tag{7.39}$$

where λ_σ is a non-decreasing homotopy between the function $\lambda : \mathbb{R} \to [1 - \epsilon, 1]$ and the constant function 1. This two-dimensional family (7.39) then satisfies all desired properties.

We can then define a parametrized moduli space

$$\mathcal{P}^A_+(x_A, x)$$

of pairs (σ, w_σ), where $\sigma \in [0, 1]$ and $w_\sigma : Z \to M$ is an inhomogeneos pseudoholomorphic map with respect to the Floer datum $(H_{A,s}, J_{A,(s,t)})$ and the Lagrangian boundary conditions $(L_{0,A,s,\sigma}, L_{1,A,s,\sigma})$. This moduli space provides the desired cobordism between the moduli spaces $\tilde{\mathcal{P}}^A(x_A, x)$ and $\mathcal{P}^A(x_A, x)$.

To obtain a cobordism between the compactified moduli spaces, we just need to compactify the moduli space $\mathcal{P}_+^A(x_A, x)$ in an appropriate way. Such a compactification can be obtained in a similar way to those for $\tilde{\mathcal{P}}^A(x_A, x)$ and $\mathcal{P}^A(x_A, x)$, as described below. For each fixed σ, we have a moduli space $\mathcal{P}_\sigma^A(x_A, x)$ of maps w_σ, such that when $\sigma = 0$, $\mathcal{P}_0^A(x_A, x) = \tilde{\mathcal{P}}^A(x_A, x)$, and when $\sigma = 1$, $\mathcal{P}_1^A(x_A, x) = \mathcal{P}^A(x_A, x)$. Each such moduli space $\mathcal{P}_\sigma^A(x_A, x)$ is compactified in the same way to $\bar{\mathcal{P}}^A(x_A, x)$ and $\bar{\tilde{\mathcal{P}}}^A(x_A, x)$. Thus we may define the compactification $\bar{\mathcal{P}}_+^A(x_A, x)$ to the the union of these:

$$\bar{\mathcal{P}}_+^A(x_A, x) = \cup_{\sigma \in [0,1]} \bar{\mathcal{P}}_\sigma^A(x_A, x). \tag{7.40}$$

This compactified moduli space then provides the desired cobordism between $\bar{\tilde{\mathcal{P}}}^A(x_A, x)$ and $\bar{\mathcal{P}}^A(x_A, x)$.

□

Corollary 7.2. *Under the assumption of Lemma 7.5, the cochain maps Π_Γ and r^1 are chain homotopic.*

Proof. Note that there is a chain homotopy equivalence

$$k_A : CW_-^*(i(\psi_U^A L_0'), i(\psi_U^A L_1'); H_A) \to CW_-^*(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_A)$$

defined by counting rigid elements in a moduli space of inhomogeneous pseudoholomorphic maps $u : Z \to M$ satisfying the equation

$$\partial_s u + J_{A,t}(\partial_t u - X_{H_A}(u)) = 0,$$

and the Lagrangian boundary conditions

$$u(s, j) \in L_{j,A,-\infty,\sigma=\rho(s)}, j = 0, 1,$$

where $L_{j,A,-\infty,\sigma=\rho(s)}$ is obtained from the family $L_{j,A,s,\sigma}$ by first specializing $s = -\infty$, and then substituting σ by $\rho(s)$. $L_{j,A,-\infty,\sigma=\rho(s)}$ can be regarded as an exact Lagrangian isotopy parametrized by $s \in \mathbb{R}$ such that for $s \ll 0$, $L_{j,A,-\infty,\sigma=\rho(s)} = L_j^{1-\epsilon}$, and for $s \gg 0$, $L_{j,A,-\infty,\sigma=\rho(s)} = i(\psi_U^A L_j')$.

We claim that $\tilde{r}_A^1$ and $k_A \circ t_A^1$ are chain homotopic. By counting rigid elements in the moduli space $\bar{\mathcal{P}}_+^A(x_A, x)$, we define a map

$$T_A : CW^*(L_0, L_1; H_M) \to CW_-^{*-1}(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_A)$$

of degree -1. By a standard gluing argument, this is a chain homotopy between $\tilde{r}_A^1$ and $k_A \circ t_A^1$.

Composing T_A with the map

$$\tau_A : CW_-^*(L_0^{1-\epsilon}, L_1^{1-\epsilon}; H_A) \to CW_{(-A\epsilon^2,\delta)}^*(L_0', L_1'; H_U),$$

we get a map

$$S_A : CW^*(L_0, L_1; H_M) \to CW_{(-A\epsilon^2,\delta)}^{*-1}(L_0', L_1'; H_U).$$

Following a similar homotopy commutativity argument as in Lemma 7.2, we may take the direct limit of the directed system of maps S_A to get

$$S = \lim_{A \to +\infty} S_A.$$

All the above maps have unique continuous extensions to the completed wrapped Floer cochain spaces, when the energy of inhomogeneous pseudoholomorphic maps is taken into account in the counting definition.

Combining Lemma 7.5 and Lemma 7.6, we conclude that Π_Γ is chain homotopic to r^1_A when the former map is restricted to given sub-complex generated by those generators whose images under Π_Γ fall within the action filtration window $(-A\epsilon^2, \delta)$. Such a chain homotopic is given by the map S_A. Taking the direct limit over A, we conclude that Π_Γ is chain homotopic to r^1, where the chain homotopy is given by the map S.

□

Since Π_Γ is chain homotopic to Θ^1_Γ, Theorem 1.4 is therefore complete.

Remark 7.4. *A point worth noting is that the Viterbo restriction functor is better defined as a colimit of continuation functors with respect to linear Hamiltonians (or the cascade definition as in [AS10]), if one wants to visualize the picture of the Hamiltonian dynamics more directly. We took the current approach simply because of the quadratic Hamiltonians are more convenient for the purpose of constructing functors from Lagrangian correspondences, so that these functors can be compared in the same setup.*

7.6. **Further questions.** It is therefore natural to ask whether the functors Θ_Γ and r as a whole are homotopic to each other, not just limited to their linear terms. While the expectation is yes, an efficient way of proving this is yet to be discovered. At least, there is a very naive case where such coincidence can be easily verified. For example, consider the case where L_0 is a closed exact Lagrangian submanifold that is contained in U_0. Then the action of $\Theta_{\mathcal{L}}$ is the identity. This can be easily proved using the maximum principle, which implies that any pseudoholomorphic disk in M_0 with boundary on L and its Hamiltonian perturbations must be contained in U_0. In the same way, one sees that the Viterbo restriction functor is also the identity functor on such Lagrangian submanifolds, which implies that the functor Θ_Γ agrees with the Viterbo restriction functor on such objects as well. To solve this problem in general, the main difficulty is to find a workable geometric construction of the functor Θ_Γ, as the definition of the cochain map Π_Γ does not seem to have a straightforward generalization to an A_∞-functor. Finding a suitable model of the moduli spaces of quilted surfaces based on which the functor Θ_Γ can be constructed directly is the key step of solving this problem.

There are of courses many other Lagrangian submanifolds which do not satisfy the geometric conditions we have just discussed. First, there are non-compact exact cylindrical Lagrangian submanifolds of M that does not have very nice restriction to U_0. A typical example is the cotangent fiber of an annulus restricted to the disjoint union of three cotangent fibers of a deformed sub-annulus, as illustrated in Example 4.2 of [AS10]. Second, there are closed exact Lagrangian submanifolds of M which are not entirely contained in U_0. In such cases, the usual construction of the Viterbo restriction functor does not yield an A_∞-functor in general. However, by analyzing the failure of it being an A_∞-functor, we expect that there is an extension of the Viterbo restriction functor to such Lagrangian submanifolds. Spectacularly, such an extension is related to deformation theory of the wrapped Fukaya category of U, and we phrase it as the following conjecture.

Conjecture 7.1. *Suppose $U_0 \subset M_0$ is a Liouville sub-domain. Let the wrapped Fukaya category of M and that of U consist of a suitable countable collection of Lagrangian submanifolds. Then there is a canonical deformation of the wrapped*

Fukaya category $\mathcal{W}(U)$ of U, denoted by $\mathcal{W}(U; B)$, where B is a collection of bounding cochains for objects in $\mathcal{W}(U)$, such that there is a natural A_∞-functor

$$r_B : \mathcal{W}(M) \to \mathcal{W}(U; B),$$

which agrees with the Viterbo restriction functor on the full sub-category $\mathcal{B}(M)$.

The study of such an extension also brings up the question when the extended Viterbo restriction functor can be identified with the functor Θ_Γ and when not. That would require more thorough understanding of the bounding cochains in both pictures.

7.7. **Extending the Viterbo functor.** Let us briefly explain the idea on how it is possible to extend the Viterbo restriction functor to an arbitrary exact cylindrical Lagrangian submanifold $L \subset M$ which does not necessarily satisfy Assumption 7.1, stated as Conjecture 7.1.

We consider $L \subset M$ satisfying the following conditions when restricted to U. $\partial L' \subset \partial U$ is a Legendrian submanifold with respect to the contact form $\alpha_- = \lambda_M|_{\partial U}$, which is disconnected and is decomposed to connected components:

$$\partial L' = \coprod_{i=1}^{N} l'_i, \tag{7.41}$$

where each $l'_i \subset \partial U$ is a connected Legendrian submanifold. The primitive f_L for L is locally constant near $\partial L'$, but takes different values c_i on l'_i.

For a pair of connected components l'_i, l'_j, if $c_i > c_j$, then there might be Reeb chords from l'_j to l'_i which have positive action. Such Reeb chords correspond to non-constant Hamiltonian chords from L' to itself starting from l'_i and ending on l'_j. Note that these cannot exist if the primitive is locally constant near ∂U, in which case $c_i = c_j$ for all i, j.

The deformation of the A_∞-structure of $CW^*(L')$ is contributed from certain kind of pseudoholomorphic disks asymptotic to these chords. This deformation is given by a bounding cochain b, or also called a Maurer-Cartan element in the case that the original A_∞-structure is non-curved. Roughly speaking, b is the count of pseudoholomorphic disks outside U_0 with boundary on $L \setminus L'_0$. Let $W_0 = M_0 \setminus int(U_0)$, and $L_0^c = L_0 \setminus int(L'_0)$. Then W_0 is a compact Liouville cobordism between contact manifolds ∂U and ∂M, and L_0^c is a compact exact Lagrangian cobordism between Legendrian submanifolds $l' \subset \partial U$ and $l \subset \partial M$. For each x with positive action and $\deg(x) = 1$, we consider inhomogeneous pseudoholomorphic disks in W_0 with boundary on L_0^c, which asymptotically converges to x. The space of such disks divided by automorphism is denoted by $\mathcal{M}_1(x; W_0, L_0^c)$. We define b to be

$$b = \sum_{\substack{x \\ \deg(x)=1, \mathcal{A}(x)>0}} \sum_{u \in \mathcal{M}_1(x; W_0, L_0^c)} o_u,$$

where $o_u \in o_x$ is the canonical element in the orientation line of x determined by u.

In general, this definition is not valid, as the moduli space is not compact, and cannot be compactified using techniques in Hamiltonian Floer theory. But there are some examples in which the moduli space is automatically compact due to geometric reasons.

