The Vanishing of the First Chern Class for Simply Connected Complex Surfaces Is a Quasiconformal Invariant

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Abstract of the Dissertation

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We prove that the vanishing of the first chern class for simply connected complex surfaces is a quasiconformal invariant, modulo some discussion. That is, a complex surface quasiconformally homeomorphic to a K3 surface is also a K3 surface. We prove this theorem using Donaldson and Sullivan's extension of Donaldson's theory of smooth 4-manifolds to the quasiconformal context, as well as Friedman and Morgan's explicit calculations of the Donaldson polynomial invariants for elliptic surfaces. It is a priori not possible to prove this result using Seiberg-Witten theory, due to the latter's deep reliance on the underlying smooth structure.

Dedication

This thesis is dedicated to my fiancée Claire, and to my parents Jenny and Charles. Thank you for all your love and support.

Contents

Chapter 1

1 The Vanishing of the First Chern Class for Simply Connected Complex Surfaces Is a Quasiconformal Invariant

1.1 Introduction

A manifold is *quasiconformal* if it is equipped with charts whose overlap maps have bounded infinitesimal distortion. In dimensions $\leqslant 3$, every topological manifold has a smooth structure unique up to diffeomorphism, but in dimensions $\geqslant 4$, the topological and smooth categories diverge. The quasiconformal category fits in between the two: in dimension $\geqslant 5$ (and indeed for dimensions $\neq 4$,) Sullivan proved in [Sul79] that every topological manifold has a quasiconformal structure unique up to quasiconformal homeomorphism. However, in dimension 4, the topological and quasiconformal categories diverge: Donaldson and Sullivan proved in [DS89] that there exist topological 4-manifolds which do not admit a quasiconformal structure, and that there exist quasiconformal structures on the same topological 4-manifold which are not quasiconformally homeomorphic.

This behavior of quasiconformal 4-manifolds mimics the behavior of smooth manifolds in dimension 4. Indeed, the tools necessary to study smooth 4-manifolds are mostly available for quasiconformal manifolds: a quasiconformal 4-manifold admits a bounded conformal structure, a theory of connections and curvature, and a differential graded Banach algebra of differential forms whose 0-forms are (just) beyond the Sobolev borderline; thus, Yang-Mills theory is possible on quasiconformal 4-manifolds. In fact, the Donaldson polynomials are quasiconformal invariants [DS89]. One might conjecture that just as in dimensions $\leqslant 3$, the quasiconformal and smooth categories agree in dimension 4. It is currently unknown if every quasiconformal 4-manifold is smoothable, however every known quasiconformal 4-manifold is already smooth. Moreover, it is unknown whether there can exist two non-diffeomorphic smooth structures on the same quasiconformal 4-manifold. In this thesis, we shall prove a partial converse to this last statement: any complex surface quasiconformally homeomorphic to a K3 surface is a K3 surface. In particular, since the K3 surfaces are diffeomorphic, any complex surface quasiconformally homeomorphic to a K3 surface X is also diffeomorphic to X. Note there exists a smooth 4-manifold homeomorphic to a K3 surface which is not diffeomorphic to any complex surface [HK93], so our theorem does not imply that the quasiconformal manifold underlying the $K3$ surfaces has a unique smooth structure.

1.2 Definitions and Background

1.2.1 Quasiconformal Structures

Let $U \subset \mathbb{R}^n$ be open and $K \geqslant 1$. A function $f : U \to \phi(U) \subset \mathbb{R}^n$ is K-*quasiconformal* if

$$\limsup_{r \to 0} \frac{\sup\{|f(x) - f(y)| : |x - y| = r\}}{\inf\{|f(x) - f(y)| : |x - y| = r\}} \leqslant K$$

for all $x \in U$. We say that f is *quasiconformal* if there exists $K \geqslant 1$ such that f is K-quasiconformal. Note that like Lipschitz and Hölder conditions, the property of being quasiconformal is only dependent on the underlying metric structure, so the definition is equally valid for an abstract metric space.

If f is a diffeomorphism then $f\lfloor_W$ is quasiconformal for every open set W compactly contained in U. Likewise, a bilipschitz homeomorphism is quasiconformal.

