The Positive Mass Theorem with Charge Outside Horizon(s)

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Abstract of the Dissertation

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We start with time-symmetric initial ($k = 0$) data for the Einstein-Maxwell equations consisting of a Riemannian asymptotically flat 3-manifold whose boundary consists of either one connected component, several disjoint components, or finitely many *cylindrical ends*. On this Riemannian manifold, we define a vector field E, representing the Electric field, whose square norm is bounded from above by half the scalar curvature of the Riemannian manifold (the dominant energy condition). Using the Inverse Mean Curvature Flow (IMCF) as outlined in [11], we prove that if a natural upper bound on the mean curvature of a single horizon is satisfied, then the ADM mass of the 3-manifold is bounded from below by the absolute value of the charge. Equality holds if and only if the spacetime arising from the initial data contains a subset isometric to a $t = 0$ slice of an extreme Reissner-Nordström spacetime. If the boundary has multiple components, the IMCF is no longer appropriate to a get a strong result. We instead use an idea outlined in a paper of M. Herzlich [6] involving spinors to prove that if a slightly modified upper bound on the mean curvature of each component is satisfied, then the ADM mass of the 3-manifold has same lower bound. This technique involves solving a *charged* Dirac equation on the 3-manifold, which is in turn is applied to proving the positive mass theorem on an asymptotically flat 3-manifold with finitely many cylindrical ends. For multiple boundary components, equality only leads to a subset of an IWP black hole spacetime, while for cylindrical ends, it leads to the standard subset of the Majumbdar-Papapetrou spacetime. In the non-time symmetric case ($k \neq 0$), the manifold has multiple asymptotically flat ends, in addition to a bounded interior domain. Around the boundary of this domain, the regularity of the metric and of k are reduced (manifold with corners). Assuming natural matching conditions for the interior and exterior mean curvatures, the normal components of the electric and magnetic vector fields, as well as certain components of k, we then prove a positive mass theorem relating the ADM mass of each end to its momentum (defined in terms of k), and electric and magnetic charges (assuming an analogous dominant energy condition is satisfied). This result extends the work of Shi and Tam [16].

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1 Introduction

Chapter 1

Let (M^3, g) be an asymptotically flat Riemannian 3-manifold containing a vector field E representing the Electric field. Here, *asymptotically flat* means that there exists a compact subset $K \subset M^3$, a real number $r_1 > 0$, and a diffeomorphism, $\Phi : \mathbb{R}^3 - B_{r_1}(0) \to M^3 - K$ such that the pull-back metric $\Phi^* g$ satisfies

$$|(\Phi^* g)_{ij} - \delta_{ij}| \leq \frac{C}{r} \tag{1}$$

and

$$|(\Phi^* g)_{ij,k}| \leq \frac{C}{r^2} \tag{2}$$

where r denotes the radial coordinate ($r^2 = x^2 + y^2 + z^2$) for the point $< x, y, z > \in \mathbb{R}^3 - B_{r_1}(0)$ and δ_{ij} the Euclidean metric, respectively, on the manifold $\mathbb{R}^3 - B_{r_1}(0)$. $B_{r_1}(0)$ denotes a ball of radius r_1 centered at the origin in $\mathbb{R}^3$.

(M^3, g, E) will serve as initial data for the Einstein-Maxwell Equations in the time-symmetric case. An important global invariant of such a manifold is the ADM mass, representing the strength of the gravitational field at infinity, defined as in [7] (p. 373, equation 1.1.32) by

$$E_{ADM} = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S(0,r)} (\partial_l g_{lj} - \partial_j g_{ll}) dS^j \tag{3}$$

where $S(0, r) = \partial(\Phi(\mathbb{R}^3 - B_r(0))$ denotes the boundary of the complement of a coordinate ball of radius $r > r_1$ in M^3. In the following sections, we will also set $B(0, r) = \Phi(\mathbb{R}^3 - B_r(0))$.

Let ν denote a unit normal vector field on $S(0, r)$, pointing towards the asymptotically flat end of M^3. We then define the *total charge*, Q, on M^3 by

$$Q = \frac{1}{4\pi} \lim_{r\to\infty} \int_{S(0,r)} E(\nu) \tag{4}$$

Further, R will denote the scalar curvature of M^3.

It is a well-known result (cf. [10]) that if the boundary of M^3 consists of only one connected component that is a *trapped* surface, then, assuming the dominant energy condition, $R \geq 2|E|_g^2$, E_{ADM} is bounded from below by $|Q|$ (this is called the *charged positive mass theorem*). The case of equality occurs if and only if the metric on M^3 is the spatial part of an *extreme Reissner-Nordström* metric, which will be defined in the next section. Note that in the time-symmetric case, $k = 0$, the trapped surface condition is equivalent to non-positive mean curvature of the boundary, $H \leq 0$ on ∂M^3. Our goal here is to extend this result to the case where the boundary is not necessarily a trapped surface, but instead, its mean curvature satisfies an appropriate upper bound. Also, we assume the boundary, ∂M^3 is *outer minimizing*, meaning that it minimizes area among all surfaces in M^3 that enclose ∂M^3. This actually implies that the mean curvature of ∂M^3 is nonnegative, $H \geq 0$. Again, equality leads to an extreme Reissner-Nordström metric. In particular, we have the following theorem:

Theorem 1. *Assume that M^3 is an asymptotically flat Riemannian 3-manifold (with boundary) with divergence free vector field E satisfying the dominant energy condition $R \geq 2|E|_g^2$. Assume further that the boundary is outer minimizing and that the mean curvature, H, of the boundary ∂M^3, satisfies the inequality:*

$$\frac{1}{2}H \leq |\sqrt{\frac{4\pi}{A}} - \frac{4\pi|Q|}{A}| \tag{5}$$

Then

$$E_{ADM} \geq |Q| \tag{6}$$

and equality holds if and only if (M^3, g, E) agrees with the initial data set for the Einstein-Maxwell equations of an extreme Reissner-Nordström spacetime.

The proof of this theorem uses the inverse mean curvature flow (IMCF) and Geroch monotonicity as outlined in [10]. The main difference is finding the appropriate upper bound for H and showing that the same rigidity result holds.

If M^3 instead has interior boundary consisting of multiple components, then the IMCF is no longer appropriate since it will give a lower bound on E_{ADM} in terms of the charge of a particular boundary component instead of the total charge. In [6], Herzlich was able to show that in the absence of the electric field, $E = 0$, $H \leq \sqrt{\frac{4\pi}{A}}$ implies that $E_{ADM} \geq 0$ and equality holds if and only if the metric on M^3 arises a spatial portion of the Schwarzchild metric. He did this by building on Witten's proof of the positive mass theorem using spinors. Including charge in in this case involves modifying the boundary Dirac operator and finding a lower bound for its first eigenvalue. This leads to a modified, albeit reasonable, upper bound for the mean curvature of each connected component. Rigidity in this case is also weakened as it only leads to an IWP metric, not necessarily the Majumbdar Papapetrou metric:

Assume that M^3 contains multiple boundary components, each of which is diffeomorphic to a sphere; $\partial M^3 = \cup_{j=1}^m N_j$, where each N_j is diffeomorphic to S^2. g will also denote the restriction of the metric on M^3 to N_j. Let A_j denote the the area of each N_j. Further, let ν denote a unit normal vector field to N_j, pointing towards the asymptotically flat end. We set $E(\nu) = g(E, \nu)$. Define the charge of each component, Q_j by the following equation:

$$Q_j = \frac{1}{4\pi} \int_{N_j} E(\nu) \tag{7}$$

Note: If each each N_j were an actual horizon ($H = 0$), then a standard result on charged black holes shows that Q_j would then satisfy the following inequality in relation to A_j:

$$4\pi (Q_j)^2 < A_j \tag{8}$$

If $E(\nu)$ were in addition constant on N_j, then $E(\nu) = \frac{4\pi Q_j}{A_j}$, and $\int_{N_j} (E(\nu))^2 = \frac{16\pi^2 Q_j^2}{A_j}$. Combined with equation (8), this reads

$$\int_{N_j} (E(\nu))^2 < 4\pi \tag{9}$$

We shall show that in fact that a slightly more relaxed upper bound on $\int_{N_j} (E(\nu))^2$ is a sufficient condition, combined with an associated upper bound for mean curvature, to prove the following positive mass theorem:

Theorem 2. *Let M^3 is an asymptotically flat Riemannian 3-manifold with vector field E satisfying the dominant energy condition $R \geq 2|E|_g^2$, with $R - 2|E|_g^2 \in L^1(M_{ext})$. Assume that the boundary, ∂M^3 of M^3 consists of finitely many connected components, N_j, each of which is diffeomorphic to a sphere (S^2). Further, assume that on each component of the boundary, N_j, the normal component of the electric field $E(\nu)$ satisfies the following inequality with respect to the charge Q_j of that boundary:*

$$\int_{N_j} (E(\nu))^2 \leq 4\pi + \frac{16\pi^2 Q_j^2}{A_j} \tag{10}$$

and the mean curvature, H_j of each horizon N_j, satisfies the inequality:

$$\frac{1}{2} H_j \leq \sqrt{\frac{4\pi}{A_j} + \frac{16\pi^2 Q_j^2}{A_j^2} - \frac{1}{A_j} \int_{N_j} (E(\nu))^2} - \frac{4\pi |Q_j|}{A_j} \tag{11}$$

Then

$$E_{ADM} \geq |Q| \tag{12}$$

Further, if equality holds then the spacetime arising from the initial data (M^3, g, E) contains a neighborhood diffeomorphic to an open subset of an IWP black hole spacetime.

Note: We will see in section 2 that if N_j is a coordinate sphere in an extreme RN spacetime, then $E(\nu)$ is constant, so that $E(\nu) = \frac{4\pi Q_j}{A_j}$ and therefore $\int_{N_j}(E(\nu))^2 = A_j \frac{16\pi^2 Q_j}{(A_j)^2} = \frac{16\pi^2 Q_j}{A_j}$. Therefore, strict inequality in (10) is satisfied. However, we will see that $\frac{1}{2}H_j = \sqrt{\frac{4\pi}{A_j}} - \frac{4\pi|Q|}{A_j}$ on any coordinate sphere of radius r in extreme RN, so that the inequality (12) is saturated. Since the horizons in a standard Majumbdar-Papapetrou spacetime have the same properties as the horizon in an extreme RN spacetime, we obtain the same results for coordinate spheres in a standard MP spacetime. Since both of these spacetimes have $E_{ADM} = |Q|$, they show that equality can hold in (13) without the inequality (11) being saturated. However, we will see using spinors that equality holding in (13) implies equality must hold in (12), provided $\sqrt{\frac{4\pi}{A_j} + \frac{16\pi^2 Q_j^2}{A_j^2} - \frac{1}{A_j}\int_{N_j}(E(\nu))^2} - \frac{4\pi|Q_j|}{A_j} > 0$ for that particular N_j. On the other hand, if $\sqrt{\frac{4\pi}{A_j} + \frac{16\pi^2 Q_j^2}{A_j^2} - \frac{1}{A_j}\int_{N_j}(E(\nu))^2} - \frac{4\pi|Q_j|}{A_j} \leq 0$ on a particular N_j then then the boundary conditions imposed on the spinor restricted to that N_j will actually force $H = 0$ on N_j (cf. [17]).

If each interior boundary component is replaced with a cylindrical end, then rigidity in stronger in the sense that it leads to a standard Majumbdar-Papapetrou spacetime:

Theorem 3. *Assume that (M^3, g, E) is initial data for the Einstein-Maxwell equations consisting of a Riemannian 3-manifold M^3 containing one asymptotically flat end and m cylindrical ends. Assume further that the scalar curvature R and the Electric field E defined on M^3 satisfy the dominant energy condition: $R \geq 2|E|_g^2$, with $R - 2|E|_g^2 \in L^1(M_{ext})$. Then*

$$E_{ADM} \geq |Q| \tag{13}$$

and equality holds if and only if the the spacetime arising from the initial data contains a neighborhoood diffeomorphic to an open subset Majumbdar-Papapetrou spactime.

All of the above cases assumed time symmetry, $k = 0$. It turns out that the boundary Dirac operator becomes too complicated to get a useful result like theorem 2 if $k \neq 0$. However, we do have the following extension of Shi and Tam's result [15], which allows allows the positive mass theorem to include a manifold with *corners* in its hypothesis, along with multiple *asymptotically flat ends*, instead of just one. On this manifold, it is possible to define a *charged* energy density, μ_{EB}, and a *charged* momentum density, $J_{EM} = (J_{EM})^i e_i$ (in some orthornormal frame e_i defined on M^3). If each asymptotically flat end is indexed by j, one can define the ADM energy, $(E_{ADM})_j$, the electric and magnetic charge, Q_j^E and Q_j^B, and the spacelike vector portion of the ADM four-momentum, p_j. On each end the inequality relating these is summarized in the following:

Theorem 4. *Assume that* (M^3, g) *is a Riemannian 3-manifold containing* m *asymptotically flat ends, indexed by* j*. Define vector fields,* $E \in \Gamma(TM^3)$*,* $B \in \Gamma(TM^3)$*, and a symmetric two-tensor* k *on* M^3 *satisfying the following:*

(1). The charged energy density, $\mu_{EM} = R - |k|_g^2 + (tr_g k)^2 - 2|E|_g^2 - 2|B|_g^2$*, satisfies the dominant energy condition,* $\mu_{EM} \geq \sqrt{|J_{EM}|_g^2 + |div\ E|^2 + |div\ B|^2}$*.*

(2). There is a bounded domain $\Omega \subset M^3$ *such that we assume that* g *is continuous on* M^3*, smooth on* $M^3 - \Omega$ *and* $\bar{\Omega}$*, and Lipschitz near* $\partial\Omega$*. Let* e_n*,* $n = 2,\ 3$ *define an orthonormal frame for* $T(\partial\Omega)$*, and let* ν *denote the outer unit normal vector field on* $\partial\Omega$ *(directed towards* $M^3 - \bar{\Omega}$*).*

(3). We assume that $H|_{\bar{\Omega}} - |Tr_{\partial\Omega}(k_{\bar{\Omega}})| \geq H|_{M^3-\bar{\Omega}} + |Tr_{\partial\Omega}(k_{M^3-\bar{\Omega}})|$*,* $k_{\bar{\Omega}}(\nu, e_n) = k_{M^3-\bar{\Omega}}(\nu, e_n)$*, for* $n = 2,\ 3$*,* $B_{\bar{\Omega}}(\nu) = B_{M^3-\bar{\Omega}}(\nu)$*, and* $E_{\bar{\Omega}}(\nu) = E_{M^3-\bar{\Omega}}(\nu)$*.*

(4). The components, E^i *of* $E = E^i e_i$ *and* B^i *of* $B = B^i e_i$ *satisfy the following decay conditions:* $E^i \in o(r^{-1})$*,* $B^i = o(r^{-1})$*. Further,* $E,\ B,\ k \in W^{1,1}_{loc}(M^3)$ *and* $E,\ B,\ k$ *and their weak partials are bounded near* $\partial\Omega$*.*

Then

$$(E_{ADM})_j > \sqrt{|p_j|_g^2 + (Q_j^E)^2 + (Q_j^B)^2} \tag{14}$$

The mean curvatures, $H|_{\bar{\Omega}}$ and $H|_{M^3-\bar{\Omega}}$ with respect to $\bar{\Omega}$ and $M^3 - \bar{\Omega}$, the *charged* momentum density, J_{EM}, and the momentum, p_j, of the jth asymptotically flat end will be defined precisely in section 9 where this theorem is proved.

2 A Manifold with a Single Connected Boundary Component and the Inverse Mean Curvature Flow

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In this section, we prove theorem 1 using the Inverse Mean Curvature Flow.

Let A denote the area of ∂M and define the charge, given in Q, of ∂M, by

$$Q = \frac{1}{4\pi} \int_{\partial M} E(\nu) \tag{15}$$

Note: If $div\ E = 0$, then by the divergence theorem, this is equal to the total charge, Q, given by (4), as well as the charge, $\int_S E(\nu)$ on any closed two-surface $S \subset M^3$ that also bounds ∂M^3, (i.e. $\partial M^3 \subset M^3 - S$):

Assume that the scalar curvature, R, of M^3 satisfies the dominant energy condition, $R \geq 2|E|_g^2$, at each point of M^3. From now on, we will let H denote the mean curvature of any two-surface $S \subset M^3$ with respect to the outward pointing unit normal vector field ν, pointing toward spatial infinity. Let $A(S)$ denote the area of S.