Example 7.2. *This is the same as Example 4.2 of [AS10]. Consider* $M_0 = DT^*S^1$*, the unit cotangent bundle of* S^1*, which is identified with an annulus in* $\mathbb{C} = \mathbb{R}^2$*. Let* U_0 *be a neighborhood of some Hamiltonian perturbation of the zero section. Note that* U_0 *itself is isomorphic to* DT^*S^1*, so we may equip it with the standard Liouville form on the cotangent bundle to make it a Liouville domain. However, this is not compatible with the standard Liouville form on* M_0*, but one may add df to the Liouville form on* M_0 *for some appropriate function* $f : M_0 \to \mathbb{R}$ *such that* U_0 *becomes a Liouville sub-domain. Let* $L_0 \subset M_0$ *be a cotangent fiber, such that* $L'_0 = L_0 \cap U_0$ *is a disjoint union of three cotangent fibers. Denote* $L' = L'_1 \coprod L'_2 \coprod L'_3$*. We then have*

$$CW^*(L) \cong R = \mathbb{Z}[x, x^{-1}],$$

with A_∞*-operations* $m^1 = 0$*,* m^2 *being the usual multiplication of polynomials, and* $m^k = 0$ *for all* $k \geq 3$*. Similarly, we can compute*

$$CW^*(L') = CW^*(L'_1 \coprod L'_2 \coprod L'_3) = \bigoplus_{i,j=1}^{3} CW^*(L'_i, L'_j) \cong M(3, R),$$

the 3×3*-matrix algebra over* R*. More precisely, we have the following description: the generators for the direct summand* $CW^*(L'_i)$ *are* x_i, x_i^{-1}*, where* x_i *is the shortest non-constant Hamiltonian chord from the* i*-th component* L'_i *to itself of positive Maslov index; while* $CW^*(L'_i, L'_j)$ *is a left-*$CW^*(L'_j)$ *and right-*$CW^*(L'_i)$ A_∞*-bimodule, generated by* y_{ij} *as a module over either* $CW^*(L'_i)$ *or* $CW^*(L'_j)$*, where* y_{ij} *is the shortest non-constant Hamiltonian chord from* L'_i *to* L'_j *located on the boundary* ∂U*, which has positive Maslov index. Thus,*

$$CW^*(L'_i, L'_j) = y_{ij} CW^*(L'_i) = CW^*(L'_j) y_{ij} \cong y_{ij}\mathbb{Z}[x_i, x_i^{-1}] \cong \mathbb{Z}[x_j, x_j^{-1}] y_{ij}.$$

Except that m^2 *is the usual multiplication of polynomials, all other* m^k*'s vanish as in the case of* L*.*

Let us fix an absolute grading such that the shortest Hamiltonian chord y_{12} *from* L'_1 *to* L'_2 *and* y_{23} *from* L'_2 *to* L'_3 *have degree 1. Note that there is a unique (up to automorphism) pseudoholomorphic disk outside* U_0 *with boundary on* $L_0 \setminus L'_0$ *which asymptotically converges to* y_{12} *(or* y_{23}*). This is because in dimension two, we have automatic transversality so that pseudoholomorphic disks are in one-to-one correspondence with isomorphism classes of immersed polygons. Moreover, the moduli space is in fact compact due to degree reasons, and therefore also smooth. In this case, we get that*

$$b = y_{12} + y_{23}.$$

Then a straightforward computation shows that the b*-deformed* A_∞*-algebra* $CW^*(L', b)$ *is naturally isomorphic to* $R \cong CW^*(L)$*. And the Viterbo restriction homomorphism*

$$r : CW^*(L) \to CW^*(L', b)$$

is the "identity", up to automorphism.

There are more general questions regarding the well-definedness of the bounding cochain b, and how it can be identified with the bounding cochain that is obtained by applying the Lagrangian correspondence Γ. For the first question, the idea is to introduce a different A_∞-algebra whose cohomology is the linearized Legendrian (co)homology [BEE12]. This A_∞-algebra is homotopy equivalent to the wrapped Floer A_∞-algebra $(CW^*(L'), m^k)$. Then we can use techniques from symplectic field theory to define a bounding cochain for that A_∞-algebra, which then yields the desired b. For the second question, we must analyze degenerations of moduli spaces of quilted maps, and "identify" those elements at infinity with the above-mentioned pseudoholomorphic disks outside the sub-domain in an appropriate sense. More detailed discussions will be given in the upcoming work [Gao].

8. Analytic details in the construction of action-restriction data

8.1. **The case of product Lagrangian submanifolds.** In this section, we provide additional analytic details that are required to make choices of action-restriction data possible, and therefore to construct the action-restriction functor (4.16) which establishes quasi-equivalence between the two versions of wrapped Fukaya categories $\mathcal{W}^s(\mathbb{L})$ and $\mathcal{W}(\mathbb{L})$ of the product manifold, completing the discussions in section 4.

For each pair of exact Lagrangian submanifolds, the symplectic action functional on the space of paths from one Lagrangian submanifold to the other, associated to each Hamiltonian as well as the Lagrangian submanifolds together with their primitives, defines a filtration on the Floer cochain complexes. We can arrange that the truncated Floer cochain complexes $CF^*_{(-\infty,a]}(L_0, L_1; H, J)$ for a fixed positive number $a > 0$ captures all essential information, because we have chosen Hamiltonians that are C^2-small in the interior part of the Liouville manifold. In particular, the action of any time-one chord for such a Hamiltonian either negative or bounded above from a given positive number, which depends only on the Lagrangian submanifolds which the chord starts from and lands on.

Let us consider the case of products of exact cylindrical Lagrangian submanifolds of M and those of N. This is more complicated than the case of exact cylindrical Lagrangian submanifolds of $M \times N$, so we first discuss it in great detail. A priori, we only have well-defined wrapped Floer theory for these product Lagrangian submanifolds with respect to split Hamiltonians and product almost complex structures. The main strategy is then to change the split Hamiltonian to an admissible one in several steps, while keeping track of the action of the extra chords that possibly appear in each step. We will make sure that the potentially newly arising chords all have sufficiently positive action, so we can eventually rule them out in the truncated Floer complexes.

For simplicity, consider a single product Lagrangian submanifold $L \times L' \subset M \times N$ be a pair of admissible Lagrangian submanifolds of the product Liouville manifold, with chosen primitive $f_{L\times L'}$ which is the sum $f_L + f_{L'}$. For each Hamiltonian H, the action functional on the space of paths from $L \times L'$ to itself is

$$(8.1)\quad \mathcal{A}_{H,L\times L'}(\gamma) = -\int_0^1 \gamma^*\lambda_{M\times N} + \int_0^1 H(\gamma(t))dt + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0)),$$

and time-one Hamiltonian chords are critical points of this functional. The action defines a filtration on the Floer cochain space $CF^*(L \times L'; H)$, provided that H satisfies certain growth condition. In particular, this is the case with H being a split Hamiltonian, or a Hamiltonian which is split outside a compact set.

Suppose we are given a sequence of split Hamiltonians $H_{M,N,i} = \pi_M^* H_{M,i} + \pi_N^* H_{N,i}$ (later we write $H_{M,i} + H_{N,i}$ for simplicity), such that $H_{M,i}$ is positive everywhere, depends only on the radial coordinate on $\partial M \times (0, +\infty)$ and is linear of slope k'_i for $r_1 \geq 1$, and $H_{N,i}$ satisfies similar properties, with slope k''_i. The sequences of slopes $\{k'_i\}$ and $\{k''_i\}$ are increasing and go to infinity as $i \to \infty$. We assume that the Hamiltonians are well-chosen such that the Fukaya A_∞-structures on the wrapped Floer complexes $CW^*(L; \{H_{M,i}\})$ and $CW^*(L'; \{H_{N,i}\})$ are both well-defined for suitable choices domain-dependent families of almost complex structures. Without loss of generality, we may choose such sequences of Hamiltonians so that the slopes agree,

$$k_i' = k_i''.$$

For the exact cylindrical Lagrangian submanifold $L \subset M$, the intersection $l = L \cap \partial M$ is a Legendrian submanifold. The set of Reeb chords on ∂ from l to itself is discrete, assuming the Reeb dynamics is generic. In particular, the length spectrum is discrete, and for any given $\lambda > 0$, there are finitely many isolated Reeb chords whose lengths are smaller than λ. Similar statements hold for L' and its Legendrian boundary $l' \subset \partial N$.

Let us go back to Hamiltonians and discussion relevant properties about action filtration. Since $H_{M,i}$ and $H_{N,i}$ are chosen to be C^2-small in the interior part M_0 and N_0 respectively, we may assume that there is some $\epsilon > 0$ such that

$$| - \int_0^1 \gamma^* \lambda + \int_0^1 H_{M,N}(\gamma(t))dt| \leq \epsilon,$$

for every $H_{M,N,i}$-chord γ that lies in the interior part $M_0 \times N_0$ of $M \times N$. Fix a positive number $a > 0$ which is bigger than $\max(|f_L| + |f_{L'}|) + \epsilon$. Then the truncated Floer cochain complex $CW^*_{(-\infty,a]}(L \times L'; H_{M,N,i}, J_{M,N,i})$ includes all Hamiltonian chords in the interior of $M \times N$ as generators. The goal is to construct an Hamiltonian K_i which depends only on the radial coordinate in the cylindrical end $\Sigma \times [1, +\infty)$ at least for r sufficiently large, as well as an almost complex structure J_i of contact type, such that

(i) (K_i, J_i) agrees with the given $(H_{M,N,i}, J_{M,N,i})$ inside the compact set $r_1 \leq A - \epsilon, r_2 \leq A - \epsilon$ for some suitable constant $A = A_i$ (depending on i) to be determined later;
(ii) K_i depends only on the radial coordinate on $\Sigma \times [1, +\infty)$ outside the compact set $r \leq B$ for another suitable constant B to be determined later (so that the compact set $r \leq B$ contains the compact set $r_1 \leq A, r_2 \leq A$);
(iii) J_i agrees with the given $J_{M,N,i}$ inside the compact set $r_1 \leq A-\epsilon, r_2 \leq A-\epsilon$, and is of contact type outside the compact set $r \leq B$;
(iv) The extra Hamiltonian chords for K_i compared to $H_{M,N,i}$ have sufficiently positive action for i sufficiently large, and therefore do not fall in the action filtration window $(-b, a]$ chosen at the beginning;
(v) For each b there is i_b such that for $i > i_b$, (K_i, J_i)-pseudoholomorphic strips coincide with $(H_{M,N,i}, J_{M,N,i})$-pseudoholomorphic strips, and are contained in a compact set inside $r \leq 1$.

With this kind of Hamiltonian and almost complex structure constructed, we can define the following sequence of cochain maps:

$$\begin{aligned} CF^*_{(-b,a]}(L \times L'; H_{M,N,i}) &\to CF^*_{(-b,a]}(L \times L'; K_i) \\ \to CF^*_{(-2b,a]}(L \times L'; H_{M,N,i}) &\to CF^*_{(-2b,a]}(L \times L'; K_i) \end{aligned} \tag{8.2}$$

with the property that the following two compositions

$$CF^*_{(-b,a]}(L \times L'; H_{M,N,i}) \to CF^*_{(-2b,a]}(\mathcal{L}_0, \mathcal{L}_1; H_{M,N,i}),$$
$$CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; K_i, J) \to CF^*_{(-2b,a]}(\mathcal{L}_0, \mathcal{L}_1; K_i)$$

are inclusions with respect to the corresponding action filtrations.

8.2. **The first step.** We first deform the Hamiltonian $H_{M,N,i}$ so that it becomes constant outside a compact set. The key is to choose the constants and the compact set carefully so that the additional chords that could possibly appear after change of Hamiltonians have sufficiently postive action, which can then be excluded from the given action filtration window $(-b, a]$.