An n-dimensional topological manifold X is *quasiconformal* if there exists an atlas $\{(U_i, \phi_i) : i \in I\}$ whose overlap maps $\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j) \subset \mathbb{R}^n$ are K-quasiconformal homeomorphisms for some fixed $K \geqslant 1$. A homeomorphism $f : X \to Y$ between quasiconformal manifolds is *quasiconformal* if it is K'-quasiconformal on charts, for some fixed $K' \geqslant 1$.

It is not hard to show that a smooth manifold, compact or otherwise, has a compatible quasiconformal structure, and the constant K can be chosen arbitrarily close to 1.

1.2.2 Invariants of Complex Surfaces

If X is a compact complex surface, let $K_X = \Lambda^2(\Omega^{1,0}X)$ denote the *canonical bundle of* X, the second exterior power of the holomorphic cotangent bundle. The bundle K_X is a holomorphic line bundle, and locally its sections are of the form $f(z, w)dzdw$. The *canonical class* $[K_X] \in H^2(X; \mathbb{Z})$ *of* X is defined to be the first chern class of K_X. Using the splitting principle, if $T^{1,0}X \sim \mathcal{L}_1 \oplus \mathcal{L}_2$, Then $K_X = \Lambda^2(\Omega^{1,0}X) \sim \mathcal{L}_1^* \otimes \mathcal{L}_2^*$, and we see that $[K_X] = c_1(\mathcal{L}_1^*) + c_1(\mathcal{L}_2^*) = -c_1(\mathcal{L}_1) - c_1(\mathcal{L}_2) = -c_1(T^{1,0}X) = -c_1(X)$.

For $n \geqslant 1$, let $P_n(X) = \dim_{\mathbb{C}}(H^0(X, K_X^{\otimes n}))$ denote the *n-th plurigenus of* X, and let $p_g(X) = P_1(X)$ denote the *geometric genus of* X. Let $\kappa(X) \in \{-\infty, 0, 1, 2\}$ denote the *Kodaira dimension of* X, which is defined to be $-\infty$ if $P_n(X) = 0$ for all $n \geqslant 1$, otherwise

$$\kappa(X) = \min\{\alpha \in \mathbb{Z} : P_n(X)/n^\alpha \text{ is a bounded function of } n\}.$$

Since complex surfaces carry a natural orientation, Poincaré duality yields a symmetric bilinear form

$$q_X : H_2(X,\mathbb{Z}) \times H_2(X,\mathbb{Z}) \to \mathbb{Z},$$

the *intersection form of* X. We also have the *betti numbers* $b_i = b_i(X) = \dim H^i(X,\mathbb{R})$, the euler characteristic $\chi(X) = \sum(-1)^i b_i$, as well as b_2^+ and b_2^-, the dimensions of any maximal positive and negative subspaces for the (real) intersection form. In particular, $b_2 = b_2^+ + b_2^-$. The *signature* $\sigma(X)$ *of* X is defined to be the quantity $b_2^+ - b_2^-$.

The geometric genus, intersection form, betti numbers, signature, and $[K_X]^2$ (the self-intersection number of the Poincaré dual of the canonical class) are all oriented homotopy invariants, but the higher plurigenera (and hence Kodaira dimension) are only deformation invariants. Moreover, under an orientation reversing homotopy equivalence, the intersection form changes sign, so b_2^+ and b_2^- trade places, and σ changes sign. If b_1 is even (e.g. if X is simply connected,) we also have the relation $b_2^+ = 2p_g + 1$.

Finally, we need the (stable, restricted) *Donaldson polynomial invariant*, which assigns to every simply connected compact complex surface X with $b_2^+ \geqslant 3$, every integer $c \geqslant (3b_2^+ + 5)/4$ and every orientation β of $H^2_+(X;\mathbb{R})$ a homomorphism $\gamma_c(X,\beta) : \mathrm{Sym}^{d(c)}(H_2(X;\mathbb{Z})) \to \mathbb{Z}$, where $d(c) = 4c - \frac{3}{2}(b_2^+ + 1)$. We can also think of $\gamma_c(X,\beta)$ as an element of $\mathrm{Sym}^{d(c)}(H^2(X;\mathbb{Q}))$ or as a homogenous polynomial of degree $d(c)$ on $H_2(X;\mathbb{Q})$. If $f : Y \to X$ is a quasiconformal homeomorphism, then $\gamma_c(Y, f^*\beta) = f^*\gamma_c(X,\beta)$ [DS89] (see also below.) Also, we have $\gamma_c(X,-\beta) = -\gamma_c(X,\beta)$.