In [10], Khuri and Disconzi introduced the charged Hawking Mass, defined on any closed two-surface S by:

$$M_{CH}(S) = \sqrt{\frac{A(S)}{16\pi}}(1 + \frac{4\pi Q^2}{A(S)} - \frac{1}{16\pi}\int_S H^2) \tag{16}$$

This charged Hawking mass is useful because it satisfies Geroch monotonicity (it is non-decreasing) under surfaces solving the inverse mean curvature flow (IMCF) (defined in [11]). In fact, if S_τ denotes a one-parameter family of surfaces solving the IMCF then (cf. inequality (4.1) in [10]), we have the following inequality:

$\frac{\mathrm{d}}{d\tau M_{CH}(S_\tau) \geq -\frac{1}{2}\sqrt{\frac{\pi}{A(S_\tau)}}Q^2 + \frac{1}{16\pi}\sqrt{\frac{A(S_\tau)}{16\pi}}\int_{S_\tau}(R + 2\frac{|\nabla_\tau H|^2}{H^2} + |II|^2 - \frac{1}{2}H^2)}$ where II denotes the second fundamental form on S_τ with respect to the induced metric and the unit normal vector (ν) pointing toward spatial infinity.

Since (cf. top of p.7 in [10]), $|II|^2 - \frac{1}{2}H^2 = \frac{1}{2}(\lambda_1 - \lambda_2)^2$, where λ_1 and λ_2 denote the two principal curvature of S_τ, the last three terms appearing in the second term of the inequality (2.18) will always be non-negative.

By the dominant energy condition $R \geq 2|E|_g^2$ and by Hölder's inequality (cf. inequality [4.2] in 10), $\int_{S_\tau}|E|_g^2 \geq \frac{16\pi^2 Q^2}{A(S_\tau)}$, and therefore $M_{CH}(S_\tau)$ is monotonically non-decreasing:

$$\frac{d}{d\tau}M_{CH}(S_\tau) \geq 0 \tag{17}$$

Further, like the Hawking mass, if S_τ denotes the one-parameter family of surfaces solving the inverse mean curvature flow, then, because, $A(S_\tau)$ grows exponentially in τ,

$$\lim_{\tau\to\infty} M_{CH}(S_\tau) = E_{ADM} \tag{18}$$

Therefore, if $M_{CH}(S_0) \geq C$ for some constant C, this inequality will extend to $E_{ADM} \geq C$. For an apparent horizon boundary, which in the time-symmetric case, corresponds to $H = 0$ on ∂M, Khuri and Disconzi [10] used (2.20) along with Geroch monotonicity (2.19) and the charged Jang transformation to prove the charged Riemannian Penrose inequality $E_{ADM} \geq \sqrt{\frac{A}{16\pi}} + \sqrt{\frac{\pi}{A}}Q^2$ assuming that the dominant energy condition is satisfied. Our goal here is to move slightly outside of the apparent horizon boundary, replacing the condition $H = 0$ in the time-symmetric case with an appropriate upper-bound for mean curvature. We derive this upper bound in the following:

Proposition 5. *Assume that the mean curvature H on S satisfies the following inequality:*

$$\frac{1}{2}|H| \leq |\sqrt{\frac{4\pi}{A(S)}} - \frac{4\pi|Q|}{A(S)}| \tag{19}$$

Then

$$M_{CH}(S) \geq |Q| \tag{20}$$

Proof. We start by rewriting the inequality, $M_{CH}(S) \geq |Q|$:

$$|Q| \leq \sqrt{\frac{A(S)}{16\pi}}(1 + \frac{4\pi Q^2}{A(S)} - \frac{1}{16\pi}\int_S H^2)) \tag{21}$$

Multiplying both sides of equation (2.23) by $\frac{16\pi}{A(S)}$, we obtain

$$\sqrt{\frac{16\pi}{A(S)}}|Q| \leq (1 + \frac{4\pi Q^2}{A(S)} - \frac{1}{16\pi}\int_S H^2) \tag{22}$$

Adding $\frac{1}{16\pi}\int_S H^2 - \sqrt{\frac{16\pi}{A(S)}}|Q|$ to both sides of the (2.24) yields:

$$\frac{1}{16\pi}\int_S H^2 \leq 1 - \sqrt{\frac{16\pi}{A(S)}}|Q| + \frac{4\pi Q^2}{A(S)} \tag{23}$$

Upon completing the square, $1 - \sqrt{\frac{16\pi}{A(S)}}|Q| + \frac{4\pi Q^2}{A(S)} = (1 - \sqrt{\frac{4\pi}{A(S)}}|Q|)^2$. Therefore, (2.25) becomes

$$\frac{1}{16\pi}\int_S H^2 \leq (1 - \sqrt{\frac{4\pi}{A(S)}}|Q|)^2 \tag{24}$$

If we multiply both sides of the above equation (2.26) by 16π, we obtain

$$\int_S H^2 \leq 16\pi(1 - \sqrt{\frac{4\pi}{A(S)}}|Q|)^2 \tag{25}$$

We have therefore shown that the inequality $M_{CH} \geq |Q|$ is equivalent to the inequality (2.27) above. Now, notice that if $H^2 \leq \frac{16\pi}{A(S)}(1-\sqrt{\frac{4\pi}{A(S)}}|Q|)^2$, then (2.27) will hold automatically.

The expression $\frac{16\pi}{A(S)}(1-\sqrt{\frac{4\pi}{A(S)}}|Q|)^2$ can be rewritten as $(\sqrt{\frac{16\pi}{A(S)}}-\sqrt{\frac{64\pi^2}{(A(S))^2}})^2$, which is equivalent to $(2\sqrt{\frac{4\pi}{A(S)}}-\frac{8\pi}{A(S)})^2 = [2(\sqrt{\frac{4\pi}{A(S)}}-\sqrt{\frac{4\pi}{A(S)}})]^2$. Taking the square root of both sides, the condition $H^2 \leq \frac{16\pi}{A(S)}(1-\sqrt{\frac{4\pi}{A(S)}}|Q|)^2$ is equivalent to $|H| \leq 2|\sqrt{\frac{4\pi}{A(S)}}-\frac{4\pi|Q|}{(A(S)}|$, or $\frac{1}{2}|H| \leq |\sqrt{\frac{4\pi}{A(S)}}-\frac{4\pi|Q|}{|A(S)|}|$, which is precisely the condition (2.21) □

Therefore, if we start an inverse mean curvature flow at the boundary, that is, if $S_0 = \partial M^3$, and if the mean curvature satisfies $\frac{1}{2}|H| \leq |\sqrt{\frac{4\pi}{A}}-\frac{4\pi|Q|}{A}|$ on ∂M^3, then $M_{CH}(S_0) \geq |Q|$ by proposition 2.5 and by Geroch monotonictiy (2.19) combined with (2.10), $E_{ADM} \geq |Q|$. If $E_{ADM} = |Q|$, then the spacetime is identical to a $t=0$ slice of an *Extreme Reissner-Nördstrom* spacetime, which is defined along the lines of [13] below:

Definition 6. Let $m > 0$ denote a positive constant and $p \in \mathbb{R}^3$ denote a point in space, and let r denote the Euclidean distance from $p \in \mathbb{R}^3$. Set $u = (1+\frac{|m|}{r})^{1/2}$. A time slice $t=0$ of the *Reissner-Nördstrom* spacetime is characterized by initial data consisting of a the topological manifold $M^3 = \mathbb{R}^3 - p$, on which we define the spatial metric

$$g = u^4(dr^2 + r^2 d\sigma^2) \tag{26}$$

and electric field

$$E = 2\nabla[\ln u] \tag{27}$$

This spatial metric will be a slice of the *Extreme Reissner-Nördstrom* spacetime precisely when $m = |Q|$.

It is important to note the following properties of extreme Reissner-Nördstrom spacetime given in the following proposition:

Proposition 7. *The second fundamental form II, charge, Q_r, and mean curvature, H of any coordinate sphere S_r in extreme Reisssner-Nördstrom spacetime are given by the following two formulas:*

$$II = \frac{1}{2}[r^2\frac{\partial_r u^4}{u^2} + 2ru^2]d\omega^2 \tag{28}$$

$$Q_r = -m = -|Q| \tag{29}$$

and

$$H(S_r) = \sqrt{\frac{4\pi}{A}} - \frac{4\pi|Q|}{A} \tag{30}$$

Proof. We calculate

$\mathrm{II} = \frac{1}{2L_{u^{-2}\partial_r} g = \frac{1}{2}(u^{-2}\partial_r(u^4)(dr^2+r^2d\omega^2)+u^4(2dr(d(u^{-2}))+2ru^{-2}d\omega^2)-2u^{-3}\partial_r u dr = \frac{1}{2}(u^{-2}\partial_r(u^4)(dr^2+r^2d\omega^2)+u^4(2dr(-2u^{-3}\partial_r u dr)+2ru}$
and therefore,

$\mathrm{H(S}_r) = Tr_{S_r} II = u^{-4}r^{-2}(r^2\frac{\partial_r u^4}{u^2} + 2ru]$
$= \frac{\partial_r u^4}{u^6} + r^{-1}u^{-3}$
$= \frac{4\partial_r u}{u^3} + \frac{2}{ru^2}$
Since $\partial_r u = \frac{1}{2}(1+\frac{m}{r})^{-1/2} \cdot \frac{-m}{r^2} = \frac{-m}{2ur^2}$, and so we have

$$H(S_r) = \frac{2}{r+m} - \frac{2m}{(r+m)^2} \tag{31}$$

Now, notice that, because $E = 2\nabla(\ln\ u)$ in extreme RN, we have the following formula for the normal component of the electric field:

$\mathrm{E}(\nu) = 2\nabla_{u^{-2}\partial_r}(\ln\ u)$
$= 2u^{-2}\frac{\partial_r u}{u}$
$= 2\frac{\partial_r u}{u^3}$
$= \frac{-m}{(r+m)^2}$
Now, notice that, with respect to the metric g, the area of S_r, $A(S_r)$, is

$$A(S_r) = \int_{S_r} r^2u^4d\omega^2 = 4\pi(r+m)^2 \tag{32}$$

and therefore the *flux* or *charge*, Q_r, through S_r is

$\mathrm{Q}_r = \frac{1}{4\pi}\int_{S_r} E(\nu) = (r+m)^2\frac{-m}{(m+r)^2} = -m$
Notice that this value does not depend on r, and in particular, $\lim_{r\to\infty} Q_r = Q = -m$, or $m = |Q|$ since $m > 0$.

Therefore, (2.33) becomes

$$H(S_r) = \sqrt{\frac{4\pi}{A}} - \frac{4\pi|Q|}{A} \tag{33}$$

$\square$

Lemma 8. *If $E_{ADM} = |Q|$, then the metric g arises as a time slice of an extreme Reissner-Nordström Spacetime. As a consequence (2.32) holds on each S_τ. Further, in this spacetime, inequality (2.19) is an equality for each coordinate sphere S_τ.*

Proof. $\mathrm{E}_{ADM} = |Q|$, then since $M_{CH}(S_\tau)$ is monotonically non-decreasing, we must have $M_{CH}(S_\tau) = |Q| = E_{ADM}$, for all τ, and therefore $\frac{d}{d\tau}M_{CH}(S_\tau) = 0$, so that (2.19) holds on each S_τ. It then follows from the inequality (2.17)-(2.18) that $\int_{S_\tau}(R+2\frac{|\nabla_\tau H|^2}{H^2}+|II|^2-\frac{1}{2}H^2) = 0$; since it was shown in [10] that $|II|^2 - \frac{1}{2}H^2$ is always non-negative, we must have:

$$\nabla_\tau H = 0 \text{ } on \text{ } S_\tau \tag{34}$$

and

$$|II|^2 = \frac{1}{2}H^2 \text{ } on \text{ } S_\tau \tag{35}$$

These two equations in turn imply that H and II are constant on S_τ. Further, equality in (2.18)-(2.19) implies that we must have

$$R = 2|E|_g^2 \tag{36}$$

and further that equality must hold in Hölder's inequality, so that $\int_{S_\tau}|E|_g^2 = \frac{16\pi Q^2}{A(S_\tau)}$. Therefore, we must have

$$E = f(\tau)\nu \tag{37}$$

for some smooth function $f(\tau)$, i.e. E (and therefore also R) is constant on each S_τ. Here, ν denotes the outer unit normal to S_τ pointing towards spatial infinity (cf. Equation (5.4) of [10]). By equation (1.3) in [11], we further have

$$\frac{\partial H}{\partial \tau} = -\Delta_{S_\tau}(H^{-1}) - (|II|^2 + Ric(\nu,\nu))H^{-1} \tag{38}$$

Since H is constant (and assumed to be nonzero in order for a smooth solution of IMCF to exist) on each S_τ, we have $\Delta_{S_\tau}(H^{-1}) = 0$ and therefore

$$\frac{\partial H}{\partial \tau} = -(|II|^2 + Ric(\nu,\nu))H^{-1} \tag{39}$$

Since both H and II depend only on τ, it follows that $Ric(\nu,\nu)$ is constant on each S_τ. A standard formula obtained by taking two traces of the Gauss equation allows us to write the Gauss curvature K of each S_τ as follows (cf. equation preceeding 5.5 in [10]):

$$K = \frac{1}{2}R - Ric(\nu,\nu) + \frac{1}{2}H^2 - \frac{1}{2}|II|^2 \tag{40}$$

Since every term on the right hand side of (2.42) is constant on S_τ, it follows that K is also constant on S_τ. Therefore, the induced metric on S_τ is identical to $r^2(\tau)d\sigma^2$ for some function $r(\tau)$ satisfying $4\pi r^2(\tau) = A(S_\tau) = A(S_0)e^\tau$ (cf. the equation following 1.1 of [11]). Using the Gauss Lemma and the fact that $d\tau = 2r^{-1}dr$, we can rewrite g as follows (cf. equation (5.5) of [10]):

$$g = \frac{4H^{-2}}{r^2}dr^2 + rd\sigma^2 \tag{41}$$

Now, note that equation (2.16) applied to S_τ (with $M_{CH} = |Q|$) can be rewritten as

$$\frac{1}{16\pi}\int_{S_\tau} H^2 = -\sqrt{\frac{16\pi}{A(S_\tau)}}|Q| + 1 + \frac{4\pi Q^2}{A(S_\tau)} \tag{42}$$

Since H^2 is constant on S_τ, and since $4\pi r(\tau)^2 = A(S_\tau)$, this equation becomes

$$H^2\frac{4\pi r^2(\tau)}{16\pi} = -\sqrt{\frac{16\pi}{4\pi r^2(\tau)}}|Q| + 1 + \frac{4\pi Q^2}{4\pi r^2(\tau)} \tag{43}$$

Solving (2.45) for H^2, and writing r for $r(\tau)$ yields

$$H^2 = \frac{4}{r^2}(-\frac{2|Q|}{r} + \frac{Q^2}{r^2} + 1) \tag{44}$$

which can be simplified further as

$$H^2 = \frac{4(r-|Q|)^2}{r^4} \tag{45}$$

Plugging the formula (2.47) for H into equation (2.43) for g gives

$$g = \frac{r^2}{(r-|Q|)^2}dr^2 + r^2d\sigma^2 \tag{46}$$

To get this into the form of a slice of standard Reissner-Nordström Spacetime, we make the coordinate transformation, $r = r' + |Q|$. Since $dr = dr'$, this turns equation (2.48) into

$$g = \frac{(r'+|Q|)^2}{(r')^2}dr'^2 + (r'+|Q|)^2d\sigma^2 \tag{47}$$

Rewriting $(r'+|Q|)^2$ as $(1+\frac{|Q|}{(r')^2})^2(r')^2$ and $\frac{(r'+|Q|)^2}{(r')^2} = (1+\frac{|Q|}{r'})^2$, equation (2.49) becomes

$$g = (1+\frac{|Q|}{r'})^2dr'^2 + r'^2d\sigma^2] \tag{48}$$

This is precisely equivalent to the induced spatial metric on the $t = 0$ slice of an extreme Reissner-Nordström Spactime with $u = (1 + \frac{|Q|}{r'})^{1/2}$ and $m = |Q|$. In this spacetime, the electric field is given by $E = 2\nabla[\ln u]$, where ∇ denotes the gradient operator acting on a function. In terms of the metric above, in this reads

$$E = \frac{-|Q|}{(r' + |Q|)^2} \frac{1}{1 + \frac{|Q|}{r'}} \partial_r \tag{49}$$

Now, notice from equation (2.39) that $E(\nu) = f(\tau)$ on S_τ in light of $A(S(\tau)) = 4\pi r^2(\tau)$, we have

$$4\pi Q = \int_{S_\tau} f(\tau) = 4\pi r^2(\tau) f(\tau) \tag{50}$$

from which we can solve for $f(\tau)$, (after making the replacement $r(\tau) = r' + |Q|$):

$$f(\tau) = \frac{Q}{(r' + |Q|)^2} \tag{51}$$

It then follows that $E = \frac{Q}{r'(\tau)+|Q|}\nu$ on S_τ. Since S_τ is simply a sphere of radius $r(\tau) = r' + |Q|$, the spatial infinity-pointing unit normal ν (with respect to the g given by (2.49)) is simply $\nu = \frac{1}{1+\frac{|Q|}{r'}} \partial_{r'}$. Hence the original electric field E defined on M^3 is identical to the one arising from the extreme Reissner-Nordström metric, with $Q = -|Q|$ (notice how this agrees with the point charge of an electron). Furthermore, equation (2.47) under the replacement of $r = r(\tau)$ with $r' + |Q|$ gives the following:

$$H = \frac{2(r')^2}{(r' + |Q|)^2} \tag{52}$$

On the other hand, note that since $A(S_\tau) = 4\pi(r' + |Q|)^2$, we have

$$\sqrt{\frac{4\pi}{A(S_\tau)}} - \frac{4\pi|Q|}{A(S_\tau)} = \frac{1}{r' + |Q|} - \frac{|Q|}{(r + |Q|)^2} \tag{53}$$

Combining the right-hand side of equation (2.55) as one fraction gives:

$$\sqrt{\frac{4\pi}{A(S_\tau)}} - \frac{4\pi|Q|}{A(S_\tau)} = \frac{(r')^2}{(r + |Q|)^2} \tag{54}$$

It thus follows that

$$\frac{1}{2}H = \sqrt{\frac{4\pi}{A(S_\tau)}} - \frac{4\pi|Q|}{A(S_\tau)} \tag{55}$$

on S_τ as claimed. □

We also need the following:

Lemma 9. *Assume that the boundary, ∂M^2, is outer minimizing. Then there exists a solution of the IMCF such that $A(S_0) = A(\partial M^3)$.*

Proof. This is simply a consequence of remark 1.4 in [18] (cf. theorem 1.2) and the the existence theorems for the inverse mean curvature flow given in [11] □

Note: If ∂M is outer minimizing, then $H \geq 0$, so absolute value sign appearing to the left hand side of inequality (2.21) is unnecessary (cf. Remark 1.2 of [18]).

