Nonexistence of Wandering Domains for Infinitely Renormalizable Hénon Maps

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Abstract of the Dissertation

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In the thesis, I proved the absence of wandering domains for strongly dissipative infinitely renormalizable Hénon-like maps with arbitrary stationary combinatorics. The theorem solves an open problem proposed by van Strien (2010) [vS10] and Lyubich and Martens (2011) [LM11], and opens a direction of studying the existence of wandering domains in higher-dimensional systems. Unimodal maps are a reduced version of Hénon-like maps in one-dimension and unimodal maps do not have wandering intervals. However, the classical proofs for unimodal maps break down in the Hénon setting. To resolve this issue, two higher-dimensional techniques, "the area argument" and "the good region and the bad region", are introduced in the thesis to prove the theorem.

The proof is split into two cases. The first case covers infinitely period-doubling renormalizable Hénon-like maps. The second case covers infinitely renormalizable Hénon-like maps with stationary combinatorics other than period-doubling. The difference between the proofs of the two cases comes from the way of how we measure the expansion of a set when it is iterated under a Hénon-like map. The prior case relies on the Euclidean metric, and the later case relies on the hyperbolic metric. The two cases are disjoint, and together solve the problem for all stationary combinatorics.

As an application, the theorem enriches our understanding about the topological structure of the heteroclinic web: the union of the stable manifolds forms a dense set in the domain.

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1. Introduction

Dynamical systems is a field in mathematics that studies how a mathematical model evolves over time. The state of a system at a certain moment is described by an element in a set which is called the phase space. We assume that the state at the next moment only depends on the state at the current moment. Thus, the evolution is determined by a map (for discrete-time) or a flow (for continuous-time).

A simplest type of discrete-time dynamical systems is the iteration of a linear map over the phase space $\mathbb{R}^n$ or $\mathbb{T}^n$. The n-tours $\mathbb{T}^n$ is identified as a quotient of $\mathbb{R}^n$ over the group of integral vectors. The dynamics of a linear map on $\mathbb{R}^n$ is simple. The system has only one periodic orbit at the origin which is a fixed point. The behavior of other points depends on the eigenvalues and eigenvectors. An eigenvalue λ is called attracting if $|\lambda| < 1$, indifferent if $|\lambda| = 1$, and repelling if $|\lambda| > 1$. Hyperbolic systems are the linear maps that do not have indifferent eigenvalues. The dynamical behavior of a hyperbolic system is completely characterized by two subspaces: the stable manifold and the unstable manifold. The stable manifold is the direct sum of all generalized eigenspaces associated with an attracting eigenvalue. The orbit of a point on the hyperplane moves toward to the fixed point. The unstable manifold is the direct sum of all generalized eigenspaces associated with a repelling eigenvalues. The orbit of a point on the hyperplane moves away from the fixed point. For non-hyperbolic systems, the dynamics on a subspace formed by the indifferent eigenvectors can still be understood by the structure of the associated Jordan blocks.

The dynamics of a linear map on $\mathbb{T}^n$ is similar to the case of $\mathbb{R}^n$. The local dynamics can still be understood by the eigenspaces. However, the dynamics becomes much more complicated in the global picture. In the case of hyperbolic systems, the stable and unstable manifolds of the origin may have more than one transversal intersections. These are called traversal homoclinic points. Traversal homoclinic points always come with a Smale's horseshoe structure which implies complexity of the dynamical behavior [Sma67]: there is an invariant Cantor set that is conjugated to the two-sided shift map over two alphabets.

For a nonlinear system, a fixed point is hyperbolic if the derivative at the point does not have an indifferent eigenvalue. The local behavior of the map around a hyperbolic fixed point (or periodic orbit) can be fully analyzed by linearizing the map. Hence, the behavior of the map around the point persists under small perturbation. This is called structurally stable. On the other hand, the local behavior around an indifferent fixed point cannot be understood by linear theory, and a small perturbation of the system gives birth to new periodic orbits. A famous example is the period-doubling bifurcation of the family of logistic maps in one-dimensional dynamics. Moreover, the global behavior of the map becomes much more complicated. A hyperbolic fixed point also has a stable manifold and an unstable manifold, but they are no longer hyperplanes as they become submanifolds of the domain. The manifolds are transversal at the fixed point, but they may have a tangential intersection away from the fixed point. A tangential homoclinic point is a tangential intersection of a stable manifold and an unstable manifold of the same fixed point (or periodic orbit); a tangential heteroclinic point is a tangential intersection of a stable manifold and an unstable manifold that come from different fixed points (or periodic orbit). A bifurcation of homoclinic point occurs when perturbing the system around a tangential homoclinic point. This gives a creation (or annihilation) of horseshoes near the homoclinic point and hence the dynamical behavior changes under a small perturbation. Both of the two issues have the same characteristic: the tangent space cannot be decomposed as a direct sum of an expanding subspace and a contracting subspace. This is called nonhyperbolic. Nonhyperbolic systems are not structurally stable. The dynamical behavior of a nonhyperbolic system does not persist under a small perturbation. To understand the dynamics of a system when tangencies occur, Hénon maps serve as a simple model in higher-dimensions for realizing the homoclinic and heteroclinic tangencies.

A Hénon-like map $F : D \subset \mathbb{R}^2 \to \mathbb{R}^2$ is a discrete-time dynamical system on a two-dimensional phase space D. It is an analytic map of the form

$$F(x,y) = (f(x) - \varepsilon(x,y), x) \tag{1.1}$$

where f is a unimodal map. A unimodal map is a differentiable map on the real line that has exactly one turning point in the interior of its domain. A Hénon-like map is dissipative if it is area contracting, i.e. the determinant of the Jacobian $|\det DF|$ is strictly lesser than one. Strongly dissipative means that the Jacobian is small. Figure 1.1 illustrates the image of a strongly dissipative Hénon-like map. These maps are a generalization of the classical Hénon family [Hén76]

$$F_{a,b}(x,y) = (1 - ax^2 - by, x) \tag{1.2}$$

to the analytic setting, and an extension of unimodal maps from one- to two-dimensions.

In this thesis, we will study the dynamical behavior of Hénon-like maps by using a dynamical object that is called wandering domain and establish the theorem:

Theorem (Main Theorem)**.** *A strongly dissipative infinitely renormalizable Hénon-like map of arbitrary stationary combinatorics does not have a wandering domain.*

The definition of wandering domain may defer on the system aiming to study. Here, we define a wandering domain to be a nonempty connected open subset that is disjoint from the stable manifolds of all periodic points. This is equivalent to the classical definition for unimodal maps (Remark 7.2). Renormalizable maps are the maps that exhibit similar dynamical behavior in different scales which will be discussed later.

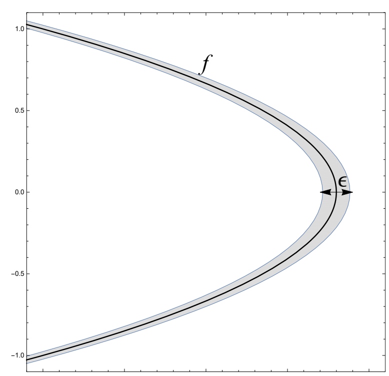


Figure 1.1.: The image of a Hénon map.

It is known that unimodal maps do not have wandering intervals [Guc79, dMvS88, dMvS89, Lyu89, BL89, MdMvS92]. The theorem shows that this property can be extended to higher-dimensions and solves an open problem proposed by van Strien [vS10] and Lyubich and Martens [LM11]. As an application, the theorem enriches our understanding of the topological structure of the heteroclinic web: the union of the stable manifolds forms a dense set in the domain.

The research on wandering domain in different systems

The problem of the existence of wandering domains has a broad interest in the field of dynamics. Points in a wandering domain have similar dynamical behavior and have orbits wander around. Excluding these kinds of domain is a popular research topic and has different aspects in different systems.

Real one-dimensional systems. In real one-dimensional systems, the absence of wandering intervals helps to solve the classification problem. Two discrete-time dynamical systems $f : X \to X$ and $g : Y \to Y$ are said to be topological semi-conjugate if there exists a continuous surjective map $h : X \to Y$ such that $h \circ f = g \circ h$. The map h is called a semi-conjugacy. The two maps are topological conjugate if h is a homeomorphism. The conjugacy h acts as a change of coordinate. In this context, a wandering interval is a nonempty open interval that has a disjoint orbit and the orbit does not approach a periodic orbit.

One example is the class of circle homeomorphisms. We consider the circle $\mathbb{S}^1$ as the quotient of the real numbers over the group of integers $\mathbb{R}/\mathbb{Z}$. The study of the dynamics of circle homeomorphism started from Poincaré [Poi86]. He classified the orientation-preserving circle homeomorphisms by a quantity that is called the rotation number $\rho(f) \in \mathbb{S}^1$: the average translation of an orbit. The quantity is independent of the choice of orbit. A rigid rotation is the special case when the amount of translation is a fixed value everywhere. He showed that

Theorem. *For any orientation-preserving circle homeomorphism with irrational rotation number, the map is topological semi-conjugated to the rigid rotation that has the same rotation number. In addition, the maps are topological conjugate if the circle homeomorphism has no wandering interval.*

Wandering intervals are exactly the gaps consisting the points that are sent to a single value by the semi-conjugacy. In terms of the long-time behavior of the system, the absence of wandering intervals is equivalent of saying that any orbit is dense in the circle. Therefore, studying the existence of wandering intervals turns out to be an important subject.

Denjoy [Den32] strengthened the theorem of Poincaré by giving the conditions for the absence of wandering intervals:

Theorem. *Assume that the map $f : S^1 \to S^1$ is an orientation-preserving C^1 diffeomorphism with irrational rotation number. If $\log(f')$ has bounded variation, then the map has no wandering interval.*

Schwartz [Sch63] gave a different proof by assuming that $\log(f')$ is Lipschitz. There are examples showing that the regularity conditions are essential. Bohl [Boh16], Kneser [Kne24], and Denjoy [Den32] produced counterexamples of C^1 circle diffeomorphisms of arbitrary irrational rotation number having wandering intervals. Herman [Her79] improved the result by constructing counterexamples of $C^{1+\alpha}$ diffeomorphisms (with the derivative having α-Hölder continuity) for any $\alpha < 1$. In addition, the techniques developed by Denjoy [Den32] and Schwartz [Sch63] cannot handle critical circle maps. Hall [Hal81] found counterexamples of C^∞ homeomorphisms with at most two critical points. On the other hand, the conditions are not sharp. The paper of Hu and Sullivan [HS97] improved the Denjoy's theorem by assuming that $\log(f')$ have bounded Zygmund variation and bounded quadratic variation. In contrast to the work of Hall, the paper by Yoccoz [Yoc84] extended the Denjoy's theorem to critical circle homeomorphisms: allowing the map to have critical points but restricting the regularity of the map to be analytic.

Another example is the class of unimodal maps. A unimodal map is a continuous endomorphism on a compact interval that has a unique turning point in the interior of the interval. We assume that the turning point is the maximal point of the map. The dynamics of a unimodal map is determined by the kneading sequence [MT88] which encodes the combinatorics of the critical orbit.

Guckenheimer [Guc79] developed an analog of the Poincaré theory for unimodal maps: if two unimodal maps have the same kneading sequence which is not periodic and both of them have no wandering interval, then the two maps are topological conjugate. In particular, he showed that C^3 unimodal maps with negative Schwarzian derivative do not have wandering intervals. Only C^3 regularity is not enough. Some additional conditions are required in order to take care of the critical points. Similar to the examples of circle homeomorphisms, there are C^∞ unimodal maps with a flat critical point exhibiting wandering intervals [SI83, dM87]. Some other papers relaxed the hypothesis or generalized the theorem to multimodal maps [dMvS88, dMvS89, Lyu89, BL89, MdMvS92], but all require some non-flatness around the critical point.

Complex one-dimensional systems. In complex one-dimension, the Fatou set and the Julia set are two complementary sets in the domain. The Fatou set contains points that have the same asymptotic behavior on their neighborhoods. The Julia set is the compliment of the Fatou set. It contains points that have wild behavior: the behavior of an orbit depends sensitively on the initial point.

A Fatou component is a connected component of the Fatou set. The domain is decomposed by the components into regions with different asymptotic behaviors. The dynamics of a periodic Fatou component is classified into several categories by the behavior of its first return map [Cre32, Fat]:

- Attracting: the orbits of the points approach an attracting periodic orbit which belongs in the interior of the component.
- Parabolic: the orbits of the points approach a parabolic periodic orbit which lies on the boundary of of the component.
- Siegel disk: the dynamics on the component is equivalent to a rigid rotation on a disc.
- Herman ring: the dynamics on the component is equivalent to a rigid rotation on an annulus.
- Baker domain: the orbits of the points approach an essential singular point which lies on the boundary of the component.

The remaining possibility is the case when a component is not eventually periodic. Those components are called wandering domains. Equivalently, a wandering domain is a Fatou component that has a disjoint orbit.

A rational map is a quotient of two polynomials defined on the Riemann sphere. The possible types of Fatou components are clear in the setting of rational maps. An attracting Fatou component occurs when there is an attracting periodic orbit. A parabolic Fatou component is observed around an indifferent periodic point with a multiplier that is a root of unity. Seigel [Sie42] showed that a holomorphic map is linearizable around an indifferent periodic point for some multipliers that are not a root of unity (irrational rotation), and its neighborhood forms a Siegel disk. Herman [Her79] found an example of a rational map with a Herman ring. A rational map does not have Baker domains. However, there are other meromorphic functions that have Baker domains. The first example with a Baker domain was given by Fatou [Fat26]. Baker domains are more complicated than other periodic Fatou components. For a meromorphic function with a finite number of poles, the dynamics of a Baker domain can be further classified into three types [Kön99]: a translation on the complex plane, a translation on the upper half plane, and a linear expansion on the upper half plane.

Sullivan [Sul85] completed the last puzzle of the classification of Fatou components for rational maps by excluding the possibility of having a wandering domain. In other words, all Fatou components are eventually periodic and belong to one of the first four categories. The main interest turns to study the existence of wandering domains for other meromorphic functions. Sullivan's theorem was extended to some larger classes of transcendental functions [GK86, EL92, BHK$^+$93, MBRG13, EL84, Kee88, Kot87, Bak84, DK89, Sta91, Ber93, BT96]. However, in general, there are entire functions that have wandering domains. Fatou [Fat] showed that if the map f has a wandering domain U, then all convergent subsequence of the iterates restricting to the wandering domain $\{f|_U^{n_j}\}_{j=1}^{\infty}$ has a constant limit. Thus, the wandering domains can be classified by the limiting function: a wandering domain U is called escaping if $\lim_{n\to\infty} f|_U^n = \infty$, called oscillating if the sequence of iterates have a convergent subsequence approaching to a finite number and a convergent subsequence approaching to infinity. There are many examples exhibiting an escaping wandering domain [Bak76, Her84, Bak84, Bak85, Dev89]. An oscillating wandering domain was first found by Eremenko and Lyubich [EL87]. A map with two orbits of oscillating wandering domains was constructed by Bishop [Bis15]. It is still an open problem whether if there is a wandering domain neither escaping nor oscillating [BH89, Problem 2.87].

Real higher-dimensional systems. In higher-dimensions, the classification problem becomes a delicate problem. Topological equivalence breaks down between any two different levels of differentiability. The work of Harrison [Har75, Har79] showed that for every d-manifold with $d \neq 1,4$ and integer $r \geq 0$, there exists a C^r diffeomorphism that is not topologically conjugate to any C^{r+1} diffeomorphism. The paper of Hazard, Martens, and Tresser [HMT18] studied the possible combinatorics behavior of a Hénon-like map with zero entropy. They showed that unlike the one-dimensional case, infinitely many parameters are needed to exhaust all the possible topological types.

Nevertheless, due to the successful of the Denjoy theory on the circle, there are attempts of generalizing the Denjoy theory to higher-dimensional systems. Bonatti , Gambaudo, Lion, and Tresser [BGLT94] studied infinitely renormalizable diffeomorphisms of the disk. They proved that a diffeomorphism has no wandering domain when the map is smooth enough (C^1 Hölder with bounded geometry), and proved the existence of a diffeomorphism consisting wandering domains when the regularity is not sufficient (C^1). Norton [Nor91] excluded the occurrence of some types of wandering domains for a C^3 diffeomorphism of a compact smooth 2-manifold. There are also some developments on a similar theory for higher-dimensional torus. Mc Swiggen [McS93, McS95] proved that for any dimension k and constant $\varepsilon > 0$, there exists a $C^{k+1-\varepsilon}$ diffeomorphism of the k-torus that acts like a rotation but has a wandering domain. He also conjectured that C^{k+1} might be the upper bound of the smoothness for having wandering domains. In the other direction, Norton and Sullivan [NS96] showed that a C^3 diffeomorphism on a 2-tours that acts like a rotation does not have circular wandering domains. And the results was extended to higher dimensional torus by Navas [Nav17]. However, the problem of existence of wandering domains is still unsolved.

Nonhyperbolic phenomena are used to construct examples exhibiting wandering domains [CV01, KS17, KNS17]. Hyperbolic systems are the maps that have uniformly controlled contraction and expansion. A relevant work by Kiriki and Soma [KS17] found examples of Hénon-like maps close to $F_{2,0}$ in (1.2) having wandering domains. The main theorem of the thesis does not overlap with their work. In the thesis, the Hénon-like maps are real analytic, and the maps are away from having a homoclinic tangency. However, in their article, the maps they found having wandering domains have only finite differentiability, and their construction relies on the existence of homoclinic tangency for some maps close to $F_{2,0}$ [KLS10, KS13].

In this thesis, we center on strongly dissipative infinitely renormalizable Hénon-like maps of arbitrary stationary combinatorics. The main theorem covers maps that are not hyperbolic. For the period-doubling combinatorics, the unique invariant measure of the Cantor set has a 0 characteristic exponent [dCLM05, Theorem 6.3]. Indeed, the main issue occurs when the expansion and contraction are out of control in the region called "the bad region". The solution is to rely on the "hyperbolicity" of area instead of the expansion or contraction of length. In particular, we show that the contraction of area is uniform bounded because the map has a universal shape around the tip [dCLM05, Theorem 7.9].

Complex higher-dimensional systems. In complex higher-dimensions, counterexamples in transcendental maps can be constructed from one-dimensional examples [FS98] by taking direct products. For polynomial maps, very little was known about the existence of wandering Fatou components until recent developments on polynomial skew-products [Lil04, ABD$^+$16, PS17, PR17, PV16] which are the maps of the form

$$F(z,w) = (f(z,w), g(w)).$$

Unlike the one-dimensional case, Astorg, Buff, Dujardin, Peters, and Raissy [ABD$^+$16] found a polynomial skew-product possessing a wandering Fatou component as the quasi-conformal methods break down. The reader can refer to the survey [Rai16] for more details about other relevant work on polynomial skew-product [Lil04, PS17, PR17, PV16].

The study of complex Hénon maps is motivated by the classification of polynomial automorphisms [FM89]. The maps have the same form as in (1.1) but covers a broader class of functions by allowing f to be any polynomial [Hub86, HOV95] or analytic map [Duj04]. A recent paper by Arosio, Benini, Fornæss, and Peters [ABFP18] found transcendental Hénon maps exhibiting a wandering domain. Those are similar to the examples found in one-dimensional transcendental maps. Nevertheless, the problem is still unsolved [Bed15] for complex polynomial Hénon maps.

Renormalization

Renormalization is an important procedure that allows people to study the dynamics on different scales. In the degenerate case, the unimodal-renormalization was introduced by Feigenbaum [Fei78, Fei79] and Coullet and Tresser [CT78, TC78] to study the period-doubling cascades for

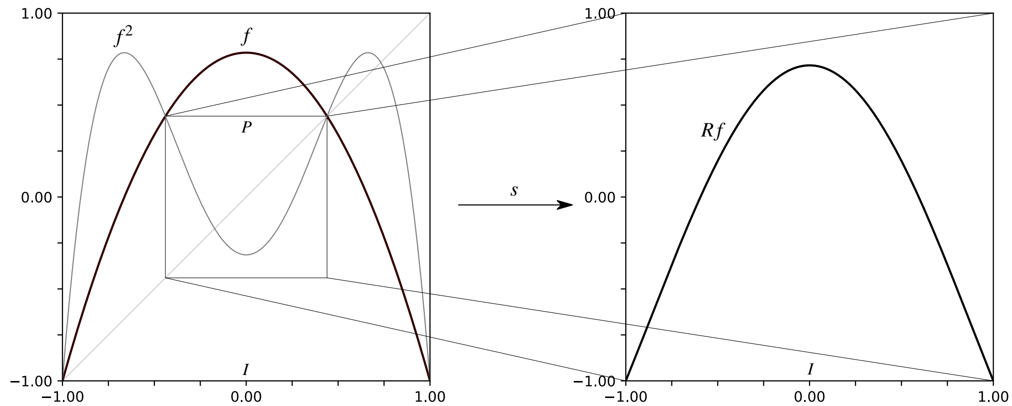


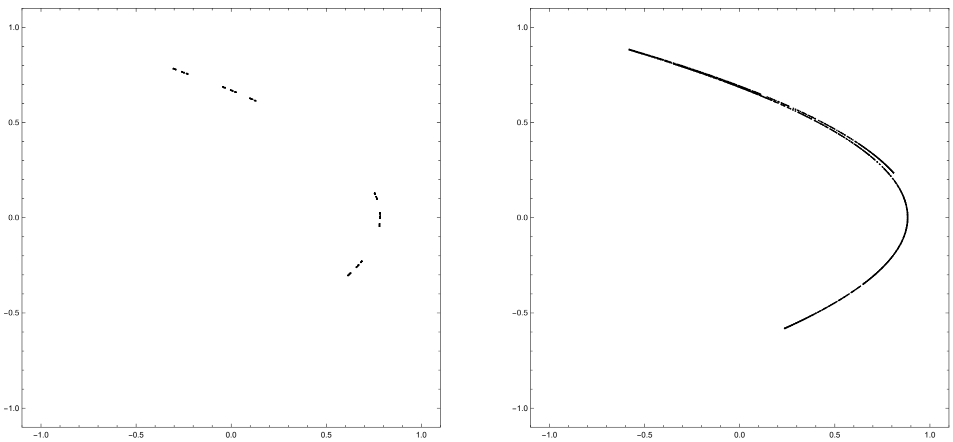
Figure 1.2.: A period-doubling renormalizable unimodal map f and its renormalization Rf.

unimodal maps. A unimodal map $f : I \to I$ is renormalizable if there exists a subinterval $P \subset I$ and an integer n such that $f^n(P) \subset P$ and the restriction of f^n to P is also a unimodal map. In other words, the map has a self-similarity in different scales. A renormalizable map acts on the orbit of the intervals $P, f(P), \cdots, f^{n-1}(P)$ like a permutation which is called the combinatorics of renormalization. For example, period-doubling means $n = 2$. Finally, the renormalization is the coordinate change of the first return map $Rf = s \circ f^n \circ s^{-1}$ that brings the domain P of the n-th iterate back to the unit interval I by the affine rescaling map $s : P \to I$ which turns Rf into a unimodal map on I. Figure 1.2 shows an illustration of the period-doubling renormalization of a unimodal map. A map is called infinitely renormalizable with stationary combinatorics if the procedure of renormalization can be applied infinitely many times and all renormalizations have the same combinatorics.

Renormalization can also be done in the Hénon setting. In this thesis, the proof will be based on the framework developed by de Carvalho, Lyubich, and Martens [dCLM05] for the period-doubling combinatorics and Hazard [Haz11] for other combinatorics. The Hénon-renormalization turns out to be much more delicate because the rescaling map is not affine. Several papers [BGLT94, MW14, MW16] in different contexts show that the condition "infinitely renormalizable" is sufficient for proving the absence of wandering domains. In this thesis, we will make use of this condition to prove the main theorem. By the hyperbolicity of the renormalization operator, the renormalization R^nF converges to the fixed point of the unimodal-renormalization operator. The size of the region, called the bad region, where the map R^nF behaves different from the degenerate case converges to 0 at a super-exponential rate. Roughly speaking, the case when the expansion and contraction are out of control becomes less likely to happen as the renormalization operator applies to the map more times. Therefore, we will show that the estimates from one-dimension (unimodal maps) also apply to Hénon-like maps except finitely many exceptions (Proposition 11.17 and Proposition 18.11).

Dynamics of Hénon-like maps

One of the important problems in dynamics is to study the asymptotic behavior of a system. An attracting set is a closed set such that many points evolve toward the set. The collection of those



(a) A six times period-doubling renormalizable Hénon map.

(b) A Hénon map that is not period-doubling renormalizable.

Figure 1.3.: The 7501 to 10000 iterates of a random point by a Hénon-like map.

points is called the basin of the attracting set. One type of attracting set is the omega limit set which characterizes the limiting behavior of a point. The omega limit set $\omega(x)$ of a point x is defined as $\omega(x) = \cap_{n=1}^{\infty} \overline{O(f^n(x))}$ where $O(x)$ stands for the forward orbit of the point x. Figure 1.3 shows two numerical experiments plotting the limiting trajectory of an orbit. A strange attractor is an attracting set having chaotic behavior (depends sensitivity on the initial condition).

Hénon maps are famous of its chaotic limiting behavior since Hénon first discovered the strange attractor in the classical Hénon family [Hén76]. For a strongly dissipative infinitely period-doubling renormalizable Hénon map, the omega limit sets are classified into two categories [GvST89, LM11]: it can be either a saddle periodic orbit (of period 2^n) or the renormalization Cantor set. The dynamics on the Cantor set is conjugated to the dyadic adding machine. Figure 1.3a shows a numerical simulation of the Cantor set. The structure of the omega limit set is similar to the hierarchy structure of unimodal maps [JR80]. From the dichotomy, the topological structure of the stable manifolds fully characterizes the long time behavior of the map. As an application of the main theorem, we show that the union of the stable manifolds is dense. In other words, the basin of the Cantor set has no interior even though it has full Lebesgue measure.

Idea of the Proof

When $\varepsilon = 0$ in (1.1), the behavior of the Hénon-like map degenerates to unimodal dynamics. This is called a degenerate Hénon-like map. One can view the class of unimodal maps as a subset of the class of the Hénon-like maps by identifying a unimodal map as a degenerate Hénon-like map. It is known that unimodal maps do not have wandering intervals. The proof for the Hénon-like maps is motivated from a proof from the unimodal case.

In the unimodal case, we prove by contradiction: assume that an infinitely renormalizable unimodal map f with stationary combinatorics has a wandering interval J. The map is renormalizable, so the first return map of f is defined on some subinterval $P(1)$ of the domain $P(0)$. One can define a rescaling map ϕ that rescales the subinterval $P(1)$ back to the original scale $P(0)$. The coordinate change of the first return map $RF = \phi \circ f^p \circ \phi^{-1}$ is also a unimodal map that is called the renormalization of the map. Since the map is infinitely renormalizable, the procedure of renormalization can be repeated infinitely many times.

Then a rescaled orbit of the wandering interval J that closest approaches to the critical value is defined by iterating and rescaling the set J. The orbit is called the J-closest approach. We study the expansion and contraction of the sizes of the orbit elements. For the period-doubling combinatorics, the sizes are measured by the Euclidean metric (Definition 2.8); for other combinatorics, the sizes are measured by the hyperbolic metric (Definition 2.9). We establish an expansion estimate: the sizes of the orbit elements expand at a definite rate (Equation (8.1) and Proposition 13.61). This leads to a contradiction because the size of the domain is bounded but the sizes of the orbit elements approach infinity.

Inspired by the degenerate case, our goal is to prove an analog of the expansion estimates for Hénon-like maps then show that the (one-dimensional) sizes of the elements in a closest approach tend to infinity. Based on how the sizes are measure, the proof is split into two parts. Part I focuses on the period-doubling combinatorics. In this case, we will measure the sizes of the elements by the horizontal size (Definition 7.9) which is a generalization of the Euclidean measurement from one- to two-dimensions. Part II covers the remaining cases but not period-doubling. In this case, we will measure the sizes by the hyperbolic size (Definition 16.3) which is a generalization of the hyperbolic length.

However, in the Hénon case, the expansion estimate breaks down because of a nonhyperbolic behavior. In the thesis, we will develop two machinery to resolve the issue:

1. The good region and the bad region
2. The area argument (thickness)

The domain of a Hénon-like map is classified into two complimentary areas: the good region and the bad region. The good region is an area where the Hénon-like map behaves like a unimodal map. In particular, the expansion estimate can be promoted to the Hénon-like map when the closest approach stays in the good region. For the period-doubling case (Part I), we will show in Chapter 10 that the horizontal sizes of the orbit elements expand at a definite rate (Proposition 10.11). For the other combinatorics (Part II), we will show in Chapter 17 that the hyperbolic sizes of the orbit elements expand at a definite rate (Proposition 17.32). We note that the proofs for the expansion estimates in the two cases are not exchangeable. The bad region is an area where the Hénon-like map behaves different from a unimodal map. The expansion or contraction of the size is out of control whenever an element enters the bad region because of nonhyperbolicity. The size of the bad region has the same order as the size of ε. The size of ε measures how far the Hénon-like map is away from the class of unimodal maps.

To take care of the nonhyperbolicity in the bad region, we rely on studying the contraction of the areas of the orbit elements. This is because that we have “hyperbolic control” on the area due to the universality around the tip: the Jacobian of the map has a uniform controlled lower bound. This is called the area argument. The area provides a good estimate for the contraction of the (one-dimensional) sizes when the expansion estimate breaks down.

In order to show that the (one-dimensional) sizes approach infinity, we prove that the contraction happens at most finitely many times. This is achieved by showing that a closest approach have at most finite entries to the bad regions. The proof based on the following ingredients:

1. When an element in a closest approach enters the bad region, the size of that element cannot exceed the size of the bad region. This means that if the size of the bad region is small, a closest approach have a small chance of entering the bad region.

2. Whenever an element in a closest approach is rescaled, then we renormalize the map and iterate the rescaled set by the renormalized map. By the hyperbolicity of the Hénon-renormalization operator, the renormalized map becomes closer to the class of unimodal maps. This means that the size of the bad region becomes smaller whenever the map is renormalized. In other words, the closest approach becomes less likely to enter the bad region whenever a rescaling is applied.

3. When an element enters the bad region, a large amount of rescaling is applied to the element.

After combining all of the ingredients, roughly speaking, we will show that the sizes of the bad regions contract faster than the contraction of the (one-dimensional) sizes of the elements in a closest approach. This relies on delicate estimations from the expansion estimate, the contraction of the area, the size of the bad region, and the relationships between the (one-dimensional) size and the area. Note that we give two different proofs for this property in Part I and Part II. The proof from Section 11.3 is more intuitive but longer; the proof from Section 18.3 is shorter but less intuitive. The two proofs are interchangeable.

Finally, since the sizes of the orbit elements expand at a definite rate with only finitely many exceptions, we show that the sizes approach infinity. This leads to a contradiction. Therefore, wandering domains cannot exist.

2. Preliminary and Notation

2.1. Sets and topology

Assume that $(X, \prec)$ is a simple (total) ordered set [Mun00, Section 3]. An *interval* I on X is a subset such that for all $a, b \in I$ with $a \prec b$ implies that $c \in I$ for all $c \in X$ with $a \prec c \prec b$. The bracket notations for intervals are used for ordered sets.

The set of real numbers is denoted as $\mathbb{R}$. The set of complex numbers is denoted as $\mathbb{C}$. The set of all positive integers is denoted as $\mathbb{N}$. Let I be an interval on $\mathbb{R}$ and $\delta > 0$. The *complex δ-neighborhood* of I is defined to be the open set $I(\delta) = \{z \in \mathbb{C}; |z - z'| < \delta \text{ for some } z' \in I\}$.

Assume that X is a topological space. The closure of a subset $A \subset X$ is denoted as $\mathrm{cl}(A)$. The interior of the set is denoted as $\mathrm{int}(A)$. For $A, B \subset X$, the relation $A \Subset B$ means $\mathrm{cl}(A) \subset \mathrm{int}(B)$.

2.2. Functions

Assume that V and W are Banach spaces. A function $f : V \to W$ is *Lipschitz continuous* with constant L if

$$|f(y) - f(x)| \leq L\,|y - x|$$

for all $x, y \in V$.

Assume that S is a set and f is a complex-valued function on S. The *sup norm* of f on S is

$$\|f\|_S = \sup\{|f(x)|; x \in S\}.$$

The subscript S is neglected whenever the context is clear.

Assume that I is an interval on $\mathbb{R}$. The space of *$C^n(I)$ functions* is the collection of functions $f : I \to \mathbb{R}$ that are n-times differentiable and the n-th derivative is bounded and continuous. The *C^n-norm* of a $C^n(I)$ function f is

$$\|f\|_{C^n(I)} = \sum_{j=0}^{n} \left\| f^{(j)} \right\|_I$$

where $f^{(j)}$ is the j-th derivative of f.

As an application from the Cauchy's integral formula [SS10, Corollary 4.3], the derivatives of a holomorphic function are bounded by the following lemma.

Lemma 2.1. *Assume that I is an interval and $\delta > 0$. There exist positive constants $c = c(\delta)$ and $\delta' = \delta'(\delta) < \delta$ such that*

$$\left\|f'\right\|_{I(\delta')} \leq c\,\|f\|_{I(\delta)}.$$

for all holomorphic maps $f : I(\delta) \to \mathbb{C}$.

Higher order derivatives and partial derivatives (of a multi-variable holomorphic map) can also be estimated by a similar inequality.

Assume that $f : X \to Y$ is a map where X and Y are simple ordered sets. The map f is *increasing (orientation preserving)* on an interval $I \subset X$ if $f(x) \prec f(y)$ whenever $x \prec y$. *Decreasing (orientation reversing)* is defined to be similar. The map f is *monotone (has a fixed orientation)* on I if it is either orientation preserving or orientation reversing on I. Two points x and y in the domain is said to have *same orientation* if f is monotone on some interval that contains both x and y.

Assume that $U \subset \mathbb{R}^2$ is open and $F : U \to \mathbb{R}^2$ is a differentiable map. The *Jacobian* of the function F is the determine of its derivative $\det DF$.

The *canonical projections* π_x and π_y on the plane are the maps $\pi_x(x,y) = x$ and $\pi_y(x,y) = y$.

2.3. Dynamics

Assume that $f : X \to X$ is a function. For any integer $n > 0$, denote the n-th iterate as $f^n = f \circ \cdots \circ f$ where f is composed n-times. Also, let $f^0 = \mathrm{Id}$ be the identity map. If f is invertible, then set $f^{-n} = f^{-1} \circ \cdots \circ f^{-1}$ where the inverse f^{-1} is composed n-times.

A point $x \in X$ is a *periodic point* of f with period p if p is the smallest positive integer such that $f^p(x) = x$. The point $x \in X$ is a *fixed point* of f if x is a periodic point of f with period 1. A point $x \in X$ is a *preperiodic point* of f if $f^n(x)$ is a periodic point of f for some integer $n \geq 0$. In addition, if $X \subset \mathbb{R}^n$ is open and the map f is differentiable, the *multipliers* of x are the eigenvalues of the derivative $D(f^p)(x)$ at the point.

Assume that X is a topological space and $x \in X$ is a periodic point of the continuous map $f : X \to X$ with period p. The *stable set* of p is the set

$$W^s(p) = W^s(f,p) = \left\{ x \in X ; \lim_{n\to\infty} f^n(x) = p \right\}.$$

If X is a manifold, the stable set is called the *stable manifold* if itself is a submanifold of X. In this article, a *local stable manifold* refers to a connected component of a stable manifold.

Assume that I is an interval on $\mathbb{R}$ and f is a $C^1(I)$ function. A point $c \in I$ is a *critical point* of f if $f'(c) = 0$. Its iterate $v = f(c)$ is called a *critical value* of f. The map f is a *unimodal map* if it has a unique maximal point and no other local extrema in the interior of I. A $C^2(I)$ unimodal map f is *nondegenerate* if $f''(c) \neq 0$ where c is the critical point.

2.4. Permutation

A *permutation* is an automorphism on a set. In this article, we will only consider permutations on a finite set. A *cyclic permutation* ν (or cycle) is a permutation that contains exactly one periodic orbit with period $p \geq 2$. The *length* $|\nu| = p$ of the cycle ν is the period of the orbit. In this article, any periodic orbit of period p on a simple ordered set can be uniquely identified with the permutation on the finite ordered set $\mathbb{Z}_p = \{1, \cdots, p\}$ that has the same order.

2.5. Schwarzian derivative

In this section, we recall the definition and the properties of Schwarzian derivative. The proof for the properties can be found in [dMvS12].

Definition 2.2 (Schwarzian Derivative)**.** Assume that $f \in C^3(I)$ where I is an interval. The *Schwarzian derivative* of f is defined by

$$(Sf)(x) = \left(\frac{f''(x)}{f'(x)}\right)' - \frac{1}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

whenever $f'(x) \neq 0$. The map f is said to have *negative Schwarizan derivative* if $Sf(x) < 0$ for all $x \in I$ with $f'(x) \neq 0$.

Negative Schwarzian derivative is preserved under iteration.

Proposition 2.3. *If $f : I \to I$ has negative Schwarzian derivative, then f^n also has negative Schwarzian derivative for all $n > 0$.*

Proposition 2.4 (Minimal Principle)**.** *Assume that J is a bounded closed interval and $f : J \to \mathbb{R}$ is a C^3 map with negative Schwarzian derivative. If $f'(x) \neq 0$ for all $x \in J$, then $|f'(x)|$ does not attain a local minimum in the interior of J.*

Corollary 2.5. *Assume that J is a bounded closed interval and $f : J \to \mathbb{R}$ is a C^3 map with negative Schwarzian derivative. If $f'(x) \neq 0$ for all $x \in J$, then*

1. *p is an inflection point if and only if $f''(p) = 0$,*
2. *if p is an inflection then $f'''(p) < 0$, and*
3. *f can have at most one inflection point in the interior of J.*

Proof. Without lose of generality, we assume that $f' > 0$. For each point $p \in \text{int}J$ with $f''(p) = 0$, since $Sf(p) < 0$, we have $f'''(p) < 0$ and hence p is a inflection point. Thus, properties 1 and 2 follows.

To prove the last property, we prove by contradiction. Assume that f has at least two inflection point in the interior of J. Property one implies that the set of inflection point is closed and property two implies that the set of inflection points does not contain any accumulation point. Thus, J can only contain at most finitely many inflection points.

Let $a < b$ be two consecutive inflection points in the interior of J. By property 2, f'' is decreasing on a neighborhood of a. Thus, $f''(x) < 0$ for all $x \in (a,b)$ because f does not have inflection point on (a,b). However, by the same reason, f'' is decreasing on a neighborhood of b and $f''(b) = 0$. This is a contradiction. Therefore, f can have at most one inflection point in the interior of J. □

Proposition 2.6 (Singer)**.** *Assume that J is an interval and $f : J \to J$ is a C^3 map with negative Schwarzian derivative. Then the immediate basin of any attracting periodic orbit contains either a critical point of f or a boundary point of the interval J.*

The next proposition shows that the property of negative Schwarizan derivative is preserved under small perturbation.

Proposition 2.7. *Assume that f is a $C^3(I)$ nondegenerate unimodal map on a compact interval I with negative Schwarzian derivative. There exists $\delta > 0$ (depending on f) such that if g is a $C^3(I)$ map with $\|g' - f'\|_{C^2(I)} < \delta$, then g is a map with negative Schwarzian derivative.*

Proof. The Schwarzian derivative can be rewritten in the form of

$$Sf(x) = \left(\frac{1}{f'(x)}\right)^2 \left[f'(x)f'''(x) - \frac{3}{2}\left(f''(x)\right)^2\right].$$

It is easy to see that a C^3 nondegenerate unimodal map f has negative Schwarzian derivative if and only if

$$f'(x)f'''(x) - \frac{3}{2}\left(f''(x)\right)^2 < 0 \tag{2.1}$$

for all $x \in I$.

Next, we claim that g has negative Schwarzian derivative. Since I is compact, we may assume that $M, N > 0$ be values such that $f'(x)f'''(x) - \frac{3}{2}\left(f''(x)\right)^2 < -M$ and $\|f'\|, \|f''\|, \|f'''\| < N$. Use (2.1), we get

$$\begin{aligned}
g'(x)g'''(x) - \frac{3}{2}\left(g''(x)\right)^2 \leq & f'(x)f'''(x) - \frac{3}{2}\left(f''(x)\right)^2 \\
& + \left|\left[g'(x)g'''(x) - \frac{3}{2}\left(g''(x)\right)^2\right] - \left[f'(x)f'''(x) - \frac{3}{2}\left(f''(x)\right)^2\right]\right| \\
\leq & f'(x)f'''(x) - \frac{3}{2}\left(f''(x)\right)^2 + \left|g'(x)\right|\left|g'''(x) - f'''(x)\right| \\
& + \left|f'''(x)\right|\left|g'(x) - f'(x)\right| + \frac{3}{2}\left|g''(x) + f''(x)\right|\left|g''(x) - f''(x)\right| \\
\leq & -M + \left[\left|f'(x)\right| + 3\left|f''(x)\right| + \left|f'''(x)\right| + \frac{5}{2}\left\|g' - f'\right\|_{C^2(I)}\right]\left\|g' - f'\right\|_{C^2(I)} \\
\leq & -M + 6N\left\|g' - f'\right\|_{C^2(I)} \\
\leq & -\frac{1}{2}M < 0
\end{aligned}$$

for all C^2 functions g with $\|f' - g'\|_{C^2(I)} < \delta$ where $\delta = \min(\frac{2}{5}N, \frac{1}{12}\frac{M}{N})$. Therefore, g has negative Schwarzian derivative. □

2.6. Hyperbolic length

In this article, we will use the hyperbolic length to measure the size of a wandering interval in this article. It will be generalized to dimension two in Chapter 16 to measure the size of a wandering domain.

First define the Euclidean length.

Definition 2.8 (Euclidean Length)**.** The *Euclidean length* of an interval $J \subset \mathbb{R}$ is

$$|J| = \sup\{|b - a|; a, b \in J\}.$$

The brackets are neglected whenever the context is clear.

Recall the definition of hyperbolic length.

Definition 2.9 (Hyperbolic Length)**.** Assume that T is an bounded interval and J is a subinterval such that $J \Subset T$. The *hyperbolic length* of J in T is

$$|J|_T = \ln \frac{(L+J)\,(R+J)}{LR} = \ln\left(1+\frac{J}{L}\right) + \ln\left(1+\frac{J}{R}\right) \tag{2.2}$$

where L and R are the left and the right components of the complement $T \backslash J$ respectively as illustrated in Figure 2.1.

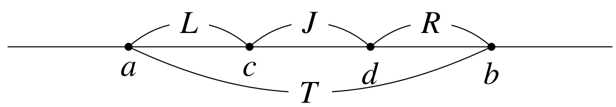


Figure 2.1.: Definition of T, L, R, and J.

Proposition 2.10. *Assume that T and T' are intervals and $f : T \to T'$ is a C^3-diffeomorphism with negative Schwarzian derivative. If J is an interval on* $\mathrm{int}(T)$*, then*

$$|f(J)|_{T'} > |J|_T\,.$$

Proof. See [dMvS12, Section IV.1]. ☐

Proposition 2.11. *Assume that s is an affine map, T and T' are nontrivial intervals, and $s(T) = T'$. If $J \Subset \mathrm{int}T$ is an interval, then*

$$|s(J)|_{T'} = |J|_T\,.$$

Proof. The equality holds because affine map is a special case of Möbius transformation. See [dMvS12, Section IV.1]. ☐

2.6.1. Relations with the Euclidean length

The first proposition allows to bound the Euclidean length by the hyperbolic length.

Proposition 2.12. *There exists a constant $c > 0$ such that the inequality*

$$|J|_T \geq \ln\left(1+\frac{J}{T}\right) \geq c\frac{J}{T}$$

holds for all intervals $J \Subset T$.

Proof. The first part of the inequality follows directly from the definition (2.2). The last part of the inequality is true because the logarithm has a linearized lower bound when $0 \leq \frac{J}{T} \leq 1$. ☐

The next proposition shows that the hyperbolic length and the Euclidean length are comparable when the two sides have definite size.

Proposition 2.13. *The inequality*

$$|J|_T \le \left(\frac{1}{L}+\frac{1}{R}\right) J = \left(\frac{T}{L}+\frac{T}{R}\right)\frac{J}{T}$$

holds for all intervals $J \Subset T$.

Proof. The inequality follows directly from the definition of hyperbolic length

$$|J|_T = \ln\left(1+\frac{J}{L}\right)+\ln\left(1+\frac{J}{R}\right) \le \frac{J}{L}+\frac{J}{R}.$$

□

2.6.2. Expansion of hyperbolic length from topological expansion

The goal of this section is to prove Proposition 2.17. The horizontal size of an interval grows when it is measured inside a smaller base interval. The proposition estimates the lower bound of the growth.

Assume that J, t, and T are intervals $J \Subset t \Subset T$. Let l and r be the left and the right components of $t \backslash J$ respectively and L and R be the left and the right components of $T \backslash t$ respectively. See Figure 2.2 for illustration.

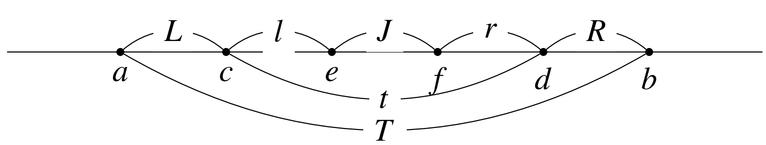


Figure 2.2.: Definition of the intervals T, t, L, l, R, r, and J.

Lemma 2.14. *The equality holds for all J*

$$|J|_t - |J|_T = \ln\left(1+\frac{J}{l}\frac{L}{l+L+J}\right)+\ln\left(1+\frac{J}{r}\frac{R}{r+R+J}\right).$$

Proof. By the definition of hyperbolic distance, compute

$$\begin{aligned}
|J|_t - |J|_T &= \ln\frac{(l+J)\,(r+J)}{lr} - \ln\frac{(l+L+J)\,(r+R+J)}{(l+L)\,(r+R)} \\
&= \ln\left(\frac{1+\frac{J}{l}}{1+\frac{J}{l+L}}\right)\left(\frac{1+\frac{J}{r}}{1+\frac{J}{r+R}}\right) \\
&= \ln\left[1+\frac{\left(\frac{1}{l}-\frac{1}{l+L}\right)J}{1+\frac{J}{l+L}}\right]\left[1+\frac{\left(\frac{1}{r}-\frac{1}{r+R}\right)J}{1+\frac{J}{r+R}}\right] \\
&= \ln\left[1+\frac{\frac{JL}{l(l+L)}}{1+\frac{J}{l+L}}\right]\left[1+\frac{\frac{JR}{r(r+R)}}{1+\frac{J}{r+R}}\right]
\end{aligned}$$

$$= \ln\left(1+\frac{J}{l}\frac{L}{l+L+J}\right)\left(1+\frac{J}{r}\frac{R}{r+R+J}\right).$$

□

Corollary 2.15. *Assume that $J \Subset t \subset T$. Then $|J|_t \geq |J|_T$.*

We also need the following lemma.

Lemma 2.16. *Assume that $0 < a < 1$. Then*

$$\ln(1+ax) > a\ln(1+x)$$

for all $x > 0$.

Proof. Let $f(x) = \ln(1+ax)$ and $g(x) = a\ln(1+x)$. Then

$$f'(x) = \frac{a}{1+ax} > g'(x) = \frac{a}{1+x}$$

for all $x > 0$. Also, $f(0) = g(0) = 0$. Therefore, the lemma follows by integrating the inequality from both sides. □

If the interval $J \subset T$ is embedded into a smaller interval t, then the hyperbolic size expands and the size of expansion can be estimated by the left L and right R intervals as follows.

Proposition 2.17. *Assume that $J \Subset t \Subset T$ and $M > 0$. If $\frac{L}{T}, \frac{R}{T} > M$, then*

$$|J|_t > \frac{1}{1-M}|J|_T.$$

Proof. By Lemma 2.16 and Lemma 2.14, we have

$$\begin{aligned}
|J|_t - |J|_T &> \frac{L}{l+L+J}\ln\left(1+\frac{J}{l}\right) + \frac{R}{r+R+J}\ln\left(1+\frac{J}{r}\right)\\
&> \frac{L}{T}\ln\left(1+\frac{J}{l}\right) + \frac{R}{T}\ln\left(1+\frac{J}{r}\right)\\
&> M|J|_t.
\end{aligned}$$

Then

$$(1-M)|J|_t > |J|_T$$

and hence the proposition follows. □

Part I.

The period doubling combinatorics

3. Outline

In this part of the article, chapters, sections, or statements marked with a star sign "*" means that the main theorem, Theorem 11.18, does not depend on them. Terminologies in the outline will be defined precisely in later chapters.

Chapters 2, 4, 5, and 6 are the preliminaries of the theorem. Most of the theorems in Chapter 5 and Section 6.1 can be found in [dCLM05, LM11].

The proof for the nonexistence of wandering domains is motivated by the proof for the degenerate case. A Hénon-like map is degenerate means that $\varepsilon = 0$ in (1.1). In this case, the dynamics of the map degenerates to the unimodal dynamics. In Chapter 8, a short proof for the nonexistence of wandering intervals for infinitely renormalizable unimodal maps is presented by identifying a unimodal map as a degenerate Hénon-like map. The proof assumes the contrapositive, there exists a wandering interval J. Then we apply the Hénon renormalization instead of the standard unimodal renormalization to study the dynamics of the rescaled orbit of J that closest approaches the critical value. The rescaled orbit is called the J-closest approach (Definition 7.1). The proof argues that the lengths of the elements in the rescaled orbit approach infinity by a length expansion estimate which leads to a contradiction. The expansion estimate motivates the proof for the Hénon case.

The proof of the main theorem is covered by Chapters 7, 9, 10, and 11.

Assume the contrapositive, a Hénon-like map has a wandering domain J. In Chapter 7, we study the rescaled orbit $\{J_n\}_{n\geq 0}$ of J that closest approaches to the tip, called the J-closest approach. Each element J_n belongs to some appropriate renormalization scale (the domain of the $r(n)$-th renormalization $R^{r(n)}F$ for some nonnegative integer $r(n)$). The transition between two consecutive orbit elements $J_n \to J_{n+1}$ is called one step. Inspired by the expansion estimate from the degenerate case, we study the expansion rate of the horizontal sizes l_n of the elements. The horizontal size of a set is the length of its projection to the first coordinate (Definition 7.9). Our final goal is to show that the horizontal sizes of the sequence elements approach infinity to obtain a contradiction.

In the degenerate case, the expansion estimate says that the horizontal sizes expand at a uniform rate, and hence the horizontal sizes of the sequence elements approach infinity. Unfortunately, the argument breaks down in the non-degenerate case. There are two features that make the non-degenerate case special:

1. The good region and the bad region.
2. Thickness.

The good region and the bad region, introduced in Chapter 9, divide the phase space into two regions by how similar the map behaves like a unimodal map. Each renormalization scale (domain of the n-th renormalization R^nF for some n) has its own good region and bad region, and the sizes of the bad regions contract super-exponentially as the renormalization operator applies more times to the map ([dCLM05, Theorem 4.1] and Definition 9.1).

When the elements in a closest approach stay in the good regions of some appropriate scales, the Hénon-like map behaves like a unimodal map. In particular, we show that the expansion estimate can be promoted to the Hénon-like maps in Chapter 10.

However, when an element J_n enters the bad region of some proper scale, the expansion estimate breaks down and the horizontal size contracts. To estimate the size of contraction, we introduce a quantity called "thickness" (Definition 11.2). The thickness w_{n+1} of the next element J_{n+1} offers a good approximation for the horizontal size l_{n+1} of the element. We will show that the contraction rate of the thicknesses has the same size as the Jacobian of the map (Proposition 11.7). For a strongly dissipative Hénon-like map, the Jacobian is small and hence the amount of contraction is large. The contraction produces the main obstruction of showing that the horizontal sizes approach infinity.

The key observation is that a closest approach can have at most finitely many entries to the bad regions (Proposition 11.17). When an element J_n enters the bad region, the horizontal size becomes smaller but the size of the bad region also contracts because the element is rescaled. Roughly speaking, we found that the sizes of the bad regions contract faster than the contractions of the horizontal sizes so that the elements cannot enter the bad regions infinitely many times. The actual proof is more delicate because the contraction of the horizontal sizes also depends on the time span in the good regions (Definition 11.10). The two-row-lemma (Lemma 11.15) relates the contraction of thicknesses, the expansion of the horizontal sizes in the good region, the time span in the good region, and the size of the bad region when the closest approach enters the bad region twice. Finally, the property is proved after applying the two-row-lemma inductively (Lemma 11.16).

In summary, the horizontal sizes of the elements in a closest approach expand in the good regions and contract in the bad regions. However, the contraction happens at most finitely many times. This shows that the horizontal sizes approach infinity and leads to a contradiction. Therefore, wandering domains cannot exist.

4. Unimodal Maps

In this chapter, we give a short review over the procedure of unimodal-renormalization. The goal is to introduce the hyperbolic fixed point of the renormalization operator (Proposition 4.28) and establish the estimates for its derivative (Subsection 4.4.3).

Definition 4.1 (Unimodal Map)**.** Let $I = [-1, 1]$. A unimodal map in this paper is a smooth map $f : I \to I$ such that

1. the point -1 is the unique fixed point with a positive multiplier,
2. $f(1) = -1$, and
3. the map f has a unique maximum at $c \in \text{int}(I)$ and the point c is a non-degenerate critical point, i.e. $f'(c) = 0$ and $f''(c) \neq 0$.

The class of analytic unimodal maps $f : I \to I$ is denoted as $\mathscr{U}$.

Definition 4.2 (Critical Orbit)**.** For a unimodal map $f \in \mathscr{U}$, let $c^{(0)} = c^{(0)}(f) \in I$ be the critical point of f. The critical orbit is denoted as $c^{(n)} = f^n(c^{(0)})$ for all $n > 0$.

Definition 4.3 (Reflection)**.** Assume that $f \in \mathscr{U}$ and $x \in I$. If $x \neq c^{(0)}$, define the reflection of x to be the point $\hat{x} \in I$ such that $f(\hat{x}) = f(x)$ and $\hat{x} \neq x$. If $x = c^{(0)}$, define $\hat{x} = c^{(0)}$.

4.1. The renormalization of a unimodal map

To define the period-doubling renormalization operator for unimodal maps, we introduce a partition on I that allows us to define the first return map for a renormalizable unimodal map.

Definition 4.4. Assume that $f \in \mathscr{U}$ has a unique fixed point $p(0) \in I$ with a negative multiplier. Let $p^{(1)} = \hat{p}(0)$ and $p^{(2)}$ be the point such that $f(p^{(2)}) = p^{(1)}$ and $p^{(2)} > c^{(0)}$. Define $A = (-1, p^{(1)}) \cup (p^{(2)}, 1)$, $B = (p^{(1)}, p(0))$, and $C = (p(0), p^{(2)})$. The sets $A = A(f)$, $B = B(f)$, and $C = C(f)$ form a partition of the domain $D \equiv I$. See Figure 4.1 for an illustration.

The property "renormalizable" is defined by using the partition elements.

Definition 4.5 (Renormalizable)**.** A unimodal map $f \in \mathscr{U}$ is (period-doubling) renormalizable if it has a fixed point $p(0)$ with a negative multiplier and $f(B) \subset C$. The class of renormalizable unimodal maps is denoted as $\mathscr{U}^r$.

Remark 4.6. Most of the articles use the critical orbit to define unimodal-renormalization. However, Hénon-like maps do not have a critical point. So here we use the preimages of the fixed points in order to be consistent with the Hénon-like maps (Definition 5.14).