Let $A > 0$ be a large constant to be determined later. We deform the Hamiltonian $H_{M,N,i}$ to a new Hamiltonian $H_{i,1} = H_{M,i,1} + H_{N,i,1}$ which is still of split type (namely $H_{M,i,1}$ depends only on M-factor and $H_{N,i,1}$ depends only on N-factor), such that

$$H_{M,i,1} = \begin{cases} H_{M,i}(r_1) & \text{if } r_1 \leq A - \epsilon, \\ C & \text{if } r_1 \geq A. \end{cases} \quad H_{N,i,1} = \begin{cases} H_N(r_2) & \text{if } r_2 \leq A - \epsilon, \\ C & \text{if } r_2 \geq A. \end{cases} \tag{8.3}$$

Here C is a constant close to the values of the functions $H_{M,i}$ and $H_{N,i}$ at $r_1 = A-\epsilon$ and respectively $r_2 = A - \epsilon$:

$$C = k_i'(A - \epsilon) + \epsilon = k_i''(A - \epsilon) + \epsilon,$$

where the second equality is true since we have assumed $k_i' = k_i''$.

To illustrate what indeed happens, we should think of both the Hamiltonians $H_{M,i}$ and $H_{N,i}$ are being deformed to the constant function C near level $r_1 = A$ and respectively $r_2 = A$. We arrange such deformation carefully to make sure the resulting Hamiltonians are still non-decreasing, but grow in a tempered way in the region $r_1, r_2 \in [A - \epsilon, A]$. For example, we may choose a smooth non-increasing cut-off function $\rho_A : [1, +\infty) \to [0, 1]$ such that $\rho_A(r) = 1$ for $r \leq A - \epsilon$, and $\rho_A(r) = 0$ for r close to A and $r \geq A$, and set

$$H_{M,i,1}(y, r) = \rho_A(r_1)H_{M,i}(y, r_1) + (1 - \rho_A(r_1))C,$$

and

$$H_{N,i,1}(y, r) = \rho_A(r_2)H_{M,i}(y, r_2) + (1 - \rho_A(r_2))C.$$

Now the new Hamiltonian $H_{i,1} = H_{M,i,1} + H_{N,i,1}$ might have addtional time-one chords compared to $H_{M,N,i}$. There are four possible types of them:

(i) Constant chords on level sets of $H_{i,1} = 2C$. Since these chords are constant, $\gamma(0) = \gamma(1)$ so the contributions from the primitives vanish. Such a chord γ has action

$$\mathcal{A}_{H_{i,1}, L\times L'}(\gamma) = 2C = 2k_i'(A - \epsilon) \tag{8.4}$$

which is sufficiently positive if i is sufficiently large.

(ii) Hamiltonian chord $\gamma = (x, y)$ in $M \times N$, such that x is a non-constant chord for $H_{M,i,1}$ in M from L to itself on levels $\partial M \times \{r_1\}$ for r_1 close to $A - \epsilon$ ($r_1 \in (A - \epsilon, A]$), y is a constant in N on levels $H_{N,i,1} = C$. These chords have action

(8.5)

$$\begin{aligned} \mathcal{A}_{H_{i,1}, L\times L'}(\gamma) =& -\frac{\partial H_{M,i,1}}{\partial r_1} r_1 + H_{M,i,1}(r_1) + C + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0)) \\ \geq& -(k_i' - \eta_i)A + k_i'(A - \epsilon) + k_i''(A - \epsilon) + \epsilon - 2c \\ =& \eta_i A - k_i'\epsilon + k_i''(A - \epsilon) + \epsilon - 2c, \end{aligned}$$

which is sufficiently positive for i large.

(iii) Hamiltonian chord $\gamma = (x, y)$ in $M \times N$, such that x is a constant chord in M on levels $H_{M,i,1} = C$, and y is a non-constant chord in N from L' to itself on levels $\partial N \times \{r_2\}$ for r_2 close to $A - \epsilon$ ($r_2 \in (A - \epsilon, A]$). This case is symmetric to the above one, and we can estimate the action

$$\mathcal{A}_{H_{i,1}, L \times L'}(\gamma) \geq \sigma_i A - k'_i \epsilon + k'_i (A - \epsilon) + \epsilon - 2c,$$

which is sufficiently positive for large i.

(iv) Hamiltonian chord $\gamma = (x, y)$ in $M \times N$, such that x is a non-constant chord in M from L to itself on levels $\partial M \times \{r_1\}$ for r_1 close to $A - \epsilon$ ($r_1 \in (A - \epsilon, A]$), and y is a non-constant chord in N from L' to itself on levels $\partial N \times \{r_2\}$ for r_2 close to $A - \epsilon$ ($r_2 \in (A - \epsilon, A]$). These chords have action

(8.6)
$$\begin{aligned}
&\mathcal{A}_{H_{i,1}, L \times L'}(\gamma) \\
=& -\frac{\partial H_{M,i,1}}{\partial r_1} r_1 + H_{M,i,1}(r_1) - \frac{\partial H_{N,i,1}}{\partial r_2} r_2 + H_{N,i,1}(r_2) + f_{L \times L'}(\gamma(1)) - f_{L \times L'}(\gamma(0)) \\
\geq& -(k'_i - \eta_i) A + k'_i (A - \epsilon) - (k''_i - \sigma_i) A + k''_i (A - \epsilon) - 2c \\
=& \eta_i A - k'_i \epsilon + \sigma_i A - k'_i \epsilon - 2c.
\end{aligned}$$

Thus, if we choose

$$A = \max\{\frac{3k'_i}{\eta_i}, \frac{3k''_i}{\sigma_i}\}, \tag{8.7}$$

the action of such a chord will be sufficiently positive for large i.

To summarize, we have deformed the split Hamiltonian $H_{M,N,i}$ to a new one $H_{i,1}$ so that the undesired Hamiltonian chords all have sufficiently positive action, which therefore are excluded from the action filtration window $(-b, a]$.

8.3. **The second step.** Now the split Hamiltonian $H_{i,1} = H_{M,i,1} + H_{N,i,1}$ on $M \times N$ is constant and equal to $2C$ in the region $\{r_1 \geq A, r_2 \geq A\}$. The second step is to deform it to a Hamiltonian that is constant outside the compact set $\{r_1 \leq B, r_2 \leq B\}$ for appropriate choice of $B > A$ to be determined.

By the first step, this is already the case in the region $\{r_1 \geq A, r_2 \geq A\}$. Consider the following two regions:

$$I = M \times (\partial N \times [A, +\infty)), \tag{8.8}$$

$$II = (\partial M \times [A, +\infty)) \times N. \tag{8.9}$$

We first deal with the region I. The other case is symmetric.

The idea is to deform $\pi_M^* H_{M,i,1}$ in an appropriate way to a new function $H_{I,i,2}$ on the region I by interpolating it with a suitable constant.

Definition 8.1. *Define a smooth cut-off function $\rho : [A, +\infty) \to [0, 1]$ satisfying the following properties.*

$$\rho = \begin{cases} 0 & \text{on } [A, A_1], \\ 1 & \text{on } [B - \epsilon, +\infty), \end{cases} \tag{8.10}$$

and is strictly increasing on $[A_1, B - \epsilon]$, where $A_1 > A$ is a big positive number to be chosen later. Moreover, we require that for $r_2 \in [A_1 + \epsilon, B - 2\epsilon]$, the derivative of ρ is constant (namely ρ is linear), and this constant satisfies

$$\rho'(r_2) \equiv \textit{constant} \in [\frac{1}{B - A_1 - \epsilon}, \frac{1}{B - A_1 - 3\epsilon}]. \tag{8.11}$$

These assumptions on the cut-off function ρ will imply the following consequences:

Lemma 8.1. *(i) The amount that ρ increases on the interval $[A_1 + \epsilon, B - 2\epsilon]$ is between $1 - \frac{2\epsilon}{B - A_1 - \epsilon}$ and 1;*

(ii) The sum of the total variation of ρ on $[A_1, A_1 + \epsilon]$ and on $[B - 2\epsilon, B - \epsilon]$ is between 0 and $\frac{2\epsilon}{B - A_1 - \epsilon}$;

(iii) For $r_2 \in [A_1, A_1 + \epsilon]$, we have that $0 \leq \rho(r_2) \leq \frac{2\epsilon}{B - A_1 - \epsilon}$. And we can choose ρ growing in a tempered way such that the derivative $\rho'(r_2) \leq \frac{3}{B - A_1 - \epsilon}$ for all $r_2 \in [A_1, A_1 + \epsilon]$;

(iv) For $r_2 \in [B - 2\epsilon, B - \epsilon]$, we have that $1 - \frac{2\epsilon}{B - A_1 - \epsilon} \leq \rho(r_2) \leq 1$. And we can choose ρ such that the derivative $\rho'(r_2) \leq \frac{3}{B - A_1 - \epsilon}$ for all $r_2 \in [B - 2\epsilon, B - \epsilon]$.

Proof. The proofs of all the four statements are elementary calculations. □

Now we define the function $H_{I,i,2}$ as follows:

$$H_{I,i,2} : M \times \partial N \times [A, +\infty) \to \mathbb{R}, \tag{8.12}$$

$$H_{I,i,2}(x, y, r_2) = (1 - \rho(r_2))H_{M,i,1}(x) + \rho(r_2)C. \tag{8.13}$$

To see what Hamiltonian chords can possibly arise and to estimate their action, first of all we need to find the Hamiltonian vector field for $H_{I,i,2}$. The symplectic form on $M \times \partial N \times [A, +\infty)$ is of the form $\omega_M \bigoplus d(r_2 \lambda_N|_{\partial N})$ where $\lambda_N|_{\partial N}$ is the contact form on ∂N obtained by restricting the Liouville form to ∂N. Thus we find that the Hamiltonian vector field of $H_{I,i,2}$ has the form:

$$X_{H_{I,i,2}}(x, y, r_2) = (1 - \rho(r_2))X_{H_{M,i,1}}(x) - (C - H_{M,i,1}(x))\rho'(r_2)Y_{\partial N}(y), \tag{8.14}$$

where $Y_{\partial N}(y)$ is the Reeb vector field for $\lambda_N|_{\partial N}$ on $\partial N \times \{1\}$.

Again, the new Hamiltonian $H_{I,i,2} + H_{N,i,1}$ might have extra time-one chords. Consider such a time-one chord $\gamma = (x, y)$ of $X_{H_{I,i,2}}$. Its projection to M is a $X_{H_{M,i,1}}$-chord x of time-$(1 - \rho(r_2))$ from L to itself, along which $H_{M,i,1}$ is constant equal to $H_{M,i,1}(x(0))$, by Hamilton's equation (since $H_{M,i,1}$ is time-independent). Since $X_{H_{I,i,2}}$ does not have $\frac{\partial}{\partial r_2}$-component, the chord γ lies on some level $\{r_2 = \text{constant}\}$. Hence its projection to $\partial N \times [A, +\infty) \subset N$ is a non-constant chord y corresponding to a Reeb chord for $Y_{\partial N}$ of length $(C - H_{M,i,1}(x(0)))\rho'(r_2)$, which is located on the level $\partial N \times \{r_2\}$ for some $r_2 \geq A$, but with opposite direction to the one determined by $Y_{\partial N}$. To summarize, a time-one $X_{H_{I,i,2}}$-chord γ corresponds to a pair (x, y) where x is a time-$(1 - \rho(r_2))$ $X_{H_{M,i,1}}$-chord in M and y is a non-constant chord corresponding to a Reeb chord for $-Y_{\partial N}$ of length $(C - H_{M,i,1}(x(0)))\rho'(r_2)$, located on the level $\partial N \times \{r_2\}$.