We therefore have the following proof of theorem 1:

Proof. If we start an inverse mean curvature flow at the boundary, that is, if $S_0 = \partial M^3$, and if the mean curvature satisfies $\frac{1}{2}H \leq |\sqrt{\frac{4\pi}{A}} - \frac{4\pi|Q|}{A}|$ on ∂M^3, then $M_{CH}(S_0) \geq |Q|$ by proposition 2.5. As a consequence of Geroch monotonicity and the asymptotic behavior of the S_τ, it follows that $\lim_{\tau\to\infty} M_{CH}(S_\tau) \leq E_{ADM}$ (cf. Lemma 7.4 of [11]). This proves the lower bound on E_{ADM} in theorem 1. The rigidity result is then simply lemma 2.8. □

Note: In [17], Hawking-Horowitz and Perry have already shown that if ∂M is an apparent horizon, which in the time-symmetric case corresponds to $H \leq 0$, then $E_{ADM} \geq |Q|$, so the above theorem only gives a new result if both H and $|\sqrt{\frac{4\pi}{A}} - \frac{4\pi|Q|}{A}|$ are positive. Since $4\pi Q^2 < A$ when ∂M is an apparent horizon, this is not an unreasonable assumption.

Second Note: In [6], M. Herzlich proved that in the absence of charge ($E = 0$, $Q = 0$), that $\frac{1}{2}H \leq \sqrt{\frac{4\pi}{A}}$ implies the uncharged positive mass theorem: $E_{ADM} \geq 0$. His proof uses spinors and is the motivation for the proofs of the theorems 2 and 3 in the next sections.

3 Outline of the Proof of Theorem 2

We break the proof of theorem 2 down into three steps:

Step 1: We write the quantity $E_{ADM} - |Q|$ in terms of spinor fields. The Schrödinger-Lichnerowicz formula combined with the divergence theorem show that this expression is an integral involving a charged Dirac operator, the charged covariant derivative, and the function $R - 2|E|_g^2$ on the ambient manifold, along with a boundary integral involving a charged boundary Dirac operator and the mean curvature on each connected component of the boundary. This expression will be nonnegative if the spinor field satisfies the charged Dirac equation on the ambient manifold and if its restriction satisfies appropriate boundary conditions depending upon whether $\sqrt{\frac{4\pi}{A_j} + \frac{16\pi^2}{A_j^2} - \frac{1}{A_j}\int_{N_j} E(\nu)} - \frac{4\pi|Q_j|}{A_j}$. This is discussed in section 5. If $\sqrt{\frac{4\pi}{A_j}+\frac{16\pi^2}{A_j^2}-\frac{1}{A_j}\int_{N_j}E(\nu)}-\frac{4\pi|Q_j|}{A_j}$ is positive. All of this is explained in section 4.

Step 2: If $\sqrt{\frac{4\pi}{A_j}+\frac{16\pi^2}{A_j^2}-\frac{1}{A_j}\int_{N_j}E(\nu)}-\frac{4\pi|Q_j|}{A_j}>0$, we use an argument of Hijazi-Bar as in [1] to show that, assuming the inequality (11) holds for $E(\nu)$ on each horizon, the absolute value of the smallest eigenvalue of the boundary Dirac operator on each N_j is greater than or equal to $\sqrt{\frac{4\pi}{A_j}+\frac{16\pi^2}{A_j^2}-\frac{1}{A_j}\int_{N_j}E(\nu)}-\frac{4\pi|Q_j|}{A_j}$. This is discussed in section 5. If $\sqrt{\frac{4\pi}{A_j}+\frac{16\pi^2}{A_j^2}-\frac{1}{A_j}\int_{N_j}E(\nu)}-\frac{4\pi|Q_j|}{A_j}$, then condition (11) will also imply that $H\leq 0$, so that we can use the boundary conditions imposed by Hawking, Horowitz, and Perry in [17] on N_j to conclude that the boundary term is nonnegative.

Step 3: Assuming the given upper bound on mean curvature for each component, N_j (inequality (11)), we prove the existence of a spinor Ψ satisfying the charged Dirac equation whose restriction to each component satisfies the appropriate boundary conditions. This is discussed in the section 6. Section 7 then applies the nontrivial solution to the Dirac equation with the results of the preceding sections to actually prove the positive mass theorem.

4 Definition of a Manifold with Finitely Many Cylindrical Ends

In this section, we consider the case where M^3 consists of one asymptotically flat end and finitely many cylindrical ends. This is of interest in mathematical general relativity because cylindrical ends can also be used to model a horizon; objects *stretch* as they enter a black hole. Let M_{ext} denote a subset of M^3 whose induced metric satisfies the decay rates of (1) and (2). We consider a Riemannian 3-manifold, M^3 containing one asymptotically flat end, M_{ext} and multiple cylindrical ends, which are defined below along the lines of [9]:

Definition 10. A manifold (M^3,g) is said to have m *cylindrical ends* if there exists m Riemannian 2-manifolds (S_j,g_{N_j}), a compact subset $K\subset M^3$, a real number $r_2>0$, $j=1,2,...,m$, and a diffeomorpshism $\Phi:\cup_{i=1}^m S_j\times(r_2,\infty)\to M^3-(M_{ext}\cup K)$ such that the pull-back metric Φ^*g on each $N_j\times(r_2,\infty)$ satisfies, for all $l\geq 0$,

$$|\nabla_j^l(\Phi^*g-[g_{S_j}+dr^2])|=O(\frac{1}{r}) \tag{56}$$

where each ∇_j^l denotes the Levi-Cevita connection on each $S_j\times(r_2,\infty)$ with respect to the product metric $g_{S_j}+dr^2$. We will denote the subset of M^3 that is diffeomorphic to $S_j\times(r_2,\infty)$ by M_j.

For any $r>r_2$, consider the subset $M_j(r)=\Phi(S_i\times(r,\infty))$ of M^3 consisting of everything of length greater than r down the jth cylindrical end. We will use $\partial M_j(r)=\partial\Phi(S_j\times(r,\infty))$ to denote the corresponding cross-section.

5 The charged Dirac Operator, Lichnerowicz Formula, and ADM Energy

To prove theorems 2 and 3, we first need to define spinors and the charged Dirac Operator. As outlined in [7], we will rewrite the ADM mass in terms of spinors. Using the Lichnerowicz formula and divergence theorem, this expression will become a volume integral minus a boundary integral. The volume integral will be nonnegative if a spinor satisfies the *charged* Dirac equation. The boundary integral will consist of a *charged* boundary Dirac operator minus a multiple of the mean curvature, and we will see using eigenvalue estimates and other appropriate boundary conditions that this term will be nonnegative if the inequalities (10) and (11) are satisfied.

5.1 The Charged Connection and Super-Covariantly Constant Spinors

Since any Riemannian 3-manifold is spin, we can choose a spin structure on M^3 (cf. chapter 1 [1] for details). This spin structure then defines a spin structure on the spacetime N^4 that arises after solving the Einstein-Maxwell equations with (M, g, E) as initial data (cf. [3]). Let S denote the spinor bundle over N^4 restricted to M^3 and let $\Gamma(S)$ denote sections of S. Elements of $\Gamma(S)$ are called *spinor fields*. Now, choose some orthonormal frame, e_i near an arbitrary $p \in M$. e_0 will denote the tangent vector in N^4 perpendicular to M^3 in the direction of forward time. Let D_i denote the covariant derivative (induced from the Levi-Cevita covariant derivative on TM^3) acting on spinor fields in the direction of e_i (cf Proposition 1.2.3d of [1]). Further, let $c : TN^4 \to End(\Gamma(S))$ denote the standard Clifford representation of the tangent bundle of N^4 When an electric field, $E \in \Gamma(TM^3)$ is defined on M^3, we can define the *charged* or *super* covariant derivative $\nabla_i : \Gamma(S) \to \Gamma(S)$ in the direction of e_i as follows:

$$\nabla_i = D_i - \frac{1}{2}E^k c(e_k)c(e_i)c(e_0) \tag{57}$$

We are now in the position to define the notion of a super-covariantly constant spinor field and the Dirac operator:

Definition 11. A section of $\psi \in \Gamma(S)$ is called a *super-covariantly* constant spinor field if for *almost every* $p \in M$ (that is for all p on M outside of a subset of measure zero) and the given orthonormal frame, e_i near p, we have

$$\nabla_i \psi = 0 \tag{58}$$

This is often just written

$$\nabla \psi = 0 \tag{59}$$

5.2 The Dirac Operator

Let e_i denote a local orthonormal frame e_i on TM^3.

Definition 12. The classical Dirac operator, denoted $\not{D}$ is defined by the following equation:

$$\not{D}\psi = c(e_i)D_i\psi \tag{60}$$

This definition is actually independent of the choice of local orthonormal frame e_i on TM^3 (cf. page 10 of [1]) The *charged* Dirac operator is defined using the *charged* covariant derivative ∇_i as follows:

Definition 13. Let e_i be an orthonormal frame around a given point $p \in M^3$. The charged Dirac operator, denoted $D_M : \Gamma(S) \to \Gamma(S)$ is a first order differential operator acting on spinor fields defined by the equation:

$$D_M\psi = c(e_i)\nabla_i\psi \tag{61}$$

Note also, that, because of the definition of the *charged* connection, ∇, the formula for the *charged* Dirac operator reduces to:

$$D_M\psi = \not{D} - \frac{1}{2}c(E)c(e_0) \tag{62}$$

where $\not{D} : \Gamma(S) \to \Gamma(S)$ is the classical Dirac operator given by definition 5.2

5.3 The Schrödinger-Lichnerowicz Formula and ADM Energy

For the purposes of proving the positive mass theorem, we need to show that $E_{ADM} - |Q|$ can be written in terms of integrals whose integrands consist of inner products of spinors. First, we have the Schrödinger-Lichnerowicz formula [equation 3.3.43 of [10]], which we summarize in the following lemma:

Lemma 14. *Assume that the vector field E has vanishing divergence, div $E = 0$, then the* Schrödigner-Lichnerowicz *formula reads:*

$$D_i < \Psi, \nabla^i\Psi + c(e_i)D_M\Psi >= |\nabla\Psi|^2 - |D_M\Psi|^2 + \frac{1}{4}(R - 2|E|_g^2)|\Psi|^2 \tag{63}$$

Note: This formula shows that the square of the Dirac operator $|D_M\Psi|^2$ is, aside from a divergence term, the square of the conformal laplacian, $|\nabla\Psi|^2$, plus a zeroth-order term (a function that is merely multiplied by the spinor field and does involve any covariant derivatives of the spinor field), namely, $\frac{1}{4}(R - 2|E|_g^2)$. This charged Dirac operator is thus, modulo addition of zeroth-order term (the function $\frac{1}{4}(R - |E|_g^2)$, a formal half-iterate of the conformal laplacian, $\nabla^*\nabla$, at least in the weak sense when integrating by parts over a manifold without boundary. Pauli Dirac originally conceived his operator as an observable producing the rest energy (mc^2), whose square needed to satisfy Einstein's equation $E^2 - |p|^2 = m^2c^4$, where $|p|$ is the absolute value of the total momentum (in quantum mechanics, the operator for the observable $E^2 - |p|^2$ is the laplacian in Minkowski spacetime, sometimes called the d'Alembert operator and denoted by $\Box$).

In the remainder of this subsection, we will assume that Ψ is constant in a chart around spatial infinity, say Ψ is identically ψ_∞ in some frame in a neighborhood of M_{ext}; assume further that $|\psi_\infty|^2 = 1$. Then a standard result, (cf. chapter 3, page 114 of [7]) shows that:

$$\lim_{r\to\infty} \int_{S(0,r)} < \Psi, (D^i + c(e^i)\not{D})\Psi >= 4\pi E_{ADM} \tag{64}$$

If we further assume that ψ_∞ satisfies, $< \psi_\infty, c(e_0)\psi_\infty >= -sgn(Q)|\psi_\infty|^2 = -sgn(Q)$, then we have the following:

$$\lim_{r\to\infty} \int_{S(0,r)} < \Psi, E^i c(e_0)\Psi > dS^i = -4\pi|Q| \tag{65}$$

Now, set $U^i =< \Psi, \nabla^i\Psi + c(e_i)D_M\psi >$. We can then conclude from lemma 5.4 [cf. 3.3.46 of [7]] that

$$\lim_{r\to\infty} \int_{S(0,r)} U^i dS^i = 4\pi(E_{ADM} - |Q|) \tag{66}$$

If M^3 contains an inner boundary consisting of multiple disjoint components as in the assumptions of theorem 2, then applying the divergence theorem to equation 5.7 in Lemma 5.4, we obtain:

Corollary 15. $\int_{M^3} |\nabla\Psi|^2 - |D_M\Psi|^2 + \frac{1}{4}(R - 2|E|_g^2)|\Psi|^2$
$- \Sigma_{j=1}^m \int_{N_j} < \Psi, (D^i + c(e^i)\not{D} + E^i c(e_0))\Psi > dS^i$
$= 4\pi(E_{ADM} - |Q|)$

If instead M^3 has finitely many cylindrical ends and one asymptotically flat end as in the assumptions of theorem 3, then the boundary term is replaced with limit of Ψ along $\partial M_i(r)$:

Corollary 16. $\int_{M^3} |\nabla\Psi|^2 - |D_M\Psi|^2 + \frac{1}{4}(R - 2|E|_g^2)|\Psi|^2$
$- \lim_{r\to\infty}[\Sigma_{j=1}^m \int_{\partial M_j(r)} < \Psi, (D^i + c(e^i)\not{D} + E^i c(e_0))\Psi > dS^i]$
$= 4\pi(E_{ADM} - |Q|)$

Both corollary 5.5 and 5.6 demonstrate that $E_{ADM}-|Q|$, or more specifically, $4\pi(E_{ADM}-|Q|)$, can be rewritten in terms integrals whose integrands consist of inner products of spinor fields. It is worth noting that the term inside the limit in corollary 5.6 will necessarily vanish if Ψ is a compactly supported spinor field.

5.4 The *charged* Boundary Dirac Operator

To deal with the boundary boundary terms $\int_{N_j} < \Psi, (D^i + c(e^i)\not{D} + E^i c(e_0))\Psi > dS^i$ in the Corollary 5.5, we first introduce, on each N_j, the *classical boundary Dirac Operator* $\not{D}_{N_j}$. Defining this requires defining the *boundary spin connection*:

Definition 17. The boundary spin connection ${}^{(2)}\nabla_A$ on N_j is defined by [c.f. 5, p. 3],

$$ {}^{(2)}\nabla_X \psi = \partial_X \psi + \frac{1}{4}\Gamma^k_{il} c(e_l)c(e_k)\psi \tag{67} $$

where Γ^k_{il}, $i, l, k = 2, 3$ denote the Christoffel Symbols on (N_j, g).

$$ \not{D}_{N_j} = -c(\nu_j)c(e^A)^{(2)}\nabla_A \tag{68} $$

where ${}^{(2)}\nabla_A$ denotes the covariant derivative on N_j inherited from the metric on M^3, ν_j denotes the outer unit normal vector field on each respective N_j (pointing toward the asymptotically flat end, M_{ext}), and where A ranges from 2 to 3, e^2 and e^3 being chosen as the two orthonormal unit vectors fields in $\Gamma(TN_j)$. These vector fields are chosen so that they agree with the orthonormal frame e_i mentioned in section 5.1, i.e. $e_1 = \nu_j$ on N_j when we choose the point p to live on N_j, i.e. $p \in N_j$. We then have the following:

Lemma 18. *Let Ψ denote a harmonic spinor, $D_M\Psi = 0$, then, on each component N_j of ∂M^3.*

$$ (D^1 + c(e^1)\not{D} + E^1 c(e_0))\Psi = (-\not{D}_{N_j} + E(\nu)c(e_0) + \frac{1}{2}H)\Psi \tag{69} $$

where H denotes the mean curvature of N_j with respect to the outer normal vector field ν_j.

