On the collapsing and convergence of Ricci flows and solitons

A Dissertation presented

by

Shaosai Huang

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2018

Stony Brook University

The Graduate School

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Abstract of the Dissertation

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2018

Perelman's no local collapsing theorem [78] says that at a finite time singularity of a Ricci flow on a fixed closed manifold, there will be no collapsing with bounded curvature, and therefore the blow-up limit at the singular time – a gradient shrinking soliton – is non-collapsing. However, this may not be the situation when considering a family of Ricci flows with collapsing initial data, and this is the direction in which the current thesis explores. We present two results, of different flavors: one concerning the existence of a weak limit – a metric space whose metric is determined by the Ricci flows; the other on the regularity of the limit space.

Our first result is in general dimensions. We prove a distance distortion estimate for a family of Ricci flows whose initial data may collapse in a controlled way, generalizing a similar estimate of Bamler-Zhang [61], which, as the best known result to date, requires uniform non-collapsing initial data.

The second result is in dimension four. We prove an ε-regularity theorem for complete non-compact gradient shrinking Ricci solitons, and establish a compactness theorem in the generalized Cheeger-Gromov sense. This confirms a decade-long conjecture of Cheeger-Tian [20].

Soli Deo gloria.

Contents

List of Notations

Throughout this paper the following notations are employed:

1. $p_0 \in M$ denotes the base point of M; also use $p_i^0 \in M_i$ for a sequence $\{M_i\}$.

2. $\mathcal{R}m_g$, $\mathcal{R}c_g$ and $\mathcal{R}_g$ denote the Riemannian curvature, the Ricci curvature, and the scalar curvature of a given Riemannian metric g, respectively. For the sake of simplicity, we will write $\mathcal{R}m$, $\mathcal{R}c$ and $\mathcal{R}$ when there is no confusion.

3. For any $E \subset M$ and $r > 0$, define
$$B(E, r) := \{x \in M : \ \exists y \in E, \ d(x, y) < r\}.$$
For any $E \subset M$ and $0 < r_1 < r_2$, define
$$A(E; r_1, r_2) := \{x \in M : \forall y \in E, \ d(x, y) > r_1, \text{ and } \exists z \in E, \ d(x, z) < r_2\}.$$
Especially, $B(x, r)$ is the geodesic ball of radius r around $x \in M$ and $A(x; r_1, r_2)$ is the geodesic annulus around $x \in M$, with inner and outer radii specified by r_1 and r_2 respectively.

4. $\Psi(\alpha, \beta \mid a, b, c)$ will denote some positive function depending on α, β, a, b, c such that for any fixed a, b, c,
$$\lim_{\alpha,\beta\to 0} \Psi(\alpha, \beta \mid a, b, c) \ = \ 0.$$
Notice that the specific value of Ψ may change from line to line.

5. We will use bold-face letter to denote a vector in $\mathbb{R}^4$, e.g. the origin is denoted by $\mathbf{0}$ and a vector is denoted by $\mathbf{v}$.

Acknowledgements

I would like to thank my advisor Xiuxiong Chen, for bringing into my attention the following areas of mathematics: the theory of Cheeger-Colding on manifolds with Ricci curvature bounds, the Cheeger-Fukaya-Gromov theory of collapsing manifolds with bounded curvature, and Perelman's pseudo-locality theorem for Ricci flows. I have gained much pleasure through studying these fields. I also thank him for his constant support during my graduate study.

During the past six years since our first discussion on Cheegr-Colding's theory, Bing Wang has been of great help and support to me. Discussions with Bing have always been exciting, inspiring, and — most importantly — fruitful. I thank him for his generosity, insights and passion in mathematics.

I thank Mike Anderson for his support in my job hunting. As a poineering figure in my field of study, his recoganition and support of my doctoral work have been a great encouragement for me.

I thank Jeff Cheeger for his friendly and patient conversation with me when part of this thesis was just finished last May. Even without this, I am grateful of him for creating the mathematical world that I am so much into.

I thank Blaine Lawson and Martin Roček for kindly agreeing to be on my thesis defense committee.

I thank my teachers back in Toronto: George A. Elliott, Marco Gualtieri, Larry Guth, Joel Kamnitzer, and my master advisor Yael Karshon. They not just taught me mathematical knowledge, but also nurtured my taste in mathematics.

I aslo thank other mathematics professors in Stony Brook, especially Eric Bedford, Christina Sormani, Dennis Sullivan and Dror Varilon, for many friendly conversations, on or off mathematics.

I also benefited and enjoyed a lot in discussions with my colleagues and friends, to whom I would like to thank: Jean-Francois Arbor, Gao Chen, Weiyong He, Demetre Kazaras, Long Li, Yu Li, Song Sun, Selin Taşkent, Yuanqi Wang, Chengjian Yao, Yu Zeng, Ruobing Zhang and Kai Zheng.

I thank my friends Zeyu Cao and Dingxin Zhang for their companion during some of my hardest time.

Finally, the gratitude to the unconditional support of my parents and my wife is beyond my expression.

Chapter 1

Introduction

1.1 Background

The study of Ricci flow starts from Richard Hamilton's seminal paper [36] in 1982, where he defines a Ricci flow on a closed Riemannian manifold (M, g) to be a family of Riemannian metrics $g(t)$ on M satisfying the tensorial equation:

$$\partial_t g(t) = -2\mathcal{R}c_{g(t)} \tag{1.1}$$

with initial data $g(0) = g$. Hamilton first showed the short time existence and then applied this flow to the study of 3-manifolds equipped with an initial metric of non-netative and non-vanishing Ricci curvature. He has shown that strikingly, the *normalized* Ricci flow, evolves any such metric on a 3-manifold to the round sphere.

The approach of deforming geometric quantities using a non-linear heat flow is not new — before the invention of Ricci flows, there have been a large amount of literatures in the study of harmonic map heat flow, and of the mean curvature flow. The Ricci flow, as an intrinsic heat flow, is more difficult to study, but it looked promising through the work of Hamilton: if one could remove the assumption on the initial curvature, evolution of the normalized Ricci flow would lead to a resolution of the long-standing Poincaré conjecture for 3-manifolds.

A program toward solving the 3-dimensional Poincaré conjecture was indeed initiated by Richard Hamilton, but as pointed out in [37], a bottleneck is the possible collapsing with locally bounded curvature at finite time singularities of the Ricci flow. About a decade later, Grisha Perelman made a breakthrough by ruling out the above mentioned possibility in [78], and consequently succeeded in proving Thurston's geometrization conjecture, having the Poincaré conjecture as an essential piece.

Major tools, introduced in [78], that enable Perelman to prove his no local collapsing theorem, are the $\mathcal{F}$- and $\mathcal{W}$-functionals, which have the Ricci flow as their gradient flow, and are thus monotone under the evolution of the Ricci flow. Perelman's no local collapsing theorem enabled us to classify the finite time singularity models of 3-dimensional Ricci flows, and perform surgeries to continue the Ricci flow.

Besides the long-time existence issue, which is settled in many interesting cases, e.g. Kähler Ricci flows starting from a nef metric, one would like to understand the limiting behavior of immortal Ricci flows — remember, our ultimate goal is to evolve the Riemannian metric along the Ricci flow to a canonical one.

The 3-dimensional picture is completed following Perelman's work, see [77]. In contrast to the finite time singularities, collapsing may occur for the large-time behavior. Therefore, it is necessary to investigate the collapsing phenomenon in the setting of Ricci flows.

In general dimensions, the above mentioned goal is far from being reached; a more realistic goal would be constructing and studying the limit of a sequence of Ricci flows $\{(M, g_i(t))\}$, with $g_i(t) := g(t + t_i)$ (or $g_i(t) := t_i^{-1} g(t + t_i)$ depending on the scenario) as $t_i \to \infty$.

This limit may not exist in the strongest sense, i.e. smooth convergence to a smooth Ricci flow (compare [38]), but a weak limit as a metric space may exist for some sequences $t_i \to \infty$.

This weak limit, once in existence, may also acquire a nicer structure, e.g. it may be smooth away from a small set of singularities, since it is resulted from the evolution of a Ricci flow.

In this thesis, we make efforts towards these directions, by proving a uniform distance distortion estimate in any dimension, and then establishing an ε-regularity theorem for gradient shrinking Ricci solitons in dimension 4.

We emphasize again that a key difference between the finite time singularities and the infinite time singularity is, as $t \to \infty$, the no local collapsing result fails to hold in general. Therefore, a key feature of our results is that we have to deal with the lack of a uniform volume ratio lower bound, which adds much more complexities into our study.

1.2 Distance distortion estimate

For a fixed Ricci flow, a fundamental question of Richard Hamilton (see Section 17 of [37]) is to obtain a uniform distance distortion estimate depending on a minimal requirement of the space-time curvature bound. A natural and non-trivial condition is to assume a uniform bound of the scalar curvature in space-time, as evidenced by Kähler-Ricci flows on Fano manifolds. The distance distortion problem in this case is completely settled by Chen-Wang in [22], and again in [24] as an important intermediate step towards their main result. The Käher condition was then dropped by Bamler-Zhang in [61]. See also the previous works of Richard Hamilton [37], Miles Simon [53] and Tian-Wang [54] for several important partial results. However, all these estimates, including the ones of Chen-Wang and Bamler-Zhang, rely on the uniform lower bound of the initial μ-entropy, a crucial condition that we will relax in this note.

As a second motivation, in studying the uniform behavior of *all* Ricci flows, one may have to encounter a family of Ricci flows without a uniform lower bound for the initial μ-entropy. A very common situation is when the family of initial data have their diameter uniformly bounded, but volume degenerating to 0, causing the initial μ-entropy to approach negative infinity. A natural question would then be whether there is a limiting metric space whose metric evolves in a way determined by the Ricci flows. Here we make efforts towards constructing such limiting metric spaces by uniformlly estimating the distance distortion along the Ricci flows:

Theorem 1.2.1. *Let $(M, g(t))$ be a complete Ricci flow solution on $[0, T]$ with initial diameter D_0 and initial volume V, and assume the following conditions:*

1. *$(M, g(0))$, as a closed Riemannian manifold, has its doubling constant uniformly bounded above by C_D, and its L^2-Poincaré constant by C_P, and*

2. *the scalar curvature is uniformly bounded in space-time: $\sup_{M\times[0,T]} |\mathcal{R}_{g(t)}| \leq C_0$.*

There exist two positive constants $\alpha = \alpha(\theta \mid C_D, C_P, C_0, D_0, n, T) < 1$ with

$$\lim_{\theta\to 0} \alpha(\theta \mid C_D, C_P, C_0, D_0, n, T) = 0,$$

and $\nu = \nu(C_D, C_P, C_0, n) < 1$, such that whenever $VD_0^{-n} \leq \nu\omega_n$, for fixed $t \in [0, T]$ and $r \in (0, \sqrt{t})$, if we set $\theta := \min\{1, r/D_0\}$, then

$$\forall x, y \in M \text{ with } d_{g(t)}(x, y) \geq r, \quad \text{and} \quad \forall s \in (t - \alpha r^2, \min\{T, t + \alpha r^2\}),$$

we have

$$\alpha(\theta) d_{g(t)}(x, y) \;\leq\; d_{g(s)}(x, y) \;\leq\; \alpha(\theta)^{-1} d_{g(t)}(x, y). \tag{1.2}$$

Remark 1.2.2. *The requirement that $VD_0^{-n} < \nu\omega_n$ indicates that the initial data is volume collapsing with bounded diameter. Notice that with ω_n being the volume of the n-dimensional Euclidean unit ball, $\nu\omega_n$ is a dimensional constant only depending on C_D, C_P and C_0.*

In the statement of the theorem, θ refers to the relative size of the scale on which we consider distance distortion compared to the initial diameter, and $\alpha \approx \theta^{8n} e^{-\theta^{-2}}$. The bound α becomes worse as the scale on which we observe becomes smaller compared to the initial diameter.

This is reasonable, as demonstrated in the case of collapsing initial data with bounded curvature and diameter: there will be no uniform estimate of the distance distortion in the fiber directions. However, we notice that such estimate is not needed for providing a rough metric structure on the collapsing limit, since eventually it is the estimates in the base directions that we will need. Therefore, regardless of how small the relative scale we are considering, a *uniform* estimate, even though depending on such scale, is indeed what we need.

The previous distance distortion estimates are based on the estimates of the volume ratio change along the Ricci flow: with uniformly bounded scalar curvature and initial entropy, the volume ratio at a point can neither suddenly decrease (no local collapsing theorem of Perelman [78]), nor suddenly increase (non-inflation property due to Chen-Wang [67] and Qi S. Zhang [85]). Discretizing the geodesic distance by the number of fix-sized geodesic balls that suitably cover the minimal geodesic, these non-collapsing and non-inflation properties together provide the desired control of the distance distortion. This type of "ball containment" argument is succinctly discribed in the third section of Chen-Wang [25].

In order to obtain uniform estimates of the change of volume ratio along the Ricci flow, in the Kähler case Chen-Wang [24] studied the Bergman kernel, while in the Riemannian case, Bamler-Zhang [61] relies on Qi S. Zhang's heat kernel estimates in [85].

Our theorem is proven along the same paths that lead to such estimates. However, we need to start from scratch: underlying the estimates of the heat kernel, a corner stone is the expression of the log-Sobolev constant in terms of the initial μ-entropy (see [81] and [83]), which, in the current note, will be replaced by a renormalized version involving the *initial global volume ratio* VD_0^{-n}.

Heuristically speaking, collapsing is a geometric phenomenon, while the behavior of the heat kernel (which reflects the volume ratio) is analytic in nature. The monotonicity of Perelman's functionals along the Ricci flow is another instance where a geometric deformation bears an analytic meaning. A basic principle in dealing with the analytic information associated with collapsing, especially the Dirichlet energy and related objects, is making a correct renormalization. This was first noticed by Kenji Fukaya [72] in the setting of collapsing with bounded curvature and diameter, and then strengthened through a series of work by Cheeger-Colding (see [64], [63], [65] and [66]) to the case with only Ricci curvature lower bound.

Another motivation of proving this estimate is therefore to demonstrate the necessity of the above renormalization principle in the setting of Ricci flows with collapsing initial data: the initial collapsing is a geometric phenomenon, yet in order to obtain the distance distortion estimate, we need to control the analytic quantities — the heat kernel bounds — which could only be made possible through a correct renormalization.

We now outline the series of estimates of the renormalized quantities that lead to the uniform distance distortion estimate. We emphasize that these inequalities are invariant under the parabolic rescaling of the Ricci flow, a crucial point for them to work in a geometric setting. Also notice that the constants involved are determined by C_D, C_P, C_0, D_0, n, but we only write explicitly their dependence on T. Our starting point is a renormalized L^2-Sobolev inequality (see [1] and [79]):

$$\forall u \in H^1(M, g(0)), \quad \left(\int_M u^{\frac{2n}{n-2}} \, \mathrm{d}V_{g(0)}\right)^{\frac{n-2}{n}} \leq C_S (VD_0^{-n})^{-\frac{2}{n}} \int_M |\nabla u|^2 + D_0^2 u^2 \, \mathrm{d}V_{g(0)}, \tag{1.3}$$

where D_0 is the initial diameter and $V := \int_M 1 \, \mathrm{d}V_{g(0)}$ is the initial volume.

Following classical arguments and the definition of the $\mathcal{W}$-functional, this gives a lower bound of the initial entropy (3.3): for any $\tau > 0$,

$$\mu(g(0), \tau) \geq \log VD_0^{-n} - (C_0 D_0^2 + D_0^{-2})\tau - \frac{n}{2}\log(8n\pi e C_S).$$

Here we would like to raise the readers' attention that it is not just the initial total volume V, but the initial global volume ratio VD_0^{-n}, that controls the lower bound of the entropy. This quantity not only technically makes the inequality scaling-correct, but also conceptually reveals the meaning of *collapsing initial data* — volume collapsing with bounded diameter.

Following Perelman's classical argument [78], we could deduce the lower bound of the renormalized volume ratio (see Proposition 3.2.2): there is a uniform $C_{VR}^+(T) > 0$, such that

$$\forall t \in (0, T], \ \forall r \in (0, \sqrt{t}], \quad (VD_0^{-n})^{-1}|B_t(x, r)| \geq C_{VR}^+(T) r^n.$$

Here we start seeing the effect of the correct renormalization: even if the volume ratio fails to have a uniform lower bound, once renormalized by $(VD_0^{-n})^{-1}$, it is indeed bounded below by $C_{VR}^+(T)$.

Further exploring the definition and monotonicity of the $\mathcal{W}$-functional, and following Qi S. Zhang's application [85] of the method of Edward Davies [70], we obtain the following rough upper bound of the renormalized heat kernel (see Proposition 3.3.1): there is a uniform $C_H^+(T) > 0$ such that

$$\forall t \in (0, T], \ \forall s \in (0, t), \ \forall x, y \in M, \quad VD_0^{-n} G(x, s; y, t) \leq C_H^+(T)(t - s)^{-\frac{n}{2}}.$$

For the definition of $G(x, s; y, t)$ see Subsection 2.3. Here we see the duality between the heat and the volume of a Riemannian manifold. Intuitively, the collapsing is an intrinsic geometric procedure, and it should not cause the addition or loss of the total heat. Therefore, if the global volume ratio behavies like $VD_0^{-n} \to 0$, then the heat density should in general behave like $(VD_0^{-n})^{-1} \to \infty$.

Up to this stage it is basically just the interplay between the Sobolev inequality and the $\mathcal{W}$-functional: purely analytic in nature. In order to estimate the distance distortion, we still need a lower bound of the renormalized heat kernel. The original argument of Chen-Wang [67] and Qi S. Zhang [85], however, will not give us the desired bound: their argument, based on the estimate of the reduced length of a space-constant curve at the base point of the heat kernel, is valid regardless of scales; but in our setting there is a drastic difference between the very small scales, which resemble the locally n-dimensional Euclidean property of the manifold, and the large scales, on which the collapsing to a lower dimensional space is observed.

We will overcome this difficulty by obtaining a positive-time diameter bound in terms of the initial diameter, and stick to our principle of keeping the heat-volume duality. The following diameter bound is deduced following an argument of Peter Topping in [80] (see Proposition 3.2.4): there exists a uniform constant $C_{diam} > 0$ such that if the initial global volume ratio is sufficiently small, i.e. $VD_0^{-n} < \nu\omega_n$ for some uniform $\nu \in (0, 1]$, then

$$\forall t \in (0, T], \quad \operatorname{diam}(M, g(t)) \;\leq\; C_{diam} e^{2C_0 t} D_0.$$

This diameter bound is of great technical importance for us, since we will soon use it to deduce an on-diagonal lower bound of the renormalized heat kernel. Conceptually, this bound tells that scales that are comparable to the initial diameter, remain comparable to the diameter at a positive time, up to a uniform factor depending on the time elapsed.

With the help of the diameter bound above, we have the following lower bound of the renormalized heat kernel (see Lemma 3.3.2): there exists a uniform constant $C_H^-(T) > 0$ and a positive function $\Psi(\theta \mid T)$ with $\lim_{\theta\to 0} \Psi(\theta \mid T) = 0$, such that if $VD_0^{-n} \leq \nu\omega_n$,

$$\forall t \in (0, T],\ \forall s \in (0, t),\ \forall x \in M, \quad VD_0^{-n} G(x, s; x, t) \;\geq\; C_{HD}^-(T)\Psi(\theta(s) | \, T)(t - s)^{-\frac{n}{2}}.$$

Here we see that the effect of scales enters into the picture via the factor $\Psi(\theta \mid T)$: for any $t \in (0, T]$ and any $s \in (0, t)$, $\theta(s) := \sqrt{t - s}/D_0$ is the ratio of the (parabolic) scale under consideration compared to the initial diameter; when the scale that we observe approaches 0, relative to the initial diameter, then the lower bound of the renormalized heat kernel will also approach 0. (Rigorously speaking, we actually have $\theta = \sqrt{t - s}/\operatorname{diam}(M, g(t))$ in our mind, but the diameter bound above allows us to compare r directly with D_0, making the definition more canonical.) This estimate naturally leads to a Gaussian type lower bound of the renormalized heat kernel, as well as the non-inflation property of the renormalized volume ratio.

The bounds of the renormalized heat kernel, together with the previous lower bound of the renormalized volume ratio, are enough to prove the desired distance distortion estimate, in view of the arguments in proving Theorem 1.1 of [61].

1.3 ε-Regularity for 4-D Ricci shrinkers

A four dimensional gradient shrinking Ricci soliton (a.k.a. 4-D Ricci shrinker) is a four dimensional Riemannian manifold (M^4, g) equipped with a potential function f satisfying the defining equation (with a fixed scaling)

$$\mathcal{R}c_g + \nabla^2 f = \frac{1}{2} g, \tag{1.4}$$

where $\mathcal{R}c_g$ denotes the Ricci curvature of the Riemannian metric g.

We intend to study uniform behaviors of complete non-compact 4-d Ricci shrinkers through their moduli spaces, whose compactification is of foundamental importance. Poineered by the work of Cao-Sesum [5], Xi Zhang [59], Brian Weber [56], Chen-Wang [22] and Zhenlei Zhang [60] in this direction, the most satisfactory compactness results to date, obtained by Robert Haslhofer and Reto Müller (né Buzano) [39] [40], assume a uniform entropy lower bound. In fact, Bing Wang has conjectured that a 4-d Ricci shrinker should have an *a priori* entropy lower bound, depending solely on some topological restrictions. To verify this, however, we need to study the degeneration of the metrics along sequences of 4-d Ricci shrinkers *without* uniform entropy lower bound, and then use contradiction arguments to rule out the potential occurrence of such a situation. (For the relation between entropy lower bound and no local collapsing property, see [78] and [77].)

The obvious analogy between Ricci solitons and Einstein manifolds brings us the foundational work of Cheeger-Tian [20], which, built on Anderson's ε-regularity with respect to collapsing [1], obtains a new ε-regularity theorem for any four dimensional Einstein manifolds. Cheeger-Tian conjectured (in Section 11 of [20]) that a similar result should hold for four dimensional Ricci solitons, moreover:

"Of particular interest is the case of shrinking Ricci solitons."

Our first theorem confirms their conjecture for 4-D Ricci shrinkers:

Theorem 1.3.1 (ε-regularity for 4-D Ricci shrinkers)**.** *Let (M^4, g, f) be a complete non-compact four dimensional shriking Ricci soliton and fix $R > 0$. Then there* *exists $r_R > 0$, $\varepsilon_R > 0$ and $C_R > 0$, such that for any $r \in (0, r_R)$ and $B(p, r) \subset B(p_0, R)$, the weighted local L^2-curvature control*

$$\int_{B(p,r)} |\mathcal{R}m_g|^2 \, e^{-f} dV_g \ \leq \ \varepsilon_R$$

implies the local boundedness of curvature

$$\sup_{B(p,\frac{1}{4}r)} |\mathcal{R}m_g| \ \leq \ C_R \, r^{-2}.$$

Here $p_0 \in M$ is a minimum point of f, see Lemma 2.2.2 for more details.

In order to further motivate our theorem, we notice that Cheeger-Tian's ε-regularity theorem could be viewed as a non-trivial localization of the fact that for a closed four dimensional Einstein manifold (M, g), its Euler characteristic can be computed as

$$\chi(M) = \frac{1}{8\pi^2} \int_M |\mathcal{R}m_g|^2 \, \mathrm{d}V_g.$$

If $\|\mathcal{R}m_g\|_{L^2(M)}$ is sufficiently small, then the integrity of $\chi(M)$ will force $\chi(M) = 0$, whence the flatness of (M, g).

Similarly, on a closed four dimensional Ricci soliton (M, g, f), we have

$$\chi(M) = \frac{1}{8\pi^2} \int_M \left(|\mathcal{R}m_g|^2 - |\mathring{\mathcal{R}}c_g|^2 \right) \, \mathrm{d}V_g,$$

where $\mathring{\mathcal{R}}c_g$ is the traceless Ricci tensor. Now if $\|\mathcal{R}m_g\|_{L^2(M)} < \pi$, as $|\mathring{\mathcal{R}}c_g|^2 \leq 2|\mathcal{R}m_g|^2$, we must have $-1 < \chi(M) < 1$, and thus $\chi(M) = 0$. It follows that (M, g, f) must be a steady or an expanding Ricci soliton: otherwise, were (M, g, f) a 4-D Ricci shrinker, $\pi_1(M)$ must be finite by [57], leading to $\chi(M) \geq 2$ that contradicts the vanishing of $\chi(M)$. But a closed steady or expanding four dimensional Ricci soliton must be Einstein, so $|\mathring{\mathcal{R}}c| \leq |\nabla^2 f| \equiv 0$ by (1.4), and $\|\mathcal{R}m_g\|^2_{L^2(M)} = 8\pi^2\chi(M) = 0$, which means that (M, g) must be flat.

So our ε-regularity theorem for 4-D Ricci shrinkers is a localization of the above rigidity of closed four dimensional Ricci solitons, and particularly suits the study of non-compact ones. Notice however, as pointed out in [39], that *"most interesting singularity models are non-compact, the cylinder being the most basic example"*.

We also need to notice that the dependence of the constants in our ε-regularity theorem is a new feature caused by the presence of the potential function: the allowence of the existence of non-compact Ricci shrinkers — in fact, they strongly resemble positive Einstein manifolds, which are compact, scaling rigid, and which, may only admit families of metrics that are either collapsing everywhere, or else nowhere. Similar phenomenon occuring for geodesic balls of fixed size centered at the base point in a non-compact 4-D Ricci shrinker, our ε-regularity theorem only applies within a fixed distance from the base point. Moreover, in our future presentations, we will fix such a distance and do not elaborate on writing down scaling invariant formulae.

The proof of Theorem 1.3.1 is based on the recent advances in the study of shrinking Ricci solitons, and the comparison geometry of Bakry-Émery Ricci curvature lower bound (see, among others, [39] [40], [4], [21], [45], [47], [57] and [58], etc.). Here we briefly outline the proof of Theorem 1.3.1, which follows the strategy of Cheeger-Tian [20] in the Einstein case. We will indicate the necessary improvements in order to deal with the lack of the Einstein's equation.

Starting point

Our starting point is a 4-D Ricci shrinker version of Anderson's ε-regularity with respect to collapsing [1], see Proposition 2.4.6. For any $r \leq 1$ and $B(p,r) \subset B(p_0,R)$, let the renormalized energy of $B(p,r)$ be defined as (see Definition 2.4.7)

$$I^f_{\mathcal{R}m}(p,f) \;:=\; \frac{\bar{\mu}_R(r)}{\mu_f(B(p,r))}\int_{B(p,r)} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f,$$

where $\mathrm{d}\mu_f := e^{-f}\mathrm{d}V_g$ and we will denote $\mu_f(U) = \int_U 1 \,\mathrm{d}\mu_f$ for any $U \subset M$. We notice that it is continuously increasing in r. Anderson's theorem asserts the existence of positive constants $\varepsilon_A(R)$ and $C_A(R)$, such that

$$I^f_{\mathcal{R}m}(p,r) \;\leq\; \varepsilon_A(R) \implies \sup_{B(p,\frac{r}{2})} |\mathcal{R}m| \;\leq\; C_A(R)\, r^{-2} I^f_{\mathcal{R}m}(p,r)^{\frac{1}{2}}.$$

However, the input of our ε-regularity theorem seems to be quite far from fulfilling the smallness of $I^f_{|Rm|}(p,r)$ required by this theorem: when collapsing happens, the smallness of the energy $\int_{B(p,r)} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f$ may be caused by the smallness of $\mu_f(B(p,r))$. This difficulty is overcome in two steps: firstly the key estimate guarantees a uniform bound of $I^f_{|\mathcal{R}m|}(p,2r)$ from the smallness of $\int_{B(p,r)} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f$, as assumed in Theorem 1.3.1; then the fast decay proposition guarantees that after a definite number, say j_R times, of bisecting the given scale r, $I^f_{|\mathcal{R}m|}(p, 2^{1-j_R}r)$ is small enough so that Anderson's theorem applies. Throughout the introduction we will let $B(U,s)$ denote the s-tubular neighborhood around any set $U \subset M$, and $A(U;s,r) = B(U,r)\backslash B(U,s)$ for $r > s > 0$.

Key estimate

The key estimate (Proposition 4.4.5) follows from an interation argument, in each step of which, the energy over a domain U is roughly bounded by the $\frac{3}{4}$-power of the energy on some s-tubular neighborhood of U, with some carefully chosen small $s \in (0, r)$:

$$\int_U |\mathcal{R}m|^2 \,\mathrm{d}\mu_f \;\leq\; C(R)\,\mu_f(B(U,s))\left(s^{-4} + \left(\frac{s^{-\frac{4}{3}}}{\mu_f(B(U,s))}\int_{B(U,s)} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f\right)^{\frac{3}{4}}\right), \quad (1.5)$$

see also the estimates (4.23) and (4.24). Now we briefly explain how to obtain this estimate.

If U is collapsing with locally bounded curvature (see Definition 4.1.10), Cheeger-Tian proved in Section 2 of [20] that a slightly larger neighborhood U' of U acquires a nilpotent structure, which implies the vanishing of the Euler characteristic of U':

$$0 \;=\; \chi(U') \;=\; \int_{U'} \mathcal{P}_\chi \;+\; \int_{\partial U'} \mathcal{TP}_\chi. \quad (1.6)$$

For 4-D Ricci shrinkers, $8\pi^2\mathcal{P}_\chi \;=\; \left(|\mathcal{R}m|^2 - |\mathring{\nabla}^2 f|^2\right)\mathrm{d}V_g$ since $\mathring{\mathcal{R}c}_g \;=\; \mathring{\nabla}^2 f$ by the defining equation (1.4); and $\mathcal{TP}_\chi$ is a three form on $\partial U'$ with coefficients determined by $\mathcal{R}m_g|_{\partial U'}$ and $II_{\partial U'}$, the second fundamental form of $\partial U'$, see (2.17). The integral of $|\mathcal{R}m|^2 - |\mathring{\nabla}^2 f|^2$ over U', using (1.6), is then pushed to the boundary integral $\int_{\partial U'} \mathcal{TP}_\chi$.

The control of $\int_{\partial U'} \mathcal{TP}_\chi$ relies on the equivariant good chopping theorem, (stated and used in Theorem 3.1 of [20], also see Appendix A for a detailed proof,) which enables us to choose U' so that $\partial U'$ is saturated by the nilpotent structure, and essentially bounds $|II_{\partial U'}|$ by $|\mathcal{R}m|^{\frac{1}{2}}$. It follows that $|\mathcal{TP}_\chi|$ is then controlled by $|\mathcal{R}m|^{\frac{3}{2}}$, which, via (1.6), improves the integration of $|\mathcal{R}m|^2$ over U' to integrating $|\mathcal{R}m|^{\frac{3}{2}}$ over $\partial U'$. Averaging on a tubular neighborhood of ∂U and using a maximal function argument, Cheeger-Tian then obtained:

$$\left|\int_{\partial U'} \mathcal{TP}_\chi\right| \leq C(R)\mu_f(A(U;0,s))\left(s^{-4} + \left(\frac{s^{-\frac{4}{3}}}{\mu_f(A(U;0,s))}\int_{A(U;\frac{1}{4}s,\frac{3}{4}s)} |\mathcal{R}m|^2 \,\mathrm{d}V_g\right)^{\frac{3}{4}}\right). \quad (1.7)$$

Invoking (1.6), we obtain a control of $\int_U |\mathcal{R}m|^2 - |\mathring{\nabla}^2 f|^2 \,\mathrm{d}\mu_f$ by the right-hand side of the above estimate, since $\mathrm{d}\mu_f$ is comparable to $\mathrm{d}V_g$ in $B(p_0, R)$ in a uniform way.

In the Einstein case, since $|\nabla^2 f| \equiv 0$, the above dominating term on the right-hand side suffices to provide the desired control of $\int_U |\mathcal{R}m|^2 \,\mathrm{d}\mu_f$ in (1.5). For 4-D Ricci shrinkers, however, $|\mathring{\nabla}^2 f|^2$ does not vanish and the control of $\int_U |\nabla^2 f|^2 \,\mathrm{d}\mu_f$ relies on the gradient estimate $|\nabla f| \;\leq\; R/2 + \sqrt{2}$ (see Lemma 2.2.3), as well as

Cheeger-Colding's cut-off function (see Lemma 2.3.3):

$$\int_U |\nabla^2 f|^2 \,\mathrm{d}\mu_f \;\le\; C(R)\,\mu_f(B(U,s))\,s^{-2}, \tag{1.8}$$

see Lemma 4.4.4. When $s > 0$ is very small, the right-hand side of this estimate is dominated by $\mu_f(B(U,s))\,s^{-4}$, so replacing $\mu_f(A(U;0,s))$ by $\mu_f(B(U,s))$ in (1.7), we could obtain the desired energy estimate (1.5) for the iteration argument.

Fast decay

The fast decay proposition (Proposition 4.4.7) asserts the existence of some gap $\eta_R \in (0,1)$, such that if the energy $\int_{B(p,r)} |\mathcal{R}m|^2\,\mathrm{d}\mu_f$ and volume $\mu_f(B(p,r))$ of a ball $B(p,r)$ is sufficiently small, then

$$I^f_{\mathcal{R}m}(p,r/2) \;\le\; (1-\eta_R)\,I^f_{\mathcal{R}m}(p,r).$$

This is proved by a contradiction argument. Suppose on the contrary, for positive $\eta \to 0$, there are counterexamples $B(p,r) \subset B(p_0,R)$ of vanishing μ_f-volume $\mu_f B(p,r) \to 0$ and

$$I^f_{\mathcal{R}m}(p,r/2) \;>\; (1-\eta)\,I^f_{\mathcal{R}m}(p,r),$$

we could use volume comparison to see that $A(p;r/2,r)$ is an almost μ_f-volume annulus (4.40). By the theory of Cheeger-Colding (Lemma 2.3.4), this property implies that a slightly smaller annular region in $A(p;r/2,r)$ is an almost metric cone, whose radial distance approximated by some smooth function $\tilde{u}$. The approximation is in the C^0-sense, as well as the *average* H^2-sense, see (4.41) – (4.43).

Moreover, the key estimate implies the almost vanishing (4.38) and regularity (4.39) of the curvature on the annulus $A(p;r/2,r)$, and all derivative controls of $\tilde{u}$, see (4.44).