1.2.3 Quasiconformal Invariance of the Donaldson Polynomial Invariants

Here we collect a few key results from [DS89], and give some auxiliary statements sufficient to prove the invariance of the Donaldson polynomials under quasiconformal homeomorphisms between smooth 4-manifolds.

First, there exists for every quasiconformal 4-manifold X a sheaf of differential graded Banach algebras

$$\hat{B}^0_{\mathrm{loc}} \to \hat{B}^1_{\mathrm{loc}} \to \hat{B}^2_{\mathrm{loc}} \to \hat{B}^3_{\mathrm{loc}} \to \hat{B}^4_{\mathrm{loc}},$$

consisting of differential forms, with the following good properties:

(a) The Banach spaces $\hat{B}^i_{\mathrm{loc}}$ are separable, reflexive, with separable dual for each $i = 0,\ldots,4$;

(b) If X is compact, there is a compact inclusion $\hat{B}^0_{\mathrm{loc}} \hookrightarrow C^0(X)$;

(c) The cohomology of the complex $(\hat{B}^*_{\text{loc}}, d)$ is naturally isomorphic to the singular cohomology of X;

(d) A quasiconformal homeomorphism $X \to Y$ between quasiconformal 4-manifolds induces an isomorphism of sheaves $\hat{B}^*_{\text{loc}}(Y) \to \hat{B}^*_{\text{loc}}(X)$.

If E is a $\hat{B}^0_{\text{loc}}$ vector bundle over X, we also have a sheaf of differential graded Banach algebras

$$\hat{B}^k_{\text{loc}}(E) := \hat{B}^k_{\text{loc}} \otimes_{\hat{B}^0_{\text{loc}}} \Gamma(E)$$

consisting of differential forms with values in E. Finally, we also have, given a bounded conformal structure μ on a compact quasiconformal 4-manifold X, the operator $d^+ : \hat{B}^1(E) \to \hat{L}^2(\Omega^+_X(E))$ consisting of exterior d followed by L^2 orthogonal projection onto the self-dual 2-forms with values in E.

In particular, we have a space $\mathcal{A}$ of $\hat{B}$ connections A on E, an affine space modeled on the Banach space $\hat{B}^1(\mathfrak{g}_E)$. The curvature F_A of A lies in $\hat{L}^2(\mathfrak{g}_E)$ and can be decomposed into its self dual and anti-self-dual components. There is also the $\hat{B}^0$ gauge group $\mathcal{G}$, a Banach Lie group with smooth action $\mathcal{G} \times \mathcal{A} \to \mathcal{A}$, and its orbit space $\mathcal{B}$. Finally, there is the moduli space $M \subset \mathcal{B}$ of equivalence classes of anti-self-dual connections.

The moduli space has the following local structure, given by [DS89] Proposition 4.16: If $A \in \mathcal{A}$ is an anti-self-dual connection, there exists a neighborhood of $[A]$ in M given by $\psi^{-1}(0)/\Gamma_A$, where ψ is a smooth, Γ_A-equivariant, Fredholm map from a neighborhood of 0 in a subspace $T_A \subset T\mathcal{A}_A$ transverse to $\operatorname{Im} d_A$ to $\hat{L}^2(\Omega^+(\mathfrak{g}_E))$, with Fredholm index $i(\mathfrak{g}_E) - \dim \operatorname{Ker} d_A$, where $i(\mathfrak{g}_E) = (8c - 3)(b_2^+ + 1)$ for a $SU(2)$ bundle E and simply connected X.

The compactification and orientation of M have likewise been set up in [DS89], via Uhlenbeck's theorems on the existence of local Coloumb gauges and removable singularities.