Proof. This is a simple calculation; in [5] and [6] it has already been shown that $D^1 + c(e^1)\not{D} = -\not{D}_{N_j}$ if $D_M\Psi = 0$. $\square$

On each N_j, define the *charged* boundary Dirac operator D_{N_j} as

$$ D_{N_j} = \not{D}_{N_j} - E(\nu)c(e_0) \tag{70} $$

Note: D_{N_j} is self-adjoint.

Then equation (5.13) in Lemma 5.8 becomes

$$(D^1 + c(e^1)\not{D} + E^1 c(e_0))\Psi = (-D_{N_j} + \frac{1}{2}H)\Psi \tag{71}$$

Applying Lemma 5.8 to Corollary 5.5, we obtain

Corollary 19. $\int_{M^3} |\nabla\Psi|^2 - |D_M\Psi|^2 + \frac{1}{4}(R - 2|E|_g^2)|\Psi|^2$
$+ \Sigma_{j=1}^m [\int_{N_j} < \Psi, D_{N_j}\Psi > - \int_{N_j} \frac{1}{2}H|\Psi|^2]$
$= 4\pi(E_{ADM} - |Q|)$

Note: Assume for now that each D_{N_j} has only positive eigenspinors; we will denote the lowest such eigenspinor by λ_{1j} and assume further that Ψ solves the *charged* Dirac equation, $D_M\Psi = 0$. Let $\Psi = a^{nj}\Psi_{nj}$, $n \geq 1$, denote the decomposition of Ψ along the eigenspaces of D_{N_j} corresponding to the eigenvalues λ_{nj}. Since the eigenspinors of a self-adjoint elliptic operator are orthogonal, we have:

$\Sigma_{j=1}^m [\int_{N_j} < \Psi, D_{N_j}\Psi > - \int_{N_j} \frac{1}{2}H|\Psi|^2]$
$= \Sigma_{j=1}^m [\lambda_{nj}(a^{nj})^2 \int_{N_{nj}} |\Psi_{nj}|^2 - \int_{N_j} \frac{1}{2}H|\Psi|^2]$
$\geq \Sigma_{j=1}^m [\lambda_{nj} - \frac{1}{2}\sup_{N_j} H](a^{nj})^2 \int_{N_j} |\Psi_{nj}|^2$
It follows that the term $\Sigma_{j=1}^m [\int_{N_j} < \Psi, D_{N_j}\Psi > - \int_{N_j} \frac{1}{2}H|\Psi|^2]$ in Corollary 5.9 will be nonnegative if $\frac{1}{2}H \leq \lambda_{1j}$ on each N_j. In the following section, we derive a lower bound for the eigenvalue λ_{1j} assuming that $\sqrt{\frac{4\pi}{A_j} + \frac{16\pi^2 Q_j^2}{A_j^2} - \frac{1}{A_j}\int_{N_j}(E(\nu))^2} - \frac{4\pi|Q_j|}{A_j} > 0$ (which actually amounts to the inequality (9))

6 A Lower Eigenvalue Bound for the *Charged* Boundary Dirac Operator

Consider the charged Dirac operator defined in section 5.4 (equation (5.12))

$$D_{N_j} = \not{D}_{N_j} - E(\nu)c(e_0) \tag{72}$$

Our goal is to find a lower bound for the eigenvalue λ_{1j} of this operator assuming $\sqrt{\frac{4\pi}{A_j} + \frac{16\pi^2 Q_j^2}{A_j^2} - \frac{1}{A_j}\int_{N_j}(E(\nu}$
$\frac{4\pi|Q_j|}{A_j} > 0$. Note that this inequality is actually equivalent to $\int_{N_j}(E(\nu))^2 < 4\pi$, which is actually equivalent to the inequality $\int_{N_j}(E(\nu))^2 < 4\pi$, which is (9).

Note: For the purposes of proving the charged positive mass theorem, what we really care about is the *restriction* of the spinor bundle on the spacetime N^4, first to M^3 and then to the boundary of M^3, whose components are N_j. However, because of the discussion in

[3], a spin structure M^3 will determine a spin structure on N^4 and likewise a spin structure ∂M^3, which amounts to a spin structure on each N_j, will determine a spin structure on M^3. Therefore, it is sufficient to consider the spin structures on each N_j and in this section S will denote the spinor bundle on each N_j, and $\Gamma(S)$ will denote its respective sections. $c : TN_j \to End(\Gamma(S))$ will denote the standard representation of the Clifford Algebra of TN_j

6.1 The Relationship Between Lower Eigenvalues of the Charged Boundary Dirac Operator and those of a Dirac-Schrödinger Operator

A *Dirac-Schrödinger operator* is simply a Dirac operator plus a function. In this subsection, we prove that analyzing the eigenvalues of D_{N_j} reduces to analyzing the eigenvalues of either the Dirac-Schrödinger operator $\not{D}_{N_j} - E(\nu)$ or the Dirac-Schrödinger operator $\not{D}_{N_j} + E(\nu)$.

Lemma 20. *Assume that $c(e_0)\psi + \psi$ is not identically zero on N_j and let ψ be an eigenspinor of either $D_{N_j} = \not{D}_{N_j} - E(\nu)c(e_0)$ with eigenvalue λ. Then λ is also an eigenvalue of $\not{D}_{N_j} - E(\nu)$.*

Proof. Note first that $c(e_0)$ commutes with ${}^{(2)}\nabla_A$ for $A = 2,\ 3$ and anti-commutes with $c(e_A)$ for $A = 2,\ 3$. For any spinor field $\psi \in \Gamma(S)$, it follows that

$$\not{D}_{N_j}(c(e_0)\psi) = c(e_0)\not{D}_{N_j}\psi \tag{73}$$

so that $c(e_0)$ commutes with $\not{D}_{N_j}$.

Now, let I denote the identity endomorphism on the spinor bundle, sending any section $\psi \in \Gamma(S)$ to itself, i.e. $I\psi = \psi$. Let λ be an eigenvalue of $\not{D}_{N_j} - E(\nu)c(e_0)$. From the relation $(I + c(e_0))c(e_0) = I + c(e_0)$, it follows that
$[\not{D}_{N_j} - E(\nu)]((I + c(e_0))\psi) = \not{D}_{N_j}[(I + c(e_0))\psi] - E(\nu)(I + c(e_0))\psi$
$= \not{D}_{N_j}[(I + c(e_0))\psi] - [E(\nu)(I + c(e_0))c(e_0)]\psi$
$= (I + c(e_0))[\not{D}_{N_j} - E(\nu)c(e_0)]\psi$
$= (I + c(e_0))D_{N_j}\psi$
$= \lambda(I + c(e_0))\psi$

Hence λ is also an eigenvalue of $[\not{D}_{N_j} - E(\nu)]$ with eigenspinor $(I + c(e_0))\psi$, which is not identically zero by the assumption.

□

Note that if $c(e_0)\psi = -\psi$, then the eigenvalue equation $\not{D}_{N_j}\psi - E(\nu)c(e_0)\psi = \lambda\psi$ becomes

$$[\not{D}_{N_j} + E(\nu)]\psi = \lambda\psi \tag{74}$$

Therefore, in this case, λ will also be an eigenalue of the Dirac-Schrödinger operator, $\not{D}_{N_j} + E(\nu)$.

6.2 Hijazi-Bär Argument

We follow an argument of Hijazi-Bär [Theorem 3.3.1 and Corollary 3.3.2 of [1]] to obtain a lower bound for the eigenvalue of this operator in terms of the area of N_j, the charge Q_j, and the average value of $(E(\nu))^2$

First, we define the Penrose operator on the spinor bundle.

Definition 21. The *Penrose operator* $P : \Gamma(S) \otimes \Gamma(TN_j) \to \Gamma(S)$ is the map given by

$$P_X\psi =^{(2)} \nabla_X\psi + \frac{1}{2}c(X)\not{D}_{N_j}\psi \ for\ all\ X \in \Gamma(TN_j) \tag{75}$$

Let λ be an eigenvalue of $\not{D}_{N_j} - E(\nu)$ with corresponding eigenspinor ψ. From the eigenvalue equation

$$\not{D}_{N_j}\psi = (\lambda + E(\nu))\psi \tag{76}$$

we have

$$\int_{N_j} |\not{D}_{N_j}\psi|^2 = \int_{N_j} (\lambda + E(\nu))^2|\psi|^2 \tag{77}$$

In [1] (Chapter 3, equation 3.3), it has been shown that

$$\int_{N_j} |\not{D}_{N_j}\psi|^2 = \int_{N_j} 2|P\psi|^2 + K|\psi|^2 \tag{78}$$

where K is the Gauss curvature of (N_j, g), from which it follows that

$$\int_{N_j} 2|P\psi|^2 + K|\psi|^2 = \int_{N_j} (\lambda + E(\nu))^2|\psi|^2 \tag{79}$$

Now, let $(N_j, \bar{g})$ denote the boundary component N_j with a conformally changed metric, $\bar{g} = e^{2u}g$ for some function or some function $u \in C^\infty(N_j)$. Notice that there is an isometry of the two tangent bundles $T(N_j, g)$ and $T(N_j, \bar{g})$ defined simply by sending $X \in T(N_j, g)$ to $e^{-u}X \in T(N_j, \bar{g})$ ([1], p. 18). This isometry induces a principal-bundle isomorphism from $SO_g(TN_j)$ to $SO_{\bar{g}}(TN_j)$ that lifts to an isomorphism from the spinor bundle S (associated to $SO_g(TN_j)$) to the spinor bundle $\bar{S}$ (associated to $SO_{\bar{g}}(TN_j)$). For a section, $\phi \in \Gamma(S)$, its image under this isomorphism will be denoted by $\bar{\phi} \in \Gamma(\bar{S})$. Notice that there is, up to equivalence, one representation, $\bar{c} : T(N_j.\bar{g}) \to \bar{S}$ ([1], proposition 1.2.3 part (b)) of the

Clifford Algebra of TN_j on the spinor bundle $\bar{S}$.

Let $\bar{\not{D}}_{N_j}$ denote the Dirac operator and connection on N_j with the conformally equivalent metric $e^{2u}g$. In particular, we choose an eigenspinor ϕ of $\not{D}_{N_j} - E(\nu)$ with eigenvalue λ (i.e. $\not{D}_{N_j}\phi = (\lambda + E(\nu))\phi$) and set $\bar{\psi} = e^{-\frac{u}{2}}\bar{\phi}$. From the conformal covariance of the Dirac Operator (c.f. [1] proposition 1.3.10), we have

$$\bar{\not{D}}_{N_j}(\bar{\psi}) = e^{-u}(\lambda + E(\nu))\bar{\psi} \tag{80}$$

Further, let $\bar{P}$ denotes the Penrose operator acting on $\Gamma(\bar{S}) \otimes \Gamma(TN_j))$, i.e.

$$P_X\psi =^{(2)} \bar{\nabla}_X\bar{\psi} + \frac{1}{2}\bar{c}(X)\bar{\not{D}}_{N_j}\psi \ for\ all\ X \in \Gamma(T\partial M) \tag{81}$$

Then equation (5.13) becomes

$$\int_{N_j} 2|\bar{P}\bar{\psi}|^2 + \bar{K}|\bar{\psi}|^2 = \int_{N_j} e^{-2u}(\lambda + E(\nu))^2|\bar{\psi}|^2 \tag{82}$$

from which we derive the inequality

$$\int_{N_j} e^{-2u}(\lambda + E(\nu))^2|\bar{\psi}|^2 \geq \int_{N_j} \bar{K}|\bar{\psi}|^2 \tag{83}$$

Since a nonzero spinor field must remain nonzero on a set of positive measure on N_j, there must exist a $p \in N_j$ such that

$$e^{-2u}(\lambda + E(\nu))^2(p) \geq \bar{K}(p) \tag{84}$$

or equivalently

$$(\lambda + E(\nu))^2(p) \geq \bar{K}e^{2u}(p) \tag{85}$$

Notice that in dimension 2, $\bar{K}e^{2u} = K + \Delta u$, so that we get

$$(\lambda + E(\nu))^2(p) \geq [K + \Delta u](p) \tag{86}$$

or

$$\lambda^2 \geq [K + \Delta u - 2\lambda E(\nu) - (E(\nu))^2](p) \tag{87}$$

Setting

$$\Delta u = \frac{1}{A_j}\int_{N_j} K - K - 2\lambda\frac{1}{A_j}\int_{N_j} E(\nu) + 2\lambda E(\nu) + \frac{1}{A_j}\int_{N_j}(E(\nu))^2 - [E(\nu)]^2 \qquad (88)$$

(which is possible by Fredholm's alternative since the R.H.S. has vanishing integral), we obtain,

$$\lambda^2 \geq \frac{1}{A_j}\int_{N_j} K - 2\lambda\frac{1}{N_j}\int_{N_j} E(\nu) - \frac{1}{N_j}\int_{N_j}(E(\nu))^2 \qquad (89)$$

By Gauss-Bonnet Theorem,

$$\frac{1}{A_j}\int_{N_j} K = \frac{4\pi}{A_j} \qquad (90)$$

so that inequality 6.23 becomes

$$\lambda^2 \geq \frac{4\pi}{A_j} - 2\lambda\frac{1}{A_j}\int_{N_j} E(\nu) - \frac{1}{A_j}\int_{N_j}(E(\nu))^2 \qquad (91)$$

Since $\frac{1}{A_j}\int_{N_j} E(\nu) = \frac{4\pi Q_j}{A}$, this becomes, after completing the square,

$$(\lambda + \frac{4\pi Q_j}{A_j})^2 \geq \frac{4\pi}{A_j} + \frac{16\pi^2 Q_j^2}{A_j^2} - \frac{1}{A_j}\int_{N_j}(E(\nu))^2 \qquad (92)$$

We can assume without loss of generality that $Q_j > 0$ and $\lambda > 0$. Assume further that $E(\nu)$ satisfies the inequality (10), which is equivalent to $\frac{1}{A_j}\int_{N_j}(E(\nu))^2 < \frac{4\pi}{A_j} + \frac{16\pi^2 Q_j^2}{A_j^2}$. Then taking the square root of both sides of the above inequality yields

$$\lambda \geq \sqrt{\frac{4\pi}{A_j} + \frac{16\pi^2 Q_j^2}{A_j^2} - \frac{1}{A_j}\int_{N_j}(E(\nu))^2} - \frac{4\pi|Q_j|}{A_j} \qquad (93)$$

The right hand side of inequality 6.27 will be non-negative precisely when $E(\nu)$ satisfies

$$\frac{1}{A_j}\int_{N_j}(E(\nu))^2 < \frac{4\pi}{A_j} \qquad (94)$$

which is precisely equivalent to the inequality (9).

Note: By Lemma 6.1, (6.27) will be a lower bound on the eigenvalue of $D_{N_j} - E(\nu)c(e_0)$ if the eigenspinor, ψ, satisfies $\psi + c(e_0)\psi$ is not identically zero on N_j. If $c(e_0)\psi = -\psi$, then, ψ is actually an eigenspinor of the Dirac-Schrödinger operator, $\not{D}_{N_j} + E(\nu)$. From the same argument (just with $E(\nu)$ replaced with $-E(\nu)$ and the assumption that $Q_j < 0$), the (5.27) is also a lower bound of an eigenvalue of $\not{D}_j + E(\nu)$. Hence (6.27) will be a lower bound on *any* eigenvalue of $D_{N_j} = \not{D}_{N_j} - E(\nu)c(e_0)$. This is summarized in this following theorem:

Theorem 22. *Assume that $E(\nu)$ satisfies the equality $\frac{1}{A_j}\int_{N_j} E(\nu) < \frac{4\pi}{A_j}$. Then any eigenvalue λ of $D_{N_j} = \not{D}_{N_j} - E(\nu)c(e_0)$ satisfies:*

$$|\lambda| \geq \sqrt{\frac{4\pi}{A_j} + \frac{16\pi^2 Q_j^2}{A_j^2} - \frac{1}{A_j}\int_{N_j} (E(\nu))^2} - \frac{4\pi|Q_j|}{A_j} \tag{95}$$

7 Solutions to the charged Dirac Equation

In this section, we solve the charged Dirac equation subject to natural boundary conditions in order to make the terms on the right hand side of the equation in corrollaries 5.6 and 5.9 positive. In corollary 5.9 this involves assuming that the mean curvature satisfies the upper bound (11). Let P_{j-} denote the L^2 projection onto the space of eigenspinors of D_{N_j} on N_j with negative eigenvalues. We wish to solve the equation

$$D_M\psi = 0$$

subject to the the following boundary conditions:

If $\int_{N_j} E(\nu) < 4\pi$, then

$$P_{j-}\psi = 0 \text{ on } N_j \tag{97}$$

(we think of P_{j-} as acting on ψ restricted to N_j). All this means is that the spinor field ψ, when restricted to each N_j and decomposed into eigenspinors of D_{N_j} will have terms consisting only of eigenspinors of D_{N_j} corresponding to *positive* eigenvalues. Solving the charged Dirac equation is done along the lines of pp. 117-120 of [7] and proposition 2.5 in [6] using the Riesz Representation Theorem/Lax-Milgrim Theorem.