Now we compute the action of γ. Note that we have only modified $H_{M,i,1} + H_{N,i,1}$ on the region $I = M \times \partial N \times [A, +\infty)$ to the new one $H_{I,i,2} + H_{N,i,1}$, so we should compute the action with respect to $H_{I,i,2} + H_{N,i,1}$. Furthermore, in the region I the function $H_{N,i,1}$ is constant, so we can replace it by its value C (the same C as in the first step). A straightforward calculation by the definition of the action gives:

(8.15)

$$
\begin{aligned}
&\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\
=&-\int_0^{1-\rho(r_2)} x^*\lambda_M - \int_0^{(C-H_{M,i,1}(x(0)))\rho'(r_2)} -r_2 y^*\lambda_N|_{\partial N}\\
&+\int_0^1 H_{I,i,2}(\gamma(t))dt + C + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0))\\
=&-\int_0^{1-\rho(r_2)} x^*\lambda_M + r_2\rho'(r_2)(C-H_{M,i,1}(x(0))) + \rho(r_2)(C-H_{M,i,1}(x(0)))\\
&+H_{M,i,1}(x(0)) + C + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0))\\
=&-\int_0^{1-\rho(r_2)} x^*\lambda_M + (1-\rho(r_2)-r_2\rho'(r_2))H_{M,i,1}(x(0))\\
&+(1+\rho(r_2)+r_2\rho'(r_2))C + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0)).
\end{aligned}
$$

There are five possible classes of $X_{H_{M,i,1}}$-chords, which are listed below. And according to the class of the projection of γ to the $X_{H_{M,i,1}}$-chord x, we estimate the action of γ. The goal is to show that the action is sufficiently positive in any case.

(i) x is a short chord in the interior of M. Here we say the chord is short because the Hamiltonian $H_{M,i,1}$ is C^2-small there. For such a chord x, we have

$$\int_0^{1-\rho(r_2)} x^*\lambda_M \leq \epsilon. \tag{8.16}$$

Now there are three sub-cases to consider, depending on the value of r_2.

(i a): $r_2 \in [A, A_1]$, where $\rho(r_2) \equiv 0, \rho'(r_2) \equiv 0$. In this case, $X_{H_{I,i,2}}$ does not have $Y_{\partial N}$-component. This implies that

$$
\begin{aligned}
&\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\
\geq&-\epsilon + H_{M,i,1}(x(0)) + C + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0))\\
\geq&C - 2\epsilon - 2c\\
=&k_i'(\frac{3k_i'}{\eta_i} - \epsilon) - \epsilon - 2c,
\end{aligned}
\tag{8.17}
$$

which is sufficiently positive for large i.

(i b): $r_2 \in [B-\epsilon, +\infty)$, where $\rho(r_2) \equiv 1, \rho'(r_2) \equiv 0$. In this case, $X_{H_{I,i,2}}$ is zero so γ is constant. Thus we have

$$
\begin{aligned}
&\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\
=&H_{M,i,1}(x(0)) + C + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0))\\
\geq&C - \epsilon + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0))\\
=&k_i'(\frac{3k_i'}{\eta_i} - \epsilon) - 2c,
\end{aligned}
\tag{8.18}
$$

which is sufficiently positive for large i.

(i c): $r_2 \in [A_1, B-\epsilon]$. Recall that $H_{M,i,1}$ is C^2-small, taking values in $[0,\epsilon]$. Therefore we have

$$r_2(C - H_{M,i,1}(x(0)))\rho'(r_2) \geq 0,$$
$$\rho(r_2)(C - H_{M,i,1}(x(0))) \geq 0$$

Thus we obtain the estimate

$$\begin{aligned}(8.19)\qquad &\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\ \geq& -2\epsilon + C + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0))\\ =& k_i'(\frac{3k_i'}{\eta_i} - \epsilon) - \epsilon - 2c,\end{aligned}$$

which is sufficiently positive for large i.

(ii) x is a non-constant Hamiltonian chord which corresponds to a Reeb chord on level $\partial M \times \{r_1\}$ for some r_1 close to 1. Here the derivative satisfies $0 < \frac{\partial H_{M,i,1}}{\partial r_1} \leq k_i'$. For these chords, we have

$$(8.20)\qquad \int_0^{1-\rho(r_2)} x^*\lambda_M \leq (k_i' - \eta_i)(1-\rho(r_2)).$$

Now there are three sub-cases to consider, depending on the value of r_2.

(ii a): $r_2 \in [A, A_1]$, where $\rho(r_2) \equiv 0, \rho'(r_2) \equiv 0$. We have

$$\begin{aligned}(8.21)\qquad &\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\ \geq& -k_i' + H_{M,i,1}(x(0)) + C + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0))\\ \geq& -k_i' + k_i'(\frac{3k_i'}{\eta_i} - \epsilon) + \epsilon - 2c,\end{aligned}$$

which is sufficiently positive for i large.

(ii b): $r_2 \in [B-\epsilon, +\infty)$, where $\rho(r_2) \equiv 1, \rho'(r_2) \equiv 0$. We have

$$\begin{aligned}(8.22)\qquad &\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\ \geq& -k_i' + 2C + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0))\\ \geq& -k_i' + 2k_i'(\frac{3k_i'}{\eta_i} - \epsilon) + 2\epsilon - 2c,\end{aligned}$$

which is sufficiently positive for i large.

(ii c): $r_2 \in [A_1, B-\epsilon]$. Since $H_{M,i,1}$ takes values in $(0, k_i]$ and $C > k_i$, we still have

$$r_2(C - H_{M,i,1}(x(0)))\rho'(r_2) \geq 0,$$
$$\rho(r_2)(C - H_{M,i,1}(x(0))) \geq 0$$

Thus we obtain the estimate

$$\begin{aligned}(8.23)\qquad &\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\ \geq& -k_i' + C + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0))\\ \geq& -k_i' + k_i'(\frac{3k_i'}{\eta_i} - \epsilon) + \epsilon - 2c,\end{aligned}$$

which is sufficiently positive for i large.

(iii) x is a non-constant chord on level $\partial M \times \{r_1\}$ for some $r_1 \in [1, A-\epsilon]$ that corresponds to a Reeb chord on ∂M from l to itself. For such a chord, we have

$$\int_0^{1-\rho(r_2)} x^*\lambda_M \leq (k_i' - \eta_i)(A-\epsilon)(1-\rho(r_2)). \tag{8.24}$$

Now there are three sub-cases to consider, depending on the value of r_2.

(iii a): $r_2 \in [A, A_1]$, where $\rho(r_2) \equiv 0, \rho'(r_2) \equiv 0$. We have

$$\begin{aligned}
&\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\
\geq& -(k_i'-\eta_i)(A-\epsilon) + H_{M,i,1}(r_1) + C + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0))\\
\geq& -(k_i'-\eta_i)(\frac{3k_i'}{\eta_i}-\epsilon) + k_i'(\frac{3k_i'}{\eta_i}-\epsilon) + \epsilon - 2c\\
=&3k_i' - \eta_i\epsilon - 2c,
\end{aligned} \tag{8.25}$$

which is sufficiently positive for i large.

(iii b): $r_2 \in [B-\epsilon, +\infty)$, where $\rho(r_2) \equiv 1, \rho'(r_2) \equiv 0$. We have

$$\begin{aligned}
&\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\
=&C - H_{M,i,1}(x(0)) + H_{M,i,1}(x(0)) + C + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0))\\
\geq&2C - 2c\\
=&2k_i'(\frac{3k_i'}{\eta_i}-\epsilon) + 2\epsilon - 2c,
\end{aligned} \tag{8.26}$$

which is sufficiently positive for i large.

(iii c): $r_2 \in [A_1, B-\epsilon]$. Since $H_{M,i,1}(x(0)) \leq C$, we still have

$$r_2(C - H_{M,i,1}(x(0)))\rho'(r_2) \geq 0,$$
$$\rho(r_2)(C - H_{M,i,1}(x(0))) \geq 0$$

Hence,

$$\begin{aligned}
&\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\
&\geq -(k_i'-\eta_i)(A-\epsilon) + C - 2c\\
&= -(k_i'-\eta_i)(\frac{3k_i'}{\eta_i}-\epsilon) + k_i'(\frac{3k_i'}{\eta_i}-\epsilon) + \epsilon - 2c\\
&= 3k_i' - \eta_i\epsilon + \epsilon - 2c,
\end{aligned} \tag{8.27}$$

which is sufficiently positive for i large.

(iv) x is a non-constant chord of time $1-\rho(r_2)$ on level $\partial M \times \{r_1\}$ for some r_1 close to A ($r_1 \in [A-\epsilon, A]$, which corresponds to a Reeb chord. So for these chords, we have

$$\int_0^{1-\rho(r_2)} x^*\lambda_M \leq (k_i' - \eta_i)A(1-\rho(r_2)). \tag{8.28}$$

Now there are five sub-cases to consider, depending on the value of r_2.

(iv a): $r_2 \in [A, A_1]$, where $\rho(r_2) \equiv 0, \rho'(r_2) \equiv 0$. We have

$$\begin{aligned}
&\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\
\geq& -(k_i'-\eta_i)A + k_i'(A-\epsilon) + C - 2c\\
\geq&
\end{aligned} \tag{8.29}$$

which is sufficiently positive for i large.

(iv b): $r_2 \in [B-\epsilon, +\infty)$, where $\rho(r_2) \equiv 1, \rho'(r_2) \equiv 0$. We have

$$\begin{aligned} &\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma) \\ =&H_{M,i,1}(x(0)) + C + f_{\mathcal{L}_1}(\gamma(1)) - f_{\mathcal{L}_0}(\gamma(0)) \\ \geq &k_i'(A-\epsilon) + k_i'(\frac{3k_i'}{\eta_i} - \epsilon) + \epsilon - 2c, \end{aligned} \tag{8.30}$$

which is sufficiently positive for i large.

(iv c): $r_2 \in [A_1, \frac{A_1+B}{2}]$. In this case, we can choose appropriate ρ such that

$$1 - r_2\rho'(r_2) - \rho(r_2) \geq 0. \tag{8.31}$$

So the action satisfies

$$\begin{aligned} &\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma) \\ \geq &-(k_i' - \eta_i)A + (1 - r_2\rho'(r_2) - \rho(r_2))k_i'(A-\epsilon) \\ &+ (1 + r_2\rho'(r_2) + \rho(r_2))C + f_{\mathcal{L}_1}(\gamma(1)) - f_{\mathcal{L}_0}(\gamma(0)) \\ \geq &-(k_i' - \eta_i)\frac{3k_i'}{\eta_i} + k_i'(\frac{3k_i'}{\eta_i} - \epsilon) + \epsilon - 2c, \\ = &3k_i' - k_i'\epsilon + \epsilon - 2c, \end{aligned} \tag{8.32}$$

which is sufficiently positive for i large.