In the smooth case, one can appeal to Uhlenbeck's generic metrics theorem to show that for generic smooth metrics, the dimension of M is exactly $i(\mathfrak{g}_E)$. If the underlying manifold is only equipped with a quasiconformal structure, the set of smooth metrics may be empty, so this theorem cannot be brought forward without modification. However, we are only concerned with quasiconformal invariance of smooth manifolds, so we may assume this set is non-empty. There are some additional considerations to deal with if the proof is to be modified for our setting, chiefly the Sobolev space of forms $\Omega^+(\mathfrak{g}_X)$ employed is a Hilbert space and the inner product is explicitly used. However, this may not actually be essential to the proof.

Alternatively, the method suggested in [DS89], of perturbing the ASD equations for each moduli space in a coherent manner compatible with convergence, might instead be used.

As a third alternative, one might show that the moduli space M obtained above is isomorphic to the standard one obtained via the smooth theory, given the inclusion of an appropriate Sobolev space of connections into $\hat{B}^1(\mathfrak{g}_E)$. This, however, is somewhat unsatisfactory, as we will need a more robust transversality statment later, namely [DS89] Proposition 7.6.

In any case, this is a gap which we shall assume for the remainder.

With transversality of the ASD equations in hand, one may proceed with the steps in [DS89] §7.4 to produce the intersections

$$M(g) \cap V_1 \cap \cdots \cap V_d.$$

There is a small gap here as well, a final step necessary to prove the quasiconformal invariance of the intersection number of

$$M(g) \cap V_1 \cap \cdots \cap V_d,$$

and that is a strengthened version of [DS89] Proposition 7.6, which is only proved in the case of a zero-dimensional moduli space, but which we need to hold in general: Given a sequence of bounded conformal structures $[g^{(i)}] \to [g]$, then if the above intersection is transverse, then there exists a $\mathcal{B}^*$ neighborhood of each point $[A]$ in $M[g] \cap V_1 \cap \cdots \cap V_d$ which contains exactly one point of $M[g^{(i)}] \cap V_1 \cap \cdots \cap V_d$ for i large enough.

1.2.4 K3 Surfaces

By a *K3 surface*, we shall mean a simply connected compact complex surface with trivial canonical bundle (or equivalently vanishing canonical class[1].) All K3 surfaces are diffeomorphic (in fact deformation equivalent, due to Kodaira) and any complex surface diffeomorphic to a K3 surface is also a K3 surface (due to Friedman and Morgan.)

If X is a K3 surface, then $\kappa(X) = 0$, $p_g(X) = 1$, $\chi(X) = 24$, $b_2^+(X) = 3$, $b_2^-(X) = 19$, $\sigma(X) = -16$ and the intersection form q_X is given by

$$3 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 2(-E_8),$$

[1]Since we are assuming the surface is simply connected, there is no ambiguity about whether "trivial" here means holomorphically trivial or only topologically trivial; in this case the two are one in the same. Indeed, the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$ of sheaves yields a connecting homomorphism $\partial : H^1(X; \mathcal{O}_X^*) \to H^2(X; \mathbb{Z})$ on cohomology, which is injective if X is simply connected. The domain of ∂ parameterizes the holomorphic line bundles over X, whose first chern class is the image under ∂ of the given line bundle.

where E_8 is the even positive definite form given by the matrix

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Since the intersection form is even, $K3$ surfaces are spin. In particular, they are *minimal* in the sense that there are no embedded 2-spheres of self-intersection -1.

1.2.5 Elliptic Surfaces

An *elliptic surface* is a pair (X, π) consisting of a compact complex surface X and a surjective holomorphic map $\pi : X \to C$ where C is a smooth curve, such that for the regular values $c \in C$ of π, the fiber $\pi^{-1}(C)$ is a smooth curve of genus 1. We say a complex surface X has an *elliptic structure* if there exists such a map π, called an *elliptic fibration*, such that the pair (X, π) is an elliptic surface. The inverse image of a critical value of π is called a *singular fiber*. Such a singular fiber F is a *multiple fiber* if there exists an integer $m > 1$ such that $\pi^*(p) = mF$ as a divisor.