If $\int_{N_j} E(\nu) \geq 4\pi$, then inequality (11) implies that $H \leq 0$ on N_j. Hence, we impose the boundary condition given in [17]:

$$c(e_1)c(e_0)\psi = \psi \ on \ N_j \tag{98}$$

In fact, assuming that the electric field of each N_j satisfies inequality (10) and that the mean curvature H of each N_j satisfies (11), it suffices to show that the bilinear form B defined by $B(\psi,\phi) = \int_{M^3} < D_{M^3}\psi, D_{M^3}\phi >$ is coercive (cf. Lemma 2.4 of [6]). Provided that B is defined on a Hilbert space containing smooth compactly supported spinors satisfying (7.2) or (7.3) on each N_j as a dense subset, we have from Corollary 5.9,
$\int_{M^3} |D_M\psi|^2 = \int_{M^3} |\nabla\psi|^2 + \frac{1}{4}(R - 2|E|_g^2)|\psi|^2$
$+ \Sigma_{j=1}^m[\int_{N_j} < \psi, D_{N_j}\psi > - \int_{N_j} \frac{1}{2}H|\psi|^2]$

The second term on the right-hand side is non-negative from the dominant energy condition. If the normal component of E satisfies (9) (or equivalently (5.28)) on N_j, then the boundary condition (6.2) will be introduced, so that condition (11) on the mean curvature of N_j together with theorem 6.3 will imply that the term $\int_{N_j} < \psi, D_{N_j}\psi > - \int_{N_j} \frac{1}{2}H|\psi|^2$ will be non-negative. If $\int_{N_j} E(\nu) \geq 4\pi$, then boundary condition (7.3) will be imposed. Now, note that $H \leq 0$ on N_j implies that N_j is an apparent horizon (notice that $k = 0$ means there is no distinction between a past or future apparent horizon). From [17] (cf. equation (28) and the following paragraph), it then follows that the boundary integral, $\int_{N_j} < \psi, [D_{N_j} - \frac{1}{2}H]\psi >$ will vanish. It thus follows that

$$\int_{M^3} |D_M\psi|^2 \geq \int_{M^3} |\nabla\psi|^2 \tag{99}$$

(c.f. lemma 2.4 of [6]).

Since we are operating under the assumption that M^3 admits no super-covariantly constant spinors, $\int_{M^3} |\nabla\psi|^2$ will define a Hilbert norm on the Hilbert space and we have thus shown that the given form is coercive.

Along the lines of proposition 2.5 of [6] and theorem 3.2.1 of [7], we then have the following:

Theorem 23. *Let M^3 denote an asymptotically flat Riemannian 3-manifold whose interior boundary consists of finitely many disjoint components, N_j, $j = 1, 2, ...m$, and assume that the electric field E defined on M^3 satisfies the dominant energy condition $R \geq 2|E|_g^2$ as well as the inequalities (10) and (11) on each N_j. Let ψ_∞ denote a smooth spinor field on M^3 satisfying (7.2) or (7.3) on each N_j, depending on whether $\int_{N_j} E(\nu) < 4\pi$ (9) or $\int_{N_j} E(\nu) \geq 4\pi$. Assume that ψ_∞ is constant in $B(0, r_3)$ for a fixed, large r_3. Assume further that $D_{M^3}\psi_\infty \in L^2(M^3)$. Then there is a nontrivial solution of the Dirac equation of the form $\psi_\infty + \psi$, where ψ lives in a Hilbert space containing compactly supported spinors as a dense subset:*

$$D_{M^3}(\psi_\infty + \psi) = 0 \tag{100}$$

We now turn to the case where M^3 has finitely many cylindrical ends. In this case, if ψ lives in Hilbert space containing smooth compactly supported spinors as a dense subset, Corollary 4.9 and continuity show that
$\int_{M^3} |D_M\psi|^2 = \int_{M^3} |\nabla\psi|^2 + \frac{1}{4}(R - 2|E|_g^2)|\psi|^2$ and again (4.3) holds. We therefore, have, assuming the conditions of theorem 3,

Theorem 24. *Let M^3 denote a Riemannian 3-manifold containing one asymptotically flat end and finitely many cylindrical ends (as defined in definition 13). Let ψ_∞ denote a smooth spinor field on M^3 that is constant in $B(0, r_3) \subset M_{ext}$ for a fixed, large r_3. Assume further that $\psi_\infty = 0$ on each $M_j(r)$ and that $D_M\psi_\infty \in L^2(M^3)$. Then there is a nontrivial solution of the Dirac equation of the form $\psi_\infty + \psi$, where ψ lives in a Hilbert space containing compactly supported spinors as a subset:*

$$D_{M^3}(\psi_\infty + \psi) = 0 \tag{101}$$

Note: In theorems 7.1 and 7.2, the Hilbert space consists of spinors ψ satisfying the following:

$$\int_{M^3} |D_M\psi| < \infty \tag{102}$$

and

$$\int_{M^3} G(r)|\psi|^2 < \infty \tag{103}$$

$G(r)$ is a function of the radial coordinate defined as follows: Let $v_{r_3} =< x_3, y_3, z_3 >\in \mathbb{R}^3 - B_{r_1}(0)$ denote a fixed point with $r_3^2 = x_3^2 + y_3^2 + z_3^2$ chosen so that $r_3 > r_1$, and let $v_r =< x, y, z >\in \mathbb{R}^3 - B_{r_1}(0)$ denote a variable point of length r, $r^2 = x^2 + y^2 + z^2$ on M_{ext}, $r > r_1$. Pick a point $v_{r_2} =< x_2, y_2, z_2 >$ with $r_2^2 = x_2^2 + y_2^2 + z_2^2$ chosen so that the strict inequality $r_1 < r_2 < r_3$ holds. Let $dist_g(\Phi(v_{r_2}), \Phi(v_r))$ denote the distance in M^3 between the point $\Phi(v_{r_2})$ and $\Phi(v_r)$ with respect to the metric g for any $r > r_3$. We will set $G(r) = [dist_g(\Phi(v_{r_2}), \Phi(v_r))]^{-2}$ for $r > r_3$ and then extend this function smoothly, so that $G(r) = 0$ for $r_1 < r \leq r_2$. If M^3 contains finitely many cylindrical ends, then we also require that $G(r) = 0$ on each cylindrical end, M_j. In this way, the decay condition,

$$\int_{M^3} G(r)|\psi|^2 < \infty$$

is really a condition characteristic on the asymptotically flat end, *not* the cylindrical ends.

The Lax-Milgrim/Riesz Representation Theorem will give us a nontrivial solution to $D_M(\psi + \psi_\infty) = 0$ precisely when $\int_{M^3} |D_M\sigma|^2 < 0$ and $D_M\sigma = 0$ (weakly) for a spinor $\sigma \in W^{1,2}_{loc}(M^2)$ implies that $\sigma = 0$ (cf. lemma 3.4 of [15]). According to the proof of lemma 3.4 of [15], this can be shown to happen when we can find a cutoff function f satisfying the following: $f = 1$ on $M^3 - B(0,r)$, $r > r_1$, $f = 0$ on $M^3 - B(0,2r)$, and $f = 1$ on $M_j(r)$, and $f = 0$ on $M_j - M_j(2r)$ for some $r \geq r_4$. Here, r_4 is a number representing the length down a cylindrical end (for convenience, we can pick this number to be the same for all cylindrical ends).

Let D denote the Levi-Cevita derivative on M^3 with respect to g acting on functions, $f : M^3 \to \mathbb{R}$ by $D_i f = \partial_i f$ (in local coordinates). Then, we further require that $|Df| \leq \frac{C}{r}$ in M_{ext} for all $r \geq r_2$ and $|Df| \leq \frac{C}{r}$ in $M_j(r)$ for all $r \geq r_4$ and some constant C. This is possible since if r is large enough, the metric on M_{ext} will look like the Euclidean metric, and the metric on M_j and/or $M_j(r)$ will look like the product metric, so that we can simply choose a bump function *depending only on* the r coordinate, whose derivative in that direction decays as $O(\frac{1}{r})$.

Now, note the following equation (cf. Corollaries 5.4 and 5.5, [4], [7], and [16]) for any *compactly supported* spinor $\eta \in W_0^{1,2}(M^3)$.

$$0 = \int_{M^3} < D_M\sigma, D_M\eta >= \int_{M^3} < \nabla\sigma, \nabla\eta > + \frac{1}{4}\int_{M^3} \frac{1}{4}[R - 2|E|_g^2] < \sigma, \eta > \tag{104}$$

Setting $\eta = f\sigma$, we obtain, in analogy to lemma 3.4 of [15],

$$\int_{M^3 - [B(0,r) \cup M_j(r)]} |\nabla\sigma|^2 \leq \frac{C}{r} \int_{M^3} |\sigma|^2 \tag{105}$$

Letting $r \to \infty$, we then see that $\int_{M^3} |\nabla\sigma|^2 = 0$, implying that $\sigma = 0$ as it was assumed that M^3 admits no super-covariantly constant spinors.

Note that, because the only term in ∇ (cf. equation equation (5.1)) affecting the cutoff function f is D, the above result will hold regardless of the assumption made on the decay rate of E and g along $M_r(r)$. This is the reason the decay condition was relaxed from $O(e^{\beta r})$) $((\beta < 0))$ in [9] to $O(\frac{1}{r})$ in definition 4.1 of section 4.

If M^3 *did* admit a nontrival supercovariantly constant spinor, i.e. a spinor σ with $\nabla\sigma = 0$ then, by definition of the charged Dirac operator, D_M, this spinor would solve $D_M\sigma = 0$. We shall see in section 8 that existance of such a spinor leads to an open subset of the IWP black hole spacetime (in the case of finitely many connection boundary componetns) or an open subset of the Majumbdar-Papapetrou spacetime (in the case of finitely many cylindrical ends).

Note: Here is the reason that $\psi_\infty + \psi$ is a nontrivial solution in theorems 7.1 and 7.2:

If ψ_∞ is constant, say $\psi_\infty = 1$ on $B(0, r_5)$ for $r_5 > r_3$ large enough, then we have

$$\int_{B(0,r_3)} \frac{|\psi_\infty|^2}{[dist_g(\Phi(v_{r_2}), \Phi(v_r))]^2} \geq \int_{B(0,r_5)} \frac{1}{[dist_g(\Phi(v_{r_2}), \Phi(v_r))]^2} \tag{106}$$

However, note that $\int_{B(0,r_5)} \frac{1}{[dist_g(\Phi(v_{r_2}),\Phi(v_r))]^2}$ cannot be finite for the following reason. Since the pull-back metric $\Phi^* g$ approaches the Euclidean metric on $\mathbb{R}^3 - B_{r_1}(0)$, $dist_g(\Phi(v_{r_2}),\Phi(v_r))$ grows like r, so that $[dist_g(\Phi(v_{r_2}),\Phi(v_r)]^{-2} = O(r^{-2})$, but the volume of $B(0,r) - B(0,r_3)$ in M^3 with respect to g grows like r^3, i.e. $Vol_g(B(0,r_3) - B(0,r)) = O(r^3)$. Therefore,

$$\int_{B(0,r_5)} \frac{1}{[dist_g(\Phi(v_{r_2}),\Phi(v_r)]^2} = \lim_{r\to\infty} \int_{B(0,r_5)-B(0,r)} \frac{c^2}{[dist_g(\Phi(v_{r_2})\Phi(v_r)]^2}$$
$$= \lim_{r\to\infty} \frac{r^3}{r^2}$$
$$= \lim_{r\to\infty} r = \infty$$

It follows that $\int_{B(0,r_3)} \frac{|\psi_\infty|^2}{[dist_g(\Phi(v_{r_2}),\Phi(v_r))]^2}$ is not finite, so that it would be impossible for ψ_∞ to be equal to $-\psi$ a.e. on M^3 as $G(r) = \frac{1}{[dist_g(\Phi(v_{r_2}),\Phi(v_r))]^2}$ on $B(0,r_3)$.

8 Proof of Theorems 2 and 3

In this section, we use the nontrivial solution of the charged Dirac equation on the ambient manifold of theorems 7.1 and 7.2 to prove theorems 2 and 3, which were stated in the introduction.

8.1 Proof of Theorem 1.51

We start proving theorem 2 for an asympotically flat Riemannian 3-manifold whose interior boundary contains finitely many disjoint components N_j:

First, notice that corollary 5.5 holds for the spinor ψ_∞ in theorem 7.1 (without loss of generality, we can assume that $|\psi_\infty|^2 = 1$ on $B(0,r_3)$. As in [7], equation 3.2.34, it can be show that the functional $F : \Gamma(S) \to \mathbb{R}$ defined by

$$\mathrm{F}(\psi) = \int_{M^3} |\nabla(\psi+\psi_\infty)|^2 - |D_M(\psi+\psi_\infty)|^2 + \tfrac{1}{4}(R - 2|E|_g^2)|\psi+\psi_\infty|^2,$$
$$-\Sigma_{j=1}^m \int_{N_j} < \psi+\psi_\infty, (D^i + c(e^i)\not{D} + E^i c(e_0))(\psi+\psi_\infty) > dS^i.$$

is continuous on the Hilbert space of the last section. In particular, if ψ_k is a sequence of compactly supported spinors converging to ψ, then $F(\psi_k)$ must converge to $F(\psi)$. Since ψ_k is compactly supported, $F(\psi_k) = F(0) = 4\pi(E_{ADM} - |Q|)$ by corollary 5.5. Therefore, if ψ is a spinor satisfying (7.5) in theorem 7.1, then we have the identity:

$$\int_{M^3} |\nabla(\psi+\psi_\infty)|^2 + \tfrac{1}{4}(R - 2|E|_g^2)|\psi+\psi_\infty|^2$$
$$-\Sigma_{j=1}^m \int_{N_j} < \psi+\psi_\infty, (D^i + c(e^i)\not{D} + E^i c(e_0))(\psi+\psi_\infty) > dS^i$$
$$= 4\pi(E_{ADM} - |Q|)$$

Assuming the dominant energy condition, $R \geq 2|E|_g^2$ holds on M^3, the normal component of the electric field E satisfies (10) on each N_j and that the mean curvature of each N_j satisfies (11), then the preceedings sections show that every term in the integrands of the left-hand side of (8.2) are non-negative. It therefore follows that $E_{ADM} - |Q| \geq 0$, or equivalently $E_{ADM} \geq |Q|$.

8.2 Proof of Theorem

We now move on to the case where M^3 contains one asymptotically flat end and finitely many cylindrical ends, as in the assumption of theorem 3. The proof follows the preceeding section, except that now the functional F of (8.1) no longer has any interior boundary terms:

$$F(\psi) = \int_{M^3} |\nabla(\psi + \psi_\infty)|^2 - |D_M(\psi + \psi_\infty)|^2 + \frac{1}{4}(R - 2|E|_g^2)|\psi + \psi_\infty|^2 \tag{107}$$

Again, an analogy to equation 3.2.34 of [7], this is a linear functional can be shown to be continuous on the Hilbert space of theorem 7.2. Since compactly supported smooth spinors are dense in this Hilbert space, we have, if ψ_k denotes compactly supported smooth spinors approaching ψ, that $F(\psi_k) = F(0) = 4\pi(E_{ADM} - |Q|)$ by corollary 5.6. Therefore, a spinor satisfying (7.6) in theorem 7.2 produces the following identity:

$\int_{M^3} |\nabla(\psi + \psi_\infty)|^2 + \frac{1}{4}(R - 2|E|_g^2)|\psi + \psi_\infty|^2$

$= 4\pi(E_{ADM} - |Q|)$

It follows that if the metric g on M^3 and the electric field E satisfy the dominant energy condition: $R \geq 2|E|_g^2$, then every term on the left-hand side of (8.4) must be non-negative, and therefore, the term $4\pi(E_{ADM} - |Q|)$ is also nonnegative, from which we can conclude, assuming $|\psi_\infty|^2 = 1$ near infity, $E_{ADM} \geq |Q|$.