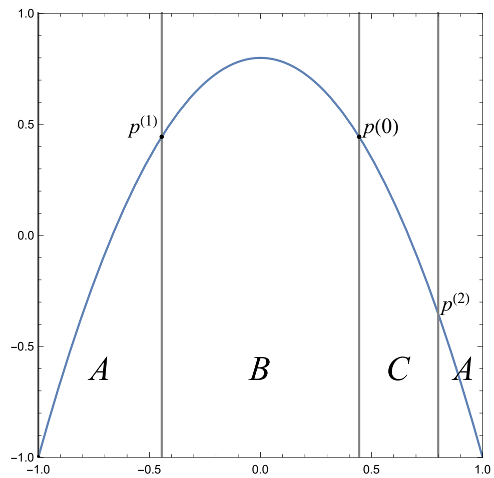


Figure 4.1.: The partition $\{A,B,C\}$ of a unimodal map. The parabola is the graph of a unimodal map. The points $p(0)$, $p^{(1)}$, and $p^{(2)}$ are defined as in Definition 4.4.

For a renormalizable unimodal map, an orbit that is not eventually periodic follows the paths in the following diagram.

$$\begin{tikzcd} A \arrow[loop left] \arrow[r] & B \arrow[r, shift left] & C \arrow[l, shift left] \end{tikzcd}$$

Figure 4.2.: The itinerary of an orbit on the partition A, B, and C.

This allows us to define the first return map on B or on C and the period-doubling renormalization.

Definition 4.7 (Renormalization Operator)**.** Assume that $f \in \mathscr{U}^r$.

1. Define the renormalization operator around the critical point $R_c : \mathscr{U}^r \to \mathscr{U}$ as $R_c f = s_c \circ f^2 \circ s_c^{-1}$ where s_c is the orientation-reversing affine rescaling that satisfies $s_c(p(0)) = -1$ and $s_c(p^{(1)}) = 1$.

2. Define the renormalization operator around the critical value $R_v : \mathscr{U}^r \to \mathscr{U}$ as $R_v f = s_v \circ f^2 \circ s_v^{-1}$ where s_v is the orientation-preserving affine rescaling that satisfies $s_v(p(0)) = -1$ and $s_v(p^{(2)}) = 1$.

With the definition of the renormalization operator, we may renormalize a renormalized unimodal map again if the renormalization is renormalizable. This gives the definition of infinitely renormalizable map.

Definition 4.8 (Infinitely Renormalizable)**.** A unimodal map $f \in \mathscr{U}$ is (period-doubling) infinitely renormalizable if for any length $n \geq 0$ there exists a sequence $\{a_j\}_{j=1}^n$ of letters $a_j \in \{c, v\}$ such that the renormalization $R_{a_n} \circ \cdots \circ R_{a_1}(f)$ is renormalizable. The class of infinitely renormalizable unimodal maps is denoted as $\mathscr{I}$.

4.2. *Structure and dynamics of infinitely renormalizable unimodal maps

In this section, we study the dynamics and rescalling on the partition. We consider infinitely renormalizable unimodal maps and renormalization about the critical value. This gives us an analog of infinitely renormalizable Henon-like map.

Assume that $f \in \mathscr{I}$. We define a sequence of unimodal map $f_n = R_v^n f$. The value n is called the renormalization scale. We use the subscript for the renormalization scale of the corresponding object. For example, the partition $A_n = A(f_n)$, $B_n = B(f_n)$, and $C_n = C(f_n)$ for the domain $D_n = D(f_n)$; the fixed point $p_n = p(f_n)$ with negative multiplier for f_n.

Define $S_n^j = s_{n+j-1} \circ \cdots \circ s_n$ for $j \geq 0$ and $n \geq 0$. Also, define $p_n(j) = \left(S_n^j\right)^{-1}(p_{n+j})$ for $j \geq 0$ and $p_n(-1) = -1$. Define a finer partition on C_n as $C_n(j) = (p_n(j-1), p_n(j))$ for $j \geq 0$. The value j is called the rescaling level. If a point belongs to $C_n(j)$, the point can be rescaled at most j times. It follows from definition and conjugation that

Proposition 4.9. *Assume that $f \in \mathscr{I}$. The following properties holds for all $n \geq 0$.*

1. *$p_n(j)$ is a periodic point of f_n with period 2^j for $j \geq 0$.*
2. *$S_n^k(p_n(j)) = p_{n+k}(j-k)$ for $j \geq k-1$.*
3. *$C_n(0) = A_n \cup \{p_n(0)\} \cup B_n$.*
4. *The affine maps $s_n : C_n \to D_n$ and $S_n^k : C_n(j) \to C_{n+k}(j-k)$ are bijective for $j \geq k$.*

The proposition and Figure 4.2 provide the following diagram for the dynamics on the finer partition. When a point enters A_n, it is mapped to A_n or B_n by f_n. When a point enters B_n, it is mapped to C_n by f_n. When a point enters C_n, the point belongs to $C_n(j)$ in the finer partition for some j so we are able to rescale the point at most j times. The value j is called the rescaling level. After rescale j times by applying S_n^j, the point enters A_{n+j} or B_{n+j} on the $n+j$ renormalization scale.

$$\begin{tikzcd}
A_n \arrow[loop right, "f_n"] \arrow[d, "f_n"] & & A_{n+j} \arrow[loop right, "f_{n+j}"] \arrow[d, "f_{n+j}"] & & \\
B_n \arrow[r, "f_n"] & C_n(j) \arrow[r, "S_n^j"] \arrow[ur, "S_n^j"] & B_{n+j} \arrow[r, "f_{n+j}"] & C_{n+j}(j') \arrow[r, "S_{n+j}^{j'}"] & \cdots
\end{tikzcd}$$

Figure 4.1.: The dynamics on the partition of a unimodal map.

4.3. *Nonexistence of wandering intervals

In this section, we include two elementary proofs of nonexistence of wandering interval for infinitely renormalizable maps with negative Schwarzian derivative. The first prove uses the renormalization operator around the critical point, while the second one uses the renormalization operator around the critical value. This will give us some ideas for the case in two dimensional Hénon-like maps.

Definition 4.10 (Wandering Interval). Assume that f is a unimodal map. We say that a closed subinterval $J \subset I$ is a wandering interval of f if J is nonempty, J is not a singleton, $\{f^n(J)\}_{n=0}^{\infty}$ does not tend to a periodic orbit, and $\{f^n(J)\}_{n=0}^{\infty}$ are disjoint.

The following important proposition allows us to generate wandering intervals by iteration and rescaling.

Proposition 4.11. *Assume that $f \in \mathscr{U}^r$ and s_c, s_v are the rescaling functions of f.*

1. *If $J \subset D(f)$ is a wandering interval of f then $f(J)$ is a wandering interval of F.*
2. *If $J \subset B(F)$ is a wandering interval of f then $s_c(J) \subset I$ is a wandering interval of R_cF.*
3. *If $J \subset C(F)$ is a wandering interval of f then $s_v(J) \subset D(R_vF)$ is a wandering interval of R_vF.*

Proof. It follows directly from the definition and conjugation. □

To prove the nonexistence of wandering interval, the strategy is to prove by contradiction. If there exists a wandering interval, we iterate and rescale the interval by the proposition to generate a sequence of wandering intervals. Our goal is to prove the length (or hyperbolic length) of the wandering interval in the sequence increases by a fixed rate. Therefore, the length approaches infinity and hence a contradiction.

4.3.1. Proof by renormalization about the critical point

For the first proof, we use the renormalization about the critical point to prove that the wandering interval does not exist by contradiction. If a wandering interval exists, we construct a sequence of wandering intervals by iteration and rescaling then prove the length of the wandering interval approaches infinity.

Lemma 4.12. *Assume that $f \in \mathscr{U}^r$ is symmetric, infinitely renormalizable about the critical point, and has negative Schwarzian derivative. If f has a wandering interval, then*

$$|f'(p)| \geq 1$$

where p is the fixed point with negative multiplier. Moreover, $|Rf'(-1)| \geq 1$.

Proof. Prove by contradiction. If p is an attracting fixed point, its immediate basin cannot contain -1 and 1 since -1 is a fixed point and 1 is the preimage of 1. By Proposition 2.6, the immediate basin must contain the critical point. Hence, the immediate basin of p contains B.

Assume that J is a wandering interval. J is not contained in B since B is in the immediate basin of p.

On the interval $(-1,0)$, f can have at most one fixed point since $f(-1) = -1$, $f(0) > 0$, and f has at most one inflection point on $(-1,0)$ by Corollary 2.5.

If f contains a fixed point p' on $(-1,0)$, then $f(x) < x$ on $(-1,p')$ and $f(x) > x$ on $(p',0)$. If J is contained in A, then $J \subset (p',0)$ since $[-1,p')$ is the immediate basin of -1. This implies that $f^n(J) \subset B$ for some n large enough which is a contradiction.

If f does not have a fixed point on $(-1,0)$, then $f(x) > x$ on $(-1,0)$. If J is contained in A, then $f^n(J) \subset B$ for some n large enough which is a contradiction.

If J is contained in C, then $f(J) \subset A$ or $f(J) \subset B$ which is impossible by the previous argument. Therefore, p can not be attracting.

Moreover, compute the renormalization directly, we get

$$\begin{aligned} Rf'(-1) &= s_c' \circ f^2 \circ s_c^{-1}(-1) f' \circ f \circ s_c^{-1}(-1) f' \circ s_c^{-1}(-1) s_c^{-1\prime}(-1) \\ &= \left[f'(p)\right]^2. \end{aligned}$$

Thus, $|Rf'(-1)| \geq 1$. $\square$

The rescaling function s_c maps $B(f)$ onto I. So $D(f)$ is too small. We extend the partition A, B, and C to I by letting $\hat{C} = (p,1)$. Then the following lemma shows that the dynamics of a wandering interval can be described by the partition A, B, and $\hat{C}$.

Lemma 4.13. *Assume that $f \in \mathscr{U}^r$ and $J \subset I$ is a wandering interval of f. Then J is a subset of one of the sets A, B, or $\hat{C}$.*

Proof. It follows by a wandering interval cannot contain any periodic point or preperiodic point. $\square$

Proposition 4.14. *Assume that $f \in \mathscr{I}$ is symmetric, infinitely renormalizable about the critical point, and has negative Schwarzian derivative. Then f does not have wandering interval.*

Proof. Without lose of generality we may assume that -1 is a repelling fixed point of f by Proposition 4.11 and Lemma 4.12. Write $f_n = R_c^n f$ and $f_{n+1} = s_n \circ f_n^2 \circ s_n^{-1}$. Then f_n has only two fixed point and the fixed points are repelling for all $n \geq 0$ by Lemma 4.12 and the assumption on f.

Assume that f has a wandering interval J. First, we show that there exists an integer $n \geq 0$ such that $f^n(J) \subset B$. Since f does not have a fixed point on $(-1,0)$, then $f(x) > x$ for all $x \in (-1,0)$. If J is contained in A, then $f^n(J) \subset B$ for some n large enough.

Define level of renormalization $r(n)$ and a closest approach J_n of $f_{r(n)}$ by induction. For $n = 0$, define $r(0) = 0$ and $J_0 = J$. Assume that $r(n)$ and J_n are defined. By Lemma 4.13, only one of the inclusion holds $J_n \subset A_{r(n)}$, $J_n \subset B_{r(n)}$, or $J_n \subset \hat{C}_{r(n)}$. If $J_n \subset A_n \cup \hat{C}_n$, define $r(n+1) = r(n)$ and $J_{n+1} = f_{r(n)}(J_n)$. If $J_n \subset B_{r(n)}$, define $r(n+1) = r(n)+1$ and $J_{n+1} = s_n(J_n)$. For both cases, J_{n+1} is a wandering interval of $f_{r(n+1)}$ by Proposition 4.11. The itinerary of the sequence follows the path of the following graph.

$$\begin{tikzcd}
 & \hat{C}_n \arrow[d, "f_n"] \arrow[dd, bend left=60, "f_n"] & \\
 & A_n \arrow[d, "f_n"] \arrow[loop right, "f_n"] & \\
B_{n-1} \arrow[uur, "s_{n-1}"] \arrow[ur, "s_{n-1}"] \arrow[r, "s_{n-1}"] & B_n \arrow[r, "s_n"] & A_{n+1} \cup B_{n+1} \cup \hat{C}_{n+1}
\end{tikzcd}$$

Our goal is to prove that $|J_n| \to \infty$ to obtain a contradiction.

If $J_n \subset A_{r(n)} \cup \hat{C}_{r(n)}$, lemma 4.12, $|f'(-1)| > 1$, and the minimal principle implies that $\left| f'_{r(n)}(x) \right| > 1$ for all $x \in J$. This yields $|J_{n+1}| > |J_n|$.

If $J_n \subset B_{r(n)}$, by definition of the rescaling, $|J_{n+1}| = \lambda_{r(n)} |J_n|$ where $\lambda_n = |s'_n(x)|$. The wandering intervals cannot stay in A_n forever since $f_n(x) > x$ for all $x \in \mathrm{int} A_n$ and -1 is the only fixed point in A_n. Hence, $J_n \subset B_{r(n)}$ for infinitely many n. Also, by the theory of infinitely renormalizable maps, $\lim_{n\to\infty} \lambda_{r(n)} = \lambda$ for some constant $\lambda > 1$. This shows that $\lim_{n\to\infty} |J_n| = \infty$ which is a contradiction. Therefore, f does not have a wandering interval. □

4.3.2. Proof by renormalization about the critical value

For the second proof, we use the renormalization about the critical value to prove that the wandering interval does not exist by contradiction. If a wandering interval exists, we construct a sequence of wandering intervals by iteration and rescaling then prove the hyperbolic length of the wandering interval approaches infinity.

First, we define a partition on the domain D for hyperbolic length. Since the hyperbolic length only increases on a injective branch of a unimodal map and f is not injective on B, the partition A, B, and C is not useful for studying hyperbolic length. We need to modify the partition. For $f \in \mathscr{U}$ that has a fixed point p of negative multiplier, define $B^l = B^l(f) = (\hat{p}, c^{(0)})$, $B^r = B^r(f) = (c^{(0)}, p)$, and $\hat{A} = A \cup \{\hat{p}\} \cup B^l = (-1, c^{(0)})$. Then $\hat{A}$, B^r, and C forms a partition of $D = (-1, c^{(1)})$ and f is injective on each subset.

The following lemma shows that the dynamics of a wandering interval can be described by this partition.

Lemma 4.15. *Assume that $f \in \mathscr{U}^r$ and $J \subset D(f)$ is a wandering interval of f that does not contain the critical point. Then J is a subset of one of the sets $\hat{A}$, B^r, or C.*

Proof. It follows by a wandering interval cannot contain any periodic point and preperiodic point. □

To apply the tool of renormalization about the critical value, we also need the following definition. For $f \in \mathscr{I}$, define $f_n = R_v^n f$ and write $f_{n+1} = s_n \circ f_n^2 \circ s_n^{-1}$. We abbreviate $\hat{A}_n = \hat{A}(f_n)$, $B_n^l = B^l(f_n)$, $B_n^r = B^r(f_n)$, and $C_n = C(f_n)$.

Lemma 4.16. *Assume that $f \in \mathscr{U}^r$ and $J \subset C$ is an interval such that $c^{(0)}(f) \notin f^n(J)$ for all $n \geq 0$. Then $c^{(0)}(R_v f) \notin (R_v f)^n \circ s_v(J)$ for all $n \geq 0$.*

Proof. Prove by contradiction. If $c^{(0)}(R_v f) \in (R_v f)^n \circ s_v(J)$ for some $n \geq 0$, by the definition of the renormalization operator, we have $c^{(0)}(R_v f) \in (R_v f)^n \circ s_v(J) = s_v \circ f^{2n}(J)$. Also by the definition of the renormalization operator, we have $f \circ s_v^{-1} \circ c^{(0)}(R_v f) = c^{(0)}(f)$. Combine the two equations, we get $c^{(0)}(f) \in f^{2n+1}(J)$ which is a contradiction. Therefore, $c^{(0)}(R_v f) \notin (R_v f)^n s_v(J)$ for all $n \geq 0$. □

From the two lemmas, if there is a wandering interval such that any iterate does not contain the critical point, then its itinerary under iteration and rescaling follows the path of the following graph.

Therefore, we construct a sequence of wandering intervals by iteration and rescaling as follows.

$$\begin{tikzcd}
 & \hat{A}_n \arrow[loop right, "f_n"] \arrow[d, "f_n"] \arrow[dd, bend left, "f_n"] & \\
 & B_n^r \arrow[d, "f_n"] & \\
C_{n-1} \arrow[uur, "s_{n-1}"] \arrow[ur, "s_{n-1}"] \arrow[r, "s_{n-1}"] & C_n \arrow[r, "s_n"] & D_{n+1}
\end{tikzcd}$$

Figure 4.1.: Itinerary for a sequence of wandering interval.

Definition 4.17. Assume that J is a wandering interval of $f \in \mathscr{I}$ such that $c^{(0)}(f) \notin f^n(J)$ for all $n \geq 0$. Define the level of renormalization $r(n)$ and a closest approach J_n of $f_{r(n)}$ by the following procedure.

1. Define $r(0) = 0$ and $J_0 = J$.
2. If $J_n \subset \hat{A}_{r(n)} \cup B^r_{r(n)}$, define $r(n+1) = r(n)$ and $J_{n+1} = f_{r(n)}(J_n)$.
3. If $J_n \subset C_{r(n)}$, define $r(n+1) = r(n) + 1$ and $J_{n+1} = s_n(J_n)$.

Next, we study the expansion of hyperbolic length for each edge on the graph.
We need the following linear approximation for logarithm when J is small.

Lemma 4.18. *For all $\varepsilon > 0$, there exists a constant $\delta > 0$ such that*

$$(1-\varepsilon)x \leq \ln(1+x) \leq x$$

for all $0 \leq x < \delta$.

Proof. The upper bound is trivial and holds for all $x \geq 0$.

To the lower bound, we know that $\frac{1}{1+x} \nearrow 1$ as $x \searrow 0$. For any given $\varepsilon > 0$, there exists a constant $\delta > 0$ such that

$$1 - \varepsilon \leq \frac{1}{1+x}$$

for all $0 \leq x < \delta$. Integrate both sides on $[0,x]$, we get

$$(1-\varepsilon)x \leq \ln(1+x).$$

$\square$

We estimate the hyperbolic distance of an interval in the following setting. Assume that $J = [e,f]$, $T' = (c,d)$, and $T = (a,b)$ are intervals such that $J \subset T' \subset T$. Let $L = (a,c)$, $R = (d,b)$, $l = (c,e)$, and $r = (f,d)$.

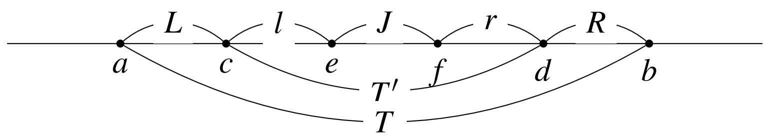


Figure 4.2.: Definition of T, T', L, l, R, r, and J.

Lemma 4.19. *For all $\varepsilon > 0$, there exists a constant $\delta > 0$ such that the following property holds. Then*

$$|J|_{T'} \geq (1-\varepsilon)\left(1+2\frac{LR}{TT'}\right)|J|_T$$

whenever $\frac{J}{l} < \delta$ and $\frac{J}{r} < \delta$.

Proof. Given $\varepsilon > 0$. Let $\delta > 0$ to be defined by Lemma 4.18.

Apply the linear approximation for logarithm to the definition of hyperbolic distance, we have

$$\begin{aligned} |J|_T &= \ln\left(1+\frac{J}{l+L}\right)+\ln\left(1+\frac{J}{r+R}\right) \\ &\leq \left(\frac{1}{l+L}+\frac{1}{r+R}\right)J \end{aligned}$$

and

$$\begin{aligned} |J|_{T'} &= \ln\left(1+\frac{J}{l}\right)+\ln\left(1+\frac{J}{r}\right) \\ &\geq (1-\varepsilon)\left(\frac{1}{l}+\frac{1}{r}\right)J. \end{aligned}$$

Combine the two inequalities, we get

$$\begin{aligned} |J|_{T'} &\geq (1-\varepsilon)\left(\frac{1}{l}+\frac{1}{r}\right)\left(\frac{1}{l+L}+\frac{1}{r+R}\right)^{-1}|J|_T \\ &= (1-\varepsilon)\left(1+\frac{\frac{1}{l}-\frac{1}{l+L}+\frac{1}{r}-\frac{1}{r+R}}{\frac{1}{l+L}+\frac{1}{r+R}}\right)|J|_T \\ &= (1-\varepsilon)\left[1+\frac{\frac{L}{l(l+L)}+\frac{R}{r(r+R)}}{\frac{1}{l+L}+\frac{1}{r+R}}\right]|J|_T \\ &= (1-\varepsilon)\left[1+\frac{\frac{L}{l}(r+R)+\frac{R}{r}(l+L)}{r+R+l+L}\right]|J|_T \\ &\geq (1-\varepsilon)\left(1+\frac{\frac{LR}{T'}+\frac{RL}{T'}}{T}\right)|J|_T \\ &= (1-\varepsilon)\left(1+2\frac{LR}{TT'}\right)|J|_T\,. \end{aligned}$$

□

Lemma 4.20. *The equality holds for all J*

$$|J|_{T'} - |J|_T = \ln\left(1+\frac{J}{l}\frac{L}{l+L+J}\right)\left(1+\frac{J}{r}\frac{R}{r+R+J}\right).$$

Proof. Apply the definition of hyperbolic distance, compute

$$\begin{aligned}
|J|_{T'} - |J|_T &= \ln\frac{(l+J)(r+J)}{lr} - \ln\frac{(l+L+J)(r+R+J)}{(l+L)(r+R)} \\
&= \ln\left(\frac{1+\frac{J}{l}}{1+\frac{J}{l+L}}\right)\left(\frac{1+\frac{J}{r}}{1+\frac{J}{r+R}}\right) \\
&= \ln\left[1+\frac{\left(\frac{1}{l}-\frac{1}{l+L}\right)J}{1+\frac{J}{l+L}}\right]\left[1+\frac{\left(\frac{1}{r}-\frac{1}{r+R}\right)J}{1+\frac{J}{r+R}}\right] \\
&= \ln\left[1+\frac{\frac{JL}{l(l+L)}}{1+\frac{J}{l+L}}\right]\left[1+\frac{\frac{JR}{r(r+R)}}{1+\frac{J}{r+R}}\right] \\
&= \ln\left(1+\frac{J}{l}\frac{L}{l+L+J}\right)\left(1+\frac{J}{r}\frac{R}{r+R+J}\right).
\end{aligned}$$

□

Corollary 4.21. *Assume that $J \Subset intT'$ and $T' \subset T$, then $|J|_{T'} \geq |J|_T$.*

Proof. The inequality follows directly from Lemma 4.20. □

Corollary 4.22. *For all $\varepsilon > 0$, there exists a constant $\delta > 0$ such that for all J one of the following inequality holds*

$$|J|_{T'} \geq (1-\varepsilon)\left(1+2\frac{LR}{TT'}\right)|J|_T$$

or

$$|J|_{T'} \geq |J|_T + \ln\left[1+\delta\frac{\min(L,R)}{T}\right].$$

Proof. Given $\varepsilon > 0$. Let $\delta > 0$ be defined in Lemma 4.19. If $\frac{J}{l}, \frac{J}{r} < \delta$, then

$$|J|_{T'} \geq (1-\varepsilon)\left(1+2\frac{LR}{TT'}\right)|J|_T.$$

If $\frac{J}{l} > \delta$ or $\frac{J}{r} > \delta$, by Lemma 4.20, we have

$$\begin{aligned}
|J|_{T'} - |J|_T &\geq \ln\left(1+\frac{J}{l}\frac{\min(L,R)}{T}\right) + \ln\left(1+\frac{J}{r}\frac{\min(L,R)}{T}\right) \\
&\geq \ln\left(1+\delta\frac{\min(L,R)}{T}\right).
\end{aligned}$$

□

From the theory of infinitely renormalizable maps, we have

Lemma 4.23. *Assume that $f \in \mathscr{I}$ has negative Schwarzian derivative. Then the ratio of any two lengths of the interval in the list D, A, B, C, $\hat{A}$, B^l, and B^r are bounded above and below by a universal constant.*

Corollary 4.24. *Assume that $f \in \mathscr{I}$ has negative Schwarzian derivative. If J is a subinterval of B, then $|J|_D$ is bounded by a universal constant.*

Proof. Compute

$$|J|_D \leq |B|_D = \ln\left(1+\frac{B}{A}\right)+\ln\left(1+\frac{B}{C}\right).$$

By the previous Lemma, the quantity is bounded above by a universal constant. □

Now we are ready to study the expansion for each path in Figure 4.1. If an interval is in $\hat{A}$ or B^r, we use the hyperbolic length in D to measure the interval. Otherwise, if an interval is in C, we use the hyperbolic length in C to measure the interval.

Lemma 4.25. *There exist constants $c > 1$ and $d > 0$ such that the following properties hold. Assume that $f \in \mathscr{I}$ has negative Schwarzian derivative and J is an interval. Then we have the following two tables for the expansion of hyperbolic distance:*

J	$f(J)$	*Expansion*
$\hat{A}$	$\hat{A}$	$\lvert f(J)\rvert_D > \lvert J\rvert_D$
$\hat{A}$	B^r	$\lvert f(J)\rvert_D > \lvert J\rvert_D$
$\hat{A}$	C	$\lvert f(J)\rvert_C > c\lvert J\rvert_D$ *or* $\lvert f(J)\rvert_C > \lvert J\rvert_D+d$
B^r	C	$\lvert f(J)\rvert_C > c\lvert J\rvert_D$ *or* $\lvert f(J)\rvert_C > \lvert J\rvert_D+d$

J	$s_v(J)$	*Expansion*
C	$\hat{A}$	$\lvert s_v(J)\rvert_{D(Rf)} = \lvert J\rvert_{C(f)}$
C	B^r	$\lvert s_v(J)\rvert_{D(Rf)} = \lvert J\rvert_{C(f)}$
C	C	$\lvert s_v(J)\rvert_{C(Rf)} \geq \lvert J\rvert_{C(f)}$

Proof. For any $\varepsilon > 0$, let $\delta > 0$ to be defined in Proposition 4.18.

Case one and two: Assume that $J \subset \hat{A}$ and $f(J) \subset \hat{A}$ or $f(J) \subset B^r$. Since $f : \hat{A} \to D$ is a diffeomorphism, apply Proposition 2.10 and Corollary 4.21, we get

$$|J|_D \leq |J|_{\hat{A}} < |f(J)|_D.$$

Case three: Assume that $J \subset \hat{A}$ and $f(J) \subset C$. Then $J \subset B^l$. Since $f : B^l \to C$ is a diffeomorphism, by Proposition 2.10, we have

$$|J|_{B^l} < |f(J)|_C.$$

Also, apply Corollary 4.22 to the sets $J \subset B^l \subset D$, we get

$$|f(J)|_C > |J|_{B^l} \geq (1-\varepsilon)\left[1+2\frac{A(B^r+C)}{DB^l}\right]|J|_D \tag{4.1}$$

or

$$|f(J)|_C > |J|_{B^l} \geq |J|_D + \ln\left[1 + \delta \frac{\min(A, B^r + C)}{D}\right]. \tag{4.2}$$

Case four: Assume that $J \subset B^r$ and $f(J) \subset C$. Since $f : B^r \to C$ is a diffeomorphism, by Proposition 2.10, we have

$$|J|_{B^r} < |f(J)|_C.$$

Also, apply Corollary 4.22 to the sets $J \subset B^r \subset D$, we get

$$|f(J)|_C > |J|_{B^r} \geq (1 - \varepsilon)\left(1 + 2\frac{\hat{A}C}{DB^r}\right)|J|_D \tag{4.3}$$

or

$$|f(J)|_C > |J|_{B^r} \geq |J|_D + \ln\left[1 + \delta \frac{\min(\hat{A}, C)}{D}\right]. \tag{4.4}$$

Case five and six follow directly from Proposition 2.11. Case seven follows from Proposition 2.11 and Corollary 4.21.

Finally by Lemma 4.23, the ratios $\frac{A(B^r+C)}{DB^l}$ and $\frac{\hat{A}C}{DB^r}$ in (4.1) and (4.3) are bounded below by a universal constant. We may assume that $\varepsilon > 0$ be so small such that $(1-\varepsilon)\left[1 + 2\frac{A(B^r+C)}{DB^l}\right] > c$ and $(1-\varepsilon)\left(1 + 2\frac{\hat{A}C}{DB^r}\right) > c$ for some universal constant $c > 1$. This fixes δ. Also, $\frac{\min(A, B^r+C)}{D}$ and $\frac{\min(\hat{A}, C)}{D}$ in (4.2) and (4.4) are bounded below by the same reason. Therefore, the Lemma is proved. $\square$

With the expansion of hyperbolic length, we are able to prove our goal for this section.

Proposition 4.26. *Assume that $f \in \mathscr{I}$ is infinitely renormalizable about the critical value and has negative Schwarzian derivative. Then f does not have wandering interval.*

Proof. Prove by contradiction. Assume that $J \subset D$ is a wandering interval of f. Without lose of generality, we may assume that $c^{(0)}(f) \notin f^n(J)$ for all $n \geq 0$ by Proposition 4.11. Define the level of renormalization $r(n)$ and a closest approach J_n of $f_{r(n)}$ by Definition 4.17.

Consider the sequence elements with the same renormalization scale $j > 1$. Let s and t be integers such that $r(s+1) = \cdots = r(s+t) = j$, $r(s) = j-1$, and $r(s+t+1) = j+1$. The path of the sequence has four different cases in the same renormalization scale by Figure 4.1:

1. $C_{j-1} \xrightarrow{s_{j-1}} \hat{A}_j \xrightarrow{f_j} B_j^r \xrightarrow{f_j} C_j$,

2. $C_{j-1} \xrightarrow{s_{j-1}} \hat{A}_j \xrightarrow{f_j} C_j$,

3. $C_{j-1} \xrightarrow{s_{j-1}} B_j^r \xrightarrow{f_j} C_j$, and

4. $C_{j-1} \xrightarrow{s_{j-1}} C_j$.

The iterations in $\hat{A}_j$ cannot occur infinitely many times because there is only one fixed point -1 in $\overline{\hat{A}_j}$.

Also, at least one of the cases 1, 2, or 3 must occur infinitely many j. Otherwise, there exists a constant $s > 0$ such that $r(t+1) = r(t) + 1$ for all $t \geq s$. Then

$$|J_{t+1}| = \lambda_{(t-s)+r(s)} |J_t|$$

for all $t \geq s$ where $\lambda_n = |s_n'(x)|$. This implies that $\lim_{t\to\infty} |J_t| = \infty$ since $\lambda_n \to \lambda > 1$ by the theory of period doubling renormalization which yields a contradiction.

Finally, by Lemma 4.25, the cases 1, 2, and 3 gives a strict expansion with fixed rate as

$$|J_{s+t}|_{C_j} > c\,|J_s|_{C_{j-1}}$$

or

$$|J_{s+t}|_{C_j} > |J|_{C_{j-1}} + d.$$

And the forth case gives also expands the hyperbolic length

$$|J_{s+1}|_{C_j} > |J_s|_{C_{j-1}}.$$

Since at least one of the cases 1, 2, or 3 must occurs infinitely many times, the hyperbolic length of the sequence approaches infinity which contradicts to Corollary 4.24 and the sequence must enters $B \subset \hat{A} \cap B^r$ before entering C. Therefore, wandering interval does not exists. □

4.4. The fixed point of the renormalization operator

In this section, we study the fixed point g of the renormalization operator about the critical point. The map g is also important for the Hénon case because it also defines the hyperbolic fixed point of the Hénon renormalization operator [dCLM05, Theorem 4.1].

To define the fixed point, [EL81, CER82] proved that

Lemma 4.27. *There exist a unique constant $\lambda = 2.5029...$ and a unique function $f : [-1,1] \to [-1,1]$ that satisfies the Cvitanović-Feigenbaum-Coullet-Tresser functional equation*

$$\begin{cases} f(x) = -\lambda f^2\left(-\frac{x}{\lambda}\right), & -1 \leq x \leq 1 \\ f(0) = 1 \end{cases}$$

The function f has the following properties:

1. *f is analytic in a complex neighborhood of $[-1,1]$.*
2. *f is even and concave: $f''(x) < 0$ for all $-1 \leq x \leq 1$.*
3. *$\lambda > 1$, $f(1) = -\frac{1}{\lambda}$, and $f'(1) = -\lambda$.*
4. *f has negative Schwarzian derivative.*

The solution in this lemma is not in the right scale of unimodal map in this article. It does not contain a fixed point with positive multiplier. The next proposition convert the solution into the correct scale that this article is using.

Proposition 4.28. *There exist a unique constant $\lambda = 2.5029...$ and a unique solution $g \in \mathscr{I}$ of the Cvitanović-Feigenbaum-Coullet-Tresser functional equation*

$$g(x) = -\lambda g^2\left(-\frac{x}{\lambda}\right) \tag{4.5}$$

for $-1 \leq x \leq 1$ with the following properties:

1. *g is analytic in a complex neighborhood of $[-1,1]$.*
2. *g is even.*
3. *g is concave on $[-c^{(1)}, c^{(1)}]$.*
4. *$g(c^{(1)}) = -\frac{1}{\lambda}c^{(1)}$ and $g'(c^{(1)}) = -\lambda$.*
5. *g has negative Schwarzian derivative.*

Proof. Let f be the solution in the previous lemma. Since $f(0) = 1$ and $f(1) = -\frac{1}{\lambda}$, it follows by the intermediate value that f has a fixed point $\alpha \in (0,1)$. The fixed point is unique on $[0,1]$ and has negative eigenvalue because f is decreasing on $[0,1]$. However, f does not have a fixed point on $[-1,0]$ because $f(-1) = -\frac{1}{\lambda} > -1$, $f(0) = 1 > 0$, and f is concave. So we need to first extend f such that the extension contains a fixed point with positive multiplier.

First we construct an extension of f. Define $\hat{f}(x) = -\lambda f^2\left(-\frac{x}{\lambda}\right)$ for $-\lambda \leq x \leq \lambda$. It satisfies the functional equation (4.5) for all $-\lambda \leq x \leq \lambda$ because

$$\hat{f}(x) = -\lambda f^2\left(-\frac{x}{\lambda}\right) = \lambda^2 f^4\left(-\frac{x}{\lambda^2}\right) = -\lambda \hat{f}^2\left(-\frac{x}{\lambda}\right).$$

Thus, $\hat{f}$ is an extension of f on $[-\lambda, \lambda]$.

The point $-\lambda\alpha \in [-\lambda, 0]$ is a fixed point of $\hat{f}$. This is because

$$\hat{f}(-\lambda\alpha) = -\lambda f^2(\alpha) = -\lambda\alpha.$$

Also, we show that $\hat{f}$ has a critical point on $(-\lambda, -\lambda\alpha)$. Apply the intermediate value theorem to $f(\alpha) > 0$ and $f(1) < 0$, there exists a constant $c' \in (\alpha, 1)$ such that $f(c') = 0$. The root is unique on $(0,1)$ since f is decreasing on the interval. By simple computation,

$$\hat{f}'(-\lambda c') = f' \circ f(c') f'(c') = 0.$$

Thus, $-\lambda c'$ is a critical point of $\hat{f}$ on $(-\lambda, -\lambda\alpha)$.

Moreover, $\hat{f}$ does not have any other critical point on $(-\lambda c', 0)$. Critical point does not exist on $[-1,0)$ because $\hat{f}$ is an extension of f and f is concave on $[-1,1]$. If $\hat{f}$ has a critical point $c'' \in (-\lambda c', -1)$, then

$$0 = \hat{f}(c'') = f' \circ f\left(-\frac{c''}{\lambda}\right) f'\left(-\frac{c''}{\lambda}\right).$$

This implies that $f\left(-\frac{c''}{\lambda}\right)=0$ since $-\frac{c''}{\lambda}\neq 0$ which contradicts to c' is the unique root on $(0,1)$. Consequently, $\hat{f}$ is increasing on $(-\lambda c',0)$.

It follows that $-\lambda\alpha$ is the only fixed point on $(-\lambda c',-1)$. This is because $\hat{f}$ has negative Schwarzian derivative. It contains only one inflection on $(-\lambda c',0)$ by Corollary 2.5. If $\hat{f}$ has more than one fixed point in $(-\lambda c',-1)$, then $\hat{f}$ will have more then one inflection point which is impossible.

Finally, we rescale $\hat{f}$ by defining $g(x)=\frac{1}{\alpha\lambda}f(\alpha\lambda x)$ for $-1\leq x\leq 1$. The map g has only two fixed points for -1 and $\frac{1}{\lambda}$. Also, by definition, g satisfies the functional equation (4.5). Hence, $g\in\mathscr{I}$. It is easy to check that g satisfies all the desired properties. □

Corollary 4.29. *The map g satisfies the following property*

$$g^{2^n}\left(\frac{1}{(-\lambda)^n}x\right)=\frac{1}{(-\lambda)^n}g(x) \tag{4.6}$$

for all $n\geq 0$ and all $x\in I$.

Proof. Prove by induction. The case $n=0$ is trivial. Assume that the lemma is true for n. For the case $n+1$, apply the induction hypothesis, we have

$$g^{2^{n+1}}\left(\frac{x}{(-\lambda)^{n+1}}\right)=g^{2^n}\circ g^{2^n}\left(\frac{1}{(-\lambda)^n}\frac{x}{-\lambda}\right)=g^{2^n}\left(\frac{1}{(-\lambda)^n}g\left(\frac{x}{-\lambda}\right)\right)=\frac{1}{(-\lambda)^n}g^2\left(\frac{x}{-\lambda}\right).$$

Also the functional equation (4.5) yields

$$g^{2^{n+1}}\left(\frac{x}{(-\lambda)^{n+1}}\right)=\frac{1}{(-\lambda)^n}g^2\left(\frac{x}{-\lambda}\right)=\frac{1}{(-\lambda)^{n+1}}g(x).$$

Therefore the lemma is proved by induction. □

In the remaining part of the section, the notations for the unimodal maps will be applied to the map g. For example, $\left\{c^{(j)}=c^{(j)}(g)\right\}_{j\geq 0}$ is the critical orbit and the sets $A=A(g)$, $B=B(g)$, and $C=C(g)$ form a partition of the domain $D=I$. In Definition 4.31, we will use another notation q (instead of p) to denote the fixed points because g is special, i.e. the points $q(-1)=-1$ and $q(0)$ are the fixed points with positive and negative multiplier respectively.

4.4.1. The periodic points of period 2^n

Lemma 4.30. *If $p\in I$ is a fixed point for g, then $\frac{p}{(-\lambda)^n}$ is a fixed point for g^{2^n}.*

Proof. The proof is straightforward from (4.6) by setting $x=p$

$$g^{2^n}\left(\frac{p}{(-\lambda)^n}\right)=\frac{1}{(-\lambda)^n}g(p)=\frac{p}{(-\lambda)^n}.$$

□

The periodic points of g can be written by an explicit formula.

Definition 4.31. Define $q^c(n) = -\left(-\frac{1}{\lambda}\right)^{n+1}$ and $q(n) = g(q^c(n))$ for all integers $n \geq -1$.

Next proposition shows that these values are the periodic points for g. The superscript c for q^c stands for the periodic points around the critical point. The other collection of points $q(n)$ are the periodic points around the critical value. The superscript is suppressed from the later notation because it is important for the Hénon renormalization.

Proposition 4.32. *For $n \geq 0$, $q^c(n)$ is a periodic point of period 2^n for g. Moreover, $q^c(-1) = q(-1)$ and $q^c(0) = q(0)$ are the fixed points for g.*

Proof. By the definition of unimodal map, $q^c(-1) = q(-1) = -1$ is the fixed point with positive multiplier.

We prove $q^c(n)$ is a periodic point of period 2^n for $n \geq 0$ by induction.

For the base case $n = 0$, we know that the rescaling function is $s_c(x) = -\lambda x$ from the functional equation (4.5). From the rescaling function, we get

$$q^c(-1) = -1 = s_c \circ q^c(0).$$

Thus, $q^c(0) = q(0) = \frac{1}{\lambda}$ is the other fixed point of g.

Assume that $q^c(n)$ is a periodic point of period 2^n. By Lemma 4.30, we see that $q^c(n+1)$ is a fixed point for $g^{2^{n+1}}$. It suffices to show that the orbit $q^c(n+1), g(q^c(n+1)), \cdots, g^{2^{n+1}-1}(q^c(n+1))$ are distinct points. By the functional equation $s_c \circ g^2 \circ s_c^{-1} = g$, we have

$$s_c \circ g^{2j}(q^c(n+1)) = g^j \circ s_c(q^c(n+1)) = g^j(q^c(n))$$

for all $j \geq 0$. Thus, the points $q^c(n+1), g^2(q^c(n+1)), \cdots, g^{2^{n+1}-2}(q^c(n+1))$ are distinct elements in B because $q^c(n)$ is a periodic point of period 2^n by the induction hypothesis. Also, $g(B) \subset C$ and $B \cap C = \phi$. Therefore, $q^c(n+1)$ is a periodic point of period 2^{n+1}. The proposition is proved by induction. □

4.4.2. The orbit of the critical point

First, we derive a formula for the forward orbit of the critical points.

Lemma 4.33. **The orbit of the critical point satisfies the equality*

$$c^{(2^n)} = \left(\frac{-1}{\lambda}\right)^n c^{(1)}$$

for all $n \geq 0$.

Proof. From the functional equation (4.6), we have

$$c^{(2^n)} = g^{2^n}(0) = \frac{1}{(-\lambda)^n} g(0) = \left(\frac{-1}{\lambda}\right)^n c^{(1)}.$$

□

Then, we study a backward orbit of the critical point. Let $b^{(1)} \in [0, c^{(1)}]$ be the point such that $g(b^{(1)}) = 0$. Set $b^{(2)} = \frac{1}{\lambda} b^{(1)}$.

Lemma 4.34. *We have*

$$g\left(b^{(2)}\right) = b^{(1)}.$$

Proof. Since g is even, the only two roots of g are $-b^{(1)}$ and $b^{(1)}$. By the functional equation (4.6), we have

$$g^2\left(b^{(2)}\right) = -\frac{1}{\lambda} g\left(-b^{(1)}\right) = 0.$$

Thus, $g\left(b^{(2)}\right) = -b^{(1)}$ or $b^{(1)}$. Also, $g\left(b^{(2)}\right) \neq -b^{(1)}$ because $b^{(2)} \in (0, b^{(1)})$ and $g(x) > 0$ on $(0, b^{(1)})$. Therefore, $g\left(b^{(2)}\right) = b^{(1)}$. □

4.4.3. Estimations for the derivative

Apply the chain rule to the functional equation (4.5), we have

$$g'(x) = g'\left(-\frac{x}{\lambda}\right) g' \circ g\left(-\frac{x}{\lambda}\right) \tag{4.7}$$

for $x \in I$. We will use this formula to derive the values for the derivative of g at some particular values.

Lemma 4.35. *The slope at the critical value is*

$$g'(c^{(1)}) = -\lambda. \tag{4.8}$$

Proof. The lemma follows either from Proposition 4.28 or from the functional equation explained below.

Take the derivative of (4.7), we have

$$g''(x) = -\frac{1}{\lambda}\left[g''\left(-\frac{x}{\lambda}\right) g' \circ g\left(-\frac{x}{\lambda}\right) + \left(g'\left(-\frac{x}{\lambda}\right)\right)^2 g'' \circ g\left(-\frac{x}{\lambda}\right)\right].$$

Substitute $x = 0$, we get

$$g''(c^{(0)}) = -\frac{1}{\lambda} g''(c^{(0)}) g'(c^{(1)}).$$

Since the critical point is non-degenerate, we solved

$$g'(c^{(1)}) = -\lambda.$$

□

Lemma 4.36. *The slope at $b^{(2)}$ is*

$$g'(b^{(2)}) = -1. \tag{4.9}$$

Proof. From (4.7) and g is even, we have

$$g'(b^{(1)}) = g'\left(-b^{(2)}\right) g' \circ g\left(-b^{(2)}\right) = -g'\left(b^{(2)}\right) g'\left(b^{(1)}\right).$$

We solve $g'(b^{(2)}) = -1$. $\square$

Recall from Definition that 4.31 $q(-1) = -1$ is the fixed point with a positive multiplier and $q(0) = \frac{1}{\lambda}$ is the fixed point with a negative multiplier.

Lemma 4.37. *The slopes at the fixed points satisfy the relation*

$$g'(q(-1)) = \left[g'(q(0))\right]^2. \tag{4.10}$$

Proof. From (4.7), compute

$$g'(q(-1)) = g'(-1) = g'\left(\frac{1}{\lambda}\right) g' \circ g\left(\frac{1}{\lambda}\right) = g'(q(0))\, g' \circ g(q(0)) = \left[g'(q(0))\right]^2.$$

$\square$

Finally, we prove that the map g is expanding on A and C.

Proposition 4.38. *The slope of g is bounded below by*

$$\left|g'(x)\right| \geq \left|g'(q(0))\right| > 1$$

for all $x \in [q(-1), \hat{q}(0)] \cup [q(0), \hat{q}(-1)]$.

Proof. It is enough to prove the case when $x \in [q(0), \hat{q}(-1)]$ since g is even.

First, we consider the interval $[b^{(2)}, c^{(1)}]$. We have $b^{(2)} < q(0) < c^{(1)}$. By (4.9) and Proposition 4.28, the derivatives of the boundaries are $g'(b^{(2)}) = -1$ and $g'(c^{(1)}) = -\lambda$. We get $|g'(q(0))| > 1$ by the minimal principle (Proposition 2.4).

Next, we consider the interval $[q(0), \hat{q}(-1)]$. From (4.10), we also get $|g'(\hat{q}(-1))| > 1$. Therefore, the proposition follows from the minimal principle (Proposition 2.4). $\square$

4.4.4. *Concavity of g

From Proposition 4.28, we learned that g is concave on $[-c^{(1)}, c^{(1)}]$. In this section, we discuses the concavity on the remaining part of the domain $[-1, -c^{(1)}] \cup [c^{(1)}, 1]$.

Lemma 4.39. *The second derivative at the fixed point -1 satisfies*

$$g''(-1) > 0.$$

Proof. Take the derivative of (4.7), we have

$$g''(x) = -\frac{1}{\lambda}\left[g'' \circ g\left(-\frac{x}{\lambda}\right)\left(g'\left(-\frac{x}{\lambda}\right)\right)^2 + g''\left(-\frac{x}{\lambda}\right) g' \circ g\left(-\frac{x}{\lambda}\right)\right].$$

Substitute $x = -1$, we get

$$g''(-1) \quad = \quad -\frac{1}{\lambda}\left[g''(q(0))\left(g'(q(0))\right)^2 + g''(q(0))\, g'(q(0))\right]$$

$$= \quad -\frac{1}{\lambda} g''(q(0))\, g'(q(0)) \left[g'(q(0)) + 1\right].$$

By Proposition 4.28, $g''(q(0)) < 0$. Also, by Proposition (4.38), $g'(q(0)) < -1$. Therefore,

$$g''(-1) > 0.$$

□

Corollary 4.40. *The map g changes concavity on $(-1, -c^{(1)})$ and $(c^{(1)}, 1)$.*

5. Hénon-like Maps

In this chapter, we give an introduction to Hénon-renormalization based on the framework developed by [dCLM05, LM11].

5.1. The class of unimodal maps

Definition 5.1 (Class of unimodal maps)**.** Assume that $\delta > 0$, $\kappa > 0$, and $I^h \Supset I \equiv [-1,1]$. Let $\mathscr{U}_{\delta,\kappa}(I^h) \subset \mathscr{U}$ be the class of analytic unimodal maps $f : I^h \to \mathbb{R}$ such that

1. f has a unique critical point c such that $f(c) \in [c+\kappa, 1-\kappa]$,
2. f has two fixed points -1 and p such that -1 has an expanding positive multiplier and p has a negative multiplier,
3. f has holomorphic extension to $I^h(\delta)$,
4. f can be factorized as $f = Q \circ \phi$ where $Q(x) = c^{(1)} - (c^{(1)}+1)x^2$, $c^{(1)}$ is the critical value, and ϕ is an $\mathbb{R}$-symmetric univalent map on $I^h(\delta)$, and
5. f has negative Schwarzian derivative.

In the remaining part of the article, we fix a small constant $\kappa > 0$ such that the class contains the renormalization fixed point g, and we suppress the subscript from the notation $\mathscr{U}_\delta(I^h) = \mathscr{U}_{\delta,\kappa}(I^h)$.

Remark 5.2. From the conditions $f(-1) = -1$ and $f(1) = -1$, this forces $\phi(-1) = -1$ and $\phi(1) = 1$. Thus, $\mathscr{U}_\delta$ forms a normal family by [Mil11, Theorem 3.2].

5.2. The class of Hénon-like maps

Definition 5.3 (Hénon-like map)**.** Assume that $I^v \supset I^h \Supset I$ are closed intervals. A Hénon-like map is a smooth map $F : I^h \times I^v \to \mathbb{R}^2$ of the form

$$F(x,y) = (f(x) - \varepsilon(x,y), x)$$

where f is a unimodal map and ε is a small perturbation. The function h will also be used to express the x-component, $h_y(x) = h(x,y) = \pi_x F(x,y)$. A representation of F will be expressed in the form $F = (f - \varepsilon, x)$.

The function spaces of the Hénon-like map are defined below.

Definition 5.4 (Class of Hénon-like maps)**.** Assume that $I^v \supset I^h \Supset I$ and $\delta > 0$.

1. Denote $\mathscr{H}_\delta(I^h \times I^v)$ to be the class of real analytic Hénon-like maps $F : I^h \times I^v \to \mathbb{R}^2$ that have the following properties:

a) It has a representation $F = (f - \varepsilon, x)$ such that $f \in \mathscr{U}_\delta(I^h)$.

b) It has a saddle fixed point $p(-1)$ near the point $(-1,-1)$. The fixed point has an expanding positive multiplier.

c) The x-component $h(x,y)$ has a holomorphic extension to $I^h(\delta) \times I^v(\delta) \to \mathbb{C}$.

2. Given $\overline{\varepsilon} > 0$ and $f \in \mathscr{U}_\delta(I^h)$. Denote $\mathscr{H}_\delta(I^h \times I^v, f, \overline{\varepsilon})$ to be the class of Hénon-like maps $F \in \mathscr{H}_\delta(I^h \times I^v)$ with the form $F = (f - \varepsilon, x)$ such that $\|\varepsilon\| < \overline{\varepsilon}$.

3. Denote $\mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon}) = \cup \mathscr{H}_\delta(I^h \times I^v, f, \overline{\varepsilon})$ where the union is taken over all $f \in \mathscr{U}_\delta(I^h)$.

Remark 5.5. The domain $I^h \times I^v$ used in this article is larger than the domain studied in the two papers [dCLM05, LM11]. Their domain is equivalent to the dynamical interval $[f^2(c), f(c)]$ for unimodal maps which does not include the fixed point with an expanding positive multiplier. The larger domain is necessary in this article to study the rescaled orbit of a point. See Proposition 5.11, Proposition 5.16, and Proposition 6.3.

Their work also holds on the extended domain $I^h \times I^v$. See for examples [dCLM05, Footnote 7, Section 3.4] and [LM11, Lemma 3.3, Proposition 3.5, Theorem 4.1]. However, reproving their theorem on the larger domain is not the aim here. Here, we assume the results from [dCLM05, LM11] also hold in the extended domain and rephrase them in the notations used in this article without reproving. See also Remarks 5.18, 5.29, and 11.19.

From the definition, it follows immediately that

Lemma 5.6. *Given $I^v \supset I^h \Supset I$, $\delta > 0$, $\overline{\varepsilon} > 0$, and $f \in \mathscr{U}_\delta(I^h)$.*

1. If $\overline{\varepsilon_1} < \overline{\varepsilon_2}$ then $\mathscr{H}_\delta(f, \overline{\varepsilon_1}) \subset \mathscr{H}_\delta(f, \overline{\varepsilon_2})$.

2. If $I \subset I^h_1 \subset I^h_2 \subset I^v$ and $f \in \mathscr{U}_\delta(I^h_2)$, then $\mathscr{U}_\delta(I^h_1) \supset \mathscr{U}_\delta(I^h_2)$ and $\mathscr{H}_\delta(I^h_1 \times I^v, f, \overline{\varepsilon}) \supset \mathscr{H}_\delta(I^h_2 \times I^v, f, \overline{\varepsilon})$.

An important property of a Hénon-like map is that it maps vertical lines to horizontal lines; it maps horizontal lines to parabola-like arcs.

Example 5.7 (Degenerate case)**.** Assume that $F(x,y) = (f(x) - \varepsilon(x,y), x)$ is a Hénon-like map. The map is called a degenerate Hénon-like map if $\frac{\partial \pi_x F}{\partial y} = \frac{\partial \varepsilon}{\partial y} = 0$; a non-degenerate Hénon-like map if $\frac{\partial \pi_x F}{\partial y} = \frac{\partial \varepsilon}{\partial y} \neq 0$.

If F is degenerate, then ε depends only on x. In this case, without lose of generality, we will assume the Hénon-like map has the representation $F(x,y) = (f(x), x)$ where $f = \pi_x F$ and $\varepsilon = 0$.

For the degenerate case, the dynamics of the Hénon-like map is completely determined by its unimodal component. So it will also be called as the unimodal case in this article.

The degenerate case is an important example in this article. A proof for the nonexistence of wandering intervals for unimodal maps will be presented in Chapter 8 by identifying a unimodal map as a degenerate Hénon-like map. The expansion estimate in the proof motivates the proof for the non-degenerate case. The difference between the degenerate case and the non-degenerate case produces the main difficulty (explained in Chapter 9 and Chapter 11) of extending the proof to the non-degenerate case.

Example 5.8 (Classical Hénon maps)**.** The classical Hénon family is a two-parameter family of the form $F_{a,b}(x,y) = (-1 + a(1-x^2) - by, x)$ where $a, b > 0$. These are Hénon-like maps $F_{a,b} \in \mathscr{H}_\delta(I^h \times I^v, -1 + a(1-x^2), b[|I^v| + 2\delta])$ for all $\delta > 0$ and $I^v \supset I^h$.

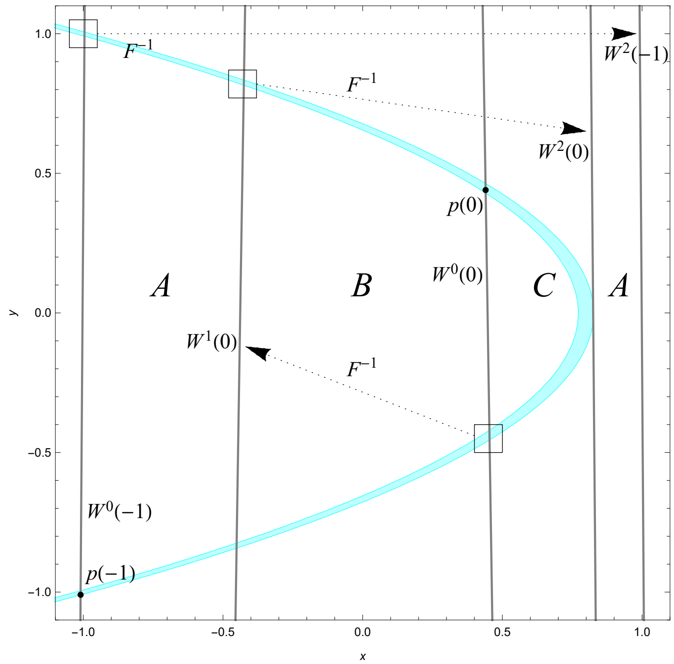


Figure 5.1.: Local stable manifolds and the partition of a Hénon-like map F. The shaded area is the image of the Hénon-like map. The vertical graphs are the local stable manifolds $W^0(-1)$, $W^1(0)$, $W^0(0)$, $W^2(0)$, and $W^2(-1)$ from left to right. The arrows illustrates the construction of each local stable manifold.