Let $W = B(x,3r/2) \cup \tilde{u}^{-1}(r/2,a)$ for some regular value $a \in (3r/4,r)$ of $\tilde{u}$. On the one hand, $\int_W |\mathcal{R}m|^2\,\mathrm{d}V_g$ is positive but very small (as assumed by the ε-regularity theorem), say

$$0 \;<\; \int_W |\mathcal{R}m|^2\,\mathrm{d}V_g \;<\; \frac{1}{4};$$

on the other hand, $\partial W = \tilde{u}^{-1}(a)$ smoothly approximates the outer boundary of an annulus A_∞ in a flat cone. Intuitively, since the cone is flat, we know that the second fundamental form of its outer boundary, $II_{\partial_+ A_\infty}$, is positive, and its boundary Gauss-Bonnet-Chern term $|\mathcal{TP}_\chi|_{\partial A_\infty}| \equiv 1$. Thus the smoothness of the approximation $\tilde{u}^{-1}(a) \to \partial_+ A_\infty$ together with the vanishing of curvature (4.38) will imply the positivity of coefficients of $\mathcal{TP}_\chi|_{\partial W} = \mathcal{TP}_\chi|_{\tilde{u}^{-1}(a)}$, and the collapsing implies the smallness of its integral, say

$$0 < \int_{\partial W} \mathcal{TP}_\chi < \frac{1}{4}. \tag{1.9}$$

In this way, for Einstein manifolds, we have obtained a smooth bounded domain W whose Euler characteristic $\chi(W)$ satisfies

$$0 < \chi(W) = \frac{1}{8\pi^2} \int_W |\mathcal{R}m|^2 \, \mathrm{d}V_g + \int_{\partial W} \mathcal{TP}_\chi < \frac{1}{2}.$$

This is impossible.

More specifically, in the Einstein case, Cheeger-Tian appealed to the theory of Cheeger-Colding-Tian (see Theorem 3.7 of [13]), which controls the average error $|II_{\tilde{u}^{-1}(a)} - II_{\partial_+ A_\infty}|$ on the level set $\tilde{u}^{-1}(a)$, see (8.14) – (8.19) of [20]. This was implemented by lifting to a local covering, which is non-collapsing, and where, since (4.42) and (4.43) are estimates of the *average*, similar estimates (4.52) and (4.53) hold.

In the case of 4-D Ricci shrinkers, the control of $\int_W |\mathcal{R}m|^2 \, \mathrm{d}\mu_f$ does not impose a control of $\int_W \mathcal{P}_\chi$ directly, due to the presence of the term $|\mathring{\nabla}^2 f|^2$. However, we could further assume $I^f_{|\mathcal{R}m|}(p, r/2) > \varepsilon_A(R)$, since otherwise there is no need of proving the proposition. This assumption gives us a lower bound of $\int_W |\mathcal{R}m|^2 \, \mathrm{d}\mu_f$ proportional to $\mu_f(B(p,r))\, r^{-4}$. On the other hand, recall that we have the estimate (1.8) of $|\nabla^2 f|^2$, with $s = 4$ and $U = B(p,r)$. Thus whenever r is sufficiently small, $\int_W |\nabla^2 f|^2 \, \mathrm{d}\mu_f$ is dominated by the energy $\int_W |\mathcal{R}m|^2 \, \mathrm{d}\mu_f$, so

$$0 < \frac{1}{8\pi^2} \int_W |\mathcal{R}m|^2 - |\mathring{\nabla}^2 f|^2 \, \mathrm{d}V_g = \int_W \mathcal{P}_\chi.$$

This mainly accounts for the bound of scale r_R in the statement of Theorem 1.3.1. The small upper bound of $\int_W \mathcal{P}_\chi$ follows easily from the assumption of the ε-regularity theorem.

In controlling the boundary Gauss-Bonnet-Chern integral $\int_{\partial W} \mathcal{TP}_\chi$, we notice that the theory of Cheeger-Colding-Tian [13] is not available for 4-D Ricci shrinkers. (Though we expect a version of this theory to hold in the case of Bakry-Émery Ricci curvature bounded below.) We turn to the regularity (4.44) of $\tilde{u}$: we could find a fine enough net $\{x_j\}$ in a slightly smaller annulus contained in $A(p; r/2, r)$, such that at each point of the net we have

$$II_{\tilde{u}^{-1}(\tilde{u}(x_j))}(x_j) > \frac{1}{2} I_3,$$

where I_3 is the 3×3-identity matrix; by the regularity of $\tilde{u}$ (4.44) and the closeness of points in the net, we could then obtain a bound

$$\forall a \in (0.7r, 0.8r), \quad II_{\partial \tilde{u}^{-1}(a)} > \frac{1}{4} I_3.$$

This, together with the vanishing of the curvature (4.38), give the desired point-wise positivity and upper bound of coefficients of $\mathcal{TP}_\chi|_{\partial\tilde{u}^{-1}(a)}$, for any $a \in (0.7r, 0.8r)$. Integrating over $\partial\tilde{u}^{-1}(a)$ (for some $a \in (0.7r, 0.8r)$) and using the volume collapsing of $\partial\tilde{u}^{-1}(a)$, we could obtain the desired bound (1.9), see (4.57) – (4.62), thus concluding the proof of the fast decay proposition.

Our ε-regularity theorem sees a few applications in understanding the moduli space of complete non-compact 4-D Ricci shrinkers. Our second theorem is in this direction:

Theorem 1.3.2. *Let (M_i, g_i, f_i), be a sequence of complete non-compact 4-D Ricci shrinkers with $\mathcal{R}_{g_i} \le \bar{S}$ and $|\chi(M_i)| \le \bar{E}$, then there exist positive numbers $\bar{R} = \bar{R}(\bar{S})$ and $\bar{J} = \bar{J}(\bar{E}, \bar{S})$ together with the following data:*

1. *a subsequence, still denoted by (M_i, g_i, f_i),*
2. *marked points $\{p_i^1, \cdots, p_i^J\} \subset B(p_i^0, \bar{R})$ with $J \le \bar{J}$, and*
3. *a length space (X, d_∞) with marked points $\{x_\infty^1, \cdots, x_\infty^J\}$,*

such that $(M_i, g_i, p_i^1, \cdots, p_i^J) \to (X, d_\infty, x_\infty^1, \cdots, x_\infty^J)$ in the sense of strong multi-pointed Gromov-Hausdorff convergence.

Here we notice that f has a global minimum point p_0 (see [6] and [39]), which will be our designated base point. For more details about the *"strong multi-pointed Gromov-Hausdorff convergence"*, see Definition 2.5.3. Notice that once we are given a global upper bound of scalar curvature, then as we will show later, $\chi(M)$ is a finite number, so our consideration is well-posed. The condition on bounded Euler characteristic is topological in nature, while the assumption on the scalar curvature, although being natural in the Kähler setting, is technical and we hope to remove in our future work.

Chapter 2

Preliminaries

2.1 Ricci flow

2.1.1 Heat equation solutions coupled with the Ricci flow

In this subsection we collect some point-wise estimates of heat equation solutions coupled with the Ricci flow. For any $x, y \in M$ and $0 \leq s < t < T$, we will let $G(x, s; y, t)$ denote the heat kernel coupled with the Ricci flow based at (x, s), i.e. fixing $(x, s) \in M \times [0, T)$, we have

$$(\partial_t - \Delta_{g(t)})G(x, s; -, -) = 0, \quad \text{and} \quad \lim_{t \downarrow s} G(x, s; -, -) = \delta_{(x,s)}, \tag{2.1}$$

where $\delta_{(x,s)}$ is the space-time Dirac delta function at $(x, s) \in M \times [0, T)$. On the other hand, fixing $(y, t) \in M \times (0, T)$ and setting (x, s) free, this same function satisfies

$$(\partial_s + \Delta_{g(s)} + \mathcal{R}_{g(s)})G(-, -; y, t) = 0, \quad \text{and} \quad \lim_{s \uparrow t} G(-, -; y, t) = \delta_{(y,t)}, \tag{2.2}$$

i.e. $G(-, -; y, t)$ is the conjugate heat kernel coupled with the Ricci flow based at (y, t).

Our heat kernel lower bound of Gaussian type will be base on the following key gradient estimate due to Qi S. Zhang, see Theorem 3.3 in [82]:

Proposition 2.1.1 (Gradient estimate)**.** *Let $(M, g(t))$ be a Ricci flow on a complete n-manifold M over time $[0, T)$ and let $u \in C^\infty(M \times [0, T))$ be a positive solution to the heat equation $(\partial_t - \Delta)u = 0$, $u(\cdot, 0) = u_0$ coupled with the Ricci flow. Then there is a constant $B < \infty$ depending only on n, such that if $u \leq a$ on $M \times [0, T]$ for some constant $a > 0$, then $\forall (x, t) \in M \times (0, T]$,*

$$\frac{|\nabla u|(x, t)}{u(x, t)} \leq \sqrt{\frac{1}{t}} \sqrt{\log \frac{a}{u(x, t)}}. \tag{2.3}$$

Note that this inequality also reads

$$\left|\nabla\sqrt{\log\frac{a}{u}}\right|(x,t)\leq\frac{1}{\sqrt{t}}$$

for any $(x,t)\in M\times(0,T]$.

Now for any fixed $(x,t_0)\in M\times[0,T)$, let $G(x,t_0;-,-)$ be the coupled heat kernel described above. Viewing $u(y,s)=G(x,t_0;y,s)$ as a coupled heat equation solution on $M\times[\frac{t_0+t}{2},t]$, and integrating the above inequality along minimal geodesics, we could get a Harnack inequality for heat equation solutions coupled with the Ricci flow, also see inequality (3.44) of [82]:

Corollary 2.1.2. *We have* $\forall(y,t),(y',t)\in M\times(t_0,T]$,

$$G(x,t_0;y,t)\ \leq\ H(n)\left(\sup_{M\times[(t_0+t)/2,t]}G(x,t_0;-,-)\right)^{\frac{1}{2}}G(x,t_0;y',t)^{\frac{1}{2}}e^{H'(n)d_t(y,y')^2/(t-t_0)},\tag{2.4}$$

where $H(n)$ *and* $H'(n)$ *are dimensional constants.*

In order to estimate the distance distortion we also need a time derivative bound of the coupled heat kernel. This is achieved by the following estimate, which is Lemma 3.1(a) in [61]:

Proposition 2.1.3. *Let* $(M,g(t))$ *be a Ricci flow on a closed n-manifold* M *over time* $[0,T]$ *and let* $u\in C^\infty(M\times[0,T])$ *be a positive solution to the heat equation* $(\partial_t-\Delta)u=0$, $u(\cdot,0)=u_0$ *coupled with the Ricci flow. Then there is a constant* $B<\infty$ *depending only on n, such that if* $u\leq a$ *on* $M\times[0,T]$ *for some constant* $a>0$, *then* $\forall(x,t)\in M\times(0,T]$,

$$\left(|\Delta u|+\frac{|\nabla u|^2}{u}-a\mathcal{R}\right)(x,t)\leq\frac{aB(n)}{t}.$$

Again, setting $u(y,t)=G(x,t_0;y,t)$ for $t>t_0$, and considering it as a coupled heat equation solution on $M\times(\frac{t_0+t}{2},t)$, we immediately obtain

$$\begin{aligned}&|\partial_tG(x,t_0;y,t)|+\frac{|\nabla_yG(x,t_0;y,t)|^2}{G(x,t_0;y,t)}\\ \leq&\sup_{M\times[(t_0+t)/2,t]}G(x,t_0;-,-)\left(\mathcal{R}_{g(t)}(x,t_0;y,t)+\frac{B(n)}{t-t_0}\right).\end{aligned}\tag{2.5}$$

2.1.2 Perelman's $\mathcal{W}$-functional

As mentioned in the introduction, the monotonicity of Perelman's $\mathcal{W}$-functional along the Ricci flow is an instance where a geometric deformation bears an analytic meaning. This connection is the foundation of the current note. We now recall Perelman's $\mathcal{W}$-functional [78]: for any $\bar{t} \in (0, T]$, any $v^2 \in C^1(M, g(\bar{t}))$ and any $\tau > 0$,

$$\mathcal{W}(g(\bar{t}), v^2, \tau) \;:=\; \int_M \tau\left(4|\nabla v|^2 + \mathcal{R}_{g(\bar{t})}v^2\right) - v^2 \log v^2 - n\left(1 + \frac{1}{2}\log(4\pi\tau)\right)v^2 \mathrm{d}V_{g(\bar{t})}. \tag{2.6}$$

If we require $\int_M v^2\,\mathrm{d}V_{g(\bar{t})} = 1$, let τ solve $\tau'(t) = -1$, and let $u(t)$ solve the conjugate heat equation along the Ricci flow: $(\partial_t + \Delta - \mathcal{R})u = 0$ with the prescribed final data $u(\bar{t}) := v^2$, then we have the monotone increasing property of the $\mathcal{W}$-functional:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(g(t), u(t), \tau) \;\geq\; 0.$$

The μ-entropy is defined as

$$\mu(g(t), \tau) \;:=\; \inf_{\int_M v^2\;\mathrm{d}V_{g(t)}=1} \mathcal{W}(g(t), v^2, \tau),$$

and letting the data varying similarly as in the $\mathcal{W}$-functional, we also obtain the monotone increasing property of the μ-entropy.

A major difficulty blocking Hamilton's program in solving the Poincaré conjecture via the Ricci flow approach lies in the understanding of the finite time singularities under the evolution of Ricci flows, the core problem of which being the possible collapsing with bounded curvature at small scales. Perelman's first major contribution in this direction is therefore his celebrated "No local collapsing theorem", which rules out the possibility of volume degeneration under bounded curvature, when approaching a singular time (finite!) slice. Perelman's main tool is the monotonicity of the $\mathcal{W}$-functional, which proves even stronger statements than just locally volume non-collapsing with respect to the full curvature bound — the volme ratio is in fact bounded in terms of a scalar curvature upper bound!

Here we present Perelman's no local collapsing theorem:

Theorem 2.1.4 (No local collapsing)**.** *Let $(M, g(t))$ be a Ricci flow solution on $[0, T)$. Assume that the scalar curvature is uniformly bounded from above by C_0 in space-time, then there is a constant $C_V(T)$ depending only on $\mu(g(T), 0)$ such that for any time $t \in [0, T)$ and any scale r such that $r^2 \in (0, t]$,*

$$\frac{|B_t(x, r)|}{r^n} \;\geq\; C_V(T). \tag{2.7}$$

2.2 Gradient shrinking Ricci soliton

Given a 4-D Ricci shrinker (M, g, f), in this section we record those properties needed later in the thesis. The results, except for the equations concerning the Euler characteristic, are valid in general dimensions, but we present them in the four dimensional setting for the sake of simplicity. A good overall reference on topics covered here is Huai-Dong Cao's Lecture notes [4].

2.2.1 Equations of the potential

We start with taking trace of the defining equation (1.4) to get

$$\mathcal{R} + \Delta f = 2. \tag{2.8}$$

We also notice the fundamental observation due to Hamilton [37] states that the quantity $\mathcal{R} + |\nabla f|^2 - f$ is a constant on M, and in this paper we will make the following normalization for the potential function:

$$\mathcal{R} + |\nabla f|^2 = f. \tag{2.9}$$

Subtracting (2.9) from (2.8), we will get an elliptic equation of f that does not involve any curvature term:

$$\Delta f - |\nabla f|^2 = 2 - f. \tag{2.10}$$

This equation is of fundamental importance for our argument to obtain various estimates in later sections, since it gives a the Weitzenböck formula of f:

$$\Delta^f |\nabla f|^2 \ = \ 2|\nabla^2 f|^2 - |\nabla f|^2, \tag{2.11}$$

where the drifted Laplacian $\Delta^f \ := \ \Delta - \nabla f \cdot \nabla$, and we used the defining equation (1.4), the elliptic equation (2.10), together with the equality $\nabla|\nabla f|^2 \cdot \nabla f \ = \ 2\nabla^2 f(\nabla f, \nabla f)$,.

2.2.2 Equations of the curvature

On the other hand, the curvature satisfies the following elliptic equations (see [50]):

$$\begin{aligned} \Delta\mathcal{R} - \nabla f \cdot \nabla\mathcal{R} \ &= \ \mathcal{R} - 2|\mathcal{R}c|^2, & (2.12)\\ \Delta\mathcal{R}c - \nabla f \cdot \nabla\mathcal{R}c \ &= \ \mathcal{R}c - 2\mathcal{R}m * \mathcal{R}c, \quad \text{and} & (2.13)\\ \Delta\mathcal{R}m - \nabla f \cdot \nabla\mathcal{R}m \ &= \ \mathcal{R}m + \mathcal{R}m * \mathcal{R}m. & (2.14) \end{aligned}$$

By the maximum principle applied to (2.12), it was observed in [21]:

Lemma 2.2.1. $\mathcal{R} > 0$ *unless* (M, g) *is flat.*

Also see [43] for a uniform lower bound only depending on the entropy.

2.2.3 Equations of the Euler characteristic

Moreover, on the topological side, the 4-Dimensional Riemannian manifold (M, g) has the localized Euler characteristic of any open subset $U \subset M$ expressed as

$$\chi(U) = \int_U \mathcal{P}_\chi + \int_{\partial U} \mathcal{T}\mathcal{P}_\chi, \tag{2.15}$$

provided that the integrals are defined. Here the Pfaffian 4-form $\mathcal{P}_\chi$ is given by

$$\mathcal{P}_\chi = \frac{1}{8\pi^2}\left(|\mathcal{W}|^2 - \frac{1}{2}\left|\mathring{\mathcal{R}c}\right|^2 + \frac{\mathcal{R}^2}{24}\right)\mathrm{d}V_g = \frac{1}{8\pi^2}\left(|\mathcal{R}m|^2 - \left|\mathring{\nabla}^2 f\right|^2\right)\mathrm{d}V_g, \tag{2.16}$$

where $\mathcal{W}$ is the Weyl tensor of $\mathcal{R}m$, $\mathring{\nabla}^2 f = \nabla^2 f - \frac{\Delta f}{4} g$ is the traceless Hessian of f, and we have used the defining equation (1.4) in the second equality. For the boundary 3-form $\mathcal{T}\mathcal{P}_\chi$, if we denote the area form of ∂U by $\mathrm{d}\sigma$, and let $\{e_i\}$ $(i = 1, 2, 3)$ be an orthonormal local frame tangent to ∂U diagonalizing its second fundamental form $II_{\partial U}$, then we have

$$\mathcal{T}\mathcal{P}_\chi = \frac{1}{4\pi^2}\left(2k_1k_2k_3 - k_1\mathcal{K}_{23} - k_2\mathcal{K}_{13} - k_3\mathcal{K}_{12}\right)\mathrm{d}\sigma, \tag{2.17}$$

where for $i, j = 1, 2, 3$, $\mathcal{K}_{ij} = \mathcal{R}m(e_i, e_j, e_j, e_i)$ is the sectional curvature along the tangent plane spanned by e_i and e_j, and $k_i = II_{\partial U}(e_i, e_i)$ is the principal curvature of ∂U, see [39].

2.2.4 Potential and volume growth

The potential function f obeys a very nice growth control by distance function both from below and above. This was first proved by Cao-Zhou [6]. Here, we shall need the improved version by Haslhofer-Müller [39]:

Lemma 2.2.2 (Potential growth)**.** *Let (M, g, f) be a 4-D Ricci shrinker such that the normalization condition (2.9) is satisfied. Then there exists a point $p_0 \in M$ where f attains its infimum and*

$$\forall x \in M, \quad \frac{1}{4}\left(\max\{\mathbf{d}(x) - 20, 0\}\right)^2 \le f(x) \le \frac{1}{4}\left(\mathbf{d}(x) + 2\sqrt{2}\right)^2, \tag{2.18}$$

where $\mathbf{d}(x) := d(x, p_0)$. Moreover, all minimum points of f is contained in the geodesic ball $B\left(p_0, 10 + 2\sqrt{2}\right)$.

From the normalization (2.9), the non-negativity of scalar curvature Lemma 2.2.1 and the above growth control of potential (2.18), we have the following gradient estimate:

Lemma 2.2.3 (Gradient estimate for potential)**.** *Let* (M, g, f) *be a 4-D Ricci shrinker such that the normalization condition (2.9) is satisfied, then*

$$|\nabla f| \leq \frac{\mathbf{d}}{2} + \sqrt{2}. \tag{2.19}$$

Moreover, the following control of volume growth is discovered by [6] and [46]:

Lemma 2.2.4 (Volume growth)**.** *Let* (M, g, f) *be a complete non-compact 4-D Ricci shrinker, then there exists some constant* $C_{CMZ} > 0$*, such that* $\forall r > 10$,

$$Vol_g(B(p_0, r)) \leq C_{CMZ} r^4.$$

Moreover, if u *is any function on* M *satisfying* $|u| \leq Ae^{\alpha \mathbf{d}^2}$ *for some* $\alpha \in [0, \frac{1}{4})$ *and* $A > 0$*, then*

$$\int_M |u| e^{-f} dV_g < \infty.$$

Especially, the weighted volume of M *is finite, i.e.* $\int_M e^{-f} dV_g < \infty$.

2.3 Comparison geometry of Bakry-Émery Ricci curvature

Compared to Einstein manifolds, one drawback of Ricci solitons comes from the lack of a uniform Ricci curvature lower bound, whence the lack of volume ratio monotonicity. However, Ricci solitons do satisfy the Bakry-Émery Ricci curvature bounds. If we define $\mathcal{R}c_f := \mathcal{R}c + \nabla^2 f$, then the defining equation (1.4) becomes

$$\mathcal{R}c_f = \frac{1}{2} g, \tag{2.20}$$

saying that the Bakry-Émery Ricci curvature of a 4-D Ricci shrinker is half the metric tensor. This subsection explores the analogy of 4-D Ricci shrinkers and manifolds with uniform Ricci lower bound, basic references being [45] and [57].

2.3.1 Weighted volume comparison

The measure compatible with the Bakry-Émery Ricci curvature is the weighted measure $\mathrm{d}\mu_g := e^{-f} \mathrm{d}V_g$, and according to [57], there stands a weighted volume comparison theorem, the counterpart of the Bishop-Gromov volume comparison for the Ricci lower bound case. For a 4-D Ricci shrinker (M, g, f) viewed as a metric measure space $(M, g, \mathrm{d}\mu_f)$, we define its comparison metric measure space as following:

Definition 2.3.1 (Metric measure space form). *For any $R > 2\sqrt{2}$ fixed, define the metric measure space $M^4_R := (\mathbb{R}^4, g_{Euc}, d\bar{\mu}_R)$, the four dimensional Euclidean space equipped with the weighted measure $d\bar{\mu}_R$. Here we define $d\bar{\mu}_R(\mathbf{x}) := e^{R|\mathbf{x}|}dx^1 \wedge \cdots \wedge dx^4$ for any $\mathbf{x} = \left(x^1, x^2, x^3, x^4\right)^T \in \mathbb{R}^4$. Also let the weighted volume of radius r ball centered at the origin of M^4_R be defined as*

$$\bar{\mu}_R(r) := \int_{B(\mathbf{0},r)} 1 \; d\bar{\mu}_R.$$

Moreover, define the area function $\bar{\mu}'_R(r) := 2\pi^2 e^{Rr} r^3$.

We immediately notice that

$$\omega_4 r^4 \leq \mu_R(r) \leq e^{R^2}\omega_4 r^4, \tag{2.21}$$

where ω_4 is the volume of the unit ball in the four dimensional Euclidean space.

The following monotonicity formula follows directly from [57].

Lemma 2.3.2 (Monotonicity of area and volume ratio). *Let (M, g, f) be a 4-D Ricci shrinker and fix $R > 2\sqrt{2}$. For any $p \in B(p_0, R)$ and any unit tangent vector $\mathbf{v}$ at p, let $\mathcal{A}(\mathbf{v}, r)$ be the area form of the geodesic sphere at* $\exp_p(r\mathbf{v})$*, then*

$$0 < s < r < d(p, \partial B(p_0, R)) \quad \Rightarrow \quad \frac{\mathcal{A}_f(\mathbf{v}, r)}{\bar{\mu}'_R(r)} \leq \frac{\mathcal{A}_f(\mathbf{v}, s)}{\bar{\mu}'_R(s)}. \tag{2.22}$$

Moreover, for any $B(p, r_1) \subset B(p, r_2) \subset B(p_0, R)$ and $B(p, s_1) \subset B(p, s_2) \subset B(p_0, R)$,

$$0 < s_1 < r_1 \text{ and } 0 < s_2 < r_2 \quad \Rightarrow \quad \frac{\mu_f(A(p; r_1, r_2))}{\bar{\mu}_R(r_2) - \bar{\mu}_R(r_1)} \leq \frac{\mu_f(A(p; s_1, s_2))}{\bar{\mu}_R(s_2) - \bar{\mu}_R(s_1)} \tag{2.23}$$

Proof. Recall that (2.19) implies

$$\sup_{B(p_0,R)} |\nabla f| \leq \frac{R}{2} + \sqrt{2}.$$

When $R > 2\sqrt{2}$, we have the radial derivative $\partial_r f \geq -R$ on $B(p_0, R)$. Now we can apply (4.8) of [57] directly to obtain (2.22). See also Theorem 3.1 and (4.3) of [57].

For (2.23), integrate (2.22) along geodesics gives the directional comparison

$$\frac{\int_{r_1}^{r_2} \mathcal{A}_f(\mathbf{v}, t)\, \mathrm{d}t}{\int_{r_1}^{r_2} \bar{\mu}'_R(t)\, \mathrm{d}t} \leq \frac{\int_{s_1}^{s_2} \mathcal{A}_f(\mathbf{v}, t)\, \mathrm{d}t}{\int_{s_1}^{s_2} \bar{\mu}'_R(t)\, \mathrm{d}t},$$

and integrating the above inequality in $\mathbf{v} \in S_pM$ gives the desired inequality. □

Notice that the doubling property of the weighted measure follows easily from the above monotonicity: for any $R > 2\sqrt{2}$ fixed,

$$B(p, 2r) \subset B(p_0, R) \ \Rightarrow\ \mu_f(B(p, 2r)) \le C_D(R)\mu_f(B(p, r)), \tag{2.24}$$

with the doubling constant $C_D(R) = 16e^{R^2}$.

2.3.2 Cheeger-Colding theory

The theory of Cheeger-Colding [10] [65] provides powerful tools in studying the structure of manifolds with uniform lower Ricci bounds. In the context of lower bounded Bakry-Émery Ricci curvature, a similar theory has been developed in [58], where the study is focused on *non-collapsing* manifolds. Yet our major concern is the *collapsing* phenomenon. Still, some of their lemmas see a few applications in our situation.

A good cut-off function

The existence of a cut-off function with controlled gradient and Laplacian will play a fundamental role in our local L^2-Ricci curvature estimate. In [58], such a cut-off function on a *unit ball* has been constructed following [10]. However, noticing that the equation (2.20) is not scaling invariant, we need a more careful argument when dealing with the general case, see also [26].

Lemma 2.3.3 (Existence of good cut-off function)**.** *For any $R > 10$, there is a constant $C(R) > 0$ such that for any $r \in (0, 1)$, and any compact $K \subset B(p_0, R - r)$, there is a smooth cut-off function φ supported on $B(K, r)$, with $\varphi \equiv 1$ on $B(K, \frac{r}{2})$, $\varphi \equiv 0$ outside $B(K, \frac{3r}{4})$, and $r|\nabla\varphi| + r^2|\Delta^f\varphi| \le C(R)$.*

Proof. Fix $r \in (0, 0.1)$. When $K = \{x_0\} \subset B(p_0, R - r)$, the construction of such a cut-off function originates in the work of Cheeger-Colding [10], and a Bakry-Émery version was constructed in [58]. For shrinking Ricci solitons, consider the rescaled metric $\tilde{g} = 4r^{-2}g$, then $\mathcal{R}c_{\tilde{g}} + \tilde{\nabla}^2 f = \frac{r^2}{8}\tilde{g}$, or the Bakry-Émery Ricci curvature satisfies $\mathcal{R}c^f_{\tilde{g}} = \frac{r^2}{8}\tilde{g} \ge 0$ as symmetric two tensors. Moreover, $|\tilde{\nabla} f| = \frac{r}{2}|\nabla f| \le R + 2$ since $r < 1$. Then we can apply Lemma 1.5 of [58] to obtain a cut-off function φ supported on $\tilde{B}(x_0, 2)$, $\varphi \equiv 1$ on $\tilde{B}(x_0, \frac{5}{4})$ and $\varphi \equiv 0$ outside $\tilde{B}(x_0, \frac{7}{4})$, moreover

$$|\tilde{\nabla}\varphi| + |\tilde{\Delta}^f\varphi| \le C(R). \tag{2.25}$$

Notice that the constant $C(R)$ depends on the lower Bakry-Émery Ricci curvature bound, which is 0, thus scaling invariant, and it also depends on an uniform upper bound of $|\tilde\nabla f|$ on $\tilde B(p_0, r^{-1}R)$, which is uniformly bounded above by $R+2$, regardless of the scaling by r as long as $r < 1$. In the original metric, (2.25) reads $r|\nabla\varphi| + r^2|\Delta^f\varphi| \le C(R)$.

Now suppose $K \subset B(p_0, R-r)$, let a maximal subset of points $\{x_i\} \subset B(K, \frac{r}{2})$ with $d(x_i, x_j) > \frac{r}{20}$. Then the maximality implies that $B(K, \frac{r}{2}) \subset \cup_i B(x_i, \frac{r}{10})$. Moreover, if $x \in \cap_{j=1}^k B(x_{i_j}, \frac{1}{5}r_{i_j})$, then by Lemma 2.3.2, relations

$$B(x, r/5) \subset B(x_{i_j}, 2r/5) \subset B(x, 3r/5), \quad \text{and} \quad B(x_{i_j}, r/40) \cap B(x_{i_{j'}}, r/40) = \emptyset,$$

bound the multiplicity of the covering $\{B(x_i, \frac{r}{10})\}$ by some $m(R)$.

Then we use the first step of the lemma on each $B(x_i, \frac{r}{5})$, to construct cutoff functions φ_i supported on $B(x_i, \frac{r}{5})$ such that $\varphi_i|_{B(x_i,\frac{r}{10})} \equiv 1$, and $r|\nabla\varphi_i| + r^2|\Delta^f\varphi_i| \le c(R)$. Let $\bar\varphi = \sum_i \varphi_i$, then $1 \le \bar\varphi \le m(R)$ on $B(K, \frac{r}{2})$, and vanishes outside $B(K, \frac{7r}{10})$. Let $u : [0, \infty) \to [0, 1]$ be a smooth function that vanishes near zero and constantly equals one on $[1, \infty)$, then $\varphi = u(\bar\varphi)$ is the desired cutoff function. □

sectionAlmost volume cone A fundamental tool of Cheeger-Colding theory is a controlled smoothing of the distance function using solutions to the Poisson equations with prescribed Dirichlet boundary conditions given by the distance function. In the case of Bakry-Émery Ricci curvature uniformly bounded below, similar estimates were obtained in [58]:

Lemma 2.3.4. *For any $\eta > 0$ and $\varepsilon > 0$, let (M, g, f) be a 4-Dimensional smooth Riemannian manifold with $\mathcal{R}c^f \ge 0$ and $|\nabla f| \le \varepsilon A$. Suppose that*

$$\frac{\mu_f(\partial B(p,s))}{\mu_f(\partial B(p,r))} \ge (1-\eta)\frac{\bar\mu'_{\varepsilon A}(r)}{\bar\mu'_{\varepsilon A}(s)},$$

and that u solves the following Poisson-Dirichlet problem

$$\Delta^f u = 4 \quad \text{on } A(p; r, s), \qquad u|_{\partial B(p,r)} = \frac{r^2}{2} \quad \text{and} \quad u|_{\partial B(p,s)} = \frac{s^2}{2}.$$

Then for $r < r_1 < r_2 < s_2 < s_1 < s$, denoting $d_p^2(x) := d^2(p, x)$ and $\tilde u := \sqrt{2u}$, then u and $\tilde u$ satisfies the following estimates:

1. $\sup_{A(p;r_1,s_1)} |\tilde u - d_p| \le \Psi(\eta, \varepsilon \,|\, A, r, s, r_1, s_1)$;
2. $⨏_{A(p;r,s)} |\nabla\tilde u - \nabla d_p|^2 \, d\mu_f \le \Psi(\eta, \varepsilon \,|\, A, r, s)$; and
3. $⨏_{A(p;r_2,s_2)} |\nabla^2 u - g|^2 \, d\mu_f \le \Psi(\eta, \varepsilon \,|\, A, r, r_1, r_2, s, s_1, s_2)$.

Basically, this lemma states that when f is approximately a constant function, the situation is reduced to the Ricci lower bound case and corresponding estimates follow from the work of Cheeger-Colding [10].

2.4 Functional inequalities

2.4.1 The segment and Poincaré inequalities on 4-D Ricci shrinkers

Another important consequence of the monotonicity (2.22) is the segment inequality, originally due to Cheeger-Colding [10] for manifolds with uniform Ricci lower bound. We will provide a proof here as this is the first time the segment inequality appears in the context of Bakry-Émery Ricci curvature bounded below.

Lemma 2.4.1 (Segment inequality)**.** *Let (M, g, f) be a 4-D Ricci shrinker, and fix $R > 0$. For any $U \subset B(p_0, R)$ and any non-negative $u \in C^0(U)$, there is a constant $C_{ChCo}(R) > 0$ such that if a subset A of U sees almost all pairs of its points connected by minimal geodesics contained in U, then*

$$\int_{A\times A} \mathcal{F}_u(x,y)\, d\mu_f(x) d\mu_f(y) \;\leq\; C_{ChCo}(R)\, \mu_f(A)\, \mathrm{diam}\, U \int_U u\, d\mu_f, \tag{2.26}$$

where

$$\mathcal{F}_u(x,y) := \inf_{\gamma_{xy}} \int_0^{d(x,y)} u(\gamma_{xy}(t))\, dt,$$

the infimum being taken over all minimal geodesics γ_{xy} connecting x and y.