9 Case of Equality

In this section, we prove that the equality, $E_{ADM} = |Q|$ in theorem 3 implies that the initial data (M^3, g, E) gives rise to a spacetime containing a neighborhood diffeomorphic to an open subset of the standard Majumdar-Papapetrou space-time. In theorem 2, equality is weaker in that it only leads to a Israel-Wilson-Perjes (IWP) black hole. It is natural to first define the metric characterizing *standard Majumbdar-Papapetrou spacetime*. Along the lines of [8], we have:

Definition 25. A *standard* Majumbdar-Papapetrou spacetime or standard MP spacetime is a spacetime that is topologically the manifold $N^4 = \mathbb{R} \times (\mathbb{R}^3 - (\cup_{i=1}^n a_i))$, where $a_i \in \mathbb{R}^3$, $i = 1, 2, ..., n$ denote n finitely many points in space. Let m_i, $i = 1, 2, ..., n$ denote finitely many positive constants. Define a function

$$u : \mathbb{R}^3 - (\cup_{i=1}^n a_i) \to \mathbb{R} \tag{108}$$

by the equation:

$$u(x) = 1 + \sum_{i=1}^{n} \frac{m_i}{|x - a_i|} \tag{109}$$

The standard MP metric, ${}^{(4)}g$, is then the Lorentzian metric given by:

$${}^{(4)}g = u^{-2}dt^2 - u^2(dx^2 + dy^2 + dz^2) \tag{110}$$

The Electromagnetic field on this spacetime is given by the Maxwell potential,

$$A = u^{-1}dt \tag{111}$$

and the associated Electromagnetic field tensor (Maxwell tensor), F, given by:

$$F = dA \tag{112}$$

First, assume that no nontrivial super-covariantly constant spinors exist on M^3. Then equality in (8.4) for ψ satisfying (7.6) of theorem 7.2, gives the equation:

$$\int_{M^3} |\nabla(\psi + \psi_\infty)|^2 = 0 \tag{113}$$

which gives a contradiction. We therefore conclude that $E_{ADM} = |Q|$ if and only if there is a nontrivial supercovariantly constant spinor $\phi = \psi + \psi_\infty$ on M^3. Note also that from the equation (8.4), we must have the dominant energy condition $R = 2|E|_g^2$.

In [8], Chrusciel, Reall, and Tod show how this super-covariantly constant spinor can be extended to a a super-covariantly constant spinor field on a neighborhood of (M^3, g, E) in the spacetime N^4 arising from it, using what is called the *Killing development*. This is defined using given *Kiling initial data*, which is defined below (cf. equation 4.3 of [8]):

Definition 26. Let (M^3, g, E) be initial data for the Einstein-Maxwell equations. Let $W : M^3 \to \mathbb{R}$ and $Y \in \Gamma(TM^3)$ be a given function and a given vector field respectively defined on M^3. (W, Y) forms what is called *Killing initial data*. The *Killing development* of (M^3, g, E) is then defined to be the topological manifold $N^4 = \mathbb{R} \times M^3$ with Lorentzian metric:

$$^{(4)}g = W^2 dt^2 - g_{ij}(dx^i + Y^i dt)(dx^j + Y^j dt) \tag{114}$$

Let $e_0 \in \Gamma(TN^4)$ be the unit normal vector field to M^3 (which is necessarliy timelike since M^3 is spacelike), and assume that this vector field is pointing in the direction of positive time. We then define the *Killing vector field* as:

$$X = We_0 + Y \tag{115}$$

Note: In the case of a supercovariantly constant spinor field, ϕ, the function W and vector field Y on M^3 are given in the spin frame e_j by $W =< \phi, \phi >$ and $Y^j =< \phi, c(e_0)c(e_j)\phi >$ respecitvley.

The authors of [8] extended the spinor field ϕ to a *super-covariantly constant* spinor field on the Killing development of M^3 as follows. They first extended the electric field to the Killing development of M^3 by requiring that the Lie derivative of the electromagnetic field tensor $F_{\mu\nu}$ vanish. In this way, they were able to construct a *charged* connection on the Killing development. By requring the Lie derivative of the spinor field ϕ along X to vanish on the Killing development of M^3, they were able to extend ϕ to the entire Killing development of M^3, and show that this ϕ is parallel with respect to the charged connection, i.e. *super-covariant.*

The associated Killing vector field is then given, with respect to the orthonormal frame e_j, by

$$X =< \phi, \phi > + < \phi, c(e_0)c(e_j)\phi > e^j \tag{116}$$

The (Dirac) spinor ϕ on the spacetime N^4 can be broken down into two-components (one right-handed and the other left-handed), each of which lives in a complex vector space of dimension 2 (ismorphic to $\mathbb{C}^2$). Let $\phi = (\alpha_A, \beta_{A'})$ be the decomposition of ϕ into its two components. For any $w =< a, b > \in \mathbb{C}^2$, let $\bar{w} =< \bar{a}, \bar{b} > \in \mathbb{C}^2$ denote its complex conjugate. Chrusciel et. al. in [8] show that the vector field $X =< \phi, \phi > e_0 + < \phi, c(e_0)c(e_j)\phi > e^j$ can be written with respect to these components as follows:

$$X = \frac{1}{\sqrt{2}}(\alpha^A \bar{\alpha}^{A'} + \bar{\beta}^A \beta^{A'})\frac{\partial}{\partial x^{AA'}} \tag{117}$$

They then set $V = \alpha_A \bar{\beta}^B$ and show that if the above Killing vector field (9.9)/(9.10) is timelike at a point $p \in M^3$, then there exists a one-form ω satisfying $curl\ \omega = i(\bar{V}^{-1}\nabla V^{-1} - V^{-1}\nabla(\bar{V}^{-1}))$ (cf. equation (2.24) of [8]) and the metric $^{(4)}g$ defined above on the *Killing* *development* of M^3 agrees with the (local) IWP metric defined below (cf. equation (2.13) of [8]):

$$^{(4)}g = V\bar{V}(dt + \omega \cdot dx)^2 - (V\bar{V})^{-1}dx \cdot dx) \tag{118}$$

on a neighborhood on $p \in M^3$) (cf. p. 5 of [8])

In section 4.1 of [8], the authors show that if the ADM four-momentum, $p^\mu = (E_{ADM}, p^i)$ of the asymptotically flat end is timelike, then the Killing vector field defined above in (9.9) and (9.10) is strictly timelike (cf. section 3 of [14]). In our case, we have assumed time-symmetry ($k = 0$), so that the p^i components of the four-momentum are zero (cf. equation equation 3.3.15 in [7]), and therefore, since E_{ADM} is the only component of the four-momentum, it is automatically timelike, as it was already assumed that $E_{ADM} = |Q| > 0$.

9.1 Equality in the case of a Boundary with Multiple Components

In [8], theorem 1.1 assumes the following two conditions:

1. The Killing vector X associated with ϕ is time-like on $M^3 - \partial M^3$.

2. X is null on ∂M^3, i.e. $^{(4)}g_{\mu\nu}X^\mu X^\nu = 0$ on ∂M^3,

and concludes that there is a neighborhood of M^3 on the spactime N^4 arising from it that is diffeomorphic to an *open subset of a standard Majumdar-Papapetrou space-time*

We have already seen above that the first condition is satisfied as a consequence of the time-symmetry of the intial data. It follows that around each point $p \in M^3$, there exists a neighborhood that is a neighborhood whose spacetime metric inherited from the Killing development of M^3 is an IWP metric. That is, we have shown the following:

If equality holds in theorem 2, then the spacetime metric on N^4 arising form the initial data (M^3, g, E) is locally an IWP spacetime.

For the second condition, note from equation (9.7), we calculate

$^{(4)}g_{00} = W^2 - g_{ij}Y^iY^j$
$^{(4)}g_{ij} = -g_{ij}$
$^{(4)}g_{i0} = -g_{ij}Y^j$
$^{(4)}g_{0j} = -g_{ij}Y^i$
and therefore

$g_{\mu\nu}X^\mu X^\nu = W^2[W^2 - g_{ij}Y^iY^j] - g_{ij}Y^iY^jW - g_{ij}Y^jW \cdot Y^i$
$= W^4 - g_{ij}Y^iY^j[W^2 + 2W + 1]$
$= W^4 - g_{ij}Y^iY^j(W + 1)^2$

$= W^4 = g(Y,Y)(W+1)^2$
The condition that X be null on ∂M^3 thus reduces to the equation:

$$W^4 = g(Y,Y)(W+1)^2 \tag{119}$$

Note that here $W =< \phi, \phi >$ and $Y^j =< \phi, c(e_0)c(e_j)\phi > e^j$.

In the case of an apparent horizon, Bartnik and Chrusciel in [16] imposed the boundary condition $c(e_0)c(e_1)\phi = \phi$, and were able to show that this implied $< \phi, c(e_0)c(e_1)\phi >= 0$, so that $Y = We_1$, and $g(Y,Y) = W^2$, so the above the equation becomes

$$W^4 = W^2(W+1)^2 \tag{120}$$

Obviously this equation does not have a solution as $W > 0$. However, in [17] (theorem 2), the Khuri and Weinstein were able to show that, assuming the boundary of M^3 is an apparent horizon, then (M^3, g, E) arises as a spacelike hypersurface in the Majumbdar-Papapetrou spacetime.

However, there is no reason to expect that equation (9.12) hold in theorem 2; there, the boundary conditions (7.2) and (7.3) were imposed were imposed on each N_j. Therefore, we can only conclude that equality in theorem 2 gives rise to spacetime arising from the initially data that is locally an IWP spacetime.

9.2 Equatlity in the case of finitely many cylindrical ends

In [8], theorem 1.2 asserts that any electro-vacuum spacetime containing a nontrivial super-covariantly spinor field and a simply connected maximal hypersurface, M^3, consisting of one asymptotically flat end and the union of finitely many *weakly* cylindrical ends contains a neighborhood of M^3 isometrically diffeomorphic to an open subset of a standaard MP spacetime. Since we have already shown that we can extend the super-covariantly constant spinor field on M^3 to the whole spacetime, N^4, arising from it, we need only check that our initial data in theorem 3 is a maximal hypersurface. This is actually a trivial consequence of the definition of a maximal hypersurface:

Definition 27. Let $(N^4, {}^{(4)}g)$ be a Lorentzian metric, and let $M^3 \subset N^4$ be a spacelike hypersurface, meaning that the induced metric g on M^3 consisting of restricting ${}^{(4)}g$ to TM^3 is positive definite (Riemannian). Let k denote the second fundamental form of M^3 in N^4 with respect to the timelike normal vector, $\frac{\partial}{\partial t}$. The spacelike hypersurface, M^3, is then said to be *maximal* if the mean curvature $H = Tr_{M^3}k$ of M^3 in N^4 vanishes.

Since we assuemd time-symmetry of the initial data, we have $k = 0$ and hence $H = 0$ on M^3, so that M^3 is indeed a maximal hypersurface. We have the following conclusion:

If equality holds in theorem 3, then the spacetime arising from the initial data (M^3, g, E) is contains an open subset diffeomorphic to a standard Majumbdar-Papapetrou spacetime

10 A Manifold with Corners in the Non-Time Symmetric Case

In all of the previous sections, we have assumed time-symmetry ($k = 0$). In [15], Shi and Tam have shown that the positive mass theorem holds for a 3-manifold with multiple asymptotically flat ends containing an interior region with boundary. The metric is allowed to be Lipschitz near the boundary, but the mean curvature of the interior must agree with the mean curvature of the exterior (cf. section and theorem 3.1 of [15]). We reiteirate these conditions in some definitions and then present a theorem extending theorem 3.1 of [15] that includes charge and a symmetric two-tensor, k.

Definition 28. Let (M^3, g) be an orientable, non-compact Riemannian 3-manifold. M^3 is said to contain finitely many (say m) asymptotically flat ends if there is a compact set $K \subset M^3$ and such that

$M^3 - K = \cup_{j=1}^{m} E_j$. Each E_j is diffeomorphic to $\mathbb{R}^3 - B_{r_1}(0)$ for some $r_1 > 0$. The diffeomoprhisms $\Psi_j : \mathbb{R}^3 - B_{r_1}(0) \to E_j$ satisfy the following property:

If g denote the Riemannian metric inherited on each E_j from the original Riemannian metric on M^3, then we can write:

$$(\Psi^* g)_{il} = \delta_{il} + b_{il} \tag{121}$$

and we require that b_{il} additionally satisfies the following decay condition:

$$|b_{il}| + r|\partial b_{il}| + r^2|\partial\partial b_{il}| = O(r^{-1}) \tag{122}$$

where r is defined by $r^2 = x^2 + y^2 + z^2$ and denotes the Euclidena distance from the origin in $\mathbb{R}^3 - B_{r_1}(0)$ and ∂ denotes the gradient operator in $\mathbb{R}^3 - B_{r_1}(0)$ (i.e. $\partial : C^1(\mathbb{R}^3 - B_{r_1}(0)) \times C^1(\mathbb{R}^3 - B_{r_1}(0)) \to C^1(\mathbb{R}^3 - B_{r_1}(0)))$ is defined by $\partial(X, f) = \partial_X f$. If the vector field X is given in local coordinates by $X = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} + a_3 \frac{\partial}{\partial x_3}$, then $\partial_X f = a_1 \frac{\partial f}{\partial x_1} + a_2 \frac{\partial f}{\partial x_2} + a_3 \frac{\partial f}{\partial x_3}$).

As in section 1, for any $r > r_1$, we will set $B_j(0, r) = \Psi_j(\mathbb{R}^3 - B_{r(0))}$ and $S_j(0, r) = \partial(\Psi_j(\mathbb{R}^3 - B_{r(0)))}$ for any $r > r_2$. We then define the ADM mass, $(E_{ADM})_j$ of the end E_j in a formula analogous to (1.3):

$$(E_{ADM})_j = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_j(0,r)} (\partial_l g_{lv} - g_v g_{ll}) dS^v \tag{123}$$

Now assume that two vector fields, $E \in \Gamma(TM^3)$ and $B \in \Gamma(TM^3)$ have been defined on M^3. Let ν denote the outer unit normal vector field pointing toward spatial infinity on $S_j(0,r)$ and set $g(E,\nu) = E(\nu)$ and $g(B,\nu) = B(\nu)$. We then define the electric charge Q_j^E, and the magnetic charge Q_j^B of each end, E_j as follows:

$$Q_j^E = \lim_{r\to\infty} \int_{S_j(0,r)} E(\nu) \tag{124}$$

and

$$Q_j^B = \lim_{r\to\infty} \int_{S_j(0,r)} B(\nu) \tag{125}$$

Further, assume that there in addition has been a symmetric two-tensor, k_{il} defined on M^3; this tensor represents the extrinsic curvature curvature of M^3 in the spacetime N^4 arising from it after solving the Einstein-Maxwell equations. We then define the spacelike components p_j^i of the the ADM 4-momentum, p_j of each end E_j

$$p_j^i = \lim_{r\to\infty} \frac{1}{2} \int_{S_j(0,r)} ((Tr_g k) g^{il} - k^{il}) dS^l \tag{126}$$

(cf. theorem 3.3.3 of [7]).

Let D_l denote the covariant derivative of the Levi-Cevita connection in the direction of e_l (for a given orthonormal frame), and let R denote the scalar curvature of M^3 with respect to g. We also define the *charged* energy density, μ_{EM} and *charged* momentum density $J_{EM} = J^i_{EM} e_i$ by the following (cf. 11.18 of [16]):

$$\mu_{EM} = R - |k|_g^2 + (tr_g k)^2 - 2|E|_g^2 - 2|B|_g^2 \tag{127}$$

and

$$J^i_{EM} = 2D^l(k_{il} - (tr_g k)\delta_{il}) + 4\epsilon_{ila} E^l B^a \tag{128}$$

The *dominant energy condition* in this case then becomes
$\mu_{EM} \geq \sqrt{|J_{EM}|_g^2 + |div\ E|^2 + |div\ B|^2}$ (cf. equation 11.24 of (15)). In order to present our theorem, we now need to define the mean curvature $H|_{\bar{\Omega}}$ of $\partial\Omega$ with respect to the unit normal of $\partial\Omega$ pointing toward the interior of $\bar{\Omega}$ and the mean curvature $H|_{M^3 - \bar{\Omega}}$ with respect to the outward unit normal of $\partial\Omega$ pointing toward the exterior of $\bar{\Omega}$ (i.e. pointing toward

$M^3 - \bar{\Omega}$).