5.3. Local stable manifolds and the partition of a Hénon-like map

To study the dynamics of a Hénon-like map, we need to find a domain $D \subset I^h \times I^v$ that turns the Hénon-like map into a self-map. Also, to renormalize a Hénon-like map, we need to find a subdomain $C \subset D$ that defines a first return map. Inspired from unimodal maps, we construct a partition of the domain $I^h \times I^v$. In the unimodal case, an orbit that maps to the fixed point $p(0)$ with an expanding multiplier splits the domain D into a partition $\{A, B, C\}$ (Definition 4.4). For a strongly dissipative Hénon-like map, the orbit becomes components of the stable manifold of the saddle fixed point $p(0)$. These components are vertical graphs that split the domain into multiple vertical strips.

Definition 5.9 (Vertical graph)**.** A set Γ is a vertical graph if there exists a continuous function $\gamma : I^v \to I^h$ such that $\Gamma = \{(\gamma(t), t); t \in I^v\}$. The vertical graph Γ is said to have Lipschitz constant L if the function γ is Lipschitz with constant L.

In this paper, a local stable manifold is a connected component of a stable manifold. Inspired by [dCLM05], the partition elements are defined by the vertical strips separated by the associated local stable manifolds which are vertical graphs.

First, we study the local stable manifolds of the saddle fixed point $p(-1)$ which contains an expanding positive multiplier.

Definition 5.10 (The local stable manifolds of $p(-1)$ and the iteration domain D)**.** Given $I^v \supset I^h \Supset I$, $\delta > 0$, and $F \in \mathscr{H}_\delta(I^h \times I^v)$. Consider the stable manifold of the saddle fixed point $p(-1)$.

1. If the connected component that contains the fixed point $p(-1)$ is a vertical graph, let $W^0(-1)$ be the component.
2. Assume that $W^0(-1)$ exists. If $F^{-1}(W^0(-1))$ has two components, one is $W^0(-1)$ and the other is a vertical graph. Let $W^2(-1)$ be the one that is disjoint from $W^0(-1)$.

If the the local stable manifolds $W^0(-1)$ and $W^2(-1)$ exists, define $D = D(F) \subset I^h \times I^v$ to be the open set bounded between the two local stable manifolds. See Figure 5.1 for an illustration.

The domain D turns the Hénon-like map into a self-map.

Proposition 5.11. *Given $\delta > 0$ and intervals I^h and I^v with $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $c > 0$ such that for all $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$ the following properties hold:*

1. *The sets $W^0(-1)$, $W^2(-1)$, and D exist. The two local stable manifolds are vertical graphs with Lipschitz constant $c\|\varepsilon\|$.*
2. *$F(D) \subset D$.*

Proof. The first property follows from the graph transformation. The techniques were developed in [LM11, Chapter 3]. See [LM11, Lemma 3.1, 3.2].

The second property follows from the definition of the local stable manifolds and $\overline{\varepsilon} > 0$ is sufficiently small. □

Next, we study the local stable manifolds of the other saddle fixed point $p(0)$ with an expanding negative multiplier to define a partition of D.

Definition 5.12 (The local stable manifolds of $p(0)$)**.** Given $I^v \supset I^h \Supset I$, $\delta > 0$, and $F \in \mathscr{H}_\delta(I^h \times I^v)$. Assume that F has a saddle fixed point $p(0)$ with an expanding negative multiplier. Consider the stable manifold of $p(0)$.

1. If the connected component that contains $p(0)$ is a vertical graph, let $W^0(0)$ be the component.
2. Assume that $W^0(0)$ exists. If $F^{-1}(W^0(0))$ has two components, one is $W^0(0)$ and the other is a vertical graph. Let $W^1(0)$ be the one that is disjoint from $W^0(0)$.
3. Assume that $W^0(0)$ and $W^1(0)$ exist. If $F^{-1}(W^1(0))$ has two components and one component is a vertical graph located to the right of $W^0(0)$. Let $W^2(0)$ be the component.

See Figure 5.1 for an illustration.

Remark 5.13. At this moment, the numbers 0 and -1 in the notation of the fixed points $p(0)$ and $p(-1)$ (and also the local stable manifolds) do not have a special meaning. After introducing infinitely renormalizable Hénon-like maps, the notation $p(k)$ will be used to define a periodic point with period 2^k. See Definition 6.2. The numbers are introduced here for consistency.

The local stable manifolds split the domain D into vertical strips. These strips define a partition of the domain.

Definition 5.14 (A, B, and C)**.** Given $I^v \supset I^h \Supset I$, $\delta > 0$, and $F \in \mathscr{H}_\delta(I^h \times I^v)$. Assume that F has a saddle fixed point $p(0)$ with an expanding negative multiplier, the local stable manifolds in Definition 5.12 exist, and D exists.

1. Define $A = A(F) \subset I^h \times I^v$ to be the union of two sets. One is the open set bounded between $W^0(-1)$ and $W^1(0)$; the other is the open set bounded between $W^2(0)$ and $W^2(-1)$.
2. Define $B = B(F) \subset I^h \times I^v$ to be the open set bounded between $W^0(0)$ and $W^1(0)$.
3. Define $C = C(F) \subset I^h \times I^v$ to be the open set bounded between $W^0(0)$ and $W^2(0)$.

Remark 5.15. The local stable manifolds $W^0(-1)$, $W^1(0)$, $W^0(0)$, $W^2(0)$, and $W^2(-1)$ are associated to the points $p(-1) = -1$, $p^{(1)}$, $p(0)$, $p^{(2)}$, and 1 respectively (Definition 4.4).

For a strongly dissipative Hénon-like map, the local stable manifolds are vertical graphs and the dynamics on the partition is similar to the unimodal case.

Proposition 5.16. *Given $\delta > 0$ and intervals I^h and I^v with $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $c > 0$ such that for all $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$ the following properties hold:*

1. *The sets $W^0(0)$, $W^1(0)$, $W^2(0)$, A, B, and C exist. The local stable manifolds are vertical graphs with Lipschitz constant $c\,\|\varepsilon\|$.*
2. *$F(A) \subset A \cup W^1(0) \cup B$.*
3. *$F(C) \subset B$.*
4. *If $z \in A$ then its orbit eventually escapes A, i.e. there exists $n > 0$ such that $F^n(z) \notin A$.*

Proof. The first property is proved by graph transformation. See [LM11, Chapter 3].

The second and third properties follows from the definition of the local stable manifolds. See also [LM11, Lemma 4.2].

The last property holds because the only fixed points are $p(-1)$ and $p(0)$ so the local unstable manifold of $p(-1)$ must extends across the whole set A. See also [LM11, Lemma 4.2]. □

By the definition of B, its iterate $F(B)$ is contained in the right component of $D \backslash W^0(0)$. With the third property of Proposition 5.16, we can define the condition "renormalizable" as follows.

Definition 5.17 (Renormalizable)**.** Assume that $\overline{\varepsilon} > 0$ is sufficiently small. A Hénon-like map $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$ is (period-doubling) renormalizable if it has a saddle fixed point $p(0)$ with an expanding negative multiplier and $F(B) \subset C$. The class of renormalizable Hénon-like maps is denoted by $\mathscr{H}^r_\delta(I^h \times I^v, \overline{\varepsilon}) \subset \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$.

Remark 5.18. The notion of "renormalizable" here is similar to [dCLM05, Section 3.4] (which they called pre-renormalization) but not exactly the same. The "renormalizable" in their paper is called CLM-renormalizable here to compare the difference. In their article, the set "C" (they named the set D) where they define the first return map is a region bounded between $W^0(0)$ and a section of the unstable manifold of $p(-1)$. In this article, the set C is defined to be the largest candidate (around the critical value) that is invariant under F^2 which only uses the local stable manifolds of $p(0)$. Thus, the sets B and C in this article is larger than theirs.

As a result, the property "renormalizable" in this article is stronger than theirs. If a Hénon-like map is renormalizable then it is also CLM-renormalizable. Although the converse is not true in general, the hyperbolicity of the renormalization operator [dCLM05, Theorem 4.1] allows us to apply the notion of renormalizable to an infinitely CLM-renormalizable map. This makes the final result, Theorem 11.18, also works for CLM-renormalizable maps. See Remarks 5.29 and 11.19 for more details.

Their definition has some advantages and disadvantages. Their notion of renormalizable does not depend on the size of the vertical domain I^v. However, their sets B and C are too small. It may requires more iterations for an orbit to enter their B and C. See the proof of [LM11, Lemma 4.2]. This is the reason for adjusting their definition.

For a renormalizable Hénon-like map, an orbit that is disjoint from the stable manifold of the fixed points follows the paths in the following diagram.

$$\text{finite iterations} \circlearrowright A \longrightarrow B \rightleftarrows C$$

Therefore, a renormalizable map has a first return map on C.

5.4. Existence and properties of the local stable manifolds

In this section, we give a review for the properties of the local stable manifolds developed in [LM11].

To prove the existence of local stable manifold, we use the graph transformation method. The following lemma allows us to generate a vertical graph by pulling back another vertical graph under the Hénon-like map. This will be used in the graph transformation.

Lemma 5.19. *Given $m > 0$, $\delta > 0$, $\overline{\varepsilon} > 0$, $I \subset I^h \subset I^v$, and $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$. Assume that $U, U' \subset I^h$ are two closed intervals such that $F(x,y) = (h_y(x), x)$ satisfies the properties for all $y \in I^v$*

1. *$h_y(U') \supset U$ and*
2. *$\left\|h'_y\right\|_{U'} = \sup_{x\in U'} \left|\frac{\partial h}{\partial x}(x,y)\right| \geq m$.*

If $\Gamma \subset U \times I^h$ is the vertical graph of some L-Lipschitz function on I^h with $L < m$, then the preimage $F^{-1}(\Gamma) \cap U' \times I^v$ is the vertical graph of some $\frac{1}{\delta(m-L)} \|\varepsilon\|$-Lipschitz function on I^v.

Proof. Assume that Γ is the vertical graph of a L-Lipschitz function $\gamma : I^h \to U$.

First, we prove that $F^{-1}(\Gamma) \cap U' \times I^v$ is a vertical graph of some function on I^v. Fixed $y \in I^v$. Our goal is to find a unique solution $\hat{\gamma}(y) \in U'$ such that

$$F(\hat{\gamma}(y), y) = (h_y(\hat{\gamma}(y)), \hat{\gamma}(y)) = (\gamma(\hat{\gamma}(y)), \hat{\gamma}(y)),$$

i.e. the fixed point of $h_y^{-1} \circ \gamma$.

By the second condition, h_y is injective on U'. So the inverse $h_y^{-1} : U \to U'$ exists. Define $T : U' \subset I^h \to U'$ by $T = h_y^{-1} \circ \gamma$. For all $x_1, x_2 \in U'$, we have

$$|T(x_2) - T(x_1)| \quad = \quad \left|h_y^{-1} \circ \gamma(x_2) - h_y^{-1} \circ \gamma(x_1)\right|$$

$$
\begin{aligned}
&\le \frac{1}{m}\left|\gamma(x_2)-\gamma(x_1)\right| \\
&\le \frac{L}{m}\left|x_2-x_1\right|.
\end{aligned}
$$

Thus, T is a strong contraction since $L < m$. By the contraction mapping principle [MH93, Theorem 5.7.1], T has a unique fixed point $\hat{\gamma}(y) \in U'$. Consequently, $F^{-1}(\Gamma) \cap U' \times I^v$ is the vertical graph of $\hat{\gamma} : I^v \to U'$.

It remains to prove that $\hat{\gamma}$ is Lipschitz. Let $y_1, y_2 \in I^v$ and $x_i = \hat{\gamma}(y_i)$ for $i = 1,2$. Then $h_{y_i}(x_i) = \gamma(x_i)$ for $i = 1,2$. We get

$$(h_{y_2}(x_2) - h_{y_2}(x_1)) - (\gamma(x_2) - \gamma(x_1)) = h_{y_1}(x_1) - h_{y_2}(x_1).$$

For the left hand side of the equality, we have

$$
\begin{aligned}
\left|(h_{y_2}(x_2) - h_{y_2}(x_1)) - (\gamma(x_2) - \gamma(x_1))\right| &\ge \left|h_{y_2}(x_2) - h_{y_2}(x_1)\right| - \left|\gamma(x_2) - \gamma(x_1)\right| \\
&\ge (m-L)\left|x_2 - x_1\right|.
\end{aligned}
$$

For the right hand side of the inequality, we have

$$\left|h_{y_1}(x_1) - h_{y_2}(x_1)\right| \le \left\|\frac{\partial \varepsilon}{\partial y}\right\|_{I^h \times I^v} \left|y_2 - y_1\right| \le \frac{1}{\delta}\left\|\varepsilon\right\|\left|y_2 - y_1\right|.$$

Combine the two inequalities, we obtain

$$\left|\hat{\gamma}(y_2) - \hat{\gamma}(y_1)\right| \le \frac{1}{\delta(m-L)}\left\|\varepsilon\right\|\left|y_2 - y_1\right|.$$

Therefore, $\hat{\gamma}$ is $\frac{1}{\delta(m-L)}$-Lipschitz. □

Now we use the graph transformation to prove the existence of local stable manifold.

Proposition 5.20. *Given $\delta > 0$, $I^v \supset I^h \Supset I$, and $f \in \mathscr{U}_\delta(I^h)$ such that f has a fixed point $\hat{p}$ with $f'(\hat{p}) < -1$. There exists a constant $\overline{\varepsilon} > 0$ such that for all $F \in \mathscr{H}_\delta(I^h \times I^v, f, \overline{\varepsilon})$, the local stable manifold $W^0(0)$ exists and $W^0(0)$ is a vertical graph of a $\frac{2}{\delta}\left\|\varepsilon\right\|$-Lipschitz function on I^v.*

Proof. We use the method of graph transform to prove the existence of the local stable manifold. Let $\overline{\varepsilon} > 0$ be a constant. We will adjust $\overline{\varepsilon}$ in the proof so that $\overline{\varepsilon}$ depends only on δ and f. Assume that $F \in \mathscr{H}_\delta(I^h \times I^v, f, \overline{\varepsilon})$ and $d > 0$ such that $f'(x) < \frac{1}{2}(f'(\hat{p}) - 1)$ for all $x \in [\hat{p} - 2d, \hat{p} + 2d]$. Also, assume that $\overline{\varepsilon}$ is small enough such that $\frac{\partial h}{\partial x}(z) < -1$ for all $z \in [\hat{p} - 2d, \hat{p} + 2d] \times I^v$. By Lemma A.2, F has a fixed point $p(0) = (q,q)$ on $(\hat{p} - d, \hat{p} + d) \times I^v$ and h_y has a fixed point on $(\hat{p} - d, \hat{p} + d)$ for all $y \in I^v$ when $\overline{\varepsilon} < d$. Then $h_y[\hat{p} - 2d, \hat{p} + 2d] \supset [\hat{p} - 2d, \hat{p} + 2d]$. Let $U = U' = [\hat{p} - 2d, \hat{p} + 2d]$. So we are now able to apply Lemma 5.19 with this setting.

Define $L = \frac{2}{\delta}\left\|\varepsilon\right\|$ and $\mathfrak{L}$ be the collection of L-Lipschitz functions $\gamma : I^v \to \mathbb{R}$ such that $\gamma(q) = q$. Then $\mathrm{Im}\gamma \subset U$ when $\overline{\varepsilon} < \frac{d\delta}{2|I^v|}$. Also, assume that $\overline{\varepsilon}$ is small enough such that $L < \frac{1}{2}$. Then

$$\frac{1}{\delta(1-L)}\left\|\varepsilon\right\| < L.$$

By Lemma 5.19, $F^{-1}(\Gamma) \cap U' \times I^v$ is the vertical graph of a L-Lipschitz function for all $\gamma \in \mathfrak{L}$ where Γ is the vertical graph of γ. This defines a graph transformation $T : \mathfrak{L} \to \mathfrak{L}$. It remains to show that T has a fixed point.

We first claim that

$$\|T\gamma_1 - T\gamma_2\|_{(q-s,q+s)\cap I^v(\delta)} \leq \frac{2}{1-2L}\|\gamma_1 - \gamma_2\|_{(q-Ls,q+Ls)\cap I^v(\delta)} \tag{5.1}$$

for all $\gamma_1, \gamma_2 \in \mathfrak{L}$ and $s > 0$. By the definition of T, we have

$$h_y(T\gamma(y)) = \gamma(T\gamma(y))$$

for all $\gamma \in \mathfrak{L}$ and $y \in I^v(\delta)$. Then

$$\begin{aligned}
|T\gamma_1(y) - T\gamma_2(y)| &\leq \left|h_y(T\gamma_1(y)) - h_y(T\gamma_2(y))\right| \\
&= |\gamma_1(T\gamma_1(y)) - \gamma_2(T\gamma_2(y))| \\
&\leq |\gamma_1(T\gamma_1(y)) - \gamma_1(T\gamma_2(y))| + |\gamma_1(T\gamma_2(y)) - \gamma_2(T\gamma_2(y))| \\
&\leq L|T\gamma_1(y) - T\gamma_2(y)| + |\gamma_1(T\gamma_2(y)) - \gamma_2(T\gamma_2(y))|
\end{aligned}$$

for all $\gamma_1, \gamma_2 \in \mathfrak{L}$ and $y \in (q-s, q+s) \cap I^v(\delta)$ since $\mathrm{Im}T\gamma_1, \mathrm{Im}T\gamma_2 \subset U'$. Thus we proved (5.1).

Now we show T has a fixed point. Let $\gamma_0(y) = q$ be the constant map. Define $\gamma_n = T^n\gamma_0$. We prove that $\{\gamma_n\}_{n=0}^{\infty}$ is a Cauchy sequence. For all $m > n$, we have

$$\begin{aligned}
\|T\gamma_m - T\gamma_n\|_{I^v(\delta)} &\leq \left(\frac{1}{1-L}\right)^n \|\gamma_{m-n} - \gamma_0\|_{(q-L^n|I^v(\delta)|, q+L^n|I^v(\delta)|)\cap I^v(\delta)} \\
&\leq \left(\frac{L}{1-L}\right)^n L|I^v(\delta)|.
\end{aligned}$$

Note that $\frac{L}{1-L} = \frac{1}{\frac{1}{L}-1} < 1$ since $L < \frac{1}{2}$. Therefore, γ_n has a limit and the Proposition is proved. □

Remark 5.21. By the stable manifold theorem, the local stable manifolds are analytic curves. The proposition gives a bound for the slope of the analytic curve.

One can also prove the existence for $W^1(0)$, $W^2(0)$, $W^0(-1)$, and $W^1(-1)$. See Proposition 6.8.

5.5. The renormalization operator

When a Hénon-like map is renormalizable, the map has a first return map on C. However, the first return map is no longer a Hénon-like map by a direct computation

$$F^2(x,y) = (h_x(h_y(x)), h_y(x)).$$

The paper [dCLM05] introduced a nonlinear coordinate change $H(x,y) \equiv (h_y(x), y)$ that turns the first return map into a Hénon-like map. The next proposition defines the renormalization operator.

Proposition 5.22 (The renormalization operator)**.** *Given $\delta > 0$ and intervals I^h, I^v with $I^v \supset I^h \Supset I$. There exist constants $\bar{\varepsilon} > 0$ and $c > 0$ so that for all $F \in \mathscr{H}^r_\delta(I^h \times I^v, \bar{\varepsilon})$ there exists an $\mathbb{R}$-symmetric* *orientation reversing affine map $s = s(F)$ that depends continuously on F such that the following properties hold:*

Let $\Lambda(x,y) = (s(x), s(y))$ and $\phi = \Lambda \circ H$.

1. *The map $x \to h_y(x)$ is injective on a neighborhood of $C(F)$ and hence ϕ is a diffeomorphism from a neighborhood of $C(F)$ to its image.*
2. *The renormalization $RF \equiv \phi \circ F^2 \circ \phi^{-1}$ is an Hénon-like map defined on $I^h_R(\delta_R) \times I^v_R(\delta_R)$ for some $\delta_R > 0$ and intervals I^h_R and I^v_R. The intervals satisfy $I^h_R \Supset [-1,1]$ and $I^v_R = s(I^v)$.*
3. *The domain $I^h_R \times I^v_R$ contains $D(RF)$, and the rescaling ϕ maps $\phi(C(F)) = D(RF)$.*
4. *The fixed points satisfy the relation $\phi(p(0)) = p_{RF}(-1)$ where $p_{RF}(-1)$ is the saddle fixed point of RF with an expanding positive multiplier.*
5. *The renormalization has a representation $RF = (f_R - \varepsilon_R, x)$ where $f_R \in \mathscr{U}$. The representation satisfies the relations*
$$\|f_R - R_c f\|_{I^h_R(\delta_R)} < c\,\|\varepsilon\|$$
and
$$\|\varepsilon_R\|_{I^h_R(\delta_R)\times I^v_R(\delta_R)} < c\,\|\varepsilon\|^2 .$$

Proof. See [dCLM05, Section 3.5]. □

Remark 5.23. The rescaling ϕ preserves the orientation along the x-coordinate and reverses the orientation along the y-coordinate.

Remark 5.24. The operator R is canonical. The renormalization RF depends continuously on the Hénon-like map F. The renormalization RF and rescaling ϕ do not depend on the representation f and ε for F. However, the representation f_R and ε_R of RF and the estimates in property 5 do depend on the choice of representation of F.

Lemma 5.25. *(to be removed)The derivatives of the nonlinear rescaling are given by*

$$DH(x,y) = \begin{bmatrix} \frac{\partial h}{\partial x}(x,y) & \frac{\partial h}{\partial y}(x,y) \\ 0 & 1 \end{bmatrix}$$

$$DH^{-1}(x,y) = \begin{bmatrix} \frac{1}{\frac{\partial h}{\partial x}(h_y^{-1}(x),y)} & -\frac{\frac{\partial h}{\partial y}(h_y^{-1}(x),y)}{\frac{\partial h}{\partial x}(h_y^{-1}(x),y)} \\ 0 & 1 \end{bmatrix}$$

Hence,
$$h_y^{-1\prime}(x) = \left(h_y' \circ h_y^{-1}(x)\right)^{-1}$$
and
$$\frac{\partial h_y^{-1}}{\partial y}(x,y) = -\frac{\frac{\partial \varepsilon}{\partial y}(h_y^{-1}(x),y)}{h_y' \circ h_y^{-1}(x)}.$$

Proof. The lemma follows directly from computation. □

A map is called infinitely renormalizable if the procedure of renormalization can be done infinitely many times. The class of infinitely renormalizable Hénon-like map is denoted as $\mathscr{I}_\delta(I^h \times I^v, \overline{\varepsilon}) \subset \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$.

Assume that $F \in \mathscr{I}_\delta(I^h \times I^v, \overline{\varepsilon})$, we define $F_n = R^n F$. The subscript n is called the renormalization scale. The subscript is also used to indicate the associated renormalization scale of an object. For example, H_n, s_n, and Λ_n are the functions in Proposition 5.22 that corresponds to F_n. The vertical domain I^v_n satisfies $I^v_0 = I^v$ and $I^v_{n+1} = s_n(I^v_n)$ for all $n \geq 0$. The vertical graphs $W^t_n(j)$ are the local stable manifolds of F_n. The sets A_n, B_n, and C_n form a partition of the dynamical domain D_n that associates to F_n. The points $p_n(-1)$ and $p_n(0)$ are the two saddle fixed points of F_n.

Also, define $\Phi^j_n = \phi_{n+j-1} \circ \cdots \circ \phi_n$ and $\lambda_n = s'_n(x)$.

Recall $g \in \mathscr{U}$ is the fixed point of the renormalization operator R_c, and λ is the rescaling constant defined in 4.28. Let $G(x,y) = (g(x), x)$ be the induced degenerate Hénon-like map.

The renormalization operator is hyperbolic. The next proposition lists the properties of infinitely renormalizable Hénon-like maps.

Proposition 5.26 (Hyperbolicity of the renormalization operator)**.** *Given $\delta > 0$ and intervals I^h, I^v with $I^v \supset I^h \Supset I$. There exists constants $\rho < 1$ (universal), $\overline{\varepsilon} > 0$, and $c > 0$ such that for all $F \in \mathscr{I}_\delta(I^h \times I^v, \overline{\varepsilon})$ there exist a constant δ_R with $0 < \delta_R < \delta$, an interval I^h_R with $I^h \supset I^h_R \Supset I$, and a constant $b = b(F) \in \mathbb{R}$ such that the following properties hold:*

Let $F_n = R^n F$ be the sequence of renormalizations of F. Then $F_n \in \mathscr{H}_{\delta_R}(I^h_R \times I^v_n)$ for all $n \geq 0$. Also, the sequence has a representation $F_n = (f_n - \varepsilon_n, x)$ with $f_n \in \mathscr{U}_{\delta_R}(I^h_R)$ that satisfies

1. $\|f_n - g\|_{I^h_R(\delta_R)} < c\rho^n \|F - G\|_{I^h_R(\delta_R) \times I^v(\delta_R)}$
2. $\|\varepsilon_{n+1}\|_{I^h_R(\delta_R) \times I^v_{n+1}(\delta_R)} < c \|\varepsilon_n\|^2_{I^h_R(\delta_R) \times I^v_n(\delta_R)}$,
3. $\left\|f_{n+1} - s_n \circ f^2_n \circ s^{-1}_n\right\|_{I^h_R(\delta_R)} < c \|\varepsilon_n\|_{I^h_R(\delta_R) \times I^v_n(\delta_R)}$,
4. $|\lambda_n - \lambda| < c\rho^n \|F - G\|_{I^h_R(\delta_R) \times I^v(\delta_R)}$, *and*
5. $\varepsilon_n(x,y) = b^{2^n} a(x) y (1 + O(\rho^n))$ *(universality)*

for all $n \geq 0$ where $a(x)$ is a universal analytic positive function. The value δ_R in the estimates can be replaced by any positive number that is smaller than δ_R.

Proof. See [dCLM05, Theorem 3.5, 4.1, 7.9, and Lemma 7.4]. □

Remark 5.27. The constant b is called the average Jacobian of F. See [dCLM05, Section 6].

Remark 5.28. The Hénon-renormailzation is an operation that renormalizes a map around the critical value. However, the renormalization F_n converges to the fixed point G of the unimodal-renormalization that renormalizes around the critical point. This is because of the nonlinear rescaling H maps the domain from C to B in the degenerate case. See Chapter 8 for a more detail explanation.

Remark 5.29. Although infinitely CLM-renormalizable in general does not imply infinitely renormalizable, the hyperbolicity provides a connection between the two notions of infinitely renormalizable. Assume that F is infinitely CLM-renormalizable. The hyperbolicity of the renormalizable operator [dCLM05, Theorem 4.1] says that R^nF converges to the fixed point G. This means that R^nF is also infinitely renormalizable for all n sufficiently large. This makes Theorem 11.18 also applies to infinitely CLM-renormalizable Hénon-like maps. See Remark 11.19 for more details.

From now on, for any infinitely renormalizable map F, we fix a representation $F_n = (f_n - \varepsilon_n, x)$ such that the maps f_n and ε_n satisfy the properties given in Proposition 5.26. Also, we neglect the subscript of the supnorms $\|f_n - g\| = \|f_n - g\|_{I_R^h(\delta_R)}$ and $\|\varepsilon_n\| = \|\varepsilon_n\|_{I_R^h(\delta_R)\times I_n^v(\delta_R)}$ whenever the context is clear.

Corollary 5.30. *There exists a constant $c > 1$ such that*

$$\|F_n - G\| < c\rho^n \|F - G\|$$

and

$$\|\varepsilon_{n+t}\| < (c\,\|\varepsilon_n\|)^{2^t}$$

for all $t \geq 1$.

Lemma 5.31. *Assume that $\overline{\varepsilon} > 0$ small enough such that Proposition 5.26 holds. There exists a constant $c_1 > 0$ such that the inequalities hold*

$$\left|\frac{\partial \varepsilon_n}{\partial x}(x,y)\right|, \left|\frac{\partial \varepsilon_n}{\partial y}(x,y)\right| \leq c_1\,\|\varepsilon_n\| \tag{5.2}$$

for all $F \in \mathscr{I}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $(x,y) \in I^h \times I_n^v$. In addition, if F is non-degenerate, there exist constants $N = N(F) \geq 0$, $\delta_R > 0$, and $c_2 > 0$ such that

$$\frac{\partial \varepsilon_n}{\partial y}(x,y) \geq \frac{c_1}{|I_n^v|}\,\|\varepsilon_n\| \tag{5.3}$$

for all $(x,y) \in I^h \times I_n^v$ and $n \geq N$.

Proof. The first inequality (5.2) follows from Lemma 2.1.

By the universality (and the proof of [dCLM05, Theorem 7.9]) of the infinitely renormalizable Hénon-like maps, the perturbation ε and its derivative has the asymptotic form

$$\varepsilon_n(x,y) = b^{2^n} a(x)y(1 + O(\rho^n))$$

and

$$\frac{\partial \varepsilon_n}{\partial y}(x,y) = b^{2^n} a(x)(1 + O(\rho^n)).$$

There exists a constant $N \geq 0$ such that $O(\rho^n) < \frac{1}{2}$ for all $n \geq N$. Then

$$\|\varepsilon_n\| \leq \frac{3}{2} b^{2^n} \max_{x \in I_R^h(\delta_R)} |a(x)|\,|I_n^v|$$

and

$$\frac{\partial \varepsilon_n}{\partial y}(z) \geq \frac{1}{2} b^{2^n} \min_{x \in I_R^h} a(x)$$

for all $z \in I^h \times I_n^v$ and $n \geq N$. We get

$$\frac{\partial \varepsilon_n}{\partial y}(z) \geq \frac{1}{3\,|I_n^v|} \frac{\min_{x \in I_R^h} a(x)}{\max_{x \in I_R^h(\delta_R)} |a(x)|} \,\|\varepsilon_n\|$$

for all $z \in I^h \times I_n^v$ and $n \geq N$. Note that $\max_{x \in I_R^h(\delta_R)} |a(x)| \neq 0$ and $\min_{x \in I_R^h} a(x) \neq 0$ since a is positive on I_R^h and δ_R can be chosen to be arbitrary small. $\square$

To study the wandering domains, it is enough to consider Hénon-like maps that are close to the hyperbolic fixed point G. By Corollary 7.4 later, for any integer $n \geq 0$, we show an infinitely renormalizable Hénon-like map F has a wandering domain in $D(F)$ if and only if F_n has a wandering domain in $D(F_n)$. Also, the maps F_n converge to the hyperbolic fixed point G as n approaches to infinity by Proposition 5.26. Thus, we focus on a small neighborhood of the fixed point G.

Definition 5.32. Given $\delta > 0$ and $I \Subset I^h \subset I^v$. If $\overline{\varepsilon}$ is small enough such that Proposition 5.26 holds, define $\hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ to be the class of non-degenerate Hénon-like maps $F \in \mathscr{I}_\delta(I^h \times I^v, \overline{\varepsilon})$ such that $F_n \in \mathscr{H}_\delta(I^h \times I_n^v, \overline{\varepsilon})$, $\|F_n - G\| < \overline{\varepsilon}$, $|\lambda_n - \lambda| < \overline{\varepsilon}$, $\|s_n(x) - (-\lambda)x\|_{I^h} < \overline{\varepsilon}$, and (5.3) holds for all $n \geq 0$.

In the remaining part of the article, we will study the dynamics and the topology of Hénon-like maps in this smaller class of maps.

6. Structure and Dynamics of Infinitely Renormalizable Hénon-Like Maps

In this chapter, we study the topology of the local stable manifolds and the dynamics on the partition for a infinitely renormalizable Hénon-like map.

6.1. Rescaling levels

This section introduces a finer partition of C, called the rescaling levels, based on the maximum possible rescalings of a point in C.

For each two consecutive levels of renormalization n and $n+1$, the maps F_n^2 and F_{n+1} are conjugated by the nonlinear rescaling ϕ_n. The rescaling ϕ_n relates the two renormalization scales by the following lemma.

Lemma 6.1. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exists a constant $\overline{\varepsilon} > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following properties hold for all $n \geq 0$:*

1. $\phi_n(p_n(0)) = p_{n+1}(-1)$,
2. $\phi_n(W_n^k(0)) = W_{n+1}^k(-1)$ *for $k = 0, 2$, and*
3. $\phi_n : C_n \to D_n$ *is a diffeomorphism.*

The itinerary of a point follows the arrows in the diagram.

$$\begin{tikzcd}
 & A_{n+1} \arrow[loop right, "F_{n+1}"] \arrow[d, "F_{n+1}"] & A_{n+2} \arrow[loop right, "F_{n+2}"] \arrow[d, "F_{n+2}"] & \\
 & B_{n+1} \arrow[d, "F_{n+1}"] & B_{n+2} \arrow[d, "F_{n+2}"] & \\
C_n \arrow[uur, "\phi_n"] \arrow[ur, "\phi_n"] \arrow[r, "\phi_n"] & C_{n+1} \arrow[uur, "\phi_{n+1}"] \arrow[ur, "\phi_{n+1}"] \arrow[r, "\phi_{n+1}"] & C_{n+2} \arrow[r, "\phi_{n+2}"] & \cdots
\end{tikzcd}$$

The diagram says, if $z_0 \in C_n$, then we can rescale the point. The rescaled point $z_1 = \phi_n(z_0)$ enters the domain D_{n+1} of the next renormalization scale $n+1$ by Lemma 6.1. On the renormalization scale $n+1$, the rescaled point z_1 belongs to one of the sets A_{n+1}, B_{n+1}, or C_{n+1} if it is disjoint from the stable manifolds. The process of rescaling stops if z_1 belongs to A_{n+1} or B_{n+1} and z_0 can be rescaled at most one time. If z_1 belongs to C_{n+1}, we can continue to rescale the point. The rescaled point $z_2 = \phi_{n+1}(z_1)$ enters the domain D_{n+2} of the next renormalization scale $n+2$. Similarly, the process of rescaling stops if z_2 belongs to A_{n+2} or B_{n+2} and z_0 can be rescaled at most two times. If z_2 belongs to C_{n+2}, we can rescale again and repeat the procedure until the rescaled point enters the sets A or B of some deeper renormalization scale.

Next, we define the finer partition $C_n(j)$ on C_n by the maximal possible rescalings.

Definition 6.2. For consistency, set $C_n(0) = A_n \cup W_n^1(0) \cup B_n$ and $B_n(0) = A_n \cup W_n^2(0) \cup C_n$. Given a positive integer j. The j-th rescaling level in C is defined as $C_n(j) = \left(\Phi_n^j\right)^{-1}(C_{n+j}(0))$ and the j-th rescaling level in B is defined as $B_n(j) = F_n^{-1}(C_n(j))$. Also, set $p_n(j) = \left(\Phi_n^j\right)^{-1}(p_{n+j}(0))$ and $W_n^t(j) = \left(\Phi_n^j\right)^{-1}(W_{n+j}^t(0))$ for $t = 0, 2$.

For each $n \geq 0$, the local stable manifolds $\{W_n^t(j)\}_{j\geq 0, t=0,2}$ cannot intersect each other. The diagram explains the definition of a rescaling level.

$$\begin{array}{ccc} C_n(j) & \xrightarrow{F_n^{2^j}} & C_n(j) \\ {\scriptstyle \Phi_n^j}\downarrow & & \downarrow{\scriptstyle \Phi_n^j} \\ D_{n+j} & \xrightarrow[F_{n+j}]{} & D_{n+j} \end{array}$$

From the definition, the rescaling levels of two renormalization scales are related by the rescaling map with the following properties. The proposition is an analog of Proposition 4.9.

Proposition 6.3. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exists a constant $\overline{\varepsilon} > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following properties hold for all $n \geq 0$:*

1. *$p_n(j)$ is a periodic point of F_n with period 2^j for $j \geq 0$.*
2. *$W_n^t(j)$ is a local stable manifold of $p_n(j)$ for $j \geq 0$ and $t = 0, 2$.*
3. *$\Phi_n^k(W_n^t(j)) = W_{n+k}^t(j-k)$ and $\Phi_n^k(p_n(j)) = p_{n+k}(j-k)$ for $j \geq k-1$ and $t = 0, 2$.*
4. *The map $\Phi_n^k : C_n(j) \to C_{n+k}(j-k)$ is a diffeomorphism for $j \geq k$, and*
5. *For each $j \geq 0$, the set $C_n(j)$ contains two components. The left component $C_n^l(j)$ is the set bounded between $W_n^0(j-1)$ and $W_n^0(j)$ and the right component $C_n^r(j)$ is the set bounded between $W_n^2(j)$ and $W_n^2(j-1)$.*

The partition and the local stable manifolds $W_n^t(j)$ are illustrated in Figure 6.1. The sets $\{C_n(j)\}_{j\geq 1}$ form a partition of C_n and the sets $\{B_n(j)\}_{j\geq 1}$ form a partition of B_n.

6.2. Existence of the local stable manifolds and the partition for all renormalization scales

Although we already proved the existence of the local stable manifolds in Proposition in 5.20, the constants may depend on the Hénon-like map. Our goal in this section is to make the constants to be independent of the Hénon-like maps when the Hénon-like maps are close to the renormalization fixed point G. The proofs are based from the methods in [LM11].

Recall $q(j)$ are the periodic points for the fixed point g by Definition 4.31 and Proposition 4.32.

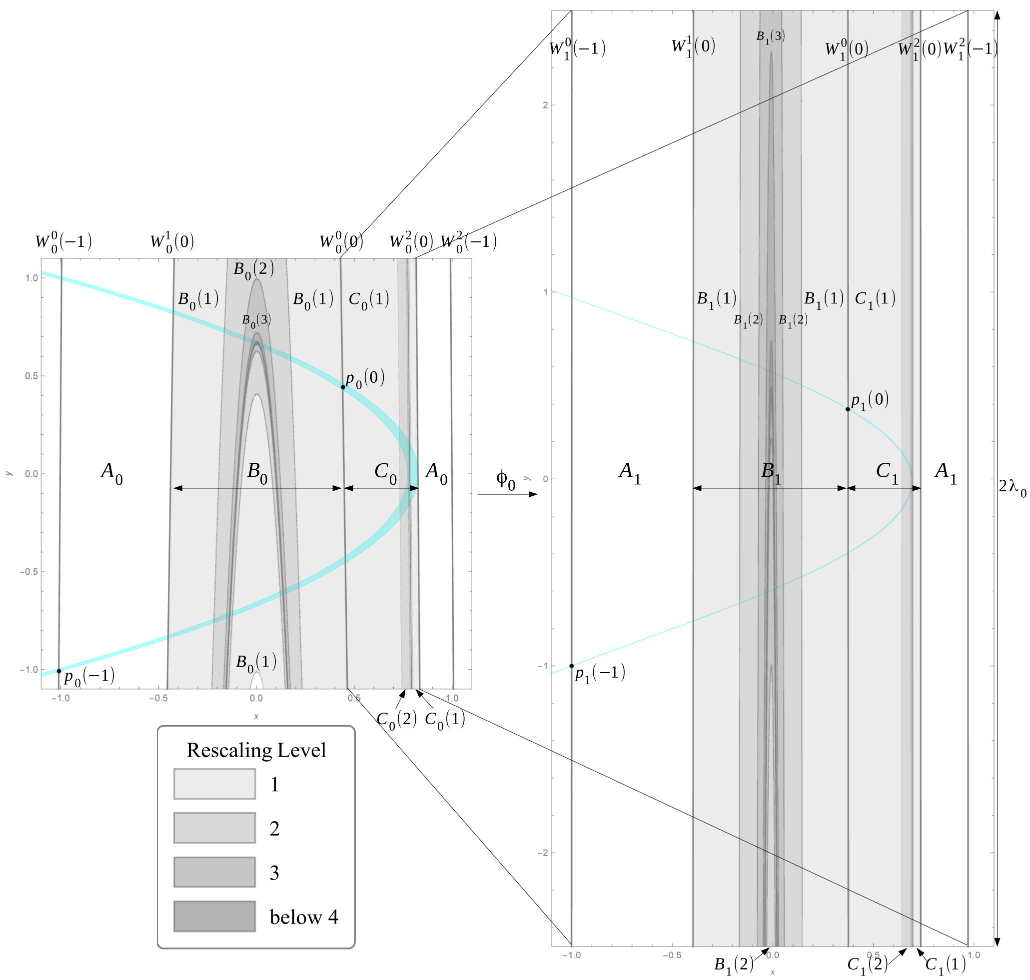


Figure 6.1.: The partition and the local stable manifolds of two renormalization scales F_0 and F_1 from the left to the right. The rescaling levels 1, 2, 3, and below 4 are shaded from light to dark as shown in the legend.

Lemma 6.4. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. For all $d > 0$ there exists a constant $\overline{\varepsilon} = \overline{\varepsilon}(d) > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ we have*

$$|\pi_x p_n(0) - q(0)| < d$$

for all $n \geq 0$.

Proof. Without lose of generality, we may assume that d is small enough such that g is decreasing on $[q(0)-d, q(0)+d]$. Let $\overline{\varepsilon} < d$ be small enough such that Proposition 5.26 holds. Then

$$\|h_n - g\|_{I^h \times I^v_n} < d$$

for all $n \geq 0$. In particular, $|g(x) - h_n(x,x)| < d$ for all $x \in [q(0)-d, q(0)+d]$. Since g is decreasing on $[q(0)-d, q(0)+d]$, we obtain

$$|\pi_x p_n(0) - q(0)| < d$$

by Lemma A.2. □

For the limiting case $F = G$, define

Definition 6.5. Define $q^C(j) = g(q(j))$ for all $j \geq 0$.

For the limiting case, the local stable manifold $W^0(j)$ is determined by the point $q^C(j)$. Precisely, $W^0(j)$ is the vertical line $x = q^C(j)$.

Corollary 6.6. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. For all $d > 0$ and $j > 0$, there exists a constant $\overline{\varepsilon} = \overline{\varepsilon}(d,j) > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ we have*

$$\left|\pi_x p_n(j) - q^C(j)\right| < d$$

for all $n \geq 0$.

To prove the existence of the local manifolds, we apply the graph transformation by Lemma 5.19. To use the Lemma, we first prove

Lemma 6.7. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $m > 0$, and $d > 0$ and closed intervals $U_1, U_2 \subset I$ such that for all $F \in \hat{\mathscr{I}}_{\hat{\delta}}(\hat{I}^h \times I^v, \overline{\varepsilon})$ the following properties hold for all $n \geq 0$ and $y \in I^v_n$:*

1. *$q(0), \pi_x p_n(0) \in \mathring{U}_1$, $\hat{q}(0) \in \mathring{U}_2$, and $\pi_x p_n(0) \notin U_2$.*
2. *$B_{\pi_x p_n(0)}(d) \subset h_n(U_j, y)$ for all $j = 1,2$.*
3. *$\left\|\frac{\partial h_n}{\partial x}(\cdot, y)\right\|_{U_j} \geq m$ for all $j = 1,2$.*
4. *$U_2 \subset h_n(U_1, y)$.*

Here the constants m, d and intervals U_1, U_2 are universal.

Proof. Let $\overline{\varepsilon} > 0$ be small enough (explain later) such that Proposition 5.26 holds. Let $c > 0$, $\rho < 1$, $\delta > 0$ be the constants defined in Proposition 5.26.

First, we define U_1. Let $\hat{m} = \min(|g'(q(0))|, |g'(\hat{q}(-1))|) = \min(|g'(q(-1))|, |g'(\hat{q}(1))|) = |g'(q(1))|$. By the minimal principle of negative Schwarzian derivative, there exists $a_1 < q(0)$ such that $g'(x) \leq -\frac{2}{3}\hat{m}$ for all $x \in U_1 = [a_1, b_1]$ where $b_1 = \hat{q}(-1) = 1$. Then $U_1 \subset I$, $q(0) \in \mathring{U}_1$, and the image $q(0) \in g(\mathring{U}_1)$. Hence, there exists $d_1 > 0$ such that $B_{q(0)}(3d_1) \subset g(U_1) \cap U_1$.

Assume that $\overline{\varepsilon} < d_1$. That is

$$\|h_n - g\|_{I^h(\delta)\times I^v_n(\delta)} < d_1 \tag{6.1}$$

for all $n \geq 0$. By Lemma 6.4,

$$|\pi_x p_n(0) - q(0)| < d_1 \tag{6.2}$$

when $\overline{\varepsilon}$ is small enough. This proves the first property for U_1.

To prove the second property for U_1, we claim that

$$h_n(b_1, y) < \pi_x p_n(0) - d < \pi_x p_n(0) + d < h_n(a_1, y) \tag{6.3}$$

for $n \geq 0$. Since $(q(0) - 3d_1, q(0) + 3d_1) \subset g(U_1)$, we have

$$g(b_1) < q(0) - 3d_1 < q(0) + 3d_1 < g(a_1).$$

It follows by (6.2) and (6.1) that (6.3) holds and $B_{\pi_x p_n(0)}(d) \subset h_n(U_1, y)$ for all $n \geq 0$ since $h_n(U_1, y)$ is connected.

To prove the third property, also assume that $\overline{\varepsilon} < \frac{\delta}{3}\hat{m}$. That is

$$\|h_n - g\|_{I^h(\delta)\times I^v_n(\delta)} < \frac{\delta}{3}\hat{m}$$

for all $n \geq 0$. Then

$$\begin{aligned} h_n'(x,y) &\leq g'(x) + \left|\frac{\partial h_n}{\partial x}(x,y) - g'(x)\right| \\ &\leq -\frac{2}{3}\hat{m} + \frac{1}{\delta}\|h_n - g\|_{I^h(\delta)\times I^v_n(\delta)} < -\frac{1}{3}\hat{m} \end{aligned}$$

for all $x \in U_1$ and $y \in I^v_n$. Thus, $\left\|\frac{\partial h_n}{\partial x}(\cdot, y)\right\|_{U_1} \geq \frac{1}{3}\hat{m}$.

To construct U_2, we recall that $\hat{q}(0) \in [q(-1), q(0)] \subset g([q(0), \hat{q}(-1)]) \subset g(\mathring{U}_1)$. By the minimal principle of negative Schwarzian derivative, there exists $d_2 > 0$ such that $g'(x) \geq \frac{2}{3}\hat{m}$ for all $x \in B_{\hat{q}(0)}(2d_2)$, $B_{\hat{q}(0)}(2d_2) \subset g(U_1)$. Let $U_2 = B_{\hat{q}(0)}([\hat{q}(0) - d_2, \hat{q}(0) + d_2])$. We can choose d_2 be small enough such that $U_2 \cap B_{q(0)}(d_1) = \phi$. This implies that $\pi_x p_n(0) \notin U_2$ for all $n \geq 0$ since $\pi_x p_n(0) \in B_{q(0)}(d_1)$. This proves the first property for U_2.

Moreover, also assume that $\overline{\varepsilon} < d_2$. That is

$$\|h_n - g\|_{I^h(\delta)\times I^v_n(\delta)} < d_2$$

for all $n \geq 0$. Then $U_2 \subset h_n(U_1, y)$ for all $y \in I^v_n$ and $n \geq 0$. This proves the fourth property.

To prove the second and third property, let $d_3 > 0$ be such that $B_{q(0)}(3d_3) \subset g(U_2)$ since $q(0) \in$

$g(\mathring{U}_2)$. By the same reason, the second and third properties hold for U_2 when $\overline{\varepsilon}$ is sufficiently small. The lemma is proved by setting $d = \min(d_1, d_3)$. □

We are ready to prove the existence of the local manifolds.

Proposition 6.8. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exists constants $\overline{\varepsilon} > 0$ and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following properties hold for all $n \geq 0$:*

The local stable manifold $W^t_n(0) \subset I \times I^v_n$ exists and $W^t_n(0)$ is a vertical graph of a $c\,\|\varepsilon_n\|$-Lipschitz function on I^v_n for $t = 0, 1, 2$. In addition, $W^0_n(0), W^2_n(0) \subset U_1 \times I^v_n$ and $W^1_n(0) \subset U_2 \times I^v_n$ where U_1 and U_2 are the closed intervals defined in Lemma 6.7.

Proof. We use the method of graph transform to prove the existence of the local stable manifold. Let $\overline{\varepsilon} > 0$ be small enough such that Proposition 5.26 holds, $m > 0$, $d > 0$, U_1, and U_2 be defined in Lemma 6.7. Also, let $d > 0$, $V_n \equiv [\pi_x p_n(0) - d, \pi_x p_n(0) + d]$. Assume that $F \in \mathscr{I}_{\hat{\delta}}(\hat{I}^h \times \hat{I}^v, \hat{f}, \overline{\varepsilon})$. Write $F_n = R^n F = (h_n, x) = (f_n - \varepsilon_n, x)$. Then $V_n \subset h_n(U_j, y)$ for all $j = 1, 2$ and $n \geq 0$ when $\overline{\varepsilon}$ is small enough.

Define $L_n = \frac{2}{\delta m}\|\varepsilon_n\|$ and $\mathfrak{L}_n$ be the collection of L_n-Lipschitz functions $\gamma : I^v_n \to \mathbb{R}$ such that $\gamma(\pi_x p_n(0)) = \pi_x p_n(0)$. First we show that when $\overline{\varepsilon}$ is sufficiently small, then $\Gamma \subset V_n \times I^v_n$ for all $\gamma \in \mathfrak{L}_n$ and $n \geq 0$ where Γ is the vertical graph of γ. Assume that $\overline{\varepsilon}$ is small enough such that

$$\frac{2}{\delta m}(c\overline{\varepsilon})^{2^n}(\lambda + \overline{\varepsilon})^n \left|I^h\right| < d$$

for all $n \geq 0$ where m is the constant defined in Lemma 6.7. Then

$$\begin{aligned}
|\gamma(y) - \pi_x p_n(0)| &= |\gamma(y) - \gamma(\pi_x p_n(0))| \\
&\leq \frac{2}{\delta m}\|\varepsilon_n\|\,|y - \pi_x p_n(0)| \\
&\leq \frac{2}{\delta m}\|\varepsilon_n\|\,|I^v_n| \\
&\leq \frac{2}{\delta m}(c\,\|\varepsilon_0\|)^{2^n}\prod_{j=0}^{n-1}\lambda_j\,|I^v| \\
&\leq \frac{2}{\delta m}(c\overline{\varepsilon})^{2^n}(\lambda + \overline{\varepsilon})^n\,|I^v| \\
&< d
\end{aligned}$$

for all $y \in I^v_n$. Thus, $\mathrm{Im}\gamma \subset V_n$ and $\Gamma \subset V_n \times I^v_n$ for all $n \geq 0$.

Moreover, assume that $\overline{\varepsilon}$ is small enough such that

$$L_n = \frac{2}{\delta m}\|\varepsilon_n\| \leq \frac{2}{\delta m}(c\,\|\varepsilon_0\|)^{2^n} \leq \frac{2}{\delta m}(c\overline{\varepsilon})^{2^n} < \frac{m}{2}$$

for all $n \geq 0$. Then

$$\frac{1}{\delta(m - L_n)}\|\varepsilon_n\| \leq L_n$$

for all $n \geq 0$. By Lemma 5.19, $F_n^{-1}(\Gamma) \cap U_1 \times I^v_n$ is the vertical graph of a L_n-Lipschitz function for all $\gamma \in \mathfrak{L}_n$ where Γ is the vertical graph of γ. Also, the preimage preserves the fixed point $p_n(0)$.

Thus, the preimage defines a graph transformation $T_n : \mathfrak{L}_n \to \mathfrak{L}_n$ such that if Γ' is the vertical graph of $T_n\gamma$ then $\Gamma' \subset U_1 \times I_n^v$. It remains to show that T_n has a fixed point.

We claim that

$$\|T_n\gamma_1 - T_n\gamma_2\|_{B(\pi_x p_n(0),s)\cap I_n^v} \le \frac{1}{m-L_n}\|\gamma_1-\gamma_2\|_{B(\pi_x p_n(0),L_n s)\cap I_n^v} \tag{6.4}$$

for all $\gamma_1, \gamma_2 \in \mathfrak{L}_n$ and $s > 0$. By the definition of T_n, we have

$$h_n(T_n\gamma(y), y) = \gamma(T_n\gamma(y))$$

for all $\gamma \in \mathfrak{L}_n$ and $y \in I_n^v$. Then

$$\begin{aligned}
|T_n\gamma_1(y) - T_n\gamma_2(y)| &\le \frac{1}{m}|h_n(T_n\gamma_1(y),y) - h_n(T_n\gamma_2(y),y)| \\
&= \frac{1}{m}|\gamma_1(T_n\gamma_1(y)) - \gamma_2(T_n\gamma_2(y))| \\
&\le \frac{1}{m}|\gamma_1(T_n\gamma_1(y)) - \gamma_1(T_n\gamma_2(y))| + \frac{1}{m}|\gamma_1(T_n\gamma_2(y)) - \gamma_2(T_n\gamma_2(y))| \\
&\le \frac{L_n}{m}|T_n\gamma_1(y) - T_n\gamma_2(y)| + \frac{1}{m}\|\gamma_1-\gamma_2\|_{B(\pi_x p_n(0),L_n s)\cap I_n^v}
\end{aligned}$$

for all $\gamma_1, \gamma_2 \in \mathfrak{L}_n$ and $y \in B_{\pi_x p_n(0)}(s) \cap I_n^v$ since $\mathrm{Im} T_n\gamma_1, \mathrm{Im} T_n\gamma_2 \subset U_1$. Note that

$$|T_n\gamma_2(y) - p_{x,n}(0)| = |T_n\gamma_2(y) - T_n\gamma_2(p_{x,n}(0))| \le L_n s$$

and so $T_n\gamma_2(y) \in B_{\pi_x p_n(0)}(L_n s)$. Thus we proved (6.4).

Now we show T_n has a fixed point. Let $\gamma_0(y) = \pi_x p_n(0)$ be the constant map. Define $\gamma_j = T_n^j\gamma_0$. We prove that $\{\gamma_j\}_{j=0}^{\infty}$ is a Cauchy sequence. For all $j > k$, we have

$$\begin{aligned}
\left\|T\gamma_j - T\gamma_k\right\|_{I_n^v} &= \left\|T\gamma_j - T\gamma_k\right\|_{B(\pi_x p_n(0),|I_n^v|)\cap I_n^v} \\
&\le \left(\frac{1}{m-L_n}\right)^k \left\|\gamma_{j-k} - \gamma_0\right\|_{B(\pi_x p_n(0),L_n^k|I_n^v|)\cap I_n^v} \\
&= \left(\frac{1}{m-L_n}\right)^k \left\|\gamma_{j-k} - \gamma_{j-k}(p_{x,n}(0))\right\|_{B_{p_{n,x}(0)}(L_n^k|I_n^v|)\cap I_n^v} \\
&\le \left(\frac{L_n}{m-L_n}\right)^k L_n|I_n^v|.
\end{aligned}$$

Note that $\frac{L_n}{m-L_n} < 1$ since $L_n < \frac{m}{2}$ for all $n \ge 0$. Therefore, γ_n has a limit and we proved $W_n^0(0) \subset U_1 \times I_n^v$ exists.

Moreover, we have $W_n^0(0) \subset V_n \times I_n^v$ and $V_n \subset h_n(U_2, y)$ for all $y \in I_n^v$ by Lemma 6.7. Apply Lemma 5.19 again, $F_n^{-1}(W_n^0(0)) \cap U_2 \times I_n^v$ is the vertical graph of a L_n-Lipschitz function. This proves that $W_n^1(0) \subset U_2 \times I_n^v$ exists.

Similarly, we have $W_n^1(0) \subset U_2 \times I_n^v$ and $U_2 \subset h_n(U_1, y)$ for all $y \in I_n^v$ by Lemma 6.7. Apply Lemma 5.19 again, $F_n^{-1}(W_n^1(0)) \cap U_1 \times I_n^v$ is the vertical graph of a L_n-Lipschitz function. This proves that $W_n^2(0) \subset U_1 \times I_n^v$ exists. □

Proposition 6.9. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ the local stable manifold $W_n^t(-1) \subset I^h \times I_n^v$ exists and $W_n^t(-1)$ is a vertical graph of a $c\,\|\varepsilon_n\|$-Lipschitz function on I_n^v for $t = 0, 2$ and $n \geq 0$.*

Proof. The proof is similar to the proof of Lemma 6.7 and Proposition 6.8. □

Since the local stable manifolds $W_n^0(-1)$, $W_n^2(-1)$, $W_n^0(0)$, $W_n^1(0)$, and $W_n^2(0)$ exists, this shows that

Corollary 6.10. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exists a constant $\overline{\varepsilon} > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ the partition A_n, B_n, and C_n on D_n exists for all $n \geq 0$. Moreover, $C_n \subset U_1 \times I_n^v$ where U_1 is the interval defined in Lemma 6.7.*

Moreover, we are able to control the local stable manifolds so that they are close to the local manifold of the limiting case when the sequence of Henon-like maps is close enough to the fixed point G.