Proof. We may consider $\mathcal{F}_u(x,y) = \mathcal{F}_u^+(x,y) + \mathcal{F}_u^-(x,y)$ where

$$\mathcal{F}_u^+(x,y) := \inf_{\{\gamma_{xy}\}} \int_{\frac{d(x,y)}{2}}^{d(x,y)} u(\gamma_{xy}(f))\, \mathrm{d}t \quad \text{and} \quad \mathcal{F}_u^-(x,y) := \inf_{\{\gamma_{xy}\}} \int_0^{\frac{d(x,y)}{2}} u(\gamma_{xy}(f))\, \mathrm{d}t.$$

Since $\mathcal{F}_u^+(x,y) = \mathcal{F}_u^-(y,x)$, by Fubini's theorem,

$$\int_{A\times A} \mathcal{F}_u^+(x,y)\, \mathrm{d}\mu_f(x)\mathrm{d}\mu_f(y) \;=\; \int_{A\times A} \mathcal{F}_u^-(x,y)\, \mathrm{d}\mu_f(x)\mathrm{d}\mu_f(y),$$

and so we only need to do the estimate for $\mathcal{F}_u^+$. For any $x \in A$ and any $\mathbf{v} \in S_xM$ fixed, define $d_{x,\mathbf{v}} := \min\{t > 0 : \exp_x(t\mathbf{v}) \in \partial U\}$, also denote $\gamma_{\mathbf{v}}(t) = \exp_x(t\mathbf{v})$. Then $\forall t \in (0, d_{x,\mathbf{v}})$, by the area ratio monotonicity (2.22),

$$\begin{aligned}
\mathcal{F}_u^+(\gamma_{\mathbf{v}}(t/2), \gamma_{\mathbf{v}}(t))\, \mathrm{d}\mu_f(\gamma_{\mathbf{v}}(t)) \;&\leq\; \left(\int_{\frac{t}{2}}^{t} u(\gamma_{\mathbf{v}}(s))\, \mathrm{d}s\right) \mathcal{A}_f(\mathbf{v}, t)\mathrm{d}t \\
&\leq\; 8e^{R^2} \left(\int_{\frac{t}{2}}^{t} u(\gamma_{\mathbf{v}}(s))\, \mathcal{A}_f(\mathbf{v}, s)\mathrm{d}s\right) \mathrm{d}t.
\end{aligned}$$

By the assumption on $A \subset U$, for almost every $y \in A$, there exists some $\mathbf{v} \in S_xM$ such that $\gamma_{\mathbf{v}}(d(x,y)) = y$, we have

$$\begin{aligned}\int_A \mathcal{F}_u^+(x,y)\, \mathrm{d}\mu_f(y) &\le \int_{S_xM}\int_0^{d_{x,\mathbf{v}}} \mathcal{F}_u^+(\gamma_{\mathbf{v}}(t/2), \gamma_{\mathbf{v}}(t))\mathcal{A}_f(\mathbf{v},t)\, \mathrm{d}t\mathrm{d}\mathbf{v} \\ &\le 8e^{R^2} \operatorname{diam} U \int_{S_xM}\int_0^{d_{x,\mathbf{v}}} u(\gamma_{\mathbf{v}}(s))\, \mathcal{A}_f(\mathbf{v},s)\, \mathrm{d}s\mathrm{d}\mathbf{v} \\ &\le 16e^{R^2} \operatorname{diam} U \int_U u\, \mathrm{d}\mu_f.\end{aligned}$$

Finally, integrate the above inequality for $x \in A$, we get

$$\int_A\int_A \mathcal{F}_u^+(x,y)\, \mathrm{d}\mu_f(y)\mathrm{d}\mu_f(x) \le 16e^{R^2}\mu_f(A) \operatorname{diam} U \int_U u\, \mathrm{d}\mu_f.$$

□

Iterating the segment inequality, one easily obtains the local L^2-Poincaré inequality, whose constants are determined by $C_{ChCo}(R)$:

Lemma 2.4.2 (Poincaré inequality)**.** *Let (M,g,f) be a 4-D Ricci shrinker and fix $R > 2\sqrt{2}$. There exists a positive constant $C_P(R) > 0$ such that for any $B(p,r) \subset B(p_0,R)$ and any $u \in C^1(B(p,r))$,*

$$\int_{B(p,r)} \left| u - \fint_{B(p,r)} u\, d\mu_f \right|^2 d\mu_f \le C_P(R)\, r^2 \int_{B(p,r)} |\nabla u|^2\, d\mu_f. \tag{2.27}$$

This Poincaré inequality could be viewed as a weighted Poincaré inequality. See also [76] for a version of the Poincaré inequality on a complete non-compact manifold with non-negative Ricci curvature and the heat kernel as a weight function.

2.4.2 Sobolev inequalities

In the situation where a uniform Ricci curvature lower bound is assumed, Michael Anderson explicitly estimated the Sobolev constant in [1] (see also [73]): for a fixed geodesic ball $B(x,r)$, its Sobolev constant is comparable to $(|B(x,r)|r^{-n})^{-\frac{2}{n}}$. The lesson is to consider explicitly the effect of a correct renormalization, when applying the Sobolev inequality to the study of Ricci flows.

In another direction, using methods in stochastic analysis and the Moser iteration technique, Laurent Saloff-Coste has shown an even more general Sobolev inequality in [79], where the Sobolev constant only depends on the doubling constant and the L^2-Poincaré constant. This is the inequality that we will employ in this note:

Proposition 2.4.3 (Renormalized L^2-Sobolev inequality)**.** *Let (M^n, g) be a Riemannian manifold such that the doubling constant and the L^2-Poincaré constant are bounded from above by C_D and C_P respectively. Then there is a constant $C_S = C_S(n, C_D, C_P)$ such that for any $B(x,r) \subset M$ and any $u \in H^1_0(B(x,r))$, the following renormalized Sobolev inequality holds:*

$$\left(\int_M u^{\frac{2n}{n-2}}\, dV_g\right)^{\frac{n-2}{n}} \;\leq\; C_S(|B(x,r)|r^{-n})^{-\frac{2}{n}} \int_M |\nabla u|^2 + r^{-2}u^2\, dV_g. \tag{2.28}$$

Remark 2.4.4. *This is of course just one version of the Sobolev inequality. We call it renormalized just to emphasize the independence of the Sobolev constant from the volume, since eventually the volume will be sent to zero.*

We also notice that inequality (1.3) is just a (weaker) global version of this inequality.

If now we are on a 4-D Ricci shrinker, then the previous segment inequality and volume comparison results provide uniform bounds on the Poincaré (2.27) and doubling (2.24) constants of the mesure $e^{-f}\mathrm{d}V_g$ within a fixed geodesic ball $B(p_0, R)$. We therefore obtain the following local Sobolev inequality, with whose constants are determined by $C_D(R)$ and $C_P(R)$ (see [52]):

Lemma 2.4.5 (Sobolev inequality)**.** *Let (M, g, f) be a 4-D Ricci shrinker and fix $R > 2\sqrt{2}$. For any $B(p,r) \subset B(p_0, R)$ and $u \in C^1_c(B(p,r))$,*

$$\left(\int_{B(p,r)} u^4\, d\mu_f\right)^{\frac{1}{2}} \leq \frac{C_S(R)\, r^2}{\mu_f(B(p,r))^{\frac{1}{2}}} \int_{B(p,r)} \left(|\nabla u|^2 + r^{-2}u^2\right)\, d\mu_f. \tag{2.29}$$

A natural consequence of the above Sobolev inequalities is an ε-regularity with respect to collapsing due to Mike Anderson [1]. It is the starting point of Cheeger-Tian's ε-regularity theorem for four dimensional Einstein manifolds [20]. By the bound on the Sobolev constant for $\mathrm{d}\mu_f$, as obtained in Lemma 2.4.5, the proof of this theorem is by now standard using Moser iteration, see [1] and [39] for the original work.

Proposition 2.4.6 (Weighted ε-regularity with respect to collapsing)**.** *Let (M, g, f) be a 4-D Ricci shrinker. There exist $\varepsilon_A(R) > 0$ and $C_A(R) > 0$ such that if $B(p,r) \subset B(p_0, R)$, then*

$$\frac{\bar{\mu}_R(r)}{\mu_f(B(p,r))} \int_{B(p,r)} |\mathcal{R}m|^2\, d\mu_f \;\leq\; \varepsilon_A(R) \tag{2.30}$$

implies that

$$\sup_{B(p,\frac{r}{2})} |\mathcal{R}m| \;\leq\; C_A(R)\, r^{-2} \left(\frac{\bar{\mu}_R(r)}{\mu_f(B(p,r))} \int_{B(p,r)} |\mathcal{R}m|^2 \, d\mu_f \right)^{\frac{1}{2}}.$$

This proposition basically says that even if a geodesic ball has no uniform volume lower bound, and consequently no uniform estimate from the Sobolov inequality, when the local energy is sufficiently small — much smaller compared to the volume — we still have uniform curvature control. Adapted to this phenomenon, we define the "renormalized energy" as following:

Definition 2.4.7. *Fix* $r \in (0,1]$. *For any* $p \in B(p_0, R)$*, define the scale r renormalized energy as*

$$I^f_{\mathcal{R}m}(p,r) := \frac{\bar{\mu}_{R+1}(r)}{\mu_f(B(p,r))} \int_{B(p,r)} |\mathcal{R}m|^2 \, d\mu_f.$$

So Proposition 2.4.6 says that for $p \in B(p_0, R)$,

$$I^f_{\mathcal{R}m}(p,r) < \varepsilon_A(R) \Rightarrow \sup_{B(p,\frac{r}{2})} |\mathcal{R}m| \;\leq\; C_A(R)\, r^{-2} I^f_{\mathcal{R}m}(p,r)^{\frac{1}{2}}. \tag{2.31}$$

Moreover, we immediately notice the following key properties of the renormalized energy:

1. $I^f_{\mathcal{R}m}$ is invariant under rescaling, so is (2.31);

2. $I^f_{\mathcal{R}m}$ is continuous and monotonically *non-decreasing* in radius r.

2.5 Convergence and collapsing of Riemannian manifolds

In this subsection, we start by introducing various convergence concepts of metric spaces, whose canonical reference is [35], then discuss Fukaya's structural results about the collapsing limit under bounded curvature, see [72] and [29]. See also [31] for a relevant result concerning the local structure of Riemannian manifolds.

2.5.1 Weak convergence

Given a sequence of metric spaces (X_i, d_i) with diameter bounded above by R, we say that $(X_i, d_i) \to_{GH} (X_{\infty, d_\infty})$ if when $i \to \infty$, the Gromov-Hausdorff distance, $d_{GH}((X_i, d_i), (X_\infty, d_\infty)) \to 0$. Recall that $d_{GH}((X_i, d_i), (X_\infty, d_\infty))$ is defined as the infimum of the Hausdorff distance between X and Y in $X \sqcup Y$, equipped with all possible metrics. If $(X_i, d_i) \to_{GH} (X_{\infty, d_\infty})$, we could then find maps $G_i : X_i \to X_\infty$ and $H_i : X_\infty \to X_i$ such that for any $\varepsilon > 0$, there exists some i_ε so that $\forall i > i_\varepsilon$, $\forall x_i, x_i' \in X_i$ and $\forall x_\infty, x_\infty' \in X_\infty$,

1. $\left| d_i(x_i, x_i') - d_i(H_i \circ G_i(x_i), H_i \circ G_i(x_i')) \right| < \varepsilon$, and
2. $\left| d_\infty(x_\infty, x_\infty') - d_\infty(G_i \circ H_i(x_\infty), G_i \circ H_i(x_\infty')) \right| < \varepsilon$.

Gromov's fundamental observation says that if $\{(X_i, d_i)\}$ has uniformly bounded Hausdorff dimension, diameter and volume doubling property, then there exists some metric space (X_∞, d_∞) with the same diameter bound, such that a subsequence Gromov-Hausdorff converges to (X, d). Notice that if $(X_i, d_i) \subset B(p_i^0, R) \subset (M_i, g_i, f_i)$ with d_i induced by $g_i|_{X_i}$, then by the uniform doubling property (2.24) for μ_{f_i}:

Lemma 2.5.1. *Suppose $\{(X_i, d_i) \subset B(p_i^0, R) \subset (M_i, g_i, f_i)\}$ is a sequence of uniformly bounded domains in 4-D Ricci shrinkers, possibly with marked points, then there exists a metric space (X_∞, d_∞) with $\operatorname{diam}_{d_\infty} X_\infty \leq R$, such that some subsequence, still denoted by $\{(X_i, d_i)\}$, Gromov-Hausdorff converges to (X_∞, d_∞).*

For a sequence of complete non-compact 4-D Ricci shrinkers, we may define the multi-pointed Gromov-Hausdorff convergence to respect the specified base point, i.e. a minimum of the potential function.

Definition 2.5.2. *We say that a sequence of complete non-compact 4-D Ricci shrinkers (M_i, g_i, f_i, p_i^0) with base points p_i^0 (a minimum of f_i) and J marked points $Mk_i := \{p_i^1, \cdots, p_i^J\}$ multi-pointed Gromov-Hausdorff converges to a metric space $(X_\infty, d_\infty, x_\infty^0)$ with J marked points $Mk_\infty = \{x_\infty^1, \cdots, x_\infty^J\}$, if for any $R > 0$, $B(p_i^0, R) \to_{GH} B(x_\infty^0, R)$, and there are maps $G_i : M_i \to X_\infty$ and $H_i : X_\infty \to M_i$ such that $G_i(p_i^j) = x_\infty^j$ and $H_i(x_\infty^j) = p_i^j$ $(j = 0, 1, \cdots, J)$. Moreover, for any $\varepsilon > 0$, there exists some $i_\varepsilon(R)$ so that $\forall i > i_\varepsilon(R)$,*

1. *$\forall p_i, p_i' \in B(p_i^0, R) \backslash Mk_i$, $\left| d_i(p_i, p_i') - d_i(H_i \circ G_i(p_i), H_i \circ G_i(p_i')) \right| < \varepsilon$, and*
2. *$\forall x_\infty, x_\infty' \in B(x_\infty^0, R) \backslash Mk_\infty$, $\left| d_\infty(x_\infty, x_\infty') - d_\infty(G_i \circ H_i(x_\infty), G_i \circ H_i(x_\infty')) \right| < \varepsilon$.*

For convenience we will also use the notation $X_i \to_{pGH} X_\infty$ and $Mk_i \to_{GH} Mk_\infty$ for such type of convergence. Also notice, it is possible that $p_i^0 \in Mk_i$.

2.5.2 Strong convergence

Gromov's compactness result provides a weak limit in the category of metric spaces. In order to extract information from a convergent sequence, we need to consider stronger convergence. For a sequence of 4-D Ricci shrinkers $\{M_i, g_i, f_i\}$, suppose $\{(X_i, d_i) \subset (M_i, g_i, f_i)\}$ multi-pointed Gromov-Hausdorff converges to a limit space (X_∞, d_∞), with marked points $Mk_i \to_{GH} Mk_\infty$. According to (a trivial generalization of) the work of [64] and [26], $X_\infty \backslash Mk_\infty = \mathcal{R}(X_\infty) \cup \mathcal{S}(X_\infty)$, with $\dim_H(\mathcal{R}(X_\infty)) \leq 4$, and $\dim_H(\mathcal{S}(X_\infty)) < \dim_H(\mathcal{R}(X_\infty))$. We define the strong convergence as following:

Definition 2.5.3 (Strong convergence)**.** *Let (M_i, g_i, f_i) be a sequence of 4-D Ricci shrinkers, whose subsets $(X_i, d_i) \to_{pGH} (X_\infty, d_\infty)$, with J marked points $Mk_i \to_{GH} Mk_\infty$. We say that the convergence is strong if there is an exhaustion of $X_\infty \backslash Mk_\infty$ by compact subsets K_j ($j = 1, 2, 3, \cdots$), such that for each j, there is an $i_j > 0$ and for all $i > i_j$,*

1. *if $\dim_H(\mathcal{R}(X_\infty)) = 4$, then $\mathcal{S}(X_\infty) = \emptyset$, X_∞ is a smooth 4-manifold, and each $H_i|_{K_j}$ can be chosen as a diffeomorphism onto its image, with $H_i^* g_i \to g_\infty$ smoothly as symmetric 2-tensor fields; or else,*
2. *if $\dim_H(\mathcal{R}(X_\infty)) < 4$, then each $G_i^{-1}(K_j)$ has uniformly bounded curvature C_j, and $G_i^{-1}(K_j) \to_{GH} K_j$ is collapsing with bounded curvature, in the sense of Cheeger-Fukaya-Gromov [14].*

We notice that the two cases in the above definition are alternatives. Case (1) above is guaranteed to happen if a sequence has uniformly locally bounded curvature and uniform volume ratio lower bound, through the work of [8]. See Theorem 4.2.2 for a more detailed description of case (2).

2.5.3 Collapsing with bounded curvature

When collapsing with bounded curvature, i.e. case (2) in Definition 2.5.3, happens, there is a rich structural theory about the Riemannian metric, mainly developed by Cheeger, Fukaya and Gromov, see [34], [51], [17], [18], [27], [29] and [14]. The following proposition gives a full account of Fukaya's results in [72] and [29] that are relevant to our argument in the following subsections:

Proposition 2.5.4 (Structure of collapsing limit)**.** *Let $X_i \subset (M_i^n, g_i)$ be bounded domains in a sequence of n-dimensional Riemannian manifolds such that*

$$|\nabla^k \mathcal{R}m_{g_i}| \leq C_k \ (k = 0, 1, 2, 3, \cdots) \quad on \quad X_i.$$

Suppose $X_i \to_{GH} X_\infty$ for some metric space (X_∞, d_∞), with $\dim_H X_\infty = m < n$, then there is a regular-singular decomposition $X_\infty = \mathcal{R}(X_\infty) \cup \mathcal{S}(X_\infty)$, such that

1. $(\mathcal{R}(X_\infty), d_\infty) \equiv (\mathcal{R}(X_\infty), g_\infty)$, *a smooth m-dimensional Riemannian manifold, such that*
$$\sup_{\mathcal{R}(X_\infty)} |\mathcal{R}m_{g_\infty}| \leq C_0;$$

2. $\mathcal{S}(X_\infty)$ *is a closed subset of* X_∞ *with* $\dim_H(\mathcal{S}(X_\infty)) = m' \leq m - 1$*;*

3. *there is a stratification* $\emptyset \subset \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{m'} = \mathcal{S}(X_\infty)$*, each strata* $\mathcal{S}_j$ *is by itself a j-dimensional smooth Riemannian manifold;*

4. *there exists some* $\iota_{X_\infty} > 0$ *such that* $inj_{\mathcal{R}(X_\infty)}\, x = \min\{\iota_{X_\infty}, d_\infty(x, \mathcal{S}(X_\infty))\}$*, for any* $x \in \mathcal{R}(X_\infty)$*.*

For all i sufficiently large, the Gromov-Hausdorff approximation $G_i : X_i \to X_\infty$ *can be chosen such that on* $U_i := G_i^{-1}(\mathcal{R}(X_\infty))$*,*
$$G_i : U_i \to \mathcal{R}(X_\infty)$$
is an almost Riemannian submersion, and for each $x \in \mathcal{R}(X_\infty)$*,* $G_i^{-1}(x)$ *is diffeomorphic to N, an infranil-manifold.*

Chapter 3

Distance distortion estimate

3.1 A uniform renormalized Sobolev inequality along the Ricci flow

A uniform Sobolev inequality along Ricci flows will enable us to do analysis on positive time slices. Notice that the lower bound of the μ-entropy reflects the upper bound of the log-Sobolev constant, and the monotone increasing property of the μ-entropy will further preserve, rather than destroying, the log-Sobolev constant. In this section, we will see that the information of initial global volume ratio is encoded in the initial μ-entropy via a log-Sobolev inequality, deduced following a classical argument, but with the renormalized Sobolev inequality (1.3) as our starting point. We will also deduce a uniform renormalized Sobolev inequality along the Ricci flow, which clearly shows how the initial global volume ratio affects the Sobolev constants on positive time slices. For previous results we refer the readers to the works of Rugang Ye [81] and Qi S. Zhang [83], [**?**], [**?**].

3.1.1 Lower bound of initial entropy via the renormalized Sobolev inequality

From (1.3), we see that if $\int_M v^2 \mathrm{d}V_{g(0)} = 1$, then

$$\left(\int_M v^{\frac{2n}{n-2}} \mathrm{d}V_{g(0)}\right)^{\frac{n-2}{n}} \leq 4C_S V^{-\frac{2}{n}} \left(D_0^2 \int_M |\nabla v|^2 \,\mathrm{d}V_{g(0)} + 1\right).$$

Due to the uniform bound of the scalar curvature, we could further obtain

$$\left(\int_M v^{\frac{2n}{n-2}} \,\mathrm{d}V_{g(0)}\right)^{\frac{n-2}{n}} \leq 4C_S V^{-\frac{2}{n}} \left(D_0^2 \int_M \left(4|\nabla v|^2 + \mathcal{R}_{g(0)} v^2\right) \,\mathrm{d}V_{g(0)} + C_0 D_0^2 + 1\right).$$

Since the logarithm function is concave, and since $v^2 \mathrm{d}V_{g(0)}$ defines a probability measure on M, by Jensen's inequality, we have

$$\forall u \in L^1(M, v^2 \mathrm{d}V_{g(0)}), \quad \int_M (\log |u|)\, v^2 \mathrm{d}V_{g(0)} \leq \log \int_M |u|\, v^2 \mathrm{d}V_{g(0)}.$$

With $u = v^{q-2}$ with $q = \frac{2n}{n-2}$ (notice that $\frac{q}{q-2} = \frac{n}{2}$), the above inequality gives

$$\begin{aligned} \int_M v^2 \log v^2 \, \mathrm{d}V_{g(0)} &= \int_M \frac{2}{q-2} \left(\log v^{q-2}\right)\, v^2 \mathrm{d}V_{g(0)} \\ &\leq \frac{2}{q-2} \log \int_M v^q \, \mathrm{d}V_{g(0)}, \end{aligned}$$

which is exactly $\frac{n}{2} \log \|v\|^2_{L^q(M)}$; furthermore,

$$\begin{aligned} &\frac{n}{2} \log \|v\|^2_{L^q(M)} \\ &\leq \frac{n}{2} \log \left(D_0^2 \int_M \left(4|\nabla v|^2 + \mathcal{R}_{g(0)} v^2\right) \mathrm{d}V_{g(0)} + C_0 D_0^2 + 1 \right) - \log V + \frac{n}{2} \log 4C_S. \end{aligned}$$

Now applying the elementary inequality $\log u \leq \alpha u - 1 - \log \alpha$ for all $\alpha > 0$ to the first term in the right-hand side of this last inequality, we obtain

$$\begin{aligned} \int_M v^2 \log v^2 \, \mathrm{d}V_{g(0)} &\leq \frac{n}{2} \log \left(D_0^2 \int_M \left(4|\nabla v|^2 + \mathcal{R}_{g(0)} v^2\right) \mathrm{d}V_{g(0)} + C_0 D_0^2 + 1 \right) \\ &\quad - \log V + \frac{n}{2} \log 4C_S \\ &\leq \frac{\alpha n}{2} \left(D_0^2 \int_M \left(4|\nabla v|^2 + \mathcal{R}_{g(0)} v^2\right) \mathrm{d}V_{g(0)} + C_0 D_0^2 + 1 \right) \\ &\quad - \log V + \frac{n}{2} \left(\log 4C_S - 1 - \log \alpha\right). \end{aligned} \tag{3.1}$$

Recalling the definition of the $\mathcal{W}$-functional (2.6), and taking $\alpha = \frac{2\tau}{nD_0^2}$ in (3.1), we immediately see

$$\mathcal{W}(g(0), v^2, \tau) \;\geq\; \log V D_0^{-n} - (C_0 D_0^2 + D_0^{-2})\tau - \frac{n}{2} \log(8n\pi e C_S). \tag{3.2}$$

Here τ, as a multiple of α, could be any positive number. Since this is valid for any function v on M with unit L^2-norm, we have, for any $\tau \in [T, 2T]$,

$$\mu(g(0), \tau) \;\geq\; \log V D_0^{-n} - (C_0 D_0^2 + D_0^{-2})\tau - \frac{n}{2} \log(8n\pi e C_S). \tag{3.3}$$

Here we notice that both sides of the inequalities above are invariant under a parabolic rescaling.

Remark 3.1.1. *It is well-know that collapsing initial data implies that there is no uniform lower bound of the $\mathcal{W}$-entropy, and here we give an explicit lower bound in terms of the initial global volume ratio.*

Now suppose we evolve v^2 at some $\bar{t}$-slice backward by the conjugate heat equation, i.e. we consider a function u such that

$$u(\bar{t}) \;=\; v^2; \quad (\partial_t + \Delta_{g(t)} - \mathcal{R}_{g(t)})u = 0; \quad \partial_t g \;=\; -2\mathcal{R}c_{g(t)},$$

then $\mathcal{W}(g(t), u(t), \tau(t))$ is increasing in t where $\tau' = -1$. Therefore, for any v^2 with unit $L^1(g(\bar{t}))$-norm, we have, by the monotone increasing property of the $\mathcal{W}$-functional, that

$$\begin{aligned}\mathcal{W}(g(\bar{t}), v^2, \tau(\bar{t})) &\geq \mathcal{W}(g(0), u(0), \tau(\bar{t}) + \bar{t}) \\ &\geq \log V D_0^{-n} - (C_0 + D_0^{-2})(\tau(\bar{t}) + \bar{t}) - \frac{n}{2}\log(8n\pi e C_S),\end{aligned}$$

or in the form of the log-Sobolev inequality,

$$\int_M \tau(4|\nabla v|^2 + \mathcal{R}_{g(\bar{t})}v^2) - v^2 \log v^2 \, \mathrm{d}V_{g(\bar{t})} \;\geq\; \log V\left(\frac{\tau}{D_0^2}\right)^{\frac{n}{2}} - (C_0 + D_0^{-2})(\tau + \bar{t}) - C_{lS}, \tag{3.4}$$

where $C_{lS} := \frac{n}{2}\log(2ne^{-1}C_S)$ and τ is any positive number.

3.1.2 Uniform renormalized Sobolev inequality along the Ricci flow

In this subsection, we establish a uniform renormalized Sobolev inequality along the Ricci flow. We will follow the exposition of [84], which is based on the argument of Edward Davies [70] in the case of a fixed Riemannian manifold. The result of this subsection will not be needed in our estimate of the distance distortion, yet we still include it here because we will later use a similar argument to prove a rough upper bound of the renormalized heat kernel in Section 5.1.

The first step would be using the uniform log-Sobolev inequality (3.4) to obtain an upper bound of the heat kernel on a fixed future time slice $(M, g(\bar{t}))$. Now let u be any solution to the equation

$$(\partial_t - \Delta_{g(\bar{t})} + \mathcal{R}_{g(\bar{t})})u \;=\; 0$$

on the *fixed* Riemannian manifold $(M, g(\bar{t}))$. Consider for any fixed $t > 0$, the exponent $p(s) := \frac{t}{t-s}$ for $s \in [0, t]$. We immediately see that $p'(s) = t(t-s)^{-2} > 0$, moreover,

$$\begin{aligned} 0 \;\le\; \frac{p(s)-1}{p'(s)} \;&=\; \frac{s(t-s)}{t} \;\le\; \frac{t}{4}, \\ \text{and} \quad \frac{p'(s)}{p^2(s)} \;&=\; \frac{1}{t}. \end{aligned} \tag{3.5}$$

We also let

$$v(x,s) \;:=\; u(x,s)^{\frac{p(s)}{2}} \|u^{\frac{p(s)}{2}}\|^{-1}_{L^2(M,g(\bar t))},$$

so that $\|v\|_{L^2(M,g(\bar t))} = 1$. Routine computations give

$$\begin{aligned} & p^2(s)\partial_s \log \|u\|_{L^{p(s)}(M,g(\bar t))} \\ &= p'(s)\int_M v^2 \log v^2 \,\mathrm{d}V_{g(\bar t)} - 4(p(s)-1)\int_M |\nabla v|^2 \,\mathrm{d}V_{g(\bar t)} - p^2(s)\int_M \mathcal{R}_{g(\bar t)} v^2 \,\mathrm{d}V_{g(\bar t)} \\ &\le p'(s)\left(\int_M v^2 \log v^2 \mathrm{d}V_{g(\bar t)} - \frac{(p(s)-1)}{p'(s)}\int_M 4|\nabla v|^2 + \mathcal{R}_{g(\bar t)} v^2 \,\mathrm{d}V_{g(\bar t)} + \frac{3t}{4}C_0\right). \end{aligned}$$

Thus if we plug $\tau = \frac{p(s)-1}{p'(s)}$ into (3.4), then the above computation, together with (3.5) give

$$\begin{aligned} & \partial_s \log \|u\|_{L^{p(s)}(M,g(\bar t))} \\ &\le \frac{1}{t}\left(-\frac{n}{2}\log\frac{s(t-s)}{t} - \log VD_0^{-n} + (C_0 + D_0^{-2})\left(\bar t + \frac{s(t-s)}{t}\right) + C_{lS} + \frac{3t}{4}C_0\right). \end{aligned}$$

Notice that $p(0) = 1$ and $p(t) = \infty$, we integrate the above inequality (with respect to s) from 0 to t to obtain for any $t > 0$,

$$\log \frac{\|u(-,t)\|_{L^\infty(M,g(\bar t))}}{\|u(-,t)\|_{L^1(M,g(\bar t))}} \;\le\; -\frac{n}{2}\log t - \log VD_0^{-n} + (C_0 - D_0^{-2})t + \tilde C_H(\bar t), \tag{3.6}$$

where $\tilde C_H(\bar t) = 2(\bar t + 1)(C_0 + D_0^{-2}) + C_{lS} + n$. Now let $G_{\bar t}(x,t,y)$ be the heat kernel of $(M, g(\bar t))$ centered at $x \in M$, then

$$\begin{aligned} u(x,t) &= \int_M G_{\bar t}(x,t,y) u(y,0) \,\mathrm{d}V_{g(\bar t)}(y), \\ \text{and} \quad \|u(-,t)\|_{L^1(M,g(\bar t))} &= \int_M u(y,t) \,\mathrm{d}V_{g(\bar t)}(y), \end{aligned} \tag{3.7}$$

we conclude that

$$(VD_0^{-n})G_{\bar t}(x,t,y) \;\le\; e^{\tilde C_H(\bar t) + (2C_0 + D_0^{-2})t} t^{-\frac{n}{2}}. \tag{3.8}$$

Now consider $\tilde{G}_{\bar{t}}(-,t,-) := e^{-(2C_0+D_0^{-2})t}G_{\bar{t}}(-,t,-)$, then $\tilde{G}_{\bar{t}}$ is the fundamental solution to the equation

$$\left(\partial_t - \Delta_{\bar{t}} + \mathcal{R}_{g(\bar{t})} + (2C_0 + D_0^{-2})\right) u = 0,$$

and by(3.8) we have the control

$$\forall t > 0, \quad \tilde{G}_{\bar{t}}(-,t,-) \leq \tilde{C}_H(\bar{t})(VD_0^{-n})^{-1}t^{-\frac{n}{2}}.$$

Notice that $\tilde{C}_H(\bar{t}) = 2\bar{t}(C_0+D_0^{-2})+C_{IS}+n$ is independent of time and space variables on the fixed manifold $(M, g(\bar{t}))$; also notice that it is invariant under the parabolic rescaling.

Now we can conclude that the operator of integrating against the kernel $\tilde{G}_{\bar{t}}$ is a contraction, and standard argument gives the L^2-Sobolev inequality on $(M, g(\bar{t}))$:

$$\|f\|^2_{L^{\frac{2n}{n-2}}(M,g(\bar{t}))} \leq C_{Sob}(\bar{t})V^{-\frac{2}{n}}D_0^2\left(\|\nabla f\|^2_{L^2(M,g(\bar{t}))} + (2C_0 + D_0^{-2})\|f\|^2_{L^2(M,g(\bar{t}))}\right), \qquad (3.9)$$

where $C_{Sob}(\bar{t}) = (2(\bar{t}+1)(C_0 + D_0^{-2}) + C_{IS} + n)^{\frac{2}{n}}$ is uniformly bounded for bounded $\bar{t}$, independent of V and the flow.

3.2 Estimating the geometric quantities along the Ricci flow

In this section we give a lower bound of the renormalized volume ratio on any scale, and a scaling invariant upper bound of the diameter along the Ricci flow. The estimates only depend on the initial doubling constant C_D, the initial L^2-Poincaré constant C_P, the initial diameter D_0, the space-time scalar curvature bound C_0, and the time elapsed from the beginning.

Both estimates are based on the idea that the $\mathcal{W}$-functional, when tested against a suitable spacial cut-off function, bounds from below the volume ratio at the given time slice, and then the monotone increasing property of the $\mathcal{W}$-functional further provides the desired renormalization by the initial total volume, as shown in (3.2).

More specifically, throughout this section, we fix a time slice $\bar{t} \in (0, T]$ and a scale r such that $r^2 \in (0, \bar{t}]$. For any fixed $x \in M$, we could define a spacial cut-off function as

$$h^2(y) = e^{-A}(4\pi r^2)^{-\frac{n}{2}}\eta^2\left(r^{-1}d_{\bar{t}}(x,y)\right),$$

with η being a smooth cut-off function supported on $[0, 1)$, constantly equal to 1 on $[0, \frac{1}{2}]$ and $-2 \leq \eta' \leq 0$ on $(\frac{1}{2}, 1)$. Moreover, A is chosen so that $\int_M h^2 \, dV_{g(\bar{t})} = 1$, and we immediately see

$$\frac{|B_{\bar t}(x,\frac{r}{2})|}{(4\pi r^2)^{\frac{n}{2}}} \;\le\; e^A \;=\; \int_M \frac{\eta^2\left(r^{-1}d_{\bar t}(x,y)\right)}{(4\pi r^2)^{\frac{n}{2}}}\,\mathrm{d}V_{g(\bar t)}(y) \;\le\; \frac{|B_{\bar t}(x,r)|}{(4\pi r^2)^{\frac{n}{2}}}. \tag{3.10}$$

Recall that the $\mathcal{W}$-functional for $(M, g(\bar t, h^2)$ is defined as

$$\mathcal{W}(g(\bar t), h^2, r^2) \;=\; \int_M 4r^2|\nabla h|^2 + r^2\mathcal{R}_{g(\bar t)}h^2 - h^2\log h^2\,\mathrm{d}V_{g(\bar t)} - \frac{n}{2}\log(4\pi r^2) - n.$$

We now roughly estimate some terms of the right-hand side of this inequality:

Since $|\nabla d_{\bar t}| \le 1$, we have

$$\begin{aligned}\int_M 4r^2|\nabla h|^2\,\mathrm{d}V_{g(\bar t)} \;&\le\; \int_{B_{\bar t}(x,r)} \frac{16r^2e^{-B}\,|\eta'\nabla d_{\bar t}(x,y)|^2}{(4\pi r^2)^{\frac{n}{2}}r^2}\,\mathrm{d}V_{g(\bar t)}(y)\\ &\le \frac{64|B_{\bar t}(x,r)|}{e^A(4\pi r^2)^{\frac{n}{2}}}\\ &\le \frac{64|B_{\bar t}(x,r)|}{|B_{\bar t}(x,\frac{r}{2})|}\end{aligned} \tag{3.11}$$

where we have used (3.10); moreover, since h^2 is supported in $B_{\bar t}(x,r)$, and since the mapping $\sigma \mapsto -\sigma\log\sigma$ is concave, we apply this to $\sigma = h^2$ and use Jensen's inequality see

$$\begin{aligned}&\int_M -h^2\log h^2\,\mathrm{d}V_{g(\bar t)} - \frac{n}{2}\log 4\pi r^2 - n\\ \le\; &-\int_{B_{\bar t}(x,r)} h^2\,\mathrm{d}V_{g(\bar t)}\left(\log ⨍_{B_{\bar t}(x,r)} h^2\,\mathrm{d}V_{g(\bar t)}\right) - \frac{n}{2}\log 4\pi r^2 - n\\ =\; &\log\left(|B_{\bar t}(x,r)|r^{-n}\right) - \frac{n}{2}\log 4\pi e^2.\end{aligned} \tag{3.12}$$

3.2.1 Lower bound of the renormalized volume ratio

It is well known, as Perelman's no local collapsing theorem tells, that the lower bound of the initial μ-entropy and the upper bound of the scalar curvature together give a lower bound of the volume ratio, see [78] and [77]. Following this classical argument, but with the more explicit lower bound (3.3) of the initial μ-entropy, we obtain a generalized lower bound of the renormalized volume ratio. We begin with the following lemma:

Lemma 3.2.1. *For the fixed time slice $\bar{t}$ and any positive $r \leq \sqrt{\bar{t}}$, suppose the doubling property*

$$|B_{\bar{t}}(x, \frac{r}{2})| \ \geq \ 3^{-n} \, |B_{\bar{t}}(x, r)| \tag{3.13}$$

holds, then there is a constant $C_{VR}(T) = C_{VR}(T)(C_0, C_S, D_0, T)$ such that

$$\frac{|B_{\bar{t}}(x, r)|}{r^n} \ \geq \ C_{VR}(T) V D_0^{-n}. \tag{3.14}$$

Proof. We examine the upper bound of $\mathcal{W}(g(\bar{t}), h^2, r^2)$ with the help of (3.13).