(iv d): $r_2 \in [\frac{A_1+B}{2}, B-2\epsilon]$. Recall that in this region, the derivative $\rho'(r_2) \equiv$ *constant* $\in [\frac{1}{B-A_1-\epsilon}, \frac{1}{B-A_1-3\epsilon}]$. So we have

$$\frac{r_2 - A_1 - \epsilon}{B - A_1 - \epsilon} \leq \rho(r_2) \leq \frac{2\epsilon}{B - A_1 - \epsilon} + \frac{r_2 - A_1 - \epsilon}{B - A_1 - 3\epsilon} \leq \frac{r_2 - A_1 + \epsilon}{B - A_1 - 3\epsilon}, \tag{8.33}$$

$$\frac{r_2}{B - A_1 - \epsilon} \leq r_2\rho'(r_2) \leq \frac{r_2}{B - A_1 - 3\epsilon}. \tag{8.34}$$

Thus we obtain

$$\begin{aligned} 1 - r_2\rho'(r_2) - \rho(r_2) &\geq 1 - \frac{2r_2 - A_1 + \epsilon}{B - A_1 - 3\epsilon} \\ &\geq 1 - \frac{2B - A_1 - 3\epsilon}{B - A_1 - 3\epsilon} \\ &= -1 - \frac{A_1 + 3\epsilon}{B - A_1 - 3\epsilon}, \end{aligned} \tag{8.35}$$

and also

$$\begin{aligned} 1 + r_2\rho'(r_2) + \rho(r_2) &\geq 1 + \frac{2r_2 - A_1 - \epsilon}{B - A_1 - \epsilon} \\ &\geq 1 + \frac{2\frac{A_1+B}{2} - A_1 - \epsilon}{B - A_1 - \epsilon} \\ &= 1 + \frac{B - \epsilon}{B - A_1 - \epsilon} \\ &= 2 + \frac{A_1}{B - A_1 - \epsilon}. \end{aligned} \tag{8.36}$$

Thus the action satisfies

(8.37)
$$\begin{aligned}
&\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\
\geq&-(k_i'-\eta_i)A\frac{B}{B-A_1-\epsilon}+(-1-\frac{A_1+3\epsilon}{B-A_1-3\epsilon})k_i'A+(2+\frac{A_1}{B-A_1-\epsilon})C\\
&+f_{L\times L'}(\gamma(1))-f_{L\times L'}(\gamma(0))\\
\geq&-(k_i'-\eta_i)A\frac{B-r_2}{B-A_1-\epsilon}+(-1-\frac{A_1+3\epsilon}{B-A_1-3\epsilon})k_i'A+(2+\frac{A_1}{B-A_1-\epsilon})(k_i'(A-\epsilon)+\epsilon)-2c\\
\geq&-k_i'A\frac{(B-A_1)/2}{B-A_1-\epsilon}+(-1-\frac{A_1+3\epsilon}{B-A_1-3\epsilon})k_i'A+(2+\frac{A_1}{B-A_1-\epsilon})(k_i'(A-\epsilon)+\epsilon)-2c\\
&\geq k_i'A(\frac{1}{2}+\frac{A_1-\epsilon/2}{B-A_1-\epsilon}-\frac{A_1+3\epsilon}{B-A_1-3\epsilon})-k_i'\epsilon(2+\frac{A_1}{B-A_1-\epsilon})+\epsilon(2+\frac{A_1}{B-A_1-\epsilon})-2c.
\end{aligned}$$

As long as A_1, B are suitably chosen so that B is much bigger compared to A_1, the action is sufficiently positive for large i.

(iv e): $r_2 \in [B-2\epsilon, B-\epsilon]$. Here we have

$$1-\frac{2\epsilon}{B-A_1-\epsilon}\leq\rho(r_2)\leq 1. \tag{8.38}$$

We can estimate the action:

(8.39)
$$\begin{aligned}
&\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\
\geq&-(k_i'-\eta_i)A(1-\rho(r_2))+(1-r_2\rho'(r_2)-\rho(r_2))k_i'(A-\epsilon)+(1+r_2\rho'(r_2)+\rho(r_2))C-2c\\
\geq&-k_i'A\frac{2\epsilon}{B-A_1-\epsilon}+2k_i'(A-\epsilon)+\epsilon-2c.
\end{aligned}$$

for suitable choice of A_1, B making the coefficient $\frac{2\epsilon}{B-A_1-\epsilon}$ small. An easy computation shows that it suffices to require that $B > 2A_1$.

(v) x is a constant chord on level $\partial M\times\{r_1\}$ for $r_1 \geq A$, where $H_{M,i,1}\equiv C$. For these chords, we have

$$\int_0^{1-\rho(r_2)} x^*\lambda_M = 0. \tag{8.40}$$

Now there are three sub-cases to consider, depending on the value of r_2.

(v a): $r_2\in[A, A_1]$, where $\rho(r_2)\equiv 0, \rho'(r_2)\equiv 0$. We have

$$\begin{aligned}
&\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\
=&H_{M,i,1}(x(0))+C+f_{L\times L'}(\gamma(1))-f_{L\times L'}(\gamma(0))\\
=&2C+f_{L\times L'}(\gamma(1))-f_{L\times L'}(\gamma(0))\\
\geq&2k_i'(\frac{3k_i'}{\eta_i}-\epsilon)+2\epsilon-2c,
\end{aligned} \tag{8.41}$$

which is sufficiently positive for i large.

(v b): $r_2\in[B-\epsilon,+\infty)$, where $\rho(r_2)\equiv 1, \rho'(r_2)\equiv 0$. We have

$$\begin{aligned}
&\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma)\\
=&2C+f_{\mathcal{L}_1}(\gamma(1))-f_{\mathcal{L}_0}(\gamma(0))\\
\geq&2k_i'(\frac{3k_i'}{\eta_i}-\epsilon)+2\epsilon-2c,
\end{aligned} \tag{8.42}$$

which is sufficiently positive for i large.

(v c): $r_2 \in [A_1, B-\epsilon]$. Again, we have

$$\begin{aligned} &\mathcal{A}_{H_{I,i,2}+H_{N,i,1},L\times L'}(\gamma) \\ =&2C + f_{L\times L'}(\gamma(1)) - f_{L\times L'}(\gamma(0)) \\ \geq&2k_i'(\frac{3k_i'}{\eta_i} - \epsilon) + 2\epsilon - 2c, \end{aligned} \tag{8.43}$$

which is sufficiently positive for i large.

We have finished deforming $H_{M,i,1} + H_{N,i,1}$ to $H_{I,i,2} + H_{N,i,1}$ whose additional time-one chords involved all have sufficiently positive action. We then perform a symmetric construction in the region $II = \partial M \times [A, +\infty) \times N$ deforming $H_{N,i,1}$ to $H_{II,i,2}$, which is constant for $r_2 \geq B$, while the addtional time-one chords all have sufficiently positive action for i large.

The upshot of this second step is that we get a Hamiltonian $H_{I,i,2} + H_{II,i,2}$ on $M \times N$ which agrees with the original split Hamiltonian $H_{M,N,i}$, and is constant equal to $2C$ outside the compact subset $\{r_1 \leq B, r_2 \leq B\}$. Also, the additional chords compared to the original Hamiltonian $H_{M,N,i}$ all have sufficiently positive action for i sufficiently large.

8.4. **The third step.** The third step is to deform the Hamiltonian $H_{I,i,2} + H_{II,i,2}$ to a Hamiltonian K_i which depends only on the radial coordinate $\Sigma \times [1, +\infty)$, and linear in the radial coordinate outside a compact set.

Note the Hamiltonian $H_{I,i,2}+H_{II,i,2}$ is constant equal to $2C$ outside the compact set $\{r_1 \leq B, r_2 \leq B\}$. In particular, this is true for $r \geq B$. We then deform it to a Hamiltonian K_i on $\Sigma \times [B, +\infty)$ by a smooth cut-off function such that the following holds:

(i) K_i agrees with $H_{I,i,2}+H_{II,i,2}$ in the region $\{r \leq B\}$. In particular, it agrees with the split Hamiltonian $H_{M,N,i}$ in the region $\{r_1 \leq A-\epsilon, r_2 \leq A-\epsilon\}$.

(ii) K_i is convex and strictly increasing with respect to the radial coordinate on $\Sigma \times [B, +\infty)$.

(iii) For $r \geq B + \epsilon$, we have

$$K(z, r) = k_i(r - B - \epsilon) + \epsilon + 2C. \tag{8.44}$$

(iv) K_i does not grow too fast in the region $B \leq r \leq B+\epsilon$. This can be achieved by requiring that the slope k_i is not too big.

With such a Hamiltonian K_i (depending on b), we can construct our desired map.

Lemma 8.2. *For such a K_b, there is a well-defined homomorphism of modules of truncated Floer cochain groups:*

$$R_b : CF^*_{(-b,a]}(L \times L'; H_{M,N,i}) \to CF^*_{(-b,a]}(L \times L'; K_i) \tag{8.45}$$

Proof. The proof is based on analyzing the action of the K-chords, so that the action of additional chords that do not agree with Hamiltonian chords for $H_{I,2} + H_{II,2}$ is sufficiently large and therefore does not fall in the action filtration window $(-b, a]$. Thus we are able to define the desired map on truncated Floer complexes, which is basically the identity map (by identifying generators).

There might be additional K_i-chords which are on the level hypersurface $\Sigma \times \{r\}$ for r close to B, say $r \in [B, B+2\epsilon]$. Let γ be such a chord, we estimate its action as

$$\begin{aligned} &\mathcal{A}_{K_i, L\times L'}(\gamma) \\ \geq& -k_i(B+\epsilon) + 2C - 2c \\ \geq& -k_i(B+\epsilon) + 2k_i'(\frac{3k_i'}{\eta_i} - \epsilon) + 2\epsilon - 2c. \end{aligned} \tag{8.46}$$

Now if we choose $k_i = \sqrt{k_i'}$, the action of such a chord is sufficiently positive for i large.

Now we specify the choice of B and A_1 to make sure the above two estimates are sufficiently positive. Note $2 < \frac{3}{\sqrt{2}} < \sqrt{5}$ and $1 < \frac{3}{4\sqrt{2}} + \frac{1}{2} < \frac{3}{2\sqrt{2}}$. Let us take

$$\begin{aligned} B &= \frac{3}{\sqrt{2}}A, \\ A_1 &= (\frac{3}{4\sqrt{2}} + \frac{1}{2})A. \end{aligned} \tag{8.47}$$

These choices ensure the action of the additional chords are sufficiently positive, and also make the previous estimate (8.39) valid. This finishes all the steps in deforming the split Hamiltonian to an admissilble one, so we obtain the desired homomorphism of modules

$$\bar{R}_{i,b} : CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; H_{M,N,i}) \to CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; K_i). \tag{8.48}$$

□

Regarding the other homomorphism

$$CF^*_{(-b,a]}(L \times L'; K_i) \to CF^*_{(-2b,a]}(L \times L'; H_{M,N,i}), \tag{8.49}$$

it is well-defined because of the fact that within the region $\{r_1 \leq A-\epsilon, r_2 \leq A-\epsilon\}$, we have not changed the Hamiltonian $H_{M,N,i}$, and the chords that lie outside this region all have action out of the given filtration window $(-b, a]$.

Note that for our particularly constructed K_i, we may find a decreasing homotopy $H_{i,w}$ from $H_{M,N,i}$ to K_i, parametrized by $w \in [0,1]$, such that it is constant homotopy inside the compact set $r_1 \leq A-\epsilon, r_2 \leq A-\epsilon$ when the two Hamiltonians agree, and during the homotopy for every $w > 0$, the extra Hamiltonian chords have sufficiently positive action and do not fall in the action filtration window $(-b, a]$. In a sense, the desired action-restriction map

$$\bar{R}_{i,b} : CF^*_{(-b,a]}(L \times L'; H_{M,N,i}) \to CF^*_{(-b,a]}(L \times L'; K_i) \tag{8.50}$$

is the continuation map associated to the decreasing homotopy $H_{i,w}$, restricted to given action filtration window. For our particular choice of homotopy, this turns out to be an isomorphism as just argued. This viewpoint allows us to generalize the definition to higher order terms.

8.5. **Almost complex structures.** The matter with almost complex structures is less subtle than that with Hamiltonians. This is a general feature of Floer theory on non-compact manifolds: even though the almost complex structures are required to be of contact type near infinity, there is still plenty of flexibility of perturbing them (for example this is how we can achieve transversality). On the other hand, the map (8.48) that we have defined in the previous subsections does not even involves choices of almost complex structures.

Suppose that we have chosen regular almost complex structures $J_{M,i}$ and $J_{N,i}$. According to our construction, K_i agrees with $H_{M,N,i}$ for $r_1 \leq A - \epsilon, r_2 \leq A - \epsilon$, and is still split $(= H_{M,i,1} + H_{N,i,1}$ when $r_1 \leq A_1, r_2 \leq A_1$. We may then choose a regular J_i for K_i which is of the form $J_{M,N,i}$ when $r_1 \leq A_1, r_2 \leq A_1$, and is of contact type on $\Sigma \times [B + \epsilon, +\infty)$.