Lemma 6.11. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. For all $d > 0$, there exists a constant $\overline{\varepsilon} > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following properties hold for all $n \geq 0$:*

1. $W_n^0(0) \in B_{q(0)}(d) \times I_n^v$,
2. $W_n^1(0) \in B_{\hat{q}(0)}(d) \times I_n^v$,
3. $W_n^0(-1) \in B_{q(-1)}(d) \times I_n^v$, *and*
4. $W_n^2(-1) \in B_{\hat{q}(-1)}(d) \times I_n^v$

Proof. By Lemma 6.4, there exists $\overline{\varepsilon} > 0$ such that

$$|\pi_x p_n(0) - q(0)| < \frac{d}{2}$$

for all $n \geq 0$. Also assume that $\overline{\varepsilon}$ is small enough such that Proposition 6.8 holds and $c' > 0$ is the constant defined by the proposition. Assume that $W_n^0(0)$ is the vertical graph of γ. Then

$$\begin{aligned}
|\gamma(y) - q(0)| &\leq |\gamma(y) - \pi_x p_n(0)| + |\pi_x p_n(0) - q(0)| \\
&\leq c'\,\|\varepsilon_n\|\,|I_n^v| + \frac{d}{2} \\
&\leq c'\,(c\,\|\varepsilon_0\|)^{2^n} \prod_{j=0}^{n-1} \lambda_j\,|I^v| + \frac{d}{2} \\
&\leq c'\,(c\overline{\varepsilon})^{2^n}\,(\lambda + \overline{\varepsilon})^n\,|I^v| + \frac{d}{2} \\
&< d.
\end{aligned}$$

Here, we also assume that c is the constant defined in Proposition 5.26 and $\overline{\varepsilon}$ is small enough such that

$$c'\,(c\overline{\varepsilon})^{2^n}\,(\lambda + \overline{\varepsilon})^n\,|I^v| < \frac{d}{2}$$

for all $n \geq 0$. This proves the first property.

To prove the second property, let m be the constant defined by Lemma 6.7, $\overline{\varepsilon}$ be small enough such that the lemma holds, and $W_n^0(0) \in B_{q(0)}(\frac{dm}{2}) \times I_n^v$ for all $n \geq 0$. By the mean value theorem, we have

$$|h_n(\gamma(y),y) - h_n(\hat{q}(0),y)| \geq m\,|\gamma(y) - \hat{q}(0)|$$

since $W_n^1(0) \in U_2 \times I_n^v$ and $\hat{q} \in U_2$. Also

$$\begin{aligned}
|h_n(\gamma(y),y) - h_n(\hat{q}(0),y)| &\leq |h_n(\gamma(y),y) - q(0)| + |g(\hat{q}(0)) - h_n(\hat{q}(0),y)| \\
&< \frac{dm}{2} + \|h_n - g\|_{\varepsilon(\delta)} \\
&< dm.
\end{aligned}$$

Here, we also assume that $\overline{\varepsilon} < \frac{dm}{2}$. That is

$$\|h_n - g\|_{I^h \times I_n^v} < \frac{dm}{2}$$

for all $n \geq 0$. Combine the two inequalities, we get

$$|\gamma(y) - \hat{q}(0)| < d$$

and hence $W_n^1(0) \in B_{\hat{q}(0)}(d) \times I_n^v$.

The third and forth properties are similar to the first two. □

6.3. Structure of the local stable manifolds for period doubling periodic points

In this section, we study the geometric properties of the local stable manifolds.

First, we study the geometry of $W_n^t(j)$ by generalizing Proposition 6.8 and Proposition 6.9. The following two lemmas allows us to pullback vertical graphs by the nonlinear rescaling ϕ_n.

Lemma 6.12. *Assume that I^h and I^v are intervals, s is an affine map such that $s(I^h) = \hat{I}^h$ and $s(I^v) = \hat{I}^v$, and $\Lambda(x,y) = (s(x), s(y))$. If $\Gamma \subset \hat{I}^h \times \hat{I}^v$ is a vertical graph of a L-Lipschitz function, then $\Lambda^{-1}(\Gamma) \subset I^h \times I^v$ is a vertical graph of a L-Lipschitz function.*

Proof. It follows directly by s is affine. □

Lemma 6.13. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $m > 0$ and $\overline{\varepsilon} > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ the following property hold for all $n \geq 0$:*

If $\Gamma \subset \Lambda_n^{-1}(D_{n+1})$ is a vertical graph of a L-Lipschitz function on I_n^v, then $H_n^{-1}(\Gamma) \subset C_n$ is a vertical graph of a $\frac{1}{m}\left(L + \frac{1}{\delta}\|\varepsilon_n\|\right)$-Lipschitz function on I_n^v.

Proof. (x) Assume that $\Gamma \subset \Lambda_n^{-1}(D_{n+1})$ is a vertical graph of a L-Lipschitz function γ on I_n^v. By direct computation, we have

$$H_n^{-1}(\gamma(y),y) = (h_y^{-1} \circ \gamma(y), y).$$

for all $y \in I_n^v$. Thus, $H_n^{-1}(\Gamma)$ is a vertical graph of the function $y \to h_y^{-1} \circ \gamma(y)$.

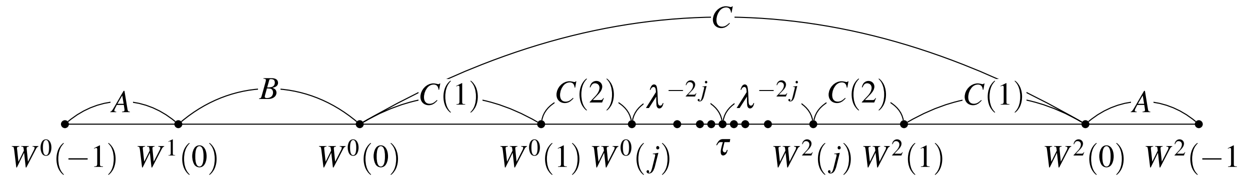


Figure 6.1.: The structure of the rescaling levels. The figure shows the partition and the local stable manifolds on the horizontal cross section that intersects the tip.

For all $y_1, y_2 \in I_n^v$, by the mean value theorem and Lemma 5.25, we have

$$
\begin{aligned}
\left|h_{y_2}^{-1} \circ \gamma(y_2) - h_{y_1}^{-1} \circ \gamma(y_1)\right| &\leq \left|h_{y_2}^{-1} \circ \gamma(y_2) - h_{y_2}^{-1} \circ \gamma(y_1)\right| + \left|h_{y_2}^{-1} \circ \gamma(y_1) - h_{y_1}^{-1} \circ \gamma(y_1)\right| \\
&\leq \left(\frac{L}{\left|h_{y_2}' \circ h_{y_2}^{-1} \circ \gamma(\xi)\right|} + \frac{\left|\frac{\partial \varepsilon_n}{\partial y}(h_\eta^{-1} \circ \gamma(y_1), \eta)\right|}{\left|h_\eta' \circ h_\eta^{-1} \circ \gamma(y_1)\right|} \right) |y_2 - y_1|
\end{aligned}
$$

for some $\xi, \eta \in (y_1, y_2)$. Since $\Gamma \subset \Lambda_n^{-1}(D_{n+1})$, we have $H_n^{-1}(\Gamma) \subset C_n \subset U_1 \times I_n^v$ where U_1 is the closed interval defined in Lemma 6.7. Thus, $h_{y_2}^{-1} \circ \gamma(\xi), h_\eta^{-1} \circ \gamma(y_1) \in U_1$. Apply Lemma 6.7, get

$$
\left|h_{y_2}^{-1} \circ \gamma(y_2) - h_{y_1}^{-1} \circ \gamma(y_1)\right| \leq \frac{1}{m} \left(L + \frac{1}{\delta} \|\varepsilon_n\| \right) |y_2 - y_1|
$$

where $m > 0$ is the constant defined in Lemma 6.7. □

Combine Lemma 6.12 and 6.13, we obtain

Corollary 6.14. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $m > 0$ and $\overline{\varepsilon} > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ the following property hold for all $n \geq 0$:*
If $\Gamma \subset D_{n+1}$ is a vertical graph of a L-Lipschitz function on I_{n+1}^v, then $\phi_n^{-1}(\Gamma) \subset C_n$ is a vertical graph of a $\frac{1}{m}\left(L + \frac{1}{\delta}\|\varepsilon_n\|\right)$-Lipschitz function on I_n^v.

We recall from [dCLM05] that

Definition 6.15 (Tip)**.** Assume that $\overline{\varepsilon} > 0$ is sufficiently small. The tip τ of an infinitely renormalizable Hénon-like map $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ is the unique point such that

$$
\{\tau\} = \cap_{j=N}^{\infty} \left(\Phi_0^j\right)^{-1} (D_j \cap I^h \times I^h)
$$

for all $N \geq 0$.

The tip is an analog of the critical value in the non-degenerate case. Roughly speaking, the tip generates the attracting Cantor set of a Hénon-like map. See [dCLM05, Chapter 5] for more information.

From Proposition 6.3, a rescaling level $C_n(j)$ contains two components which are both bounded by two local stable manifolds. The following proposition lists the geometric properties of the local stable manifolds.

Proposition 6.16. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $c > 0$, and $c' > 1$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following properties hold for all $n \geq 0$:*

1. *$W_n^t(j)$ is a vertical graph with Lipschitz constant $c\,\|\varepsilon_n\|$ for all $j \geq -1$ and $t = 0,2$.*
2. *$\frac{1}{c'}\left(\frac{1}{\lambda}\right)^{2j} < \left|z_n^{(t)}(j) - \tau_n\right| < c'\left(\frac{1}{\lambda}\right)^{2j}$ for all $j \geq -1$ and $t = 0,2$ where $z_n^{(t)}(j)$ is the intersection point of $W_n^t(j)$ with the horizontal line through τ_n. See Figure 6.1.*

Proof. By Proposition 6.8 and Proposition 6.9, let $c'' > 0$ be a constant such that $W_n^0(0)$, $W_n^1(0)$, $W_n^2(0)$, $W_n^0(-1)$, and $W_n^2(-1)$ are vertical graphs of a $c''\,\|\varepsilon_n\|$-Lipschitz function on I_n^v for all $n \geq 0$ when $\overline{\varepsilon}$ is sufficiently small. Also let c' be the constant defined in Proposition 5.26. Set $c = \max(c'', \frac{2}{m\delta})$.

We prove the second property by induction on j. For the cases $j = -1, 0$, it is clear that $W_n^0(0)$, $W_n^1(0)$, $W_n^2(0)$, $W_n^0(-1)$, and $W_n^2(-1)$ are vertical graphs of $c\,\|\varepsilon_n\|$-Lipschitz functions for all $n \geq 0$.

Assume that for some $j \geq 0$, $W_n^t(j)$ is a vertical graph of a $c\,\|\varepsilon_n\|$-Lipschitz function on I_n^v for all $n \geq 0$ and $t = 0,2$. Then $W_n^t(j+1) = \phi_n^{-1}(W_{n+1}^t(j))$. By Corollary 6.14 and the induction hypothesis, $W_n^t(j+1)$ is a vertical graph of a Lipschitz function.

Finally, compute the Lipschitz constant by the formula from Corollary 6.14,

$$\begin{aligned}
\frac{1}{m}\left(c\,\|\varepsilon_{n+1}\|_{\varepsilon(\delta)} + \frac{1}{\delta}\,\|\varepsilon_n\|_{\varepsilon(\delta)}\right) &\leq \frac{1}{m}\left(cc'\,\|\varepsilon_n\|^2 + \frac{1}{\delta}\,\|\varepsilon_n\|\right) \\
&\leq \frac{1}{m}\left(cc'\overline{\varepsilon} + \frac{1}{\delta}\right)\|\varepsilon_n\| \\
&\leq \frac{2}{m\delta}\,\|\varepsilon_n\| \\
&\leq c\,\|\varepsilon_n\|.
\end{aligned}$$

Here, we assume that $\overline{\varepsilon}$ is small enough such that $cc'\overline{\varepsilon} < \frac{1}{\delta}$. Therefore, the second property is proved by induction.

The third property comes from [LM11, Proposition 3.5]. □

Now we study the topology of $B_n(j)$. For a fixed rescaling level j, we will show that the set $B_n(j)$ has two components, each component is bounded by two local stable manifolds that are vertical graphs when $\overline{\varepsilon}$ is sufficiently small. To show this, we study the pullback of the local stable manifold $W_n^0(t)$. When $\overline{\varepsilon}$ is sufficiently small, the pullback $F_n^{-1}(W_n^0(j))$ consists of two components on B_n.

First we define the points that are associated to the limiting case $F = G$. Recall from Definition 4.31 that $q^c(j)$ is the periodic point of g with period 2^j around the critical point.

Definition 6.17. Define $q^l(j) = -|q^c(j)|$ and $q^r(j) = |q^c(j)|$ for all $j \geq 0$.

For the limiting case, the two components of $G^{-1}(W^0(j))$ are determined by the points in the definition. Precisely, $G^{-1}(W^0(j))$ is the union of the two vertical lines $x = q^l(j)$ and $x = q^r(j)$.

The next lemma provides a required condition for applying Lemma 5.19 to pullback the local stable manifold.

Lemma 6.18. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. For all $j \geq 0$, there exist constants $\overline{\varepsilon} = \overline{\varepsilon}(j) > 0$, $m = m(j) > 0$, and $d' = d'(j) > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ we have*

$$\left|\frac{\partial h_n}{\partial x}(x,y)\right| \geq m$$

for all $x \in [q^l(j) - d', q^l(j) + d'] \cup [q^r(j) - d', q^r(j) + d']$, $y \in I_n^v$, and $n \geq 0$.

Proof. We prove the case for $x \in [q^l(j) - d', q^l(j) + d']$. The other case is similar.

Since g is continuous and $\left|g'(q^l(j))\right| > 0$, there exist $d > 0$ and $m > 0$ such that

$$\left|g'(x)\right| > 2m$$

for all $x \in [q^l(j) - d, q^l(j) + d]$. Also assume that $\overline{\varepsilon} < \delta m$. That is

$$\left\|\frac{\partial h_n}{\partial x} - g'\right\|_{I^h \times I_n^v} \leq \frac{1}{\delta}\left\|h_n - g\right\|_{I^h(\delta) \times I_n^v(\delta)} < m$$

for all $n \geq 0$. We get

$$\begin{aligned} \left|\frac{\partial h_n}{\partial x}(x,y)\right| &\geq \left|g'(x)\right| - \left\|\frac{\partial h_n}{\partial x} - g'\right\|_{I^h \times I_n^v} \\ &\geq m \end{aligned}$$

for all $x \in [q^l(j) - d, q^l(j) + d]$, $y \in I_n^v$, and $n \geq 0$. □

The last proposition provides the topology of the local stable manifolds in B_n when $\overline{\varepsilon}$ is sufficiently small.

Proposition 6.19. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. For all $j \geq 0$ and $d > 0$, there exist constants $\overline{\varepsilon} = \overline{\varepsilon}(j,d) > 0$ and $c = c(j) > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following properties hold for all $n \geq 0$:*

1. *$F_n^{-1}(W_n^0(j))$ has exactly two components $W_n^l(j) \subset [q^l(j) - d, q^l(j) + d] \times I_n^v$ and $W_n^r(j) \subset [q^r(j) - d, q^r(j) + d] \times I_n^v$.*
2. *Both components $W_n^l(j)$ and $W_n^r(j)$ are vertical graphs with Lipschitz constant $c\left\|\varepsilon_n\right\|$.*

Proof. First, we prove the existence of $W_n^l(j)$.

To pullback the local stable manifold $W_n^0(j)$, we check the conditions that are required for Lemma 5.19.

Assume that $d' < d$ is small enough such that Lemma 6.18 holds and m be the constant in the lemma. Then

$$\begin{aligned} h_n(q^l(j) - d', y) &\leq h_n(q^l(j), y) - md' \\ &\leq g(q^l(j)) + \left\|h_n - g\right\|_{I^h(\delta) \times I_n^v(\delta)} - md' \\ &\leq q(j) - \frac{1}{2}md' \end{aligned} \tag{6.5}$$

for all $y \in I_n^v$ and $n \geq 0$. Here we assume that $\overline{\varepsilon} > 0$ is sufficiently small such that Lemma 6.18 holds and $\|h_n - g\|_{I^h(\delta)\times I_n^v(\delta)} < \frac{1}{2}md'$. Similarly,

$$\begin{aligned}
h_n(q^l(j)+d',y) &\geq h_n(q^l(j),y)+md' \\
&\geq g(q^l(j)) - \|h_n - g\|_{I^h(\delta)\times I_n^v(\delta)} + md' \\
&\geq q(j)+\frac{1}{2}md' \qquad (6.6)
\end{aligned}$$

for all $y \in I_n^v$ and $n \geq 0$ when $\overline{\varepsilon}$ is small enough. The two inequalities (6.5) and (6.6) yields

$$h_n([q^l(j)-d',q^l(j)+d'],y) \supset [q(j)-\frac{1}{2}md',q(j)+\frac{1}{2}md'].$$

Also, by Lemma 6.18, we have

$$\left|\frac{\partial h_n}{\partial x}([q^l(j)-d',q^l(j)+d'],y)\right| \geq m$$

for all $y \in I_n^v$ and $n \geq 0$ when $\overline{\varepsilon}$ is small enough.

Finally, we need to check that $W_n^0(j)\cap I^h \times I^h \subset [q(j)-\frac{1}{2}md',q(j)+\frac{1}{2}md'] \times I^h$. By Corollary 6.6, we assume that $\overline{\varepsilon} = \overline{\varepsilon}(j,d) > 0$ is sufficiently small such that $|\pi_x p_n(j) - q(j)| < \frac{1}{4}md'$ since j is fixed. Also, by Proposition 6.16, $W_n^0(j)$ is a vertical graph of a $c\,\|\varepsilon_n\|$-Lipschitz function. Thus,

$$|\pi_x z - \pi_x p_n(j)| \leq c\,\|\varepsilon_n\|\left|I^h\right|$$

for all $z \in W_n^0(j)\cap I^h \times I^h$ and $n \geq 0$. We obtain

$$|\pi_x z - q(j)| \leq |\pi_x z - \pi_x p_n(j)| + |\pi_x p_n(j) - q(j)| < c\,\|\varepsilon_n\|\left|I^h\right| + \frac{1}{4}md' < \frac{1}{2}md'$$

for all $z \in W_n^0(j)\cap I^h \times I^h$ and $n \geq 0$ when $\overline{\varepsilon}$ is small enough. Hence, $W_n^0(j)\cap I^h \times I^h \subset [q(j)-\frac{1}{2}md',q(j)+\frac{1}{2}md'] \times I^h$ for all $n \geq 0$.

By Lemma 5.19, $W_n^l(j) \equiv F^{-1}(W_n^0(j)) \cap [q^l(j)-d',q^l(j)+d'] \times I_n^v$ is the vertical graph of a Lipschitz function on I_n^v. Apply the formula from Lemma 5.19 to compute the Lipschitz constant, we get

$$\frac{1}{\delta(m - c\,\|\varepsilon_n\|)}\,\|\varepsilon_n\| \leq \frac{2}{\delta m}\,\|\varepsilon_n\|$$

for all $n \geq 0$ when $\overline{\varepsilon}$ is small enough such that $c\overline{\varepsilon} \leq \frac{m}{2}$. Consequently, the local stable manifold $W_n^l(j)$ is the vertical graph of a $\frac{2}{\delta m}\,\|\varepsilon_n\|$-Lipschitz function.

Similarly, the local stable manifold $W_n^r(j) \equiv F^{-1}(W_n^0(j)) \cap [q^r(j)-d',q^r(j)+d'] \times I_n^v$ is also the vertical graph of a $\frac{2}{\delta m}\,\|\varepsilon_n\|$-Lipschitz function.

It remains to show that $W_n^l(j)$ and $W_n^r(j)$ are the only components of $F^{-1}(W_n^0(j))$. From 6.5, we have $h_n((-\infty,q^l(j)-d']\cap I^h,y)) \subset (-\infty,q(j)-\frac{1}{2}md']$. Similarly, one can prove $h_n([q^l(j)+d,q^r(j)-d]\cap I^h,y)) \subset [q(j)+\frac{1}{2}md',\infty)$ and $h_n([q^r(j)+d',\infty)\cap I^h,y)) \subset (-\infty,q(j)-\frac{1}{2}md']$. This shows that the sets $F_n(((-\infty,q^l(j)-d']\cap I^h)\times I_n^v)$, $F_n([q^l(j)+d',q^r(j)-d']\times I_n^v)$, and $F_n(([q^r(j)+$

$d',\infty)\cap I^h)\times I^v_n)$ does not intersect $W^0_n(j)$ when $\overline{\varepsilon}$ is small enough. This proves that $W^l_n(j)$ and $W^r_n(j)$ are the only components in $F_n^{-1}(W^0_n(j))$. □

Remark 6.20. Unlike Proposition 6.16, here the constant $\overline{\varepsilon}$ is not uniform on $j\geq 0$. For a non-degenerate Hénon-like map, the structure of the local stable manifolds is similar to degenerate case when j is large. The local stable manifold $W^0_n(j)$ is far away from the tip and hence the pullback $F_n^{-1}(W^0_n(j))$ is the union of two vertical graphs in B_n. However, the structure turns to be different when j is large. The local stable manifold is close to the tip and the vertical line argument in Chapter 9 shows that the pullback $F_n^{-1}(W^0_n(j))$ is a concave curve in B_n.

6.4. Asymptotic behavior near G

In this section, we estimate the derivatives of a Hénon-like map that is close to the hyperbolic fixed point G. Define $v_n\in I^h$ to be the critical point of f_n and $w_n=f_n(v_n)$ be the critical value.

The first lemma proves that a Hénon-like map acts like a quadratic map on B.

Lemma 6.21. *Given $\delta>0$ and $I^v\supset I^h\Supset I$. There exist constants $a>0$, $\overline{\varepsilon}>0$, and an interval $I^B\subset I^h$ such that for all $F\in\hat{\mathscr{I}}_\delta(I^h\times I^v,\overline{\varepsilon})$, the following properties hold for all $n\geq 0$:*

The interior of I^B contains $\hat{q}(0)$ and $q(0)$, $I^B\times I^v_n\supset B_n$,

$$\frac{1}{a}|x-v_n|\leq\left|f_n'(x)\right|\leq a|x-v_n|,$$

and

$$\frac{1}{2a}(x-v_n)^2\leq|f_n(x)-f_n(v_n)|\leq\frac{a}{2}(x-v_n)^2$$

for all $x\in I^B$.

Proof. By Proposition 4.28, the map g is concave on $[-c^{(1)},c^{(1)}]$. Let $d>0$ be small enough such that $[-q(0)-d,q(0)+d]\subset[-c^{(1)},c^{(1)}]$. Define $I^B=[-q(0)-d,q(0)+d]$. Then $\min_{x\in I^B}|g''(x)|>a'$ for some constant $a'>0$ when $d>0$ is small enough. Assume that $\overline{\varepsilon}<\frac{a'\delta^2}{4}$. In particular,

$$\left\|f_n''-g''\right\|_{I^h}<\frac{2}{\delta^2}\|f_n-g\|_{I^h(\delta)}<\frac{a'}{2}$$

for all $n\geq 0$. Also assume that $\overline{\varepsilon}$ is small enough so that $B_n\subset I^B\times I^v_n$ for all $n\geq 0$ by Lemma 6.11 because that B_n is bounded by $W^1_n(0)$ and $W^0_n(0)$.

By the mean value theorem, there exists $\xi\in(x,v_n)$ such that

$$f_n'(x)=f_n'(x)-f_n'(v_n)=f_n''(\xi)(x-v_n).$$

We obtain the lower bound for the first inequality as

$$\begin{aligned}\left|f_n'(x)\right| &= \left|f_n''(\xi)\right||x-v_n|\\ &\geq \left(\left|g''(\xi)\right|-\left|f_n''(\xi)-g''(\xi)\right|\right)|x-v_n|\\ &\geq \left(\min_{x\in I^B}\left|g''(y)\right|-\frac{a'}{2}\right)|x-v_n|\end{aligned}$$

$$= \frac{a'}{2}|x-v_n|.$$

Similarly, for the upper bound, we have

$$\begin{aligned}
\left|f_n'(x)\right| &= \left|f_n''(\xi)\right||x-v_n| \\
&\leq \left(\left|g''(\xi)\right| + \left|f_n''(\xi)-g''(\xi)\right|\right)|x-v_n| \\
&\leq \left(\max_{x\in I^B}\left|g''(y)\right| + \frac{a'}{2}\right)|x-v_n| \\
&= \frac{3a'}{2}|x-v_n|
\end{aligned}$$

This proves the first inequality.

To prove the second inequality, we prove the case for $x > v_n$. By the fundamental theorem of calculus, we have

$$|f_n(x)-f_n(v_n)| = f_n(v_n)-f_n(x) = \int_{v_n}^{x} -f_n'(t)dt.$$

After applying the first inequality and evaluate the integration, we obtain

$$\frac{1}{2a}(x-v_n)^2 \leq |f_n(x)-f_n(v_n)| \leq \frac{a}{2}(x-v_n)^2.$$

The case for $x < v_n$ is similar. □

The next lemma shows that a Hénon-like map is expanding on A and C in the x-coordinate when it is close enough to the fixed point G.

Lemma 6.22. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $E > 1$, $\overline{\varepsilon} > 0$, and a union of two intervals $I^{AC} \subset I^h$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ the following properties hold for all $n \geq 0$:*

The interior of I^{AC} contains $q(-1)$, $\hat{q}(0)$, $q(0)$, and $\hat{q}(-1)$, $I^{AC} \times I_n^v \supset A_n \cup W_n^2(0) \cup C_n$, and

$$\left|\frac{\partial h_n}{\partial x}(x,y)\right| \geq E$$

for all $(x,y) \in I^{AC} \times I_n^v$.

Proof. By Proposition 4.38, there exists $E' > 1$ such that

$$\left|g'(x)\right| \geq E'$$

for all $x \in [q(-1),\hat{q}(0)] \cup [q(0),\hat{q}(-1)]$. Let $\Delta m > 0$ be small enough such that $E \equiv E' - \Delta m > 1$ and $d' > 0$ be small enough such that

$$\left|g'(y)-g'(x)\right| < \Delta m/2$$

for all $|y-x| \leq d'$ since g is uniform continuous on I^h. Define $I^{AC} = [q(-1)-d', \hat{q}(0)+d'] \cup [q(0)-d', \hat{q}(-1)+d'] \subset I^h$. Also, let $\overline{\varepsilon} > 0$ be small enough such that $A_n \cup W_n^2(0) \cup C_n \subset I^{AC} \times I_n^v$

by Lemma 6.11 because $A_n \cup W_n^2(0) \cup C_n$ is a union of two regions, one region is bounded by $W_n^0(-1)$ and $W_n^1(0)$ and the other region is bounded by $W_n^0(0)$ and $W_n^2(-1)$. Then

$$\left|g'(x)\right| \geq E' - \Delta m/2$$

for all $x \in I^{AC}$.

Moreover, also assume that $\overline{\varepsilon}$ is small enough such that

$$\left\| \frac{\partial h_n}{\partial x} - g' \right\|_{I^h \times I_n^v} \leq \frac{1}{\delta} \left\| h_n - g \right\|_{I^h(\delta) \times I_n^v(\delta)} < \frac{\Delta m}{2}$$

for all $n \geq 0$. Then

$$\left| \frac{\partial h_n}{\partial x}(x,y) \right| \geq \left|g'(x)\right| - \left\| \frac{\partial h_n}{\partial x} - g' \right\|_{I^h \times I_n^v} \geq E$$

for all $(x,y) \in I^{AC} \times I_n^v$. □

The last lemma provides a upper bound for the expansion rate of a Hénon-like map in the x-coordinate.

Lemma 6.23. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $K > 0$ and $\overline{\varepsilon} > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ we have*

$$\left| \frac{\partial h_n}{\partial x}(x,y) \right| \leq K$$

for all $(x,y) \in I^h \times I_n^v$ and $n \geq 0$.

Proof. By the compactness of I^h, there exists $K > 1$ such that

$$\left|g'(x)\right| \leq \frac{K}{2}$$

for all $x \in I^h$. Also, there exists $\overline{\varepsilon} > 0$ small enough such that

$$\left\| \frac{\partial h_n}{\partial x} - g' \right\|_{I^h \times I_n^v} \leq \frac{1}{\delta} \left\| h_n - g \right\|_{I^h(\delta) \times I_n^v(\delta)} \leq \frac{K}{2}$$

for all $n \geq 0$. Then

$$\left| \frac{\partial h_n}{\partial x}(x,y) \right| \leq \left|g'(x)\right| + \left\| \frac{\partial h_n}{\partial x} - g' \right\|_{\varepsilon(\delta)} \leq K.$$

for all $(x,y) \in I^h \times I_n^v$ and $n \geq 0$. □

6.5. Relation between the tip and the critical value

In Lemma 6.21, we proved that a Hénon-like map behaves like a quadratic map when a point is close to the critical point v_n of f_n for the representation $F_n = (f_n - \varepsilon_n, x)$. However, the critical point v_n and the critical value w_n in the estimates depend on the representation.

In this section, we show that the critical value w_n (for any representation) is $\|\varepsilon_n\|$-close to the tip τ_n in Proposition 6.27. This allows us to replace v_n and w_n by the representation independent quantity τ_n. This makes the quadratic estimates in Lemma 6.21 useful when a point is $\|\varepsilon_n\|$-away from the tip.

To estimate the distance from the tip to the critical value, we write $\tau_n = (a_n, b_n)$. Since the rescaling ϕ_n maps a horizontal line to a horizontal line, we focus on the horizontal slice that intersects the tip in each renormalization scale. Define the restriction of the rescaling map ϕ to the slice as

$$\eta_n(x) = \pi_x \circ \phi_n(x, b_n) = s_n \circ h_n(x, b_n).$$

By the definition of the tip, the quantities satisfy the recurrence relations $\phi_n(\tau_n) = \tau_{n+1}$, $\eta_n(a_n) = a_{n+1}$, and $s_n(b_n) = b_{n+1}$.

First, we prove a lemma that allows us to compare the critical value between two renormaliztion levels.

Lemma 6.24. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, we have*

$$|w_{n+1} - \eta_n(w_n)| < c\,\|\varepsilon_n\|$$

for all $n \geq 0$.

Proof. First, we compare the critical points v_n and v_{n+1}. By Proposition 5.26, we have

$$\left\|f'_{n+1} - (s_n \circ f_n^2 \circ s_n^{-1})'\right\|_{I^h} < c\left\|f_{n+1} - s_n \circ f_n^2 \circ s_n^{-1}\right\|_{I^h(\delta)} < c\,\|\varepsilon_n\|$$

for some constant $c > 0$ when $\overline{\varepsilon} > 0$ is sufficiently small. Also, there exists a constant $a > 0$ independent of F such that $\left|f''_{n+1}(x)\right|, \left|(s_n \circ f_n^2 \circ s_n^{-1})''(x)\right| > a$ for all $x \in [\hat{q}(0), q(0)]$ and $n \geq 0$ because the critical point of g is non-degenerate. Apply Lemma A.3 to the roots v_{n+1} and $s_n(v_n)$ of the functions f'_{n+1} and $(s_n \circ f_n^2 \circ s_n^{-1})'$, there exists a constant $c' > 0$ such that

$$|v_{n+1} - s_n(v_n)| \leq c'\,\|\varepsilon_n\|$$

for all $n \geq 0$.

Moreover, by the quadratic estimates in Lemma 6.21, we get

$$\begin{aligned}
\left|f_{n+1}(v_{n+1}) - s_n \circ f_n^2(v_n)\right| &\leq |f_{n+1}(v_{n+1}) - f_{n+1}(s_n(v_n))| + \left|f_{n+1}(s_n(v_n)) - s_n \circ f_n^2(v_n)\right| \\
&\leq \frac{a}{2}|v_{n+1} - s_n(v_n)|^2 + \left|f_{n+1}(s_n(v_n)) - s_n \circ f_n^2 \circ s_n^{-1}(s_n(v_n))\right| \\
&\leq \frac{ac'^2}{2}\|\varepsilon_n\|^2 + c\,\|\varepsilon_n\| \\
&\leq c''\,\|\varepsilon_n\|
\end{aligned}$$

for some constant $c'' > 0$. In the equation, one should think $s_n \circ f_n^2(v_n) = (s_n \circ f_n) \circ f_n(v_n)$ as the rescaled critical value of f_n.

Finally, we compare the critical values w_n and w_{n+1}. Compute

$$|w_{n+1} - \eta_n(w_n)| = \left|f_{n+1}(v_{n+1}) - s_n(f_n^2(v_n) - \varepsilon_n(f_n(v_n), b_n))\right|$$

$$\begin{aligned}
&\leq \left| f_{n+1}(v_{n+1}) - s_n \circ f_n^2(v_n) \right| + \lambda_n \left| \varepsilon_n(f_n(v_n), b_n) \right| \\
&\leq c'' \|\varepsilon_n\| + 2\lambda \|\varepsilon_n\| \\
&= (c'' + 2\lambda) \|\varepsilon_n\|
\end{aligned}$$

for all $n \geq 0$ whenever $\overline{\varepsilon}$ is small enough such that $\lambda_n \leq 2\lambda$. □

The rescaling maps $\{\eta_n\}_{n\geq 0}$ can be viewed as a non-autonomous dynamical system (system that depends on time). An orbit is defined as follows.

Definition 6.25 (Orbit of Non-Autonomous Systems)**.** Let Y_n be a complete metric space, $X_n \subset Y_n$ be a closed subset, and $f_n : X_n \to Y_{n+1}$ be a continuous map for all $n \geq 1$. A sequence $\{x_n\}_{n=1}^{\infty}$ is an orbit of the non-autonomous system $\{f_n\}_{n=1}^{\infty}$ if $x_n \in X_n$ and $x_{n+1} = f_n(x_n)$ for all $n \geq 1$. A sequence $\{x_n\}_{n=1}^{\infty}$ is an ε-orbit of the non-autonomous system $\{f_n\}_{n=1}^{\infty}$ if $x_n \in X_n$ and $|x_{n+1} - f_n(x_n)| < \varepsilon$ for all $n \geq 1$.

Next, we prove an analog of the shadowing theorem for non-autonomous systems.

Lemma 6.26 (Shadowing Theorem for Non-Autonomous Systems)**.** *For each $n \geq 1$, let Y_n be a complete metric space equipped with a metric d (the metric depends on n), $X_n \subset Y_n$ be a closed subset, and $f_n : X_n \to Y_{n+1}$ be a homeomorphism. Also assume that the non-autonomous system $\{f_n\}_{n=1}^{\infty}$ has a uniform expansion. That is, there exists a constant $L > 1$ such that $|f_n(a) - f_n(b)| \geq L|a - b|$ for all $a, b \in X_n$ and $n \geq 1$.*

If $\{x_n\}_{n=1}^{\infty}$ is an ε-orbit of $\{f_n\}_{n=1}^{\infty}$, there exists a unique orbit $\{u_n\}_{n=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ such that

$$d(x_n, u_n) \leq \frac{\varepsilon}{L-1}$$

for all $n \geq 1$. In addition, if $\{X_n\}_{n=1}^{\infty}$ is uniformly bounded, then the non-autonomous system $\{f_n\}_{n=1}^{\infty}$ has exactly one orbit $\{u_n\}_{n=1}^{\infty}$. For any sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in X_n$, we have

$$u_n = \lim_{j\to\infty} \left(f_{n+j-1} \circ \cdots \circ f_{n+1} \circ f_n \right)^{-1} (x_{n+j})$$

for all $n \geq 1$.

Proof. Given an ε-orbit $\{x_n\}_{n=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$. The uniqueness of the orbit $\{u_n\}_{n=1}^{\infty}$ follows from expansion.

To prove existence, define $x_n^{(0)} = x_n$ and $x_n^{(j)} = f_{n-j}^{-1}(x_n^{(j-1)}) \in X_{n-j}$ for $j = 1, \cdots, n-1$ by induction. From the expansion of f_n, we have

$$d(x_{n+j}^{(0)}, x_{n+j+1}^{(1)}) \leq \frac{1}{L} d(f_{n+j}(x_{n+j}), x_{n+j+1}) < \frac{\varepsilon}{L}$$

and

$$d(x_m^{(j+1)}, x_n^{(k+1)}) \leq \frac{1}{L} d(x_m^{(j)}, x_n^{(k)})$$

for all $j, k \geq 0$. By the triangular inequality, we get

$$d(x_{n+j}^{(j)}, x_{n+k}^{(k)}) \quad \leq \quad d(x_{n+j}^{(j)}, x_{n+j+1}^{(j+1)}) + \cdots + d(x_{n+k-1}^{(k-1)}, x_{n+k}^{(k)})$$

$$
\begin{aligned}
&\leq \frac{1}{L^j} d(x^{(0)}_{n+j}, x^{(1)}_{n+j+1}) + \cdots + \frac{1}{L^{k-1}} d(x^{(0)}_{n+k-1}, x^{(1)}_{n+k}) \\
&< \frac{\varepsilon}{L^{j+1}} \left(1 + \frac{1}{L} + \cdots + \frac{1}{L^{k-j-1}}\right) \\
&< \frac{\varepsilon}{(L-1)L^j}
\end{aligned}
$$

for all $j < k$. Therefore, $\left\{x^{(j)}_{n+j}\right\}_{j=0}^{\infty}$ is a Cauchy sequence in Y_n and hence $u_n = \lim_{j\to\infty} x^{(j)}_{n+j}$ exists.

Similarly, by triangular inequality and the expansion of f, we have

$$
\begin{aligned}
d(x_n, u_n) &\leq d(x^{(0)}_n, x^{(1)}_{n+1}) + \cdots + d(x^{(j-1)}_{n+j-1}, x^{(j)}_{n+j}) + d(x^{(j)}_{n+j}, u_n) \\
&\leq \frac{1}{L^0} d(x^{(0)}_n, x^{(1)}_{n+1}) + \cdots + \frac{1}{L^{j-1}} d(x^{(0)}_{n+j-1}, x^{(1)}_{n+j}) + d(x^{(j)}_{n+j}, u_n) \\
&\leq \frac{\varepsilon}{L} \left(\frac{1}{L^0} + \cdots + \frac{1}{L^{j-1}}\right) + d(x^{(j)}_{n+j}, u_n) \\
&< \frac{\varepsilon}{L-1} + d(x^{(j)}_{n+j}, u_n)
\end{aligned}
$$

for all $j \geq 0$. Take the limit $j \to \infty$, we obtain

$$
d(x_n, u_n) \leq \frac{\varepsilon}{L-1}.
$$

To prove that $\{u_n\}_{n=1}^{\infty}$ is an orbit of $\{f_n\}_{n=1}^{\infty}$, we evaluate the limit directly. We get

$$
f_n(u_n) = \lim_{j\to\infty} f_n(x^{(j)}_{n+j}) = \lim_{j\to\infty} x^{(j-1)}_{n+j} = \lim_{j\to\infty} x^{(j)}_{n+1+j} = u_{n+1}.
$$

In addition, if X_n is uniformly bounded and $x_n \subset X_n$ for all $n \geq 1$. Assume that the diameter of X_n is bounded by $d > 0$ for all $n \geq 1$. Let $y_n = f_n^{-1}(x_{n+1})$. Then $\{y_n\}_{n=1}^{\infty}$ is a d-orbit for $\{f_n\}_{n=1}^{\infty}$ and hence

$$
u_n = \lim_{j\to\infty} \left(f_n \circ f_{n+1} \circ \cdots f_{n+j-2}\right)^{-1} (y_{n+j-1}) = \lim_{j\to\infty} \left(f_n \circ f_{n+1} \circ \cdots f_{n+j-1}\right)^{-1} (x_{n+j})
$$

exists by the proof of the previous part. The uniqueness of the sequence $\{u_n\}_{n=1}^{\infty}$ follows from the expansion of $\{f_n\}_{n=1}^{\infty}$. □

The result from Lemma 6.24 shows that the sequence of critical values w_n is an ε-orbit of the expanding non-autonomous system η_n. With the help from the Shadowing Theorem, we are able to obtain the goal for this section.

Proposition 6.27. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, we have*

$$
|f_n(v_n) - \pi_x(\tau_n)| < c\, \|\varepsilon_n\|
$$

for all $n \geq 0$.

Proof. Fix $n \geq 0$ and consider the sequence $\left\{w_{n+j}\right\}_{j=0}^{\infty}$. By Lemma 6.24, there exists $c > 0$ such that

$$\left|w_{n+j+1} - \eta_{n+j}(w_{n+j})\right| < c\left\|\varepsilon_{n+j}\right\| \leq c\left\|\varepsilon_n\right\|$$

for some fixed $j \geq 0$ when $\overline{\varepsilon} > 0$ is small enough. Hence, the sequence $\left\{w_{n+j}\right\}_{j=0}^{\infty}$ is a $c\left\|\varepsilon_n\right\|$-orbit for the perturbed maps $\left\{\eta_{n+j}\right\}_{j=0}^{\infty}$.

To prove the non-autonomous system is uniform expanding, we evaluate the derivative

$$\eta'_{n+j}(x) = -\lambda_n \frac{\partial h_{n+j}}{\partial x}(x, b_{n+j}).$$

Assume that $\overline{\varepsilon} > 0$ is also small enough such that $\lambda_n \geq \sqrt{\lambda} > 1$. Also, by Lemma 6.22, there exists $E > 1$ such that

$$\left|\frac{\partial h_{n+j}}{\partial x}(x, b_{n+j})\right| > E$$

for all $x \in \overline{\pi_x\left(B_{n+j} \cap \{y = b_{n+j}\}\right)}$ and $j \geq 0$ when $\overline{\varepsilon}$ is small enough. Thus, there exists $L > 1$ such that

$$\left|\eta'_{n+j}(x)\right| > L$$

for all $x \in \overline{\pi_x\left(B_{n+j} \cap \{y = b_{n+j}\}\right)}$ and $j \geq 0$.

Finally, we apply Lemma 6.26. There exists an orbit $\left\{u_j\right\}_{j=0}^{\infty}$ of $\left\{\eta_{n+j}\right\}_{j=0}^{\infty}$ such that

$$\left|w_{n+j} - u_j\right| \leq \frac{c}{L-1}\left\|\varepsilon_n\right\|$$

for all $j \geq 0$. Also, by the definition of η_n and the tip, we have $a_{n+j+1} = \eta_{n+j}(a_{n+j})$ for all $j \geq 0$. The uniqueness of the orbit for bounded domains I^{AC} yields $u_j = a_{n+j}$ for all $j \geq 0$. Consequently,

$$|f_n(v_n) - \pi_x(\tau_n)| = |w_n - a_n| \leq \frac{c}{L-1}\left\|\varepsilon_n\right\|$$

for all $n \geq 0$. □

In addition, we can also estimate the distance from the critical point to the preimage of the tip.

Corollary 6.28. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $c > 0$ such that for all $F \in \mathscr{I}_{\hat{\delta}}(\hat{I}^h \times I^v, \overline{\varepsilon})$ we have*

$$\left|v_n - \pi_y(\tau_n)\right| < c\sqrt{\left\|\varepsilon_n\right\|}$$

for all $n \geq 0$.

Proof. Assume that $\overline{\varepsilon} > 0$ is small enough such that Lemma 6.21 and Proposition 6.27 hold. By Lemma 6.21, there exists $a > 1$ such that

$$\left|f_n(\pi_y\tau_n) - f_n(v_n)\right| \geq \frac{1}{a}\left|\pi_y\tau_n - v_n\right|^2.$$

Also, since $\tau_n \in \mathrm{Im}F_n$, there exists $y \in I_n^v$ such that $F_n(\pi_y \tau_n, y) = \tau_n$. We get

$$\begin{aligned}\left|f_n(\pi_y \tau_n) - f_n(v_n)\right| &\leq \left|f_n(\pi_y \tau_n) - h_n(\pi_y \tau_n, y)\right| + \left|\pi_x \tau_n - f_n(v_n)\right| \\ &\leq (1+c)\,\|\varepsilon_n\|\end{aligned}$$

by Proposition 6.27. Combine the two inequalities, we obtain

$$\left|\pi_y \tau_n - v_n\right| \leq \sqrt{a(1+c)}\sqrt{\|\varepsilon_n\|}.$$

$\square$

7. Closest Approach

Starting from this chapter, we will prove the nonexistence of wandering domains. Assume the contrapositive: there exists a wandering domain J.

In this chapter, we introduce a rescaled orbit $\{J_n\}_{n=0}^{\infty}$ of a wandering domain J which is called the J-closest approach. Then we define the horizontal size l_n, the vertical size h_n, and the rescaling level k_n of an element J_n. In the remaining part of this paper, we will study the expansion and contraction of the horizontal sizes of the orbit elements.

Recall the definition of wandering domain.

Definition 7.1 (Wandering Domain)**.** Assume that $F \in \mathscr{H}_\delta(I^h \times I^v)$, $D(F)$ exists, and F is an open map (diffeomorphism from $D(F)$ to the image). A nonempty connected open set $J \subset D(F)$ is a wandering domain of F if the orbit $\{F^n(J)\}_{n\geq 0}$ is disjoint from the stable manifolds of the periodic points.

Remark 7.2. The classical definition of wandering intervals includes one additional condition: the orbit elements are disjoint. This condition is redundant for case of the unimodal maps. Assume that J is an nonempty open interval that does not contain points from the basin of a periodic orbit. If the elements in the orbit of J intersect, then take a connected component U of the union of the orbit that contains at least two elements from the orbit. Then, there exists a positive integer n such that $f^n(U) \subset U$. It is easy to show that f^n has a fixed point in the interior of U by applying the Brouwer fixed-point theorem several times which leads to a contradiction. Therefore, the orbit elements of J are disjoint.

The following proposition allow us to generate wandering domains by iteration and rescaling.

Proposition 7.3. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exists a constant $\overline{\varepsilon} > 0$ such that for all open maps $F \in \mathscr{H}_\delta^r(I^h \times I^v)$, the following properties hold:*

1. *A set $J \subset D(F)$ is a wandering domain of F if and only if $F(J)$ is a wandering domain of F.*
2. *A set $J \subset C(F)$ is a wandering domain of F if and only if $\phi(J) \subset D(RF)$ is a wandering domain of RF.*

Proof. The proposition is true because the stable manifold of a periodic orbit is invariant under iteration and the rescaling of a stable manifold is also a stable manifold. □

Corollary 7.4. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exists a constant $\overline{\varepsilon} > 0$ such that for all open maps $F \in \mathscr{H}_\delta^r(I^h \times I^v, \overline{\varepsilon})$, F has a wandering domain in $D(F)$ if and only if RF has a wandering domain in $D(RF)$.*

Proof. Assume that $J \subset D(F)$ is a wandering domain. If $J \subset C$, then RF has a wandering domain by Proposition 7.3. If $J \subset A$, there exists $n \geq 1$ such that $F^n(J) \subset B$ by Proposition 5.16. If $J \subset B$, then $F(J) \subset C$ by Proposition 5.16. Thus, RF has a wandering domain by Proposition 7.3.

The converse follows from the second property of Proposition 7.3. □

Also, we define the rescaling level of a wandering domain in B.

Definition 7.5 (Rescaling level)**.** Assume that $U \subset A_n \cup B_n$ is a connected set that does not intersect any of the stable manifolds. The rescaling level $k(U)$ of U is a nonnegative integer such that $U \subset B_n(k(U))$.

To study the dynamics of a wandering domain, we apply the procedure of renormalization. If a wandering domain is contained in A_0 or B_0, then its orbit will eventually leave A_0 and B_0 and enter C_0. If the orbit of the wandering domain enters C_0, we rescale the orbit element by ϕ_0, ϕ_1, $\cdots$ as many times as possible until it lands on either the set A_n or the set B_n of some renormalization scale n, then study the dynamics of the rescaled orbit by the renormalized map F_n. If the rescaled orbit enters C_n again, then we rescale it and repeat the same procedure. A rescaled orbit is constructed by this procedure.

Definition 7.6 (Closest approach)**.** Assume that $\overline{\varepsilon} > 0$ is sufficiently small and $F \in \mathscr{I}_\delta(I^h \times I^v, \overline{\varepsilon})$. Let $J \subset A \cup B$ be a connected set that is disjoint from the stable manifolds. Define a rescaled orbit of sets $\{J_n\}_{n=0}^\infty$ and the associate renormalization scales $\{r(n)\}_{n=0}^\infty$ by induction such that $J_n \subset A_{r(n)} \cup B_{r(n)}$ for all $n \geq 0$:

1. Set $J_0 = J$ and $r(0) = 0$.
2. Write the rescaling level of J_n as $k_n = k(J_n)$ whenever J_n is defined.
3. If $J_n \subset A_{r(n)}$, set $J_{n+1} = F_{r(n)}(J_n)$ and $r(n+1) = r(n)$.
4. If $J_n \subset B_{r(n)}$, set $J_{n+1} = \Phi_{r(n)}^{k_n} \circ F_{r(n)}(J_n)$ and $r(n+1) = r(n) + k_n$.

The map from one orbit element to the next one is called one step. It is a composition of one iteration together with possibly some rescalings. The sequence $\{J_n\}_{n=0}^\infty$ is called the rescaled iterations of J that closest approaches the tip, or J-closest approach for short.

Remark 7.7. The papers [GvST89, LM11] showed that the orbit of a point x has two types of limiting behavior: the omega limit set $\omega(x)$ is either a periodic orbit or the renormalization Cantor set. The closest approach of a set (or a point) in the basin of the renormalization Cantor set is exactly showing the itinerary of how the orbit approaches to the Cantor set.

The itinerary of a closest approach is summarized by the following diagram.

$$\begin{tikzcd}
A_{r(n)} \arrow[loop right, "F_{r(n)}"] \arrow[d, "F_{r(n)}"] & & A_{r(n+1)} \\
B_{r(n)} \arrow[r, "\Phi_{r(n)}^{k_n} \circ F_{r(n)}"] & A_{r(n+1)} \cup B_{r(n+1)} \arrow[ur] \arrow[r] & B_{r(n+1)}
\end{tikzcd}$$

Example 7.8. In this example, we explain the construction of a closest approach and demonstrate the idea of proving the nonexistence of wandering domains. Let $F = (f - \varepsilon, x)$ be a Hénon-like map such that $f(x) = 1.7996565(1+x)(1-x) - 1$ and $\varepsilon(x,y) = 0.025y$. The map F is numerically checked to be seven times renormalizable. Given a set $J = (-0.950, -0.947) \times (0.042, 0.045) \subset A$. We show that the set is not a wandering domain by contradiction.

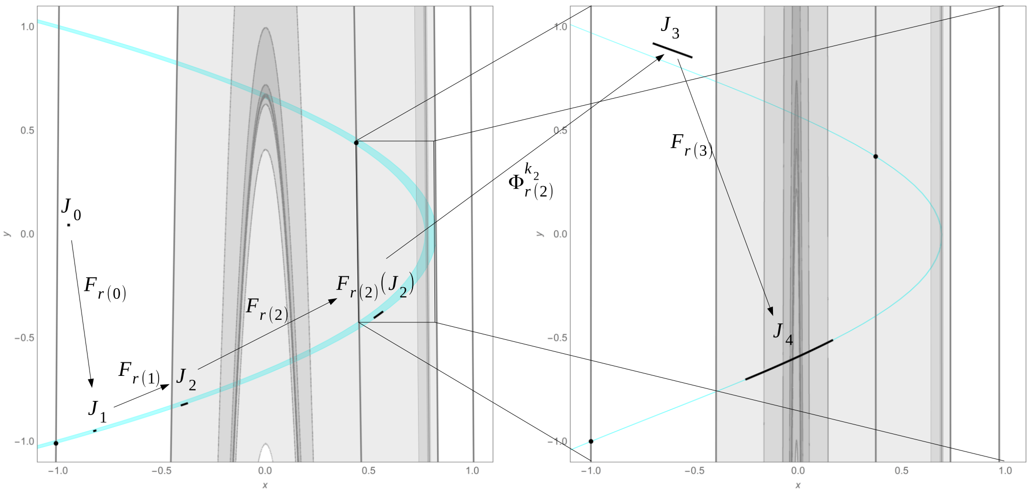


Figure 7.1.: The construction of a closest approach J_n. The graphs are the domains and the partitions of F_0 and F_1 from the left to the right.

If J is a wandering domain, we construct a J-closest approach as shown in Figure 7.1. Set $J_0 = J$ and $r(0) = 0$. The set J_0 is contained in $A_{r(0)}$. The next element is defined to be $J_1 = F_{r(0)}(J_0)$ and $r(1) = r(0) = 0$. The set J_1 is also contained in $A_{r(1)}$. Set $J_2 = F_{r(1)}(J_1)$ and $r(2) = r(1) = 0$. The set J_2 is contained in $B_{r(2)}(1)$. Set $k_2 = 1$, $r(3) = r(2) + k_2 = 1$, and $J_3 = \Phi^{k_2}_{r(2)} \circ F_{r(2)}(J_2) = \phi_0 \circ F_0(J_2)$. The set J_3 is contained in $A_{r(3)}$. Set $J_4 = F_{r(3)}(J_3)$ and $r(4) = r(3) = 1$.

From the figure, the sizes of the elements $\{J_n\}$ grow as the construction continues and the size of $J_4 \subset B_1$ becomes too large that the set intersects some stable manifolds. This leads to a contradiction. Therefore, J cannot be a wandering domain.

Motivated from the example, we study the growth of horizontal size and prove the sizes of the elements approach infinity to obtain a contradiction.

Definition 7.9 (Horizontal and Vertical size)**.** Assume that $J \subset \mathbb{R}^2$. The horizontal size of J is

$$l(J) = \sup\{|x_1 - x_2|\,; (x_1, y_1), (x_2, y_2) \in J\} = |\pi_x J|\,.$$

The vertical size of J is

$$h(J) = \sup\{|y_1 - y_2|\,; (x_1, y_1), (x_2, y_2) \in U\} = \left|\pi_y J\right|.$$

If J is compact, a pair of horizontal endpoints are two points in the set that determines $l(J)$.

Figure 11.1 shows a comparison of the horizontal size and the vertical size of a set J. For a Hénon-like map $F \in \mathscr{H}_\delta(I^h \times I^v)$, it follows from the definition that

$$h(F(J)) = l(J)$$

for all $J \subset I^h \times I^v$.

For simplicity, we start from a closed subset J of a wandering domain such that $\overline{\operatorname{int}(J)} = J$. Then consider the J-closest approach $\{J_n\}_{n\geq 0}$ instead to ensure the horizontal endpoints exist. Note that the sequence element J_n is also a subset of a wandering domain of $F_{r(n)}$. For elements in a closest approach, set $l_n = l(J_n)$ and $h_n = h(J_n)$. Our final goal is to show that the horizontal sizes $\{l_n\}_{n\geq 0}$ approach infinity and hence wandering domains cannot exist.

8. *The Degenerate Case

In this chapter, we present a proof for the nonexistence of wandering intervals for infinitely period-doubling renormalizable unimodal maps. It is known that a unimodal map (under some regularity condition) does not have wandering intervals [Guc79, dMvS88, dMvS89, Lyu89, BL89, MdMvS92]. The readers can refer to the article by Guckenheimer [Guc79] for a classical proof. The proof presented here is different from those. We identify a unimodal map as a degenerate Hénon-like map then use the Hénon-renormalization to prove the nonexistence of wandering intervals. This motivates the proof for the nondegenerate case.