Since $\sup_{M\times[0,2T]} |\mathcal{R}_{g(t)}| \leq C_0$, we have

$$\int_M r^2 \mathcal{R}_{g(\bar{t})} h^2 \, \mathrm{d}V_{g(\bar{t})} \ \leq \ 2C_0 T; \tag{3.15}$$

moreover, from (3.11) and (3.13) we have

$$\int_M 4r^2 |\nabla h|^2 \, \mathrm{d}V_{g(\bar{t})} \ \leq \ 3^{n+4}.$$

These estimates, together with (3.12) give

$$\mathcal{W}(g(\bar{t}), h^2, r^2) \ \leq \ \log \frac{|B_{\bar{t}}(x, r)|}{r^n} + 2C_0 T + 3^{n+5} - \frac{n}{2} \log 4\pi e^2. \tag{3.16}$$

On the other hand, since $\|h\|_{L^2(M, g(\bar{t}))} = 1$, we could evolve h^2 by the conjugate heat equation along the Ricci flow $\partial_t g(t) = -2\mathcal{R}c_{g(t)}$, i.e. we solve $(\partial_t + \Delta - \mathcal{R})u = 0$ with final value $u(\bar{t}) = h^2$.

By the monotone increasing property of $\mathcal{W}(g(t), u(t), \tau)$ in t (with $\tau'(t) = -1$), we may apply the initial lower bound (3.2) to see

$$\begin{aligned} \mathcal{W}(g(\bar{t}), h^2, r^2) \ &\geq \mathcal{W}(g(0), u(0), \bar{t} + r^2) \\ &\geq \ \log V D_0^{-n} - \frac{(C_0 D_0^2 + 1)}{D_0^2} (\bar{t} + r^2) - \frac{n}{2} \log(8n\pi e C_S D_0^2), \end{aligned}$$

therefore by (3.16), we have the following lower bound of the log volume ratio:

$$\log \frac{|B_{\bar{t}}(x, r)|}{r^n} \ \geq \ \log V D_0^{-n} - \frac{2T(2C_0 D_0^2 + 1)}{D_0^2} - 3^{n+5} - \frac{n}{2} \log(2ne^{-1} C_S D_0^2),$$

which is

$$\frac{|B_{\bar{t}}(x, r)|}{r^n} \ \geq \ C_{VR}(T)(C_0, D_0, T) V D_0^{-n}, \tag{3.17}$$

where $C_{VR}(T) := (2ne^{-1}C_S)^{-\frac{n}{2}} \exp(2T(2C_0 + D_0^{-2}) - 3^{n+5})$, which ultimately also depends on the L^2-Poincaré constant C_P and the doubling constant C_D of the initial metric, as encoded in C_S. Again, $C_{VR}(T)$ is invariant under the parabolic rescaling of the Ricci flow. □

We now prove the local volume doubling property (3.13), which follows directly from the original contradiction argument due to Perelman, if we notice that the constant $C_{VR}(T)$ is independent of $\bar{t}$ and r^2 as long as $\bar{t} \leq T$ and $r^2 \leq \bar{t}$.

Now suppose (3.14) fails for some scale $r \in (0, \sqrt{\bar{t}})$ at time $\bar{t} \leq T$ and a point $x \in M$, then (3.13) must fail for this r, and it will also fail at scale $\frac{r}{2}$: otherwise, the above argument applied to the $\frac{r}{2}$-ball around $x \in M$ will produce

$$\begin{aligned} |B_{\bar{t}}(x, \frac{r}{2})| &\geq 2^{-n} C_{VR}(T)(C_0, D_0, T) V r^n \\ &\geq 2^{-n} |B_{\bar{t}}(x, r)|, \end{aligned}$$

where we have used the converse of (3.14), but contradicts the failure of (3.13). Therefore, if the converse of (3.13) is observed at any point and scale, then it will pass down to all smaller scales at that point, i.e. the converse of (3.13) implies for any $k \geq 1$,

$$|B_{\bar{t}}(x, 2^{-k} r)| \leq 3^{-nk} |B_{\bar{t}}(x, r)|,$$

which is impossible for k sufficiently large, since $(M, g(\bar{t}))$ is locally Euclidean. Therefore, we have the following

Proposition 3.2.2 (Lower bound of renormalized volume ratio)**.** *Let $(M, g(t))$ be a Ricci flow solution on $[0, T]$ with initial diameter D_0 and initial volume V. Assume that the scalar curvature is uniformly bounded by C_0 in space-time, then there is a constant $C_{VR}(T)$ depending on the initial doubling constant C_D, the initial L^2-Poincaré constant C_P, the initial diameter D_0, the scalar curvature bound C_0 and T, such that for any time $t \in [0, T]$ and any scale r such that $r^2 \in (0, t]$,*

$$\frac{|B_t(x, r)|}{r^n} \geq C^+_{VR}(T) V D_0^{-n}. \tag{3.18}$$

Moreover, $C^+_{VR}(T)$ is invariant under the parabolic rescaling of the Ricci flow.

3.2.2 Diameter upper bound

In the same vein, but with the straightforward estimate (3.15) replaced by a more delicate maximal function argument, Peter Topping [80] proved a diameter upper bound in terms of the integral of the scalar curvature (see also [86]). When the scalar curvature is uniformly bounded in space-time, we notice that Topping's estimates depend on the initial volume, a factor that we hope to avoid in our estimates. However, once the quantities involved are correctly renormalized and the initial entropy lower bound (3.2) is used, Topping's argument still leads to a diameter upper bound which is independent of the initial volume. In the current subsection we discuss this in detail.

To begin with, we recall that the total volume changes as following:

$$V(t) \ := \ |M|_{g(t)} \ = \ Ve^{-\int_0^t \int_M \mathcal{R}_{g(s)} \mathrm{d}V_{g(s)} \mathrm{d}t} \ \leq \ Ve^{C_0 t}. \tag{3.19}$$

Moreover, as discussed above, we evolve the cut-off function h^2 backward by the conjugate heat equation along the Ricci flow. From (3.2), (3.10), (3.11) and (3.12) we get

$$\begin{aligned} &\log VD_0^{-n} + C_1(T) \\ \leq\ & \frac{64|B_{\bar t}(x,r)|}{|B_{\bar t}(x,\frac{r}{2})|} + \frac{r^2}{|B_{\bar t}(x,\frac{r}{2})|}\int_{B_{\bar t}(x,r)} |\mathcal{R}_{g(\bar t)}|\, \mathrm{d}V_{g(\bar t)} + \log\left(\frac{|B_{\bar t}(x,r)|}{r^n}\right), \end{aligned} \tag{3.20}$$

where $C_1(T) := -2T(C_0 + D_0^{-2}) - \frac{n}{2}\log(2ne^{-1}C_S)$, a constant only depending on the initial doubling and L^2-Poincaré constants, the initial diameter and the space-time scalar curvature bound; especially it is independent of the initial volume.

Now we define the maximal function of the scalar curvature following [80]:

$$M\mathcal{R}(x,r,\bar t) := \sup_{s\in(0,r]} \frac{|B_{\bar t}(x,s)|}{s}\left(⨍_{B_{\bar t}(x,s)} |\mathcal{R}_{g(\bar t)}|\, \mathrm{d}V_{g(\bar t)}\right)^{\frac{n-1}{2}}. \tag{3.21}$$

We also define $C_2 = \min\left\{\frac{\omega_n D_0^n}{2V}, e^{C_1(T)-2^{n+1}}\right\}$, where ω_n is the volume of n-dimensional Euclidean unit ball. Notice that since we are dealing with the case as $V \to 0$, the constant C is in fact independent of V, and we put it here just for the convenient of statement.

The key property of $M\mathcal{R}(x,r,\bar t)$, as described in [80], is that "*we cannot simultaneously have small curvature and small volume ratio*", but in our context we should consider the renormalized volume ratio instead, and this is described in the following proposition:

Lemma 3.2.3. *Let $(M, g(t))$ be a Ricci flow solution on $[0,T]$ with initial diameter D_0 and initial volume V. Assume that the scalar curvature is uniformly bounded by C_0 in space-time, then for any $\bar t \in (0,T]$ and $r > 0$ such that $r^2 \leq \bar t$, we have*

$$|B_{\bar t}(x,r)| \ \leq \ C_2 VD_0^{-n} r^n \ \Rightarrow \ M\mathcal{R}(x,r,\bar t) \ \geq \ C_2 VD_0^{-n},$$

where the constant C_2 is defined as above.

Proof (following [80]). We first claim that if $M\mathcal{R}(x,r,\bar{t}) \leq C_2 VD_0^{-n}$ then for any $s \in (0,r]$,

$$|B_{\bar{t}}(x,s)| \leq C_2 VD_0^{-n} s^n \Rightarrow |B_{\bar{t}}(x,\frac{s}{2})| \leq 2^{-n} C_2 VD_0^{-n} s^n.$$

Suppose otherwise, then we could fix some $s \in (0,r]$ that contradicts the claim, i.e.

$$|B_{\bar{t}}(x,\frac{s}{2})| > (C_2 VD_0^{-n})^{\frac{2}{n-1}} 2^{-n} s^{\frac{2n}{n-1}} |B_{\bar{t}}(x,s)|^{\frac{n-3}{n-1}},$$

so that we have

$$\begin{aligned}\int_{B_{\bar{t}}(x,s)} |\mathcal{R}_{g(\bar{t})}| \,\mathrm{d}V_{g(\bar{t})} &\leq (M\mathcal{R}(x,r,\bar{t}))^{\frac{2}{n-1}} s^{\frac{2}{n-1}} |B_{\bar{t}}(x,s)|^{\frac{n-3}{n-1}} \\ &\leq (C_2 VD_0^{-n})^{\frac{2}{n-1}} s^{\frac{2}{n-1}} |B_{\bar{t}}(x,s)|^{\frac{n-3}{n-1}} \\ &< 2^n s^{-2} |B_{\bar{t}}(x,\frac{s}{2})|.\end{aligned}$$

By (3.20), we could further deduce

$$\begin{aligned}\log VD_0^{-n} + C_1(T) &\leq \frac{64|B_{\bar{t}}(x,s)|}{|B_{\bar{t}}(x,\frac{s}{2})|} + \frac{s^2}{|B_{\bar{t}}(x,\frac{s}{2})|} \int_{B_{\bar{t}}(x,s)} |\mathcal{R}_{g(\bar{t})}| \,\mathrm{d}V_{g(\bar{t})} + \log\left(\frac{|B_{\bar{t}}(x,s)|}{s^n}\right) \\ &\leq \frac{64|B_{\bar{t}}(x,s)|}{|B_{\bar{t}}(x,\frac{s}{2})|} + 2^n + \log VD_0^{-n} + \log C_2,\end{aligned} \tag{3.22}$$

so that $|B_{\bar{t}}(x,s)| \geq 2^n |B_{\bar{t}}(x,\frac{s}{2})|$ by the choice of C_2, whence the claim.

Now by the claim, if there were any $x \in M$ that has some scale $s \in (0,r]$ contradicting the statement of the proposition, i.e. $|B_{\bar{t}}(x,s)| \leq C_2 VD_0^{-n} s^n$ and simultaneously $M\mathcal{R}(x,s,\bar{t}) \leq C_2 VD_0^{-n}$, then for any $m \in \mathbb{N}$, we have, by the choice of C_2, that

$$|B_{\bar{t}}(x,2^{-m}s)| \leq 2^{-mn} s^n C_2 VD_0^{-n} \leq \frac{\omega_n (2^{-m}s)^n}{2},$$

which is impossible for *all m* sufficiently large, since as a smooth Riemannian manifold, $(M, g(\bar{t}))$ is locally Euclidean of dimension n. □

Now we define $\nu := \min\{(ne^{-1}C_S)^{\frac{n}{2}}, 1\}$. Notice that ν only depends on the initial data: the initial doubling and L^2-Poincaré constants. We prove the following diameter bound:

Proposition 3.2.4. *Let $(M, g(t))$ be a Ricci flow solution on $[0, T]$ with initial diameter D_0 and initial volume V, and assume that the scalar curvature is uniformly bounded by C_0 in space-time.*

Then there is a constant $C_{diam} > 0$ such that if $VD_0^{-n} < \nu\omega_n$, then

$$\forall t \in [0, T], \quad \mathrm{diam}(M, g(t)) \leq C_{diam} e^{2C_0 t} D_0, \tag{3.23}$$

where the constants only depend on C_D, C_P, C_0, D_0 and are invariant under the parabolic rescaling of the Ricci flow.

Proof (following [80]). By the assumption on V, we have, by its definition, $C_2 = e^{C_1(T) - 2^{n+1}}$. For fixed $\bar{t} \in [0, T]$, let γ be a minimal geodesic in M with $|\gamma|_{g(\bar{t})} = \mathrm{diam}(M, g(\bar{t}))$, and let $\{x_i\}$ be a maximal set of points on γ such that

1. $B_{\bar{t}}(x_i, D_0/10)$ are mutually disjoint; and
2. $|B_{\bar{t}}(x_i, D_0/10)| > 10^n C_2 V$ for each i.

Let $N := |\{x_i\}|$, then clearly $N \leq V(\bar{t})/(10^n C_2 V) \leq e^{C_0 \bar{t}}/C_2$.

Now the set $\gamma \backslash \cup_{i=1}^N B_{\bar{t}}(x_i, D_0/5)$ has at most $N+1$ connected components, and let σ be one of these components with largest length. We either have $\mathrm{diam}(M, g(\bar{t})) = |\gamma|_{g(\bar{t})} = |\sigma|_{g(\bar{t})}$ if $N = 0$; or else, if $N \geq 1$, we then have

$$\begin{aligned}\mathrm{diam}(M, g(\bar{t})) &\leq (N+1)|\sigma|_{g(\bar{t})} + 2ND_0/5 \\ &\leq 2N(|\sigma|_{g(\bar{t})} + D_0/5) \\ &\leq 2C_2^{-1} e^{C_0 \bar{t}} (|\sigma|_{g(\bar{t})} + D_0/5).\end{aligned} \tag{3.24}$$

In any case, we will need to estimate $|\sigma|_{g(\bar{t})}$ in terms of the initial diameter D_0:

For any $x \in Im(\sigma)$, the maximality of $\{x_i\}$ guarantees that $|B_{\bar{t}}(x, D_0/10)| \leq 10^n C_2 V$. Now by Lemma 3.2.3, we know that

$$\forall x \in Im(\sigma), \quad M\mathcal{R}(x, D_0/10, \bar{t}) \;\geq\; C_2 V D_0^{-n}.$$

Therefore we could find some $s(x) \in (0, D_0/10]$ such that

$$\begin{aligned}C_2 V D_0^{-n} &\leq \frac{|B_{\bar{t}}(x, s)|}{s} \left(\fint_{B_{\bar{t}}(x,s)} |\mathcal{R}_{g(\bar{t})}| \, \mathrm{d}V_{g(\bar{t})} \right)^{\frac{n-1}{2}} \\ &\leq \frac{1}{s} \int_{B_{\bar{t}}(x,s)} |\mathcal{R}_{g(\bar{t})}|^{\frac{n-1}{2}} \, \mathrm{d}V_{g(\bar{t})}.\end{aligned}$$

Now we could apply Lemma 5.2 of [80] to pick a set of points $\{y_j\} \subset Im(\sigma)$ such that $\{B_{\bar{t}}(y_j, s(y_j))\}$ are mutually disjoint, and that $|\sigma| \le 6\sum_j s(y_j)$. We could now estimate

$$\begin{aligned} |\sigma|_{g(\bar{t})} &\le 6\sum_j \frac{D_0^n}{C_2 V}\int_{B_{\bar{t}}(y_j,s(y_j))} |\mathcal{R}_{g(\bar{t})}|^{\frac{n-1}{2}}\, \mathrm{d}V_{g(\bar{t})} \\ &\le \frac{6D_0^n}{C_2 V}\int_M |\mathcal{R}_{g(\bar{t})}|^{\frac{n-1}{2}}\, \mathrm{d}V_{g(\bar{t})} \\ &\le 6C_2^{-1}e^{C_0\bar{t}}D_0^n C_0^{\frac{n-1}{2}}. \end{aligned} \tag{3.25}$$

Putting the estimates (3.24) and (3.25) together we obtain

$$\mathrm{diam}(M, g(\bar{t})) \le C_{diam}e^{2C_0\bar{t}}D_0,$$

where, recalling the definition of C_2, we have $C_{diam} = 4^{n+4}D_0^{n-1}C_0^{\frac{n-1}{2}}$.

We notice that C_{diam} is invariant under the parabolic rescaling of the Ricci flow, and is independent of time. □

Remark 3.2.5. *When the scalar curvature is uniformly bounded, the previous renormalized volume ratio lower bound, and the upper bound of the total volume, actually provide a diameter upper bound. This naive estimate, however, fails to provide constants that are invariant under the parabolic rescaling of the Ricci flow.*

3.2.3 A weak compactness result

We now state a proposition that corresponds to our second motivation of the paper: to constuct, from a sequence of Ricci flows with collapsing initial data, Gromov-Hausdorff limits of the positive time-slices. Compare also a result of Chen-Yuan [68, Theorem 1] where lower bounds of the Ricci curvature and the unit ball volume, uniform in space-time, are assumed.

Proposition 3.2.6 (Weak compactness for positive time slices)**.** *Let $\{(M_i, g_i(t))\}$ be a sequence of Ricci flows defined for $t \in [0, T]$, such that they satisfy the same assumptions as in Theorem 1.3.1.*

Then for each $t \in (0, T)$, there is a subsequence of $\{(M_i, g_i(t))\}$, a compact metric space (X_t, d_t), to which the subsequence converges in the Gromov-Hausdorff topology.

Proof. This is a simple consequences of the estimates we proved previously in this section. Recall that in [74, Chapter 5, A] a quantity $N(\varepsilon, R, X)$ is defined for each complete metric space X, to denote the maximal number of disjoint ε-balls that could be possibly fitted into an R-ball in the metric space X. As shown in [74, Proposition 5.2], as long as $N(\varepsilon, R, X_i)$ is uniformly bound for all $\varepsilon \in (0, R)$, $R \in (0, \operatorname{diam} X_i)$ and X_i, the sequence $\{X_i\}$ is precompact in the pointed-Gromov-Hausdorff topology.

In our situation, $\forall t \in (0, T)$, since we have a uniform diameter upper bound (3.23), we only need to control $N\left(\varepsilon, C_{diam} e^{2C_R t} D_0, (M_i, g_i(t))\right)$. In fact, we could easily see that $\forall \varepsilon \in (0, C_{diam} e^{2C_R t} D_0)$, the total volume upper bound (3.19) together with the lower bound of renormalized volume ratio (3.18) gives: denoting $V_i := Vol(M_i, g_i(0))$, we have

$$N\left(\varepsilon, C_{diam} e^{2C_R t} D_0, (M_i, g_i(t))\right) \leq \frac{V_i e^{C_R t}}{C_{VR}^+(T) V_i D_0^{-n} \varepsilon^n} = \frac{e^{C_R t}}{C_{VR}^+} \left(\frac{D_0}{\varepsilon}\right)^n.$$

This bound is uniform on the sequence $\{(M_i, g_i(t))\}$ and therefore there is a metric space (X_t, d_t) to which the sequence subconverges in the Gromov-Hausdorff sense.

Clearly, $\operatorname{diam}(X_t, d_t) \leq C_{\mathrm{diam}} e^{2C_R t}$. □

Remark 3.2.7. *Especially, we may assume* $\lim_{i\to\infty} V_i D_0^{-n} = 0$*, and this justifies us calling "collapsing initial data".*

3.3 Estimating the analytic quantities along the Ricci flow

In this section we prvide a rough upper bound of the renormalized heat kernel, following Davies' argument as discussed in Qi S. Zhang's book and paper; we then apply this rough upper bound, the Harnack inequality (2.4), and the diameter upper bound (3.23) to obtain an on-diagonal lower bound lower bound of the renormalized heat kernel, when the initial global volume ratio is sufficiently small. As consequences, we also deduce a Gaussian type lower bound of the renormalized heat kernel, as well as the non-inflation property for the renormalized volume ratio.

3.3.1 Rough upper bound of the renormalized heat kernel in space time

The technique used to show the uniform Sobolev inquality in Subsection 3.2, i.e. the method of Davies, could be further applied to obatin a rough upper bound of the heat kernel coupled with the Ricci flow. This was first noticed by Qi S. Zhang in [85] and we will follow the exposition there. Notice that recently, Meng Zhu also extended Davies' method to Ricci flows with a uniform Ricci curvature lower bound in space-time, see [**?**].

We fix $(x_0, t_0) \in M\times[0,T)$, and consider the heat kernel based at (x_0, t_0), coupled with the Ricci flow, as introduced in Subsection 2.3. More specifically, we denote the heat kernel by

$$K(x,t) = G(x_0, t_0; x, t),$$

i.e. for any $(x,t) \in M \times (t_0, T]$, $(\partial_t - \Delta_{g(t)})K(x,t) = 0$, and $\lim_{t\downarrow t_0} K(x,t) = \delta_{x_0}(x)$.

Now fix any $t \in (t_0, T]$, let $p(s) := (t-t_0)/(t-s)$ for $s \in (t_0, t]$. Besides $p(t_0) = 1$ and $\lim_{s\uparrow t} p(s) = \infty$, we also notice the following relations:

$$\begin{aligned} 0 \le \frac{p(s)-1}{p'(s)} &= \frac{(s-t_0)(t-s)}{t-t_0} \le t - t_0, \\ 0 < \frac{1}{p'(s)} &= \frac{(t-t_0)^2}{t} \le t, \\ \text{and} \quad p'(s)p^{-2}(s) &= \frac{1}{t-t_0}. \end{aligned}$$

Defining for any $(x,s) \in M \times (t_0, t]$,

$$v(x,s) := K(x,s)^{\frac{p(s)}{2}} \|K^{\frac{p(s)}{2}}\|^{-1}_{L^2(M,g(s))},$$

we could compute as before to obtain

$$\begin{aligned} \partial_s \log \|K\|_{L^{p(s)}(M,g(s))} &= \frac{p'(s)}{p^2(s)} \int_M v^2 \log v^2 - \frac{4(p(s)-1)}{p'(s)}|\nabla v|^2 - \frac{p(s)}{p'(s)}\mathcal{R}_{g(s)}v^2 \, \mathrm{d}V_{g(s)} \\ &\le \frac{p'(s)}{p^2(s)} \int_M v^2 \log v^2 - \frac{p(s)-1}{p'(s)}\left(4|\nabla v|^2 + \mathcal{R}_{g(s)}v^2\right) \mathrm{d}V_{g(s)} + C_0. \end{aligned}$$

We now plug $\tau = \frac{p(s)-1}{p'(s)}$ and $\bar{t} = s$ into the uniform log-Sobolev inequality (3.4) (a bit abusing of notation), and obtain from the above calculations:

$$\begin{aligned} &\partial_s \log \|K\|_{L^{p(s)}(M,g(s))} \\ &\le \frac{p'(s)}{p^2(s)}\left(-\frac{n}{2}\log\frac{p(s)-1}{p'(s)} - \log VD_0^{-n} + (C_0 + D_0^{-2})\left(\bar{t} + \frac{p(s)-1}{p'(s)}\right) + C_{lS}\right) + C_0. \end{aligned}$$

Integrating s from t_0 to t, we see that

$$\log \frac{\|K(-,t)\|_{L^\infty(M,g(t))}}{\|K(-,t_0)\|_{L^1(M,g(t_0))}} \le -\log VD_0^{-n}(t-t_0)^{\frac{n}{2}} + 2t(C_0 + D_0^{-2}) + C_0(t-t_0) + C_{lS} + n.$$

Since the $K(-,s)$ acquires the Delta function property as s descends to t_0, we clearly have

$$\|K(-,t_0)\|_{L^1(M,g(t_0))} = 1.$$

Therefore, exponentiating both sides of the above estimate we obtain

$$VD_0^{-n}G(x_0, t_0; x, t) \;\leq\; C_H^+(T)(t - t_0)^{-\frac{n}{2}},$$

where $C_H^+(T) = \exp(2T(C_0 + D_0^{-2}) + C_0T + C_{lS} + n)$ is a universal constant independent of $t - t_0$, and is invariant under the parabolic rescaling of the Ricci flow. We collect the result in the following

Proposition 3.3.1 (Rough upper bound of the renormalized heat kernel)**.** *Let $(M, g(t))$ be a Ricci flow solution on $[0, T]$ with initial diameter D_0 and initial volume V, and assume that the scalar curvature is uniformly bounded by C_0 in space-time.*

Then there is a constant $C_H^+(T) = C_H^+(C_D, C_P, C_0, D_0, n, T)$ such that for any $(y, t) \in M \times (0, T]$, the conjugate heat kernel $G(-, -; y, t)$ based at (y, t) obeys the following estimate:

$$\forall x \in M,\ \forall s \in (0, t), \quad VD_0^{-n}G(x, s; y, t) \;\leq\; C_H^+(T)(t - s)^{-\frac{n}{2}}. \tag{3.26}$$

3.3.2 Gaussian type lower bound of the renormalized heat kernel

In [67] and [85], the non-inflation property of volume ratio was proven based on an on-diagonal heat kernel lower bound, which was obtained by estimating the reduced length of a constant curve in space, and Perelman's estimate. Such on-diagonal heat kernel lower bound, together with the gradient estimate Theorem 2.1.1, then gives a Gaussian lower bound of the heat kernel. This lower bound is essential in Bamler-Zhang's estimate of the distance distortion.

We notice that this lower bound, however, could be not applied in the case of collapsing initial data, basically because of the lack of a correct renormalization. In this subsection, our major task is then to obtain a Gaussian type lower bound of the renormalized heat kernel. We will start similarly with an on-diagonal lower bound, and then apply the gradient estimate to obtain the desired Gaussian lower bound of the renormalized heat kernel.

First let us recall the volume upper bound:

$$V(t) := |M|_{g(t)} = Ve^{-\int_0^t \int_M \mathcal{R}(s)\mathrm{d}V_{g(s)}\mathrm{d}t} \leq Ve^{C_0 t}. \tag{3.27}$$

We also recall the bound of the total heat on the whole manifold: for any $x, y \in M$ and any $0 \leq s < t \leq T$, recall that $G(x, s; y, t)$ satisfies the heat equation in the variable (y,t), and it satisfies the conjugate heat equation in the variable (x,s); fixing (x,s) and integrating $y \in M$ we have $\forall t' < t \leq T$,

$$\left|\partial_t \int_M G(x,t';y,t)\,\mathrm{d}V_{g(t)}\right| = \left|\int_M \Delta_y G(x,t';y,t) - \mathcal{R}_{g(t)}(y) G(x,t';y,t)\,\mathrm{d}V_{g(t)}(y)\right|$$
$$\leq C_0 \int_M G(x,t';y,t)\,\mathrm{d}V_{g(t)}(y),$$

therefore integrating in time we see

$$e^{-C_0(t-s)} \;\leq\; \int_M G(x,s;y,t)\,\mathrm{d}V_{g(t)}(y) \;\leq\; e^{C_0(t-s)}. \tag{3.28}$$

As discussed in the introduction, the collapsing procedure is an intrinsic geometric phenomenon, and it should not cause the addition or loss of the total heat. Therefore, the heat-volume duality should be preserved. If the heat kernel at a future time fails to have a pointwise lower bound of order $(VD_0^{-n})^{-1}$, then the duality between the heat and volume will be contradicted, due to the rough bound of the renormalized heat kernel, the Harnack inequality (2.4), and the volume and diameter upper bounds of the whole manifold.

More specifically, the diameter upper bound in Proposition 3.2.4, the rough pointwise upper bound of $G(x,s;-,-)$ in Propostition 3.3.1, joint with the Harnack inequality (2.4) give, for any $(y,t) \in M \times (s,T]$,

$$G(x,s;y,t) \;\leq\; H(n)\left(\frac{C_H^+(T)G(x,s;x,t)}{(VD_0^{-n})(t-s)^{\frac{n}{2}}}\right)^{\frac{1}{2}} \exp\left(\frac{H'(n)C_{diam}^2 e^{4C_0T}D_0^2}{(t-s)}\right),$$

whenever the initial golbal volume ratio is bounded as $VD_0^{-n} \leq \nu\omega_n$.

Now we let $\theta := \sqrt{t-s}/D_0$ denote the ratio of the parabolic scale to the initial diameter. Integrating $y \in M$ and involking the lower bound of total heat in (3.28), we see

$$\begin{aligned}
e^{-C_0(t-s)} \;&\leq\; \int_M G(x,s;y,t)\,\mathrm{d}V_{g(t)}(y)\\
&\leq\; H(n)Ve^{C_0t}\left(\frac{C_H^+(T)G(x,s;x,t)}{V\theta^n}\right)^{\frac{1}{2}} \exp\left(\frac{H'(n)C_{diam}^2 e^{4C_0T}}{\theta^2}\right)\\
&=\; H(n)e^{C_0t}\theta^{-\frac{n}{2}}\left(C_H^+(T)VG(x,s;x,t)\right)^{\frac{1}{2}} \exp\left(\frac{H'(n)C_{diam}^2 e^{4C_0T}}{\theta^2}\right),
\end{aligned}$$

and thus

$$VG(x,s;x,t) \;\geq\; H(n)^{-2}e^{-C_0(3t-s)}(C_H^+(T))^{-1}\theta^n \exp\left(-\frac{2H'(n)C_{diam}^2 e^{4C_0T}}{\theta^2}\right).$$

Multiplying D_0^{-n} on both sides of this inequality we get

$$VD_0^{-n}G(x,s;x,t) \;\geq\; C_{HD}^{-}(T)\Psi(\theta \mid T)(t-s)^{-\frac{n}{2}},$$

where

$$C_{HD}^{-}(T) \;:=\; H(n)^{-2}e^{-3C_0T}C_H^{+}(T)^{-1}, \quad \text{and} \quad \Psi(\theta \mid T) \;:=\; \theta^{2n}\exp\left(-2H'(n)C_{diam}^{2}e^{4C_0T}\theta^{-2}\right).$$

Especially, we notice that $C_{HD}^{-}(T)$ only depends on the initial diameter, the doubling and L^2-Poincaré constants, the space-time scalar curvatur upper bound, and the time elapsed from the beginning. On the other hand, we notice that $\Psi(\theta \mid T)$ depends, besides θ and T, only on the initial diameter and the space-time scalar curvature bound, especially it is *independent* of the initial doulbing and L^2-Poincaré constants. We clearly see that

$$\lim_{\theta\to 0}\Psi(\theta \mid T) \;=\; 0, \tag{3.29}$$

indicating that the renormalization is only valid on scales comparable to the initial diameter. Moreover, both constants $C_{HD}^{-}(T)$ and $\Psi(\theta \mid T)$ are invariant under the parabolic rescaling of the Ricci flow. Summarizing, we have the following

Lemma 3.3.2 (On-diagonal lower bound of the renormalized heat kernel)**.** *Let $(M, g(t))$ be a Ricci flow solution on $[0,T]$ with initial diameter D_0 and initial volume V, and assume that the scalar curvature is uniformly bounded by C_0 in space-time.*

Then there are positive constants

$$C_{HD}^{-}(T) = C_{HD}^{-}(C_D, C_P, C_0, D_0, n, T) \quad \textit{and} \quad \Psi(\theta \mid T) = \Psi(\theta \mid C_0, D_0, n, T)$$

such that for any $(x,t) \in M\times(0,T]$, the conjugate heat kernel $G(-,-;x,t)$ based at (x,t) obeys the following estimate: for any $s \in (0,t)$, setting $\theta := \sqrt{t-s}/D_0$, then

$$VD_0^{-n}G(x,s;x,t) \;\geq\; C_{HD}^{-}(T)\Psi(\theta \mid T)(t-s)^{-\frac{n}{2}}, \tag{3.30}$$

whenever $VD_0^{-n} \leq \nu\omega_n$. Moreover, the constants $C_{HD}^{-}(T)$ and $\Psi(\theta \mid T)$ are invariant under the parabolic rescaling of the Ricci flow, and $\lim_{\theta\to 0}\Psi(\theta \mid T) = 0$.

Recall that the positive constant ν is defined right above Proposition 3.2.4.

Once this on-diagonal estimate is obtained, we could easily apply the Harnack inequality (2.4) again to obtain a Gaussian lower bound:

Proposition 3.3.3 (Gaussian type lower bound of the renormalized heat kernel)**.** *Let $(M, g(t))$ be a Ricci flow solution on $[0, T]$ with initial diameter D_0 and initial volume V, and assume that the scalar curvature is uniformly bounded by C_0 in space-time.*

Then there are positive constants

$$C_H^-(T) = C_H^-(C_D, C_P, C_0, D_0, n, T) \quad \text{and} \quad \Psi(\theta \mid T) = \Psi(\theta \mid C_0, D_0, n, T)$$

such that for any $(y, t) \in M \times (0, T]$, the conjugate heat kernel $G(-, -; y, t)$ based at (y, t) obeys the following estimate: for any $s \in (0, t)$, setting $\theta := \sqrt{t-s}/D_0$, then

$$VD_0^{-n} G(x, s; y, t) \;\geq\; C_H^-(T)\Psi(\theta \mid T)^2 (t-s)^{-\frac{n}{2}} \exp\left(-2H'(n)\frac{d_t(x, y)^2}{t-s}\right), \qquad (3.31)$$

whenever $VD_0^{-n} \leq \nu\omega_n$. Moreover, the constants $C_H^-(T)$ and $\Psi(\theta \mid T)$ are invariant under the parabolic rescaling of the Ricci flow, and $\lim_{\theta\to 0} \Psi(\theta \mid T) = 0$.

Here the constant is defined as $C_H^-(T) := (C_{HD}^-(T))^2 (H(n)^2 C_H^+(T))^{-1}$.