Definition 29. Let ν denote the outer-unit normal of Ω pointing toward the exterior of $\bar{\Omega}$ (i.e,. pointing *into* $M^3 - \bar{\Omega}$). Further let D_X denotes the Levi-Cevita conecction on M^3 with respect to the metric g in the direction of $X \in T(\partial\Omega)$. Also, let $g|_{M^3-\bar{\Omega}}$ denote the metric g restricted to $M^3 - \bar{\Omega}$ and let $g|_{\bar{\Omega}}$ denote the metric g restricted to $\bar{\Omega}$. As in the preceeding sections, let e_2, e_3 be a frame for $T\partial\Omega$. We then define $H|_{M^3-\bar{\Omega}}$ and $H|_{\bar{\Omega}}$ and by the following equations:

$$H|_{M^3-\bar{\Omega}} = g|_{M^3-\bar{\Omega}}(\nu, D_{e_2}e_2) + g|_{M^3-\bar{\Omega}}(\nu, D_{e_3}e_3) \tag{129}$$

and

$$H|_{\bar{\Omega}} = g_{\bar{\Omega}}(\nu, D_{e_2}e_2) + g_{\bar{\Omega}}(\nu, D_{e_3}e_3) \tag{130}$$

Before proceeding, it is important to introduce the notion of an *adapted* orthonormal frame near $\partial\Omega$, defined in pp. 19-20 of [15].

Definition 30. Let $x \in \partial\Omega$, and define a *signed* distance function, $\rho : M^3 - \partial\Omega \to \mathbb{R}$, by $\rho(p) = d(x,p)$ if $p \in M^3 - \bar{\Omega}$ and $\rho(p) = -d(x,p)$ if $p \in \Omega$. Let $\nu = \frac{\partial}{\partial\rho}$ and let $e_2,\ e_3$ denote an orthornomal basis for $T_x(\partial\Omega)$. Extend $e_2,\ e_3$ to a neighborhood of $\partial\Omega$ by parallel transport along the integral curves of $\frac{\partial}{\partial\rho}$. $\nu,\ e_2,\ e_3$ is said to be an *adapted* orthornomal frame for $\partial\Omega$.

The goal of this section is to prove theorem 4, which is reiterated below:

Theorem 31. *Assume that (M^3, g) is a Riemannian 3-manifold containing m asymptotically flat ends as above. Define vector fields, $E \in \Gamma(TM^3)$ and $B \in \Gamma(TM^3)$, and a symmetric two-tensor k on M^3 satisfying the following:*

(1). The function $\mu_{EM} = R - |k|_g^2 + (tr_g k)^2 - 2|E|_g^2 - 2|B|_g^2)$ satisfies te dominant energy condition, $\mu_{EM} \geq \sqrt{|J_{EM}|_g^2 + |div\ E|^2 + |div\ B|^2}$.

(1). There is a bounded domain $\Omega \subset M^3$ such that we assume that g is continuous on M^3, smooth on $M^3 - \Omega$ and smooth on $\bar{\Omega}$ and is Lipschitz near $\partial\Omega$. Let e_n, $n = 2,\ 3$ defines an orthonormal frame for $T(\partial\Omega)$, and let ν denote the outer unit normal vector, i.e. directed towards $M^3 - \bar{\Omega}$.

(2). We assume that $H|_{\bar{\Omega}} - |Tr_{\partial\Omega}(k_{\bar{\Omega}})| \geq H|_{M^3-\bar{\Omega}} + |Tr_{\partial\Omega}(k_{M^3-\bar{\Omega}})|$ (where the two mean curvatures are defined by (8.11) and (8.12) respectively). Further, we assume that $k_{\bar{\Omega}}(\nu, e_n) =$

$k_{M^3-\bar{\Omega}}(\nu, e_n)$ and that $B_{\bar{\Omega}}(\nu) = B_{M^3-\bar{\Omega}}(\nu)$ and $E_{\bar{\Omega}}(\nu) = E_{M^3-\bar{\Omega}}(\nu)$.

(3). The components, E^i of $E = E^i e_i$ and B^i of $B = B^i e_i$ satisfy the following decay conditions: $E^i \in o(r^{-1})$, $B^i = o(r^{-1})$. Further, $E,\ B,\ k \in W^{1,2}_{loc}(M^3)$ and $E,\ B,\ k$ and their weak partials are bounded near $\partial\Omega$.

Then

$$(E_{ADM})_j > \sqrt{|p_j|^2 + (Q^E_j)^2 + (Q^B_j)^2} \tag{131}$$

To prove this theorem, we need to show that the analogies to lemma 3.1-3.4 and corollary 3.1 of [15] hold. The statement analogous to [15]'s lemma 3.1 holds simply as a consequence of of the definition of an *adapted* orthornomal frame. This allows us to show that μ_{EM} is well-defined in the distribution sense, which in turns allows us to write a Lichnerowicz formula for the operator D_M. However, proving the lemmas analogous to 3.2, 3.3, and corollary 3.1 requires running through the calculations for the new operator, D_M and connection, ∇. These calculations are done in the next subsection *Three Lemmas* Lemma 3.4 then follows from a reiteration of the argument in [15] and an application of the analogous lemmas. Below is the analogy to lemma 3.1 of [15]:

Lemma 32. *Let $\nu,\ e_2,\ e_3$ denote an* adapted *orthonormal frame for M^3 at a given $x \in \partial\Omega$. Let Γ^b_{ia} denote the Christofell symbols of the metric g. Then Γ^b_{ia} is Lipschitz for $2 \le a, b \le 3$ and all i and $\Gamma^b_{1a} = 0$ for* all $1 \le a,\ b \le 3$, *where it is understood that $e_1 = \nu = \frac{\partial}{\partial\rho}$. Furthermore, the scalar curvature R of M^3 is related to the scalar curvature R^ρ of a surface a distance ρ from $\partial\Omega$, as well as the mean curvature, H, and second fundamental form, II_{ab} $a = 2,\ 3$ of the same surface by the following equation (cf. p. 20 of [15])*

$$R = -\frac{\partial H}{\partial\rho} - \Sigma_{ab}[II_{ab}]^2 + R^\rho. \tag{132}$$

If $H_{M^3-\bar{\Omega}} = H|_{\bar{\Omega}}$, then H is Lipschitz, so that R (and hence also the charged energy distribution, $\mu_{EB} = R - |k|^2_g + (tr_g k)^2 - 2|E|^2_g - 2|B|^2_g)$) is well defined in the distribution sense.
(132)

The next three lemmas are based on the following space-time Einstein-Maxwell spin connection and associated Dirac Operator. This connection is a modification of (5.1) to include the vector field B and the tensor k;

$$\nabla_i = D_i - k_{il}(e^l)c(e_0) - \frac{1}{2}E^l c(e_l)c(e_i)c(e_0) - \frac{1}{4}\epsilon_{lab}B^l c(e^a)c(e_b)c(e_i) \tag{133}$$

(cf. equation 11.19 of [16]). This connection gives rise the associated Dirac-Operator,

$$D_M = \not{D} - \frac{1}{2}(tr_g k - c(E))c(e_0) + \frac{1}{4}\epsilon_{lab}B^l c(e^a)c(e_b) \tag{134}$$

where $\not{D} = c(e^i)D_i$ (cf. the equations at the bottom of page (69) of [16]).

Note: The adjoint of this operator (cf. bottom of page (69) of [16]):

$$D_M^* = \not{D} - \frac{1}{2}(tr_g k - c(E))c(e_0) - \frac{1}{4}\epsilon_{lab}B^l c(e^a)c(e_b) \tag{135}$$

Note: This is operator is not formally self-adjoint, but differs only in replacing B with $-B$.

It has been shown ([7], [4] [16]) that the following Lichernorowicz formula holds for D_M:

$$D_M^* D_M = \nabla^*\nabla + \frac{1}{4}(\mu_{EM} + J^i_{EM}c(e_0)c(e_i)) \tag{136}$$

10.1 Three Lemmas

In this subsection, we state and prove the three lemmas corresponding to lemma 3.2, lemma 3.3, and corollary 3.1 (in that order) in [15]:

Lemma 33. *The Schrodinger-Lichnerowicz Formula for a Manifold with Corners*

Let U be any open set of M^3 and let η denote a spinor living in $W^{1,2}_0(U)$ and Φ denote a spinor living $W^{1,2}_{loc}(U)$. Then

$\int_U < D_M\Phi, D_M\eta >= \frac{1}{2}\int_{\partial\Omega\cap U} < [H|_{\bar{\Omega}} + Tr_{\partial\Omega}(k_{\bar{\Omega}})c(\nu)c(e_0)]\Phi, \eta >$
$- \frac{1}{2}\int_{\partial\Omega\cap U} < [H|_{M^3-\bar{\Omega}} + Tr_{\partial\Omega}(k_{M^3-\bar{\Omega}})c(\nu)c(e_0)]\Phi, \eta >$
$+ \int_U < \nabla\Phi, \nabla\eta > + \frac{1}{4}\int_U < (\mu_{EM} + J^i_{EM}c(e_0)c(e_i))\Phi, \eta >$

Proof. This formula follows as a direct conseconsequnece of the Schrodinger-Lichnerowicz formula ([4], [5], [16]) on any open set U that does not intersect $\partial\Omega$ because in this case $U \cap \partial\Omega = \emptyset$. If U does intersect $\partial\Omega$, then let $T = U \cap \partial\Omega$, and applying the Licherowicz formula (8.21) to $U \cap \Omega$, we obtain, after multiplying by η and integrating by parts:

$\int_{\Omega\cap U} < D_M\Psi, D_M\eta > + \int_T < \nu \cdot D_M\Phi + \nabla_\nu\Phi, \eta >$
$= \int_{\Omega\cap U} < \nabla\Phi, \nabla\eta > + \frac{1}{4}\int_{\Omega\cap U} < (\mu_{EM} + J^i_{EM}c(e_0)c(e_i))\Phi, \eta >$ We can readily calculate

c$(\nu)D_M\Phi + \nabla_\nu\Phi = c(\nu)\sum_{a=2}^3 \nabla_{e_i}\Phi$
$= c(\nu)\sum_{a=2}^3 c(e_a)(\nabla_{e_a}\Psi^I)\sigma_I + \frac{1}{4}c(\nu)(\sum_{a,b,l=2}^3 \Gamma^l_{ab}c(e_b)c(e_l)) + 2k(e_a,\nu)c(\nu)c(e_0)\Phi$
$- [\frac{1}{2}H|_{\bar{\Omega}}\Phi + (Tr_{\partial\Omega}k_{\bar{\Omega}})c(\nu)c(e_0)]\Phi + c(E)c(\nu)c(e_0) - \sum_{a=2}^3 \frac{1}{2}\epsilon_{aml}B^a c(\nu)c(e_i)c(e_m)c(e_l)$

$= c(\nu)\sum_{a=2}^{3} c(e_a)(\nabla_{e_a}\Phi^I)\sigma_I + \frac{1}{4}c(\nu)\sum_{b,l=2}^{3}\Gamma_{ab}^{l}c(e_b)c(e_l)\Phi - \sum_{a=2}^{3}\frac{1}{2}k(e_a,\nu)c(e_0)\Phi$
$- \frac{1}{2}[H|_{\bar{\Omega}}\Phi + (Tr_{\partial\Omega}k_{\bar{\Omega}})c(\nu)c(e_0)]\Phi + c(E)c(\nu)c(e_0)\Phi + 2B(\nu)c(\nu)c(e_3)c(e_2)\Phi$

Note: In the above formula, we thought of $T = \partial\Omega \cap U$ as being the boundary of $\Omega \cap U$. If instead, we think of $T = \partial\Omega \cap U$ as being the boudnary of $U - \bar{\Omega}$, then we need to replace the $c(\nu)$ in front of $\Sigma_{a=2}^{3}\nabla_{e_i}\Phi$ with $c(-\nu) = -c(\nu)$ in the above formula. This results in the term $\frac{1}{2}H|_{\bar{\Omega}}\Phi$ being replaced with $-H|_{M^3-\bar{\Omega}}\Phi$, since we are now calculating the mean curvature with respect to $-\nu$ and $g|_{M^3-\bar{\Omega}}$ instead of with respect to ν and $\bar{\Omega}$. Therefore, the term $-\frac{1}{2}H|_{\bar{\Omega}}\Phi$ gets replaced with $\frac{1}{2}H|_{M^3-\bar{\Omega}}\Phi$.

Furthermore, we assumed that $k(e_a,\nu)$ on $\bar{\Omega}$ agrees with $k(e_a,\nu)$ on $M^3 - \bar{\Omega}$, so that the terms $\frac{1}{4}c(\nu)2[k(e_a,\nu)c(\nu)]\Phi = 2k(e_a,\nu)c(e_0)\Phi$ are replaced with
$\frac{1}{4}c(-\nu)2[k(e_a,\nu)c(\nu)]\Phi = -\frac{1}{2}k(e_a,\nu)c(e_0)\Phi$.

Since we have assumed that $E(\nu)$ and $B(\nu)$ on $\bar{\Omega}$ agree with $E(\nu)$ and $B(\nu)$ on $M^3 - \bar{\Omega}$, it follows that $c(E)$ and $B(\nu)$ on $\bar{\Omega}$ agree with $c(E)$ and $B(\nu)$ on $M^3 - \bar{\Omega}$ respectively (because e_2 and e_3 are tangential to $\partial\Omega$ on $M^3 - \bar{\Omega}$, so that $c(E) = E(\nu)c(\nu) + E_2c(e_2) + E_3c(e_3)$ on $\bar{\Omega}$ will agree with $c(E)$ on $M^3 - \bar{\Omega}$ precisely when $E(\nu)$ on $\bar{\Omega}$ agrees with $E(\nu)$ on $M^3 - \bar{\Omega}$).

Therefore, we have

$c(-\nu)D_M\Phi + \nabla_{-\nu}\Phi = -c(\nu)\sum_{a=2}^{3} c(e_a)(\nabla_{e_a}\Phi^I)\sigma_I - \frac{1}{4}c(\nu)\sum_{b,l=2}^{3}\Gamma_{ab}^{l}c(e_b)c(e_l)\Phi + \sum_{a=2}^{3}\frac{1}{2}k(e_a,\nu)c(\nu)c(e_0)$
$+ \frac{1}{2}H|_{M^3-\bar{\Omega}}\Phi + (Tr_{\partial\Omega}k_{M^3-\bar{\Omega}})c(\nu)c(e_0)\Phi + c(E) - c(\nu)c(e_0)\Phi - 2B(\nu)c(\nu)c(e_3)c(e_2)\Phi$
Adding equation (10.19) to equation (10.20), we obtain

$$c(-\nu)D_M\Phi + \nabla_{-\nu}\Phi + c(\nu)D_M\Phi + \nabla_{\nu}\Phi = \frac{1}{2}[H|_{M^3-\bar{\Omega}} + (Tr_{\partial\Omega}(k_{M^3-\bar{\Omega}}))c(\nu)c(e_0) - (H|_{\bar{\Omega}} + (Tr_{\partial\Omega}(k_{\bar{\Omega}}))c(\nu)c(e_0))] \quad (137)$$

The Schrödinger-Lichnerowicz formula applied to $U - \bar{\Omega}$ is

$\int_{U-\bar{\Omega}} < D_M\Phi, D_M\eta > + \int_T < c(-\nu)D_M\Phi + \nabla_{-\nu}\Phi, \eta >$
$= \int_{U-\bar{\Omega}} < \nabla\Phi, \nabla\eta > + \frac{1}{4}\int_{U-\bar{\Omega}} < (\mu_{EM} + J_{EM}^{i}c(e_0)c(e_i))\Phi, \eta >$
Adding (10.18) to (10.22) we obtain, we obtain, since $U = [U \cap \bar{\Omega}] \cup [U - \bar{\Omega}]$,
$\int_U < D_M\Phi, D_M\eta > + \int_T < c(\nu)D_M\Phi + \nabla_{\nu}\Phi + c(-\nu)D_M\Phi + \nabla_{-\nu}\Phi, \eta >$
$= \int_U < \nabla\Phi, \nabla\eta > + \frac{1}{4}\int_U < (\mu_{EM} + J_{EM}^{i}c(e_0)c(e_i))\Phi, \eta >$
Replacing the second term of the above equation with the right hand side of equation (10.21), (10.23) becomes:
$\int_U < D_M\Phi, D_M\eta > + \frac{1}{2}\int_T < [H|_{M^3-\bar{\Omega}} + (Tr_{\partial\Omega}(k_{M^3-\bar{\Omega}}))c(\nu)c(e_0) - (H|_{\bar{\Omega}} + (Tr_{\partial\Omega}k_{\bar{\Omega}})c(\nu)c(e_0))]\Phi, \eta >$

$= \int_U < \nabla\Phi, \nabla\eta > + \frac{1}{4}\int_U < \mu_{EM} + J_{EM}^{i}c(e_0)c(e_i)\Phi, \eta >$ Moving the second term of (10.24) to the right hand side, we obtain

$\int_U < D_M\Phi, D_M\eta >= \frac{1}{2}\int_T < [H|_{\bar{\Omega}} + (Tr_{\partial\Omega}(k_{\bar{\Omega}})c(\nu)c(e_0) - (H|_{M^3-\bar{\Omega}} + (Tr_{\partial\Omega}(k_{M^3-\bar{\Omega}}))c(\nu)c(e_0))]\Phi, \eta >$