In fact, it does no harm even if we require J to be a product almost complex structure when $r_1 \leq B, r_2 \leq B$. Regularity results still hold if we have chosen J_M and J_N generically. But we do not quite need to make such an asumption here. In any case, we do not have to worry too much about almost complex structures, as long as they are compatible with the symplectic form, and are of contact type outside a compact set.

8.6. **A homotopy argument.** There is a minor unsatisfactory point, which is that the function K_i we obtained is only linear outside a compact set $r \leq B = B_i$, whose size also depends on i (more specifically the slope). This is not convenient for defining A_∞-structures on the wrapped Floer complex. However, from the viewpoint of homotopy invariance of Floer theory, this defect is irrelevant. And that suggests that there is a way to make changes accordingly.

The way we construct this homotopy is as follows. Note that the time-$(-\ln B)$ (the minus sign means backward) Liouville flow rescales the Hamiltonian K_i to be one that is linear outside a small neighborhood of the compact set $\{r \leq 1\}$. We define the homotopy $K_{i,w} = \frac{K_i}{wB} \circ \psi^{-wB}$. Define a homotopy of almost complex structures in a similar way. The continuation map associated to this homotopy gives a cochain homotopy equivalence:

$$CF^*(L \times L'; K_i, J_i) \to CF^*(\mathcal{L}_0, \mathcal{L}_1; K'_{i,1}, J'_{i,1}). \tag{8.51}$$

Note that $K_{i,1}$ is linear for $r \geq 1$, but for different values of b this might behave slightly differently inside of this region. Since the continuation map associated to compactly-supported homotopy of Hamiltonians/almost complex structures is a cochain homotopy equivalence, we may compose such with the previous cochain homotopy equivalence (8.51) yields a cochain homotopy equivalence:

$$h_{i,b} : CF^*(L \times L'; K_i, J_i) \to CF^*(L \times L'; K_{i,1}, J_{i,1}) \tag{8.52}$$

for a single $K_{i,1}$ and single $J_{i,1}$ independent of b, though the map $h_{i,b}$ might depend on b.

Moreover, this cochain map increases action of Hamiltonian chords, hence preserves the action filtrations of the form $(-b, a]$ for a fixed at the beginning and large enough for the Floer complex to be independent of a. This is because of the action-energy identity (applied to the family of Hamiltonians). This we get a cochain homotopy equivalence

$$h_{i,b} : CF^*_{(-b,a]}(L \times L'; K_i, J_i) \to CF^*_{(-b,a]}(L \times L'; K_{i,1}, J_{i,1}) \tag{8.53}$$

on the truncated wrapped Floer complexes.

8.7. **Intertwining the Floer differentials.** By the previous construction and estimates, we obtain the homomorphisms of modules (8.48). In order to prove that the module homomorphism (8.48) is a cochain map, for which purpose we need to understand how Floer differential and the map (8.2) affect the action of Hamiltonian chords. We know that going along an inhomogeneous pseudoholomorphic strip (Floer trajectory) decreases the action, the differential increases the action (because we are using cohomology), but there cannot be chords of action greater than a. So these two truncated graded modules are indeed cochain complexes with respect to the Floer differentials. Thus we get a diagram:

$$\begin{array}{ccc} CF^*_{(-b,a]}(L \times L'; H_{M,N,i}, J_{M,N,i}) & \xrightarrow{R} & CF^*_{(-b,a]}(L \times L'; K_i, J_i) \\ \downarrow m^1 & & \downarrow m^1 \\ CF^*_{(-b,a]}(L \times L'; H_{M,N,i}, J_{M,N,i}) & \xrightarrow{R} & CF^*_{(-b,a]}(L \times L'; K_i, J_i) \end{array} \tag{8.54}$$

Lemma 8.3. *For each b there is some i_b such that for all $i > i_b$, the above diagram (8.54) is commutative.*

Proof. The proof is based on the observation that the map R is an inclusion (usually identity) map and does not change action of the chords. Since running along a Floer trajectory decreases the action, the differential increases the action (because we are using cohomology). It follows that the image of a generator in $CF^*_{(-b,a]}(L \times L'; H_{M,N,i}, J_{M,N,i})$ (which corresponds to a chord of action between $-b$ and a) under the Floer's differential $m^1 = m^1_{H_{M,N,i}, J_{M,N,i}}$ is a $\mathbb{Z}$-linear combination of $H_{M,N,i}$-chords of action still bigger than $-b$. Moreover, there are no chords of action greater than a, so these chords still have action between $-b$ and a, which under the action-restriction map R go to K-chords of action between $-b$ and a.

Suppose that the two $H_{M,N,i}$-chords γ_0, γ_1 under the map R, which are identical to themselves but regarded as two K-chords, are connected by a Floer trajectory (u, v) for (K, J). We have to show that they are in fact connected by a uniquely corresponding Floer trajectory for $(H_{M,N,i}, J_{M,N,i})$. Thus it suffices to prove that the Floer trajectory for (K, J) is indeed a Floer trajectory for $(H_{M,N,i}, J_{M,N,i})$, i.e. the Floer trajectory does not escape from the region $\{r_1 \leq A - \epsilon, r_2 \leq A - \epsilon\}$, where (K, J) agrees with $(H_{M,N,i}, J_{M,N,i})$. Suppose the contrary, namely that the projection of (u, v) to some factor (either u or v) escapes outside of the level $A - \epsilon$. Let us suppose this is the case with u. Since the almost complex structure J is split when $r_1 \leq A_1, r_2 \leq A_1$, the part of u where it lies below the level $r_1 = A_1$ satisfies Floer's equation defined by the datum $(H_{M,i,1}, J_{M,i})$. So maximum principle implies that u has to escape to some place where $r_1 > A_1$, where u might not satisfy Floer's equation as the Hamiltonian and almost complex structure are not of split type.

Without loss of generality, we may assume that the Hamiltonian chords γ_0, γ_1 are non-constant, otherwise they are contained in the compact domain $M_0 \times N_0$ and any Floer trajectory cannot at all escape from that (possibly slightly larger) domain. Also, under the genericity assumption, the projections of the two chords γ_0, γ_1 to M are H_M-chords x_0, x_1, which can be assumed to be non-trivial. Otherwise if they are H_M-chords in the interior of M, then no inhomogeneous pseudoholomorphic strip connecting them can even escape outside the boundary ∂M. In that case, there is nothing to prove.

So let us suppose that x_0, x_1 are non-trivial, and correspond to Reeb chords on some level hypersurfaces. This implies that in the place whenever u satisfies Floer's equation, the intersection $u(Z) \cap (\partial M \times \{r_1\})$ is either empty or a non-trivial arc. Since we have assumed that u escapes outside of the level $r_1 = A_1$, we know that for every $r_1 \in [A, A_1]$, $u \cap (\partial M \times \{r_1\})$ is a non-trivial arc. In particular, this is the case with $u\cap(\partial M\times\{A\})$ and $u\cap(\partial M\times\{A_1\})$. On the other hand, $u\cap\{A\le r_1\le A_1\}$ is a J_M-holomorphic curve, because the Hamiltonian $H_{M,i,1}$ is constant there. Recall that $A_1=(\frac{3}{4\sqrt{2}}+\frac{1}{2})A$. The following lemma implies that if i is sufficiently large so that $A=A_i=\frac{3k_i'}{\delta_i}$ is large, then the energy of u is very large, which is not possible, because the energy of u is bounded by that of the original Floer trajectory (u,v) in $M\times N$ connecting the given two chords with fixed amount of action. □

Lemma 8.4. *Let J_M be an almost complex structure on M of contact type over the cylindrical end, and $d>1$ a constant. Then there exists a constant $c=c(J_M,d)>0$ depending only on the almost complex structure J_M, the constant d, such that the following holds. Let S be a compact connected Riemann surface with boundary and corners $\partial S=\partial_l S\cup\partial_n S$, where the two boundary portions $\partial_l S$ and $\partial_n S$ can meet at the corner points. Let $f:S\to M$ be any J_M-holomorphic curve, which satisfies*

(i) $f(S)\subset\partial M\times[A,dA]$;
(ii) $f(\partial_n S)\cap(\partial M\times\{A\}), f(\partial_n S)\cap(\partial M\times\{dA\})$ are both non-empty;
(iii) There exists a connected component C of $\partial_l S$, such that $f(C)$ is an arc in $\partial M\times[A,dA]$, with its two endpoints lying on $\partial M\times\{A\}$ and $\partial M\times\{dA\}$ respectively.

Then we have

$$Area(f)=E_{J_M}(f)=\int_S\frac{1}{2}|df|^2_{J_M}\ge cA. \tag{8.55}$$

Proof. This essentially follows from Gromov's monotonicity lemma. Alternatively, this can be proved using inverse isoperimetric inequality. □

Now summarizing the above discussion, we have the following diagram (omitting the obvious choices of almost complex structures):

(8.56)

$$\begin{array}{ccccc}
CF^*_{(-b,a]}(L\times L';H_{M,N,i}) & \xrightarrow{R_b} & CF^*_{(-b,a]}(L\times L';K_i) & \xrightarrow{h_{i,b}} & CF^*_{(-b,a]}(L\times L';K_{i,1})\\
\downarrow i & & \downarrow i & & \downarrow i\\
CF^*_{(-2b,a]}(L\times L';H_{M,N,i}) & \xrightarrow{R_{2b}} & CW^*_{(-2b,a]}(\mathcal{L}_0,\mathcal{L}_1;K_i) & \xrightarrow{h_{i,2b}} & CF^*_{(-2b,a]}(L\times L';K_{i,1})\\
\downarrow i & & \downarrow i & & \downarrow i\\
\cdots & & & &
\end{array}$$

where $R_b, R_{2b},\cdots$ are cochain isomorphisms, $h_{i,b}, h_{i,2b},\cdots$ are cochain homotopy equivalences, the vertical arrows are all natural inclusions. By the nature of our construction, we have:

Lemma 8.5. *The first square strictly commutes, the second square homotopy commutes.*

Proof. The commutativity of the first square is clear because the extra chords have sufficiently action - if they are already not in $(-b,a]$, they must be outside of $(-2b,a]$ as well. The homotopy commutativity of the second square is a general feature of continuation maps. □

Consider the composition $h_{wb}\circ R_{wb}$, which is a cochain homotopy equivalence for every $w=1,2,\cdots$. They fit into a diagram of directed systems of cochain complexes, where all the maps homotopy commute. Therefore the homotopy direct limit homomorphism exists and is a cochain homotopy equivalence, which gives the desired action-restriction map (4.9). This algebraic clain is discussed in more details back in section 4.

8.8. **Constructing action-restriction data on disks with multiple punctures.** Suppose we are given a finite collection of products of exact cylindrical Lagrangian submanifolds $\mathbb{L}_p = \{L_1 \times L'_1, \cdots, L_d \times L'_d\}$ of $M \times N$. Let $j_i \in \{1, \cdots, d\}$ be indices, for $i = 0, \cdots, k$. For simplicity, denote $\mathcal{L}_j = L_j \times L'_j$. In $\mathcal{W}^s(M \times N)$, the k-th structure map μ^k is the combination of all sorts of maps

(8.57)

$$\mu^{k;F;i_0,\cdots,i_k} : CF^*(\mathcal{L}_{j_{k-1}}, \mathcal{L}_{j_k}; H_{M,N,i_k}, J_{M,N,i_k}) \otimes \cdots \otimes CF^*(\mathcal{L}_{j_0}, \mathcal{L}_{j_1}; H_{M,N,i_1}, J_{M,N,i_1}) \to CW^*(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; H_{M,N,i_0}, J_{M,N,i_0}).$$

The moduli space of inhomogeneous pseudoholomorphic disks defined using this choice of Floer datum is smooth. In particular, we can use it to define the above-mentioned multiplication (8.57).