As a direct geometric consequence, we could also deduce the non-inflation property of the volume ratio:

Corollary 3.3.4 (Non-inflation of the renormalized volume ratio)**.** *Let $(M, g(t))$ be a Ricci flow solution on $[0, T]$ with initial diameter D_0 and initial volume V, and assume that the scalar curvature is uniformly bounded by C_0 in space-time.*

Then there are positive constants

$$C_{VR}^-(T) = C_{VR}^-(C_D, C_P, C_0, D_0, n, T) \quad \text{and} \quad \Psi(\theta \mid T) = \Psi(\theta \mid C_0, D_0, n, T)$$

such that for any $(x, t) \in M \times (0, T]$ and any $r \in (0, \sqrt{t})$, setting $\theta = r/D_0$, then

$$(VD_0^{-n})^{-1} |B_t(x, r)| \;\leq\; \frac{C_{VR}^-(T)}{\Psi(\theta \mid T)^2} r^n,$$

whenever $VD_0^{-n} \leq \nu\omega_n$. Moreover, the constants $C_{VR}^-(T)$ and $\Psi(\theta \mid T)$ are invariant under the parabolic rescaling of the Ricci flow, and $\lim_{\theta\to 0} \Psi(\theta \mid T) = 0$.

Proof. Fix $(x, t) \in M \times (0, T]$ and $r \in (0, \sqrt{t})$. Let $G(x, t-r^2; -, -)$ be the fundamental solution to the conjugate heat equation coupled with the Ricci flow on $M \times (t-r^2, T]$, based at $(x, t-r^2)$, i.e. $\lim_{s\downarrow t-r^2} G(x, t-r^2; -, s) = \delta_{(x,t-r^2)}(-)$. By the Gaussian type lower bound (3.31) of the renormalized heat kernel, we have

$$\forall y \in B_t(x, r), \quad VD_0^{-n} G(x, t-r^2; y, t) \;\geq\; C_H^-(T)\Psi(\theta \mid T)^2 e^{-2H'(n)} r^{-n}.$$

On the other hand, by (3.28), we have an upper bound of the total heat. Therefore, integrating over $B_t(x,r)$ we have

$$\begin{aligned} e^{C_0 r^2} &\geq \int_{B_t(x,r)} G(x, t-r^2; y, t)\, \mathrm{d}V_{g(t)}(y) \\ &\geq C_H^-(T)\Psi(\theta \mid T)^2 e^{-2H'(n)} |B_t(x,r)| (VD_0^{-n} r^n)^{-1}, \end{aligned}$$

or equivalently,

$$(VD_0^{-n})^{-1} |B_t(x,r)| \leq C_{VR}^-(T)\Psi(\theta \mid T)^{-2} r^n,$$

with $C_{VR}^-(T) := e^{2H'(n)+C_0T}/C_H^-(T)$. □

Remark 3.3.5. *Again, we see that the bound becomes worse as r/D_0 becomes smaller. However, for any fixed positive scale, we have a uniform estimate.*

3.4 Estimating the distance distortion

In this section we prove the main results of our note: the distance distortion estimate. Once the lower bound of the renormalized volume ratio (3.14) matches with that of the renormalized heat kernel (3.30), the classical argument of counting geodesic balls suitably covering a minimal geodesic carries over; see Section 5.3 of [24] and Section 3 of [61]. See also Section 3 of Chen-Wang [25] for a thorough exposition. However, we reproduce a detailed proof here, following [61], for the sake of completeness and readers' convenience.

Before the commencement of the proof, we would like to emphasize the importance of the parabolic-scaling invariance of the constants in our previous estimates: for fixed small scales, we will dialate them to unit size and work with the rescaled quantities.

Proof of Theorem 1.3.1. Fix $t_1 \in (0, T]$, and suppose that $d_{t_1}(x_0, y_0) = r$. Let $\theta = r/D_0$. Then we rescale r to 1 parabolically and denote the rescaled time slice as $\bar{t}$. Also denote the rescaled metric as $\bar{g}$.

Let $\gamma : [0,1] \to M$ be a unit speed $\bar{g}(\bar{t})$-minimal geodesic that connects x_0 to y_0. Let

$$K(x,t) := G(x_0, \bar{t} - \frac{1}{2}; x, t)$$

be a heat kernel coupled with the Ricci flow, with initial data the Delta function at $(x_0, \bar{t} - \frac{1}{2})$, recall immediately that we have the bound (3.28) of the total heat:

$$\forall t \in [\bar{t} - \frac{1}{2}, \bar{t} + \frac{1}{2}], \quad \int_M K(-,t)\, \mathrm{d}V_{\bar{g}(t)} \leq e^{C_0 r^2}. \tag{3.32}$$

By the renormalized heat kernel upper bound (3.26), we have

$$\forall t \in [\bar{t} - \frac{1}{4}, \bar{t} + \frac{1}{4}], \quad (VD_0^{-n})K(-, t) \ \leq \ C_H^+ 2^n; \tag{3.33}$$

on the other hand, by the Gaussian type lower bound (3.31), we have

$$\forall s \in [0, 1], \quad (VD_0^{-n})K(\gamma(s), \bar{t}) \ \geq \ C_H^- e^{-4H'(n)} 2^{\frac{n}{2}} \Psi(\theta \mid T)^2. \tag{3.34}$$

Time derivative bound (2.5) together with (3.33) imply that

$$\forall (s, t) \in [0, 1] \times [\bar{t} - \frac{1}{4}, \bar{t} + \frac{1}{4}], \quad |\partial_t (VD_0^{-n})K(\gamma(s), t)| \ \leq \ C_H^+ 2^n (C_0 r^2 + 4B(n));$$

therefore, setting

$$\alpha_0(\theta) \ := \ \min \left\{ \frac{1}{8}, \frac{C_H^- e^{-4H'(n)} \Psi(\theta \mid T)^2}{2^{n+1} C_H^+ (C_0 T + 4B(n))} \right\}$$

and integrating the above time derivative bound we obtain from (3.34) that

$$\forall (s, t) \in [0, 1] \times [\bar{t} - \alpha_0(\theta), \bar{t} + \alpha_0(\theta)], \quad (VD_0^{-n})K(\gamma(s), t) \ \geq \ \frac{1}{2} C_H^- e^{-4H'(n)} 2^{\frac{n}{2}} \Psi(\theta \mid T)^2.$$

Now by the Harnack inequality (2.4), we could estimate

$$\forall (s, t) \in [0, 1] \times [\bar{t} - \alpha_0, \bar{t} + \alpha_0], \quad \inf_{B_t(\gamma(s), 1)} (VD_0^{-n})K(-, t) \ \geq \ C_3(T) \Psi(\theta \mid T)^4, \tag{3.35}$$

where $C_3(T) := C_H^-(T) e^{-16H'(n)} / (4H(n)^2 C_H^+(T))$ is a constant only depending on the initial diameter, the initial doubling and L^2-Poincaré constants, and the space-time scalar curvature bound. Moreover, $C_3(T)$ is invariant under the parabolic rescaling of the Ricci flow.

Now fix any $t \in [\bar{t} - \alpha_0(\theta), \bar{t} + \alpha_0(\theta)]$, and cover $Im(\gamma) \subset M$ by a minimal number of unit $\bar{g}(t)$-geodesic balls $\{B_t(\gamma(s_i), 1)\}$, $i = 1, \cdots, N$. It is easily seen that $|\gamma|_{\bar{g}(t)} \leq 2N$. Therefore, in order to obtain an upper bound of $d_t(x_0, y_0)$, it suffices to control N from above.

By the minimality of the covering, we see that the collection $\{B_t(\gamma(s_i), 1/2)\}$ are pairwise disjoint. We could therefore combine the upper bound (3.32) of the total heat, the renormalized lower bound (3.35) of the local heat, together with the lower bound (3.18) of the renormalized volume ratio, to estimate:

$$\begin{aligned} e^{C_0 r^2} \ &\geq \ \int_M K(x, t) \, \mathrm{d}V_{\bar{g}(t)} \\ &\geq \ \sum_{i=1}^{N} \int_{B_t(\gamma(s_i), \rho/2)} K(x, t) \, \mathrm{d}V_{\bar{g}(t)} \\ &\geq NC_3(T) 2^{-n} \Psi(\theta \mid T)^4. \end{aligned}$$

Therefore $N \leq 2^n e^{C_0 T} C_3(T)^{-1}\Psi(\theta \mid T)^{-4}$, a constant independent of specific Ricci flow, especially its initial entropy. On the other hand, recalling that $d_{\bar{t}}(x_0, y_0) = 1$, we get

$$\forall t \in [\bar{t} - \alpha_1(\theta), \bar{t} + \alpha_1(\theta)], \quad d_t(x_0, y_0) \ \leq \ \alpha_1(\theta)^{-1} d_{\bar{t}}(x_0, y_0), \tag{3.36}$$

where

$$\alpha_1(\theta) := \min\left\{\alpha_0(\theta), \frac{C_3(T)\Psi(\theta \mid T)^4}{2^{n+1}e^{C_0 T}}\right\}.$$

This proves one side of the desired distance distortion estimate. To see the other side, we notice that the estimate (3.36) is independent of specific time slice $\bar{t}$. Therefore, letting $\alpha(\theta) = \frac{1}{2}\alpha_1(\theta)$, and applying the previous argument at the t-slice for any $t \in [\bar{t} - \alpha, \bar{t} + \alpha]$, we see

$$\forall s \in [t - \alpha_1(\theta), t + \alpha_1(\theta)], \quad d_s(x_0, y_0) \leq \alpha_1(\theta)^{-1} d_t(x_0, y_0).$$

Especially, since $\bar{t} \in [t - \alpha_1, t + \alpha_1]$, plugging $s = \bar{t}$ into the the above inequality we get the desired estimate (1.2) with $\alpha(\theta)$ in place of $\alpha_1(\theta)$. □

Here we emphasize again that $\alpha(\theta) \to 0$ as $\theta \to 0$, reflecting the fact that when we look at smaller scales compared to the initial diameter, the estimate will be less effective.

We could also enhance the above distance distortion estimate in the following

Corollary 3.4.1. *Let $(M, g(t))$ be a complete Ricci flow solution on $[0, T]$ with initial diameter D_0 and initial volume V, and assume the following conditions:*

1. *$(M, g(0))$, as a closed Riemannian manifold, has its doubling constant uniformly bounded above by C_D, and its L^2-Poincaré constant by C_P, and*
2. *the scalar curvature is uniformly bounded in space-time: $\sup_{M\times[0,T]} |\mathcal{R}_{g(t)}| \leq C_0$.*

There exist two positive constants $\alpha = \alpha(\theta \mid C_D, C_P, C_0, D_0, n, T) < 1$ with

$$\lim_{\theta\to 0} \alpha(\theta \mid C_D, C_P, C_0, D_0, n, T) = 0,$$

and $\nu = \nu(C_D, C_P, C_0, n) < 1$, such that whenever $VD_0^{-n} \leq \nu\omega_n$, we have,

$$\forall t \in [0, T], \quad \forall x, y \in M \text{ with } d_t(x, y) =: r \leq \sqrt{t}$$
$$\text{and} \quad \forall s \in [r^2, T] \text{ with } |s - t| \leq \alpha(\theta)\min\{C_0^{-1}, t\} + r^2,$$

the following estimate:

$$d_s(x, y)^2 \ \leq \ \alpha(\theta)^{-1}(d_t(x, y)^2 + |s - t|).$$

The proof is identical to that of Corollary 1.2 of [61], and we will omit it here.

Chapter 4

Regularity and convergence 4-D Ricci shrinkers

4.1 Collapsing and local scales

The collapsing of Riemannian manifolds could mean different things in different contexts. Our original concern (as stated in introduction) is about *volume collapsing*, i.e. the manifold admitting a family of Riemannian metrics under which the volume of fix-sized metric balls approaches zero. If we assume uniformly bounded Riemannian curvature, then the volume collapsing is equivalent to *collapsing with uniformly bounded curvature*, meaning that the injectivity radius of each point, under the family of metrics, approaches zero. When collapsing with bounded curvature happens, the structure theory of Cheeger-Fukaya-Gromov [14] will be of great help in studying the underlying manifold.

4.1.1 Curvature scale

In general, however, no *a priori* uniform curvature bound could be assumed. One then realizes that the above mentioned structural theory about *collapsing with uniformly bounded curvature* could be localized if the metrics in consideration are regular. This is because the curvature scale, is locally 1-Lipschitz. See Section 3 for a detailed discussion about Cheeger-Tian's localization adopted to the 4-D Ricci shrinkers, and here we will focus on the basic properties of the curvature scale. See also [9] for an exposition of the theory of locally bounded curvature and the curvature scale.

Definition 4.1.1 (Curvature scale)**.** *For any $p \in M$, define*

$$r_{\mathcal{R}m}(p) := \sup\left\{ r > 0 : B(p,s) \text{ has compact closure in } B(p,r), \text{ and } \sup_{B(p,s)} |\mathcal{R}m| \leq s^{-2} \right\}.$$

Equivalently, $r_{\mathcal{R}m}(p)$ is the maximal scale such that if one rescales the metric to make it unit size, then the rescaled curvature will have its norm uniformly bounded by 1 on the resulting unit ball around $p \in M$.

In fact, $\forall x \in B(p, r_{\mathcal{R}m}(p))$, we have $B(x, r_{\mathcal{R}m}(p) - d(p,x)) \subset B(p, r_{\mathcal{R}m}(p))$, so

$$\sup_{B(x, r_{\mathcal{R}m}(p) - d(p,x))} |\mathcal{R}m| \leq r_{\mathcal{R}m}(p)^{-2} \leq (r_{\mathcal{R}m}(p) - d(p,x))^{-2}, \tag{4.1}$$

and thus $d(x,p) < r_{\mathcal{R}m}(p)$ implies that $r_{\mathcal{R}m}(x) \geq r_{\mathcal{R}m}(p) - d(p,x)$. Reversing the role of x and p, we have shown that the curvature scale is locally 1-Lipschitz as mentioned above:

Lemma 4.1.2. *Either $r_{\mathcal{R}m} \equiv \infty$ and $\mathcal{R}m \equiv 0$, or $r_{\mathcal{R}m}$ is locally Lipschitz with*

$$Lip\ r_{\mathcal{R}m} \leq 1. \tag{4.2}$$

In order to facilitate our local arguments, it is also convenient to truncate the curvature scale:

Definition 4.1.3 (Truncated curvature scale)**.** *For any fixed $0 < r \leq 1$, we put*

$$l_a := \min\{r_{\mathcal{R}m}, a\}.$$

Clearly l_a is locally 1-Lipschitz.

4.1.2 Elliptic regularity at the curvature scale

Besides the fact that $r_{\mathcal{R}m}$ is locally Lipschitz, another key ingredient in Cheeger-Tian's localization is that the higher regularities of Einstein metrics follow directly from local curvature bounds. This essentially follows from elliptic regularity theory and is independent of non-collapsing assumptions.

In the case of 4-D Ricci shrinkers, equations (2.8) and (2.14) form an elliptic system, which could be bootstrapped to give higher regularities of both the metric and the potential function, once a local curvature bound assumed. Also notice that according to (2.18) and (2.19), we already have a local C^1-bound of the potential function f.

Lemma 4.1.4. *(Local elliptic regularity) Let $p \in B(p_0, R)$, then there exists $C_k(R), D_k(R)$ such that*

$$\sup_{B\left(p,\frac{1}{2}l_a(p)\right)} |\nabla^k \mathcal{R}m| \leq C_k(R) l_a(p)^{-2-k}, \quad \textit{and} \quad \sup_{B\left(p,\frac{1}{2}l_a(p)\right)} |\nabla^k f| \leq D_k(R) l_a(p)^{-1-k}, \tag{4.3}$$

for $k = 0, 1, 2, 3, \cdots$.

Proof. Fix $p \in B(p_0, R)$, then $B(p, l_a(p)) \subset B(p_0, R+1)$. Since $\sup_{B(p,l_a(p))} |\mathcal{R}m| \leq l_a(p)^{-2}$, the conjugate radius r_{conj} has a definite lower bound on $B(p, l_a(p))$:

$$\inf_{B(p,l_a(p))} r_{\text{conj}} \geq \pi l_a(p).$$

This means that the exponential map $\exp_p : B(\mathbf{0}, l_a(p)) \to B(p, r_a(p))$ is well-defined and has no singularity. We can pull the manifold metric back to $B(\mathbf{0}, l_a(p)) \subset \mathbb{R}^4$, denote $\tilde{g} := \exp_p^* g$ and $\tilde{f} := \exp_x^* f$. Then the pull-back metric and potential function still satisfy the defining equation (1.4)

$$Rc_{\tilde{g}} + \tilde{\nabla}^2 \tilde{f} = \frac{1}{2}\tilde{g},$$

understood as matrix equations on an open subset of $\mathbb{R}^4$, with $\tilde{\nabla}^2$ the Hessian defined by the metric $\tilde{g}$. Notice that the equations (2.8) and (2.14) now become the elliptic system

$$\tilde{\Delta}\tilde{f} = 2 - \mathcal{R}_{\tilde{g}} \quad \text{and} \quad \tilde{\Delta}\mathcal{R}m_{\tilde{g}} = \tilde{\nabla}\tilde{f} * \mathcal{R}m_{\tilde{g}} + \mathcal{R}m_{\tilde{g}} + \mathcal{R}m_{\tilde{g}} * \mathcal{R}m_{\tilde{g}}, \tag{4.4}$$

defined on an open subset of $\mathbb{R}^4$, as equations of functions and of 4-tensors, respectively. Here $\tilde{\nabla}$ is the gradient under $\tilde{g}$ and $\tilde{\Delta} := tr_{\tilde{g}}\tilde{\nabla}^2$ is the Laplacian of $\tilde{g}$. Moreover, since $\exp_p$ is an isometry, the local C^1-bounds (2.18) and (2.19) of f translates as $\|\tilde{f}\|_{C^1(B(\mathbf{0},l_a(p)))} \leq (R+1)^2$.

On the other hand, as in [33] and [2], on $B(\mathbf{0}, l_a(p)) \subset \mathbb{R}^4$ we can use harmonic coordinates to deduce that $|\mathcal{R}m_{\tilde{g}}| \leq l_a(p)^{-2}$ implies $\|\tilde{g}\|_{C^{1,\alpha}} \leq C l_a(p)^{-1}$ on $B(\mathbf{0}, C l_a(p)/2)$.

Then we can bootstrap to get that $\|\tilde{f}\|_{C^{k,\alpha}} \leq D_k(R) l_a(p)^{-1-k}$ and $\|\mathcal{R}m_{\tilde{g}}\|_{C^{k,\alpha}} \leq C_k(R) l_a(p)^{-2-k}$ under harmonic coordinates. Since $\exp_p : (B(\mathbf{0}, l_a(p)), \tilde{g}) \to (B(p, l_a(p)), g)$ is an isometry, these estimates prove (4.3). □

Remark 4.1.5. *As explained in [2], given the results of [7], the passage from a lower bound on the harmonic radius to a corresponding compactness theorem is immediate.*

It is straightforward to obtain the following elliptic regularity under rescaling:

Lemma 4.1.6 (Rescaling)**.** *Given $\lambda \in (0,1)$. The rescaling $g \mapsto \tilde{g} := \lambda^{-2} g$ gives the equation $Rc_{\tilde{g}} + \nabla^2 f = \frac{\lambda^2}{2}\tilde{g}$. Moreover, $r_{\mathcal{R}m_{\tilde{g}}} = \lambda^{-1} r_{\mathcal{R}m_g}$ and $\forall p \in B(p_0, R)$ we have:*

$$\sup_{\tilde{B}\left(p, \frac{1}{2\lambda} l_a(p)\right)} |\tilde{\nabla}^k \mathcal{R}m_{\tilde{g}}|_{\tilde{g}} \ \leq\ C_k(R) \left(\frac{l_a(p)}{\lambda}\right)^{-2-k} \quad \text{and} \quad \sup_{\tilde{B}\left(p, \frac{1}{2\lambda} l_a(p)\right)} |\tilde{\nabla}^k f|_{\tilde{g}} \ \leq\ D_k(R) \left(\frac{l_a(p)}{\lambda}\right)^{-1-k}.$$

Moreover, for a general function solving the Poisson equation on a 4-D Ricci shrinker, we can argue similarly and obtain the following interior estimates under locally bounded curvature:

Lemma 4.1.7. *Suppose $u \in C^2(B(p, l_a(p))) \subset B(p_0, R)$ solves $\Delta^f u = c$ for some constant c, then there are constants $C''_k(R, c)$ for $k = 1, 2, 3, \cdots$, such that*

$$\sup_{B(p, \frac{1}{2} l_a(p))} |\nabla^k u| \ \leq\ C''_k(R, c)\, l_a(p)^{-k}.$$

4.1.3 Energy scale

Associated to Anderson's theorem (Proposition 2.4.6) is another local scale, called the energy scale. This scale is particularly well-adapted to the *analytical* side of the problem, and its interaction with the curvature scale, responsible for the *geometric* side of the problem, consists of the technical core of Cheeger-Tian's argument.

Definition 4.1.8. *The energy scale $\rho_f(p)$ is defined by*

$$\rho_f(p) := \min\left\{\sup\left\{r \in (0, R) : I^f_{\mathcal{R}m}(p, r) \leq \varepsilon_A(R)\right\}, 1\right\}.$$

Moreover, we could assume $\varepsilon_A(R) < 4C_A(R)^{-2}$ in Anderson's theorem (Proposition 2.4.6), so that $I^f_{\mathcal{R}m}(p, \rho_f(p)) \leq \varepsilon_A(R)$, and Proposition 2.4.6 tells that

$$\rho_f(p) \leq 2 r_{\mathcal{R}m}(p), \tag{4.5}$$

since

$$\sup_{B(p, \frac{1}{2}\rho_f(p))} |\mathcal{R}m| \leq C_A(R)\varepsilon_A(R)\rho_f(p)^{-2} \leq \left(\frac{1}{2}\rho_f(p)\right)^{-2}.$$

4.1.4 Volume collapsing and collapsing with locally bounded curvature

As mentioned above, we are concerned with the phenomenon of volume collapsing defined as:

Definition 4.1.9 (δ-volume collapsing)**.** *$U \subset B(p_0, R)$ is δ-volume collapsing if $\forall p \in U$, $\mu_f(B(p,1)) \le \delta$.*

However, volume collapsing does not give much information of the underlying geometry. The concept associated to localizing the structural theory of Cheeger-Fukaya-Gromov in [14] is (δ, a)-collapsing with locally bounded curvature:

Definition 4.1.10 ((δ, a)-collapsing with locally bounded curvature)**.** *$U \subset B(p_0, R)$ is (δ, a)-collapsing with locally bounded curvature if $\forall p \in U$, $\mu_f(B(p, l_a(p))) \le \delta\, l_a(p)^4$.*

Anderson's ε-regularity with respect to collapsing bridges these two concepts:

Lemma 4.1.11. *Suppose for some $\delta \in (0,1)$, and $\forall p \in U \subset B(p_0, R) \subset M$,*

$$\mu_f(B(p,1)) \le \frac{\delta}{16\bar{\mu}_R(1)} \quad \textit{and} \quad \int_{B(p,1)} |\mathcal{R}m|^2 \, d\mu_f \le \frac{\varepsilon_A(R)\,\delta}{16\bar{\mu}_R(1)},$$

then U is (δ, a)-collapsed with locally bounded curvature, i.e. $\forall p \in U$

$$\mu_f\,(B(p, l_a(p))) \;\le\; \delta\, l_a(p)^4. \tag{4.6}$$

Proof (following Cheeger-Tian). Without loss of generality we only need to consider points with $r_{\mathcal{R}m} \le 1$. If $\rho_f(p) = 1$, then

$$\begin{aligned}\mu_f(B(p, r_{\mathcal{R}m}(p))) \le \mu_f(B(p, 2r_{\mathcal{R}m}(p))) \le \frac{16\mu_f(B(p,1))\bar{\mu}_R(r_{\mathcal{R}m}(p))}{\bar{\mu}_R(1)} \;&\le\; \frac{\delta\,\bar{\mu}_R(r_{\mathcal{R}m}(p))}{\bar{\mu}_R(1)^2} \\ \le\; \frac{\delta\, r_{\mathcal{R}m}(p)^4}{\bar{\mu}_R(1)} \;\le\; \delta\, r_{\mathcal{R}m}(p)^4.&\end{aligned}$$

Otherwise, if $\rho_f(p) < 1$, and by continuity of $I^f_{\mathcal{R}m}(p, r)$ in r, $I^f_{\mathcal{R}m}(p, \rho_f(p)) = \varepsilon_A(R)$, and we can estimate

$$\begin{aligned}\mu_f(B(p, r_{\mathcal{R}m}(p))) \;&\le\; \frac{16\mu_f(B(p, \rho_f(p)))\bar{\mu}_R(r_{\mathcal{R}m}(p))}{\bar{\mu}_R(\rho_f(p))} \\ &=\; \frac{16\bar{\mu}_R(r_{\mathcal{R}m}(p))}{\varepsilon_A(R)} \int_{B(p,\rho_f(p))} |\mathcal{R}m|^2 \, \mathrm{d}\mu_f \\ &\le\; \frac{\delta\,\bar{\mu}_R(r_{\mathcal{R}m}(p))}{\bar{\mu}_R(1)} \;\le\; \delta\, r_{\mathcal{R}m}(p)^4,\end{aligned}$$

in the case $r_{\mathcal{R}m}(p) < a$, and a similar argument for $r_{\mathcal{R}m}(p) \ge a$ implies (4.6). □

This lemma says that if we have sufficiently small energy, *local volume collapsing* of a region does imply *collapsing with locally bounded curvature*.

4.2 Nilpotent structure and locally bounded curvature

When collapsing with bounded curvature happens, Cheeger-Fukaya-Gromov [14] gives a complete structural theory of the underlying manifold, one important consequence being the vanishing of the Euler characteristics. When the metric is locally regular, a similar structural theory could be obtained when collapsing with only *locally* bounded curvature happens on a domain. This observation was essentially discovered in [18], in the context of F-structures, and was made of full use in [20]. The vanishing of the Euler characteristic of the domain and (2.15) then help obtain an improved energy bound (Proposition 4.4.1), which will be crucial for the iteration argument for the key estimate (Proposition 4.4.5) later. In this section we will follow the expositions of Sections 2 and 3 of Cheeger-Tian [20] to see why their theory also works for 4-D Ricci shrinkers. The equivariant good chopping for sets collapsing with locally bounded curvatre (Theorem 4.3.1), which is the main theorem of Section 3 in [20], is proved in the next section.

In this subsection, we will discuss why the main theorems of Sections 2 and 3 of [20] also work for 4-D Ricci shrinkers.

We start with constructing a good covering, which sees a nice partition into sub-collections that makes the gluing arguments in [14] and [19] possible:

Lemma 4.2.1 (Existence of a good covering)**.** *Fix $a \leq 1$. There is a covering of $E \subset M$ by geodesic balls with radius being a uniform multiple of the curvature scale, such that it can be partitioned into at most N sub-collections S_j ($j = 1, \cdots, N$) of mutually disjoint balls in the covering, with any ball in a sub-collection intersecting at most one ball from another.*

Proof. Let $\{p_i\}$ $(i = 1, 2, 3, \cdots)$ be a maximal subset of E satisfying

$$d(p_i, p_j) \;\geq\; \zeta \min\{l_a(p_i), l_a(p_j)\}, \tag{4.7}$$

then for suitably chosen $\zeta \in (0, 1)$, $\{B(p_i, 2\zeta l_a(p_i))\}$ is a locally finite covering with uniformly bounded multiplicity. If $B(p_i, 2\zeta l_a(p_i)) \cap B(p_j, 2\zeta l_a(p_j)) \neq \emptyset$, then

$$d(p_i, p_j) \;\leq\; 4\zeta \max\{l_a(p_i), l_a(p_j)\}.$$

Assuming $\zeta < \frac{1}{4}$, then as done in (4.1),

$$\min\{l_a(p_i), l_a(p_j)\} \;\leq \max\{l_a(p_i), l_a(p_j)\} \;\leq\; \min\{l_a(p_i), l_a(p_j)\} + d(p_i, p_j),$$

so we can estimate the distance

$$d(p_i, p_j) \;\le\; \frac{4\zeta}{1-2\zeta}\min\{l_a(p_i), l_a(p_j)\}, \tag{4.8}$$

and thus

$$\min\{l_a(p_i), l_a(p_j)\} \;\le\; \max\{l_a(p_i), l_a(p_j)\} \;\le\; \frac{1+2\zeta}{1-2\zeta}\min\{l_a(p_i), l_a(p_j)\}. \tag{4.9}$$

Now if $B(p_{i_0}, 2\zeta l_a(p_{i_0}))\cap B(p_{i_j}, 2\zeta l_a(p_{i_j})) \neq \emptyset$ for $j = 1, \cdots, N(i_0)$, then by (4.8) and (4.9),

$$d(p_{i_0}, p_{i_j}) \;\le\; \frac{4\zeta}{1-2\zeta} l_a(p_{i_0}) \quad \text{and} \quad l_a(p_j) \;\ge\; \frac{1-2\zeta}{1+2\zeta} l_a(p_{i_0}),$$

and thus we have the following containment relations: $\forall j = 1, \cdots, N(i_0)$,

$$B\left(p_{i_0}, \zeta l_a(p_{i_0})\right) \;\subset\; B\left(p_{i_j}, \frac{5\zeta-2\zeta^2}{1-2\zeta} l_a(p_{i_0})\right) \;\subset\; B\left(p_{i_0}, \frac{9\zeta-2\zeta^2}{1-2\zeta} l_a(p_{i_0})\right), \tag{4.10}$$

while for $1 \le j_1 < j_2 \le N(i_0)$, (4.7) gives

$$B\left(p_{j_1}, \frac{\zeta(1-2\zeta)}{2(1+2\zeta)} l_a(p_{i_0})\right) \bigcap B\left(p_{j_2}, \frac{\zeta(1-2\zeta)}{2(1+2\zeta)} l_a(p_{i_0})\right) \;=\; \emptyset. \tag{4.11}$$

Let $\zeta < \frac{1}{20}$, and do the rescaling $g \mapsto l_a(p_{i_0})^{-2} g =: \tilde{g}$, then since $a \le 1$,

$$\sup_{\tilde{B}\left(p_{i_0}, \frac{9\zeta}{1-2\zeta}\right)} |\mathcal{R}m_{\tilde{g}}|_{\tilde{g}} \;\le\; 1.$$

Now apply (4.10), (4.11) and volume comparison on $\tilde{B}(p_{i_0}, 10\zeta)$ we get

$$\begin{aligned} Vol_{\tilde{g}}\left(\tilde{B}\left(p_{i_0}, \zeta\right)\right) \;&\le\; \frac{1}{N(i_0)} \sum_{j=1}^{N(i_0)} Vol_{\tilde{g}}\left(\tilde{B}\left(p_{i_j}, \frac{5\zeta-2\zeta^2}{1-2\zeta}\right)\right) \\ &\le\; \frac{1}{N(i_0)} \Lambda_{-1}\left(\frac{5\zeta-2\zeta^2}{1-2\zeta}\right) \Lambda_{-1}\left(\frac{\zeta(1-2\zeta)}{2(1+2\zeta)}\right)^{-1} Vol_{\tilde{g}}\left(\tilde{B}\left(p_0, \frac{9\zeta-2\zeta^2}{1-2\zeta}\right)\right) \\ &\le\; \Lambda_{-1}\left(\frac{5\zeta-2\zeta^2}{1-2\zeta}\right) \Lambda_{-1}\left(\frac{9\zeta-2\zeta^2}{1-2\zeta}\right) \Lambda_{-1}\left(\frac{\zeta(1-2\zeta)}{2(1+2\zeta)}\right)^{-1} \frac{Vol_{\tilde{g}}\left(\tilde{B}\left(p_{i_0}, \zeta\right)\right)}{\Lambda_{-1}(\zeta)\, N(i_0)}, \end{aligned}$$

where $\Lambda_{-1}(r)$ is the volume of radius r ball in a space form of constant curvature -1, and thus $N(i_0) \le N'$, a dimensional constant once we fix $\zeta \in (0, \frac{1}{20})$.

Now start with a maximal subset of $\{p_i\}$ with $d(p_i, p_j) > 10\zeta \max\{l_a(p_i), l_a(p_j)\}$ denoted by S_1; then choose S_2 as a maximal subset of $\{p_i\}\backslash S_1$, etc. In this way we could obtain $S_1, \cdots, S_N$. Notice that for $k = 1, 2$, if there exist $p_{i_k} \in S_i$ and $p_j \in S_j$ satisfying $B(p_{i_k}, 2\zeta l_a(p_{i_k})) \cap B(p_j, 2\zeta l_a(p_j)) \neq \emptyset$, then by (4.8) we have

$$10\zeta \max\{l_a(p_{i_1}), l_a(p_{i_2})\} \leq d(p_{i_1}, p_{i_2}) \leq \frac{8\zeta}{1-2\zeta} \max\{l_a(p_{i_1}), l_a(p_{i_2})\},$$

impossible for $\zeta < \frac{1}{20}$. Thus the ball centered at any element of S_j can intersect with at most one ball centered at some element of a different S_i.

On the other hand, by the maximality of each S_j ($j = 1, \cdots, N$), if $p_{i_0} \notin S_1 \cup \cdots \cup S_N$, then as observed in [15], there exist $p_{i_j} \in S_j$ for *each* $j = 1, \cdots, N$ (note that there may be more than one p_{i_j} from a single S_j, but we just pick one of them), such that

$$d(p_{i_0}, p_{i_j}) < 10\zeta \max\{l_a(p_{i_0}), l_a(p_{i_j})\} \quad \text{(compare (4.8))}$$

implying as before, since $\zeta < \frac{1}{20}$, that

$$\max\{l_a(p_{i_0}), l_a(p_{i_j})\} \leq \frac{\min\{l_a(p_{i_0}), l_a(p_{i_j})\}}{1-10\zeta} \quad \text{and} \quad d(p_{i_0}, p_{i_j}) \leq \frac{10\zeta}{1-10\zeta} l_a(p_{i_0}).$$

So we have the following containment relations

$$B(p_{i_0}, \zeta l_a(p_{i_0})) \subset B\left(p_{i_j}, \frac{11\zeta - 10\zeta^2}{1-10\zeta} l_a(p_{i_0})\right) \subset B\left(p_{i_0}, \frac{21\zeta - 10\zeta^2}{1-10\zeta} l_a(p_{i_0})\right),$$

and by (4.7), the mutual disjointness of $B\left(p_{i_j}, \frac{1}{2}\zeta(1-10\zeta) l_a(p_{i_0})\right)$ for $j = 1, \cdots, N$. Now we fix some $\zeta \in (0, \frac{1}{40})$, and do the same rescaling as before $g \mapsto l_a(p_{i_0})^{-2} g$. The unit curvature bound on the rescaled unit ball around p_{i_0}, the containment relations and mutual disjointness, together with the multiplicity estimate, give a dimensional bound on N, as argued by volume comparison within $\tilde{B}(p_{i_0}, 1)$ above. □

The fact that the number of partitions of the covering is independent of specific manifold, together with the elliptic regularity (4.3), ensure that the work of Cheeger-Fukaya-Gromov [14] go through. Thus we have arrived at

Theorem 4.2.2 (Cheeger-Fukaya-Gromov [14], Cheeger-Tian [20])**.** *For any $\varepsilon > 0$ and $r \in (0, 1)$, there exists a $\delta_{CFGT}(\varepsilon) > 0$ and $\alpha_0, k > 0$, such if $U \subset M$ is (δ, a)-collapsing with locally bounded curvature, for some $\delta < \delta_{CFGT}$, then there is an approximating metric g^ε on some open subset W with $U \subset W \subset B(U, \frac{a}{2})$, together with an a-standard N-structure on W, such that:*

1. *g^ε is $(\alpha_0\, l_a, k)$-round in the sense of (1.1.1)-(1.1.6) of [14];*

2. *the approximation satisfies*

$$e^{-\varepsilon} g^\varepsilon\, l_a^2 \;\leq\; g \leq\; e^\varepsilon g^\varepsilon\, l_a^2,$$
$$|\nabla^g - \nabla^{g^\varepsilon}| \;<\; \varepsilon\, l_a^{-1},$$
$$\text{and} \quad |\nabla^k \mathcal{R}m_{g^\varepsilon} - \nabla^k \mathcal{R}m_g| \;<\; \Psi(\varepsilon \,|\, k)\, l_a^{-2-k};$$

3. *g^ε is invariant under the local nilpotent actions of the N-structure;*

4. *$\forall x \in W$, its orbit, $\mathcal{N}(x)$ is compact with* $\operatorname{diam}_{g^\varepsilon} \mathcal{N}(x) \;\leq\; \varepsilon\, l_a(x)$*; and*

5. *$W = \cup_{x\in W} \mathcal{N}(x)$, i.e. W is* saturated.