8.1. Local stable manifolds and partition

In this section, we identify a unimodal map as a degenerate Hénon-like map then study the relationships between the two maps.

Let F be a degenerate Hénon-like map

$$F(x,y) = (f(x),x).$$

The super-scripts "u" and "h" are used to distinguish the difference between the notations of unimodal maps and Hénon-like maps to avoid confusion. For example, $p^u(-1) = -1$ and $p^u(0)$ are the fixed points of f; $p^h(-1) = (-1,-1)$ and $p^h(0)$ are the saddle fixed points of F; $A^u, B^u, C^u \subset I$ is the partition defined for f; $A^h, B^h, C^h \subset I^h \times I^v$ is the partition defined for F.

The next lemma relates the local stable manifolds of a degenerate Hénon-like map with the fixed points and their preimages of its unimodal component. Recall that $p^{(1)}$ and $p^{(2)}$ are the points such that $f(p^{(2)}) = p^{(1)}$, $f(p^{(1)}) = p^u(0)$, and $p^{(1)} < p^u(0) < p^{(2)}$ (Definition 4.4); $W^0(-1)$ and $W^2(-1)$ are the local stable manifolds of $p^h(-1)$ (Definition 5.10); $W^0(0), W^1(0), W^2(0)$ are the local stable manifolds of $p^h(0)$ (Definition 5.12).

Lemma 8.1 (Fixed points and their local stable manifolds)**.** *Assume that $F \in \mathscr{H}_\delta(I^h \times I^v)$ is a degenerate Hénon-like map. Then*

1. *$p^h(j) = (p^u(j), p^u(j))$ for $j = -1, 0$,*
2. *the local stable manifold $W^0(j)$ is the vertical line $x = p^u(j)$ for $j = -1, 0$,*
3. *the local stable manifold $W^2(-1)$ is the vertical line $x = \hat{p}^u(-1)$,*
4. *the local stable manifold $W^1(0)$ is the vertical line $x = p^{(1)}$,*
5. *the local stable manifold $W^2(0)$ is the vertical line $x = p^{(2)}$,*
6. *$A^h = A^u \times I^v$, $B^h = B^u \times I^v$, $C^h = C^u \times I^v$, and $D^h = I \times I^v$.*

8.2. Renormalization operator

Next we relate the Hénon-renormaliztion operator with the unimodal-renormalization operator about the critical point. Recall the definitions of the rescaling maps. For a degenerate renormalizable Hénon-like map F, the rescaling map is the composition $\phi = \Lambda \circ H$ where $\Lambda(x,y) = (s^h(x), s^h(y))$, s^h is the affine rescaling map, and $H(x,y) = (f(x),y)$ is the nonlinear rescaling. The renormalization is $RF = \phi \circ F^2 \circ \phi^{-1}$. For a renormalizable unimodal map f, s^u is the affine rescaling and $R_c f = s^u \circ f^2 \circ (s^u)^{-1}$ is the renormalization about the critical point.

Although the Hénon-renormaliation rescales the first return map around the "critical value", the operation acts like the unimodal renormalization that rescales the first return map around the "critical point". This is because of the nonlinear rescaling H for the Hénon-renormalization. Let $\{A^h_0, B^h_0, C^h_0\}$ be the partition of F and D^h_1 be the domain of RF. The rescaling map $\phi(x,y) = (s^h \circ f(x), s^h(y))$ maps C^h_0 to D^h_1. In the x-component, the operation f is a bijection from C^u_0 to B^u_0 and the affine map s^h rescales B^u_0 back to the unit size I. Thus, the two affine maps s^u and s^h are the same and

$$H \circ F^2 \circ H^{-1}(x,y) = (f^2|_{B^u_0}(x), x)$$

is the first return map on B^h_0. Therefore, the two notions of renormalization coincide

$$RF(x,y) = (s^u \circ f^2 \circ (s^u)^{-1}(x), x) = (R_c f(x), x).$$

This also explains why $R^n F$ converges to the fixed point g of R_c but not the fixed point of R_v.

The observation is summarized in the following lemma.

Proposition 8.2 (Renormalization operator)**.** *Assume that $F \in \mathscr{H}_\delta(I^h \times I^v)$ is a degenerate Hénon-like map. Then F is Hénon-renormalizable if and only if f is unimodal-renormalizable. When the map is renormalizable, we have*

1. *$s^h = s^u$ and*
2. *$RF(x,y) = (R_c f(x), x)$.*

In addition, if F is infinitely renormalizable, then the affine term $\Lambda_n : B_n(j) \to B_{n+1}(j-1)$ is a bijection for all $n \geq 0$ and $j \geq 1$ where $B_n(0) \equiv A_n \cup W^2_n(0) \cup C_n$.

From now on, we remove the super-script from s because the maps are the same.

For an infinitely renormalizable Hénon-like map, we also adapt the subscript used for the renormalization scales to the degenerate case. Assume that a degenerate Hénon-like map $F(x,y) = (f(x),x)$ is infinitely renormalizable. Let $F_n = R^n F$ and $f_n = R^n_c f$. Then $F_n(x,y) = (f_n(x),x)$ by the second property of Proposition 8.2.

The next proposition gives an important equality which will be used to prove the nonexistence of wandering intervals for infinitely renormalizable unimodal maps. The expansion estimate is derived from this equality.

Proposition 8.3 (Rescaling trick)**.** *Assume that $f \in \mathscr{I}$. Then*

$$(s_{n+j-1} \circ f_{n+j-1}) \circ \cdots \circ (s_n \circ f_n) \circ f_n = f_{n+j} \circ s_{n+j-1} \circ \cdots \circ s_n$$

for all integers $n \geq 0$ and $j \geq 0$.

Proof. Prove by induction on j. It is clear that the equality holds when $j=0$.
Assume that the equality holds for some j. Then

$$\begin{aligned}
&(s_{n+j}\circ f_{n+j})\circ(s_{n+j-1}\circ f_{n+j-1})\circ\cdots\circ(s_n\circ f_n)\circ f_n\\
=&(s_{n+j}\circ f_{n+j})\circ f_{n+j}\circ s_{n+j-1}\circ\cdots\circ s_n\\
=&(s_{n+j}\circ f_{n+j}\circ f_{n+j}\circ s_{n+j}^{-1})\circ s_{n+j}\circ s_{n+j-1}\circ\cdots\circ s_n\\
=&f_{n+j+1}\circ s_{n+j}\circ s_{n+j-1}\circ\cdots\circ s_n.
\end{aligned}$$

Therefore, the lemma is proved by induction. □

By Proposition 8.3, we get

Corollary 8.4. *Assume that $F\in\mathscr{I}_\delta(I^h\times I^v)$ is a degenerate Hénon-like map. Then*

$$\Phi_n^j\circ F_n=F_{n+j}\circ\Lambda_{n+j-1}\circ\cdots\circ\Lambda_n$$

for all integers $n\geq 0$ and $j\geq 0$.

Proof. By direct computation and the previous proposition

$$\begin{aligned}
\Phi_n^j\circ F_n(x,y) &= \big((s_{n+j-1}\circ f_{n+j-1})\circ\cdots\circ(s_n\circ f_n)\circ f_n(x), s_{n+j-1}\circ\cdots\circ s_n(x)\big)\\
&= \big(f_{n+j}\circ s_{n+j-1}\circ\cdots\circ s_n(x), s_{n+j-1}\circ\cdots\circ s_n(x)\big)\\
&= F_{n+j}\circ\Lambda_{n+j-1}\circ\cdots\circ\Lambda_n(x,y).
\end{aligned}$$

□

8.3. Nonexistence of wandering intervals

In this section, we present a proof for the nonexistence of wandering intervals for infinitely renormalizable unimodal maps by identify a unimodal map as a degenerate Hénon-like map and use the Hénon-renormalization instead of the unimodal-renormalization. A wandering interval is a nonempty interval such that its orbit elements are disjoint and the omega limit set does not contain a periodic point.

Proposition 8.5. *An infinitely renormalizable unimodal map does not have a wandering interval.*

Proof. Prove by contradiction. Assume that f is an infinitely renormalizable unimodal map that has a wandering interval J^u. Without lose of generality, we may assume that the map is close to the fixed point g of the renormalization operator because the sequence of renormalizations $R_c^n f$ converges to g as n approaches infinity. Let $F=(f,x)$. Then F is a degenerate infinitely renormalizable Hénon-like map. Assume that $J^u\subset I$ is a wandering interval of f_0. Let $J^h=J^u\times\{0\}$ and $J_n\subset A_{r(n)}\cup B_{r(n)}$ be the J^h-closest approach. The projection $\pi_x J_n$ is a wandering interval of $f_{r(n)}$ and the horizontal size l_n is the length of the projection. Our goal is to show that horizontal size expands at a definite rate

$$l_{n+1}>El_n \tag{8.1}$$

for some constant $E>1$.

If $J_n \subset A_{r(n)}$, the inequality (8.1) holds because g is expanding on $A(g)$ by Proposition 4.38 and the map $f_{r(n)}$ is close to g.

If $J_n \subset B_{r(n)}(k_n)$, then $J_{n+1} = F_{r(n+1)} \circ \Lambda_{r(n)+k_n-1} \circ \cdots \circ \Lambda_{r(n)}(J_n)$ by Corollary 8.4. Horizontal size expands when the set J_n is mapped under the rescaling maps $\Lambda_{r(n)+k_n-1} \circ \cdots \circ \Lambda_{r(n)}$. Horizontal size also expands when the rescaled set is mapped under $F_{r(n+1)}$. This is because the map $f_{r(n+1)}$ is close to g, g is expanding on $A(g) \cup C(g)$ by Proposition 4.38, and the rescaled set is in $A_{r(n+1)} \cup C_{r(n+1)}$. Thus, the inequality (8.1) also holds.

The expansion estimate (8.1) shows that the horizontal sizes $\{l_n\}_{n \geq 0}$ approach infinity which yields a contradiction. Therefore, wandering intervals cannot exist. □

In the proof, we showed that the horizontal sizes expand at a definite rate. This inspires the proof for the non-degenerate case. In the remain part of the article, we will study the growth rate and contraction rate of the horizontal sizes. We will show in Chapter 10 that the expansion estimate can be promoted to non-degenerate maps under some conditions.

9. The Good Region and the Bad Region

In this chapter, we group the sub-partitions $\{B_n(j)\}_{j=1}^{\infty}$ and $\{C_n(j)\}_{j=1}^{\infty}$ into two regions by the value of the rescaling level j. The two regions are called the good region and the bad region. Then we will study the geometric properties of the two regions.

When j is small, the rescaling level $B_n(j)$ in B is far away from the center of the domain and the rescaling level $C_n(j)$ in C is far away from the tip τ_n. The topological structure of $B_n(j)$ is similar to the structure of a degenerate map. The boundaries of $B_n(j)$ are vertical graphs with a small Lipschitz constant by Proposition 9.15. Also, the map F_n behaves like a unimodal map on $B_n(j)$. In particular, we will show in Chapter 10 that the expansion estimate (8.1) can be promoted to non-degenerate maps: the horizontal sizes of the elements in a closest approach expand at a uniform rate. The region containing the rescaling levels with small values of j is called "the good region".

When j is large, the rescaling level $B_n(j)$ in B is close to the center of the domain and the rescaling level $C_n(j)$ in C is close to the tip τ_n. The topological structure of $B_n(j)$ rescaling different from the structure of the degenerate case. The rescaling level $B_n(j)$ has only one component and looks like an arch-like domain. In fact, the map F_n behaves different from a unimodal map on $B_n(j)$. In particular, the expansion estimate breaks down: a strong contraction applies to the horizontal sizes whenever an element in a closest approach enters the rescaling levels. The area containing these rescaling levels is called "the bad region".

The vertical line argument explains why the expansion estimate breaks down in a rescaling level $B_n(j)$ when j is large. See Figure 9.1 for an illustration. First, draw a vertical line (dashed vertical line in the figure) close to the tip such that the intersection with the image of F_n have only one component. Apply the inverse F_n^{-1} to the intersection. Unlike the case when j is small, the preimage is not a vertical graph but a concave curve close to the center of the domain. Assume that there is a wandering domain J close to the preimage. If the line $\overleftrightarrow{UV}$ connecting horizontal endpoints U and V of J is parallel to the preimage (Figure 9.1a), then the line connecting the iterated horizontal endpoints $F_n(U)$ and $F_n(V)$ is also parallel to the vertical line (Figure 9.1b). This shows that the horizontal size of the iterated set can be as small as possible. Therefore, the expansion estimate breaks down in a rescaling level $B_n(j)$ when j is large.

Motivated from the vertical line argument, we group the rescaling levels in C by how close a rescaling level to the tip is. The size of the image of F_n is $\|\varepsilon_n\|$. To avoid the case of the intersection of a vertical line with the image having only one component, the line has to be $\|\varepsilon_n\|$ away from the tip. This suggests the definition of the good region and the bad region.

Definition 9.1 (The Good Region and The Bad Region)**.** Fix a constant $b > 0$. Assume that $\overline{\varepsilon} > 0$ is small so that Proposition 6.16 holds and $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$. For each $n \geq 0$, define $K_n = K_n(b)$ to be the largest positive integer such that

$$\left|\pi_x z_n^{(0)}(K_n) - \pi_x \tau_n\right| > b\,\|\varepsilon_n\|$$

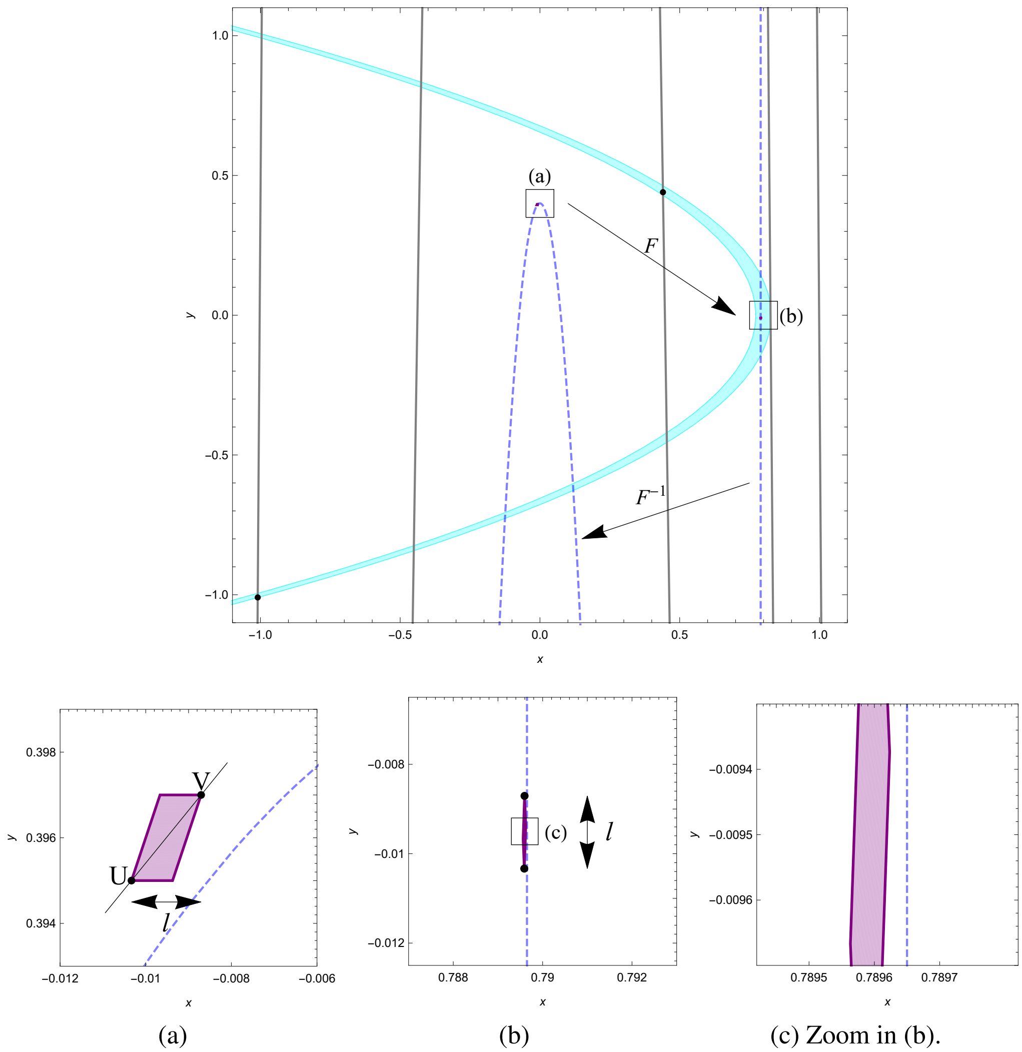


Figure 9.1.: The vertical line argument. The scales of the graphs in (a) and (b) are the same.

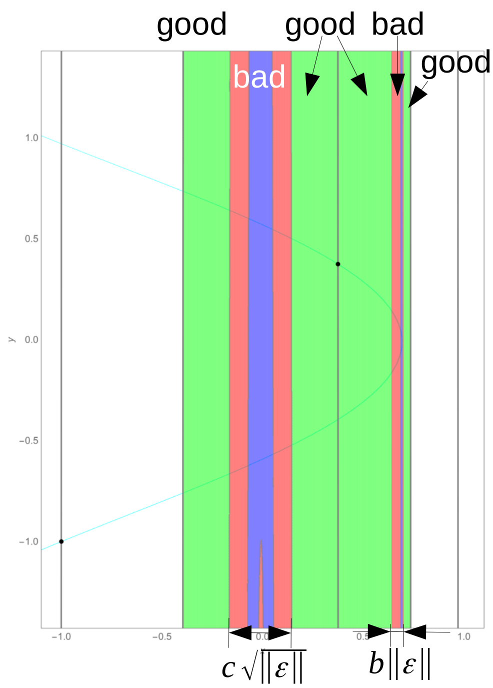


Figure 9.2.: The good region and the bad region. The rescaling levels 1, 2, and below are the shaded area colored from light to dark. In this example, one can see a tiny light area on the center bottom part of the graph. This is because $C^r(2)$ intersects the image $F(D)$. Hence, the boundary is $K = 2$ and the good region contains the lightest part and the bad region contains the two darker parts.

where $z_n^{(0)}(j)$ is the intersection point of $W_n^0(j)$ with the horizontal line through τ_n.

The rescaling level $C_n(j)$ (resp. $B_n(j)$) is in the good region if $j \le K_n$; in the bad region if $j > K_n$. The integer K_n is called the boundary of the good region and the bad region. See Figure 9.2 for an illustration.

Remark 9.2. In the definition, we make the constant b to be flexible and the boundary sequence $\{K_n\}_{n\geq 0}$ depends on the constant b. In the theorems of this chapter, we will show that the properties hold for all b sufficiently large. At the end, we will fix the constant b sufficiently large that makes all theorems work. So the sequence $\{K_n\}_{n\geq 0}$ will be fixed in the remaining part of the article.

Remark 9.3. The bad region is a special feature for the non-degenerate case. For the degenerate case, $\varepsilon_n = 0$ and hence $K_n = \infty$. This means that a degenerate Hénon-like map does not have a bad region.

In the remaining part of this chapter, we will study the geometric properties of the good region and the bad region. The properties are summarized in the next proposition.

Proposition 9.4 (Geometric properties of the good region and the bad region)**.** *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $\overline{b} > 0$, and $c > 1$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $b > \overline{b}$ the following properties hold for all $n \geq 0$:*

The upper and lower bounds of K_n are controlled by

$$\frac{1}{c}\frac{1}{\sqrt{b\,\|\varepsilon_n\|}} \leq \lambda^{K_n} \leq c\frac{1}{\sqrt{b\,\|\varepsilon_n\|}}. \tag{9.1}$$

For the rescaling levels $1 \leq j \leq K_n$ in the good region, we have

1. $C_n^r(j) \cap F_n(D_n) = \phi$,
2. $|\pi_x z - \pi_x \tau_n| > \frac{b}{c}\,\|\varepsilon_n\|$ *for all* $z \in C_n(j) \cap F_n(D_n)$,
3. $|\pi_x z - v_n| > \frac{1}{c}\sqrt{b\,\|\varepsilon_n\|}$ *for all* $z \in B_n(j)$,
4. $\frac{1}{c}\left(\frac{1}{\lambda}\right)^{2j} < |\pi_x z - \pi_x \tau_n| < c\left(\frac{1}{\lambda}\right)^{2j}$ *for all* $z \in C_n(j) \cap F_n(D_n)$, *and*
5. $\frac{1}{c}\left(\frac{1}{\lambda}\right)^{j} < |\pi_x z - v_n| < c\left(\frac{1}{\lambda}\right)^{j}$ *for all* $z \in B_n(j)$.

For the rescaling levels $j > K_n$ in the bad region, we have

1. $|\pi_x z - \pi_x \tau_n| < cb\,\|\varepsilon_n\|$ *for all* $z \in C_n(j) \cap F_n(D_n)$ *and*
2. $|\pi_x z - v_n| < c\sqrt{b\,\|\varepsilon_n\|}$ *for all* $z \in B_n(j)$.

The properties in this proposition will be proved by the lemmas in this chapter.
First, we estimate the bounds for the boundary K_n.

Lemma 9.5. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $c > 1$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $b > 0$, we have*

$$\frac{1}{c}\frac{1}{\sqrt{b\,\|\varepsilon_n\|}} \leq \lambda^{K_n} \leq c\frac{1}{\sqrt{b\,\|\varepsilon_n\|}}$$

for all $n \geq 0$.

Proof. To prove the lower bound, we apply Proposition 6.16 to the definition of K_n. Assume that $\overline{\varepsilon} > 0$ is sufficiently small. We have

$$c\left(\frac{1}{\lambda}\right)^{2(K_n+1)} \leq \left|z_n^{(0)}(K_n+1) - \tau_n\right| \leq b\,\|\varepsilon_n\|.$$

for some constant $c > 1$. Thus,

$$\lambda^{K_n} \geq \sqrt{\frac{c}{b\lambda^2}}\frac{1}{\sqrt{\|\varepsilon_n\|}}.$$

The proof for the upper bound is similar. □

9.1. Properties of the good region

To prove the properties, the strategy is to first estimate the distance from the local stable manifolds to the tip τ_n. Since the rescaling level $C_n(j)$ is the union of two components bounded between the local stable manifolds $W_n^t(j-1)$ and $W_n^t(j)$, the location of a point in the level can be estimated by using the local stable manifolds. After proving the properties of the levels in C, the properties of the levels in B holds because the Hénon-like map behaves like a quadratic map near the center of the domain (Lemma 6.21 and Proposition 6.27).

First, we estimate the x-coordinate of the points on $W_n^t(j)$.

Lemma 9.6. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $\overline{b} > 0$, and $c > 1$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $b > \overline{b}$, we have*

$$\frac{1}{c}\left(\frac{1}{\lambda}\right)^{2j} \leq |\pi_x z - \pi_x \tau_n| \leq c\left(\frac{1}{\lambda}\right)^{2j}$$

for all $z \in W_n^t(j) \cap \left(I^h \times I^h\right)$ with $t \in \{0,2\}$, $0 \leq j \leq K_n$, and $n \geq 0$.

Proof. Assume that $\overline{\varepsilon} > 0$ is sufficiently small. We only prove the lower bound of the estimate for the case $t = 0$. The upper bound and the other case $t = 2$ are similar.

To prove the lower bound, we apply Proposition 6.16. Let $z \in W_n^0(j) \cap \left(I^h \times I^h\right)$. Then

$$\begin{aligned}
|\pi_x z - \pi_x \tau_n| &\geq \left|z_n^{(0)}(j) - \tau_n\right| - \left|\pi_x z - \pi_x z_n^{(0)}(j)\right| \\
&\geq \frac{1}{c}\left(\frac{1}{\lambda}\right)^{2j} - c\,\|\varepsilon_n\|\left|I^h\right| \\
&\geq \left(\frac{1}{c} - c\left|I^h\right|\|\varepsilon_n\|\,\lambda^{2K_n}\right)\left(\frac{1}{\lambda}\right)^{2j}
\end{aligned}$$

for some constant $c > 1$. By Lemma 9.5, there exists $c' > 1$ such that

$$\begin{aligned}
|\pi_x z - \pi_x \tau_n| &\geq \left(\frac{1}{c} - \frac{cc'^2\left|I^h\right|}{b}\right)\left(\frac{1}{\lambda}\right)^{2j} \\
&\geq \frac{1}{2c}\left(\frac{1}{\lambda}\right)^{2j}
\end{aligned}$$

whenever $b \geq 2c^2 c'^2 \left|I^h\right|$. □

We prove the first property of the good region.

Lemma 9.7. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $\overline{b} > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $b > \overline{b}$, we have*

$$C_n^r(j) \cap F_n(D_n) = \phi$$

for all $1 \leq j \leq K_n$ and $n \geq 0$.

Proof. Since the right component of the good region $\overline{\cup_{j=1}^{K_n} C_n^r(j)}$ is the set bounded between the local manifolds $W_n^2(K_n)$ and $W_n^2(0)$, it suffices to show that the local stable manifold $W_n^2(K_n)$ is far away form the image. We have

$$f_n(v_n) - \|\varepsilon_n\| \le \sup_{z' \in D_n} h_n(z') = \sup_{z' \in D_n} \left(f_n(\pi_x z') + \varepsilon_n(z')\right) \le f_n(v_n) + \|\varepsilon_n\|.$$

By Proposition 6.27, Lemma 9.5, and Lemma 9.6, there exist constants $c > 0$ and $a > 1$ such that

$$\begin{aligned}
\pi_x z - \sup_{z' \in D_n} h_n(z') &\ge (\pi_x z - \pi_x \tau_n) - |\pi_x \tau_n - f_n(v_n)| - |f_n(v_n) - \sup_{z' \in D_n} h_n(z')| \\
&\ge \frac{1}{a}\left(\frac{1}{\lambda}\right)^{2K_n} - c\,\|\varepsilon_n\| - \|\varepsilon_n\| \\
&\ge \left(\frac{b}{a^3} - c - 1\right)\|\varepsilon_n\|
\end{aligned}$$

for all $z \in W_n^2(K_n) \cap \left(I^h \times I^h\right)$. The coefficient on the right hand side is positive when $b > 0$ is large enough. Consequently, $C_n^r(j) \cap F_n(D_n) = \phi$ for all $1 \le j \le K_n$. □

By the lemma, it is enough to only consider the left component of the rescaling levels $C_n^l(j)$. The second property shows that the good region is $\|\varepsilon_n\|$ away from the tip.

Lemma 9.8. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $\overline{b} > 0$, and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $b > \overline{b}$, we have*

$$|\pi_x z - \pi_x \tau_n| > cb\,\|\varepsilon_n\|$$

for all $z \in C_n(j) \cap F_n(D_n)$ with $1 \le j \le K_n$ and $n \ge 0$.

Proof. The left component of the good region in C is the set bounded between the local stable manifolds $W_n^0(0)$ and $W_n^0(K_n)$. Thus, the estimate follows from Lemma 9.5 and Lemma 9.6. □

The third property is an analog of the lemma in B.

Corollary 9.9. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $\overline{b} > 0$, and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $b > \overline{b}$ so that the following property hold for all $n \ge 0$:*

If $z \in I^B \times I_n^v$ satisfies $|h_n(z) - \pi_x \tau_n| \ge cb\,\|\varepsilon_n\|$, then

$$|\pi_x z - v_n| \ge c\sqrt{b\,\|\varepsilon_n\|}. \tag{9.2}$$

In particular, (9.2) holds for all $z \in B_n(j)$ with $1 \le j \le K_n$.

Proof. Assume that $z \in I^B \times I_n^v$ such that $|h_n(z) - \pi_x \tau_n| > cb\,\|\varepsilon_n\|$. By Proposition 6.27, we get

$$\begin{aligned}
|f_n(\pi_x z) - f_n(v_n)| &\ge |h_n(z) - \pi_x \tau_n| - |f_n(\pi_x z) - h_n(z)| - |\pi_x \tau_n - f_n(v_n)| \\
&\ge (cb - 1 - c')\,\|\varepsilon_n\| \\
&> \frac{cb}{2}\,\|\varepsilon_n\|
\end{aligned}$$

for some $c' > 0$ when $b > 2(1+c')/c$.

Moreover, by Lemma 6.21, there exists a constant $a > 1$ such that

$$|f_n(\pi_x z) - f_n(v_n)| \le \frac{a}{2}(\pi_x z - v_n)^2$$

for all $n \ge 0$ when $\overline{\varepsilon} > 0$ is small enough. Therefore,

$$|\pi_x z - v_n| \ge \sqrt{\frac{c}{a}}\sqrt{b\,\|\varepsilon_n\|}.$$

□

The next property gives an estimate of the distance from the rescaling level $C_n(j)$ to the tip τ_n.

Lemma 9.10. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $\overline{b} > 0$, and $c > 1$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $b > \overline{b}$, we have*

$$\frac{1}{c}\left(\frac{1}{\lambda}\right)^{2j} < |\pi_x z - \pi_x \tau_n| < c\left(\frac{1}{\lambda}\right)^{2j}$$

for all $z \in C_n(j) \cap F_n(D_n)$ with $1 \le j \le K_n$ and $n \ge 0$.

Proof. For all $z \in C_n^l(j) \cap F_n(D_n)$ with $1 \le j \le K_n$, there exist $z_1 \in W_n^0(j-1) \cap \left(I^h \times I^h\right)$ and $z_2 \in W_n^0(j) \cap \left(I^h \times I^h\right)$ such that $\pi_y z = \pi_y z_1 = \pi_y z_2$ because the local stable manifolds are vertical graphs. By Lemma 9.6, we obtain

$$\frac{1}{c}\left(\frac{1}{\lambda}\right)^{2j} \le |\pi_x z_2 - \pi_x \tau_n| \le |\pi_x z - \pi_x \tau_n| \le |\pi_x z_1 - \pi_x \tau_n| \le c\lambda^2\left(\frac{1}{\lambda}\right)^{2j}.$$

This proves the corollary because $C_n^l(j)$ is the component bounded between $W_n^0(j-1)$ and $W_n^0(j)$.

□

One can deduce an analog of the lemma for the rescaling levels in B. The proof is similar to Corollary 9.9. The details are left to the reader.

Corollary 9.11. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $\overline{b} > 0$, and $c > 1$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $b > \overline{b}$, we have*

$$\frac{1}{c}\left(\frac{1}{\lambda}\right)^{j} < |\pi_x z - v_n| < c\left(\frac{1}{\lambda}\right)^{j}$$

for all $z \in B_n(j)$ with $1 \le j \le K_n$ and $n \ge 0$.

9.2. Properties of the bad region

We prove the first property of the bad region by applying Lemma 9.6 to the boundary local stable manifolds $W_n^0(K_n)$ and $W_n^2(K_n)$ of the bad region.

Lemma 9.12. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $\overline{b} > 0$, and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $b > \overline{b}$, we have*

$$|\pi_x z - \pi_x \tau_n| < cb\,\|\varepsilon_n\|$$

for all $z \in C_n(j) \cap F_n(D_n)$ with $j > K_n$ and $n \geq 0$.

Proof. Assume that $z \in W_n^t(K_n) \cap \left(I^h \times I^h\right)$ with $t \in \{0,2\}$. By Lemma 9.5 and Lemma 9.6, there exists $c > 1$ such that

$$|\pi_x z - \pi_x \tau_n| \leq c\left(\frac{1}{\lambda}\right)^{2K_n} \leq c^3 b\,\|\varepsilon_n\|$$

for all $b > 0$ sufficiently large. Therefore, the estimate holds because the bad region is bounded by the local stable manifolds $W_n^0(K_n)$ and $W_n^2(K_n)$. □

The second property of the bad region follows from Lemma 6.21.

Corollary 9.13. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $\overline{b} > 0$, and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $b > \overline{b}$, we have*

$$|\pi_x z - v_n| < c\sqrt{b\,\|\varepsilon_n\|}$$

for all $z \in B_n(j)$ with $j > K_n$ and $n \geq 0$.

Proof. Assume that $z \in B_n(j)$. Then $F_n(z) \in C_n(j) \cap F_n(D_n)$. By Proposition 6.27 and Lemma 9.12, there exists $c > 0$ such that

$$\begin{aligned}
|f_n(\pi_x z) - f_n(v_n)| &\leq |h_n(z) - \pi_x \tau_n| + |f_n(\pi_x z) - h_n(z)| + |\pi_x \tau_n - f_n(v_n)| \\
&\leq (cb + 1 + c)\,\|\varepsilon_n\| \\
&< 2cb\,\|\varepsilon_n\|
\end{aligned}$$

for all $b > 0$ sufficiently large. Also, by Lemma 6.21, we have

$$|f_n(\pi_x z) - f_n(v_n)| \geq \frac{1}{2a}(\pi_x z - v_n)^2$$

for some constant $a > 0$. Combine the two inequalities, we obtain

$$|\pi_x z - v_n| \leq \sqrt{4ac}\sqrt{b\,\|\varepsilon_n\|}.$$

□

9.3. *Local stable manifolds in B_n

In this section, we show in the good region $j \leq K_n$ the rescaling level $B_n(j) \subset B_n$ has two components, the left component $B_n^l(j)$ and the right component $B_n^r(j)$. Each component is bounded by two local stable manifolds that are vertical graphs.

Lemma 9.14. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $\overline{b} > 0$, and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $b > \overline{b}$, the following properties hold for all $n \geq 0$:*

There exist closed intervals $I_n^{Bl} = I_n^{Bl}(b, F)$, $I_n^{Br} = I_n^{Br}(b, F)$, and $I_n^{Cg} = I_n^{Cg}(b, F)$ such that

1. *$W_n^0(j) \cap \left(I^h \times I^h\right) \subset I_n^{Cg} \times I^h$ for all $1 \leq j \leq K_n$,*
2. *$F_n^{-1}(W_n^0(j)) \subset (I_n^{Bl} \cup I_n^{Br}) \times I_n^v$,*
3. *$h_n(I_n^{Bl}, y) \supset I_n^{Cg}$, $h_n(I_n^{Br}, y) \supset I_n^{Cg}$, and*
4. *$\left|\frac{\partial h_n}{\partial x}(x,y)\right| \geq c\sqrt{\|\varepsilon_n\|}$ for all $(x,y) \in (I_n^{Bl} \cup I_n^{Br}) \times I_n^v$.*

Proof. Let $c > 0$ be the constant defined by Corollary 9.9, $I_n^{Cg} = [\pi_x p_n(0), \pi_x \tau_n - b\|\varepsilon_n\|]$, $I_n^{Bl} = I^B \cap [-1, v_n - c\sqrt{b\|\varepsilon_n\|}]$, $I_n^{Br} = I^B \cap [v_n + c\sqrt{b\|\varepsilon_n\|}, 1]$. Also, let $\overline{\varepsilon} > 0$ be small enough such that $W_n^0(1) \cap \left(I^h \times I^h\right) \subset I_n^{Cg} \times I^h$ for all $n \geq 0$. Then the first property holds.

The second property follows from Corollary 9.9.

The third property follows from Corollary 9.9 and the definitions of I_n^{Cg}, I_n^{Bl}, and I_n^{Br}.

To prove the last property, let $(x,y) \in I_n^{Bl} \times I_n^v$. Then

$$\begin{aligned}
\left|\frac{\partial h_n}{\partial x}(x,y)\right| &\geq \left|f_n'(x)\right| - \|\varepsilon_n\| \\
&\geq \frac{1}{a}|x - v_n| - \|\varepsilon_n\| \\
&\geq \frac{c}{a}\sqrt{b\|\varepsilon_n\|} - \|\varepsilon_n\| \\
&\geq \frac{c}{2a}\sqrt{b\|\varepsilon_n\|}
\end{aligned}$$

by Lemma 6.21 and the definition of I^{Bl}. Here we assume that $\overline{\varepsilon}$ is also small enough such that $\sqrt{\|\varepsilon_n\|} < \frac{c}{2a}$ for all $n \geq 0$. The other case $I_n^{Br} \times I_n^v$ is similar. □

Proposition 9.15. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $\overline{b} > 0$, and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $b > \overline{b}$ the following properties hold for all $n \geq 0$:*

For all $1 \leq j \leq K_n$, the set $F_n^{-1}(W_n^0(j))$ is the union of two components, left component $W_n^l(j)$ and right component $W_n^r(j)$. Both components are vertical graphs of $c\sqrt{\|\varepsilon_n\|}$-Lipschitz functions in B_n. For $1 \leq j \leq K_n$, let $B_n^l(j)$ be the set bounded by $W_n^l(j-1)$ and $W_n^l(j)$ in $I^h \times I_n^v$; and $B_n^r(j)$ be the set bounded by $W_n^r(j)$ and $W_n^r(j-1)$ in $I^h \times I_n^v$. Then $B_n(j) = B_n^l(j) \cup B_n^r(j)$ for all $1 \leq j \leq K_n$.

Proof. By Proposition 6.16, the local stable manifold $W_n^0(j)$ is the vertical graph of a $c\|\varepsilon_n\|$-Lipschitz function for some constant $c > 0$. Let $c' > 0$ be the constant in Lemma 9.14. Then

$$\frac{1}{\delta\left(c'\sqrt{b\|\varepsilon_n\|} - c\|\varepsilon_n\|\right)}\|\varepsilon_n\| \leq \frac{2}{\delta c'}\sqrt{\|\varepsilon_n\|}$$

for all $n \geq 0$. Here we assume that $\overline{\varepsilon}$ is small enough such that $c\sqrt{\|\varepsilon_n\|} < \frac{c'}{2}$ for all $n \geq 0$. By Lemma 5.19 and Lemma 9.14, $W_n^l(j) = F_n^{-1}(W_n^0(j)) \cap I_n^{Bl} \times I_n^v$ and $W_n^r(j) = F_n^{-1}(W_n^0(j)) \cap I_n^{Br} \times I_n^v$

are vertical graphs of $\frac{2}{\delta c'}\sqrt{\|\varepsilon_n\|}$-Lipschitz functions for all $0 \leq j \leq K_n$. By the second property in Lemma 9.14, $F_n^{-1}(W_n^0(j)) = W_n^l(j) \cup W_n^r(j)$.

The property for $B_n(j)$ follows from definition. $\square$

10. The Good Region and the Expansion Estimate

In this chapter, we generalize the expansion estimate (8.1) from unimodal maps to Hénon-like maps: the horizontal sizes expand at a definite rate when the elements in a closest approach stay in the good regions (Proposition 10.11). From now on, we fix a constant $b > 0$ sufficiently large so that Proposition 9.4 holds and the boundaries of the good regions and the bad regions $\{K_n\}_{n\geq 0}$ depends only on F.

To prove horizontal size expands, we impose a technical condition "regular" to the elements in a closest approach. In terms of notations from the vertical line argument (Figure 9.1), this condition ensures that the line connecting a pair of horizontal endpoints is far from being parallel to the preimage of a vertical line.

Definition 10.1 (Regular)**.** Let $R > 0$. A set $U \subset D(F)$ is R-regular if

$$\frac{h(U)}{l(U)} \leq R \frac{1}{\|\varepsilon\|^{1/4}}. \tag{10.1}$$

To see R-regular implies not parallel, we estimate the slope of the preimage of a vertical line. Assume that $\gamma : I^v \to I^h$ is the vertical graph of the preimage of some vertical line $x = x_0$ by the Hénon-like map F_n and the vertical graph is in the good region. Then

$$h_n(\gamma(y), y) = x_0.$$

Apply the derivative in terms of y to the both sides, we solved

$$\gamma'(y) = \frac{\frac{\partial \varepsilon_n}{\partial y}(\gamma(y), y)}{f_n'(\gamma(y)) - \frac{\partial \varepsilon_n}{\partial x}(\gamma(y), y)}.$$

By Lemma 6.21 and Proposition 9.4, we get

$$\begin{aligned} \left| f_n'(y) - \frac{\partial \varepsilon_n}{\partial x}(\gamma(y), y) \right| &\geq \frac{1}{a} |\gamma(y) - v_n| - \frac{1}{\delta} \|\varepsilon_n\| \\ &\geq \frac{c}{a} \sqrt{\|\varepsilon_n\|} - \frac{1}{\delta} \|\varepsilon_n\| \\ &\geq \frac{c}{2a} \sqrt{\|\varepsilon_n\|} \end{aligned}$$

when $\overline{\varepsilon}$ is small enough. This yields

$$|\gamma'(y)| \leq c' \sqrt{\|\varepsilon_n\|} \tag{10.2}$$

for some constant $c' > 0$.

The condition R-regular says that the vertical slope of the line determined by the horizontal endpoints (x_1, y_1) and (x_2, y_2) of J is bounded by

$$\frac{|x_2 - x_1|}{|y_2 - y_1|} \geq \frac{l(J)}{h(J)} \geq \frac{1}{R} \|\varepsilon_n\|^{1/4}. \tag{10.3}$$

From (10.2) and (10.3), we get

$$\frac{|x_2 - x_1|}{|y_2 - y_1|} \gg |\gamma'(y)|.$$

This concludes that the line connecting the horizontal endpoints is not parallel to the preimage of a vertical line if the wandering domain is R-regular.

Assume that $J \subset A_n \cup B_n$ is a wandering domain in the good region. We will prove the expansion estimate, Proposition 10.11, in three different cases:

1. the case when the element J is in A_n (Section 10.1),
2. the case when the element J is in $B_n(j)$ with $1 \leq j < \overline{K}$ for some positive integer $\overline{K}$ (Section 10.3), and
3. the case when the element J is in $B_n(j)$ with $\overline{K} \leq j \leq K_n$ (Section 10.2).

The proof is technical but the result is not surprising when an element is far away from the bad region (case 1 and 2) because the Hénon-like map F inherits the properties from g when F is close to g. In short, the proof is just showing that the expansion estimate for g holds on a neighborhood of the partition elements of g, the partition elements of a Hénon-like map F are close to the partition elements of g, and the expansion estimate survives under a small perturbation. On the other hand, the expansion estimate breaks down in the bad region. The intermediate region (case 3) turns out to be the nontrivial part of the proof because the rescaling trick (Proposition 8.3) does not apply to the Hénon case. The proof depends on the properties of the good region and relies on the estimation of contraction and expansion of the iteration and rescaling. In other words, the good region is defined to be the condition that makes the expansion estimate work in the intermediate region.

10.1. Case $J_n \subset A_{r(n)}$

In this section, we estimate the expansion rate of horizontal size when a wandering domain J_n is in $A_{r(n)}$. We show that the expansion estimate for g (Proposition 4.38) also applies to Hénon-like maps when F_n is close to the degenerate map G.

Lemma 10.2. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. For all $R > 0$, there exist $\overline{\varepsilon} = \overline{\varepsilon}(R) > 0$ and $E > 1$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following properties hold for all $n \geq 0$:*

Assume that $J \subset A_n$ is a closed R-regular set. Then its iterate $J' = F_n(J)$ is an R-regular set in $A_n \cup W_n^1(0) \cup B_n$ and

$$l(J') \geq E\, l(J).$$

Proof. Let $E > 1$ be the constant defined in Lemma 6.22 and $\Delta E > 0$ be small enough such that $E' \equiv E - \Delta E > 1$. Assume that $\overline{\varepsilon} > 0$ is small enough such that Lemma 6.22 holds and

$$\frac{R}{\delta} \|\varepsilon_n\|^{3/4} < \Delta E \tag{10.4}$$

for all $n \geq 0$.

To prove the inequality, let $(x_1, y_1), (x_2, y_2) \in J$ such that $x_2 - x_1 = l(J)$. Then $h(J) \geq |y_2 - y_1|$. Compute

$$\begin{aligned} l(J') &\geq |\pi_x [F_n(x_2, y_2) - F_n(x_1, y_1)]| \\ &\geq |\pi_x [F_n(x_2, y_2) - F_n(x_1, y_2)]| - |\pi_x [F_n(x_1, y_2) - F_n(x_1, y_1)]| . \end{aligned}$$

By the mean value theorem, there exist $\xi \in (x_1, x_2)$ and $\eta \in (y_1, y_2)$ such that

$$\pi_x [F_n(x_2, y_2) - F_n(x_1, y_2)] = \frac{\partial h_n}{\partial x}(\xi, y_2)(x_2 - x_1)$$

and

$$\pi_x [F_n(x_1, y_2) - F_n(x_1, y_1)] = \frac{\partial \varepsilon_n}{\partial y}(x_1, \eta)(y_2 - y_1).$$

Since $(x_1, y_1), (x_2, y_2) \in A_n \subset I^{AC} \times I_n^v$, we have $(\xi, y_2) \in I^{AC} \times I_n^v$. By Lemma 5.31 and Lemma 6.22, we get

$$\begin{aligned} l(J') &\geq E l(J) - \frac{1}{\delta} \|\varepsilon_n\| h(J) \\ &= \left(E - \frac{1}{\delta} \|\varepsilon_n\| \frac{h(J)}{l(J)} \right) l(J). \end{aligned}$$

Also, by J is R-regular and (10.4), this yields

$$\begin{aligned} l(J') &\geq \left(E - \frac{R}{\delta} \|\varepsilon_n\|^{3/4} \right) l(J) \\ &\geq E' l(J). \end{aligned} \tag{10.5}$$

To prove that J' is R-regular, we apply (10.5) and $h(J') = l(J)$. We get

$$\frac{h(J')}{l(J')} \leq \frac{1}{E'}.$$

Also assume that $\overline{\varepsilon}$ is small enough such that $\frac{1}{E'} \leq R \|\varepsilon_n\|^{-1/4}$ for all $n \geq 0$. This proves that J' is R-regular. □

10.2. Case $J_n \subset B_{r(n)}(k_n)$, $\overline{K} \leq k_n \leq K_{r(n)}$

In this section, we prove that horizontal size expands when a wandering domain $J_n \subset B_{r(n)}(k)$ is iterated then rescaled in the intermediate region $\overline{K} \leq k \leq K_{r(n)}$ (Lemma 10.3). We first show that the amount of contraction is well controlled when the set J_n is iterated by $F_{r(n)}$ in the good region (Lemma 10.4). Then we estimate the size of expansion when the iterated set $F_{r(n)}(J_n)$ is rescaled by $\Phi_{r(n)}^{k_n}$ (Lemma 10.5). Finally, we show that the expansion is larger than the contraction:

Lemma 10.3. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. For all $R > 0$, there exist constants $\overline{\varepsilon} = \overline{\varepsilon}(R) > 0$, $E > 1$, $R' > 0$, and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following properties hold for all $n \geq 0$:*

Assume that $J \subset B_n(k)$ is a closed R-regular set and $1 \leq k \leq K_n$, then the set $J' = \Phi_n^k \circ F_n(J)$ is an R'-regular set in $C_{n+k}(0) = A_{n+k} \cup W^1_{n+k}(0) \cup B_{n+k}$ and

$$l(J') \geq cE^k l(J). \tag{10.6}$$

The constants E, R', and c do not depend on R.

After this section, we fix $\overline{K} > 0$ to be a large integer so that the lemma produces a definite expansion for all $k \geq \overline{K}$. We will also set $R = R'$ to make the property regular to be invariant under the construction of a closest approach.

First, we estimate the amount of contraction when a wandering domain is iterated.

Lemma 10.4. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. For all $R > 0$, there exist constants $\overline{\varepsilon} = \overline{\varepsilon}(R) > 0$, $c > 1$, and $R' > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following properties hold for all $n \geq 0$:*

Assume that $J \subset B_n(k)$ is a closed R-regular set and $k \leq K_n$. Let $J' = F_n(J)$. Then

$$l(J') \quad \geq \quad c\lambda^{-k} l(J) \tag{10.7}$$

and

$$\frac{h(J')}{l(J')} \leq R' \frac{1}{\sqrt{\|\varepsilon_n\|}}.$$

The constants c and R' do not depend on R.

Proof. Let $\overline{\varepsilon} > 0$ be a small number such that Lemma 5.31, Lemma 6.21, and Proposition 9.4 hold. Let $(x_1, y_1), (x_2, y_2) \in J$ be a pair of horizontal endpoints and $x \in \{x_1, x_2\}$ be such that $|x - v_n| = \min_{i=1,2} |x_i - v_n|$. By the triangular inequality, we have

$$\begin{aligned} l(J') &\geq |\pi_x (F_n(x_2, y_2) - F_n(x_1, y_1))| \\ &\geq |\pi_x (F_n(x_2, y_2) - F_n(x_1, y_2))| - |\pi_x (F_n(x_1, y_2) - F_n(x_1, y_1))| \end{aligned} \tag{10.8}$$

Apply the mean value theorem, there exist $\xi \in (x_1, x_2)$ and $\eta \in (y_1, y_2)$ such that

$$\pi_x (F_n(x_2, y_2) - F_n(x_1, y_2)) = \left[f_n'(\xi) - \frac{\partial \varepsilon_n}{\partial x}(\xi, y_2) \right] (x_2 - x_1) \tag{10.9}$$

and

$$\pi_x (F_n(x_1, y_2) - F_n(x_1, y_1)) = -\frac{\partial \varepsilon_n}{\partial y}(x_1, \eta)(y_2 - y_1). \tag{10.10}$$

Then $\xi \in I^B$ since $(x_1, y_1), (x_2, y_2) \in B_n \subset I^B \times I^v_n$. By Lemma 6.21, (10.9) yields

$$\begin{aligned} |\pi_x (F_n(x_2, y_2) - F_n(x_1, y_2))| &\geq \left(|f_n'(\xi)| - \left| \frac{\partial \varepsilon_n}{\partial x}(\xi, y_2) \right| \right) l(J) \\ &\geq \left(\frac{1}{a} |x - v_n| - \frac{1}{\delta} \|\varepsilon_n\| \right) l(J). \end{aligned} \tag{10.11}$$

Also, since J is R-regular, (10.10) yields

$$|\pi_x(F_n(x_1,y_2)-F_n(x_1,y_1))| \leq \frac{1}{\delta}\|\varepsilon_n\|\,h(J) \leq \frac{R}{\delta}\|\varepsilon_n\|^{3/4}\,l(J). \tag{10.12}$$

Combine (10.8), (10.11), and (10.12), we get

$$l(J') \geq \left(\frac{1}{a}|x-v_n| - \frac{1}{\delta}\left(\|\varepsilon_n\|^{1/2}+R\|\varepsilon_n\|^{1/4}\right)\sqrt{\|\varepsilon_n\|}\right) l(J)$$

for some constant $a>1$. By Proposition 9.4 (3rd property of the good region), we have

$$c\,|x-v_n| > \sqrt{\|\varepsilon_n\|} \tag{10.13}$$

for some constant $c>1$. Also, assume that $\overline{\varepsilon}>0$ is small enough such that $\frac{c}{\delta}\left(\|\varepsilon_n\|^{1/2}+R\|\varepsilon_n\|^{1/4}\right) < \frac{1}{2a}$ for all $n \geq 0$. We obtain

$$l(J') \geq \frac{1}{2a}|x-v_n|\,l(J) \tag{10.14}$$

when $\overline{\varepsilon}>0$ is small. Then (10.7) follows from Proposition 9.4 (5th property of the good region).

Moreover, by (10.13), (10.14), and $h(J')=l(J)$, we get

$$\frac{h(J')}{l(J')} \leq \frac{2a}{|x-v_n|} \leq R'\frac{1}{\sqrt{\|\varepsilon_n\|}}$$

where $R'=2ac$. □

Then we estimate the size of expansion when a wandering domain is rescaled.

Lemma 10.5. *Given $\delta>0$ and $I^v \supset I^h \Supset I$. For all $R>0$, there exist constants $\overline{\varepsilon}=\overline{\varepsilon}(R)>0$ and $E>1$ such that for all $F\in\hat{\mathscr{I}}_\delta(I^h\times I^v,\overline{\varepsilon})$, the following properties hold for all $n\geq 0$:*

Assume that $J\subset C_n(k)$ is a closed set and $\frac{h(J)}{l(J)} \leq R\frac{1}{\sqrt{\|\varepsilon_n\|}}$, then

$$l(\Phi_n^j(J)) \;\; \geq \;\; (\lambda E)^j\,l(J) \tag{10.15}$$

and

$$\frac{h(\Phi_n^j(J))}{l(\Phi_n^j(J))} \leq R\frac{1}{\sqrt{\|\varepsilon_n\|}} \tag{10.16}$$

for all integer j with $0\leq j\leq k$. The constant E does not depend on R.

Proof. Let $E'>1$ be the expansion factor defined in Lemma 6.22 and $E=\sqrt{E'}$. We prove the lemma by induction on j. The statement is trivial for the case $j=0$.

Assume that the lemma is true for some integer $j\leq k$. We show the lemma also holds for $j+1\leq k$. Let $(x_1,y_1),(x_2,y_2)\in\Phi_n^j(J)$ be a pair of horizontal endpoints. By the mean value theorem, there exist $\xi_j\in(x_1,x_2)\subset I^{AC}$ and $\eta_j\in(y_1,y_2)$ such that

$$\pi_x\left(\phi_{n+j}(x_2,y_2)-\phi_{n+j}(x_1,y_2)\right) = -\lambda_{n+j}\frac{\partial h_n}{\partial x}(\xi_j,y_2)\,(x_2-x_1) \tag{10.17}$$

and

$$\pi_x\left(\phi_{n+j}(x_1,y_2)-\phi_{n+j}(x_1,y_1)\right)=\lambda_{n+j}\frac{\partial \varepsilon_{n+j}}{\partial y}(x_1,\eta_j)\,(y_2-y_1)\,. \tag{10.18}$$

Apply Lemma 6.22 to (10.17), we have

$$\left|\pi_x\left(\phi_{n+j}(x_2,y_2)-\phi_{n+j}(x_1,y_1)\right)\right| \;\geq\; \lambda_{n+j}E' l(\Phi_n^j(J)) \tag{10.19}$$

for some constant $E'>1$. Also apply the induction hypothesis to (10.18), we have

$$\begin{aligned}\left|\pi_x\left(\phi_{n+j}(x_1,y_2)-\phi_{n+j}(x_1,y_1)\right)\right| &\leq \frac{\lambda_{n+j}}{\delta}\left\|\varepsilon_{n+j}\right\| h(\Phi_n^j(J))\\ &\leq \frac{\lambda_{n+j}R'}{\delta}\sqrt{\|\varepsilon_n\|}\,l(\Phi_n^j(J)).\end{aligned} \tag{10.20}$$

By the triangular inequality, (10.19), and (10.20), we get

$$\begin{aligned}&l(\Phi_n^{j+1}(J))\\ \geq&\left|\pi_x\left(\phi_{n+j}(x_2,y_2)-\phi_{n+j}(x_1,y_2)\right)\right|-\left|\pi_x\left(\phi_{n+j}(x_1,y_2)-\phi_{n+j}(x_1,y_1)\right)\right|\\ \geq&\lambda_{n+j}\left(E'-\frac{R'}{\delta}\sqrt{\|\varepsilon_n\|}\right)l(\Phi_n^j(J))\\ \geq&\lambda E\, l(\Phi_n^j(J))\end{aligned} \tag{10.21}$$

Here we assume that $\bar{\varepsilon}$ is sufficiently small such that $\lambda_{n+j}\left(E'-\frac{R'}{\delta}\sqrt{\|\varepsilon_n\|}\right)>\lambda E$ for all $n\geq 0$ and $j\geq 0$ since $E'>E>1$ and $|\lambda_n-\lambda|<\bar{\varepsilon}$. Then (10.15) follows from the induction hypothesis.

Moreover, the vertical sizes are related by $h(\Phi_n^{j+1}(J))=\lambda_{n+j}h(\Phi_n^j(J))$. By (10.21) and the induction hypothesis, we get

$$\frac{h(\Phi_n^{j+1}(J))}{l(\Phi_n^{j+1}(J))}\leq\frac{1}{E'-\frac{1}{\delta}R\sqrt{\|\varepsilon_n\|}}\frac{h(\Phi_n^j(J))}{l(\Phi_n^j(J))}<R\frac{1}{\sqrt{\|\varepsilon_n\|}}$$

when $\bar{\varepsilon}>0$ is small since $E'-\frac{1}{\delta}R'\sqrt{\|\varepsilon_n\|}>1$.