Choose the filtration number $a > 0$ greater than the action of any Hamiltonian chord between Lagrangians from the collection. Thus the truncated Floer cochain complexes $CF^*_{(-b,a]}(\mathcal{L}_{j_0}, \mathcal{L}_{j_1}; H_{M,N,i}, J_{M,N,i})$ includes all Hamiltonian chords in the interior of $M \times N$, for every $j_0, j_1 \in \{1, \cdots, d\}$. Consider any $(k+1)$-tuple $(b_0, b_1, \cdots, b_k)$ of numbers which satisfy $b_0 \geq \sum_{i=1}^k b_i$. Because of the action-energy relation, the map m^k induces the following map on truncated Floer complexes:

(8.58)

$$m^k : CW^*_{(-b_k,a]}(\mathcal{L}_{j_{k-1}}, \mathcal{L}_{j_k}; H_{M,N,i}, J_{M,N,i}) \otimes \cdots \otimes CW^*_{(-b_1,a]}(\mathcal{L}_{j_0}, \mathcal{L}_{j_1}; H_{M,N}, J_{M,N}) \to CW^*_{(-b_0,a]}(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; H_{M,N,i}, J_{M,N,i}).$$

The action of every output chord is still less than or equal to a, because there are no chords having higher action by our choice of H_M and H_N.

We define a sequence of multilinear maps on the truncated Floer complex with respect to Hamiltonians K_i for various i:

(8.59)

$$m^{k;F;i_0,\cdots,i_k} : CF^*_{(-b_k,a]}(\mathcal{L}_{j_{k-1}}, \mathcal{L}_{j_k}; K_{i_k}, J_{i_k})[q] \otimes \cdots \otimes CF^*_{(-b_1,a]}(\mathcal{L}_{j_0}, \mathcal{L}_{j_1}; K_{i_1}, J_{i_1})[q] \to CF^*_{(-b_0,a]}(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; K_{i_0}, J_{i_0})[q].$$

using the cochain homotopy equivalences $h_{i,b}$. Recall that these cochain homotopy equivalences are defined on the whole wrapped Floer complexes, and have restrictions to the truncated Floer complexes as:

$$h_{i,b} : CF^*_{(-b_i,a]}(\mathcal{L}_{j_{i-1}}, \mathcal{L}_{j_i}; K_i, J_i) \to CF^*_{(-2b,a]}(\mathcal{L}_{j_{i-1}}, \mathcal{L}_{j_i}; K_{i,1}, J_{i,1})$$

is defined, and is also a cochain homotopy equivalence. Consider also the cochain homotopy equivalence

$$h_{b_0} : CF^*_{(-b_0,a]}(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; K_{b_0}, J_{b_0}) \to CF^*_{(-b_0,a]}(\mathcal{L}_{j_0}, \mathcal{L}_{j_k}; K_{i_k,1}, J_{i_k,1}).$$

Choose a canonical cochain homotopy inverse $g_{i,b}$ of $h_{i,b}$, which is defined by using the backward homotopy of Hamiltonians that is used to define $h_{i,b}$, i.e. by reversing the parameter. Then we define a multilinear map

(8.60)

$$\begin{aligned} m_{\vec{b}}^{k;F;i_0,\cdots,i_1} :& CF^*_{(-b_k,a]}(\mathcal{L}_{j_{k-1}},\mathcal{L}_{j_k};K_{b_k},J_{b_k})\otimes\cdots\otimes CF^*_{(-b_1,a]}(\mathcal{L}_{j_0},\mathcal{L}_{j_1};K_{b_1},J_{b_1}) \\ &\to CF^*_{(-b_0,a]}(\mathcal{L}_{j_0},\mathcal{L}_{j_k};K_{b_0},J_{b_0}) \end{aligned}$$

by composing $m^{k;F;i_0,\cdots,i_k}$ for $(K_{i,1},J_{i,1})$ with these various cochain homotopy equivalences. That is, consider the usual k-th order multiplication map restricted to truncated Floer complexes:

$$\begin{aligned} m^{k;F;i_0,\cdots,i_k} :& CF^*_{(-b_k,a]}(\mathcal{L}_{j_{k-1}},\mathcal{L}_{j_k};K_{i_k,1},J_{i_k,1})\otimes\cdots\otimes CF^*_{(-b_1,a]}(\mathcal{L}_{j_0},\mathcal{L}_{j_1};K_{i_1},J_{i_1}) \\ &\to CF^*_{(-b_0,a]}(\mathcal{L}_{j_0},\mathcal{L}_{j_k};K_{i_0},J_{i_0}), \end{aligned}$$

and define $m_{\vec{b}}^{k;F;i_0,\cdots,i_k}$ by

$$m_{\vec{b}}^{k;F;i_0,\cdots,i_k} = g_{b_0}\circ m^{k;F;i_0,\cdots,i_k}\circ(h_{i_k,b_k}\otimes\cdots\otimes h_{i_1,b_1})$$

We shall now explain how to obtain the families of Hamiltonians and almost complex structures involved in the action-restriction data that will be used to define multilinear maps

(8.61)

$$\begin{aligned} R_{\vec{b}}^{k;F;i_0,\cdots,i_k} :& CF^*_{(-b_k,a]}(\mathcal{L}_{j_{k-1}},\mathcal{L}_{j_k};H_{M,N,i},J_{M,N,i})\otimes\cdots\otimes CF^*_{(-b_1,a]}(\mathcal{L}_{j_0},\mathcal{L}_{j_1};H_{M,N,i},J_{M,N,i}) \\ &\to CF^*_{(-b_0,a]}(\mathcal{L}_{j_0},\mathcal{L}_{j_k};K_{b_0},J_{b_0})[1-k], \end{aligned}$$

where $[1-k]$ means the map has degree $1-k$.

Recall that the Hamiltonian K_i and the almost complex structure J_i is chosen such that the map

$$CF^*_{(-b_l,a]}(\mathcal{L}_{j_{l-1}},\mathcal{L}_{j_l};H_{M,N,i_l},J_{M,N,i_l})\to CF^*_{(-b_i,a]}(\mathcal{L}_{j_{l-1}},\mathcal{L}_{j_l};K_{i_l},J_{i_l})$$

is a cochain isomorphism. This can be achieved provided i_l is sufficiently large.

To find a family of Hamiltonians $\tilde{H}_{(S,w);F;i_0,\cdots,i_k;\vec{b}}$ that is required in an action-restriction datum, we use the homotopy between the two Hamiltonian functions $H_{M,N,i}$ and K_i. Note that after suitable compactly-supported homotopy, K_i agrees with $H_{M,N,i}$ inside a compact set, and differs by a decreasing homotopy $H_{i,w}$ outside. Then for every $S\in\mathcal{M}_{k+1}$ disk with $k+1$ punctures, label F and weights $i_0,\cdots,i_k$, we consider $(F,i_0,\cdots,i_k)$-flavored popsicles which are domains of inhomogeneous pseudoholomorphic curves defining $m^{k;F;i_0,\cdots,i_k}$ and the new operation $m_{\vec{b}}^{k;F;i_0,\cdots,i_k}$. Near each strip-like end we have the homotopy $H_{i_l,w}$, we then extend these to the whole disk underlying the popsicle.

The construction of a family of almost complex structures $\tilde{J}_{(S,w);F;i_0,\cdots,i_k;\vec{b}}$ follows the same pattern. We choose a small infinitesimal deformation Y of the almost complex structure in the space of admissible almost complex structures compatible, and add it (via the exponential map) to the product almost complex structure, such that the perturbed almost complex structure is generic, for the purpose of achieving transversality.

Finally, we mention that it is possible to construct such families in a consistent way based on the inductive structure of the compactification $\bar{\mathcal{N}}_{k+1}$. Such kind of inductive argument is basically well established so we shall not give a proof.

8.9. **The case of cylindrical Lagrangian submanifolds.** Now let us consider cylindrical Lagrangian submanifolds of $M \times N$, for which we define wrapped Floer theory with respect to Hamiltonians linear at infinity and cylindrical almost complex structures with respect to the cylindrical structure $\Sigma \times [1, +\infty)$. We shall show that wrapped Floer theory for such Lagrangian submanifolds are also well-defined with respect to split Hamiltonians and product almost complex structures, by constructing these Floer data and comparing to the given admissible Floer data.

For simplicity, let us first discuss the case of a single exact cylindrical Lagrangian submanifold $\mathcal{L} \subset M \times N$. We will explain later how to deal with multiple exact cylindrical Lagrangian submanifolds from a given finite collection.

Let $f_{\mathcal{L}}$ be a primitive for $\mathcal{L}$ which is locally constant in the cylindrical end of $\mathcal{L}$. Let c be a universal constant such that $|f_{\mathcal{L}}(\gamma(1)) - f_{\mathcal{L}}(\gamma(0))| < c$ for any Hamiltonian chord γ.

Note that the intersection $\mathcal{L} \cap \Sigma \subset \Sigma$ is a Legendrian submanifold, which we call the Legendrian boundary of $\mathcal{L}$. The set of Reeb chords on Σ from the Legendrian boundary of $\mathcal{L}$ to itself is discrete, assuming the Reeb dynamics is generic. In particular, the length spectrum is discrete, and for any given $\lambda > 0$, there are finitely many isolated Reeb chords whose lengths are smaller than λ.

Now for any Hamiltonian H on $M \times N$ which depends only on the radial coordinate on $\Sigma \times (0, +\infty)$, $H(y, r) = h(r)$ for $(y, r) \in \Sigma \times (0, +\infty)$, we see that time-one H-chords from $\mathcal{L}$ to itself which are contained in the cylindrical end $\Sigma \times [1, +\infty)$ are in a natural one-to-one correspondence with Reeb chords on Σ from the Legendrian boundary of $\mathcal{L}$ to itself. These H-chords occur on level hypersurfaces $\Sigma \times \{r\}$ when $h'(r)$ is equal to the length of some Reeb chord.

8.10. **From radial Hamiltonians to split Hamiltonians.** Let $\{H_i\}$ be a sequence of Hamiltonians such that $H_i > 0$ everywhere, $H_i(y, r) = h_i(r)$ for $(y, r) \in \Sigma \times (0, +\infty)$, and h_i is linear of slope $k_i > 0$ with $k_i \to \infty$ as $i \to \infty$ for $r \geq 1$. Suppose that wrapped Floer complex and A_∞-structure maps are well-defined for $\mathcal{L}$ with respect to $\{H_i\}$, for a suitable choice of domain-dependent family of almost complex structures of contact type.

Consider the action filtration on each Floer complex $CF^*(\mathcal{L}_0, \mathcal{L}_1; H_i)$ and fix an action filtration window $(-b, a]$, with b arbitrary and a fixed such that the truncated Floer complex $CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; H_i)$ is independent of a, such that $CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; H_i) = CF^*_{(-b,\infty)}(\mathcal{L}_0, \mathcal{L}_1; H_i)$.