We will need several general facts about elliptic surfaces, namely:

Lemma 1 (Theorem 2.3 Chapter II [FM94]). *A minimal elliptic surface (X, π) is simply connected if and only if the base C is $\mathbb{C}P^1$ and there are at most two multiple fibers whose multiplicities are relatively prime.*

Lemma 2 (Theorem 2.1 Chapter 1, Theorem 7.6 Chapter I). *Let (X, π) and (Y, π') be minimal simply connected elliptic surfaces and let (m_1, m_2) (resp. (m_1', m_2')) denote the multiplicities of the multiple fibers of X (resp. Y,) listed in ascending order and equal to 1 in the cases of zero and one multiple fiber. The pairs (m_1, m_2) and (m_1', m_2') are equal and $\chi(X) = \chi(Y)$ if and only if X and Y are deformation equivalent (and therefore diffeomorphic.)*

Lemma 3. *There exists a K3 surface that admits an elliptic structure. Moreover, there are no multiple fibers.*

Proof. Consider the quotient of the product of two elliptic curves $C_1 \times C_2$ by the Kummer involution $x \mapsto -x$. The surface X that results from resolving the 16 singular points (which locally look like cones on $\mathbb{R}P^3$) is a K3 surface and has an elliptic structure, since the map $\pi_i : X \to C_i/\sim \simeq \mathbb{P}^1$ induced by projection onto one of the factors C_i is an elliptic fibration. □

As part of the classification of compact complex surfaces, Kodaira showed:

Lemma 4. *Every minimal complex surface of Kodaira dimension 1 has an elliptic structure.*

Finally, if (X, π) is a minimal simply connected elliptic surface with multiple fibers of relatively prime multiplicity m_1 and m_2, there exists by (2.9) Chapter II [FM94] a unique primitive integral class $\kappa_X \in H^2(X;\mathbb{Q})$ such that $[f] = m_1 m_2 \kappa_X$, where $[f]$ is the class dual to a general fiber. We shall need a calculation of the Donaldson polynomial $\gamma_c \in \mathrm{Sym}^*(H^2(X;\mathbb{Q}))$ in terms of the intersection form $q_X \in \mathrm{Sym}^2(H^2(X;\mathbb{Q}))$ and the class $\kappa_X \in \mathrm{Sym}^1(H^2(X;\mathbb{Q})) \simeq H^2(X;\mathbb{Q})$.

Theorem 5 (Theorem 2.1 Chapter VII [FM94])**.** *Suppose (X, π) is a minimal simply connected elliptic surface with $b_2^+(X) \geqslant 3$. Let $c \geqslant (3b_2^+ + 5)/4$ be an integer, let $d(c) = 4c - \frac{3}{2}(b_2^+ + 1)$ and $n(c) = 2c - b_2^+$. If β is an orientation of $H_+^2(X;\mathbb{R})$, then*

$$\gamma_c(X, \beta) = \pm \sum_{i=0}^{n} a_i q_X^i \kappa_X^{d-2i},$$

where

$$a_n = \frac{d!}{2^n n!}(m_1 m_2)^{p_g}.$$

Moreover, the coefficients a_i are unique[2]*.*

1.3 Main Theorem

We now state our main result:

Theorem 6. *Suppose X is a K3 surface and Y is a complex surface quasiconformally homeomorphic to X. Then Y is also a K3 surface.*

Note that the above quasiconformal homeomorphism between X and Y may be orientation reversing, and that as a corollary of the theorem, the surfaces X and Y are in fact diffeomorphic. It is unknown whether the given quasiconformal homeomorphism can be smoothed via a small isotopy into a diffeomorphism; all that we can say is that a diffeomorphism between the surfaces exists.

[2]This last statement is not explicitly included in Theorem 2.1 Chapter VII [FM94], but can be found as a corollary of Lemma 2.6 Chapter VI [FM94].

1.3.1 Proof of Main Theorem

Suppose X is a K3 surface and that Y is a compact complex surface homotopy equivalent to X. Note that since X is spin and the intersection forms of X and Y are isomorphic, Y is also spin, hence minimal. We first rule out the possibility that the Kodaira dimension of Y is $-\infty$ or 2:

Since $b_2^+(Y) = 2p_g(Y) + 1$, we know that if the homotopy equivalence preserves orientation, then $p_g(Y) = 1$ and if the homotopy equivalence reverses orientation, then $p_g(Y) = 9$. In either case, $\kappa(Y) \neq -\infty$, since $p_g = 0$ for such surfaces.