We immediately have:

Corollary 4.2.3 (Vanishing Euler characteristics)**.** *If $U \subset B(p_0, R)$ is (δ, a)-collapsing with locally bounded curvature, then $\chi(W) = 0$.*

Proof. By the existence of an a-standard N-structure of positive rank over W, we have a topological fibration $\mathbb{S}^1 \hookrightarrow W \to B$ where B is the collection of all orbits of the $\mathbb{S}^1$ action, induced by the action associated to the N-structure. Thus $\chi(W) = \chi(\mathbb{S}^1)\chi(B) = 0$. □

The construction of the N-structure and approximating metric g^ε starts on geodesic balls of scale l_a. Once we do the rescaling $g \mapsto l_a(p)^{-2} g$, we can carry out the constructions of Section 2 and 5 of [14] to obtain local fibrations. In order to glue the local fibration and group actions, as done in Section 6 and 7 of [14], we need Lemma 4.2.1 which tells, essentially, that one can carry out the gluing procedure by adjusting within a single ball at a time. Finally, notice that once two balls intersect non-trivially, then (4.9) is in effect, and rescaling one ball to unit curvature bound will ensure the rescaled metric having curvature norm bounded by 2 on the union of both balls, and Proposition A2.2 of [14] works for the gluing.

4.3 Collapsing and equivariant good chopping

The equivariant good chopping theorem when collapsing with locally bounded curvature happens, as stated and used in [20], is a generalization of the original work of Cheeger-Gromov [19] in two directions: in one direction, the global curvature bound is relaxed to locally bounded curvature, as carried out by Cheeger-Tian in the proof of Theorem 3.1 of [20]; in the other, since the collapsing does not imply the existence of an isometry group action — the action being only by a sheaf of local isometries — more elaborations are needed to reduce the situation to the case considered in [19]. In this appendix, with respect to the proof given in [20], we provide additional details that were indicated but not written out explicitly.

Fix $a \in (0,1)$ throughout this appendix. For the sake of simplicity, we will assume the given metric to be locally regular under curvature scale, i.e.

(R) there exist $A_k > 0$ for $k = 0, 1, 2, \cdots$, such that

$$\sup_{B(p,l_a(p))} |\mathcal{R}m_g| \leq l_a(p)^{-2} \quad \Longrightarrow \quad \sup_{B(p,\frac{1}{2}l_a(p))} |\nabla^k \mathcal{R}m_g| \leq A_k\, l_a(p)^{-2-k}.$$

Theorem 4.3.1. *Let (M, g) be an n-dimensional Riemannian manifold satisfying property (R). There exist constants $\delta_{GC} > 0$ and $C_{GC}(n) > 0$ such that if $E \subset M$ is (δ, a)-collapsing with locally bounded curvature for some $\delta < \delta_{GC}$ and $a \in (0,1)$, then there is an open subset $U \subset B(E, \frac{a}{2})$ that contains E, saturated by some a-standard N-structure, and has a smooth boundary ∂U with*

$$|II_{\partial U}| \leq C_{GC} l_a^{-1}.$$

Fukaya's frame bundle argument [29] enables us to overcome this difficulty. Basically, we first lift to the frame bundle, where the collapsing can only produce mutually diffeomorphic nilpotent orbits with controlled second fundamental form. Then we apply the equivariant good chopping theorem of Cheeger-Gromov [19] to obtain a good neighborhood that is both invariant under the nilpotent structure and the $O(n)$-actions. Taking the quotient of this neighborhood by $O(n)$, we get the desired neighborhood on the original manifold, because the $O(n)$-action commutes with the local actions of the nilpotent structure.

We remark that the proof of this theorem utilizes Sections 3-7 of Cheeger-Fukaya-Gromov's structural theory about the geometry of collapsing with bounded curvature developed in [14], and its generalization to the case of collapsing with locally bounded curvature by Cheeger-Tian [20]: to begin with, we need the existence of a regular approximating metric on the frame bundle, invariant under the nilpotent action resulted from the collapsing. See for a detailed description.

Regularity of the frame bundle

Consider the frame bundle $FB(E, a)$, with each fiber diffeomorphic to $O(n)$ and $\pi : FB(E, a) \to B(E, a)$ the natural projection. We follow the conventions of Notation

1.3 in [29]. Let $\bar{g}$ denote the Riemannian metric on $FB(E,a)$, as defined in 1.3 of [29]. Moreover, for any object o associated to $B(E,a)$, we will let $\bar{o}$ denote the corresponding object associated to $FB(E,a)$.

For any $p \in E$, do the rescaling $\bar{g} \mapsto l_a(p)^{-2}\bar{g} =: \bar{g}_p$, then by (R) we can control, for $\bar{p} \in \pi^{-1}(p)$,

$$\sup_{FB(\bar{p},\frac{1}{2})} |\nabla^k \mathcal{R}m_{\bar{g}_p}|_{\bar{g}_p} \ \leq\ B_k'(n, A_{\leq k}, l_a(p)) \ \leq\ B_k(n, A_{\leq k}),$$

where we use $A_{\leq k}$ to denote $A_1, \cdots, A_k$. This because for $a < 1$ the rescaling will stretch the fiber metric on $O(n)$, making it less curved. This means, in the original metric,

(R1) $\quad \sup_{FB(p,\frac{1}{2}l_a(p))} |\nabla^k \mathcal{R}m_{\bar{g}}|_{\bar{g}} \ \leq\ B_k(n, A_{\leq k})\, l_a(p)^{-2-k}$ for $k = 0, 1, 2, 3, \cdots$.

Now we use Lemma 4.2.1 to construct a good covering of $B(E, \frac{a}{2})$, by $B_i := B(p_i, 2\zeta l_a(p_i))$ contained in $B(E,a)$. Clearly $FB(E, \frac{a}{2}) \subset \cup_i FB_i$.

Fibration and invariant metric of the frame bundle

We first assume $\delta < \delta_{CFGT}$. Arguing as before, we notice that if $B_i \cap B_j \neq \emptyset$, then (4.9) ensures that the curvature of the frame bundle also satisfies for $k = 0, 1, 2, \cdots$,

$$\sup_{FB_i \cup FB_j} |\nabla^k \mathcal{R}m_{\bar{g}}|_{\bar{g}} \ \leq\ \left(\frac{1-2\zeta}{1+2\zeta}\right)^{-2-k} B_k(n, A_{\leq k}) \max\{l_a(p_i), l_a(p_j)\}^{-2-k}.$$

Thus rescaling $\bar{g} \mapsto l_a(p_i)^{-2}\bar{g} =: \bar{g}_{ij}$ on $B_i \cup B_j$ will ensure for $k = 0, 1, 2, \cdots$,

$$\sup_{FB_i \cup FB_j} |\nabla^k \mathcal{R}m_{\bar{g}_{ij}}|_{\bar{g}_{ij}} \ \leq\ B_k(n, A_{\leq k}) \left(\frac{1-2\zeta}{1+2\zeta}\right)^{-2-k},$$

so that we can think as on $FB_i \cup FB_j$ there is a uniformly regular Riemannian metric $\bar{g}_{ij}$.

By Lemma 4.2.1, we notice that in each step of carrying out the procedure of Sections 3-7, especially applying Proposition A2.2 of [14], we only need to deal with the case of smoothing within a single FB_i. Thus the above regularity of the metric restricted to intersecting balls is sufficient, and we can construct the following data:

(F1) there is a global fibration $f : FB(U, 2a/3) \to Y$;

(F2) Y is a smooth Riemannian manifold of dimension $m' < \dim FE$;

There is a simply connected nilpotent Lie group $\bar{N}$ of dimension $n-m$, and a co-compact lattice Λ, such that:

(N1) $\bar{N}$ acts on $\cup_i FB_i$ so that each orbit $\mathcal{N}(\bar{x})$ at some $\bar{x} \in \cup_i FB_i$ is a compact submanifold, and up to a finite covering,

$$\bar{N}/\Lambda \approx \mathcal{N}(\bar{x}) = f^{-1}(f(\bar{x}));$$

(N2) the action of $\bar{N}$ commutes with the $O(n)$-action;

(N3) the action of $\bar{N}$ on $FB(E,a)$, after taking the $O(n)$ quotient, descends to the a-standard N-structure on $B(E, \frac{a}{2})$, as described in Theorem 4.2.2.

Moreover, for any positive ε which could be arbitrarily small, there is a smooth metric $\bar{g}^{\varepsilon}$ on $FB(E,a)$ and a constant $\alpha_0 = \alpha_0(n,a) > 0$ such that:

(G1) $\bar{g}^{\varepsilon}$ is a regular ε-approximation of $l_a(p_i)^{-2}\bar{g}|_{FB_i}$ for each i, see Theorem 4.2.2;

(G2) $\bar{g}^{\varepsilon}$ is invariant under both the actions of $\bar{N}$ and of $O(n)$;

(G3) $\forall \bar{x} \in FB_i$, $\operatorname{diam}_{\bar{g}^{\varepsilon}} \mathcal{N}(\bar{x}) < \varepsilon l_a(p_i)$ for each i.

(G4) $\forall \bar{x} \in FB_i$, the normal injectivity radius $\operatorname{inj}^{\perp}_{\bar{g}^{\varepsilon}} \bar{x} \geq \frac{2}{3}\alpha_0 l_a(p_i)$;

(G5) $\forall \bar{x} \in FB_i$, $|II_{\mathcal{N}(\bar{x})}| \leq C(B_{\leq 2}(n, A_{\leq 2}))\, l_a(p_i)^{-1}$.

Without loss of generality, we may assume that $B_k(n, A_{\leq k}) \geq 1$ and $\alpha_0 \leq 1$.

Here we make a simple convention: $\forall X \subset FB(E, a/2)$, let $\mathcal{N}(X)$ and $\mathcal{O}(X)$ denote the orbits of X under the action of $\bar{N}$ and $O(n)$, respectively. Since both actions are local isometries (G2), and they commute (N2), we have:

(G6) the operations $\mathcal{N}(-)$, $\mathcal{O}(-)$ and $B(-,r)$ (with respect to $\bar{g}^{\varepsilon}$) for $r \in (0, a/2)$ on subsets of $FB(E, a/2)$ commute.

We need to further notice that for $\varepsilon > 0$ arbitrarily small, we can choose δ small enough so that $B(E,a)$ being (δ,a)-collapsing with locally bounded curvature implies the existence of the approximating metric above, with the given ε. Notice that as long as $\delta < \delta_{CFGT}$, the existence of α_0 and the $\bar{N}$-structure is guaranteed. Here we fix $\varepsilon = 10^{-10}\alpha_0$, and let $\delta_{GC} < \delta_{CFGT}$ be one that works for the fixed ε. In practice, once there exists some $\delta' < \delta_{CFGT}$, then there exists a family of Riemannian metrics that are (δ,a)-collapsing with locally bounded curvature with $\delta \to 0$ (see [18] and [30]), so eventually $\delta < \delta_{GC}$.

Distance to orbits

Recall that we hope to smooth the boundary of E. This smoothing will be obtained by taking certain level set of a smoothing of the distance function to $\mathcal{N}(FE)$. Here for any $O(n)$-invariant $\bar{U} \subset \cup_i FB_i$, we define the "distance to orbits of $\bar{U}$" as following:

$$\bar{\rho}_{\bar{U}} : \cup_i FB_i \to [0, \infty) \quad \bar{x} \mapsto d_{\bar{g}^\varepsilon}\left(\bar{x}, \mathcal{N}(\bar{U})\right).$$

Notice that by (N2), $\mathcal{N}(\bar{U})$ is invariant under the $O(n)$-action:

$$\forall \gamma \in O(n), \quad \gamma\mathcal{N}(\bar{U}) = \mathcal{N}(\gamma\bar{U}) = \mathcal{N}(\bar{U}).$$

Then $\bar{\rho}_{\bar{U}}$ immediately satisfies the following properties:

(D1) $\bar{\rho}_{\bar{U}}$ is invariant under the actions of $\bar{N}$ and $O(n)$ by (G2);

(D2) $\forall \bar{x} \in \cup_i FB_i$, $\bar{\rho}_{\bar{U}}(\bar{x}) \leq d_{\bar{g}^\varepsilon}(\bar{x}, \bar{U})$, and thus

$$\bar{\rho}_{\bar{U}}^{-1}([0, a/4]) \subset B(\bar{U}, a/4).$$

The possible non-smoothness is caused by the behavior of $\partial\bar{U}$, since the distance to a single orbit is smooth within the normal injectivity radii, by (R1), (G1) and (G4): defining $d^{\bar{x}_0}(\bar{x}) := d_{\bar{g}^\varepsilon}(\bar{x}, \mathcal{N}(\bar{x}_0))$ for some fixed $\bar{x}_0 \in \cup_i FB_i$, then for $k = 0, 1, 2, \cdots$, we have

(D3) $\sup_{B\left(\mathcal{N}(\bar{x}_0), \frac{\alpha_0}{10} l_a(p_i)\right)} |\nabla^k d^{\bar{x}_0}| \leq C_k l_a(p_i)^{1-k}$;

(D4) $\bar{d}^{\bar{x}_0}$ is both $\bar{N}$-and $O(n)$-invariant;

(D5) $\bar{\rho}_{\bar{U}} = \inf_{\bar{x}_0 \in \bar{U}} \bar{d}^{\bar{x}_0}$.

Local parametrization of the frame bundle

For each $\bar{q} \in FB_i \cap FE$, we start with setting $\bar{H}^0 := B\left(\mathcal{N}(\bar{q}), \frac{\alpha_0}{2} l_a(p_i)\right)$ and $\bar{H} := O(\bar{H}^0)$. Notice that by (G4), the normal injectivity radius, constant on $\mathcal{N}(\bar{q})$, satisfies $\mathrm{inj}^\perp_{\bar{q}} \geq \frac{2\alpha_0}{3} l_a(p_i)$. We can deduce that $f(\bar{H}^0)$ is contractible and $\bar{H}^0$ deformation retracts to $\mathcal{N}(\bar{q})$, therefore we can find, possibly after lifting to a finite covering, a global orthonormal frame, consisting of left invariant vector fields $\xi_1, \cdots, \xi_{m'} \perp \mathcal{N}(\bar{q})$, so that $\forall \gamma \in \bar{N}$, the map

$$\exp^\perp_{\gamma\bar{q}} : B\left(\mathbf{0}, \frac{\alpha_0}{2} l_a(p_i)\right) \to \bar{H}^0, \quad \mathbf{v} = (v^1, \cdots, v^{m'})^T \mapsto \exp_{\gamma\bar{q}}\left(\sum_{s=1}^{m'} v^s \xi_s(\gamma\bar{q})\right)$$

is injective and diffeomorphic onto its image, where $B\left(\mathbf{0}, \frac{\alpha_0}{2} l_a(p_i)\right) \subset \mathbb{R}^{m'}$.

According to (G2) and the definition, we immediately notice that

(P1) $\forall\gamma \in \bar{N}$ and $\forall \bar{x} \in \bar{H}^0$, $Image(\exp^{\perp}_{\gamma\bar{q}}) \perp \mathcal{N}(\bar{x})$;

(P2) $\forall\gamma \in \bar{N}$, $(\exp^{\perp}_{\bar{q}})^*\bar{g}^{\varepsilon} \ = \ (\exp^{\perp}_{\gamma\bar{q}})^*\bar{g}^{\varepsilon}$ on $B\left(\mathbf{0}; \frac{\alpha_0}{2} l_a(p_i)\right)$;

(P3) $\forall\gamma \in \bar{N}$ and $\forall \bar{x} \in \bar{H}$, $\exists!\mathbf{v}_{\bar{x}} \in B\left(\mathbf{0}, \frac{\alpha_0}{2} l_a(p_i)\right)$ such that $\bar{\rho}_{\bar{U}}(\gamma\bar{x}) \ = \ \bar{\rho}_{\bar{U}}(\exp^{\perp}_{\bar{q}}(\mathbf{v}_{\bar{x}}))$.

We can consider the pull-back metric $h_i := (\exp^{\perp}_{\bar{q}})^*\bar{g}^{\varepsilon}$ on $B\left(\mathbf{0}, \frac{\alpha_0}{2} l_a(p_i)\right)$, as a positive definite 2-tensor field, so that:

(P4) according to (G1) and (G2), for any multi-index I with $|I| = k = 0, 1, 2, \cdots$,

$$\left|\frac{\partial^{|I|}}{\partial v^I} h_i\right| \ \leq \ C_k l_a(p_i)^{-k};$$

(P5) $B\left(\mathbf{0}, \frac{\alpha_0}{2} l_a(p_i)\right)$ is geodesically convex under the metric d_{h_i} defined by h_i;

(P6) $\forall \mathbf{v} \in B(\mathbf{0}, \frac{\alpha_0}{2} l_a(p_i))$, $d_{h_i}(\mathbf{v}, \mathbf{0}) \ = \ d^{\bar{q}}(\exp^{\perp}_{\bar{q}}(\mathbf{v}))$.

Local smoothing and chopping

In order to smooth $\bar{\rho}_{\bar{U}}$, we mollify it by a smooth cut-off function within the normal injectivity radius of $\bar{q}$, following [16]. Let $0 \leq \varphi_i \leq 1$ be a smooth function such that for some $\zeta' > 0$ to be determined later,

(S1) φ_i is supported on $[0, \frac{\zeta'\alpha_0}{100} l_a(p_i))$ and $\varphi_i(t) \equiv 1$ for $t \in [0, \frac{\zeta'\alpha_0}{200} l_a(p_i)]$;

(S2) $\varphi_i^{(k)}(t) \leq C_k(\zeta') l_a(p_i)^{-k}$ for $k = 0, 1, 2, \cdots$ and $t \in [0, \frac{\zeta'\alpha_0}{100} l_a(p_i))$.

Now we focus on an $O(n)$-invariant $\bar{U} \subset \bar{H}^0$, and define on $\bar{H}^0$:

$$\bar{\rho}^{\sharp}_{\bar{U}}(\bar{x}) := \frac{1}{\mu_i(\bar{x})} \int_{B\left(\mathcal{N}(\bar{p}), \frac{\alpha_0}{2} l_a(p_i)\right)} \bar{\rho}_{\bar{U}}(\bar{z})\varphi_i(d^{\bar{x}}(\bar{z}))\, \mathrm{d}V_{\bar{g}^{\varepsilon}}(\bar{z}),$$

where

$$\mu_i(\bar{x}) := \int_{B\left(\mathcal{N}(\bar{q}), \frac{\alpha_0}{2} l_a(p_i)\right)} \varphi_i(d^{\bar{x}}(\bar{z}))\, \mathrm{d}V_{\bar{g}^{\varepsilon}}(\bar{z}).$$

Notice that in the definition of $\bar{\rho}^{\sharp}_{\bar{U}}$ we have taken average by dividing $\mu_i(\bar{x})$, thus the numerical value of $\bar{\rho}^{\sharp}_{\bar{U}}$ is not affected if we lift the original neighborhood to a finite covering.

By the invariance of $\bar{\rho}_{\bar{U}}$ and that $\exp^{\perp}_{\bar{x}}$ being a diffeomorphism onto its image, we can reduce $\bar{\rho}^{\sharp}_{\bar{U}}$ to a function $\tilde{\rho}^{\sharp}_{\bar{U}}$ on $B\left(\mathbf{0}, \frac{\alpha_0}{2} l_a(p_i)\right)$:

$$\tilde{\rho}^{\sharp}_{\bar{U}}(\mathbf{v}) \ := \ \bar{\rho}^{\sharp}_{\bar{U}}(\exp^{\perp}_{\bar{q}}(\mathbf{v})).$$

The most important property of $\tilde{\rho}^{\sharp}_{\bar{U}}$ is:

(S3) $\quad |\nabla^k_\perp \bar{\rho}^\sharp_{\bar{U}}|(\bar{x}) \;=\; |\nabla^k \tilde{\rho}^\sharp_{\bar{U}}|\Big((\exp^\perp_{\bar{q}})^{-1}(\bar{x})\Big) \quad$ for $k = 0, 1, 2, 3, \cdots$.

Then by Fubini's theorem and the invariance of $\bar{\rho}_{\bar{U}}$ under the $\bar{N}$-action,

$$\tilde{\rho}^\sharp_{\bar{U}}(\mathbf{v}) = \frac{1}{\mu_i(\mathbf{v})}\int_{B\left(\mathbf{0},\frac{\alpha_0}{2}l_a(p_i)\right)} \bar{\rho}_{\bar{U}}(\exp^\perp_{\bar{q}}(\mathbf{w}))\varphi_i\left(d_{h_i}(\mathbf{v},\mathbf{w})\right)\,\psi(\mathbf{w})\,\mathrm{d}V_{h_i}(\mathbf{w}),$$

where

$$\psi(\mathbf{w}) := Vol_{\bar{g}^\varepsilon}(\mathcal{N}(\exp_{\bar{q})^\perp(\mathbf{w})}) \quad \text{and} \quad \mu_i(\mathbf{v}) := \int_{B\left(\mathbf{0},\frac{\alpha_0}{2}l_a(p_i)\right)} \varphi_i\left(d_{h_i}(\mathbf{v},\mathbf{w})\right)\,\psi(\mathbf{w})\,\mathrm{d}V_{h_i}(\mathbf{w}).$$

The smoothness of $\tilde{\rho}^\sharp_{\bar{U}}(\mathbf{v})$ then follows from differentiating $\varphi_i\left(d_{h_i}(\mathbf{v},\mathbf{w})\right)$ with respect to $\mathbf{v} \in B\Big(\mathbf{0}, \frac{\alpha_0}{2}l_a(p_i)\Big)$, and the derivative bounds are guaranteed by (S2) and (P4):

(S4) $\quad \sup_{B\left(\mathbf{0},\frac{\alpha_0}{2}l_a(p_i)\right)} |\nabla^k \tilde{\rho}^\sharp_{\bar{U}}| \;\leq\; C_k l_a(p_i)^{1-k} \quad$ for $k = 1, 2, 3, \cdots$.

Now we can apply Yomdin's quantitative Morse Lemma (see [3] and [44] for proofs) to the function $\tilde{\rho}^\sharp_{\bar{U}}$, to find, for small $\eta > 0$, some interval $J_{\bar{U}} \subset [0, \frac{\alpha_0}{4}l_a(p_i)]$ of length $|J_{\bar{U}}| \approx \Psi_{YM}(a, n, \eta)l_a(p_i) > 0$ such that

$$\forall t \in J_{\bar{U}}, \quad |\nabla\tilde{\rho}^\sharp_{\bar{U}}| \;>\; \eta \quad \text{on} \quad (\tilde{\rho}^\sharp_{\bar{U}})^{-1}(t).$$

Here we may assume the definite constant $\Psi_{YM}(a, n, \eta) < 10^{-2}$.

By the definition of $\tilde{\rho}^\sharp_{\bar{U}}$ and (S3), we then have

(Y1) $\quad \forall t \in J_{\bar{U}}, \quad |\nabla\bar{\rho}^\sharp_{\bar{U}}| \;\geq\; |\nabla_\perp \bar{\rho}^\sharp_{\bar{U}}| \;>\; \eta \quad$ on $\quad (\bar{\rho}^\sharp_{\bar{U}})^{-1}(t)$.

Let $\bar{W}^0_{\bar{U}} := (\bar{\rho}^\sharp_{\bar{U}})^{-1}([0, t])$ for some $t \in J_{\bar{U}}$. We then have

(C1) $\bar{W}^0_{\bar{U}}$ is invariant under the actions of $\bar{N}$, by (A3.1) above, and $O(n)$ acts as local isometry on $\bar{W}^0_{\bar{U}}$;

(C2) $\bar{U} \cap \bar{H}^0 \subset \bar{W}^0_{\bar{U}} \subset B(\bar{U}, \frac{\alpha_0}{2}l_a(p_i))$;

Define $\Sigma_{\bar{U}} := \exp^\perp_{\bar{q}}\left((\tilde{\rho}^\sharp_{\bar{U}})^{-1}([0, t_i])\right)$, then $\bar{W}^0_{\bar{U}} = \mathcal{N}(\Sigma_{\bar{U}})$. We immediately have the bound

(C3) $\quad |II_{\partial\Sigma_{\bar{U}}}| \;\leq\; \frac{|\nabla^2\tilde{\rho}^\sharp_{\bar{U}}|}{|\nabla\tilde{\rho}^\sharp_{\bar{U}}|} \;\leq\; C(\eta)l_a(p_i)^{-1}$.

This, together with $|II_{\mathcal{N}(\bar{x})}| \leq Cl_a(p_i)^{-1}$, and the fact that $\forall\gamma \in \bar{N}$, $\Sigma_{\bar{U}} \mapsto \gamma\Sigma_{\bar{U}}$ is an isometry, give

(C4) $|II_{\partial \bar{W}^0_{\bar{U}}}| \le C(\eta) l_a(p_i)^{-1}$.

We notice that the function $\bar{\rho}^{\sharp}_{\bar{U}}$ is locally $O(n)$-invariant, therefore it extends smoothly from $\bar{H}^0$ to $\gamma \bar{H}^0$ for any $\gamma \in O(n)$. Thus we see that $\bar{W}_{\bar{U}} := O(\bar{W}^0)$ has a smooth boundary on $\bar{H} = O(\bar{H}^0)$, whence an $O(n)$-invariant neighborhood of $\bar{U} \subset \bar{H}$. Moreover, since each $\gamma \in O(n)$ acts as an isometry, we have

(C5) $|II_{\partial \bar{W}_{\bar{U}}}| = |II_{\partial \bar{W}^0_{\bar{U}}}| \le C(\eta) l_a(p_i)^{-1}$.

A refined good covering of the frame bundle

We start with fixing $\zeta = 10^{-2}$ (see Lemma 4.2.1) and

$$\zeta' := \min\{0.1, \zeta/\alpha_0\}.$$

Choose a maximal set of points $\{q_j\} \subset E$, such that

$$d_g(q_j, q_{j'}) \ \ge \ \zeta' \alpha_0 \min\{l_a(p_{i_j}), l_a(p_{i_{j'}})\},$$

and obtain a covering of E by $\left\{B\left(q_j, 2\zeta'\alpha_0 l_a(p_{i_j})\right)\right\}$ (with respect to the original metric g on M). Here for each q_j, p_{i_j} is chosen as any B_i containing q_j. Then by Lemma 4.2.1, we can find a finite number of sub-collections S'_j $(j = 1, \cdots, N)$, such that $E_{j,k} := B(q_{j,k}, 2\zeta'\alpha_0 l_a(p_{i_{j,k}}))$ is disjoint from any $E_{j,k'}$, and intersects with at most one $E_{j',k''}$ for $j' \neq j$.

Now the sets $\bar{E}_{j,k} = \pi^{-1}(E_{j,k})$ cover FE, and each $\bar{E}_{j,k}$ is obviously $O(n)$ invariant. Fix $\bar{q}_{j,k} \in \pi^{-1}(q_{j,k})$ for each (j,k). Since by (G1), $\bar{g}^{\varepsilon}$ is a regular ε approximation of the original metric on $FB(E,a)$, with $\varepsilon < 10^{-5}\zeta'\alpha_0$ as defined, we can *redefine*

$$\bar{E}_{j,k} := O\left(B\left(\bar{q}_{j,k}, 2\zeta'\alpha_0 l_a(p_{i_{j,k}})\right)\right) \subset FB(E,a),$$

so that the covering property and the partition into finitely many sub-collections are still satisfied.

We further define $\bar{D}^0_{j,k} := \mathcal{N}(\bar{E}_{j,k})$, then by (G6) and (G3), we have

$$\bar{D}^0_{j,k} \ = \ O\left(B\left(\mathcal{N}(\bar{q}_{j,k}), 2\zeta'\alpha_0 l_a(p_{i_{j,k}})\right)\right) \ \subset \ B\left(\bar{E}_{j,k}, \frac{(1+\zeta)\,\alpha_0}{10^{10}(1-\zeta)} l_a(p_{i_{j,k}})\right),$$

therefore $\{\bar{D}^0_{j,k}\}$ still forms a covering, and could be divided into finitely many sub-collections $S'_1, \cdots, S'_N$ as obtained above.

The point of constructing this new covering is that the original covering is with respect to the original metric g, and we need to refinish it so that each open set of the new covering is saturated by the nilpotent and orthogonal group actions, yet the whole collection of open sets could still be divided into N disjoint sub-collections, a necessity for our future step-by-step gluing to obtain the global chopping.

Now we define $\bar{D}^m_{j,k} := B\left(\mathcal{N}(\bar{q}_{j,k}), r^m_{j,k}\right)$, with $r^m_{j,k} := (2+\frac{1}{6}m)\zeta'\alpha_0 l_a(p_{i_{j,k}})$, for each $m \in \{0,1,2,3,4,5,6\}$. We need this fattening of open sets in the covering since later we will need to "glue" the local smoothings, see the forthcoming claim.

Global chopping

We now do the final step, the global chopping. The method we follow is briefly given in [19], where the curvature is assumed to be uniformly bounded, here we take the (changing) truncated curvature scale into consideration.

For the collections $S'_1, \cdots, S'_N$, we first do the above local chopping for each $FE \cap \bar{D}_{j,k}$ to obtain $\bar{W}_{j,k}$ with $t_{1,k} \approx 2^{-1}\Psi_{YM}(a,n,\eta) l_a(p_{i_{1,k}})$, and define $\bar{U}_1 := \cup_k \bar{W}_{1,k}$ as an open subset of $\cup_i FB_i$.

For the second step, we modify members of S'_2. Notice that if some $\bar{D}_{2,k}$ intersects some $\bar{W}_{1,k'}$ non-trivially, then we have the estimates of the truncated curvature scales as before

(H1) $\quad \min\{l_a(p_{i_{1,k'}}), l_a(p_{i_{2,k}})\} \leq \max\{l_a(p_{i_{1,k'}}), l_a(p_{i_{2,k}})\} \leq \frac{1+\zeta'}{1-\zeta'} \min\{l_a(p_{i_{1,k'}}), l_a(p_{i_{2,k}})\}.$

Renormalizing $g \mapsto l_a(p_{i_{2,k}})^{-2} g =: g_{2,k}$ will ensure that

$$\sup_{\bar{W}_{1,k'} \cup \bar{D}^5_{2,k}} |\mathcal{R}m_{\bar{g}_{2,k}}|_{\bar{g}_{2,k}} \leq C\frac{1+\zeta'}{1-\zeta'},$$

with corresponding bounds on $|\nabla^k \mathcal{R}m_{\bar{g}_{2,k}}|_{\bar{g}_{2,k}}$.

Now we can chop locally within $\bar{D}^6_{2,k}$ (see [19]): first chop $\bar{Z}_{2,k} := (\bar{W}_{1,k'} \cup \bar{D}^0_{2,k}) \cap \bar{D}^3_{2,k}$ to obtain some $\bar{Q}^0_{2,k}$, then choose a smooth interpolation to glue the newly chopped piece to the previously chopped ones. More specifically, we have the following

Claim 4.3.2 (Gluing the local choppings)**.** *There is a smooth interpolation between the boundaries $\partial\bar{W}_{1,k'} \cap (\bar{D}^4_{2,k} \backslash \bar{D}^3_{2,k})$ and $\partial\bar{Q}^0_{2,k} \cap (\bar{D}^2_{2,k} \backslash \bar{D}^1_{2,k})$, so that we could obtain some $\bar{R}^0_{2,k} \subset \bar{D}^4_{2,k}$, with the property*

$$\bar{R}^0_{2,k} = \bar{W}^0_{1,k'} \quad on \quad \bar{D}^4_{2,k}\backslash\bar{D}^3_{2,k}, \quad and \quad \bar{R}^0_{2,k} = \bar{Q}^0_{2,k} \quad on \quad \bar{D}^2_{2,k}.$$

Proof of claim. By the proof of the local smoothing, within $\bar{D}^6_{2,k}$, we have $\bar{\rho}^\sharp_{\bar{Z}_{2,k}}$ as the smoothed distance to $\mathcal{N}(\bar{Z}_{2,k})$, and $\bar{Q}^0_{2,k} = (\bar{\rho}^\sharp_{\bar{Z}_{2,k}})^{-1}([0, t_{2,k}])$ for some $t_{2,k} \in I_{2,k}$ with

$t_{2,k} \approx 2^{-2}\Psi_{YM}(a,n,\eta)\ l_a(p_{i_{2,k}})$. In addition, we could set $\bar{Z}'_{2,k} := \bar{W}_{1,k'} \cap (\bar{D}^5_{2,k} \backslash \bar{D}^0_{2,k})$, with the smoothing of the distance to the orbit of which being $\bar{\rho}^\sharp_{\bar{Z}'_{2,k}}$. Notice that

$$\bar{\rho}_{\bar{Z}_{2,k}} \equiv \bar{\rho}_{\bar{Z}'_{2,k}} \quad \text{on} \quad \bar{D}^3_{2,k} \backslash \bar{D}^0_{2,k},$$

therefore

$$\bar{\rho}^\sharp_{\bar{Z}_{2,k}} \equiv \bar{\rho}^\sharp_{\bar{Z}'_{2,k}} \quad \text{on} \quad \bar{D}^2_{2,k} \backslash \bar{D}^1_{2,k},$$

and thus

$$\bar{Q}^0_{2,k} \cap (\bar{D}^2_{2,k} \backslash \bar{D}^1_{2,k}) = (\bar{\rho}^\sharp_{\bar{Z}_{2,k}})^{-1}([0, t_{2,k}]) = (\bar{\rho}^\sharp_{\bar{Z}'_{2,k}})^{-1}([0, t_{2,k}]).$$

On the other hand,

$$\bar{W}_{1,k'} \cap (\bar{D}^4_{2,k} \backslash \bar{D}^1_{2,k}) = (\bar{\rho}^\sharp_{\bar{Z}'_{2,k}})^{-1}(0),$$

and now the existence of a controlled interpolation required above is easily seen: choose a smooth cut-off function $\lambda_{2,k} : [r^1_{2,k}, r^4_{2,k}] \to [0, t_{2,k}]$ with controlled derivatives, such that $\lambda_{2,k}|_{[r^1_{2,k}, r^2_{2,k}]} = t_{2,k}$ and $\lambda_{2,k}|_{[r^3_{2,k}, r^4_{2,k}]} = 0$, and the desired region $\bar{R}^0_{2,k}$ is defined as

$$\bar{R}^0_{2,k} := \left(\bar{D}^2_{2,k} \cap \bar{Q}^0_{2,k}\right) \cup \left(\bar{\rho}^\sharp_{\bar{Z}'_{2,k}} \left(\lambda_{2,k}(d^{\bar{q}_{2,k}})\right)\right)^{-1} ([r^1_{2,k}, r^4_{2,k}]).$$

□

Clearly $\bar{R}^0_{2,k}$ is $\bar{N}$-invariant and has the expected smooth boundary whose second fundamental form has control $|II_{\partial \bar{R}^0_{2,k}}| \leq C\ l_a(p_{i_{2,k}})^{-1}$.