Let $\epsilon > 0$ be small and let A be a suitable large constant, to be determined later. We are going to modify the Hamiltonian H_i outside the compact set $r = r_1 + r_2 \leq A - \epsilon$. Choose a cut-off function $\rho_{A,i} : [1, +\infty) \to [0, 1]$ such that $\rho_{A,i}(r) = 1$ for $r \leq A - \epsilon$, and $\rho_{A,i}(r) = 0$ for r near A and all $r \geq A$. Let C be a suitable constant slightly bigger than the value of h_i at $r = A - \epsilon$ but less than the value at $r = A$. Let us choose $C = k_i(A - \epsilon) + \epsilon$. Define $H_{i,1}$ to be the new Hamiltonian such that it agrees with H_i in the compact domain bounded by Σ, and

$$H_{i,1}(y, r) = h_{i,1}(r) = \rho_{A,i}(r) h_i(r) + (1 - \rho_{A,i}(r)) C, \tag{8.62}$$

for any $(y, r) \in \Sigma \times [1, +\infty)$. This Hamiltonian $H_{i,1}$ agrees with H_i for $r \leq A - \epsilon$, and becomes constant C for $r \geq A$.

The choice of A should be such that the extra Hamiltonian chords for $H_{i,1}$ have sufficiently positive action and therefore do not fall in the action filtration window

$(-b, a]$. So let us analyze the extra Hamiltonian chords. There are two kinds of them:

(i) constant chords near level $r = A$. Let γ be such a chord. Since $\gamma(0) = \gamma(1)$, the contributions from the primitive vanish. Then its action is

$$\mathcal{A}(\gamma) = C = k_i(A - \epsilon) + \epsilon, \tag{8.63}$$

which tends to $+\infty$ as $i \to \infty$, for any choice of A sufficiently large;

(ii) Hamiltonian chords near level $r = A - \epsilon$ (but not on the level $r = A - \epsilon$), which correspond to Reeb chords on the contact manifold Σ from the Legendrian boundary of $\mathcal{L}$ to itself. Since the lengths of Reeb chords starting from and landing on given Legendrian submanifolds are discrete, we may find a gap between the slope and the lengths of Reeb chords:

$$\delta_i = \inf_{\substack{\lambda \\ \lambda \text{ is the length of a Reeb chord}}} |k_i - \lambda| > 0. \tag{8.64}$$

By passing to sub-sequences of the sequence of Hamiltonians $\{H_i\}$ if necessary, we may assume without loss of generality that

$$\frac{k_i}{\delta_i} > 2. \tag{8.65}$$

Then we may estimate the action of such a Hamiltonian chord:

$$\begin{aligned} &\mathcal{A}(\gamma) \\ =& - h'_{i,1}(r)r + h_{i,1}(r) + f_{\mathcal{L}}(\gamma(1)) - f_{\mathcal{L}}(\gamma(0)) \\ &\geq -(k_i - \delta_i)A + k_i(A - \epsilon) + \epsilon - 2c \\ &= \delta_i A - k_i\epsilon + \epsilon - 2c. \end{aligned} \tag{8.66}$$

Thus, if we choose

$$A = \frac{3k_i}{\delta_i}, \tag{8.67}$$

we have that for such a chord, its action satisfies

$$\mathcal{A}(\gamma) \gg 0,$$

for i sufficiently large.

The new Hamiltonian $H_{i,1}$ is already constant outside the compact set $r \leq A$. To make it into a Hamiltonian which is split outside a compact set, it suffices to add to it a Hamiltonian of the form $H_{M,i,2} + H_{N,i,2}$ such that

(1) $H_{M,i,2} = 0$ for $r_1 \leq A$, and $H_{M,i,2} = h_{M,i,2}(r_1) = k'_i(r_1 - A - \epsilon)$ for $r_1 \geq A + \epsilon$. Moreover, we require that

$$|h_{M,i,2}(r_1)| \leq \epsilon, \text{ if } r_1 \in [A, A + \epsilon].$$

(2) $H_{N,i,2} = 0$ for $r_2 \leq A$, and $H_{N,i,2} = h_{N,i,2}(r_2) = k''_i(r_2 - A - \epsilon)$ for $r_2 \geq A + \epsilon$. Moreover, we require that

$$|h_{N,i,2}(r_2)| \leq \epsilon, \text{ if } r_2 \in [A, A + \epsilon].$$

We then set

$$K_i = H_{i,1} + H_{M,i,2} + H_{N,i,2}. \tag{8.68}$$

The choices of the slopes k_i', k_i'' should be such that the extra Hamiltonian chords for K_i from $\mathcal{L}$ to itself have sufficiently large action. The extra Hamiltonian chords for K_i can possibly be of the following kinds:

(i) constant chords in the region $r \geq A, r_1 \leq A + \epsilon, r_2 \leq A + \epsilon$. The action of such a constant chord satisfies

$$\begin{aligned}&\mathcal{A}_{H_{i,1}+H_{M,i,2}+H_{N,i,2}}(\gamma) = C\\ =&k_i(A - \epsilon) + \epsilon\\ =&k_i(\frac{3k_i}{\delta_i} - \epsilon) + \epsilon,\end{aligned}$$

which is sufficiently positive for large i.

(ii) Hamiltonian chords $\gamma = (x, y)$, where x is a non-constant Hamiltonian chord for $h_{M,i,2}$ near level $r_1 = B$, and y is a constant chord. The action of such a chord satisfies

(8.69)

$$\begin{aligned}\mathcal{A}_{H_{i,1}+H_{M,i,2}+H_{N,i,2}}(\gamma) =& -\frac{\partial h_{M,i,2}}{\partial r_1}(r_1)r_1 + h_{M,i,2}(r_1) + h_{N,i,2}(r_2) + C + f_{\mathcal{L}}(\gamma(1)) - f_{\mathcal{L}}(\gamma(0))\\ \geq& - k_i'(A + \epsilon) + C - \epsilon - 2c\\ =& - k_i'(\frac{3k_i}{\delta_i} + \epsilon) + k_i(\frac{3k_i}{\delta_i} - \epsilon) + \epsilon - 2c.\end{aligned}$$

If we choose

$$k_i' = \frac{k_i}{3},$$

then for large i the action is sufficiently positive.

(iii) Hamiltonian chords $\gamma = (x, y)$, where x is a constant chord, and y is a non-constant Hamiltonian chord for $h_{N,i,2}$ near level $r_2 = B$. This case is symmetric to case (ii). By a similar estimate we find that it suffices to choose the slope

$$k_i'' = \frac{k_i}{3},$$

such that the action of such a chord is sufficiently positive for i large;

(iv) Hamiltonian chords $\gamma = (x, y)$, where both x and y are non-constant. We can similarly estimate the action of such a chord:

(8.70)

$$\begin{aligned}&\mathcal{A}_{H_{i,1}+H_{M,i,2}+H_{N,i,2}}(\gamma)\\ =& - \frac{\partial h_{M,i,2}}{\partial r_1}r_1 + h_{M,i,2}(r_1) - \frac{\partial h_{N,i,2}}{\partial r_2}r_2 + h_{N,i,2}(r_2) + C + f_{\mathcal{L}}(\gamma(1)) - f_{\mathcal{L}}(\gamma(0))\\ \geq& - k_i'(A + \epsilon) - k_i''(A + \epsilon) - 2\epsilon + C - 2c\\ =& - k_i'(\frac{3k_i}{\delta_i} + \epsilon) - k_i''(\frac{3k_i}{\delta_i} + \epsilon) + k_i(\frac{3k_i}{\delta_i} - \epsilon) - 2\epsilon - 2c.\end{aligned}$$

Thus if we have chosen $k_i' = k_i'' = \frac{k_i}{3}$ as in the previous two cases, the action is bigger than or equal to

$$\frac{k_i^2}{\delta_i} - \frac{5k_i\epsilon}{3} - 2\epsilon - 2c.$$

Since $\frac{k_i}{\delta_i} > 2$, this is sufficiently positive for large i.

Now we have obtained a Hamiltonian K_i which is split outside the compact set $r_1 \leq B$ and $r_2 \leq B$ such that the extra chords compared to the original linear Hamiltonian H_i have sufficiently positive action, and in particular does not fall in the action filtration window $(-b, a]$.

8.11. **Involving multiple cylindrical Lagrangian submanifolds.** The key observation is that the estimates required to rule out certain chords depend on the gaps δ_i between the slopes and the lengths of Reeb chords. The gaps are the only crucial parameters, while the other parameters can be chosen freely or dependent of the gaps.

We first observe that:

Lemma 8.6. *For any fixed finite collection $\mathbb{L}_c$ of cylindrical Lagrangian submanifolds of $M \times N$, then there is a discrete subset $S_{\mathbb{L}_c} \subset \mathbb{R}_+$, such that for any pair $\mathcal{L}_0, \mathcal{L}_1 \in \mathbb{L}_c$, the length of any Reeb chord from the Legendrian boundary of $\mathcal{L}_0$ to that of $\mathcal{L}_1$ belongs to $S_{\mathbb{L}_c}$.*

Proof. Let $S_{\mathbb{L}_c}$ be the union of the length spectra of Reeb chords between the Legendrian boundaries of every pair of Lagrangian submanifolds in $\mathbb{L}_c$. This is a finite union of discrete subsets of $\mathbb{R}_+$, hence discrete as well. □

Based on this, we can rewrite the construction/estimates in the previous subsection where we replace δ_i by the following

$$\delta_i = \inf_{\lambda \in S_{\mathbb{L}_c}} |k_i - \lambda|. \tag{8.71}$$

Also, the differences of the primitives between Lagrangian submanifolds have to be taken into account. But once we fix the finite collection of exact cylindrical Lagrangian submanifolds (together with their primitives), there is a universal constant $c > 0$ which is bigger than the absolute value of the difference of two values of the primitives:

$$c > |f_{\mathcal{L}_0}(p) - f_{\mathcal{L}_1}(q)|, \forall p \in \mathcal{L}_0, q \in \mathcal{L}_1. \tag{8.72}$$

Using this new δ_i and the new c, we can follow the same construction to produce Hamiltonians K_i which are split outside a compact set, for any pair of Lagrangian submanifolds, so that the Hamiltonians for each pair of Lagrangian submanifolds are the same. This proves:

Proposition 8.1. *Fix a finite collection $\mathbb{L}_c$ of exact cylindrical Lagrangian submanifolds of $M \times N$. Given a sequence of radial Hamiltonians $\{H_i\}$ which is linear of slope k_i for $k \geq 1 + \epsilon$, such that the wrapped Floer cohomology is well-defined for any pair of objects in $\mathbb{L}_c$, and for any b, there is i_b such that for all $i > i_b$, we can construct a Hamiltonian K_i for $i > i_b$ such that the following conditions are satisfied:*

- *(i)* *K_i agrees with H_i inside the compact set $r = r_1 + r_2 \leq \frac{3k_i}{\delta_i}$;*
- *(ii)* *K_i is split outside the compact set $r_1 \leq \frac{3k_i}{\delta_i}, r_2 \leq \frac{3k_i}{\delta_i}$;*
- *(iii)* *K_i is the sum of two linear Hamiltonians on M and N of slope $k_i' = \frac{k_i}{3}$ and $k_i'' = \frac{k_i}{3}$ respectively, when $r_1 \geq \frac{3k_i}{\delta_i} + \epsilon, r_2 \geq \frac{3k_i}{\delta_i} + \epsilon$;*
- *(iv)* *The additional Hamiltonian chords for K_i compared to H_i have action sufficiently positive (much bigger than the given fixed $a > 0$), and thus do not contribute to any of the truncated Floer complexes $CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; H_i)$ and $CF^*_{(-b,a]}(\mathcal{L}_0, \mathcal{L}_1; K_i)$.*

This gives the desired sequence of Hamiltonians K_i. To obtained $K_{i,1}$, we may perform a further homotopy using the previous trick for radial Hamiltonians. That works because when $r_1 \geq A, r_2 \geq A$, the diagonal Liouville flow $\psi^t_M \times \psi^t_M$ on $M \times N$ rescales split linear Hamiltonians to split linear Hamiltonians.

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