Therefore, the lemma is proved by induction. □

Finally, we prove the expansion estimate for the case of the intermediate region.

Proof. The expansion estimate (10.6) follows from Lemma 10.4 and Lemma 10.5.

To show that J' is a regular set, we apply Lemma 10.4, Lemma 10.5, and Proposition 5.26. Then

$$\frac{h(J')}{l(J')}\leq R'\frac{1}{\sqrt{\|\varepsilon_n\|}}\leq R'c\frac{1}{\|\varepsilon_{n+1}\|^{1/4}}\leq R'c\frac{1}{\|\varepsilon_{n+k}\|^{1/4}}$$

where $c>0$ is a constant. Therefore, the set J' is $R'c$-regular. □

10.3. Case $J_n \subset B_{r(n)}(k_n)$, $1 \le k_n < \overline{K}$

In this section, we prove the expansion estimate holds when the wandering domain is inside B but far away from the bad region. Although the rescaling trick does not work in the non-degenerate case, we still can apply it to the limiting degenerate Hénon-like map G to prove the expansion estimate for G. Then we show that the estimate can be promoted to Hénon-like maps that are close to G because of continuity and $\overline{K}$ is a fixed number.

Observe in the limiting case, we have

$$\lim_{n\to\infty} F_n(x,y) = (g(x),x)$$

and

$$\lim_{n\to\infty} \phi_n(x,y) = (-\lambda)(g(x),y).$$

Then

$$\lim_{n\to\infty} \Phi_n^j(x,y) = ([(-\lambda)g]^j(x), (-\lambda)^j y)$$

and

$$\lim_{n\to\infty} \Phi_n^j \circ F_n(x,y) = ([(-\lambda)g]^j \circ g(x), (-\lambda)^j x)$$

where $[(-\lambda)g]^j$ means the function $x \to (-\lambda)g(x)$ is composed repeatedly j times.

The next lemma is a version of the rescaling trick (Lemma 8.3) in the limiting case.

Lemma 10.6 (Rescaling trick)**.** *Assume that $j \ge 0$ is an integer. Then*

$$[(-\lambda)g]^j \circ g(x) = g((-\lambda)^j x) \tag{10.22}$$

for all x with $q^l(j-1) \le x \le q^r(j-1)$.

Proof. Prove by induction on j. The base case $j = 0$ is clear.

Assume the equality holds for j. For the case $j+1$, assume that $|x| \le \left(\frac{1}{\lambda}\right)^{j+1}$. The induction hypothesis yields

$$[(-\lambda)g]^{j+1} \circ g(x) = (-\lambda)g \circ g((-\lambda)^j x) = (-\lambda)g^2\left(-\frac{(-\lambda)^{j+1}x}{\lambda}\right)$$

since $|x| \le \left(\frac{1}{\lambda}\right)^j$. By the functional equation (4.5), we get

$$[(-\lambda)g]^{j+1} \circ g(x) = g((-\lambda)^{j+1}x)$$

since $\left|(-\lambda)^{j+1}x\right| \le 1$.

Therefore, the equality is proved by induction. □

Then we estimate the size of expansion in each rescaling level.

Lemma 10.7. *There exist constants $E, E' > 1$ such that*

$$E\lambda^j \le \left|\frac{\mathrm{d}\,[(-\lambda)g]^j \circ g}{\mathrm{d}x}(x)\right| \le E'\lambda^j \tag{10.23}$$

for all $x \in B_g(j)$ and $j \geq 0$.

Proof. By the rescaling trick and chain rule, we have

$$\frac{\mathrm{d}\,[(-\lambda)g]^j \circ g}{\mathrm{d}x}(x) = (-\lambda)^j g'((-\lambda)^j x)$$

for all $|x| \leq \left(\frac{1}{\lambda}\right)^j$. By Proposition 4.38, there exists $E > 1$ such that

$$\left|g'(x)\right| \geq E$$

for all $\frac{1}{\lambda} \leq |x| \leq 1$. Also, by compactness, there exists $E' > 0$ such that

$$\left|g'(x)\right| \leq E'$$

for all $x \in I$. This yields (10.23) since $\frac{1}{\lambda} \leq \left|(-\lambda)^j x\right| \leq 1$ for all $\left(\frac{1}{\lambda}\right)^{j+1} \leq |x| \leq \left(\frac{1}{\lambda}\right)^j$. □

Assume that $J \subset B_n(j)$ is a wandering domain. As usual, we estimate the size of expansion by iterating a pair of horizontal endpoints. In order to apply the mean value theorem to the horizontal endpoints, the map $\Phi_n^j \circ F_n$ has to be defined on a convex (rectangular) neighborhood of $B_n(j)$. To promote the expansion estimate to the Hénon-like map F_n, we also need to show that the partition element $B_n(j)$ is close to the partition element of the limiting case G when $\overline{\varepsilon}$ is small. This is true because of the hyperbolicity of the saddle periodic points. The technical details are left to the reader.

Lemma 10.8. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. For all $d > 0$ and integer $j \geq 1$, there exist a constant $\overline{\varepsilon} = \overline{\varepsilon}(d, j) > 0$ and two closed intervals $U^l \subset [q^l(j-1) - d, q^l(j) + d]$ and $U^r \subset [q^r(j) - d, q^r(j-1) + d]$ such that the following properties hold:*
Let $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $U = U^l \cup U^r$. Then for all $n \geq 0$, we have

1. *$B_n^l(j) \subset U^l \times I_n^v$, $B_n^r(j) \subset U^r \times I_n^v$, and*
2. *the map $\Phi_n^j \circ F_n$ is defined on $U \times I_n^v$ where $U = U^l \cup U^r$.*

We also show that the expansion estimate, Lemma 10.7, also applies to the nondegenerate case.

Lemma 10.9. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. For all integer $j \geq 1$, there exist constants $\overline{\varepsilon}(j) > 0$, $E > 1$, and $E' > 1$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the estimate*

$$E\lambda^j \leq \left|\frac{\partial \pi_x \circ \Phi_n^j \circ F_n}{\partial x}(x,y)\right| \leq E'\lambda^j$$

holds for all $(x, y) \in U \times I_n^v$ and $n \geq 0$ where U is the set given in Lemma 10.8.

Proof. By analytic continuation, the map $x \to [(-\lambda)g]^j \circ g(x)$ has an analytic extension to a neighborhood of $B_g(j)$. By continuity, the expansion estimate (10.23) (with possibly different constants) holds on some neighborhood of $B_g(j)$. That is, there exist $E > 1$, $E' > 1$, and $d = d(j) > 0$ such that

$$E\lambda^j \leq \left| \frac{\mathrm{d}\,[(-\lambda)g]^j \circ g}{\mathrm{d}x}(x) \right| \leq E'\lambda^j$$

for all $x \in [q^l(j-1)-d, q^l(j)+d] \cup [q^r(j)-d, q^r(j-1)+d]$. Also, the map $\pi_x \circ \Phi_n^j \circ F_n$ (and its derivative) depends continuously on the sup-norm $\|F-G\|$ since we are considering a class of analytic maps that has a holomorphic extension to a small neighborhood of the domain. Together with Lemma 10.8, there exist $\overline{\varepsilon} = \overline{\varepsilon}(j) > 0$ and a union U of two open intervals such that $B_n(j) \subset U \times I_n^v$ and

$$\sqrt{E}\lambda^j \leq \left| \frac{\partial \pi_x \circ \Phi_n^j \circ F_n}{\partial x}(x,y) \right| \leq (E')^2\lambda^j$$

for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ and $(x,y) \in U \times I_n^v$. □

Finally, we estimate the size of expansion.

Lemma 10.10. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. For all $\overline{K} > 0$ and $R > 0$, there exist constanrs $\overline{\varepsilon} = \overline{\varepsilon}(\overline{K}, R) > 0$ and $E > 1$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following properties hold for all $n \geq 0$:*

Assume that $J \subset B_n(k)$ is a connected closed R-regular set and $k \leq \min(\overline{K}, K_n)$. Then $J' = \Phi_n^k \circ F_n(J)$ is an R-regular set in $C_{n+k}(0) = A_{n+k} \cup W^1_{n+k}(0) \cup B_{n+k}$ and

$$l(J') \geq E\lambda^k l(J). \tag{10.24}$$

Proof. Let $\overline{\varepsilon} = \overline{\varepsilon}(\overline{K}) > 0$ be sufficiently small such that the expansion estimate in Lemma 10.9 holds for all $j \leq \overline{K}$. Let $J \subset B_n(k)$ be a connected closed R-regular set with $k \leq \min(\overline{K}, K_n)$ and $n \geq 0$. Also, set $G = \Phi_n^k \circ F_n$ and $G_x = \pi_x \circ G$. Then $J' = G(J)$. We prove the lemma for the case of $J \subset B_n^l(k)$. The other case $J \subset B_n^r(k)$ is similar.

Let $(x_1,y_1),(x_2,y_2) \in J$ be a pair of horizontal endpoints. By Lemma 10.9, G is defined on $U^l \times I_n^v$. We can apply the mean value theorem. There exist $\xi \in (x_1,x_2)$ and $\eta \in (y_1,y_2)$ such that

$$G_x(x_2,y_2) - G_x(x_1,y_2) = \frac{\partial G_x}{\partial x}(\xi, y_2)(x_2 - x_1)$$

and

$$G_x(x_1,y_2) - G_x(x_1,y_1) = \frac{\partial G_x}{\partial y}(x_1, \eta)(y_2 - y_1).$$

By triangular inequality and J is R-regular, we get

$$\begin{aligned} l(J') &\geq |G_x(x_2,y_2) - G_x(x_1,y_2)| - |G_x(x_1,y_2) - G_x(x_1,y_1)| \\ &\geq \left| \frac{\partial G_x}{\partial x}(\xi, y_2) \right| l(J) - \left| \frac{\partial G_x}{\partial y}(x_1, \eta) \right| h(J) \\ &\geq \left(\left| \frac{\partial G_x}{\partial x}(\xi, y_2) \right| - \left| \frac{\partial G_x}{\partial y}(x_1, \eta) \right| R \, \|\varepsilon_n\|^{-1/4} \right) l(J). \end{aligned} \tag{10.25}$$

The first term can be estimated by Lemma 10.9. To bound the second term $\frac{\partial G_x}{\partial y}(x_1,\eta)$, we compute

$$\frac{\partial G_x}{\partial y}(x_1,\eta) = \frac{\partial \pi_x \circ \Phi_n^k}{\partial x} \circ F_n(x_1,\eta) \frac{\partial \varepsilon_n}{\partial y}(x_1,\eta).$$

By compactness, one can find a constant $c > 0$ that bounds $\frac{\partial \pi_x \circ \Phi_n^k}{\partial x}$ for all $k \leq \overline{K}$ and all Hénon-like maps that are close to the limiting map G. Then

$$\left| \frac{\partial G_x}{\partial y}(x_1,\eta) \right| \leq cc' \left\| \varepsilon_n \right\|$$

for some $c' > 0$ by Lemma 2.1. The inequality 10.25 becomes

$$l(J') \geq E\lambda^k \left(1 - \frac{cc'}{E\lambda^k} R \left\| \varepsilon_n \right\|^{3/4} \right) l(J) \geq \sqrt{E}\lambda^k l(J) \tag{10.26}$$

for some constant $E > 1$ when $\overline{\varepsilon} > 0$ is small.

To prove that J' is R-regular, we apply (10.26) and $h(J') = \left(\prod_{j=0}^{k(J)-1} \lambda_{j+n} \right) l(J)$. Assume that $\overline{\varepsilon} = \overline{\varepsilon}(\overline{K})$ is small enough such that $\prod_{j=0}^{i-1} \lambda_{j+n} \leq 2\lambda^i$ for all $1 \leq i \leq \overline{K}$ and $n \geq 0$. Then

$$\frac{h(J')}{l(J')} \leq \frac{2\lambda^k l(J)}{\sqrt{E}\lambda^k l(J)} = \frac{2}{\sqrt{E}} \leq R \left\| \varepsilon_{n+k} \right\|^{-1/4}$$

when $\overline{\varepsilon} = \overline{\varepsilon}(R)$ is small enough. □

10.4. The expansion estimate for a closest approach

Finally, we establish the expansion estimate.

Proposition 10.11. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $E > 1$, and $R > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following properties hold:*

Assume that $J \subset A \cup B$ is a connected closed R-regular subset of a wandering domain of F and $\{J_n\}_{n=0}^\infty$ is the J-closest approach. If $k_n \leq K_{r(n)}$ for all $n \leq m$, then J_n is R-regular for all $n \leq m+1$ and

$$l_{n+1} \geq E l_n \tag{10.27}$$

for all $n \leq m$.

Proof. We fix a constant $R > 0$ from Lemma 10.3 that makes the property regular to be invariant under the construction of a closest approach. Also set $\overline{K} > 0$ be a large number such that (10.7) gives a strict expansion. Then the expansion estimate (10.27) follows from Lemma 10.2, Lemma 10.10, and Lemma 10.3. □

11. The Bad Region and the Thickness

In this chapter, we cover the case of the bad region then prove the nonexistence of wandering domains.

In the good regions, we showed that the horizontal sizes of the elements in a closest approach expand at a definite rate by estimating the expansion of the horizontal endpoints (Proposition 10.11). However, the horizontal size is out of control when an element enters the bad region. Even worse, the situation of having a wandering domain in the bad region is unavoidable. The next lemma shows that an infinitely renormalizable Hénon-like map must have a wandering domain in the bad region if it has any wandering domain. This produces the main difficulty of proving the nonexistence of wandering domains.

Lemma 11.1. **Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exists $\overline{\varepsilon} > 0$ such that for all non-degenerate Hénon-like maps $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following property holds.*

If F has a wandering domain in D then F has a wandering domain in the bad region of B and a wandering domain in the bad region of C.

Proof. Recall that K_0 is the boundary of the good region and the bad region for $F_0 = F$ (Definition 9.1). Let $j > K_0$.

If F has a wandering domain in D, then F_j also has a wandering domain J' in D_j by Corollary 7.4. By iterating the wandering domain, we can assume without lose of generality that $J' \subset F_j(D_j)$. Set $J_C = \left(\Phi_0^j\right)^{-1}(J')$. Then J_C is a wandering domain of F in the bad region of C.

Moreover, since $J' \subset F_j(D_j)$, we have $J_C \subset \left(\Phi_0^j\right)^{-1}(F_j(D_j)) \subset F(D)$. Let $J_B = F^{-1}(J_C)$. Then J_B is a wandering domain of F in the bad region of B. □

To proceed, we first introduce the quantity "thickness" which gives a good estimate for the lower bound of the horizontal size when the expansion estimate breaks down.

After studying the relationships between horizontal size and thickness, we are able to control the lower bounds of the horizontal sizes of all elements in a closest approach. However, Proposition 11.13 shows that a strong contraction applies to the horizontal size whenever the closest approach enters the bad region. The whole proof leads to a dead-end if the contraction occurs infinitely many times. The breakthrough is the observation that a closest approach can have at most finitely many entries to the bad region (Proposition 11.17) and hence the total amount of contraction is bounded. This is done by first proving the two-row-lemma (Lemma 11.15) then applying the lemma inductively (Lemma 11.16).

To summarize, the total amount of contraction is bounded but the amount of expansion is unbounded. This leads to a contradiction. Therefore, a wandering domain cannot exist (Theorem 11.18).

11.1. Thickness and largest square subset

When an element J_n enters the bad region, the expansion estimate breaks down. It is not possible to estimate the horizontal size l_{n+1} of J_n from the horizontal size l_n of J_n because of the vertical line argument (Figure 9.1). However, in higher-dimensional systems, a wandering domain is an open set. It has area. At this moment, the length of a horizontal cross-section gives a good lower bound for the horizontal size. In this section, we introduce "thickness" to quantify the length of a horizontal cross-section.

When a wandering domain J_n is contained in a bad region, we cannot continue to estimate the horizontal size of the proceeding elements in the closest approach because of the following two reasons. First, the horizontal size of the next element J_{n+1} cannot be estimated by the horizontal size of the previous element J_n because the expansion estimate in Chapter 10 breaks down. At this moment, the horizontal size l_{n+1} is dominated by the size of its horizontal cross-section. Second, the proceeding elements are no longer regular sets. The expansion estimate (Proposition 10.11) does not apply to the elements even if they all stay in the good regions.

To resolve the two issues, it requires the following:

1. A quantity to approximate the size of a horizontal cross-section of a wandering domain, called the thickness.
2. Keep track of the thickness of the elements in a closest approach. When an element J_n enters the bad region, the horizontal size of the next element J_{n+1} will be estimated by its thickness.
3. A method to select a subset S from the wandering domain J_{n+1} that makes S an R-regular set with horizontal size close to the size of J_{n+1}. The subset will be defined by a largest square subset of J_{n+1}.

In this section, we define the thickness and a largest square subset of a wandering domain then study the properties of these two objects in a closest approach.

Definition 11.2 (Square, Largest square subset, and Thickness)**.** A set $S \subset \mathbb{R}^2$ is a square if $S = [x_1, x_2] \times [y_1, y_2]$ and $|x_2 - x_1| = |y_2 - y_1|$. This means that S is a closed square with horizontal and vertical sides. The thickness of a set $J \subset \mathbb{R}^2$ is the quantity $w(J) = \sup\{l(S)\}$ where the supremum is evaluated over all square subsets $S \subset J$. A square subset $S \subset J$ is a largest square subset if $l(S) = w(J)$. Figure 11.1 shows a comparison between the horizontal size, the vertical size, and the thickness of a set.

Lemma 11.3. *A largest square subset of a compact set exists.*

Proof. The lemma follows from compactness.

Let $J \subset \mathbb{R}^2$ be a compact set. Also let $\{I_n\}_{n\geq 1}$ be a sequence of square subsets $I_n \subset J$ such that $\{l(I_n)\}_{n\geq 1}$ is increasing and converge to $w(J)$. Write $I_n = \left[x_1^{(n)}, x_2^{(n)}\right] \times \left[y_1^{(n)}, y_2^{(n)}\right]$. Without loss of generality, we may assume that $(x_1, y_1) = \lim_{n\to\infty}(x_1^{(n)}, y_1^{(n)})$ and $(x_2, y_2) = \lim_{n\to\infty}(x_2^{(n)}, y_2^{(n)})$ exist by the compactness of J. Define $I = [x_1, x_2] \times [y_1, y_2]$. Then

$$l(I) = x_2 - x_1 = \lim_{n\to\infty} x_2^{(n)} - x_1^{(n)} = \lim_{n\to\infty} l(I_n) = w(J).$$

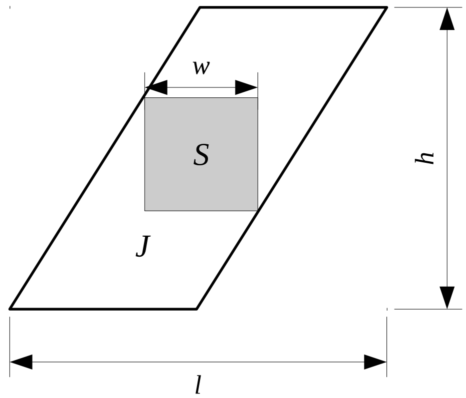


Figure 11.1.: The horizontal size l, the vertical size h, and the thickness w of a set J. In this picture, S is a largest square subset of J.

and I is a square by the similar reason.

It remains to show that $I \subset J$. Given $(x,y) \in I$.

For the case that $x_1 < x < x_2$ and $y_1 < y < y_2$. There exists n large enough such that $x_1^{(n)} < x < x_2^{(n)}$ and $y_1^{(n)} < y < y_2^{(n)}$. Then $(x,y) \in I_n \subset J$.

For the case that $x = x_1$ and $y_1 < y < y_2$. There exists N large enough such that $y_1^{(n)} < y < y_2^{(n)}$ for all $n \geq N$. Then $(x_1^{(n)}, y) \in I_n \subset J$ for all $n \geq N$. By the compactness of J, we get $(x,y) = \lim_{n\to\infty}(x_1^{(n)}, y) \in J$.

The other cases are similar. □

Next, we study the contraction of thickness when a wandering domain is iterated. The size of contraction is similar to the contraction of area: the contraction rate has the same size as the Jacobian of the map.

Lemma 11.4. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following property holds for all $n \geq 0$:*

If $S \subset D_n$ is a square, there exists a square $S' \subset F_n(S)$ such that

$$l(S') \geq c \frac{\|\varepsilon_n\|}{|I_n^v|} l(S).$$

Proof. The lemma is trivial when F is degenerate. We assume that F is non-degenerate.

Assume that $b > 0$ is a small constant which will be determined later. Let $S = [x,a] \times [y_1, y_2]$, $(x_1', x) = F_n(x, y_2)$, $(x_2', x) = F_n(x, y_1)$, and $W = b(x_2' - x_1') = b[\varepsilon_n(x,y_2) - \varepsilon_n(x,y_1)] > 0$. Define $x' = \frac{x_1'+x_2'}{2}$ and $S' = \left[x' - \frac{1}{2}W, x' + \frac{1}{2}W\right] \times [x, x+W]$.

The idea is to adjust the constant b to make the square S' fit inside the image $F_n(S)$. We will show from the proof that the constant b can be chosen to be uniform. Then

$$l(S') = W = b \frac{\partial \varepsilon_n}{\partial y}(x,\eta) l(S).$$

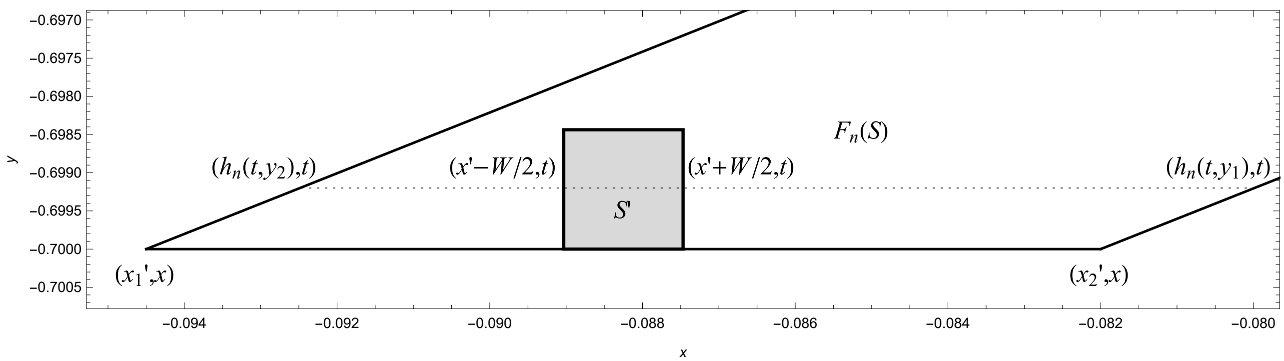


Figure 11.2.: Four points on the cross section $y = t$.

for some $\eta \in (y_1, y_2)$ by the mean value theorem. By (5.3), we get

$$l(S') \geq \frac{bc}{|I_n^y|} \|\varepsilon_n\| \, l(S)$$

for some constant $c > 0$. This proves the lemma.

To prove $S' \subset F_n(S)$, we show that the inequality

$$h_n(t, y_2) < x' - \frac{1}{2}W < x' + \frac{1}{2}W < h_n(t, y_1) \tag{11.1}$$

holds for all $t \in [x, x+W]$. The four components in the inequality are four points on the horizontal cross section $y = t$. See Figure 11.2 for an illustration. We prove the first half of the inequality (11.1)

$$h_n(t, y_2) < x' - \frac{1}{2}W.$$

By the mean value theorem and Lemma 6.23, there exist $\xi \in (x, t)$ and $E > 1$ such that

$$\left|h_n(t, y_2) - x_1'\right| = |h_n(t, y_2) - h_n(x, y_2)| = \left|\frac{\partial h_n}{\partial x}(\xi, y_2)\right| |t - x| \leq EW.$$

We get

$$\begin{aligned} \left(x' - \frac{1}{2}W\right) - h_n(t, y_2) &= \left[\left(x' - \frac{1}{2}W\right) - x_1'\right] - \left[h_n(t, y_2) - x_1'\right] \\ &\geq \left(\frac{x_2' - x_1'}{2} - \frac{1}{2}W\right) - EW \\ &= \left[\frac{1}{2} - \left(\frac{1}{2} + E\right) b\right] (x_2' - x_1') \\ &> 0 \end{aligned}$$

when $b < \frac{1}{1+2E}$. Therefore, the first half of the inequality is proved. Similarly, we prove the second half of the inequality (11.1)

$$x' + \frac{1}{2}W < h_n(t, y_1).$$

By the mean value theorem, there exists $\xi \in (x, t)$ such that

$$\left|h_n(t, y_1) - x_2'\right| = |h_n(t, y_1) - h_n(x, y_1)| = \left|\frac{\partial h_n}{\partial x}(\xi, y_1)\right| |t - x| \leq EW.$$

Similarly, we get

$$\begin{aligned} h_n(t, y_1) - \left(x' + \frac{1}{2}W\right) &= \left[x_2' - \left(x' + \frac{1}{2}W\right)\right] - \left[x_2' - h_n(t, y_1)\right] \\ &\geq \left(\frac{x_2' - x_1'}{2} - \frac{1}{2}W\right) - EW \\ &= \left[\frac{1}{2} - \left(\frac{1}{2} + E\right) b\right] (x_2' - x_1') \\ &> 0. \end{aligned}$$

Thus, the second half of the inequality is proved. □

Remark 11.5. The lemma depends on the universality around the tip (Proposition 5.26 item 5, [dCLM05, Theorem 7.9]). The property provides a uniform lower bound for the Jacobian. In other words, it ensures that the thickness (area) does not contract too fast.

Finally, we estimate the expansion of the thickness when a wandering domain is rescaled.

Lemma 11.6. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exists $\overline{\varepsilon} > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following property holds for all $n \geq 0$:*

If $S \subset C_n$ is a square, there exists a square $S' \subset \phi_n(S)$ such that

$$l(S') = \lambda_n l(S).$$

Proof. Let $S = [x_1, x_2] \times [y_1, y_2]$, $W = l(S)$, $x = \frac{1}{2}[h_n(x_2, y_1) + h_n(x_1, y_1)]$, and $S'' = [x - \frac{1}{2}W, x + \frac{1}{2}W] \times [y_1, y_2]$. Then S'' is a square with $l(S'') = l(S)$.

The idea is to show that $S'' \subset H_n(S)$ by proving the inequality

$$h_n(x_2, t) < x - \frac{1}{2}W < x + \frac{1}{2}W < h_n(x_1, t) \tag{11.2}$$

holds for all $t \in [y_1, y_2]$. The four components in the inequality are four points on the horizontal cross section $y = t$. See Figure 11.3 for an illustration.

We prove the first half of the inequality (11.2). By the mean value theorem, there exist $\xi \in (x_1, x_2)$ and $\eta \in (y_1, t)$ such that

$$h_n(x_1, y_1) - h_n(x_2, y_1) = \left|\frac{\partial h_n}{\partial x}(\xi, y_1)\right| (x_2 - x_1)$$

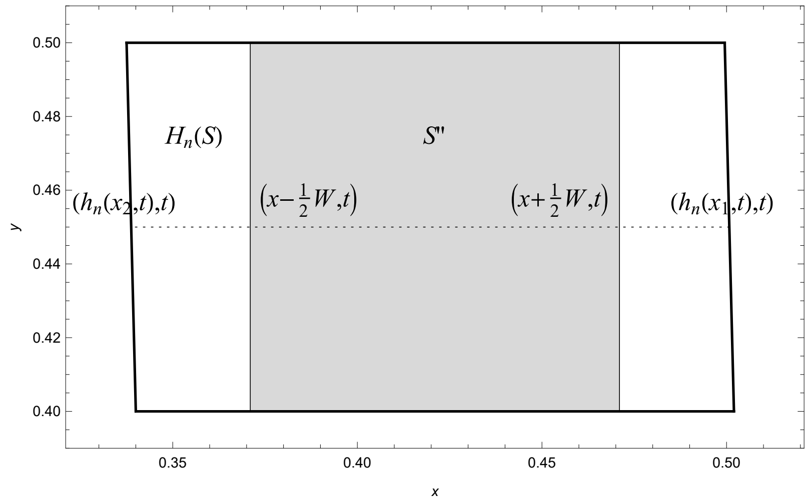


Figure 11.3.: Four points on the cross section $y=t$.

and

$$h_n(x_2,t)-h_n(x_2,y_1)=\frac{\partial \varepsilon_n}{\partial y}(x_2,\eta)(t-y_1).$$

By Lemma 6.22, there exists $E>1$ such that

$$\begin{aligned}\left(x-\frac{1}{2}W\right)-h_n(x_2,t) &= [x-h_n(x_2,y_1)]-[h_n(x_2,t)-h_n(x_2,y_1)]-\frac{1}{2}W\\ &\geq \frac{1}{2}\left|\frac{\partial h_n}{\partial x}(\xi,y_1)\right|(x_2-x_1)-\left|\frac{\partial \varepsilon_n}{\partial y}(x_2,\eta)\right|(t-y_1)-\frac{1}{2}W\\ &\geq \left(\frac{E}{2}-\frac{1}{\delta}\|\varepsilon_n\|-\frac{1}{2}\right)W\\ &> 0\end{aligned}$$

when $\overline{\varepsilon}>0$ is small. Thus, the first half of the inequality is proved.

Similarly, we prove the second half of the inequality (11.2). By the mean value theorem, there exists $\eta'\in(y_1,t)$ such that

$$\varepsilon_n(x_1,t)-\varepsilon_n(x_1,y_1)=\frac{\partial \varepsilon_n}{\partial y}(x_1,\eta')\,(t-y_1)\,.$$

Compute

$$\begin{aligned}h_n(x_1,t)-\left(x+\frac{1}{2}W\right) &= [h_n(x_1,y_1)-x]-[\varepsilon_n(x_1,t)-\varepsilon_n(x_1,y_1)]-\frac{1}{2}W\\ &\geq \frac{1}{2}\left|\frac{\partial h_n}{\partial x}(\xi,y_1)\right|(x_2-x_1)-\left|\frac{\partial \varepsilon_n}{\partial y}(x_1,\eta')\right|(t-y_1)-\frac{1}{2}W\\ &\geq \left(\frac{E}{2}-\frac{1}{\delta}\|\varepsilon_n\|-\frac{1}{2}\right)W\end{aligned}$$

$$> \quad 0.$$

Thus, the right inequality is proved.

Finally, let $S' = \Lambda_n(S'')$. Then S' is a square subset of $\phi_n(S)$ and

$$l(S') = \lambda_n l(S'') = \lambda_n l(S).$$

□

As before, we adapt the subscript n to the notation of the thickness $w_n = w(J_n)$. By Lemma 11.4 and Lemma 11.6, the contraction rate of the thicknesses for elements in a closest approach is estimated by the following.

Proposition 11.7. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following property holds:*

Assume that $J \subset A \cup B$ is a compact subset of a wandering domain of F and $\{J_n\}_{n=0}^\infty$ is the J-closest approach. Then

$$w_{n+1} \geq c \frac{\left\|\varepsilon_{r(n)}\right\|}{\left|I^v_{r(n)}\right|} w_n$$

for all $n \geq 0$.

Proof. Let $\overline{\varepsilon} > 0$ be small enough such that Lemma 11.4 and Lemma 11.6 holds. The sets $\{J_n\}_{n=0}^\infty$ are compact by the continuity of Hénon-like maps and rescaling.

For the case that $J_n \subset A_{r(n)}$, let I be a largest square of J_n. By Proposition 11.4, there exists a square $I' \subset F_{r(n)}(I) \subset J_{n+1}$ such that

$$l(I') \geq c \frac{\left\|\varepsilon_{r(n)}\right\|}{\left|I^v_{r(n)}\right|} l(I).$$

We get

$$w_{n+1} \geq l(I') \geq c \frac{\left\|\varepsilon_{r(n)}\right\|}{\left|I^v_{r(n)}\right|} l(I) = c \frac{\left\|\varepsilon_{r(n)}\right\|}{\left|I^v_{r(n)}\right|} w_n.$$

For the case that $J_n \subset B_{r(n)}$, let I be a largest square of J_n. By Proposition 11.4, there exists a square $I_0 \subset F_{r(n)}(I) \subset F_{r(n)}(J_n)$ such that

$$l(I_0) \geq c \frac{\left\|\varepsilon_{r(n)}\right\|}{\left|I^v_{r(n)}\right|} l(I).$$

Also by Proposition 11.6, there exists a square $I_{j+1} \subset \phi_{r(n)+j}(I_j) \subset \Phi^j_{r(n)} \circ F_{r(n)}(J_n)$ such that

$$l(I_{j+1}) = \lambda_{r(n)+j} l(I_j)$$

for all $0 \leq j < k_n$. We get

$$w_{n+1} \geq l(I_{k_n}) = \left(\prod_{j=0}^{k_n-1} \lambda_{r(n)+j}\right) l(I_0) \geq c\frac{\left\|\varepsilon_{r(n)}\right\|}{\left|I^v_{r(n)}\right|} l(I) = c\frac{\left\|\varepsilon_{r(n)}\right\|}{\left|I^v_{r(n)}\right|} w_n.$$

□

Remark 11.8. The original proof was based on the area and horizontal cross-section estimates briefly mentioned in the beginning of this chapter instead of tracking the sizes of the largest square subsets. However, the area argument is discarded by two reasons. First, to estimate the horizontal cross-section of a set, we need to find the lower bound of a/l. This means that we need to repeat the arguments in Chapter 10 to find the upper bound for l and the lower bound for a. This makes the argument several times longer than the current one. Second, to select a subset from the wandering domain after it enters the bad region, the area approach makes it hard to find the upper bound of l for the subset.

Since $\|\varepsilon_n\|$ decreases super-exponentially and $|I^v_n|$ increases exponentially as the number of renormalizations n approaches to infinity, we can simplify the estimate.

Corollary 11.9. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $c > 0$ such that for all $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following property holds:*

Assume that $J \subset A \cup B$ is a compact subset of a wandering domain of F and $\{J_n\}_{n=0}^{\infty}$ is the J-closest approach. Then

$$w_{n+1} \geq c\left\|\varepsilon_{r(n)}\right\|^{3/2} w_n$$

for all $n \geq 0$.

11.2. Double sequence

In this section, we study the relationships between the horizontal sizes and the thicknesses of the elements in a closest approach. We first define a sequence with two indices, called a double sequence. A double sequence consists of rows. Each row is a closest approach (Definition 7.6) and represents an entry to the bad region.

Definition 11.10 (Double sequence, Row, and Time span in the good regions)**.** Let $\delta > 0$ and $I^v \supset I^h \Supset I$. Assume that $\overline{\varepsilon} > 0$ is small and $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$ is a non-degenerate open map.

Given a square subset $J \subset A \cup B$ of a wandering domain of F. Define $\{J^{(j)}_n\}_{n\geq 0, 0\leq j\leq \overline{j}}$, $\{F^{(j)}_n = (f^{(j)}_n - \varepsilon^{(j)}_n, x)\}_{n\geq 0, 0\leq j\leq \overline{j}}$, and $\{n^{(j)}\}_{0\leq j\leq \overline{j}}$ for some $\overline{j} \in \mathbb{N} \cup \{0, \infty\}$[1] by induction on j.

Base case: For $j = 0$, set $J^{(0)}_0 = J$ and $F^{(0)}_0 = F$.

Row: The super-script j is called *row*. The first set $J^{(j)}_0$ of a row is a square subset of a wandering domain of $F^{(j)}_0$ in $A(F^{(j)}_0) \cup B(F^{(j)}_0)$. The elements $\left\{J^{(j)}_n\right\}_{n=0}^{\infty}$ in the row form a $J^{(j)}_0$-closest approach. See Definition 7.6.

[1] For the case $\overline{j} = \infty$, this means that the sequence is defined for all finite positive integers j.

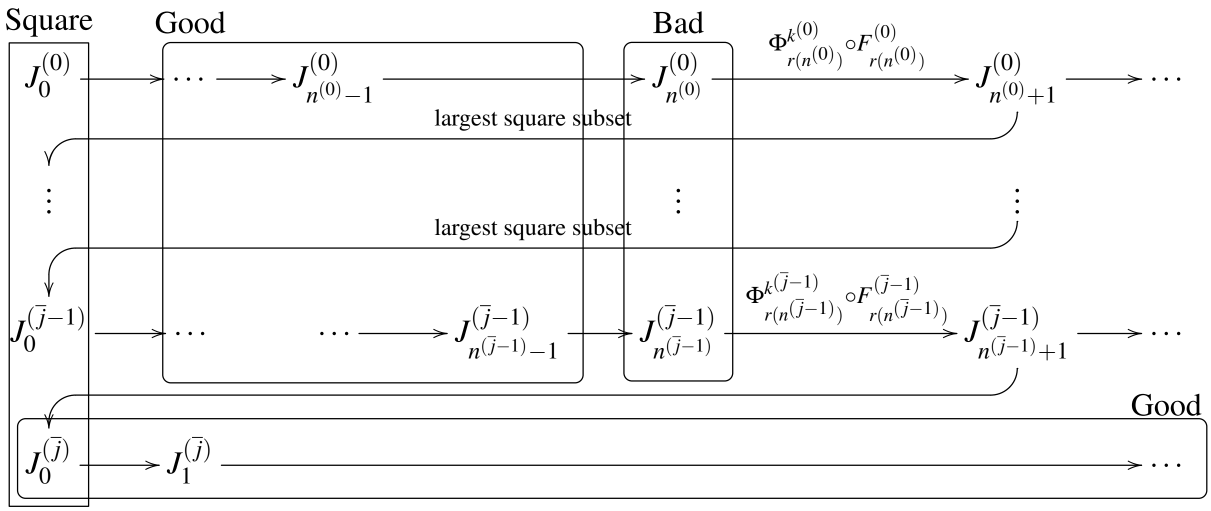


Figure 11.1.: The construction of a double sequence.

Induction step: Consider a row j. If an element in the row enters the bad region, i.e. $k_n^{(j)} > K_{r^{(j)}(n)}^{(j)}$ for some $n \geq 0$, let $J_{n^{(j)}}^{(j)}$ be the first element. The nonnegative integer $n^{(j)}$ is called the time span in the good regions. Define the first element $J_0^{(j+1)}$ of the next row $j+1$ to be a largest square subset of $J_{n^{(j)}+1}^{(j)}$ and set $F_0^{(j+1)} = F_{r^{(j)}(n^{(j)}+1)}^{(j)}$. If the elements in the row stay in the good regions forever, then the construction stops, set $\bar{j} = j$ and $n^{(j)} = \infty$. If the construction never stops, set $\bar{j} = \infty$.

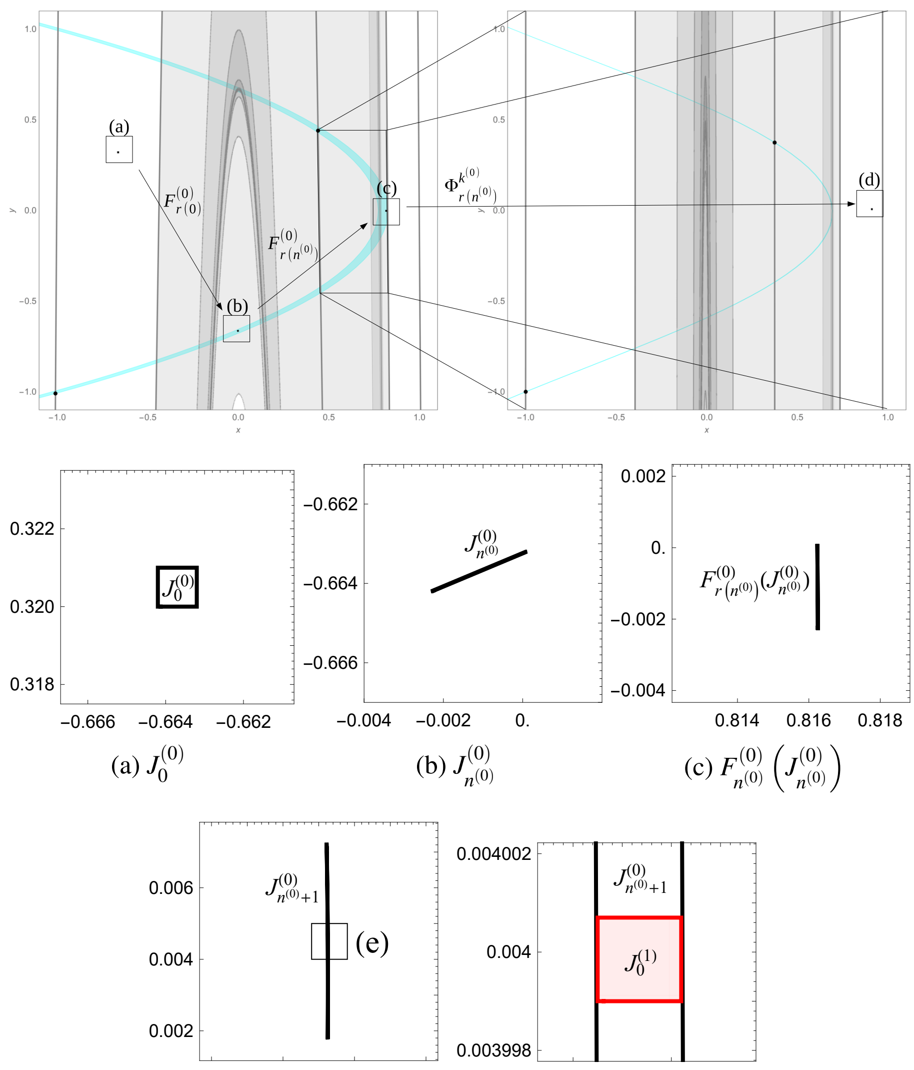
The sequence $\left\{J_n^{(j)}\right\}_{n\geq 0, 0\leq j\leq \bar{j}}$ with two indices is called a double sequence generated by J or a J-double sequence. The integer $\bar{j}$ is the total number of rows (enters the bad region $\bar{j}$ times). Figure 11.1 shows an illustration of the construction.

Remark 11.11. Unlike the J-closest approach, a J-double sequence may not be unique because the way of selecting a largest square subset of a set may not be unique.

To be consistent, the superscript is assigned to the row and the subscript is assigned to the renormalization scale or the index of an element in a closest approach. For example, the superscript is introduced to the notations: $A_n^{(j)} = A(F_n^{(j)})$, $B_n^{(j)} = B(F_n^{(j)})$, $C_n^{(j)} = C(F_n^{(j)})$, $D_n^{(j)} = D(F_n^{(j)})$, $l_n^{(j)} = l(J_n^{(j)})$, $h_n^{(j)} = h(J_n^{(j)})$, $w_n^{(j)} = w(J_n^{(j)})$, and $k_n^{(j)} = k(J_n^{(j)})$. In the following, we write $r(n) = r^{(j)}(n)$ when the context is clear. For example $F_{r(n^{(j)}+1)}^{(j)} = F_{r^{(j)}(n^{(j)}+1)}^{(j)}$. Also, let $\varepsilon^{(j)} = \varepsilon_{r(n^{(j)})}^{(j)}$, $K^{(j)} = K_{r(n^{(j)})}^{(j)}$, and $k^{(j)} = k_{n^{(j)}}^{(j)}$. For convenience, set $m^{(j)} = n^{(j)} + 1$.

Example 11.12. Figure 11.2 shows an example of the construction of a double sequence. In this example, the Hénon-like map is the same map as in Example 7.8. Given a square $J_0^{(0)} = [-0.6642, -0.6632] \times [0.320, 0.321] \subset A_0^{(0)}$ and let $r^{(0)}(0) = 0$.

According to the definition, $J_1^{(0)} = F_{r(0)}^{(0)}(J_0^{(0)})$ and $r^{(0)}(1) = r^{(0)}(0) = 0$. From the figure, we see



(a) $J^{(0)}_0$ (b) $J^{(0)}_{n^{(0)}}$ (c) $F^{(0)}_{n^{(0)}}\left(J^{(0)}_{n^{(0)}}\right)$

(d) $J^{(0)}_{n^{(0)}+1}$ (e) The beginning of a new row.

Figure 11.2.: An example of a double sequence. The graphs are the domains of $F_0^{(0)}$ and $F_1^{(0)} = F_0^{(1)}$ from the left to the right. The arrows indicate an iteration or a rescaling from one element to the other one. The sub-figures (a), (b), (c), (d), and (e) are elements of the double sequence in a zoomed scale. The scales of (a), (b), (c), and (d) are the same.

that $J_1^{(0)} \subset B_{r(0)}(1)$. In this example, the norm $\|\varepsilon\|$ is large and $C_0^r(1)$ has an nonempty intersection with the image $F_0(D_0)$. Thus $K_{r(1)}^{(0)} = K_0^{(0)} = 0$ and $J_1^{(0)}$ is contained in the bad region. Set $n^{(0)} = 1$.

Then we begin a new row $j = 1$. Set $J_{n^{(0)}+1}^{(0)} = \Phi_{r(n^{(0)})}^{k^{(0)}} \circ F_{r(n^{(0)})}^{(0)}(J_{n^{(0)}}^{(0)}) = \phi_0^{(0)} \circ F_0^{(0)}(J_1^{(0)})$. At this moment, the thickness $w_{n^{(0)}+1}^{(0)}$ gives a good estimate for the horizontal size $l_{n^{(0)}+1}^{(0)}$ as shown in Figure 11.2d. Let $J_0^{(1)}$ be a largest square subset of $J_{n^{(0)}+1}^{(0)}$ (Figure 11.2e) and $F_0^{(1)} = F_{n^{(0)}+1}^{(0)}$.

We continue to add new rows until the sequence stops to enter the bad regions.

Next, we study the relationships between the horizontal sizes and the thicknesses of the elements in a double sequence. Consider a row j, the first element $J_0^{(j)}$ of the row is a square. So $l_0^{(j)} = w_0^{(j)}$. When the elements stay in the good regions ($n < n^{(j)}$), the horizontal sizes of the elements expand at a definite rate by the expansion estimate $l_{n+1}^{(j)} \geq E l_n^{(j)}$ (Proposition 10.11). The horizontal sizes keep growing until an element $J_{n^{(j)}}^{(j)}$ enters the bad region. At this moment, the horizontal size $l_{n^{(j)}+1}^{(j)}$ of the next element $J_{n^{(j)}+1}^{(j)}$ can no longer be estimated by the horizontal size $l_{n^{(j)}}^{(j)}$ of the current element $J_{n^{(j)}}^{(j)}$. We have to use the thickness $w_{n^{(j)}+1}^{(j)}$ to bound the horizontal size $l_{n^{(j)}+1}^{(j)}$ from below. Then we apply Proposition 11.7 to relate the thickness $w_{n^{(j)}+1}^{(j)}$ with the horizontal size $l_0^{(j)}$ of the first element. Finally, by definition, the horizontal size $l_0^{(j+1)}$ and the thickness $w_0^{(j+1)}$ of the first element $J_0^{(j+1)}$ of the next row $j+1$ equal to the thickness $w_{n^{(j)}+1}^{(j)}$.

From the discussion, we are able to estimate the horizontal size of any element in the double sequence.

Proposition 11.13. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $E > 1$ such that for all non-degenerate open maps $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following properties hold:*

Let $J \subset A \cup B$ be a square subset of a wandering domain of F and $\left\{J_n^{(j)}\right\}_{n \geq 0, 0 \leq j \leq \overline{j}}$ be a J-double sequence. Then

1. $\ln l_0^{(j+1)} \geq 2m^{(j)} \ln \left\|\varepsilon^{(j)}\right\| + \ln l_0^{(j)}$ *for all* $0 \leq j \leq \overline{j} - 1$ *and*
2. $l_{n+1}^{(j)} \geq E l_n^{(j)}$ *for all* $n < n^{(j)}$ *and all* $0 \leq j \leq \overline{j}$.

Proof. Let $\overline{\varepsilon} > 0$ be small enough such that Proposition 10.11 and Corollary 11.9 hold.

By Corollary 11.9, we relate $l_0^{(j+1)}$ with $l_0^{(j)}$ by the inequality

$$\begin{aligned}
l_0^{(j+1)} &= w_{n^{(j)}+1}^{(j)} \\
&\geq \left(\prod_{n=0}^{n^{(j)}} c \left\|\varepsilon_{r(n)}^{(j)}\right\|^{3/2}\right) w_0^{(j)} \geq \left(c^{\frac{2}{3}} \left\|\varepsilon_{r(n^{(j)})}^{(j)}\right\|\right)^{\frac{3}{2}(n^{(j)}+1)} w_0^{(j)} \\
&= \left(c^{\frac{2}{3}} \left\|\varepsilon^{(j)}\right\|\right)^{\frac{3}{2}m^{(j)}} l_0^{(j)}
\end{aligned}$$

where $c > 0$ is a constant. Apply the nature logarithm to both sides, we get

$$\begin{aligned}\ln l_0^{(j+1)} &\geq \frac{3}{2}m^{(j)}\left(\ln\left\|\varepsilon^{(j)}\right\| + \frac{2}{3}\ln c\right) + \ln l_0^{(j)} \\ &\geq 2m^{(j)}\ln\left\|\varepsilon^{(j)}\right\| + \ln l_0^{(j)}.\end{aligned}$$

Here we assume that $\overline{\varepsilon}$ is small enough such that $\frac{2}{3}\ln c \geq \frac{1}{3}\ln\left\|\varepsilon^{(j)}\right\|$ for all $0 \leq j \leq \overline{j}-1$ to assimilate the constants.

The second inequality follows from Proposition 10.11, the definition of $n^{(j)}$, and a square is R-regular when $\overline{\varepsilon}$ is small. □

The next proposition relates the perturbation ε of two consecutive rows.

Proposition 11.14. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $\alpha > 0$ such that for all non-degenerate open maps $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, we have*

$$\left\|\varepsilon^{(j+1)}\right\| \leq \left\|\varepsilon^{(j)}\right\|^{\left\|\varepsilon^{(j)}\right\|^{-2\alpha}} \tag{11.3}$$

for all $0 \leq j \leq \overline{j}-1$.

Proof. By Proposition 5.26, we have

$$\left\|\varepsilon^{(j+1)}\right\| = \left\|\varepsilon^{(j+1)}_{r(n^{(j+1)})}\right\| \leq \left\|\varepsilon^{(j+1)}_0\right\| = \left\|\varepsilon^{(j)}_{r(n^{(j)}+1)}\right\| \leq c\left(\left\|\varepsilon^{(j)}_{r(n^{(j)})}\right\|\right)^{2^{k^{(j)}}} = c\left(\left\|\varepsilon^{(j)}\right\|\right)^{2^{k^{(j)}}}$$

for some constant $c > 0$. Apply the logarithm to the both sides, we get

$$\ln\left\|\varepsilon^{(j+1)}\right\| \leq 2^{k^{(j)}}\ln\left\|\varepsilon^{(j)}\right\| + \ln c \leq 2^{k^{(j)}-1}\ln\left\|\varepsilon^{(j)}\right\| \tag{11.4}$$

when $\overline{\varepsilon} > 0$ is small enough.

Consider the entry to the bad region in row j, we have $k^{(j)} > K^{(j)}$. By Proposition 9.4 and the change base formula, we get

$$2^{k^{(j)}} > 2^{K^{(j)}} = \left(\lambda^{K^{(j)}}\right)^{\frac{\ln 2}{\ln\lambda}} \geq c'\left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{\frac{\ln 2}{2\ln\lambda}} \tag{11.5}$$

for some constant $c' > 0$. Let $\alpha = \frac{\ln 2}{6\ln\lambda} > 0$. Combine (11.4) and (11.5), we obtain

$$\ln\left\|\varepsilon^{(j+1)}\right\| \leq \frac{c'}{2}\left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{3\alpha}\ln\left\|\varepsilon^{(j)}\right\| < \left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{2\alpha}\ln\left\|\varepsilon^{(j)}\right\|$$

when $\overline{\varepsilon} > 0$ is small enough. Note that $\ln\left\|\varepsilon^{(j)}\right\| < 0$. Here we also assume that $\overline{\varepsilon}$ is small enough such that

$$\frac{c'}{2}\left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{\alpha} \geq \frac{c'}{2}\left(\frac{1}{\|\varepsilon\|}\right)^{\alpha} > 1$$

for all $j \geq 0$. This proves the proposition. □

11.3. A closest approach have only finite entries to the bad region

According to Proposition 11.13, a strong contraction applies to the horizontal size whenever an element in a closest approach enters the bad region. This causes an obstruction toward our final goal of showing that the horizontal sizes approach infinity. On the other hand, whenever an element enters the bad region, a restriction also applies to the element: the size of the element cannot exceed the size of the bad region. If the sizes of the bad regions are small, then the closest approach will be less likely to enter the bad regions. In this section, we will use this restriction to show that a double sequence can have at most finitely many rows (Proposition 11.17) and conclude that the total amount of contraction is bounded.

We first prove the two-row-lemma.

Lemma 11.15 (Two-row-lemma). *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$, $E > 1$, and $\alpha > 0$ such that for all non-degenerate open maps $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following property holds:*

Let $J \subset A \cup B$ be a square subset of a wandering domain of F and $\left\{J_n^{(j)}\right\}_{n\geq 0, 0\leq j\leq \overline{j}}$ be a J-double sequence. Then the time span in the good regions $n^{(j)} = m^{(j)} - 1$ of row j is bounded below by

$$m^{(j)} > \frac{\ln E}{-2\ln\left\|\varepsilon^{(j)}\right\|} m^{(j+1)} + \left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{\alpha} + \frac{1}{-2\ln\left\|\varepsilon^{(j)}\right\|}\ln l_0^{(j)} \tag{11.6}$$

for all $0 \leq j \leq \overline{j} - 2$.

The two-row-lemma is the key lemma. It relates the size of the bad region with the contraction and the expansion of the horizontal sizes of the elements in two consecutive rows, the j-th and the $(j+1)$-th row. See Figure 11.1 for an illustration. The right hand side of (11.6) contains three terms. The first term comes from the expansion of the elements in the $(j+1)$-th row, equation (11.8) from the proof. If the expansion is large, then the contraction of the j-th row is strong because the size of the element $J_{n^{(j+1)}}^{(j+1)}$ cannot exceed the size of the bad region. The second term comes from the size of the bad region of the $(j+1)$-th row, equation (11.7) from the proof. The quantity is large since the size of the bad region $\left\|\varepsilon^{(j+1)}\right\|$ of the $(j+1)$-th row is much smaller than the contraction rate for the thickness $\left\|\varepsilon^{(j)}\right\|$ of row j (Proposition 11.14). This is because a large amount of rescalings $k^{(j)}$ was applied to the step $J_{n^{(j)}}^{(j)} \to J_{n^{(j)}+1}^{(j)}$ during the first entry to the bad region. The last term comes from the size of the initial element $J_0^{(j)}$.