Let $\bar{R}_{2,k} := O(\bar{R}^0_{2,k})$, then the isometric action of $O(n)$ and the invariance of $\bar{\rho}^\sharp_{\bar{Z}_{2,k}}$, $\bar{\rho}^\sharp_{\bar{Z}'_{2,k}}$ under such actions ensure that $\partial \bar{R}_{2,k}$ is smooth with controlled second fundamental form $|II_{\partial \bar{R}_{2,k}}| \leq C\ l_a(p_{i_{2,k}})^{-1}$. Do such adjustments for each $\bar{D}^0_{2,k} \in S_2$ and let $\bar{U}_2 := \bar{U}_1 \cup (\cup_k \bar{R}_{2,k})$, we have finished the second step.

Iterate the above procedure for N steps. At the j-th step ($j \geq 2$), we modify members of $\bar{S}_j$ with $t_{j,k} \approx 2^{-j}\Psi_{YM}(a,n,\eta)\ l_a(p_{i_{j,k}})$ for each k. By the Harnack inequality of (H1) for intersecting balls, we could produce a neighborhood $\bar{U}_j$ of FE, which is contained in $FB(E, \frac{a}{2})$, invariant under both $\bar{N}$- and $O(n)$-actions, and has a smooth, controlled boundary

$$|II_{\partial \bar{U}_j}| \leq C\ l_a^{-1}.$$

By (N3) and the invariance of $\bar{U}_N$ under the $O(n)$-action, define $U := \bar{U}_N / O(n)$, then $E \subset U \subset B(E, \frac{a}{2})$, and U is saturated by the a-standard N-structure on $B(E, \frac{a}{2})$, with a smooth and controlled boundary

$$|II_{\partial U}| \leq C_{GC}\ l_a^{-1}.$$

4.4 Proof of the ε-regularity theorem for 4-D Ricci shrinkers

The foundation of the proof is Anderson's ε-regularity with respect to collapsing, which basically asserts that the smallness of the renormalized energy $I^f_{\mathcal{R}m}$ (see Definition 2.4.7) at certain scale guarantees the uniform curvature bound at half of that scale. However, the (more natural) input of our theorem is the smallness of the local energy

$$E(p,r) := \int_{B(p,r)} |\mathcal{R}m|^2 \, \mathrm{d}\mu_f \; < \; \varepsilon,$$

which, when collapsing happens, may well be caused by the smallness of $\mu_f(B(p,r))$, and it is not obvious at all that small local energy implies the smallness of the renormalized energy. However, we will follow the strategy of Cheeger-Tian [20] to find that for 4-D Ricci shrinkers, the above smallness of energy indeed implies the smallness of the renormalized energy, at a much smaller, but definite scale.

4.4.1 The key estimate for 4-d Ricci shrinkers

Combining the above propositions, Cheeger-Tian [20] obtain the following estimates of the boundary Gauss-Bonnet-Chern term:

Proposition 4.4.1. *Let (M,g,f) be a 4-D Ricci shrinker and fix $a \in (0,1)$. There exist positive constants $\delta_{CT}(R) \le \delta_{GC}(R)$ and $C_{CT}(R) > 0$ such that for any $K \subset B(p_0, R-a)$ with $B(K,a)$ being (δ,a)-collapsing with locally bounded curvature for some $\delta < \delta_{CT}(R)$, then there exists an open subset Z, saturated with respect to the associated N-structure of an approximating metric, such that*

1. $B(K, \frac{1}{4}a) \; \subset \; Z \; \subset \; B(K, \frac{3}{4}a)$,
2. $|II_{\partial Z}| \; \le \; C_{CT}(R)(a^{-1} + r^{-1}_{\mathcal{R}m})$, *and*
3. $\left|\int_{\partial Z} \mathcal{T}\mathcal{P}_\chi\right| \; \le \; C_{CT}(R)\, a^{-1} \int_{A(K,\frac{1}{4}a,\frac{3}{4}a)} \left(a^{-3} + r^{-3}_{\mathcal{R}m}\right) dV_g$.

The proof of this proposition only used, in addition to the previous propositions, the volume comparison, and this is available within $B(p_0,R)$ by Lemma 2.3.2.

Recall that the curvature can be controlled by

$$|\mathcal{R}m|^2 \; \le \; 8\pi^2 |\mathcal{P}_\chi| + |\mathring{\nabla}^2 f|^2.$$

The main task is to obtain an average control of $|\mathcal{P}_\chi|$. This is done by an induction process, which is based on Proposition 4.4.1 and the vanishing of the Euler characteristics on subsets that are (δ,a)-collapsing with locally bounded curvature. In order to better extract information from Proposition 4.4.1, we start with a maximal function argument.

For each $u \in L^1(M, g, \mathrm{d}\mu_f)$, we can define

$$M_u^f(x, s) := \sup_{s' \leq s} \frac{1}{\mu_f(B(x, s'))} \int_{B(x,s')} u \, \mathrm{d}\mu_f.$$

Recall the volume doubling property (2.24) and applying Lemma 4.1 of [20], we get

Lemma 4.4.2. *There is a constant $C_{4.1}(R, \alpha) > 0$, for each $R, \alpha > 0$, such that for any $d\mu_f$-measurable subset $W \subset B(p_0, R)$,*

$$\left(\frac{1}{\omega} \int_W M_u^f(x, s)^\alpha \, d\mu_f\right)^{\frac{1}{\alpha}} \leq \frac{C_{4.1}(R, \alpha)}{\mu_f(W)} \int_{B(W,6s)} |u| \, d\mu_f. \tag{4.12}$$

From Proposition 4.4.1, we can estimate:

Lemma 4.4.3. *Fix $r \in (0, 1)$ and $\delta < \min\{\delta_{CFGT}, \delta_{GC}\}$. There exists a $C_{4.2}(R) > 0$, such that if some compact set $K \subset B(p_0, R - r)$ has its r-neighborhood $B(K, r)$ being (δ, r)-collapsing with locally bounded curvature, then we have some saturated open set $Z \subset B(K, \frac{1}{2}r)$ with smooth boundary, containing $B(K, \frac{1}{4}r)$ such that*

$$\left|\int_Z \mathcal{P}_\chi\right| \leq C_{4.2}(R)\mu_f(A(K; 0, r))r^{-1}\left(r^{-3} + \left(\frac{1}{\mu_f(A(K; 0, r))} \int_{A(K; \frac{1}{4}r, \frac{3}{4}r)} |\mathcal{R}m|^2 \, d\mu_f\right)^{\frac{3}{4}}\right). \tag{4.13}$$

Proof. By the measure equivalence (2.21) and Proposition 4.4.1, we get

$$\begin{aligned} \left|\int_{\partial Z} \mathcal{T}\mathcal{P}_\chi\right| &\leq C_{CT}(R)r^{-1} \int_{A(K; \frac{1}{3}r, \frac{2}{3}r)} (s^{-3} + r_{\mathcal{R}m}^{-3}) \, \mathrm{d}V_g & (4.14) \\ &\leq C_{CT}(R)e^{(2R+\sqrt{2})^2} r^{-1} \int_{A(K; \frac{1}{3}r, \frac{2}{3}r)} (s^{-3} + r_{\mathcal{R}m}^{-3}) \, \mathrm{d}\mu_f. & (4.15) \end{aligned}$$

Now we notice that for $s \in (0, 1]$,

$$\rho_f(p)^{-1} \leq c_{4.2.2}(R) \max\left\{M_{|\mathcal{R}m|^2}^f(p, s)^{\frac{1}{4}}, s^{-1}\right\}.$$

This is because if $\rho_f(p) < s \leq 1$, then

$$\frac{\bar{\mu}_R(\rho_f(p))}{\mu_f(B(p, \rho_f(p)))} \int_{B(p,\rho_f(p))} |\mathcal{R}m|^2 \, \mathrm{d}\mu_f = \varepsilon_A(R),$$

which gives

$$M^f_{|\mathcal{R}m|^2}(p,s) \geq \frac{1}{\mu_f(B(p,\rho_f(p)))} \int_{B(p,\rho_f(p))} |\mathcal{R}m|^2 \, \mathrm{d}\mu_f \tag{4.16}$$

$$= \frac{\varepsilon_A(R)}{\bar{\mu}_R(\rho_f(p))} \geq \frac{e^{-(2R+\sqrt{2})}\varepsilon_A(R)}{\bar{\mu}_R(1)} \rho_f(p)^{-4}, \tag{4.17}$$

and thus

$$\rho_f(p)^{-1} \leq c_{4.2.2}(R) M^f_{|\mathcal{R}m|^2}(p,s)^{\frac{1}{4}},$$

where $c_{4.2.2}(R) := (e^{-(2R+\sqrt{2})}\varepsilon_A(R)/\bar{\mu}_f(1))^{-\frac{1}{4}}$.

Now for $s \leq r \leq 1$ we have

$$r_{\mathcal{R}m}(p)^{-3} \leq 8\rho_f(p)^{-3} \leq c_{4.2.3}(R)\left(s^{-3} + \left(M^f_{|\mathcal{R}m|^2}(p,s)\right)^{\frac{3}{4}}\right), \tag{4.18}$$

with $c_{4.2.3}(R) := 8\max\{1, c_{4.2.2}(R)^3\}$.

Now we can choose $s = \frac{r}{512}$ and apply Lemma 4.4.2 to the function $|\mathcal{R}m|^2$ with $\alpha = \frac{3}{4}$ to obtain

$$\int_{A(K;\frac{1}{3}r,\frac{2}{3}r)} \left(M^f_{|\mathcal{R}m|^2}(\cdot,s)\right)^{\frac{3}{4}} \, \mathrm{d}\mu_f \ \leq \ \mu_f(A(K;0,r))\left(\frac{C_{4.1}(R,\frac{3}{4})}{\mu_f(A(K;0,r))} \int_{A(K;\frac{1}{4}r,\frac{3}{4}r)} |\mathcal{R}m|^2 \, \mathrm{d}\mu_f\right)^{\frac{3}{4}}. \tag{4.19}$$

Then the estimates (4.15), (4.18) and (4.19) together give

$$\left|\int_{\partial Z} \mathcal{T}\mathcal{P}_\chi\right| \ \leq \ C_{4.2}(R)\mu_f(A(K;0,r))\left(r^{-4} + \left(\frac{r^{-\frac{4}{3}}}{\mu_f(A(K;0,r))} \int_{A(K;\frac{1}{4}r,\frac{3}{4}r)} |\mathcal{R}m|^2 \, \mathrm{d}\mu_f\right)^{\frac{3}{4}}\right). \tag{4.20}$$

Since there exists an r-standard N-structure on Z, $\chi(Z) = 0$, and we can employ the Gauss-Bonnet-Chern formula on Z to finish the proof, i.e. $\int_Z \mathcal{P}_\chi = -\int_{\partial Z} \mathcal{T}\mathcal{P}_\chi$. □

Recall that our purpose is to use (4.13) together with the special relation (2.16) between $\mathcal{P}_\chi$ and $|\mathcal{R}m|^2$ in dimension four to estimate $\|\mathcal{R}m\|_{L^2_{loc}}$. In the Einstein case $\mathring{\mathcal{R}c} \equiv 0$ but for non-trivial 4-D Ricci shrinkers, $\mathring{\mathcal{R}c} = \mathring{\nabla}^2 f$ does not vanish identically. However, we could employ the good cut-off function constructed in Lemma 2.3.3 to obtain a local L^2-control of the full Hessian of f by its energy. This is the content of the following lemma:

Lemma 4.4.4. *Given $K \subset B(p_0, R-r)$, we have estimate (4.21) for the potential function f.*

Proof. By Lemma 2.3.3, we have a cut-off function φ such that $0 \leq \varphi \leq 1$, $\operatorname{supp}\varphi \subset B(K,r)$, $\varphi \equiv 1$ on $B(K, r/4)$ and $r|\nabla\varphi| + r^2|\Delta^f\varphi| \leq C_{2.10}(R)$, then we can use the Weitzenböck formula (2.11) to compute

$$\begin{aligned}\int_{B(K,\frac{1}{2}r)} 2|\nabla^2 f|^2 \, \mathrm{d}\mu_f \;&\leq\; \int_{B(K,r)} 2\varphi|\nabla^2 f|^2 \, \mathrm{d}\mu_f \\ &=\; \int_{B(K,r)} \varphi\left(\Delta^f|\nabla f|^2 + |\nabla f|^2\right) \, \mathrm{d}\mu_f \\ &\leq\; \int_{A(K;0,r)} |\Delta^f\varphi||\nabla f|^2 \, \mathrm{d}\mu_f + \int_{B(K,r)} |\nabla f|^2 \, \mathrm{d}\mu_f \\ &\leq\; c(R)\left(2R+\sqrt{2}\right)^2 r^{-2}\mu_f(A(K;0,r)) + (2R+\sqrt{2})^2\mu_f(B(K,r)),\end{aligned}$$

and thus

$$\int_{B(K;\frac{1}{2}r)} 2|\nabla^2 f|^2 \, \mathrm{d}V_g \;\leq\; C_{4.3}(R)\left(2R+\sqrt{2}\right)^2 e^{(2R+\sqrt{2})^2}\left(r^{-2}\mu_f(A(K;0,r)) + \mu_f(B(K;r))\right). \tag{4.21}$$

□

From now on, we fix $\delta_{KE} := \frac{1}{2}\{\delta_{CFGT}, \delta_{GC}\}$. Now we can generalize the following key estimate of [20] to 4-D Ricci shrinkers:

Proposition 4.4.5 (Key estimate)**.** *Fix $r \in (0,1)$ and $R > 0$. There exist constants $\varepsilon_{KE}(R) > 0$ $C_{KE}(R) > 0$, such that any $B(E,r) \subset B(p_0,R)$ which is δ-volume collapsing for any $\delta < \delta_{KE}$ sufficiently small, and with*

$$\int_{B(E,r)} |\mathcal{R}m|^2 \, d\mu_f \;\leq\; \varepsilon_{KE}(R), \tag{4.22}$$

has the estimate

$$\int_E |\mathcal{R}m|^2 \, d\mu_f \;\leq\; C_{KE}(R)\mu_f(B(E;r))\, r^{-4}.$$

Proof. The estimates (4.20) and (4.21) (with (2.16)) show that $\forall K \subset B(p_0, R-s)$ that is (δ, s)-collapsing with locally bounded curvature (assume $s \in (0,1)$),

$$\begin{aligned}\int_{B(K,\frac{1}{4}s)} |\mathcal{R}m|^2 \, \mathrm{d}\mu_f \;\leq C_{4.2}(R)\mu_f(A(K;0,s))&\left(s^{-4} + \left(\frac{s^{-\frac{4}{3}}}{\mu_f(A(K;0,s))}\int_{A(K;\frac{1}{4}s,\frac{3}{4}s)} |\mathcal{R}m|^2 \, \mathrm{d}\mu_f\right)^{\frac{3}{4}}\right) \\ &+ C_{4.3}(R)\mu_f(B(K,s)).\end{aligned} \tag{4.23}$$

Here the point is that even in practice we have $\delta \to 0$, but the threshold, δ_{KE}, for the theory developed in Section 3 to be applied to obtain (4.20), is universal.

Define $E_1 := B(E, r)$; for $i = 2, 3, 4, \cdots$, set $E_i := A(E; 2^{-i}r, r - 2^{-i}r)$,

$$D_i := \{x \in E_i : r_{\mathcal{R}m}(x) \leq 2^{-(i+1)}r\}, \quad \text{and} \quad F_i := E_i \backslash D_i.$$

Clearly $B(D_i, 2^{-(i+1)}r) \subset E_{i+1}$ and in fact we have:

Claim 4.4.6. *$B(D_i, 2^{-(i+1)}r)$ is $(\delta_{KE}, 2^{-(i+1)}r)$-collapsing with locally bounded curvature.*

Proof of claim. If $x \in B(D_i, 2^{-(i+1)}r)$ has $r_{\mathcal{R}m}(x) < 2^{-(i+1)}r$ then this follows from Lemma 4.1.11, if we assume $\varepsilon_{KE}(R) \leq \frac{\varepsilon_A(R)\delta_{KE}}{16\bar{\mu}_R(1)}$.

Otherwise, if $x \in B(D_i, 2^{-(i+1)}r)$ has $r_{\mathcal{R}m}(x) \geq 2^{-(i+1)}r$, then since $Lip\ r_{\mathcal{R}m} \leq 1$ and $\sup_{B(D_i, 2^{-(i+1)}r)} r_{\mathcal{R}m} \leq 2^{-i}r$, we have $\rho_f(x) \leq 2^{-(i-1)}r$, and

$$\begin{aligned}
\mu_f(B(x, 2^{-(i+1)}r)) \;&\leq\; \mu_f(B(x, 2^{-(i-1)}r)) \;\leq\; \frac{\mu_f(B(x, \rho_f(x)))\bar{\mu}_R(1)}{\bar{\mu}_R(\rho_f(x))2^{4(i-1)}r^{-4}} \\
&= \frac{\bar{\mu}_R(1)\ r^4}{\varepsilon_A(R)2^{4(i-1)}} \int_{B(x,\rho_f(x))} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f \;\leq\; \frac{\bar{\mu}_R(1)\ r^4}{\varepsilon_A(R)2^{4(i-1)}} \int_{B(x,r)} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f \\
&\leq \delta_{KE} 2^{-4(i+1)} r^4,
\end{aligned}$$

provided $\varepsilon_{KE}(R) \leq \frac{\varepsilon_A(R)\ \delta_{KE}}{256\bar{\mu}_R(1)}$. □

Here we could clearly see how the energy threshold $\varepsilon_{KE}(R)$ is determined by δ_{KE}.

Now we can apply (4.23) to $K = D_i$, $s = 2^{-(i+1)}r$ to obtain

$$\frac{1}{\mu_f(B(E, r))} \int_{B(D_i, 2^{-(i+3)}r)} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f \;\leq\; c(R)\left(\frac{2^{4i}}{r^4} + \frac{2^i}{r}\left(\frac{1}{\mu_f(B(E, r))} \int_{E_{i+1}} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f\right)^{\frac{3}{4}}\right), \tag{4.24}$$

where we need to notice that

$$A(D_i; r/2^{i+3}, 3r/2^{i+3}) \subset A(D_i; 0, 2^{-(i+1)}r) \subset B(D_i, 2^{-(i+1)}r) \subset E_{i+1}.$$

On F_i, we have $|\mathcal{R}m| \leq 4^{i+1}r^{-2}$, so $\int_{F_i} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f \leq c(R)2^{4(i+1)}r^{-4}\mu_f(A(E; 0, r))$. Now we can estimate

$$\begin{aligned}
\int_{E_i} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f \;&\leq\; \int_{D_i} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f + \int_{F_i} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f \\
&\leq \int_{B(D_i, 2^{-(i+3)}r)} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f + c(R)2^{4(i+1)}r^{-4}\mu_f(B(E, r)) \\
&\leq c(R)\mu_f(B(E, r))\left(\frac{2^{4i}}{r^4} + \frac{2^i}{r}\left(\frac{1}{\mu_f(B(E, r))} \int_{E_{i+1}} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f\right)^{\frac{3}{4}}\right).
\end{aligned}$$

Similarly, (4.23) directly implies that

$$\int_{E_1} |\mathcal{R}m|^2 \, d\mu_f \;\leq\; c(R)\mu_f(B(E,r))\left(16r^{-4} + 2r^{-1}\left(\frac{1}{\mu_f(B(E,r))}\int_{E_2} |\mathcal{R}m|^2 \, d\mu_f\right)^{\frac{3}{4}}\right).$$

Therefore, we could set $a_i := c(R)r^{-4}16^i$, $b_i := c(R)r^{-1}2^i$, and $x_i := \frac{1}{\mu_f(B(E,r))}\int_{E_i} |\mathcal{R}m|^2 \, d\mu_f$ for $i = 1, 2, 3, \cdots$, then a_i, b_i, x_i satisfy the relations

$$x_i \leq a_i + b_i x_{i+1}^{\frac{3}{4}}, \quad \text{and} \quad \limsup_{i\to\infty} x_i^{(\frac{3}{4})^i} = 1.$$

Notice that $\sum_{j=0}^{\infty}\left(\frac{3}{4}\right)^j = 4$, we can apply Lemma 5.1 of [20] to obtain

$$\frac{1}{\mu_f(B(E,r))}\int_{E} |\mathcal{R}m|^2 \, d\mu_f \;=\; x_1 \;\leq\; C_{KE}(R)r^{-4}.$$

□

As mentioned in the Introduction, (4.21) gives a bound that blows up in the induction process. However, the blow up rate is of second order in the inductive scale, which is absorbed by the controlling terms, i.e. the right-hand side of (4.23), blowing up of fourth order in the same scale. This observation will also be crucial for our arguments in the next ub-section.

4.4.2 The fast decay proposition.

As the key estimate tells, as long as the energy is sufficiently small at a given scale, the renormalized energy at that scale is bounded. In order to find a *uniform* scale, reducing to which the renormalized energy is small enough to apply Anderson's ε-regularity theorem, we need the following proposition:

Proposition 4.4.7. *Let (M, g, f) be a 4-D Ricci shrinker and fix $R > 2\sqrt{2}$. There exists some $r_{FD}(R) > 0$, $\varepsilon_{FD}(R) > 0$, $\delta_{FD}(R) > 0$ and $\eta_R > 0$, such that for $B(p, 2r) \subset B(p_0, R)$ with $r < r_{FD}(R)$, if*

$$\frac{\mu_f(B(p,r))}{\bar{\mu}_R(r)} \;<\; \delta_{FD}(R), \tag{4.25}$$

and

$$\int_{B(p,2r)} |\mathcal{R}m|^2 \, d\mu_f \;\leq \varepsilon_{FD}(R), \tag{4.26}$$

then

$$\frac{\bar{\mu}_R(r)}{\mu_f(B(p,r))}\int_{B(p,r)}|\mathcal{R}m|^2\, d\mu_f \leq (1-\eta_R)\frac{\bar{\mu}_R(2r)}{\mu_f(B(p,2r))}\int_{B(p,2r)}|\mathcal{R}m|^2\, d\mu_f. \quad (4.27)$$

Remark 4.4.8. *Abusing notations, we will always denote a possible subsequence by the original one.*

In this subsection we will take several steps to prove this proposition. Essentially, the proof reduces the problem, by blowing up the radius r, to a situation similar to the Einstein case. But this principle works on two levels: on the level of $|\nabla f|$, its smallness after rescaling will directly give a comparison geometry picture similar to the Einstein case; however, on the level of $|\nabla^2 f|$, we notice that $\int_{B(p,r)}|\mathring{\nabla}^2 f|^2\, \mathrm{d}\mu_f$ is scaling invariant, and we need to use the Weitzenböck formula (2.11) to give it a local L^2-control of order lower than that of $\int_{B(p,r)}|\mathcal{R}m|^2\, \mathrm{d}\mu_f$. This is in the same spirit as Lemma 4.4.4.

Moreover, our argument avoids appealing to the theory of Cheeger-Colding-Tian, see Theorem 3.7 of [13]. This is unavailable in the context of shrinking Ricci solitons since the Ricci curvature lower bound is not satisfied. However, we expect there to be a version of Cheeger-Colding-Tian's theory for manifolds with Bakry-Émery Ricci curvature bounded below.

We wish to point out that our argument is under the framework of Cheeger-Tian's in [20], whose key observation is that the estimates (4.41) – (4.43) of the approximating functions are in the average sense. Our new input is the elliptic regularity (4.44) of the approximating functions that produces smooth annuli where we have *global point-wise* derivative control, see Sub-sub-section (4.3.10). We would also like to thank Jeff Cheeger for pointing out the paper [42] for an alternative treatment in a different context.

Control of Pfaffian form.

In fact, we can assume

$$\int_{B(p,r)}|\mathcal{R}m|^2\, \mathrm{d}V_g > \varepsilon_A(R)\frac{\mu_f(B(p,r))}{e^{R^2}\omega_4\, r^4}, \quad (4.28)$$

because otherwise we could have directly applied Anderson's ε-regularity theorem to obtain the desired curvature bound, and there is no need to prove this proposition. Now we use Lemma 2.3.3 to obtain a cut-off function φ supported on $B(p,2r)$, constantly equal to 1 on $B(p, 1.6r)$, and having uniform control $r|\nabla\varphi| + r^2|\Delta^f\varphi| \leq C_{2.12}(R)$. Then we could estimate as in Lemma 4.4.4:

$$\begin{aligned}\int_{B(p,1.6r)} |\nabla^2 f|^2 \, \mathrm{d}V_g &\leq \frac{e^{R^2}}{2}\int_{B(p,2r)} (|\Delta^f\varphi| + 1)|\nabla f|^2 \, \mathrm{d}\mu_f \\ &\leq C(R)\mu_f(B(p,r))\, r^{-2}.\end{aligned}$$

As long as $r < \sqrt{\frac{\varepsilon_A(R)e^{-R^2}}{2C(R)\omega_4}}$, for any open set $B(p,r) \subset U \subset B(p, 1.6r)$ with smooth boundary, the expression of Pfaffian (2.16) gives

$$8\pi^2 \int_U \mathcal{P}_\chi \geq \int_{B(p,r)} |\mathcal{R}m|^2 \, \mathrm{d}V_g - \int_{B(p,1.6r)} |\mathring{\nabla}^2 f|^2 \, \mathrm{d}V_g > 0.$$

Let $\varepsilon_{FD}(R) \leq \pi^2 e^{-R^2}$, then the above inequality, together with (4.26) , gives

$$0 < \int_U \mathcal{P}_\chi \leq \frac{3e^{R^2}\, \varepsilon_{FD}(R)}{8\pi^2} < \frac{1}{2} \tag{4.29}$$

for any open subset U with smooth boundary such that $B(p,r) \subset U \subset B(p, 1.6r)$.

Setting up a contradiction argument.

We prove the proposition by a contradiction argument. Were the proposition false, then there exist 4-d Ricci shrinkers (M_i, g_i, f_i), sequences $r_i \to 0$, $\delta_i \to 0$ and $\eta_i \to 0$ as $i \to \infty$, such that for some $B(p_i, 4r_i) \subset B(p_i^0, R)$ (p_i^0 denoting the base point of M_i),

$$\int_{B(p_i,2r_i)} |\mathcal{R}m_{g_i}|^2 \, \mathrm{d}\mu_{f_i} \leq \varepsilon_{FD}(R), \tag{4.30}$$

$$\text{and} \quad \frac{\mu_{f_i}(B(p_i, 2r_i))}{\bar{\mu}_R(2r_i)} < \delta_i \tag{4.31}$$

but (4.27) is violated for each i.

We will find, for each i large enough, some open subset U_i with smooth boundary such that $B(p_i, r_i) \subset U_i \subset B(p_i, 2r_i)$ and that

$$0 < \int_{\partial U_i} \mathcal{T}\mathcal{P}_\chi < \frac{1}{2}. \tag{4.32}$$

Since $r_i \to 0$, (4.29) holds for all i sufficiently large, so adding (4.29) and (4.32) gives

$$0 < \chi(U_i) < 1,$$

contradicting the integrality of $\chi(U_i)$.

Rescaling.

Consider the rescaled sequence $(M_i, r_i^{-2}g_i, f_i)$. Denote $\tilde{g}_i := r_i^{-2}g_i$, then the scaling invariance of the energy and (4.26) implies that for each i,

$$\int_{\tilde{B}(p_i,2)} |\mathcal{R}m_{\tilde{g}_i}|^2 \, \mathrm{d}\tilde{\mu}_{f_i} \leq \varepsilon_{FD}(R), \tag{4.33}$$

where we add a tilde to an object to denote its rescaled correspondence. Moreover, the scaling invariance of volume ratio and the converse of (4.27) implies that

$$\frac{\bar{\mu}_{r_iR}(1)}{\tilde{\mu}_{f_i}(\tilde{B}(p_i,1))}\int_{\tilde{B}(p_i,1)} |\mathcal{R}m_{\tilde{g}_i}|^2 \, \mathrm{d}\tilde{\mu}_{f_i} > (1-\eta_i)\frac{\bar{\mu}_{r_iR}(2)}{\tilde{\mu}_{f_i}(\tilde{B}(p_i,2))}\int_{\tilde{B}(p_i,2)} |\mathcal{R}m_{\tilde{g}_i}|^2 \, \mathrm{d}\tilde{\mu}_{f_i}. \tag{4.34}$$

These two inequalities will be the starting point of our future arguments. Moreover, the rescaled metrics and potential functions satisfy

$$\mathcal{R}c_{\tilde{g}_i} + \tilde{\nabla}^2 f_i = \frac{r_i^2}{2}\tilde{g}_i, \tag{4.35}$$

which implies the non-negativity of the rescaled Bakry-Émery-Ricci curvature

$$\mathcal{R}c_{\tilde{g}_i}^{f_i} = \frac{r_i^2}{2}\tilde{g}_i \geq 0, \tag{4.36}$$

and the potential function has the gradient estimates

$$|\tilde{\nabla} f_i|_{\tilde{g}_i} \leq r_iR. \tag{4.37}$$

Finally, we denote the distance to the given point p_i by $d_{p_i}(x) := d(p_i, x)$, then its rescaled version is denoted by $d_i := r_i^{-1}d_{p_i}$.

Regularity on annuli.

On the one hand, since

$$\frac{\bar{\mu}_{r_iR}(1)\tilde{\mu}_{f_i}(\tilde{B}(p_i,2))}{\bar{\mu}_{r_iR}(2)\tilde{\mu}_{f_i}(\tilde{B}(p_i,1))} \leq 1$$

by (2.23), we have

$$\int_{\tilde{A}(p_i;1,2)} |\mathcal{R}m_{\tilde{g}_i}|^2 \, \mathrm{d}\tilde{\mu}_{f_i} \leq \frac{\eta_i}{1-\eta_i}\int_{\tilde{B}(p_i,1)} |\mathcal{R}m_{\tilde{g}_i}|^2 \, \mathrm{d}\tilde{\mu}_{f_i}.$$

Let $\varepsilon_{FD}(R) > 0$ be sufficiently small (and fixed from now on), so that we can apply the key estimate (notice the correct order of the scaling there) to obtain

$$\int_{\tilde{B}(p_i,1)} |\mathcal{R}m_{\tilde{g}_i}|^2 \, \mathrm{d}\tilde{\mu}_{f_i} \ \le \ c(R)\tilde{\mu}_{f_i}(\tilde{B}(p_i,2)),$$

and it follows that

$$\int_{\tilde{A}(p_i;1,2)} |\mathcal{R}m_{\tilde{g}_i}|^2 \, \mathrm{d}\tilde{\mu}_{f_i} \ \le \ \frac{c(R)\eta_i}{1-\eta_i}\tilde{\mu}_{f_i}(\tilde{B}(p_i,2)).$$

Now for any $x \in \tilde{A}(p_i;1.1,1.9)$, $\tilde{B}(x,0.1) \subset \tilde{A}(p_i;1,2) \subset \tilde{B}(x,4)$ and

$$\int_{\tilde{B}(x,0.1)} |\mathcal{R}m_{\tilde{g}_i}|^2 \, \mathrm{d}\tilde{\mu}_{f_i} \ \le \ \frac{c(R)\eta_i}{1-\eta_i}\tilde{\mu}_{f_i}(\tilde{B}(x,4)) \le \frac{c(R)\eta_i}{1-\eta_i}\frac{\tilde{\mu}_{f_i}(\tilde{B}(x,0.1))}{\bar{\mu}_{r_iR}(0.1)}\bar{\mu}_{r_iR}(4),$$

so by the scaling invariance of the renormalized energy, we have

$$I^{f_i}_{\mathcal{R}m_{\tilde{g}_i}}(x,0.1) \ \le \ \frac{c(R)\eta_i}{1-\eta_i}.$$

For all i sufficiently large, Anderson's ε-regularity theorem gives $|\mathcal{R}m_{\tilde{g}_i}|^2_{\tilde{g}_i}(x) \le c(R)\eta_i$, thus

$$\sup_{\tilde{A}(p_i;1.1,1.9)} |\mathcal{R}m_{\tilde{g}_i}|^2_{\tilde{g}_i} \le c(R)\eta_i \to 0 \quad \text{as } i \to \infty. \tag{4.38}$$

Notice that the above curvature estimate enables us to apply Lemma 4.1.6 and obtain uniform bounds for each $k \ge 0$:

$$\sup_{\tilde{A}(p_i;1.2,1.8)} |\tilde{\nabla}^k \mathcal{R}m_{\tilde{g}_i}|_{\tilde{g}_i} \le c(k,R), \quad \text{and} \quad \sup_{\tilde{A}(p_i;1.2,1.8)} |\tilde{\nabla}^k f_i|_{\tilde{g}_i} \le c'(k,R). \tag{4.39}$$

Almost volume annulus and smoothing distance function.

On the other hand, since

$$\int_{\tilde{B}(p_i,1)} |\mathcal{R}m_{\tilde{g}_i}|^2 \, \mathrm{d}\tilde{\mu}_{f_i} \ \le \ \int_{\tilde{B}(p_i,2)} |\mathcal{R}m_{\tilde{g}_i}|^2 \, \mathrm{d}\tilde{\mu}_{f_i},$$

then (4.34) implies that

$$\frac{\bar{\mu}_{r_iR}(1)}{\tilde{\mu}_{f_i}(\tilde{B}(p_i,1))} \ \ge \ (1-\eta_i)\frac{\bar{\mu}_{r_iR}(2)}{\tilde{\mu}_{f_i}(\tilde{B}(p_i,2))},$$

i.e. $\tilde{A}(p_i; 1, 2)$ is an annulus in an almost f_i-weighted volume cone for i sufficiently large. By weighted volume comparison (2.23), for any $r \in (1.05, 1, 95)$,

$$\frac{\tilde{\mu}_{f_i}(\partial \tilde{B}(p_i, r))}{\bar{\mu}'_{r_i R}(r)} \;\geq\; (1 - \Psi(\eta_i |\, r)) \frac{\tilde{\mu}_{f_i}(\tilde{B}(p_i, r))}{\bar{\mu}_{r_i R}(r)}, \tag{4.40}$$

where $\Psi(\eta_i|\, r)$ denotes some positive function that approaches 0 as $\eta_i \to 0$.