Now suppose $\kappa(Y) = 2$, i.e. that Y is a surface of general type. The Bogomolov-Miyaoka-Yau inequality implies that surfaces of general type must obey the inequality

$$\sigma(Y) \leqslant \frac{\chi(Y)}{3}.$$

Since $\chi(Y) = \chi(X) = 24$, we must have $\sigma(Y) \leqslant 8$. If the homotopy equivalence reverses orientation, then $\sigma(Y) = 16$, a contradiction. Thus, Y has the same oriented homotopy type as X, and so $[K_Y]^2 = [K_X]^2 = 0$. However, minimal surfaces of general type must obey the inequality $[K_Y]^2 \geqslant 1$, a contradiction.

If $\kappa(Y) = 0$ then Y is a $K3$ surface, since any simply connected surface of Kodaira dimension 0 is a $K3$ surface by the Enriques-Kodaira classification.

Thus by Lemmas 3 and 4, to prove the main theorem it suffices to show the following:

Theorem 7. *Suppose X is an elliptic $K3$ surface and that Y is an elliptic surface quasi-conformally homeomorphic to X. Then Y is also a $K3$ surface.*

Let $f : Y \to X$ be a quasiconformal homeomorphism. We may assume f is orientation preserving, since $\sigma(Y) \leqslant \frac{-2}{3}\chi(Y) \leqslant 0$ (Lemma 2.4 Ch I [FM94]) and $\sigma(X) = -16$. Since p_g is an oriented homotopy invariant, it follows that $p_g(Y) = p_g(X) = 1$.

By Lemmas 2 and 3, it suffices to show that Y has no multiple fibers. Let $c \geqslant 4$ and $n = 2c - 3$. By Theorem 5 we have:

$$\gamma_c(Y, f^*\beta) = \sum_{i=0}^{n} b_i q_Y^i \kappa_Y^{2n-2i}, \tag{1}$$

where $b_n = \frac{(2n)!}{2^n n!}(m_1 m_2)$ and β is chosen so that the overall choice of sign is positive. On the other hand,

$$\gamma_c(Y, f^*\beta) = f^*\gamma_c(X, \beta) = \pm \sum_{i=0}^{n} a_i q_Y^i (f^*\kappa_X)^{2n-2i}, \tag{2}$$

where $a_n = \frac{(2n)!}{2^n n!}$, noting that $q_Y = f^* q_X$. If $\kappa_Y = \pm f^* \kappa X$, then we may equate coefficients, and in particular $b_n = \pm a_n$, hence $m_1 m_2 = 1$, hence $m_1 = m_2 = 1$ and we are done.

On the other hand, suppose $\kappa_Y \neq \pm f^* \kappa_X$. Since κ_Y and $f^*\kappa_X$ are primitive integral classes, they are linearly independent as elements of $H^2(Y; \mathbb{C}) = H_2(Y; \mathbb{C})^*$.

In this case, we can write $\gamma_c(Y, f^*\beta)$ in terms of q_Y only[3]. In other words, the coefficients b_i for $i < n$ are identically zero and $b_n = \pm a_n$ as before and we are done. Indeed, this follows from the following linear algebra lemma, by extending γ_c complex-multilinearly to an element of $\mathrm{Sym}^{2n}(H^2(Y; \mathbb{C}))$ or equivalently a complex-valued polynomial on $H_2(Y; \mathbb{C})$, and setting $V = H_2(Y; \mathbb{C})$ and $q = q_Y$:

Lemma 8 (Lemma 2.5, Chapter VI [FM94])**.** *If V is a complex vector space of dimension $3 \leqslant$ dim $V < \infty$ and q is a non-degenerate quadratic form, and $k, k' \in V^*$ are linearly independent, then* $\mathbb{C}[q] = \mathbb{C}[q, k] \cap \mathbb{C}[q, k']$.

[3] This is in fact the case: the Donaldson polynomial for a K3 surface X is $\gamma_c = \frac{(2n)!}{2^n n!} q_X^n$ for $c \geqslant 4$ and $n = 2c - 3$ [FM94].

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