$+\int_U < \nabla\Phi, \nabla\eta > + \frac{1}{4}\int_U < (\mu_{EM} + J^i_{EM}c(e_0)c(e_i))\Phi, \eta >$ which is precisely (10.17)

□

For the next lemma (the analogy of Lemma 3.3 in [15]), we need to define what it means for a spinor Φ to be a weak solution of $D^*_M D_M\Phi = 0$:

Definition 34. A spinor $\Phi \in W^{1,2}_{loc}(U)$ is a *weak* solution of $D^*_M D_M\Phi = 0$ on an open subset $U \subset M^3$ if for every $\eta \in W^{1,2}_0(U)$,

$$\int_U < D_M\Phi, D_M\eta >= 0 \tag{138}$$

We then have the following regularity lemma analogous to lemma 3.3 of [15]:

Lemma 35. *Let $\Phi \in W^{1,2}_{loc}(U)$ be a weak solution to $D^*_M D_M\Phi = 0$. Then $\Phi \in W^{2,2}_{loc}(U)$.*

Proof. This is simply a repeat of the argument used to prove lemma 3.3 of [15], but with the terms $\omega_{al}(e_i)c(e_a)c(e_l) = \Gamma^l_{ia}c(e_i)c(e_a)c(e_l)$ replaced with $\Gamma^l_{ia}c(e_a)c(e_l) + 2k_{il}c(e_a)c(e_0) - 2c(E)c(e_i(c(e_0)) - \epsilon_{aml}B^a c(e_m)c(e_l)c(e_i)$. We obtain the same result because k_{ia}, B, and E are bounded, assuming they satisfy th asymptotic fall off conditions, $E = o(r^{-1})$, $B = o(r^{-1})$, and since $e_i(k_{ia})$, $e_i(B)$, $e_i(E)$ exist weakly on $M^3 - \Omega$ and Ω, and are in $L^1_{loc}(M^3)$. □

Lemma 10.6 allows us to prove an analogy of corollary 3.1 of [15] giving regularity for a weak solution of $D^*_M D_M\Phi = 0$:

Lemma 36. *Let U denote any open set in M^3 and let $\Phi \in W^{1,2}_{loc}(U)$ be a weak solution of $D^*_M D_M\Phi = 0$. Then $D_M\Phi \in W^{1,2}_{loc}(U)$.*

Proof. By equation (8.19),

$$D_M\Phi = \not{D}\Phi - \frac{1}{2}(tr_g k - c(E))c(e_0)\Phi + \frac{1}{4}\epsilon_{lab}B^l c(e^a)c(e_b)\Phi \tag{139}$$

By corollary 3.1 of [15], and lemmas 8.16 and 8.46, the $\not{D}\Phi \in W^{1,2}_{loc}(U)$. It therefore remains to check that $\frac{1}{2}(tr_g k - c(E))c(e_0)\Phi + \frac{1}{4}\epsilon_{lab}B^l c(e^a)c(e_b)\Phi$ is in $W^{1,2}_{loc}(M^3)$. This follows since we have assumed that the weak partials of k, E, and B are bounded near $\partial\Omega$ and since $\Phi \in W^{2,2}_{loc}$ by lemma 8.46. □

Note that the assumption that the weak partials of k, E, B are bounded near $W^{1,2}_{loc}(M^3)$ is necessary because if we assumed only $W^{1,1}_{loc}(M^3)$ regularity, then weak derivatives of terms like $c(E)c(e_0)\Phi$ would include terms such as $e_i(E)c(e_i)c(e_0)\Phi$, which would not necessary be in $L^2_{loc}(U)$.

By lemma 8.22, 8.46, and lemma 8.47, we obtain the following analogy to lemma 3.4 in [15] in the absence of super-covariantly constant spinors:

Lemma 37. *Let $\Phi \in W^{1,2}_{loc}(U)$ be a spinor field such that $D_M\Phi = 0$ and $\int_{M^3} |\Phi|^2 < \infty$. Assume further that M^3 admits no non-trivial supercovariantly constant spinors. Then $\Phi = 0$.*

Proof. By lemma 8.22, for any $\eta \in W^{1,2}_0(M^3)$, we have:
$0 \ = \int_{M^3} < D_M\Phi, D_M\eta >$
$= \frac{1}{2}\int_{\partial\Omega} < [H|_{\bar{\Omega}} + Tr_{\partial\Omega}(k_{\bar{\Omega}})c(\nu)c(e_0)]\Phi, \eta >$
$- \frac{1}{2}\int_{\partial\Omega} < [H|_{M^3-\bar{\Omega}} + Tr_{\partial\Omega}(k_{M^3-\bar{\Omega}})c(\nu)c(e_0)]\Phi, \eta >$
$+ \int_{M^3} < \nabla\Phi, \nabla\eta > + \frac{1}{4}\int_{M^3} < (\mu_{EM} + J^i_{EM}c(e_0)c(e_i))\Phi, \eta >$ Pick a point $p \in \Omega$. Let $B_r(p)$ denote a ball of radius $r > 0$ centered at p. As in lemma 3.4 of [15], we $\eta = f^2\Phi$, where f is a cutoff function satisfying the following $f = 1$ in $B_r(p)$, $f = 0$ in $B_{2r}(p)$ and there is a constant $C > 0$ such that

$$|Df| \leq \frac{C}{r} \tag{140}$$

where D denotes the Levi-Cevita connection acting on any function $f : M^3 \to \mathbb{R}$ by $D_i f = \partial_i f$. Assume that $r > 0$ is large enough so that $\bar{\Omega} \subset B_r(p)$. Then, plugging $\eta = f^2\Phi$ back into to (8.50)-(8.53), we obtain

$\int_{M^3} < \nabla\Phi, \nabla(f^2\Phi) >= -\frac{1}{2}\int_{\partial\Omega} < [H|_{\bar{\Omega}} + Tr_{\partial\Omega}(k_{\bar{\Omega}})c(\nu)c(e_0)]\Phi, \Phi >$
$+ \frac{1}{2}\int_{\partial\Omega} < [H|_{M^3-\bar{\Omega}} + Tr_{\partial\Omega}(k_{M^3-\bar{\Omega}})c(\nu)c(e_0)]\Phi, \Phi >$
$- \frac{f^2}{4}\int_{M^3} < (\mu_{EM} + J^i_{EM}c(e_0)c(e_i))\Phi, \Phi >$ Assuming the dominant energy condition, $\mu_{EM} \geq |J_{EM}|_g$, and the inequality, $H|_{\bar{\Omega}} - |Tr_{\partial\Omega})k_{\bar{\Omega}}| \geq H|_{M^3-\bar{\Omega}} - |Tr_{\partial\Omega}(k_{M^3-\bar{\Omega}})|$, each term on the right-hand side of (8.55)-(8.57) is negative. We deal with the left-hand-side as follows:
$\int_{M^3} < \nabla\Phi, \nabla(f^2\Phi) >= \int_{B_{2r}(p)} < \nabla\Phi, \nabla(f^2\Phi) >$
$= \int_{B_r(p)} |\nabla\Phi|^2 + \int_{B_{2r}(p)-B_r(p)} f^2|\nabla\Phi|^2$
$+ \int_{B_{2r}(p)-B_r(p)} D^i(f^2) < \nabla_i\Phi, \Phi >$ Note that second term of (8.59) is strictly positive. If we perform integration by parts to the term (8.60), we find that the asymptotic fall off conditions for E, B, and k imply that

$$\int_{B_{2r}(p)-B_r(p)} D^i(f^2) < \nabla_i\Phi, \Phi > \leq \frac{C_1}{r}\int_{B_{2r}(p)-B_r(p)} |\Phi|^2 \leq \frac{C_1}{r}\int_{M^3} |\Phi|^2 \tag{141}$$

It then follows letting $r \to \infty$ that $\nabla\Phi = 0$ on M^3. Since we have assume that there are no nontrivial supercovariantly constant spinors on M^3, it follows that $\Phi = 0$. $\square$

We can now begin with the proof that $(E_{ADM})_j > 0$ assuming the conditions (1)-(3) in theorem 35 hold. First , we pick a spinor ψ_∞ that is zero in a nrighborhood of

$\bar{\Omega}$, constant on the jth asymptotic end, and satisfies the following asymptotics on that end, $|D_M\psi_\infty| = O(r^{-2})$ and $|D_M^*D_M\psi_\infty| = O(r^{-3})$. We can then solve for a spinor $\psi_{r_5} \in W^{1,2}(B_p(r_5))$ (for some fixed $p \in M^3$ and large coordinate r_5 on the jth asymptotic end) solving $D_M^*D_M\psi_{r_5} = 0$ and $\psi_{r_5} = \psi_\infty$ on $\partial B_p(r_5)$. This turns out to be equivalent to solving $D_M^*D_M\sigma_{r_5} = -D_M^*D_M\psi_\infty$ and $\sigma_{r_5}|_{\partial B_p(R)} = 0$. We define a bilinear form, $A : W_0^{1,2}(B_p(r_5)) \times W_0^{1,2}(B_p(r_5)) \to \mathbb{R}$ by $A(\phi,\psi) = \int_{B_p(R)} < D_M\phi, D_M\psi >$. Notice by lemma 8.6 that A is coercive, i.e. there exists a constant C such that $A(\psi,\psi) \geq C|\psi|_{W_0^{1,2}}$, where $|\psi|_{W_0^{1,2}}$ denotes the norm of the spinor ψ with respect to the norm on the Hilbert space $W_0^{1,2}(B_p(r_5))$ Define a linear functional, $F : W_o^{1,2}(B_p(r_5)) \to \mathbb{R}$ by $F(\phi) = -\int_{B_p(r_5)} < D_M\phi, D_M\psi_\infty >$. Note that by Hölder's inequality, $|F(\psi)| \leq ||D_M\phi||_{L^2}||D_M\psi_\infty||_{L^2}$. Since we may assume that ψ_∞ is L^2 integrable on the jth asymptotically flat end of M^3, it then follows , setting $C_1 = ||\psi_\infty||_{L^2}$,that $|F(\phi)| \leq C_1||D_M\phi||_{L^2}$, and by lemma 8.6, $||D_M\phi||_{L^2}$ is bounded from above by a constant multiple of $|\phi|_{W_0^{1,2}}$. By the Lax-Milgrim theorem, we then see that there exists a $\sigma_{r_5} \in W_0^{1,2}(B_p(r_5))$ such that $A(\sigma_{r_5},\phi) = F(\phi)$ for all $\phi \in W_0^{1,2}(B_p(r_5))$. since the equation $A(\sigma_{r_5},\phi) = F(\phi)$ can be rewritten as

$$\int_{B_p(r_5)} < D_M\sigma_5, D_M\phi >= -\int_{B_p(r_5)} < D_M\psi_\infty, D_M\phi > \tag{142}$$

or

$$\int_{B_p(r_5)} < D_M(\sigma_{r_5} + \psi_\infty), D_M\phi >= 0 \tag{143}$$

This shows that, by defintion, $\psi_{r_5} = \sigma_{r_5} + \psi_\infty$ weakly satisfies $D_M^*D_M\psi_{r_5} = 0$ on $B_p(r_5)$ and has $\psi_{r_5} = \psi_\infty$ on $\partial B_p(r_5)$. Our goal here is then to let $r_5 \to \infty$ and use this obtain a weak solution

$$\Phi$$

of $D_M^*D_M\Phi = 0$ such that Φ is ψ_∞ on the jth asymptotically flat end, $\int_{M^3} |D_M\Phi|^2 < 0$ and then use lemma 8.10 to conclude that $D_M\Phi = 0$ on M^3. By the regularity lemma 8.9, ψ_{r_5} is bounded, so that $|\psi_{r_5}|^2 \in W^{1,2}(B_p(r_5))$.

To do this, note that if f is a compactly supported function on $B_p(r_5)$, with $f \geq 0$, then by lemma 8.6,
$\int_{B_p(r_5)} < \nabla|\psi_{r_5}|^2, \nabla f >= 2\int_{B_p(r_5)} < \nabla\psi_{r_5}, \nabla(f\psi_{r_5} > -2\int_{B_p} |\nabla\psi_{r_5}|^2$
$= \int_{B_p(r_5)} f < [H|_{M^3-\bar{\Omega}} + Tr_{\partial\Omega}(k_{M^3-\bar{\Omega}})c(\nu)c(e_0)]\psi_{r_5}, \psi_{r_5} >$
$- \int_{B_p(r_5)} f < [H|_{\bar{\Omega}} + Tr_{\partial\Omega}(k_{\bar{\Omega}})c(\nu)c(e_0)]\psi_{r_5}, \psi_{r_5} >$
$- 2\int_{B_p(r_5)} f|\nabla\psi_{r_5}|^2$
$- \frac{1}{2}\int_{B_p(r_5)} f < (\mu_{EM} + J_{EM}^i c(e_0)c(e_i))\psi_{r_5}, \psi_{r_5} >$
≤ 0

Therefore, $|\psi_{r_5}|^2$ is weakly sub-harmonic and since ψ_∞ is uniformly bounded, so is ψ_{r_5} by the maximum principle. Therefore, there exists a sequence $r_i \to \infty$ such that the sequence

$\psi_i = \psi_{r_i}$ converges in $W^{1,2}_{loc}(M^3)$ and solves $D^*_M D_M \psi_{r_i} = 0$ on $B_p(r_i)$ and $\psi_{r_i} = \psi_\infty$ on $\partial B_p(r_i)$. The spinor, ψ to which ψ_i converges in $W^{1,2}_{loc}(M^3)$ solves $D^*_M D_M \psi = 0$.

It can then be shown that this ψ is asymptotically close to ψ_∞. More precisely, we show that

$$|\psi - \psi_\infty| \leq C \frac{\ln r}{r} \tag{144}$$

and that

$$\int_{M^3} |\nabla(\psi - \psi_\infty)|^2 + |D_M(\psi - \psi_\infty)|^2 < \infty \tag{145}$$

To prove this, let $\bar{\Omega} \subset B_p(r_5)$ and $r_i > r_5$. Then, since $\psi_i = \psi_\infty$ on $\partial B_p(r_i)$, $\int_{B_p(r_i)} < D_M \psi_i, D_M(\psi_i - \psi_\infty >= 0$. Therefore,
$\int_{Bp(r_i)} < D_M(\psi_i - \psi_\infty), D_M(\psi_i - \psi_\infty) >$
$= \int_{B_p(r_i)} < D_M \psi_i, D_M(\psi_i - \psi_\infty) > - \int_{B_p(r_i)} < D_M \psi_\infty, D_M(\psi_i - \psi_\infty) >$
$= 0 - \int_{B_p(r_i)} < D_M \psi_\infty, D_M(\psi_i - \psi_\infty) >$
$= \int_{B_p(r_i)} |D_M \psi_\infty|^2 - \int_{B_p(r_i)} < D_M \psi_\infty, D_M \psi_i >$
$= \int_{B_p(r_i)} |D_M \psi_\infty|^2 - \int_{B_p(r_i)} |D_M \psi_i|^2$ From (8.62)-(8.65), we can easily conclude that

$$\int_{B_p(r_i)} |D_M(\psi_i - \psi_\infty)|^2 \leq \int_{M^3} |D_M \psi_\infty|^2 < \infty \tag{146}$$

This proves that the first term in (8.61) is positive. To prove that the first term (in (8.61)) is positive, note that by the dominant energy condition, $\mu \geq |J_{EM}|$, and lemma 8.6, $\int_{B_p(r_i)} |\nabla(\psi - \psi_i)|^2 \leq \int_{B_p(r_i)} |D_M(\psi - \psi_i)|^2 \leq \int_{M^3} |D_M \psi_\infty|^2 < \infty$. (8.61) then follows a simple consequence of letting $r_i \to \infty$ and noting that ψ_i converges to ψ_∞.

To prove (10.48), note that since both ψ_i and ψ_∞ are uniformly bounded, there is a constant C_1 such that $|\psi_i - \psi_\infty| \leq C_1$. We can choose $u \geq 0$ to be a solution of $\Delta u \leq -|D^2_M \psi_\infty| + \epsilon(E, B, k)$ outside $B_p(r_5)$ and such that $u \geq C_1$ on $\partial B_p(r_6)$ and $u(x) \leq C_2 r^{-1} \ln r$ for some C_2. The function $\epsilon(E, B, k)$ is a small function of E, B, and k chosen so that the Lichnerowicz formula will make $\Delta|\psi_i - \psi_\infty| \geq -|D_{M^2} \psi_\infty|$ hold weakly on $B_p(r_i) - B_p(r_5)$. Since $\psi_i \to \psi$ pointwise outside $B_p(r_5)$, (10.48) holds.

Therefore, $D_M \psi \in W^{1,2}_{loc}(M^3)$, and we can apply lemma 10.9 with $\Phi = D_M \psi$ to conclude that $D_M \psi = 0$. This solution can then be used as in section 6 to conclude that

$$(E_{ADM})_j > \sqrt{|p_j|^2 + (Q^E_j)^2 + (Q^B_j)^2} \tag{147}$$

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