Proof. (Proof of Lemma 11.15) First, we consider the $(j+1)$-th row. The horizontal size of $J_{n^{(j+1)}}^{(j+1)}$

$$\text{row } j \qquad \overbrace{w_0^{(j)} \xrightarrow{\|\varepsilon^{(j)}\|} w_1^{(j)} \xrightarrow{\|\varepsilon^{(j)}\|} \cdots \xrightarrow{\|\varepsilon^{(j)}\|} w_{n^{(j)}}^{(j)} \xrightarrow{\qquad\|\varepsilon^{(j)}\|\qquad} w_{n^{(j)}+1}^{(j)}}^{\text{contraction } \left\|\varepsilon^{(j)}\right\|^{n^{(j)}+1}}$$

$$\text{row } j+1 \qquad \underbrace{l_0^{(j+1)} \xrightarrow{E} l_1^{(j+1)} \xrightarrow{E} \cdots \xrightarrow{E} l_{n^{(j+1)}}^{(j+1)}}_{\text{expansion } E^{n^{(j+1)}}} \qquad < \sqrt{\left\|\varepsilon^{(j+1)}\right\|} \sim \left\|\varepsilon^{(j)}\right\|^{\left\|\varepsilon^{(j)}\right\|^{-\alpha}}$$

Figure 11.1.: The contraction, the expansion, and the sizes of the elements in the j-th and the $(j+1)$-th rows.

is bounded by the size of the bad region. By Proposition 9.4, there exists a constant $c > 0$ such that

$$l_{n^{(j+1)}}^{(j+1)} < 2c\sqrt{\left\|\varepsilon^{(j+1)}\right\|}. \tag{11.7}$$

Also, the horizontal sizes of the elements in the $(j+1)$-th row expands at a definite rate. There exists a constant $E > 1$ such that

$$E^{n^{(j+1)}} l_0^{(j+1)} \le l_{n^{(j+1)}}^{(j+1)} \tag{11.8}$$

by Proposition 11.13. After combining the equations (11.7) and (11.8), we get

$$\ln l_0^{(j+1)} < -m^{(j+1)} \ln E + \frac{1}{2} \ln \left\|\varepsilon^{(j+1)}\right\| + (\ln E + \ln 2c). \tag{11.9}$$

Then, we consider the j-th row. The thickness of the elements in the j-th row contracts. By Proposition 11.13 and (11.9), we have

$$\begin{aligned} 2m^{(j)} \ln \left\|\varepsilon^{(j)}\right\| &\le \ln l_0^{(j+1)} - \ln l_0^{(j)} \\ &< -m^{(j+1)} \ln E + \frac{1}{2} \ln \left\|\varepsilon^{(j+1)}\right\| + (\ln E + \ln 2c) - \ln l_0^{(j)}. \end{aligned}$$

Since $\ln \left\|\varepsilon^{(j)}\right\| < 0$, we solve

$$m^{(j)} > \frac{\ln E}{-2\ln\left\|\varepsilon^{(j)}\right\|} m^{(j+1)} + \frac{1}{4} \frac{\ln\left\|\varepsilon^{(j+1)}\right\|}{\ln\left\|\varepsilon^{(j)}\right\|} + \frac{\ln E + \ln 2c}{2\ln\left\|\varepsilon^{(j)}\right\|} + \frac{\ln l_0^{(j)}}{-2\ln\left\|\varepsilon^{(j)}\right\|}.$$

Finally we apply Proposition 11.14 to simplify the second term. Compute

$$m^{(j)} > \frac{\ln E}{-2\ln\left\|\varepsilon^{(j)}\right\|} m^{(j+1)} + \frac{1}{4} \left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{2\alpha} + \frac{\ln E + \ln 2c}{2\ln\left\|\varepsilon^{(j)}\right\|} + \frac{1}{-2\ln\left\|\varepsilon^{(j)}\right\|} \ln l_0^{(j)}$$

$$
\begin{aligned}
&= \frac{\ln E}{-2\ln\left\|\varepsilon^{(j)}\right\|}m^{(j+1)} + \left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{\alpha}\left[\frac{1}{4}\left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{\alpha} + \frac{\ln E + \ln 2c}{2\ln\left\|\varepsilon^{(j)}\right\|}\left\|\varepsilon^{(j)}\right\|^{\alpha}\right] \\
&\quad + \frac{1}{-2\ln\left\|\varepsilon^{(j)}\right\|}\ln l_0^{(j)} \\
&> \frac{\ln E}{-2\ln\left\|\varepsilon^{(j)}\right\|}m^{(j+1)} + \left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{\alpha} + \frac{1}{-2\ln\left\|\varepsilon^{(j)}\right\|}\ln l_0^{(j)}.
\end{aligned}
$$

Here we assume that $\overline{\varepsilon}$ is sufficiently small such that

$$
\frac{1}{4}\left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{\alpha} + \frac{\ln E + \ln 2c}{2\ln\left\|\varepsilon^{(j)}\right\|}\left\|\varepsilon^{(j)}\right\|^{\alpha} > 1
$$

for all $j \geq 0$ to assimilate the constants. □

Next, we use the two-row-lemma to prove that a double sequence have at most finitely many rows. Before carrying out a careful proof, we first use a reduced version

$$
m^{(j)} > \frac{1}{-\ln\left\|\varepsilon^{(j)}\right\|}m^{(j+1)} + \left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{\alpha} \tag{11.10}
$$

of the inequality (11.6) to give an intuition of how the two-row-lemma works in the proof.

First, we consider the first two rows: the 0-th and 1-st rows. At this moment, we do not have any information about the time span in the good regions $m^{(1)}$ of row 1. The recurrence relation (11.10) produces a lower bound from only the second term

$$
m^{(0)} > 0 + \left(\frac{1}{\left\|\varepsilon^{(0)}\right\|}\right)^{\alpha}. \tag{11.11}
$$

The lower bound is large because that the size of the bad region $\left\|\varepsilon^{(1)}\right\|$ and the contraction rate of thickness $\left\|\varepsilon^{(0)}\right\|$ come from the "ε" in two different rows (Proposition 11.14).

Then, we include one additional row into the estimation. Consider the first three rows: the 0-th, 1-st, and 2-nd rows. We apply recurrence relation (11.10) to the 1-st and 2-nd rows. By the same reason, we have

$$
m^{(1)} > \left(\frac{1}{\left\|\varepsilon^{(1)}\right\|}\right)^{\alpha}. \tag{11.12}
$$

Then, we apply the recurrence relation (11.10) to the 0-th and 1-st rows

$$
m^{(0)} > \underbrace{\frac{1}{-\ln\left\|\varepsilon^{(0)}\right\|}\left(\frac{1}{\left\|\varepsilon^{(1)}\right\|}\right)^{\alpha}}_{\text{improvement from the new row}} + \left(\frac{1}{\left\|\varepsilon^{(0)}\right\|}\right)^{\alpha}. \tag{11.13}
$$

Unlike the lower bound (11.11) estimated from only two rows, the additional row gives an improvement to the lower bound by knowing that $m^{(1)}$ is large. In fact, the improvement (the first term) is much larger than the original estimate (the second term) because of Proposition 11.14.

Then we continue to add more rows. Each time when we include another row, we improve the lower bound of the time span in the good regions. By induction, we will show that the lower bound approaches infinite if a double sequence has infinitely many rows. Roughly speaking, the whole argument works because the sizes of the bad regions contract much more faster than the contraction of the thicknesses. Therefore, a double sequence have at most finitely many rows.

The complete estimate is done by the following lemma.

Lemma 11.16. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exist constants $\overline{\varepsilon} > 0$ and $\alpha > 0$ such that for all non-degenerate open maps $F \in \hat{\mathscr{I}}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following property holds:*

Let $J \subset A \cup B$ be a square subset of a wandering domain of F and $\left\{J_n^{(j)}\right\}_{n\geq 0, 0\leq j\leq \overline{j}}$ be a J-double sequence. Then the time span in the good regions $n^{(j)}$ of row j is bounded below by

$$m^{(j)} = n^{(j)} + 1 > \frac{2^k}{\left\|\varepsilon^{(j)}\right\|^\alpha} + \frac{1}{-2\ln\left\|\varepsilon^{(j)}\right\|}\ln l_0^{(j)} \tag{11.14}$$

for all j and k with $0 \leq j \leq \overline{j}-2$ and $0 \leq k \leq (\overline{j}-2)-j$. In particular for the case $j = 0$

$$m^{(0)} = n^{(0)} + 1 > \frac{2^k}{\left\|\varepsilon^{(0)}\right\|^\alpha} + \frac{1}{-2\ln\left\|\varepsilon^{(0)}\right\|}\ln l_0^{(0)} \tag{11.15}$$

for all $0 \leq k \leq \overline{j}-2$.

Proof. In the lemma, the value $k+2$ is the number of rows that we use to estimate the lower bound. We prove that (11.14) holds for all $0 \leq j \leq \overline{j}-k-2$ by induction on $k \leq \overline{j}-2$. Let $\overline{\varepsilon}$ be small enough such that Proposition 11.13, Proposition 11.14, and Lemma 11.15 hold.

Consider the base case $k = 0$. By (11.6), we have

$$\begin{aligned} m^{(j)} &> \frac{\ln E}{-2\ln\left\|\varepsilon^{(j)}\right\|} m^{(j+1)} + \left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^\alpha + \frac{1}{-2\ln\left\|\varepsilon^{(j)}\right\|}\ln l_0^{(j)} \\ &> \frac{1}{\left\|\varepsilon^{(j)}\right\|^\alpha} + \frac{1}{-2\ln\left\|\varepsilon^{(j)}\right\|}\ln l_0^{(j)} \end{aligned}$$

for all j with $0 \leq j \leq \overline{j}-2$.

Assume that there exists k with $1 \leq k \leq \overline{j}-2$ such that (11.14) holds for all j with $0 \leq j \leq \overline{j}-k-2$. If $k+1 \leq \overline{j}-2$ and $0 \leq j < \overline{j}-(k+1)-2$, then $k \leq \overline{j}-2$ and $1 \leq j+1 \leq \overline{j}-k-2$. By the induction hypothesis, we have

$$m^{(j+1)} > \frac{2^k}{\left\|\varepsilon^{(j+1)}\right\|^\alpha} + \frac{1}{-2\ln\left\|\varepsilon^{(j+1)}\right\|}\ln l_0^{(j+1)}. \tag{11.16}$$

Substitute (11.16) into (11.6), we get

$$m^{(j)} > \frac{\ln E}{-2\ln\left\|\varepsilon^{(j)}\right\|}\frac{2^k}{\left\|\varepsilon^{(j+1)}\right\|^\alpha} + \frac{\ln E}{-2\ln\left\|\varepsilon^{(j)}\right\|}\frac{1}{-2\ln\left\|\varepsilon^{(j+1)}\right\|}\ln l_0^{(j+1)} \\ + \frac{1}{-2\ln\left\|\varepsilon^{(j)}\right\|}\ln l_0^{(j)}. \tag{11.17}$$

To simplify the first term of (11.17), we apply the inequality $\ln x < x$ and Proposition 11.14. Then

$$\frac{\ln E}{-2\ln\left\|\varepsilon^{(j)}\right\|}\frac{2^k}{\left\|\varepsilon^{(j+1)}\right\|^\alpha} > 2^k\left[\frac{\ln E}{2}\left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{\alpha\left\|\varepsilon^{(j)}\right\|^{-2\alpha}-1}\right] > 2^{k+2}\left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^\alpha.$$

Here, we assume that $\overline{\varepsilon}$ is small enough such that

$$\frac{\ln E}{8} > \left\|\varepsilon^{(j)}\right\|$$

and

$$\alpha\left\|\varepsilon^{(j)}\right\|^{-2\alpha} - 2 > \alpha$$

for all $j \geq 0$.

For the second term of (11.17), we apply Proposition 11.13. Compute

$$\frac{\ln E}{-2\ln\left\|\varepsilon^{(j)}\right\|}\frac{1}{-2\ln\left\|\varepsilon^{(j+1)}\right\|}\ln l_0^{(j+1)} \\ > \frac{\ln E}{2\ln\left\|\varepsilon^{(j+1)}\right\|}m^{(j)} + \frac{\ln E}{-2\ln\left\|\varepsilon^{(j)}\right\|}\frac{1}{-2\ln\left\|\varepsilon^{(j+1)}\right\|}\ln l_0^{(j)}.$$

Combine the results to (11.17), we obtain

$$m^{(j)} > 2^{k+2}\left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^\alpha + \frac{\ln E}{2\ln\left\|\varepsilon^{(j+1)}\right\|}m^{(j)} + \frac{1}{-2\ln\left\|\varepsilon^{(j)}\right\|}\left(1 + \frac{\ln E}{-2\ln\left\|\varepsilon^{(j+1)}\right\|}\right)\ln l_0^{(j)}.$$

Then

$$\left(1 + \frac{\ln E}{-2\ln\left\|\varepsilon^{(j+1)}\right\|}\right)m^{(j)} \\ > 2^{k+2}\left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^\alpha + \frac{1}{-2\ln\left\|\varepsilon^{(j)}\right\|}\left(1 + \frac{\ln E}{-2\ln\left\|\varepsilon^{(j+1)}\right\|}\right)\ln l_0^{(j)}.$$

Solve for $m^{(j)}$, we get

$$m^{(j)} \quad > \quad 2^{k+2}\left(1+\frac{\ln E}{-2\ln\left\|\varepsilon^{(j+1)}\right\|}\right)^{-1}\left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{\alpha}+\frac{1}{-2\ln\left\|\varepsilon^{(j)}\right\|}\ln l_0^{(j)}.$$

To simplify the inequality, we assume that $\overline{\varepsilon}$ is small enough such that

$$\frac{\ln E}{-2\ln\left\|\varepsilon^{(j+1)}\right\|} \le \frac{\ln E}{-2\ln\overline{\varepsilon}} < 1$$

for all $j \ge 0$. Therefore, we showed that the inequality also holds for $k+1$

$$m^{(j)} > \frac{2^{k+1}}{\left\|\varepsilon^{(j)}\right\|^{\alpha}}+\frac{1}{-2\ln\left\|\varepsilon^{(j)}\right\|}\ln l_0^{(j)}$$

and the lemma is proved by induction. □

The lemma shows that the contraction of the size of the bad regions beats the contraction of thicknesses because lower bound of (11.15) tends to infinity as k approaches infinity. This proves that

Proposition 11.17. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exists a constant $\overline{\varepsilon} > 0$ such that for all non-degenerate open maps $F \in \hat{\mathscr{I}}_{\delta}(I^h \times I^v, \overline{\varepsilon})$, the following property holds:*

Let $J \subset A \cup B$ be a square subset of a wandering domain of F and $\left\{J_n^{(j)}\right\}_{n\ge 0, 0\le j\le \overline{j}}$ be a J-double sequence. Then the number of rows $\overline{j}$ is finite.

11.4. Nonexistence of wandering domains

Finally, we prove the main theorem.

Theorem 11.18. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exists a constant $\overline{\varepsilon} > 0$ such that a open map $F \in \mathscr{I}_{\delta}(I^h \times I^v, \overline{\varepsilon})$ does not have wandering domains.*

Proof. Assume that $\overline{\varepsilon} > 0$ is small enough such that Proposition 5.26 holds and $F \in \mathscr{I}_{\delta}(I^h \times I^v, \overline{\varepsilon})$. There exist $0 < \delta_R < \delta$ and $I \Subset I_R^h \subset I^h$ such that $F_n \in \mathscr{H}_{\delta_R}(I_R^h \times I_n^v, \overline{\varepsilon})$ for all $n \ge 0$.

Prove by contradiction. Assume that F has a wandering domain. Let $\overline{\varepsilon}' > 0$ be small enough such that Proposition 11.13 and Proposition 11.17 holds for δ_R and $I_R^h \times I_R^h$. By Proposition 5.26, there exists $N \ge 0$ such that $F_N \in \hat{\mathscr{I}}_{\delta_R}(I_R^h \times I_R^h, \overline{\varepsilon}')$. Set $\hat{F} = F_N|_{I_R^h \times I_R^h}$.

By Corollary 7.4, F_N has a wandering domain J in $D(F_N) \subset I^h(F_N) \times I_N^v$. If $J \subset B(F_N)$, then $J \subset I_R^h \times I_N^v$ and so $F^2(J) \subset B(F_N) \cap \left(I_R^h \times I_R^h\right)$. If $J \subset A(F_N)$, there exists $n > 0$ such that $F^n(J) \subset B(F_N)$ by Proposition 5.16. If $J \subset C(F_N)$, then $F(J) \subset B(F_N)$. Without lose of generality, we may assume that $J \subset B(F_N) \cap \left(I_R^h \times I_R^h\right)$. Hence, $J \subset B(\hat{F})$ is a wandering domain of the restriction $\hat{F}$.

Let $\hat{J}$ be a nonempty square subset of J and $\left\{J_n^{(j)}\right\}_{n\geq 0, 0\leq j\leq \overline{j}}$ be a $\hat{J}$-double sequence. By Proposition 11.17, $\overline{j}$ is finite. Then the second property of Proposition 11.13 implies that

$$\lim_{n\to\infty} l_n^{(\overline{j})} = \infty$$

which is a contraction. Therefore, F does not have wandering domains. $\square$

Remark 11.19. The theorem also applies to infinitely CLM-renormalizable maps. This is because, without loss of generality, we can always start from a Hénon-like map that is close to the map G by the hyperbolicity of the renormalization operator, and all maps that are close to G are renormalizable.

As an immediate consequence, we have

Corollary 11.20. *Given $\delta > 0$ and $I^v \supset I^h \Supset I$. There exists a constant $\overline{\varepsilon} > 0$ such that for any non-degenerate open map $F \in \mathscr{I}_\delta(I^h \times I^v, \overline{\varepsilon})$, the union of the stable manifolds for the period doubling periodic points is dense in the domain.*

From the classification of the ω-limit sets [GvST89, LM11], almost all orbits approach to the renormalization Cantor set which is conjugated to the dyadic adding machine. However, the theorem shows that the orbits that do not approach to the Cantor set form a dense set in the domain.

Nomenclature

Notation	**Description**	**Page**
A, B, C	Partition of the domain D for unimodal map	21
A, B, C	Partition of the domain D for Hénon-like map	43
$c^{(n)}$	Critcal point and its orbit	21
$C_n(j)$	Subpartition for C_n with rescaling level j	52
$C_n^l(j)$	Left component of $C_n(j)$	52
$C_n^r(j)$	Right component of $C_n(j)$	52
D	The Hénon-like map is defined to be a self-map on $D \subset I^h \times I^v$	42
ε	Perturbation component for Hénon-like map	39
F	Hénon-like map	39
f	Unimodal component for Hénon-like map	39
G	Fixed point for R	48
g	Fixed point for R_c	33
H	Nonlinear part of the Hénon rescaling	46
h	x-component for Hénon-like map	39
$\hat{}$	Reflection point	21
$\mathscr{H}$	Class of Hénon-like maps	39
h	Vertical size	74
I^h	Horizontal domain for a Hénon-like map	39
I^v	Vertical domain for a Hénon-like map	39
$\mathscr{I}$	Class of infinite renormalizable unimodal maps	22
$\mathscr{I}_\delta$	Class of infinite renormalizable Hénon-like maps.	48
J_n	J-closest approach	73
k	Level of rescaling	73
K_n	Boundary for good and bad regions	80
l	Horizontal size	74
Λ	Affine part of the Hénon rescaling	47
λ	Universal constant $2.5029\ldots$	32
λ_n	s'_n	48
$n^{(j)}$	Time span in the good region for row j in a double sequence of wandering domain	108
$p_n(j)$	Periodic point with period 2^j for the unimodal map f_n	23

Notation	**Description**	**Page**
$p_n(j)$	Periodic point with period 2^j for the Hénon-Like map F_n	52
ϕ	Hénon rescaling	47
Φ_n^j	Nonlinear rescaling from renormalization level n to $n+j$	48
$q(n)$	Periodic point in C with period 2^n for g	34
$q^C(j)$	$g(q(j))$	54
$q^c(n)$	Periodic point in B with period 2^n for g	34
$q^l(j)$	Negative value of $q^c(j)$	61
$q^r(j)$	Positive value of $q^c(j)$	61
$r(n)$	Level of renormalization of the sequence of wandering domain J_n	73
R_c	Renormalization operator about the critical point	22
R_v	Renormalization operator about the critical value	22
s	Affine part of the Hénon rescaling	47
s_c	Affine rescaling about the critical point	22
s_v	Affine rescaling about the critical value	22
τ	The tip of an infinite renormalizable Hénon-like map	60
$\mathscr{U}$	Class of unimodal maps	21
$\mathscr{U}^r$	Class of renormalizable unimodal maps	21
$\mathscr{U}_\delta$	Class of unimodal maps with holomorphic extension on a δ-neighborhood	39
w	Thickness	101
$W^t(0)$	Local stable manifolds of $p(0)$	42
$W^t(-1)$	Local stable manifolds of $p(-1)$	41
$W_n^t(j)$	Local stable manifold of $p_n(j)$	52

Part II.

Other combinatorics

12. Outline

The proof for the nonexistence of wandering domains is motivated from the case of unimodal maps. In Chapter 13, we present a proof for infinitely renormalizable unimodal maps with stationary combinatorics other than period-doubling combinatorics. The theorem is first proved for the case of admissible combinatorics (Section 13.4) then the result is extended to other stationary combinatorics (except the period-doubling combinatorics) by the shifting trick (Example 13.24). The main goal of this chapter is to introduce a Markov partition formed by gaps and trapping sets (Section 13.3) which is designed to visualize the expansion of the topology. From the expansion of the topology, we prove an estimate on the hyperbolic length (Proposition 13.61): the hyperbolic length of a rescaled orbit of a wandering interval, called the closest approach, expands at a definite rate. The partition and the expansion estimate will be generalized to Hénon-like maps (Section 14.6, 17). As a result, if a wandering interval exists, then the hyperbolic lengths of the orbit elements tend to infinity which leads to a contradiction. Therefore, wandering intervals cannot exist.

In Chapter 14, we give an introduction to the renormalization of Hénon-like maps. Separators, vertical strips, and induced unimodal maps (Section 14.2) are the tools which allow us to study the topology of Hénon-like maps by unimodal maps. The topology of a Hénon-like map is characterized by the local stable manifolds of the periodic points. For a strongly dissipative renormalizable Hénon-like map, the structure of how the local stable manifolds are allocated is similar to a renormalizable unimodal map of the same combinatorics type in the macroscopic scale. Therefore, the Markov partition and its dynamical properties for unimodal maps can be generalized to Hénon-like maps (Section 14.6) by using the same definitions and proofs. Then we give a review of the Hénon-renormalization operator based on the framework developed by Hazard [Haz11]. Most of the other materials in this chapter can be found from the papers [dCLM05, LM11, Haz11].

The proof of the main theorem is covered by Chapters 15, 16, 17, and 18. The idea of the proof is described as follows.

Assume the contrapositive, a Hénon-like map has a wandering domain J. We define a rescaled orbit $\{J_n\}_{n\geq 0}$ of J that closest approaches to the tip by iterating and rescaling J. The orbit is called the J-closest approach (Definition 14.45). Each element is also a wandering domain of some renormalizations of the Hénon-like map. The transition from one to the next sequence element $J_n \to J_{n+1}$ is called one step. The two elements are related by one iteration plus possibly many rescalings. Motivated from the period-doubling case (Part I) and the unimodal case (Chapter 13), our goal is to show that the sizes of the orbit elements tends to infinity and hence wandering domains cannot exist.

From the proof of the period-doubling case, we define the good region and the bad region in Chapter 15. The good region is an area in the domain where the Hénon-like map behaves like an unimodal map; the bad region is an area where it behaves different from a unimodal map. In particular, we will show in Chapter 16 that the topological arguments and the expansion estimates from Chapter 13 can be promoted to Hénon-like maps. On the other hand, a strong contraction occurs whenever an element from a closest approach enters the bad region. The definition and properties of the good region and the bad region from the period-doubling case can be adopted to other stationary combinatorics because the geometric properties used in the definition and proofs are universal for all combinatorics.

In Chapter 16, we define hyperbolic size to study the expansion of the elements from a closest approach. The hyperbolic size measures the relative horizontal size of a set in some larger base set by using the hyperbolic metric. It is a generalization of hyperbolic length to two-dimensions. A class of C^3 curves, called regular curves, is used to measure the hyperbolic size. For each regular curve, the hyperbolic size of a set on the curve is the hyperbolic length of the intersection of the set with the curve. The hyperbolic size of a set is the supremum over the measurements on all regular curves (Definition 16.3 and Figure 16.1).

After defining the hyperbolic size, in Chapter 17, we study the expansion of the orbit elements in the good regions by using hyperbolic size. This replaces the Euclidean expansion estimates in the proof of the period-doubling case. We proved that the restriction of the Hénon-like map or the rescaling map to a regular curve is a map with negative Schwarzian derivative when the Hénon-like map is close to a unimodal map with negative Schwarzian derivative. We also show that the class of regular curves is invariant under iteration and rescaling. The two results show that the hyperbolic size of a set expands under iteration. However, only expansion is not enough to make the whole proof works. To get uniform expansion, we applying the tools of induced unimodal maps (Section 14.2) to generalize the expansion of topology from unimodal maps (Section 13.6) to the Hénon-like maps. Therefore, the hyperbolic sizes of the sequence elements in a closest approach expand uniformly when the elements stay in the good region (Proposition 17.32).

Finally, in Chapter 18, we take care of the contraction of hyperbolic size when a sequence element enters the bad region. The properties from the period-doubling case also applies to arbitrary stationary combinatorics. When a sequence element J_n enters the bad region, the (horizontal) size of the next element J_{n+1} is determined its horizontal cross-section, and the cross-section can be estimated in terms of the area. Thus, a strong contraction on hyperbolic size is applied to the step $J_n \to J_{n+1}$ because the Jacobian of a strongly dissipative Hénon-like map is small. The key observation is the sequence can enter the bad region at most finitely many times (Proposition 18.11) and hence the total amount of contraction is bounded. Of course, the reader can follow the original proof from the period-doubling case to reproduce Proposition 18.11. But here we present a different proof for Proposition 18.11.

To summarize, we study the hyperbolic sizes of the elements in a closest approach. The expansion argument shows that the hyperbolic size expands uniformly when the elements stay in the good regions. However, the size contracts whenever an element enters the bad region. We show that contraction is bounded and hence the hyperbolic size approach infinity. Therefore, a wandering domain cannot exist.

13. Expansion Estimate for Unimodal Maps

In this chapter, we study the topological structure of an infinite renormalizable unimodal map with stationary combinatorics. The topological structure of a map is characterized by its periodic orbits. We will use the periodic orbits to define three types of intervals: cyclic intervals (Section 13.2), trapping intervals, and gaps (Section 13.3). The periodic intervals allows us to define unimodal renormalization. The trapping intervals and gaps form a partition on the domain. We will study the dynamics of the unimodal map on these intervals.

The main goal of this chapter is to introduce an expansion estimate for hyperbolic length (Proposition 13.61). It can be used to reproduce a classical theorem: an infinite renormalizable unimodal maps with stationary combinatorics does not have a wandering interval (Theorem 13.63). For a strongly dissipative infinite renormalizable Hénon-like map with stationary combinatorics, the topological structure is similar to an infinite renormalizable unimodal map with the same combinatorics type in the macroscopic scale. Hence, the three types of intervals and the expansion estimate can be generalize to Hénon-like maps under some proper conditions. The generalization of the expansion estimate is one of the key ingredients for proving the nonexistence of wandering domains for Hénon-like maps.

13.1. Class of Unimodal Maps

Definition 13.1 (Class of unimodal maps)**.** Assume that $\kappa > 0$, $\delta > 0$, and I^h is an interval that contains $[-1,1]$. Denote $\mathscr{U}_{\delta,\kappa}(I^h)$ to be the class of real analytic maps $f : I^h \to I^h$ that has the following properties:

1. The map has a unique critical point c such that $c \in [-1, v-\kappa]$ where $v = f(c)$ is the critical value. The critical point is nondegenerated.

2. The point -1 is the unique expanding fixed point with positive multiplier (derivative) $\lambda \geq 1+\kappa$. It satisfies the identity $f(1) = f(-1) = -1$.

3. The map can be factorized as $f = Q \circ \phi$ where $Q(x) = v-(v+1)x^2$ and ϕ is a $\mathbb{R}$-symmetric univalent map on $I^h(\delta)$.

4. The map has negative Schwarzian derivative.

Denote $\mathscr{U}_\kappa = \mathscr{U}_{0,\kappa}([-1,1])$. This means that the map ϕ in the third condition is a real analytic map on $[-1,1]$.

Given $\overline{\varepsilon} > 0$ and a unimodal map $g \in \mathscr{U}_{\delta,\kappa}(I^h)$. An open $\overline{\varepsilon}$-ball $\mathscr{U}_{\delta,\kappa}(I^h, g, \overline{\varepsilon})$ is defined to be the set of unimodal maps $f \in \mathscr{U}_{\delta,\kappa}(I^h)$ with $\|f-g\|_{I^h(\delta)} < \overline{\varepsilon}$.

The notations defined in this chapter will also be adopted to Hénon-like maps. In the remaining part of the article, we fix $\kappa > 0$ to be a small number and suppress it from the subscript $\mathscr{U}_\delta(I^h) = \mathscr{U}_{\delta,\kappa}(I^h)$. If there are multiple unimodal maps in the discussion, the subscript of will be used to distinguish the objects that belongs to a specific unimodal map. For example, the value c_g is defined to be the critical point of $g \in \mathscr{U}_\delta(I^h)$.

A unimodal map in interest will always have two fixed points, one has a positive multiplier and the other one has a negative multiplier. Denote α to be the fixed point with positive multiplier and β to be the fixed point with negative multiplier. The preimages of the fixed points will contain a "bar" in their notations. In particular, the point $\alpha = -1$ and $\overline{\alpha} = 1$ from the convention in the definition and $\overline{\beta}$ is the point (on the other side of the critical point) such that $f(\overline{\beta}) = \beta$ and $\overline{\beta} \neq \beta$. For an infinitely renormalizable map which is introduced later, there will also be two collections of periodic points. Each collection is associated to the fixed point α and the fixed point β respectively. Those periodic points will also be named by $\alpha(j)$ and $\beta(j)$. It means the periodic points associated to the j-th renormalized map. See definition 13.9. For consistency, set $\alpha(0) = \alpha$, $\overline{\alpha(0)} = \overline{\alpha}$, $\beta(0) = \beta$, and $\overline{\beta(0)} = \overline{\beta}$. In the case of Hénon-like maps, there will also be two types of local stable manifolds that are similar to the period orbits. The two types of local stable manifolds will inherit the same name α and β.

Another object $P(j)$ is defined for the intervals that are invariant under some number of iterates. Similarly, the value j means the invariant interval is associated to the j-th renormalized map. For the class of unimodal maps in the definition, set $P(0) = [\alpha(0), \overline{\alpha(0)}] = [-1, 1]$. This is an interval invariant under one iteration. For a renormalizable unimodal map, the interval $P(1)$ will be defined to be an interval that is invariant under some number of iterations. See Definition 13.2. In the case of Henon-like maps, each of these intervals will be an area bounded by two vertical local stable manifolds which is called a vertical strip.

13.2. Renormalization

To define the renormalization of a unimodal map, we need to find an interval P to define the self-return map on that interval. The construction is as follows.

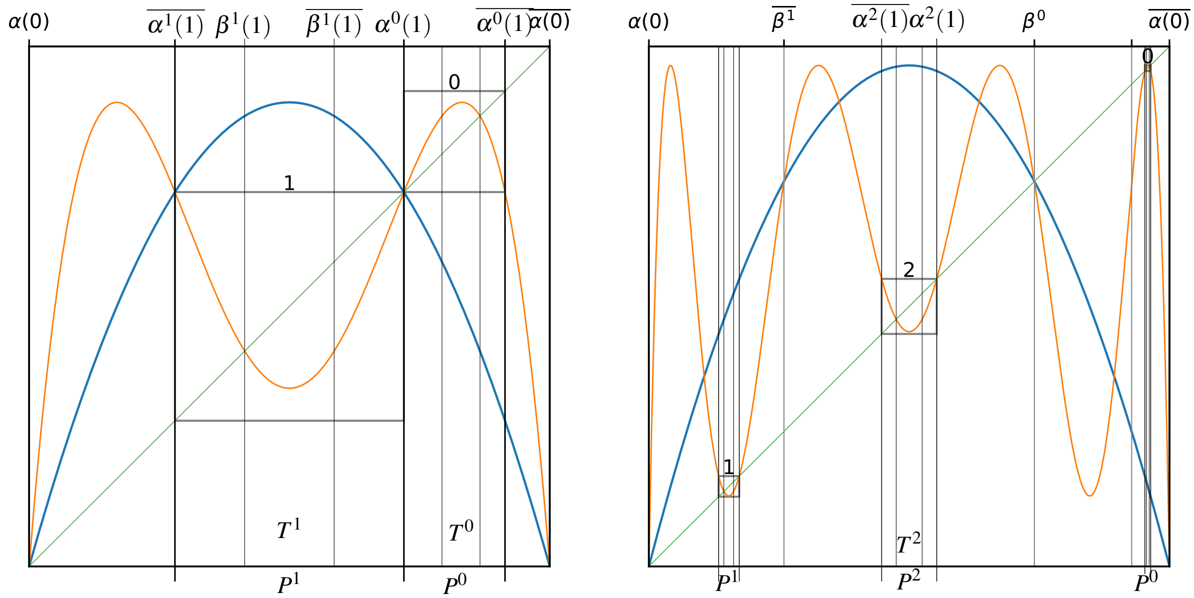
Definition 13.2 (Cycle)**.** Assume that $f \in \mathscr{U}$ has a unique fixed point $\beta = \beta(0)$ with negative multiplier. Also assume that the fixed point β is noncontracting. Set $\overline{\beta(0)}$ be the solution of $f(x) = \beta(0)$ with orientation opposite to $\beta(0)$. There are two cases: $p = 2$ and $p \geq 3$.

For the case $p = 2$, we focus on the fixed point $\beta(0)$. Assume that the multiplier λ of $\beta(0)$ satisfies $\lambda^2 \geq 1 + \kappa$. Set σ be the two cycle on $\mathbb{Z}_2$. Let $\alpha(1) = \alpha^0(1) = \alpha^1(1) = \beta(0)$.

For the case $p \geq 3$, assume that the unimodal map has a periodic orbit of period p with expanding positive multiplier $\lambda \geq 1 + \kappa$. The unimodal map f acts on the periodic orbit like a cyclic permutation σ on $\mathbb{Z}_p$. Let $\alpha(1) \in (\beta(0), \overline{\alpha(0)})$ be the largest point in the orbit and $\alpha^t(1) = f^t(\alpha(1))$ for $t = 0, \cdots, p-1$.

For both cases, define the sequence $\{\overline{\alpha^t(1)}\}_{t=0}^{p-1}$ of orbit such that

1. $f(\overline{\alpha^t(1)}) = \overline{\alpha^{t+1}(1)}$ for $t = 0, \cdots, p-2$,
2. $f(\overline{\alpha^{p-1}(1)}) = \alpha^0(1)$,
3. the map has same orientation at $\alpha^t(1)$ and $\overline{\alpha^t(1)}$ for $t = 0, \cdots, p-2$, and
4. the map has opposite orientation at $\alpha^{p-1}(1)$ and $\overline{\alpha^{p-1}(1)}$.



(a) Period-doubling renormalizable

(b) Period-tripling renormalizable

Figure 13.1.: The cyclic intervals of renormalizable unimodal maps

Set $\overline{\alpha(1)} = \overline{\alpha^0(1)}$.

The sequence $\{P^t = [\alpha^t(1), \overline{\alpha^t(1)}]\}_{t=0}^{p-1}$ is a *cycle* of period p (or of combinatorial type σ) if the interior of the intervals are disjoint and $f(P^{p-1}) \subset P^0$. An interval in a cycle is called a *cyclic interval*. Set $P(1) = P^0$. See Figures 13.1 and 13.1 for illustration.

Remark 13.3. From the definition, a two-cycle exists if and only if $f([\beta(0), v]) \subset [\overline{\beta(0)}, \beta(0)]$.

Assume that $\{P^t\}_{t=0}^{p-1}$ is a sequence of cycle. By definition, the restriction of f to $P^t \to P^{t+1}$ is a diffeomorphism for $t = 0, \cdots, p-2$. Hence, f defines a self-return map on the cycle. This yields the definition of renormalizable.

Definition 13.4 (Renormalizable)**.** A unimodal map $f \in \mathscr{U}$ is said to be *renormalizable* with combinatorial type σ (or σ-renormalizable) if there exists a cycle with combinatorial type σ. The class of renormalizable unimodal maps with combinatorial type σ is denoted as $\mathscr{U}^\sigma$.

A cyclic permutation σ is called a *unimodal permutation* if there is a renormalizable unimodal map of combinatorial type σ.

Two types of renormalization are introduced here. One is the usual unimodal renormalization about the critical point. The cyclic interval P^{p-1} contains the critical point. The renormalization operator is defined to be the affine rescaled first return map $f^p : P^{p-1} \to P^{p-1}$ on P^{p-1}. Another is the Hénon renormalization. The cyclic interval P^0 contains the critical value. The renormalization operator is defined to be the nonlinear rescaled first return map $f^p : P^0 \to P^0$ on P^0. This will be used in Section 14.4 later for Hénon-like maps because Hénon-like maps cannot be renormalized around the critical point. The definitions are stated below.

Definition 13.5. Assume that σ is a unimodal permutation and $f \in \mathscr{U}^\sigma$.

1. The *unimodal renormalization* $R^u : \mathscr{U}^\sigma \to \mathscr{U}$ is defined to be $R^u f = s^u \circ f^p \circ (s^u)^{-1}$ where $s^u : P_f^{p-1} \to P_{R^u f}(0)$ is the affine map such that $s^u(\alpha_f^{p-1}(1)) = \alpha_{R^u f}(0)$ and $s^u(\overline{\alpha_f^{p-1}(1)}) = \overline{\alpha_{R^u f}(0)}$.

2. The *Hénon renormalization* $R^h : \mathscr{U}^\sigma \to \mathscr{U}$ is defined to be $R^h f = \phi \circ f^p \circ \phi^{-1}$ where $\phi : P_f(1) \to P_{R^h f}(0)$ is a diffeomorphism defined by $\phi = s^h \circ f^{p-1}$ and s^h is the affine map that satisfies $\phi(\alpha_f(1)) = \alpha_{R^h f}(0)$ and $\phi(\overline{\alpha_f(1)}) = \overline{\alpha_{R^h f}(0)}$.

One can easily verify from the definition that the two notions of renormalization coincides for unimodal maps.

Proposition 13.6. *Assume that σ is a unimodal permutation and $f \in \mathscr{U}^\sigma$. Then*

1. $s^u = s^h$ *and*

2. $R^u f = R^h f$.

From now on, we remove the super scripts u and h because the two definitions are equivalent.

From the definition, it can be possible that a unimodal map is simultaneously renormalizable under two different combinatorial types. For example, we will show in Section 13.4 that renormalizable maps of combinatorial type $(1,4,3,5,2,6)$ is also period-doubling renormalizable and the renormalization operator is a composition of a period-doubling renormalization with a period-tripling renormalization. Define *prime combinatorics* to be the case when the permutation cannot be decomposed into two nontrivial permutations. If the unimodal map is simultaneous renormalizable under two combinatoric types σ and ρ, then the super-scripts R^σ and R^ρ will be used to distinguish the two different renormalizations if necessary.

Given a unimodal permutation σ and a positive integer j. Denote $\mathscr{U}^{\sigma^j} = \cap_{n=0}^{j-1} R^{-n}\mathscr{U}^\sigma$ to be the class of j-renormalizable unimodal maps with stationary combinatorics σ.

If a unimodal map can be renormalized recursively infinitely many times, then it is called *infinitely renormalizable*. In this paper, we will only consider infinitely renormalizable maps f such that $f, Rf, R^2 f, \cdots$ are all renormalizable with the same combinatorial type. This is called infinitely renormalizable with *stationary combinatorics*. The class of infinitely renormalizable map with stationary combinatorics σ is denoted as $\mathscr{U}^{\sigma^\infty}$. Both prime and periodic combinatorics are covered in the case of stationary combinatorics. When the combinatorics σ is not a prime, it is infinitely renormalizable with *periodic combinatorics*. For a unimodal map $f \in \mathscr{U}^{\sigma^\infty}$, write $f_n = R^n f$. The value n is called the *renormalization level*. The subscript n is used to indicate the associate renormalization level of an object. For example, β_n is the fixed point with negative multiplier for f_n. Also define $\Phi_n^j = \phi_{n+j-1} \circ \cdots \circ \phi_n$ for $j \geq 1$ and $\Phi_n^0 = \text{id}$. The map Φ_n^j is a diffeomorphism that maps from the renormalization level n to level $n+j$.

For each unimodal permutation σ, there exists a unique fixed point of the renormalization operator R with combinatorial type σ. The existence of such map is provided by [EL81, CER82] for period-doubling case and [Sul92] for other combinatorics. The fixed point will be denoted as f_σ with the associate affine rescaling map $s_\sigma(x) = \lambda_\sigma x$ where $|\lambda_\sigma| > 1$. By the hyperbolicity of the renormalization operator [Lyu99], the renormalization $R^n f$ converges to the fixed point geometrically.

For an infinitely renormalizable unimodal map with stationary combinatorics, the critical point and the critical value satisfy the relations:

Proposition 13.7. *Assume that $f \in \mathscr{U}^{\sigma^\infty}$ where σ is a unimodal permutation and $p = |\sigma|$. Then*

1. $c_n = f_n^{p-1} \circ \phi_n^{-1}(c_{n+1})$ *and*
2. $\phi_n(v_n) = v_{n+1}$

for all $n \geq 0$.

Proof. Since c_{n+1} is the critical point of f_{n+1}, we have

$$0 = f_{n+1}'(c_{n+1}) = \left(\phi_n \circ f_n^p \circ \phi_n^{-1}\right)'(c_{n+1}).$$

Then $f_n' \circ f_n^t \circ \phi_n(c_{n+1}) = 0$ for some $t \in \{0, \cdots, p-1\}$ by the chain rule. In fact, $t = p-1$ because f_n is a diffeomorphism from $P_n^0, \cdots, P_n^{p-1}$ to their images. This proves the first equality.

The second equality follows directly from the first equality and the definition of renormalization. □

As a consequence of the first equality, we have the following.

Corollary 13.8. *Assume that $f \in \mathscr{U}^{\sigma^\infty}$ where σ is a unimodal permutation and $p = |\sigma|$. Then c_n belongs to the forward orbit of $\left(\Phi_n^j\right)^{-1}(c_{n+j})$.*

13.3. Topological Structure of a Renormalizable Map

The topological structure of a renormalizable unimodal map is studied in this section. The topology is determined by the orbit of its periodic points. There are two types of periodic point in interested. Each type is related to the fixed point α and the fixed point β respectively. A Markov partition will be defined by using those periodic points. The dynamics of wandering interval will be studied inside the partition.

Definition 13.9 (Periodic Points from deeper levels)**.** Assume that $f \in \mathscr{U}^{\sigma^{N+1}}$ where σ is a unimodal permutation and $N \in \{0, 1, \cdots, \infty\}$.

Define $\alpha(j) = \left(\Phi^j\right)^{-1}(\alpha_j)$, $\overline{\alpha(j)} = \left(\Phi^j\right)^{-1}(\overline{\alpha_j})$, $\beta(j) = \left(\Phi^j\right)^{-1}(\beta_j)$, $\overline{\beta(j)} = \left(\Phi^j\right)^{-1}(\overline{\beta_j})$, and $P(j) = \left(\Phi^j\right)^{-1}(P_j(0))$ for integers $0 \leq j \leq N$. The definition is consistent with the previous definition of $\alpha(j)$, $\overline{\alpha(j)}$, and $P(j)$ for $j = 0$ and 1.

Remark 13.10. For $p \neq 2$, the points $\alpha(j)$ and $\beta(j)$ are both periodic points of f with period p^j.

For $p = 2$, the points coincide $\alpha(1) = \beta(0)$. Unlike the $p \neq 2$ case, the points $\alpha(0)$ and $\alpha(1)$ are both fixed points of f and $\alpha(j)$ is a periodic point of f with period 2^{j-1} for $j \geq 1$.

Remark 13.11. The rescaling map Φ_n^j is a differomorphism from $P_n(j)$ to $P_{n+k}(j-k)$ for $0 \leq k \leq j$.

Next, we focus on the periodic point $\beta(1) \in \operatorname{int}(P^0)$ and its orbit. The point $\beta(1)$ is also a periodic point of period p for f. The orbits $\{\beta^t(1)\}_{t=0}^{p-1}$ and $\{\overline{\beta^t(1)}\}_{t=0}^{p-1}$ are defined to be similar to the orbits $\{\alpha^t(1)\}_{t=0}^{p-1}$ and $\{\overline{\alpha^t(1)}\}_{t=0}^{p-1}$ as follows.

Definition 13.12 (Period orbit $\beta^t(1)$). Assume that $f \in \mathscr{U}^{\sigma^2}$ where σ is a unimodal permutation and $p = |\sigma|$.

Define $\beta^t(1) = f^t(\beta(1))$ and $\overline{\beta^t(1)} = f^t(\overline{\beta(1)})$ for $t = 0, \cdots, p-1$.

For the point $\overline{\beta(1)}$, the preimage $f^{-1}(\overline{\beta(1)})$ contains two points. Define θ^L and θ^R to be the left and right point of the preimage respectively.

Remark 13.13. By definition, the points form an orbit $\theta^L, \theta^R \to \overline{\beta^0(1)} \to \cdots \to \overline{\beta^{p-1}(1)} \to \beta^0(1) \to \cdots \to \beta^{p-1}(1) \to \beta^0(1)$.

By using the periodic orbits, we define a partition on the domain by trapping intervals and gaps.

Definition 13.14 (Trapping Interval and Gap). Assume that $f \in \mathscr{U}^{\sigma^2}$ where σ is a unimodal permutation and $p = |\sigma|$.

A *trapping interval* for f is an interval of the form $T^t = [\beta^t(1), \overline{\beta^t(1)}]$ where $0 \le t \le p-1$.

A *gap* is an interval between two neighboring trapping intervals. Precisely, it is a connected component of $[\overline{\beta^1(1)}, \overline{\beta^0(1)}] \setminus \cup_{t=2}^{p-1} T^t$.

The *center trapping interval* is T^{p-1} and the *center cyclic interval* is P^{p-1}.

See Figure 13.1 for an illustration of the cases $p = 3$ and $p = 5$.

Whether a trapping interval or gap contains its boundaries or not is not important for studying the dynamics of a wandering interval because a wandering interval cannot contain any periodic or pre-periodic point. In the remaining discussion, we will be careless about the boundaries of those intervals.

The next proposition summarizes the topological properties of the trapping intervals, cyclic intervals, and gaps.

Proposition 13.15 (Topology of trapping intervals, cyclic intervals, and gaps). *Assume that $f \in \mathscr{U}^{\sigma^2}$ where σ is a unimodal permutation and $p = |\sigma|$.*

1. *Each cyclic interval P^j contains a unique trapping interval T^j and $T^j \Subset P^j$.*

2. *Any two distinct trapping intervals contains at least one gap in between.*

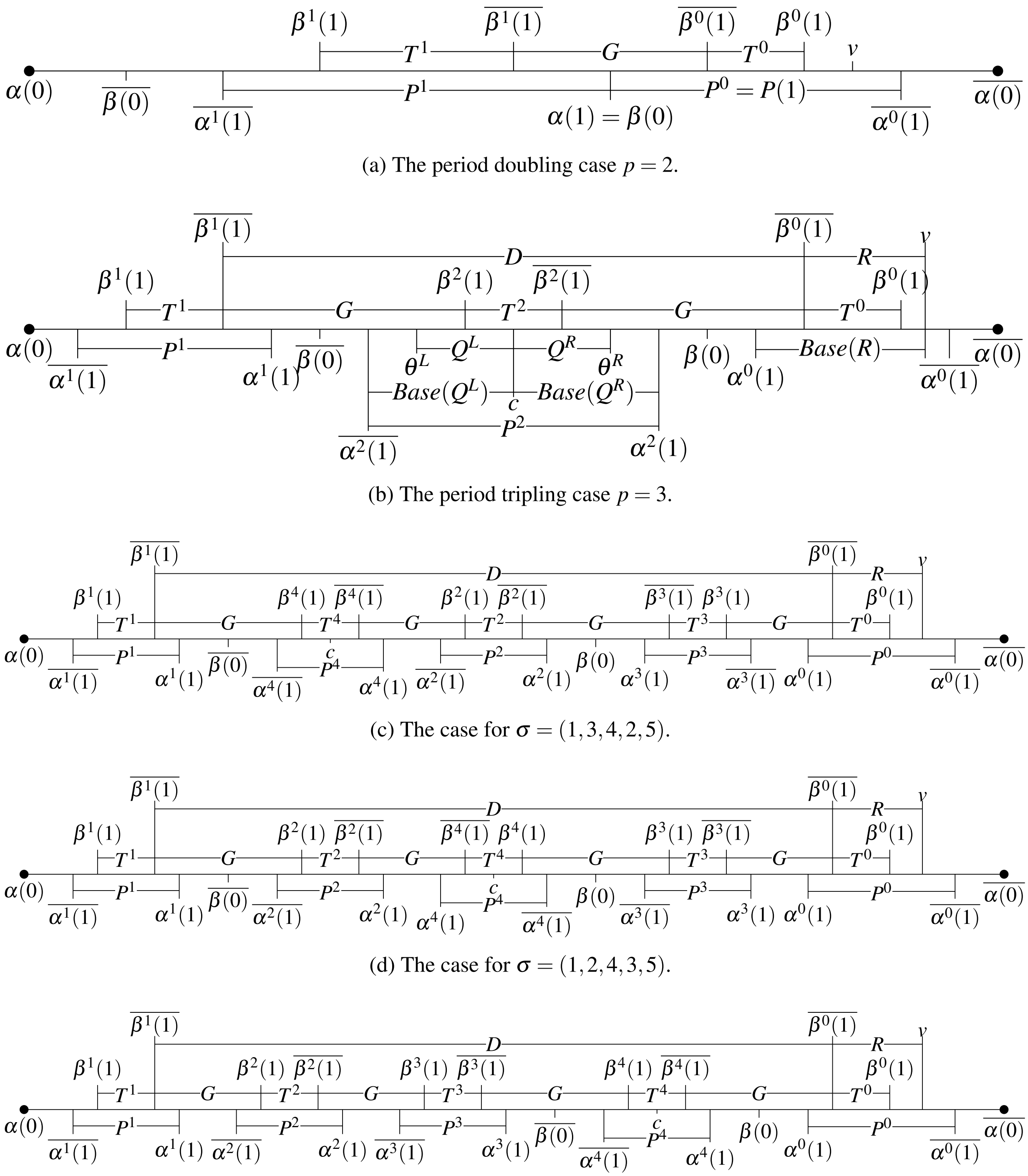
3. *The leftmost cyclic interval is P^1 and the rightmost cyclic interval is P^0. The order of the points in those intervals are $\overline{\alpha^1(1)} < \beta^1(1) < \overline{\beta^1(1)} < \alpha^1(1)$ and $\alpha(1) < \overline{\beta(1)} < \beta(1) < \overline{\alpha(1)}$.*

4. *The unimodal map is monotone on each gap, trapping interval, and cyclic interval except the center trapping interval and the center cyclic interval.*

Proof. The proposition is left to the reader. □

13.4. Admissible Permutation

Admissible permutation is a restriction applying to the combinatorics σ. The condition includes all prime combinatorics with $p \ge 3$ and all non-prime combinatorics with odd order. To proof the nonexistence of wandering domain for Hénon-like maps (or nonexistence of wandering intervals for unimodal maps), we will first prove the case for admissible permutations then extend the result to non admissible permutations with $p \ge 3$ by applying a shifting trick. See Example 13.24 and the proof of Theorem 13.63. The condition is defined as follows.



(a) The period doubling case $p = 2$.

(b) The period tripling case $p = 3$.

(c) The case for $\sigma = (1,3,4,2,5)$.

(d) The case for $\sigma = (1,2,4,3,5)$.

(e) The case for $\sigma = (1,2,3,4,5)$.

Figure 13.1.: The cyclic intervals, trapping intervals, and gaps.

Definition 13.16 (Admissible Permutation)**.** Assume that $\sigma \in \mathbb{Z}_p$ is a unimodal permutation of order p. The permutation σ is *not admissible* if the order $p = 2n$ is even and $\sigma(\{1,\cdots,n\}) = \{n+1,\cdots,2n\}$. This condition automatically implies that $\sigma(\{n+1,\cdots,2n\}) = \{1,\cdots,n\}$. The converse is called *admissible*.

Example 13.17.

1. The first nontrivial period is $p = 4$. The period has two unimodal permutations. The first one is $(1,3,2,4)$. This is not an admissible permutation and the renormalization operator is a composition of two period-doubling renormalization operators. The second one is $(1,2,3,4)$. This is an admissible prime permutation.

2. The second nontrivial period is $p = 6$. The period has five unimodal permutations. The first one is $(1,4,3,5,2,6)$. This is not an admissible permutation and the renormalization operator is a composition of the period-doubling renormalization operator with the period-tripling renormalization operator. The second one is $(1,3,5,2,4,6)$. This is an admissible permutation and the renormalization operator is a composition of the 3-renormalization operator with the 2-renormalization operator. The remaining permutations are $(1,2,4,5,3,6)$, $(1,2,3,5,4,6)$, and $(1,2,3,4,5,6)$. The permutations are all admissible and prime.

The goal of this section is to prove Proposition 13.22. The proposition gives the relation between non admissible permutations and period-doubling renormalization. It's corollary also gives a topological property that will be used in Proposition 13.30 to prove that a rescaled orbit of a wandering interval will always stays in the domain in interested.

The proof of the proposition will be separated into several lemmas. The first lemma proves a topological property for non admissible permutation.

Lemma 13.18. *Assume that σ is a unimodal permutation. If the permutation σ is not admissible then $\overline{\beta_f(0)} < \alpha_f^1(1)$ for all $f \in \mathscr{U}^\sigma$.*

Proof. The lemma is true for $p = 2$ by the definition of period-doubling renormalization. The unimodal permutation for $p = 3$ is admissible. We will only consider the case for $p \geq 4$.

If the conclusion is not true, then $\alpha^1(1) < \overline{\beta(0)}$. The unimodal map is increasing on $[\alpha(0), c]$. After iterating the inequality, we get $\alpha^1(1) < \alpha^2(1) < \beta(0)$. Since $f(x) > x$ for all $x < \beta(0)$ and the cyclic intervals are disjoint, we get $\alpha^2(1) < \alpha^3(1)$. However, this contradicts to σ is not admissible because $\alpha^1(1) < \alpha^2(1) < \alpha^3(1) < \alpha^0(1)$. Therefore, the inequality $\overline{\beta(0)} < \alpha^1(1)$ holds. $\square$

The next lemma proves that the topological property implies period-doubling renormalizable.

Lemma 13.19. *Assume that σ is a unimodal permutation. If $\overline{\beta_f(0)} < \alpha_f^1(1)$ for some unimodal map $f \in \mathscr{U}^\sigma$, then $f([\beta(0),v]) \subset [\overline{\beta(0)},\beta(0)]$. In particular, the unimodal map f is period-doubling renormalizable.*

Proof. By the definition of the fixed point β, we have $f([\overline{\beta(0)},\beta(0)]) = [\beta(0),v]$.

The interval $[\beta(0),v]$ can be decomposed into two subintervals $[\beta(0),v] \subset [\beta(0),\alpha^0(1)] \cup [\alpha^0(1), \overline{\alpha^0(1)}]$. The iteration of the first subinterval becomes $f([\beta(0),\alpha^0(1)]) = [\alpha^1(1),\beta(0)] \subset [\overline{\beta(0)}, \beta(0)]$ by the assumption $\overline{\beta(0)} < \alpha^1(1)$. The iteration of the second interval is $f([\alpha^0(1),\overline{\alpha^0(1)}]) = [\overline{\alpha^1(1)},\alpha^1(1)]$. Since the interior of the interval $[\overline{\alpha^1(1)},\alpha^1(1)]$ cannot contain $\overline{\beta(0)}$ and it is the left most cyclic interval, we get $\overline{\beta(0)} \leq \overline{\alpha^1(1)} < \alpha^1(1) \leq \alpha^{p-1}(1) \leq \beta(0)$ and hence $f([\alpha^0(1), \overline{\alpha^0(1)}]) \subset [\overline{\beta(0)},\beta(0)]$. Therefore, $f([\beta(0),v]) \subset [\overline{\beta(0)},\beta(0)]$. $\square$

Lemma 13.20. *Assume that $f \in \mathscr{U}^\sigma \cap \mathscr{U}^\mu$ where μ is the two cycle, then $P^{p-1} \subset [\overline{\beta(0)}, \beta(0)]$.*

Proof. The lemma is true because the two points $\alpha^{p-1}(1)$ and $\overline{\alpha^{p-1}(1)}$ are the preimage of $\alpha(1)$, $\alpha(1) \subset [\beta(0), v)$, the restrictions $f : [\overline{\beta(0)}, c] \to [\beta(0), v]$ and $f : [c, \beta(0)] \to [\beta(0), v]$ are homeomorphisms, and the intermediate value theorem. $\square$

The last lemma proves that period-doubling renormalizable implies non admissible. In addition, the renormalization operator of a non admissible permutation can be factorized into a period-doubling renormalization with another renormalization.