Now we smooth the square of the distance function $\frac{d_i^2}{2}$. For each i, we will solve the Dirichlet problem

$$\Delta^{f_i}_{\tilde{g}_i} u_i = 4 \quad \text{and} \quad u_i|_{\partial \tilde{A}(p_i;1,2)} = \frac{d_i^2}{2}.$$

In view of (4.36), (4.37) and (4.40), we can estimate u_i and $\tilde{u}_i := \sqrt{2u_i}$ by applying Lemma 2.3.4:

$$\sup_{\tilde{A}(p_i;1.2,1.8)} |\tilde{u}_i - d_i| \;\leq\; \Psi(\eta_i, r_i \mid R); \tag{4.41}$$

$$⨍_{\tilde{A}(p_i;1.1,1.9)} |\nabla \tilde{u}_i - \nabla d_i|^2 \; \mathrm{d}\tilde{\mu}_{f_i} \;\leq\; \Psi(\eta_i, r_i \mid R); \tag{4.42}$$

$$⨍_{\tilde{A}(p_i;1.3,1.7)} |\nabla^2 u_i - \tilde{g}_i|^2 \; \mathrm{d}\tilde{\mu}_{f_i} \;\leq\; \Psi(\eta_i, r_i \mid R). \tag{4.43}$$

Moreover, the elliptic regularity Lemma 4.1.7, estimates (4.39) and the C^0 bound (4.41) ensures that each $\tilde{u}_i$ and u_i are regular:

$$\sup_{\tilde{A}(p_i;1.3,1.7)} |\nabla^k \tilde{u}_i| + |\nabla^k u_i| \;\leq\; c''(k; R). \tag{4.44}$$

The collapsing limit

According to Proposition 2.5.4,$\tilde{A}(p_i; 1.2, 1.8) \to_{GH} (X, d_\infty)$ (after passing to a subsequence), with $X = \mathcal{R}(X) \cup \mathcal{S}(X)$. Here $\mathcal{R}(X)$ is a lower dimensional Riemannian manifold equipped with a smooth Riemannian metric g_∞ with bounded curvature (invoking (4.39)), such that $d_\infty|_{\mathcal{R}(X)}$ is induced by g_∞. $\mathcal{S}(X)$ is a stratified collection of subsets of X, each strata of $\mathcal{S}(X)$ by itself being a Riemannian manifold of dimension even lower than that of $\mathcal{R}(X)$. There is a constant $\iota_X > 0$ such that

$$\forall x_\infty \in \mathcal{R}(X), \quad \mathrm{inj}\; x_\infty \geq \min\left\{d_\infty(x_\infty, \mathcal{S}(X)), d_\infty(x_\infty, \partial X), \iota_X\right\}.$$

Local average control of u_i

We will study the behavior of u_i at each point of $\tilde{A}(p_i; 1.3, 1.7)$ by taking limit. Fix $x_\infty \in \mathcal{R}(X)$ such that $x_i \to_{GH} x_\infty$ for some sequence $x_i \in \tilde{A}(p_i; 1.3, 1.7)$. Fix a scale $\alpha = \alpha(x_\infty) < \min\{0.001, \frac{1}{2}d_\infty(x_\infty, \mathcal{S}(X)), \iota_X\}$, we have, by volume comparison,

$$\frac{\tilde{\mu}_{f_i}(\tilde{B}(x_i,\alpha))}{\tilde{\mu}_{f_i}(\tilde{A}(p_i;1.3,1.7))} \;\geq\; \frac{\bar{\mu}_{r_iR}(\alpha)}{\bar{\mu}_{r_iR}(4)}. \tag{4.45}$$

Now we can localize the estimates (4.42) and (4.43):

$$\fint_{\tilde{B}(x_i,\alpha)} |\nabla\tilde{u}_i - \nabla d_i|^2 \,\mathrm{d}\tilde{\mu}_{f_i} \;\leq\; \Psi(\eta_i, r_i \,|\, R, \alpha); \tag{4.46}$$

$$\fint_{\tilde{B}(x_i,\alpha)} |\nabla^2 u_i - g_i|^2 \,\mathrm{d}\tilde{\mu}_{f_i} \;\leq\; \Psi(\eta_i, r_i \,|\, R, \alpha). \tag{4.47}$$

Limit local covering geometry

Let $\pi_i : \tilde{B}_i \to \tilde{B}(x_i,\alpha)$ be the universal covering of $\tilde{B}(x_i,\alpha)$, with lifted base point $\tilde{x}_i$ and deck transformation group Γ_i. Recall that the scale $\alpha = \alpha(x_\infty)$ is chosen so that $B(x_\infty,\alpha)$ is away from $\mathcal{S}(X)$ and simply connected. This means, by Fukaya's fibration theorem, that for all i sufficiently large, $\tilde{B}(x_i,\alpha)$ is topologically a torus bundle over $B(x_\infty,\alpha)$, whence topologically

$$\tilde{B}_i \;\approx\; \mathbb{R}^{4-\dim_H X} \times B(x_\infty,\alpha). \tag{4.48}$$

We equip $\tilde{B}_i$ with the pull-back metric $h_i := \pi_i^*\tilde{g}_i$ and potential function $\tilde{f}_i := \pi_i^* f_i$. Clearly $(B(\tilde{x}_i,\alpha), h_i)$ is non-collapsing, and on $B(\tilde{x}_i,\alpha)$, estimates (4.38) and (4.39) hold for $\mathcal{R}m_{h_i}$ and $\tilde{f}_i$. This ensures that $\{B(\tilde{x}_i,\alpha)\}$ converges, after passing to a subsequence, to $B(\tilde{x}_\infty,\alpha)$, a 4-Dimensional Riemannian manifold with limiting Riemannian metric h_∞. Moreover, by (4.38), possibly passing to a subsequence, h_i smoothly converges to the flat metric $h_\infty = g_{Euc}$ on $B(\tilde{x},\alpha)$. We will denote the pull-back measure by $\nu_i := \pi_i^*(\mathrm{d}\tilde{\mu}_{f_i})$, and by $\tilde{d}_i := \pi_i^* d_i$.

Recall that by (4.37), $|\nabla\tilde{f}_i|_{h_i} = |\nabla f_i|_{\tilde{g}_i} \leq r_i R \to 0$ as $i \to \infty$, and that $\{\tilde{f}_i\}$ has uniform derivative control (4.39), the drifted Laplace operators $\Delta_{h_i}^{\tilde{f}_i}$ converge smoothly to $\Delta = \sum_{j=1}^4 \partial_j\partial_j$, the standard Laplace operator for $(\mathbb{R}^4, g_{Euc})$.

Moreover, each pull-back smooth function $v_i := \pi_i^*\tilde{u}_i$ satisfies the elliptic equation

$$\Delta_{h_i}^{\tilde{f}_i} v_i^2 \;=\; 8.$$

The smooth convergence of the drifted Laplace operators $\Delta_{h_i}^{\tilde{f}_i}$ further gives, for i large enough, uniform elliptic estimates

$$\sup_{B(\tilde{x}_\infty, 0.09)} |\nabla^k v_i^2|_{h_i} \leq c''(k, R). \tag{4.49}$$

The uniform boundedness (4.41) and the regularity estimates (4.49) ensure that $v_i \to v_\infty$ in $C^\infty(B(\tilde{x}_\infty, 0.9\alpha)$ (after possibly passing to a further subsequence), the limiting equation being

$$\Delta v_\infty^2 \ = \ 8 \quad \text{on} \quad B(\tilde{x}_\infty, 0.9\alpha). \tag{4.50}$$

To summarize, when $i \to \infty$ and after passing to subsequences, we have smooth convergence on $B(\tilde{x}_\infty, 0.9\alpha)$, of the sequence of metrics $h_i \to g_{Euc}$, of the sequence of potential functions $\tilde{f}_i \to c(R)$ (whence the smooth convergence of the elliptic operators $L_i \to \Delta$) and of the sequence of Poisson solutions $v_i \to v_\infty$.

Local point-wise control of u_i

Now we will discuss the effect of the estimates (4.46) and (4.47) on the local coverings. Let $B_i \ni \tilde{x}_i$ be a fundamental domain of $\tilde{B}_i$, then for each sufficiently large i, in view of (4.48), we have $B(\tilde{x}_i, \alpha) \subset \tilde{U}_i \subset B(\tilde{x}_i, 2\alpha)$ where

$$\tilde{U}_i := \cup\{\gamma B_i : \gamma \in \Gamma_i, \gamma B_i \cap B(\tilde{x}, \alpha) \neq \emptyset\}. \tag{4.51}$$

Notice that estimates (4.46) and (4.47) on the local covering, for each $\gamma \in \Gamma_i$, read

$$\begin{aligned} \int_{\gamma B_i} |\nabla v_i - \nabla \tilde{d}_i|^2 \, \mathrm{d}\nu_i \ &\leq \ \Psi(\eta_i, r_i \mid R, \alpha)\nu_i(\gamma B_i); \\ \int_{\gamma B_i} |\nabla v_i - h_i|^2 \, \mathrm{d}\nu_i \ &\leq \ \Psi(\eta_i, r_i \mid R, \alpha)\nu_i(\gamma B_i). \end{aligned}$$

Then by Bishop-Gromov volume comparison on $\tilde{B}_i$, we have

$$\begin{aligned} \int_{B(\tilde{x}_i, \alpha)} |\nabla v_i - \nabla \tilde{d}_i|^2 \, \mathrm{d}\nu_i \ &\leq \ \int_{\tilde{U}_i} |\nabla v_i - \nabla \tilde{d}_i|^2 \, \mathrm{d}\nu_i \\ &\leq \ \sum_{\gamma B_i \cap B(\tilde{x}_i, \alpha) \neq \emptyset} \int_{\gamma B_i} |\nabla v_i - \nabla \tilde{d}_i|^2 \, \mathrm{d}\nu_i \\ &\leq \ \Psi(\eta_i, r_i \mid R, \alpha) \sum_{\gamma B_i \cap B(\tilde{x}_i, \alpha) \neq \emptyset} \nu_i(\gamma B_i) \\ &\leq \ \Psi(\eta_i, r_i \mid R, \alpha)\nu_i(B(\tilde{x}_i, 2\alpha)); \end{aligned}$$

whence

$$\fint_{B(\tilde{x}_i,\alpha)} |\nabla v_i - \nabla \tilde{d}_i|^2 \, \mathrm{d}\nu_i \;\le\; \Psi(\eta_i, r_i \,|\, R, \alpha), \tag{4.52}$$

and similarly,

$$\fint_{B(\tilde{x}_i,\alpha)} |\nabla^2 v_i^2 - h_i|^2 \, \mathrm{d}\nu_i \;\le\; \Psi(\eta_i, r_i \,|\, R, \alpha). \tag{4.53}$$

When passing to the limit, these estimates, together with the regularity (4.49) give

$$|\nabla v_\infty| \;\equiv\; 1 \quad \text{and} \quad \nabla^2 v_\infty^2 \;\equiv\; 2\, g_{Euc} \quad \text{in } B(\tilde{x}_\infty, 0.7\alpha). \tag{4.54}$$

Thinking of $B(\tilde{x}_\infty, 0.7\alpha)$ as a region in $\mathbb{R}^4$ with $\tilde{x}_\infty = \mathbf{0}$, we see that $v_\infty^2(\mathbf{x}) = |\mathbf{x} - \mathbf{x}_0|^2$ for $\mathbf{x} \in B(\mathbf{0}, 0.7\alpha)$ and some $\mathbf{x}_0 \in \mathbb{R}^4$. Moreover, $v_\infty(\mathbf{0}) = \lim_{i\to\infty} \tilde{u}_i(x_i)$.

For any $i > i_{x_\infty}$, since the local covering is equipped with the pull-back metric, the smoothness of the convergence (4.49) then gives

$$|\nabla \tilde{u}_i| \;\ge\; 1 - 10^{-10} \quad \text{and} \quad |\nabla^2 u_i - \tilde{g}_i| \;\le\; 10^{-10} \quad \text{in} \quad B(x_i, 0.6\alpha). \tag{4.55}$$

Now we consider the second fundamental form of $\tilde{u}_i^{-1}(\tilde{u}_i(x_i))$: since at x_i,

$$\lim_{i\to\infty} \frac{\nabla^2 v_i}{|\nabla v_i|}(\tilde{x}_i) \;=\; \frac{1}{v_\infty(\tilde{x}_\infty)}(g_{Euc} - \nabla r \otimes \nabla r), \tag{4.56}$$

where r is the Euclidean distance function to the origin, we have, especially, the principal curvatures of $\tilde{u}_i^{-1}(\tilde{u}_i(x_i))$ at x_i, satisfies

$$\left| \kappa_i^k(x_i) - \frac{1}{\tilde{u}_i(x_i)} \right| \;<\; 10^{-10} \quad \text{for all } i > i_{x_\infty}. \tag{4.57}$$

This further implies a control of the boundary Gauss-Bonnet-Chern term for $\tilde{u}_i^{-1}(\tilde{u}_i(x_i))$ at x_i: since

$$\mathcal{TP}_\chi(x_i) \;=\; \frac{1}{4\pi^2}\left(2 \prod_{k=1,2,3} \kappa_i^k(x_i) - \sum_{k=1,2,3} \kappa_i^k(x_i) \mathcal{K}_{\tilde{g}_i}^{\hat{k}}(x_i) \right) \mathrm{d}\sigma_{\tilde{u}_i^{-1}(\tilde{u}_i(x_i))},$$

where $\hat{k}$ is a pair of numbers in $\{1, 2, 3\}$ not containing k, we have, by (4.38) and (4.57), for all $i > i_{x_\infty}$,

$$\left| \mathcal{TP}_\chi(x_i) - \frac{1}{2\pi^2 \tilde{u}_i(x_i)^3} \, \mathrm{d}\sigma_{\tilde{u}_i^{-1}(\tilde{u}_i(x_i))} \right| \;<\; 10^{-10}. \tag{4.58}$$

Global point-wise control of u_i

Notice that (4.55) and (4.57) are actually point-wise controls, since the scale α depends on specific $x_\infty = \lim_{GH} x_i \in \mathcal{R}(X)$; especially, from the argument above we could not obtain any control as we approach $\mathcal{S}(X)$. Luckily, u_i has very nice regularity (4.44), so that we can choose a *uniform* scale $\alpha_0 > 0$ sufficiently small such that for any $x', x'' \in \tilde{A}(p_i; 1.3, 1.7)$, and $\kappa_i^k(x)$ being the k-th principal vector of $\tilde{u}_i^{-1}(\tilde{u}_i(x))$,

$$d(x', x'') < 3\alpha_0 \Rightarrow \|\nabla\tilde{u}_i|(x') - |\nabla\tilde{u}_i|(x'')| + \sum_{k=1,2,3} \left|\kappa_i^k(x') - \kappa_i^k(x'')\right| < 10^{-10}. \tag{4.59}$$

Now let $\{x_\infty^j\} \subset \mathcal{R}(X)$ be a minimal α_0-net of $\mathcal{R}(X)$, and $\{x_i^j\} \subset \tilde{A}(p_i; 1.3, 1.7)$ such that $x_i^j \to_{GH} x_\infty^j$. Obviously $j < J$ for some absolute constant J. For large enough i, $\{B(x_i^j, 2\alpha_0)\}$ covers $\tilde{A}(p_i; 1.3, 1.7)$. Then (4.55), (4.57) and (4.58) work for each x_i^j, when $i > i_0 := \max\{i_{x_\infty^j} : j = 1, \cdots, J\}$ is large enough. We could further estimate, by (4.38) and (4.59), that

$$\inf_{\tilde{B}(x_i^j, 2\alpha_0)} |\nabla\tilde{u}_i| > 1 - 10^{-5} \quad \text{and} \quad \sup_{\tilde{B}(x_i^j, 2\alpha_0)} \sum_{k=1,2,3} \left|\kappa_i^k - \frac{1}{\tilde{u}_i}\right| < 10^{-5}, \tag{4.60}$$

whence the same estimate globally on $\tilde{A}(p_i; 1.3, 1.7)$, for all $i > J$ sufficiently large.

Especially, since $(1.4, 1.6) \subset Image(\tilde{u}_i)$ by (4.41), this implies that $\tilde{u}_i^{-1}(a)$ is a smooth hyper-surface in $\tilde{A}(p_i; 1.3, 1.7)$, for all $a \in (1.4, 1.6)$ and large enough i. Furthermore, we can control the boundary Gauss-Bonnet-Chern form of $\tilde{u}_i^{-1}$ by (4.38), (4.39), (4.44) and (4.60): for all $i > i_0$ large enough and $a \in (1.4, 1.6)$,

$$\left|\mathcal{TP}_\chi - \frac{1}{2\pi^2 a^3}\, \mathrm{d}\sigma_{\tilde{u}_i^{-1}(a)}\right| < 10^{-4}, \tag{4.61}$$

since $|\nabla\mathcal{TP}_\chi| \leq C(R)$.

Level sets of u_i

From the co-area formula, (4.42) and the scaling invariance of (4.30), we can estimate

$$\begin{aligned}
\int_{1.4}^{1.6} \frac{\tilde{\mu}_{f_i}(\tilde{u}_i^{-1}(s))}{2\pi^2 s^3}\, \mathrm{d}s \;&\leq\; \frac{\int_{1.4}^{1.6} \tilde{\mu}_{f_i}(\tilde{u}_i^{-1}(s))\, \mathrm{d}s}{\int_{1.4}^{1.6} 2\pi^2 s^3\, \mathrm{d}s}\\
&=\; C(R)\frac{\int_{\tilde{u}^{-1}([1.4,1.6])} |\nabla \tilde{u}_i|\, \mathrm{d}\tilde{\mu}_{f_i}}{\bar{\mu}_{r_iR}(1.6) - \bar{\mu}_{r_iR}(1.4)}\\
&\leq\; C(R)\frac{\tilde{\mu}_{f_i}(\tilde{A}(p_i; 1.1, 1.9))(1 + \Psi(\eta_i, r_i \mid R))}{\bar{\mu}_{r_iR}(1.6) - \bar{\mu}_{r_iR}(1.4)}\\
&\leq\; C(R)\frac{\tilde{\mu}(\tilde{B}(p_i, 2))}{\bar{\mu}_{r_iR}(2)} \;<\; C(R)\delta_i.
\end{aligned}$$

Thus for all $i > i_0$ sufficiently large, by (4.44) and (4.60), there is some $a_i \in (1.4, 1.6)$ such that

$$\frac{\tilde{\mu}_{f_i}(\tilde{u}_i^{-1}(a_i))}{2\pi^2 a_i^3} \;<\; \frac{1}{6} e^{-R^2 - r_i R}, \tag{4.62}$$

whenever i is large enough (so that $\delta_i < \frac{1}{6} e^{-2R^2} C(R)^{-1}$).

The contradiction

We fix any $i > i_0$ sufficiently large, and set $U_i := \tilde{B}(p_i, 1.4) \cup \tilde{u}_i^{-1}(1.3, a_i)$, and notice the smoothness of $\partial U_i = \tilde{u}_i^{-1}(a_i)$ by (4.60). Moreover, we have

$$0 \;<\; \int_{\partial U_i} \mathcal{TP}_\chi \;<\; \frac{3Vol_{\tilde{g}_i}(\partial U_i)}{4\pi^2 a_i^3}, \tag{4.63}$$

by (4.61), but then by (4.62)

$$\frac{3Vol_{\tilde{g}_i}(\partial U_i)}{4\pi^2 a_i^3} \;\leq\; \frac{3}{2} e^{R^2 + r_i R} \frac{\tilde{\mu}_{f_i}(\tilde{u}_i^{-1}(a_i))}{2\pi^2 a_i^3} \;\leq\; \frac{1}{4}.$$

Further notice that $\int_{\partial U_i} \mathcal{TP}_\chi$ is a topological constant, invariant under rescaling, so the above two estimates confirm (4.32).

Remark 4.4.9. *As kindly pointed out by Ruobing Zhang, (4.51) and the estimates that follow do not require the specific topological structure, thus we don't have to work within the injectivity radii at regular points, but instead, estimates (4.52) and (4.53) work for balls centered at any point. We wrote the estimates (4.52) and (4.53) only at the scale of injectivity radii because (4.48) gives a more intuitive explanation.*

4.4.3 Conclusion of the proof

With the help of the key estimate (Proposition 4.4.5) and the fast decay of renormalized energy (Proposition 4.4.7), we can now prove Theorem 1.3.1:

Proof of Theorem 1.3.1. Let $r_R := \frac{1}{10} r_{FD}(R)$ and let $\varepsilon_R = \min\{\varepsilon_{KE}(R), \varepsilon_{FD}(R)\}$. Fix some $r < r_R$, and assume that $B(p,r) \subset B(p_0,R)$ has small energy $\int_{B(p,r)} |\mathcal{R}m|^2 \, d\mu_f < \varepsilon_R$. It then follows from Proposition 4.4.5 that $I^f_{\mathcal{R}m}(p,r) < C_{KE}(R)$. If $I^f_{\mathcal{R}m}(p,r) < \varepsilon_A(R)$ we can apply Anderson's ε-regularity theorem directly, or if

$$\frac{\mu_f(B(p,r/2))}{\bar{\mu}_R(r/2)} \geq \delta_{KE}(R),$$

we are reduced to the known non-collapsing case, see [39]. Otherwise, we can apply Proposition 4.4.7 so that $I^f_{\mathcal{R}m}(p,r/2) < (1-\eta_R) C_{KE}(R)$. Performing the same process at most $k_R := \log_{1-\eta_R} \frac{\varepsilon_A(R)}{2C_{KE}(R)}$ many times, we will have $I^f_{\mathcal{R}m}(p, 2^{-k_R} r) < \varepsilon_A(R)$, whence

$$\sup_{B(p,2^{-k_R-1}r)} |\mathcal{R}m| \;\leq\; C_A(R) 4^{k_R} r^{-2} I^f_{\mathcal{R}m}(p, 2^{-k_R} r)^{\frac{1}{2}}. \tag{4.64}$$

Now cover $B(p,r/4)$ by balls of radius $2^{-k_R-1}r$, we have

$$B(p,r/4) \;\subset\; \bigcup_{q \in B(p,r/4)} B(q, 2^{-k_R-1} r) \;\subset\; \bigcup_{q \in B(p,r/4)} B(q, r/2) \;\subset\; B(p,r),$$

applying the argument above for each $q \in B(p,r/4)$, we obtain by (4.64),

$$\sup_{B(p,\frac{1}{4}r)} |\mathcal{R}m| \;\leq\; C_R \, r^{-2},$$

with $C_R := C_A(R) \sqrt{\varepsilon_A(R)}\, 4^{k_R+1}$. □

4.5 Strong convergence of 4-D Ricci shrinkers

In this section we will apply our ε-regularity theorem to obtain structural results concerning the convergence and degeneration of the soliton metrics. We first have a straightforward application of Theorem 1.3.1:

Proposition 4.5.1. *Let $\{(M_i, g_i, f_i)\}$ be a sequence of complete non-compact 4-D Ricci shrinkers. Suppose*

$$\int_{B(p_i^0,R)} |\mathcal{R}m_{g_i}|^2 \, d\mu_{f_i} \;\leq\; C(R). \tag{4.65}$$

Then for each $R > 0$ fixed, it sub-converges to some length space (X_R, d_∞) in the strong multi-pointed Gromov-Hausdorff sense (see Definition 2.5.3), with $J \le J(R)$ marked points.

Proof. Fix any $R > 0$. By the assumption (4.65), there are only finitely many points $p_i^1, \cdots, p_i^{J_R} \in B(p_i^0, R)$, around which there is a curvature concentration

$$\int_{B(p_i^j, r_i)} |\mathcal{R}m_{g_i}|^2 \, \mathrm{d}\mu_{f_i} \ \ge\ \varepsilon_{R+1}, \tag{4.66}$$

with $r_i \to 0$ and $j_R \le C(R+1)\varepsilon_{R+1}^{-1}$. On the other hand, for any $q \in B(p_i^0, R)$ outside $\cup_{j=1}^{J_R} B(p_i^1, 2r_i)$, we have

$$|\mathcal{R}m_{g_i}|(q) \ \le\ C_{R+1} r_i^{-2}.$$

By Lemma 2.3.2 and Gromov's compactness [35], there is a compact length space (X, d_∞) such that after passing to a subsequence, $(\overline{B(p_i^0, R)}, g_i) \to_{GH} (X, d_\infty)$. Clearly $\mathrm{diam}_{d_\infty} X \le R$. Moreover, by compactness of $\overline{B(p_i^0, R)}$, possibly passing to a further subsequence, the set of points $\{p_i^1, \cdots, p_i^{J_R}\}$ also Gromov-Hausdorff converge to a set of marked points $\{x_\infty^1, \cdots, x_\infty^{J_R}\} \subset X$.

Now fix $x \in X \backslash \{x_\infty^1, \cdots, x_\infty^{J_R}\}$ and assume $B(p_i^0, R) \ni p_i \to_{GH} x$. Fix

$$d_x := \min_{1 \le j \le J_R} d_\infty(x, x_\infty^j),$$

then for any i sufficiently large, $d_x > 10 r_i$ ($j = 1, \cdots, J_R$), and we can conclude as above

$$\sup_{B(p_i, \frac{1}{4} d_x)} |\mathcal{R}m_{g_i}| \ \le\ C_{R+1} d_x^{-2},$$

a uniform constant for the sequence $\{B(p_i^0, R)\}$. Thus the Gromov-Hausdorff convergence to any $x \ne x_\infty^j$ ($j = 1, \cdots, J_R$) is improved to strong convergence in Definition 2.5.3. □

Presumably, as $R \to \infty$, $j_R \to \infty$ and the selection of the subsequence of $\{M_i, g_i, f_i\}$ depends on R. This is a feature of Ricci solitons different from the Einstein case. However, assuming weighted L^2-bound of curvature is much more realistic for non-compact 4-D Ricci shrinkers, compared to non-compact Ricci flat manifolds. For instance, as we will see in the following proof of Theorem 1.3.2, a global weighted L^2-curvature bound by the Euler characteristics could be easily obtained if we further assume a uniform scalar curvature bound, see also [39] and [47].

From (2.18) and (2.19), we notice that a uniform bound on the scalar curvature, eliminates singularities of f outside a definite ball. It will then be convenient to use (sub-)level sets of f instead of geodesic balls centered at p_0. Therefore we use the following notations:

Definition 4.5.2. *Let (M, g, f) be a 4-D Ricci shrinker such that the normalization condition (2.9) is satisfied, and fix $R > 0$, we define*

$$D(R) := \{x \in M : f(x) < R\} \quad \text{and} \quad \Sigma(R) := \{x \in M : f(x) = R\} = \partial D(R).$$

Proof of Theorem 1.3.2. Fix $R_0 > 1$ so that there is no critical value of f outside $D(R)$.

Curvature bound outside some $D(R_{MW})$. We will start by examining the work of Munteanu-Wang [47] carefully, and obtain a *uniform* curvature control outside a fixed sized ball around the base point. (We cannot directly quote their results because their estimates involve the curvature of specific manifolds, but we need uniform estimates.) After a detailed study of the level sets $\Sigma(R)$, Munteanu-Wang observed, in Proposition 1.1 of [47], the following fundamental estimate for 4-D Ricci shrinkers: there is an absolute constant $c_{1.0}$ such that

$$c_{1.0}|\mathcal{R}m| \leq \frac{|\nabla \mathcal{R}c|}{\sqrt{t}} + \frac{|\mathcal{R}c|^2 + 1}{f} + |\mathcal{R}c|$$

outside $D(R_{1.0})$. This estimate then enables them to obtain an elliptic inequality about the positive function $u := |\mathcal{R}c|^2 \mathcal{R}^{-a}$ for some $a \in (0, 1)$ (see Lemma 1.2 of [47]): there exists some absolute constant $c_{1.1} > 0$ such that

$$\Delta_f u \geq \left(2a - \frac{c_{1.1}}{1-a}\frac{\mathcal{R}}{f}\right) u^2 \mathcal{R}^{a-1} - c_{1.1} u^{\frac{3}{2}} \mathcal{R}^{\frac{a}{2}} - c_{1.1} u$$

outside $D(R_{1.1})$. For any $R > 2\max\{R_0, R_{1.0}, R_{1.1}\}$, as done in Proposition 1.3 of [47], one can construct a cut off function φ supported on $D(3R)\backslash D(R/2)$ such that $\varphi \equiv 1$ on $D(2R)\backslash D(R)$ and $|\nabla\varphi| + |\Delta^f \varphi| \leq c_{1.2}$ ($c_{1.2} > 0$ being some absolute constant, especially independent of R). Now choose $R_{1.2} > R$ and $a \in (0, 1)$ such that

$$2a - \frac{c_{1.2}}{1-a}\frac{\mathcal{R}}{f} \;\geq\; 1$$

outside $D(R)$, for any $R > R_{1.2}$, then a becomes an absolute constant. We then obtain inequality (1.14) of [47]:

$$\varphi^2 \Delta^f G \;\geq\; S^{a-1} G^2 - c_{1.3} G^{\frac{3}{2}} - c_{1.3} G + 2\nabla G \cdot \nabla \varphi^2,$$

where $G := u\varphi^2$. Applying maximum principle to this inequality we see $G \leq c_{1.4}$, i.e. $|\mathcal{R}c| \leq c_{1.4}\mathcal{R}^a \leq c_{1.5}$ on $D(2R)\backslash D(R)$, for any $R > R_{1.2}$. Munteanu-Wang then applied the cut off function and maximum principle to the elliptic inequality (see (1.17) and (1.18) of [47])

$$\Delta_f(|\mathcal{R}m| + |\mathcal{R}c|^2) \geq |\mathcal{R}m|^2 - c_{1.6} \geq \frac{1}{2}(|\mathcal{R}m| + |\mathcal{R}c|^2)^2 - c_{1.7},$$

whence

$$\sup_{M\setminus D(R_{MW})} |\mathcal{R}m| + |\mathcal{R}c|^2 \;\le\; C_{MW}, \tag{4.67}$$

for some absolute constants $R_{MW} > 1000$ and $C_{MW} > 0$, depending only on $\bar{S}$. From this estimate, we notice (as pointed out in [47]), that under the assumption of uniform scalar curvature bound, the main concern of controlled geometry is about a bounded region $D(R_{MW})$ around the base point.

Global weighted L^2-curvature bound. By the non-degeneration of f outside $D(R)$ for any $R > R_{MW}$, we see that $D(R)$ is a smooth retraction of M, hence $\chi(M) = \chi(D(R))$ as the Euler characteristic is a homotopy invariant. Recall that for $\Sigma(R)$, the boundary Gauss-Bonnet-Chern term can be estimated as

$$\left|\mathcal{T}\mathcal{P}_\chi\right| \le \frac{1}{4\pi^2}\left(\frac{2\left|\det\nabla^2 f\right|}{|\nabla f|^3} + 3\frac{|\nabla^2 f|}{|\nabla f|}|\mathcal{R}m|\right),$$

since $|\nabla f| > 1$ and $|\mathcal{R}m| \le C_{MW}$ outside $D(R_{MW})$, we then have

$$\int_{D(R)} |\mathcal{R}m|^2 \,\mathrm{d}\mu_f \;\le\; \bar{E} + c_{2.0}\int_{\Sigma(R)} (|\nabla^2 f|^3 + |\nabla^2 f|)\,\mathrm{d}\sigma_{\Sigma(R)}. \tag{4.68}$$

The defining equation (1.4) then gives

$$\int_{\Sigma(R)} |\nabla^2 f|^3 + |\nabla^2 f|\,\mathrm{d}\sigma_{\Sigma(R)} \;\le\; c_{2.1}\int_{\Sigma(R)} |\mathcal{R}c|^3 + |\mathcal{R}c|\,\mathrm{d}\sigma_{\Sigma(R)} \;\le\; c_{2.2}Vol(\Sigma(R)).$$

On the other hand, (2.18) and Lemma 2.2.4 gives control

$$Vol(D(3R_{MW})) - Vol(D(2R_{MW})) \le c_{2.3}R_{MW}^2.$$

For some $R_2 \in [2R_{MW}, 3R_{MW}]$ such that $Vol(\Sigma(R_2)) = \min_{2R_{MW}\le R\le 3R_{MW}} Vol(\Sigma(R))$, we can apply the coarea formula and (2.19) to estimate

$$Vol(\Sigma(R_2)) \;\le\; \frac{1}{R_{MW}}\int_{D(3R_{MW})\setminus D(2R_{MW})} |\nabla f|\,\mathrm{d}V \;\le\; c_{2.4}R_{MW}^{\frac{3}{2}}.$$

These inequalities together give:

$$\begin{aligned}
\int_M |\mathcal{R}m|^2\,\mathrm{d}\mu_f \;&\le\; \int_{D(R_2)} |\mathcal{R}m|^2\,\mathrm{d}V + C_{MW}^2\int_{M\setminus D(R_2)} 1\,\mathrm{d}\mu_f \\
&\le\; \chi(D(R_2)) + c_{2.2}Vol(\Sigma(R_2)) + c_{2.3}\sum_{k=1}^{\infty} e^{-2^k R_{MW}}8^k R_{MW}^2 \\
&\le\; \chi(M) + c_{2.5}R_{MW}^{\frac{3}{2}} + c_{2.6} \\
&\le\; C(\bar{E},\bar{S}),
\end{aligned}$$

since all the constants involved are solely determined by $\bar{E}$ and $\bar{S}$. Here we recall that $\bar{E} > 0$ and $\bar{S} > 0$ are the prescribed upper bounds of the Euler characteristics (in absolute value) and the scalar curvature, respectively.

With this bound at one hand, we can apply Proposition 4.5.1 to $\{D_i(2R_{MW}) \subset M_i\}$ and obtain a convergent subsequence, to some metric space $(X_\infty(2R_{MW}), d_\infty)$ with marked points $\{x_\infty^1, \cdots, x_\infty^J\}$ and $J \leq J(2R_{MW})$. On the other hand, we have a uniform curvature bound outside $D_i(2R_{MW})$, whence a non-compact length space (X_∞, d_∞) as the Gromov-Hausdorff limit. The convergence will preserve the finitely many marked points, and away from these points, the Gromov-Hausdorff convergence is improved, by the locally uniform curvature bound, to strong multi-pointed Gromov-Hausdorff convergence in the sense of Definition 2.5.3. □

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