Lemma 13.21. *Assume that σ is a unimodal permutation. If there exists a σ-renormalizable map $f \in \mathscr{U}^\sigma$ that is also period doubling renormalizable, then σ is not admissible.*

In addition, if $p \neq 2$, then there exists a unique unimodal permutation ρ with order $|\sigma|/2$ such that ρ acts like σ^2 on the even numbers. The period-doubling renormalization $R^\mu f$ is a ρ-renormalizable map and the renormalization operator has a factorization $R^\sigma f = R^\rho \circ R^\mu f$.

Proof. Let μ be the two cycle and $\overline{\overline{\beta(0)}} = \overline{\alpha^{\mu,0}(1)}$. For a period-doubling renormalizable map, the interval $P^{\mu,0} \cup P^{\mu,1} = [\overline{\beta(0)}, \overline{\overline{\beta(0)}}]$ is invariant under iteration. Let $\{P^t = [\alpha^t(1), \overline{\alpha^t(1)}]\}_{t=0}^{p-1}$ be a σ-cycle. Since the center cyclic interval P^{p-1} is contained inside the interval by Lemma 13.20, the whole cycle must stays inside the interval under iteration. Also, the interior of a cyclic interval cannot contain the fixed point $\beta(0)$. Thus, a cyclic interval must belongs to either $P^{\mu,1} = [\overline{\beta(0)}, \beta(0)]$ or $P^{\mu,0} = [\beta(0), \overline{\overline{\beta(0)}}]$. Since the two intervals are mapped to each other under iteration, the permutation σ is not admissible. In fact, one can check that all the odd cyclic intervals $P^1, \cdots, P^{2p-1}$ are in $P^{\mu,1}$ and all the even cyclic intervals $P^0, \cdots, P^{2p-2}$ are in $P^{\mu,0}$.

Assume that $p \neq 2$. Let ϕ^μ be the rescaling for the period-doubling renormalization. Set $\alpha^t_{R^\mu f}(1) = \phi^\mu(\alpha^{2t}(1))$ and $\overline{\alpha^t_{R^\mu f}(1)} = \phi^\mu(\overline{\alpha^{2t}(1)})$ for $t = 0, \cdots, p/2$. The points $\{\alpha^t_{R^\mu f}(1)\}_{t=0}^{p/2}$ form a periodic orbit for $R^\mu f$ and the renormalized map $R^\mu f$ acts on the orbit as a cyclic permutation ρ on $\mathbb{Z}_{p/2}$ that preserves the natural order. One can easily check that the orbit defines a ρ-cycle and $R^\sigma f = R^\rho \circ R^\mu f$. $\square$

By combining the lemmas, we obtain the equivalent relations for non admissible permutations.

Proposition 13.22. *Let σ be a unimodal permutation and μ be the two cycle. The following conditions are equivalent:*

1. *The permutation σ is not admissible.*
2. *For all $f \in \mathscr{U}^\sigma$, the inequality $\overline{\beta_f(0)} < \alpha^1_f(1)$ holds.*
3. *There exists a map $f \in \mathscr{U}^\sigma$ such that the inequality $\overline{\beta_f(0)} < \alpha^1_f(1)$ holds.*
4. *The inclusion $\mathscr{U}^\sigma \subset \mathscr{U}^\mu$ holds.*

In addition, if $p \neq 2$ and one of the conditions holds, there exists a unique unimodal permutation ρ with order $|\sigma|/2$ such that ρ acts like σ^2 on the even numbers. The inclusion $R^\mu \mathscr{U}^\sigma \subset \mathscr{U}^\rho$ holds and the renormalization operator has the factorization $R^\sigma = R^\rho \circ R^\mu$.

Proof. The second property follows from the first property by Lemma 13.18. The second property automatically implies the third property. The last property follows from the second property by Lemma 13.19. And the first property and the factorization of the renormalization operator follows from the last property by Lemma 13.21 or follows from the third property by Lemmas 13.19 and 13.21. □

Similarly, the converse also satisfies the properties.

Corollary 13.23. *Let σ be a unimodal permutation and μ be the two cycle. The following conditions are equivalent:*

1. *The permutation σ is admissible.*
2. *For all $f \in \mathscr{U}^{\sigma}$, the inequality $\alpha_f^1(1) < \overline{\beta_f(0)}$ holds.*
3. *There exists a map $f \in \mathscr{U}^{\sigma}$ such that the inequality $\alpha_f^1(1) < \overline{\beta_f(0)}$ holds.*
4. *The two classes of renormalizable maps are disjoint $\mathscr{U}^{\sigma} \cap \mathscr{U}^{\mu} = \phi$.*

Example 13.24 (Shifting trick)**.** For an infinitely renormalizable unimodal map with non admissible stationary combinatorics $(1,4,3,5,2,6)$ (periodic renormalizable of the two-three type), the proposition says that the period-doubling renormalization of the map is an infinitely renormalizable unimodal map with admissible stationary combinatorics $(1,3,5,2,4,6)$ (periodic renormalizable of the three-two type). Thus, if a unimodal map is not infinitely period-doubling renormalizable, the problem of non admissible combinatorics can always be reduced to the problem of admissible combinatorics by applying some number of period-doubling renormalizations. This is called the *shifting trick*.

13.5. Dynamics of Wandering Intervals

The goal of this section is to study the dynamics of wandering intervals in the Markov partition formed by trapping intervals and gaps. We will focus on the case of admissible permutations due to the second property of Proposition 13.29 and the second property of Proposition 13.30.

First, recall the definition of wandering interval.

Definition 13.25 (Wandering Interval)**.** A *wandering interval* J of a unimodal map $f \in \mathscr{U}$ is a nontrivial interval $\mathrm{int}(J) \neq \phi$ such that

1. the orbit intervals J, $f(J)$, $\cdots$ are pairwise disjoint and
2. the orbit intervals do not tend to a periodic orbit.

From the definition, a wandering interval can be constructed by iteration and rescaling as follows.

Proposition 13.26. *Assume that $f \in \mathscr{U}^{\sigma}$ where σ is a unimodal permutation.*

1. *If $J \subset P(0)$ is a wandering interval of f, then $f(J) \subset P(0)$ is also a wandering interval of f.*

2. *If $J \subset P_f(1)$ is a wandering interval of f, then $\phi(J) \subset P_{Rf}(0)$ is also a wandering interval of Rf.*

Unlike the period-doubling case, the dynamics of a wandering interval will be studied inside the smaller intervals, iteration interval and rescaling interval, instead of the larger intervals, $P(0)$ and $P(1)$. The reason is to visualize the expansion from the topology to gain uniform expansion on the hyperbolic length. The intervals are defined as

Definition 13.27. Assume that $f \in \mathscr{U}^{\sigma^2}$ where σ is a unimodal permutation.

The *iteration interval* is $D = [\overline{\beta^1(1)}, \overline{\beta^0(1)}]$. The *rescaling interval* is $R = [\overline{\beta^0(1)}, v]$.

The *prerescaling interval* is $Q = [\theta^L, \theta^R]$. Its left and right components are $Q^L = [\theta^L, c]$ and $Q^R = [c, \theta^R]$.

See Figure 13.1b for an illustration of the period-tripling case.

Remark 13.28. The iteration interval D and rescaling interval R here are smaller than the sets defined for the period-doubling case (Definition 4.4). In the period-doubling case, the iteration interval is $P(0) \backslash P(1)$ and the rescaling interval is $P(1)$. The adjustments are necessary to obtain uniform expansion of the hyperbolic length.

The rescaling interval R and the prerescaling interval Q satisfy the following properties:

Proposition 13.29. *Assume that $f \in \mathscr{U}^{\sigma^2}$ where σ is a unimodal permutation.*

1. *The restrictions $f : Q^L \to R$ and $f : Q^R \to R$ are homeomorphisms.*

2. *The inclusions $T^{p-1}(1) \Subset Q \Subset P^{p-1}(1)$ hold. In addition, if $p \geq 3$, then $P^{p-1}(1) \Subset D$.*

We will consider wandering intervals that belongs to the iteration interval D or the iteration interval R. A wandering interval will be iterated or rescaled by the following rules:

1. If the wandering interval is in the iteration interval D, then it is iterated by f.

2. If the wandering interval is in the rescaling interval R, then it is rescaled by ϕ.

The next proposition shows that the rescaled orbit of a wandering interval that follows the rule always stays inside the iteration interval and the rescaling interval.

Proposition 13.30. *Assume that $f \in \mathscr{U}^{\sigma^\infty}$ where σ is a unimodal permutation. The iteration and rescaling intervals satisfies the following properties:*

1. *$f_n(D_n) = D_n \cup R_n$.*

2. *In addition, if σ is admissible, then $\phi_n(R_n) \subset D_{n+1} \cup R_{n+1}$.*

Proof. The iteration interval D_n contains two branches $[\overline{\beta_n^1(1)}, c]$ and $[c, \overline{\beta_n^0(1)}]$. The unimodal map f_n maps $[\overline{\beta_n^1(1)}, c]$ homeomorphically to $[\overline{\beta_n^2(1)}, v]$ and maps $[c, \overline{\beta_n^0(1)}]$ homeomorphically to $[\overline{\beta_n^1(1)}, v]$. Thus, the first property follows.

The second property follows from Proposition 13.7 and the second equivalent condition of Corollary 13.23. □

$$
\begin{tikzcd}
 & D_n \arrow[loop above, "f_n"] \arrow[r, hook] & Q_n \arrow[dr, "f_n"] & & D_{n+1} \arrow[loop above, "f_{n+1}"] \arrow[r, hook] & Q_n \arrow[dr, "f_n"] & & \\
R_{n-1} \arrow[ur, "\phi_{n-1}"] \arrow[rrr, "\phi_{n-1}"] & & & R_n \arrow[ur, "\phi_n"] \arrow[rrr, "\phi_n"] & & & R_{n+1} \arrow[r, "\phi_{n+1}"] & \cdots
\end{tikzcd}
$$

(a) Brief diagram of the dynamics in different levels.

$$
\begin{tikzcd}
 & D_n \arrow[rrr, hook] \arrow[drrr] & & & G_n \arrow[loop above, "f_n"] \arrow[r, hook] \arrow[d, Rightarrow, "f_n"] & G_n \cap Q_n \arrow[drr, hook] & & & \\
 & & T_n^2 \arrow[r, "f_n"] & \cdots \arrow[r] & T_n^j \arrow[r, "f_n"] & \cdots \arrow[r, "f_n"] & T_n^{p-1} \arrow[r, hook] & Q_n \arrow[dr, "f_n"] & & \\
R_{n-1} \arrow[uur, Rightarrow, "\phi_{n-1}"] \arrow[rrrrrrrr, "\phi_{n-1}"] & & & & & & & & R_n \arrow[r, "\phi_n"] & \cdots
\end{tikzcd}
$$

(b) Details of the step $D_n \xrightarrow{f_n} D_n$ are illustrated in the rectangle. The double arrows $\Rightarrow$ are the steps where uniform expansion occurs.

Figure 13.1.: The dynamics of a wandering interval on the partition when the permutation is admissible. The arrow $\xrightarrow{f_n}$ means iterated by f_n, $\xrightarrow{\phi_n}$ means rescaled by ϕ_n, and $\hookrightarrow$ means belong to the target interval without iteration or rescaling.

Under the rules, the dynamics of a wandering interval on the partition follows the diagram in Figure 13.1. Figure 13.1a is the diagram obtained from Proposition 13.29 and Proposition 13.30. Figure 13.1b contains the details of the step $D \to D$.

Trapping intervals and gaps form a partition on the iteration interval D. A wandering interval cannot intersect the boundaries of those intervals because it cannot contain periodic or pre-periodic points. Thus, it must be contained fully inside a trapping interval or a gap. In the remaining part of this chapter, we will study the expansion of hyperbolic length on the partition.

If a wandering interval is inside a gap, then it can be mapped either to a gap or to a trapping interval because the boundaries of a gap are mapped to boundaries of gaps and trapping intervals. If a wandering interval is simultaneously inside a gap and the prerescaling interval Q_n, then the gap must be adjacent to the center trapping interval T_n^{p-1}.

If a wandering interval is inside a trapping interval T^j, then the orbit must follows the path $T_n^2 \to \cdots \to T_n^{p-1} \subset Q_n \to R_n$ by definition.

If a wandering interval is inside the prerescaling interval Q_n, first it is mapped into the rescaling interval R_n. Then it gets rescaled into either the iteration interval D_{n+1} or the rescaling interval R_{n+1} of the next level of renormalization. If it enters the rescaling interval again, then we repeat rescaling the wandering interval. Finally, the rescaling stops when it enters the iteration interval D_{n+k} of some renormalization level. The whole process then repeats. The maximal number of possible rescaling is called the rescaling level:

Definition 13.31 (Rescaling level)**.** Assume that $f \in \mathscr{U}^{\sigma^\infty}$ and $J \subset D_n \cup R_n$ is a wandering interval where σ is a unimodal permutation. Define the *rescaling level* $k(J)$ as follows. If $J \subset R_n$, define

$k(J) \geq 1$ to be the maximal integer such that $\Phi_n^{k(J)}(J) \subset D_{n+k(J)}$; if $J \subset Q_n$, define $k(J) = k \circ f_n(J)$; otherwise if $J \subset D_n \backslash Q_n$, define $k(J) = 0$.

It is sufficient to consider the problem of wandering intervals in the smaller domains D and R due to the following proposition.

Proposition 13.32. *Assume that σ is a unimodal permutation and $f \in \mathscr{U}^{\sigma}$. If f has a wandering interval in $P(0)$, then f has a wandering interval in $[\overline{\beta(0)}, \beta(0)]$. In particular, if σ is an admissible permutation and $f \in \mathscr{U}^{\sigma^2}$, then f has a wandering interval in D.*

Proof. The interval $P(0)$ can be partitioned into three subintervals: $[\alpha(0), \overline{\beta(0)}]$, $[\overline{\beta(0)}, \beta(0)]$, and $[\beta(0), \overline{\alpha(0)}]$. The last interval $[\beta(0), \overline{\alpha(0)}]$ is mapped into the first two by definition. The first interval $[\alpha(0), \overline{\beta(0)}]$ is mapped into the first two. By the definition of unimodal maps, we have $\min_{x \in [y, \overline{\beta(0)}]} (f(x) - x) > 0$ for all $y \in (\alpha(0), \overline{\beta(0)}]$. Thus, the orbit of a wandering interval cannot stay in the first interval forever and the proposition is proved by applying Proposition 13.26.

If σ is an admissible permutation, then $[\overline{\beta(0)}, \beta(0)] \subset D$ by the second equivalent condition of Corollary 13.23. $\square$

Finally, we define a rescaled orbit of a wandering interval, called closest approach, by following the rules of iteration and rescaling.

Definition 13.33 (Closest approach)**.** Assume that σ is an admissible unimodal permutation, $f \in \mathscr{U}^{\sigma^\infty}$, and $J \subset D$ does not contain any periodic point and preimages of a periodic point. Define a sequence $\{J_n\}_{n=0}^{\infty}$ and the associate renormalization level $\{r(n)\}_{n=0}^{\infty}$ by induction such that $J_n \subset D_{r(n)}$ for all $n \geq 0$ as follows.

1. Set $J_0 = J$ and $r(0) = 0$.
2. Abbreviate the rescaling level $k_n = k(J_n)$ whenever J_n is defined.
3. If $J_n \subset D_{r(n)} \backslash Q_{r(n)}$, set $J_{n+1} = f_{r(n)}(J_n)$ and $r(n+1) = r(n)$.
4. If $J_n \subset Q_{r(n)}$, set $J_{n+1} = \Phi_{r(n)}^{k_n} \circ f_{r(n)}(J_n)$ and $r(n+1) = r(n) + k_n$.

The transition between two constitutive sequence element $J_n \to J_{n+1}$, one iteration together plus some number of rescaling if possible, is called *one step*. The sequence $\{J_n\}_{n=0}^{\infty}$ is called the rescaled iterations of J that closest approaches to the critical value, or *J-closest approach* for short.

In the remaining sections, we will study the expansion of hyperbolic length for the elements in a closest approach.

13.6. Expansion Estimate

In this section, we will introduce a way of measuring the hyperbolic lengths of the elements in a closest approach. While the elements stay in the trapping intervals and gaps by definition, we will measure the hyperbolic length in a larger interval that is called the base interval. We will show that the base intervals are designed to visualize the expansion of the topology and produces a definite expansion to the hyperbolic length. Our final goal is to show that the hyperbolic lengths of the sequence elements expand at a uniform rate (Proposition 13.61). This will be generalized to Hénon-like maps in Chapter 17.

13.6.1. Expansion from iteration

We first study the expansion of hyperbolic length while the sequence elements in a closest approach stays in the iteration interval D. The iteration interval D is partitioned by trapping intervals and gaps. To measure the hyperbolic length of a wandering interval, an interval, called the base interval, is assigned to each partition element. The hyperbolic length of a wandering interval will be measured inside the base instead of the partition element that it belongs to. It is defined as follows.

Definition 13.34 (Base interval)**.** Assume that $f \in \mathscr{U}^{\sigma^2}$ where σ is a unimodal permutation.

1. If T^j is a trapping interval with $2 \leq j \leq p-1$, define its *base interval* to be $Base(T^j) = P^j$.

2. If G is a gap, define its *base interval* to be $Base(G) = T_L \cup G \cup T_R$ where T_L and T_R are the two adjacent trapping intervals of G.

A *side* of the base interval $Base(S)$ is a connected component of $Base(S)\backslash S$. Each base interval has two sides, one on the left and the other on the right of the partition element S.

Remark 13.35. From Propositions 2.12 and 2.13, the hyperbolic length measured inside the base set is comparable with the Euclidean length because the two sides have definite length.

Definition 13.36. If $J \subset D_n$ is a wandering interval, the *base interval* of the wandering interval is defined to be $Base(J) = Base(S)$ where S a partition element containing J. Denote the *hyperbolic length* of the wandering interval as

$$l(J) = |J|_{Base(J)}.$$

Uniform expansion of hyperbolic length comes from the expansion of the unimodal map's topology. The partition and the base intervals are designed to produce the expansion. A base element contains its associate partition element (trapping interval or gap) and an extra spacing on each side of the partition element. The reason to include the two extra spacing is to gain uniform expansion from the expansion of the topology. When a partition element S is iterated several times, the iterated set expands and not only covers another partition element but also contains its base set (Lemmas 13.42 and 13.44). The two extra spacings from the base set of the original partition element visualizes the expansion of the topology. We will show that the number of iterations for this to happen is bounded and the embedding of a wandering interval from the base set of the original partition element $Base(S)$ to the partition element S yields uniform expansion to the hyperbolic length by applying Proposition 2.17.

Assume the case when a wandering interval belongs to a gap. The image of the gap is partitioned by trapping intervals and gaps which has the form of $GT\cdots GTG$ or $GT\cdots GT^0$ where G represents a gap and T represents a trapping interval. This is because of Lemma 13.44 later and the boundaries of a gap are mapped to boundaries of the trapping intervals and gaps. For the case $G \xrightarrow{f} T^0$, the wandering interval inside G is also contained in Q because $T^0 \subset R$. This is the case when the iterated wandering interval leaves the iteration interval D and will be studied later in Section 13.6.2 for the case of $Q \xrightarrow{f} R$. The case $G \xrightarrow{f} T^1$ cannot happen because the iteration interval does not contain T^1 and Proposition 13.30. Thus, there are only two possible itineraries when a wandering interval is inside a gap: $G \xrightarrow{f} G$ and $G \xrightarrow{f} T^j$ for $j = 2, \cdots, p-1$.

$$
\begin{tikzcd}
 & G_0 \arrow[r, Rightarrow, "f_0"] & \cdots \arrow[r, Rightarrow, "f_0", "\text{Cor 13.41\&13.43}"'] & G_0 \arrow[d, Rightarrow, "f_0", "\text{Cor 13.45}"'] \arrow[rrr, hook, "\text{Cor 13.52}"] & & & Q_0 \\
D_0 \arrow[ur, hook] \arrow[rrr, hook] & & & T_0^j \arrow[r, "f_0"] & \cdots \arrow[r, "f_0", "\text{Cor 13.47}"'] & T_0^{p-1} \arrow[r, hook, Rightarrow, "\text{Cor 13.54}"'] & Q_0
\end{tikzcd}
$$

$$
\begin{tikzcd}
Q_0 \arrow[r, "f_0", "\text{Cor 13.56}"'] & R_0 \arrow[r, "\phi_0"] & \cdots \arrow[r, "\phi_{n-2}", "\text{Cor 13.58}"'] & R_{n-1} \arrow[r, Rightarrow, "\phi_{n-1}", "\text{Cor 13.60}"'] & D_n \arrow[r] & \cdots
\end{tikzcd}
$$

Figure 13.1.: A summary of all expansion estimates. The arrow $\to$ represents expansion and the arrow $\Rightarrow$ represents uniform expansion.

Assume the case that a wandering interval belongs to a trapping interval T^j with $j \neq 0, 1, p-1$. The map f maps $T^j \to T^{j+1}$ bijectively by definition (Lemma 13.46).

In the remaining part of this section, the expansion of topology will be studied separately in the three cases:

1. $G \xrightarrow{f} G$,
2. $G \xrightarrow{f} T^j$ with $j = 2, \cdots, p-1$, and
3. $T^j \xrightarrow{f} T^{j+1}$ with $j = 2, \cdots, p-2$.

The expansion estimates are summarized in Figure 13.1. The goal of this section is to build of the proof for the next two propositions.

Proposition 13.37 (Expansion for one iteration)**.** *Assume that $f \in \mathscr{U}^{\sigma^2}$ where σ is a unimodal permutation and J is a wandering interval of f. If the two intervals J and $f(J)$ are both in D, then*

$$l(f(J)) \geq l(J).$$

Proof. The proof follows by Corollary 13.41 ($G \xrightarrow{f} G$), Corollary 13.45 ($G \overset{f}{\Rightarrow} T^j$), and Corollary 13.47 ($T^j \xrightarrow{f} T^{j+1}$) later. $\square$

Proposition 13.38 (Uniform expansion for iterations)**.** *Assume that σ is a unimodal permutation, $f \in \mathscr{U}^{\sigma^2}$, $K > 0$ is a constant such that $\left|T^j\right| \geq K$ for all $0 \leq j \leq p-1$, and J is a wandering interval of f. If the intervals $J, f(J), \cdots, f^p(J)$ are all in D, then*

$$l(f^p(J)) \geq E \cdot l(J) \tag{13.1}$$

for some constant $E > 1$ that depends only on K.

Proof. Assume the expansion theorems in the later subsections.

If $J, f(J), \cdots, f^p(J)$ all belong to gaps, then (13.1) follows by Corollary 13.43 ($G \overset{f^p}{\Rightarrow} G$).

If $f^t(J) \subset T^j$ for some $0 \leq t \leq p-1$, let t be the smallest integer. The integer $t \neq 0$ because T^0 and T^1 are not in D. This means that the prior intervals $J, \cdots, f^{t-1}(J)$ belongs to some gaps and the later intervals $f^t(J), \cdots, f^p(J)$ belong to some trapping intervals. Then

$$l(f^{t-1}(J)) \geq \cdots \geq l(J) \tag{13.2}$$

by Corollary 13.41 ($G \xrightarrow{f} G$),

$$l(f^t(J)) \geq E \cdot l(f^{t-1}(J)) \tag{13.3}$$

by Corollary 13.45 ($G \overset{f}{\Rightarrow} T$), and

$$l(f^p(J)) \geq \cdots \geq l(f^t(J)) \tag{13.4}$$

by Corollary 13.47 ($T^j \xrightarrow{f} T^{j+1}$). The inequality (13.1) follows by combining (13.2), (13.3), and (13.4).

It follows from the proposition that □

Corollary 13.39. *Given an admissible unimodal permutation σ and a unimodal map $f \in \mathscr{U}^{\sigma^2}$. If the unimodal map f has a wandering interval in the iteration interval D, then it also has a wandering interval in the rescaling interval R.*

Proof. Assume that f has an wandering interval $J \subset D$. Set $K = \min_{0 \leq j \leq p-1} \left|T^j\right|$.

By Proposition 13.38, the hyperbolic length of the orbit of J diverges to infinity if the orbit stays in the iteration interval D forever. This cannot happen because the hyperbolic length of the partition elements are uniformly bounded in their base intervals. Thus, the orbit of the wandering interval eventually leaves D, i.e. there exists an integer $n > 0$ such that $f^n(J) \subset R \subset P(1)$. By Proposition 13.26, the set $f^n(J)$ is a wandering interval in R. □

The case $G \xrightarrow{f} G$ The first lemma shows the expansion of the topology for the case $G \xrightarrow{f} G$ in one iteration.

Lemma 13.40 (Topological expansion for $G \xrightarrow{f} G$)**.** *Assume that σ is a unimodal permutation, $f \in \mathscr{U}^{\sigma^2}$, and the intervals G^1 and G^2 are gaps. If $f(G^1)$ contains G^2, then there exists an interval $I \subset Base(G^1)$ such that the interval is disjoint from T^{p-1} and $Base(G^2) \subset f(I)$.*

Proof. Let T_L and T_R be the two trapping intervals adjacent to G^1.

If both T_L and T_R are not T^{p-1}, then set $I = Base(G^1)$.

Otherwise, without lose of generality, assume that $T_R = T^{p-1}$. Set $I = T_L \cup G^1$. Then $f(I)$ is bounded between the two trapping intervals $f(T_L)$ and T_0. See Figure 13.2. Thus, $f(I) \supset Base(G^2)$. □

From the topological expansion, one deduce the expansion of hyperbolic length as follows.

Corollary 13.41 (Expansion for $G \xrightarrow{f} G$)**.** *Assume that σ is a unimodal permutation, $f \in \mathscr{U}^{\sigma^2}$, and the intervals G^1 and G^2 are gaps. If $J \subset G^1$ and $f(J) \subset G^2$, then*

$$|f(J)|_{Base(G^2)} \geq |J|_{Base(G^1)}.$$

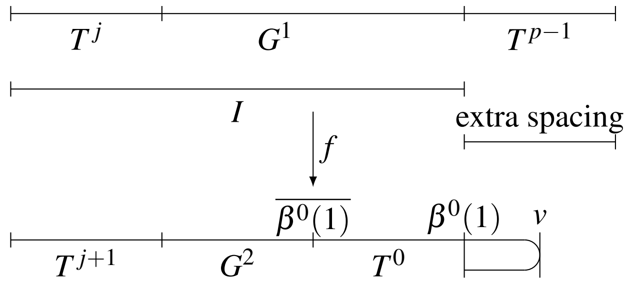


Figure 13.2.: The case of $G \xrightarrow{f} G$ when T^{p-1} is adjacent to one of the boundaries.

Proof. Let I be given by Lemma 13.40. Since f has negative Schwarzian derivative and f is injective on I, we get

$$|f(J)|_{Base(G^2)} \geq |f(J)|_{f(I)} \geq |J|_I \geq |J|_{Base(G^1)}$$

by Proposition 2.10 and Corollary 2.15. □

Uniform expansion may not be guaranteed from one iteration. The proof of Lemma 13.40 only shows the possibility of getting a extra spacing from one side of the base interval. To obtain uniform expansion, it requires extra spacing from both sides of the base set by Proposition 2.17. We will see that extra spacings on both sides can be obtained in a bounded number of iterations by tracking the orbit of the boundaries.

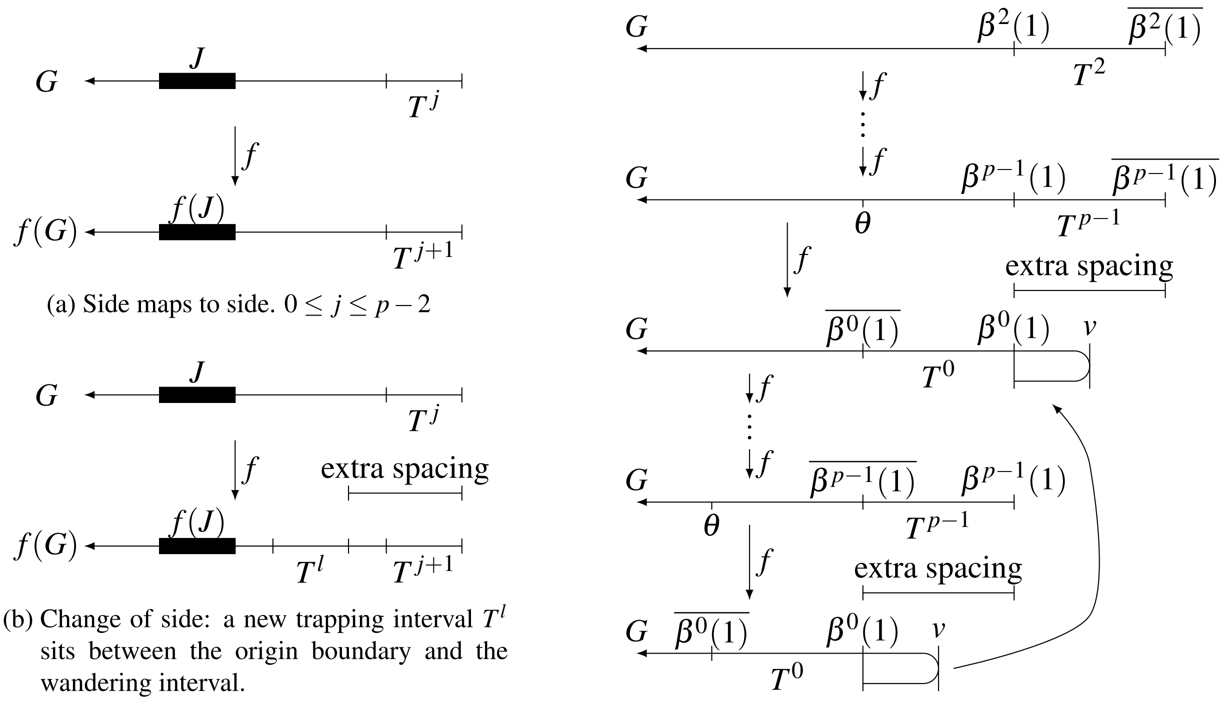
Assume that a wandering interval J belongs to a gap G. The two sides of its base set $Base(G)$ are trapping intervals. Fix a side T^j and study the orbit of that side. Under iteration, there are two cases:

1. no trapping interval sits between $f(J)$ and the side T^{j+1} (Figure 13.3a) and
2. a new trapping interval T^l sits between $f(J)$ and the side T^{j+1} (Figure 13.3b).

For the first case, the adjacent trapping interval T^{j+1} of the gap that contains $f(J)$ is inherited from the adjacent trapping interval T^j of the original gap G. No extra spacing is gained from the base interval $Base(G)$ on that side.

For the second case, the side of $Base(J)$ is changed. The new adjacent trapping interval T^l sits between the wandering interval and the inherited adjacent trapping interval T^{j+1}. The side T^j can be excluded when comparing the hyperbolic length between J and $f(J)$. Thus, an extra spacing T^j is obtained from the base interval $Base(G)$ whenever the side is changed (comes from a different orbit).

Furthermore, both sides are changed within p iterations as shown in Figure 13.3c. The figure shows the orbit of a fixed side of a gap. The orbit starts from the gap containing the boundary $\beta^2(1)$ because no gap contains the boundary $\beta^1(1)$. The side that shares the boundary $\beta^2(1)$ is T^2. After $p-3$ iterations, the side becomes T^{p-1} (second row of Figure 13.3c). Then we iterate the gap and the trapping interval T^{p-1} again. In this step, the trapping interval T^{p-1} is mapped to $\left[\beta^0(1), v\right] \subset R$ and T^0 is adjacent to the interval. If the orbit of a wandering interval still stays inside a gap, then the new trapping interval T^0 sits between the wandering interval and the inherited side $\left[\beta^0(1), v\right]$. Thus, an extra spacing is gained from T^{p-1}. This procedure repeats in



(a) Side maps to side. $0 \le j \le p-2$

(b) Change of side: a new trapping interval T^l sits between the origin boundary and the wandering interval.

(c) Orbit of a side. The superscript of θ is not labeled because it depends on the combinatorics σ.

Figure 13.3.: Iteration on one side of a gap.

every p-iterations if no trapping interval T^j with $j \neq 0$ sits between the wandering interval and the boundary. Therefore, an extra spacing from a side is gained within p iterations.

This argument visualizes the expansion of topology in a bounded number of iterations. It is summarized as follows.

Lemma 13.42 (Topological expansion for $G \overset{f^p}{\Rightarrow} G$)**.** *Assume that σ is a unimodal permutation, $f \in \mathscr{U}^{\sigma^2}$, and $\{G^j\}_{j=0}^{p}$ is a sequence of gaps. If $f(G^j) \supset G^{j+1}$ for all $0 \le j \le p-1$ then there exists a sequence of intervals $\{I^j\}_{j=0}^{p}$ such that*

1. *the intervals I^j are disjoint from the center trapping interval T^{p-1},*

2. *$f(I^j) \supset I^{j+1}$ for $j = 0, \cdots, p-1$,*

3. *$I^0 = G^0$, $I^j \supset G^j$ for $j = 1, \cdots, p-1$, and $I^p = Base(G^p)$.*

Proof. First, we define the intervals I^j. Write $G^j = [a^j, b^j]$ for $0 \le j \le p$ so that $f(a^j) \le a^{j+1} < b^{j+1} \le f(b^j)$ if $f(a^j) < f(b^j)$ and $f(a^j) \ge a^{j+1} > b^{j+1} \ge f(b^j)$ if $f(a^j) > f(b^j)$. Also, set T_L^j and T_R^j be the trapping intervals adjacent to G^j such that a^j is the common boundary of T_L^j and G^j

and b^j is the common boundary of G^j and T_R^j. Let

$$L^j = \begin{cases} T_L^j & \text{if } a^{j+1} = f(a^j), \cdots, a^p = f(a^{p-1}), \text{ and} \\ \phi & \text{otherwise} \end{cases}$$

and

$$R^j = \begin{cases} T_R^j & \text{if } b^{j+1} = f(b^j), \cdots, b^p = f(b^{p-1}), \text{ and} \\ \phi & \text{otherwise.} \end{cases}$$

Define $I^j = L^j \cup G^j \cup R^j$.

By definition, $I^j \subset Base(G^j)$ for $j = 0, \cdots, p-1$ and $I^p = Base(G^p)$.

To show that the intervals I^j are disjoint from the center trapping interval T^{p-1}, it suffice to prove that L^j and R^j are not the center trapping interval T^{p-1}. If $T_L^j = T^{p-1}$, then $a^j \in \{\beta^{p-1}(1), \overline{\beta^{p-1}(1)}\}$ and

$$a^{j+1} \leq \overline{\beta^0(1)} \lneq \beta^0(1) = f(a^j).$$

Thus, $L^j = \phi$ by definition. The case when $T_R^j = T^{p-1}$ is similar.

Now we prove the second property $I^{j+1} \subset f(I^j)$. If $T_L^{j+1} = f(T_L^j)$, then either $L^{j+1} = f(L^j) \neq \phi$ or $L^{j+1} = L^j = \phi$. If $T_L^{j+1} \neq f(T_L^j)$, then $T_L^{j+1} \subset f(G^j)$ because $G^{j+1} \subset f(G^j)$. For any of those cases, we get $L^{j+1} \subset f(L^j \cup G^j)$. Similarly, $R^{j+1} \subset f(R^j \cup G^j)$. Consequently, $I^{j+1} \subset f(I^j)$.

Finally, the last property follows from $L^0 = R^0 = \phi$ because the change of side $a^{j+1} \neq f(a^j)$ or $b^{j+1} \neq f(b^j)$ (Figure 13.3b) happens within p iterations. □

From the expansion of topology, the uniform expansion of hyperbolic length is obtained as follows.

Corollary 13.43 (Uniform expansion for $G \overset{f^p}{\Rightarrow} G$)**.** *Assume that σ is a unimodal permutation, $f \in \mathscr{U}^{\sigma^2}$, $K > 0$ is a constant such that $\left|T^j\right| \geq K$ for all $0 \leq j \leq p-1$, and $\left\{G^j\right\}_{j=0}^{p}$ is a sequence of gaps. If J is a set such that $f^j(J) \subset G^j$ for $0 \leq j \leq p$, then*

$$|f^p(J)|_{Base(G^p)} > E\,|J|_{Base(G^0)}$$

for some constant $E > 1$ that depends only on K.

Proof. Let $\left\{I^j\right\}_{j=0}^{p}$ be given by Lemma 13.42. Let L and R be the left and the right component of $Base(G^0) \backslash I^0$ respectively. Then, $\frac{L}{Base(G^0)}, \frac{R}{Base(G^0)} > \frac{K}{2}$. By Proposition 2.17, we get

$$|J|_{I^0} > E\,|J|_{Base(G^0)} \tag{13.5}$$

for some constant $E > 1$ that depends on K.

Since f has negative Schwarzian derivative and f is injective on I_j for all j, we obtain

$$|f^p(J)|_{Base(G^p)} \geq |f^p(J)|_{f(I^{p-1})} \geq \left|f^{p-1}(J)\right|_{I^{p-1}} \geq \left|f^{p-1}(J)\right|_{f(I^{p-2})} \geq \cdots \geq |J|_{I^0} \tag{13.6}$$

by Proposition 2.10 and Corollary 2.15.

The corollary follows by combining (13.5) and (13.6). □

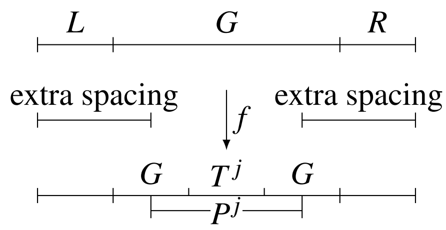


Figure 13.4.: The case $G \to T^j$.

The case $G \xrightarrow{f} T$. The next lemma shows if the image of a gap contains a trapping interval T^j with $j \neq 0$, then it also contains its associated cyclic interval P^j. This also shows that the allocation of the image of a gap must be either $GT \cdots GTG$ or $GT \cdots GTGT^0$ where G represents a gap and T represents a trapping interval.

Lemma 13.44 (Topological expansion for $G \overset{f}{\Rightarrow} T$)**.** *Assume that σ is a unimodal permutation, $f \in \mathscr{U}^{\sigma^2}$, and G is a gap. If $f(G)$ contains a trapping interval T^j with $j \neq 0$, then $f(G) \supset P^j$.*

Proof. First we prove that $f(G)$ and T^j does not share a common boundary by contradiction. Assume that $f(G)$ and T^j has a common boundary.

If the common boundary has the form $\beta^j(1)$ for $1 \leq j \leq p-1$, then $\beta^{j-1}(1)$ must be a boundary of G. This is because the boundaries of G belongs to $\{\beta^0(1), \cdots, \beta^{p-1}(1), \overline{\beta^0(1)}, \cdots, \overline{\beta^{p-1}(1)}\}$ and the points follow the orbit

$$\overline{\beta^0(1)} \to \cdots \overline{\beta^{p-1}(1)} \to \beta^0(1) \to \cdots \to \beta^{p-1}(1) \to \beta^0(1).$$

Thus, G and T^{j-1} shares the same boundary $\beta^{j-1}(1)$ and $j-1 \neq p-1$. By Proposition 13.15, f has the same orientation on the both intervals G and T^{j-1}. It implies that the gap G contains T^{j-1} because $f(G)$ contains T^j. This is impossible by the definition of gaps. Thus, $f(G)$ and T cannot share a common boundary.

The case when $f(G)$ and T has a common boundary of the form $\overline{\beta^j(1)}$ is similar.

Therefore, $f(G) \Supset T^j$. It follows that $f(G) \supset P^j$ because the only trapping interval that intersects P^j is T^j and the boundaries of T^j is in the interior of $f(G)$. $\square$

The corollary provides the uniform expansion to the hyperbolic length as follows.

Corollary 13.45 (Uniform expansion for $G \overset{f}{\Rightarrow} T$)**.** *Assume that σ is a unimodal permutation, $f \in \mathscr{U}^{\sigma^2}$, K is a constant such that $\left|T^j\right| \geq K$ for all $0 \leq j \leq p-1$, and G is a gap. If $J \subset G$ and $f(J) \subset T^j$ with $j \neq 0$, then*

$$|f(J)|_{Base(T^j)} > E\,|J|_{Base(G)}$$

for some constant $E > 1$ that depends only on K.

Proof. Let L and R be the left and the right component of $Base(G) \backslash G$ respectively. See Figure 13.4. Then, $\frac{L}{Base(G)}, \frac{R}{Base(G)} > \frac{K}{2}$. By Proposition 2.17, we get

$$|J|_G > E\,|J|_{Base(G)}$$

for some constant $E > 1$ that depends on K.

Also, since f has negative Schwarzian derivative and f is injective on G, we obtain

$$|f(J)|_{Base(T^j)} \geq |f(J)|_{f(G)} \geq |J|_G$$

by Proposition 2.10, Corollary 2.15, and Lemma 13.44. This completes the proof of the corollary. □

The case $T \xrightarrow{f} T$ The next lemma studies the steps $T^1 \xrightarrow{f} \cdots \xrightarrow{f} T^{p-1}$ in Figure 13.1.

Lemma 13.46. *Assume that $f \in \mathscr{U}^{\sigma^2}$ where σ is a unimodal permutation. The following iterations are diffeomorphisms.*

$$T^0 \xrightarrow{f} T^1 \xrightarrow{f} \cdots \xrightarrow{f} T^{p-1}$$

$$P^0 \xrightarrow{f} P^1 \xrightarrow{f} \cdots \xrightarrow{f} P^{p-1}$$

Proof. The lemma follows from the definition of the intervals. □

Corollary 13.47 (Expansion for $T \xrightarrow{f} T$)**.** *Assume that $f \in \mathscr{U}^{\sigma^2}$ where σ is a unimodal permutation. If $J \subset T^j$ where $0 \leq j \leq p-2$, then*

$$|f(J)|_{Base(T^{j+1})} > |J|_{Base(T^j)}\,.$$

Proof. The corollary follows from Lemma 13.46 and Proposition 2.10. □

13.6.2. Expansion from rescaling

We then study the expansion of hyperbolic length when a sequence element of a closest approach enters the rescaling interval R. To study the hyperbolic length in the rescaling and prerescaling interval, the hyperbolic length will also be measured inside of the base interval instead of the rescaling and prerescaling interval itself. The base intervals are defined as follows.

Definition 13.48 (Base interval)**.** Assume that $f \in \mathscr{U}^{\sigma^2}(I^h)$ where σ is a unimodal permutation.

1. For the prerescaling interval Q, define the *base intervals* to be $Base(Q^L) = [\alpha^{p-1,L}(1), c]$ and $Base(Q^R) = [c, \alpha^{p-1,R}(1)]$ where $\alpha^{p-1,L}(1), \alpha^{p-1,R}(1) \in \{\alpha^{p-1}(1), \overline{\alpha^{p-1}(1)}\}$ such that $\alpha^{p-1,L}(1) < \alpha^{p-1,R}(1)$.
2. For the rescaling interval R, define its *base interval* to be $Base(R) = [\alpha^0(1), v]$.

See Figure 13.1b for an illustration of the period tripling case.

Remark 13.49. The definition of base interval for the prerescaling and rescaling interval cannot be generalized directly to Hénon-like maps. It is because a Hénon-like map does not have critical point. See Definition 17.16 and Remark 17.17 later for Hénon-like maps.

When an element enters the prerescaling interval, the step follows the path

$$D_n \cap Q_n \hookrightarrow Q_n \xrightarrow{f_n} R_n \xrightarrow{\phi_n} \cdots \xrightarrow{\phi_{n+k-2}} R_{n+k-1} \overset{\phi_{n+k-1}}{\Rightarrow} D_{n+k}$$

where k is the rescaling level of the element. The step that involves rescaling is separated into three itineraries

1. $G, T^{p-1} \hookrightarrow Q$,
2. $Q \xrightarrow{f} R$,
3. $R_n \xrightarrow{\phi_n} R_{n+1}$, and $R_n \overset{\phi_n}{\Rightarrow} D_{n+1}$.

In the remaining part of this section, the expansion of topology will be analyzed separately in the three cases. The expansion theorems are summarized in Figure 13.1. The goal of this section is to prove the following proposition.

Proposition 13.50 (Uniform expansion for rescaling)**.** *Assume that σ is a unimodal permutation, $f \in \mathscr{U}^{\sigma^2}$, $K \le \left|\beta^1_{n+1}(0) - \alpha_{n+1}(0)\right|, \left|v_{n+1} - \beta^0_{n+1}(1)\right|$ for some constant $K > 0$, and J is a wandering domain of f_n. If $J \subset Q_n$ and $c_n \notin J$, then*

$$l(\Phi_n^{k(J)} \circ f_n(J)) \ge E \cdot l(J)$$

for some constant $E > 1$ that depends only on K.

Proof. Assume the expansion theorems in the later subsections.

Since $c_n \notin J$, we have $J \subset Q^i_n$ where $i = L$ or R. By Corollary 13.52 ($G \hookrightarrow Q$) and Corollary 13.54 ($T^{p-1} \hookrightarrow Q$), we have

$$|J|_{Base(Q^i_n)} \ge l(J). \tag{13.7}$$

By Corollary 13.56 ($Q \xrightarrow{f} R$), we have

$$|f_n(J)|_{Base(R_n)} \ge |J|_{Base(Q^i_n)}. \tag{13.8}$$

Also by Corollary 13.58 ($R_n \xrightarrow{\phi_n} R_{n+1}$) and Corollary 13.60 ($R_n \xrightarrow{\phi_n} D_{n+1}$), we get

$$l(\Phi_n^{k(J)} \circ f_n(J)) \ge E \cdot |f_n(J)|_{Base(R_n)} \tag{13.9}$$

where the constant $E > 1$ is obtained by Corollary 13.60 ($R_n \overset{\phi_n}{\Rightarrow} D_{n+1}$).

The lemma follows by combining (13.7), (13.8), and (13.9). □

The case $G \hookrightarrow Q$. The following lemma allows us to convert the hyperbolic length from the base of G to Q.

Lemma 13.51. *Assume that σ is a unimodal permutation, $f \in \mathscr{U}^{\sigma^2}$, and G is a gap. If $G \cap Q^i \neq \phi$ where $i = L$ or R, then $Base(Q^i) \subset Base(G)$.*

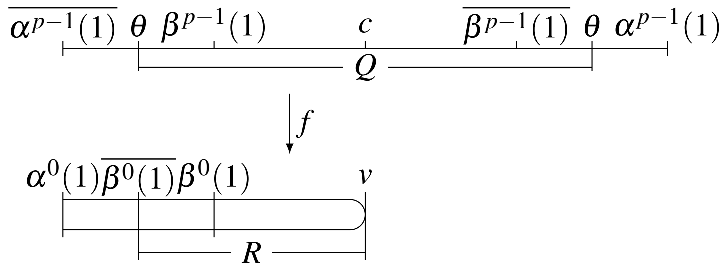


Figure 13.5.: The case $Q \to R$.

Proof. If $G \cap Q^i \neq \phi$, then T^{p-1} is one of the neighbor trapping interval of G and $G \supset [\alpha^{p-1,i}(1), \beta^{p-1,i}(1)]$. Thus

$$Base(Q^i) = [\alpha^{p-1,i}(1), \beta^{p-1,i}(1)] \cup [\beta^{p-1,i}(1), c] \subset G \cup T^{p-1} \subset Base(G).$$

□

Corollary 13.52 (Expansion for $G \hookrightarrow Q$)**.** *Assume that σ is a unimodal permutation, $f \in \mathscr{U}^{\sigma^2}$, and G is a gap. If $J \subset G \cap Q^i$ where $i = L$ or R, then*

$$|J|_{Base(Q^i)} \geq |J|_{Base(G)}.$$

Proof. The corollary follows directly by Lemma 13.51 and Corollary 2.15. □

The case $T^{p-1} \hookrightarrow Q$. The following lemma allows us to convert the hyperbolic length from the base of T^{p-1} to Q.

Lemma 13.53. *Assume that σ is a unimodal permutation and $f \in \mathscr{U}^{\sigma^2}$. Then $Base(Q^L), Base(Q^R) \subset Base(T^{p-1})$.*

Proof. The lemma follows by the definition of the intervals. □

Corollary 13.54 (Expansion for $T^{p-1} \hookrightarrow Q$)**.** *Assume that σ is a unimodal permutation and $f \in \mathscr{U}^{\sigma^2}$. If $J \subset T^{p-1} \cap Q^i$ where $i = L$ or R, then*

$$|J|_{Base(Q^i)} \geq |J|_{Base(T^{p-1})}.$$

Proof. The corollary follows directly by Lemma 13.53 and Corollary 2.15. □

The case $Q \xrightarrow{f} R$.

Lemma 13.55. *Assume that σ is a unimodal permutation and $f \in \mathscr{U}^{\sigma^2}$. The following maps are homeomorphisms*

$$Q^L \xrightarrow{f} R, Q^R \xrightarrow{f} R$$

$$Base(Q^L) \xrightarrow{f} Base(R), Base(Q^R) \xrightarrow{f} Base(R)$$

Proof. The lemma follows by the definition of the intervals. See Figure 13.5. ☐

Corollary 13.56 (Expansion for $Q \xrightarrow{f} R$)**.** *Assume that σ is a unimodal permutation and $f \in \mathscr{U}^{\sigma^2}$. If $J \subset Q^i$ where $i = L$ or R, then*

$$|f(J)|_{Base(R)} \geq |J|_{Base(Q^i)} .$$

Proof. The corollary follows from Lemma 13.55 and Proposition 2.10. ☐

The case $R_n \xrightarrow{\phi_n} R_{n+1}$

Lemma 13.57. *Assume that σ is a unimodal permutation and $f \in \mathscr{U}^{\sigma^2}$. Then $Base(R_{n+1}) \subset \phi_n(Base(R_n))$.*

Proof. The Lemma is true because $\phi_n(Base(R_n)) = [\alpha_{n+1}(0), v_{n+1}] \supset Base(R_{n+1}) = [\alpha^0_{n+1}(1), v_{n+1}]$. ☐

Corollary 13.58 (Expansion for $R_n \xrightarrow{\phi_n} R_{n+1}$)**.** *Assume that σ is a unimodal permutation and $f \in \mathscr{U}^{\sigma^2}$. If J is a wandering interval of f_n such that $J \subset R_n$ and $\phi_n(J) \subset R_{n+1}$, then*

$$|\phi_n(J)|_{Base(R_{n+1})} \geq |J|_{Base(R_n)} .$$

Proof. The corollary follows from Lemma 13.57, Proposition 2.10, and Corollary 2.15. ☐

The case $R_n \overset{\phi_n}{\Rightarrow} D_{n+1}$

Lemma 13.59 (Topological expansion for $R_n \overset{\phi_n}{\Rightarrow} D_{n+1}$)**.** *Assume that σ is an admissible unimodal permutation, $f \in \mathscr{U}^{\sigma^2}$, and $S \subset D_{n+1}$ is a trapping interval or a gap. Then $Base(S) \subset \phi_n(Base(R_n))$. In fact, $\phi_n(Base(R_n)) \backslash Base(S)$ has two components adjacent to the two sides of $Base(S)$: the left component contains $\left[\alpha_{n+1}(0), \beta^1_{n+1}(0)\right]$ and the right component contains $\left[\beta^0_{n+1}(1), v_{n+1}\right]$.*

Proof. The Lemma is true because $\phi_n(Base(J)) = [\alpha_{n+1}(0), v_{n+1}]$ contains the base of any trapping interval and gap in D_{n+1}. The two intervals $\left[\alpha_{n+1}(0), \beta^1_{n+1}(0)\right]$ and $\left[\beta^0_{n+1}(1), v_{n+1}\right]$ are contained in $\phi_n(Base(R_n))$ but disjoint from $Base(S)$. ☐

Corollary 13.60 (Uniform expansion for $R_n \overset{\phi_n}{\Rightarrow} D_{n+1}$)**.** *Assume that σ is an admissible unimodal permutation, $f \in \mathscr{U}^{\sigma^2}$, and $K \leq \left|\beta^1_{n+1}(0) - \alpha_{n+1}(0)\right|, \left|v_{n+1} - \beta^0_{n+1}(1)\right|$ for some constant $K > 0$. If $J \subset R_n$ and $\phi_n(J) \subset D_n$, then*

$$|\phi_n(J)|_{Base(\phi_n(J))} > E\,|J|_{Base(R_n)}$$

for some constant $E > 1$ that depends only on K.

Proof. Let L and R be the left and the right component of $\phi_n(Base(R_n))\backslash Base(\phi_n(J))$ respectively. Then, $\frac{L}{Base(\phi_n(J))}, \frac{R}{Base(\phi_n(J))} > \frac{K}{I}$ by Lemma 13.59 since L contains $[\alpha_{n+1}(0), \beta^1_{n+1}(0)]$ and R contains $[\beta^0_{n+1}(1), v_{n+1}]$. By Proposition 2.17, we get

$$|\phi_n(J)|_{Base(\phi_n(J))} > E\,|\phi_n(J)|_{\phi_n(Base(R_n))}$$

for some constant $E > 1$ determined by K.

Also, since $\phi_n = s_n \circ f_n^{p-1}$ has negative Schwarzian derivative and ϕ_n is a diffeomorphism from $Base(R_n)$ to its image, we obtain

$$|\phi_n(J)|_{\phi_n(Base(R_n))} \geq |J|_{Base(R_n)}$$

by Proposition 2.10. This completes the proof of the corollary. □

13.6.3. The expansion estimate for a closest approach

Finally, we summarize the expansion estimates by Proposition 13.61. The proposition says that the hyperbolic lengths of the elements in a closest approach expand at a definite rate. It can be used to prove the absence of wandering intervals (Theorem 13.63).

Proposition 13.61 (Uniform expansion). *Assume that σ is an admissible unimodal permutation. There exist $\overline{\varepsilon} > 0$ and $E > 1$ such that for all $f \in \mathscr{U}^{\sigma^\infty}$ with $\|f_n - f_\sigma\| < \overline{\varepsilon}$ for $n \geq 0$, the following property holds:*

Assume that $J \subset D$ is a wandering interval of f and $\{J_n\}_{n=0}^\infty$ is the closest approach of J. If $c_{r(n)} \notin J_n$ for all $n \geq 0$, then

$$l_n \geq E^{n-p} \cdot l_0$$

for all $n \geq 0$.

Proof. Let K be a positive constant such that $K < \inf_f \min\{|T_f^0|, \cdots, |T_f^{p-1}|, |\beta_f^1(0) - \alpha_f(0)|, |v_f - \beta_f^0(1)|\}$ where the infimum is evaluated over all unimodal maps $f \in \mathscr{U}^{\sigma^2}$ with $\|f - f_\sigma\| < \overline{\varepsilon}$. The constant K can be chosen to be positive when $\overline{\varepsilon} > 0$ is small enough.

The proposition follows by Proposition 13.37 ($D_n \xrightarrow{f_n} D_n$), Proposition 13.38 ($D_n \overset{f_n^p}{\Rightarrow} D_n$), and Proposition 13.50 ($Q_n \overset{\Phi_n^k \circ f_n}{\Rightarrow} D_{n+j}$).

In any p steps $J_t \to \cdots \to J_{t+p}$, if the rescaling $Q_n \overset{\Phi_n^k \circ f_n}{\Rightarrow} D_{n+j}$ occurs, then uniform expansion happens in the p steps by Proposition 13.50 ($Q_n \overset{\Phi_n^k \circ f_n}{\Rightarrow} D_{n+j}$). Otherwise, the wandering domain are all in the iteration interval D_n for some n. Then Proposition 13.38 ($D_n \overset{f_n^p}{\Rightarrow} D_n$) provides uniform expansion for the p steps. □

13.7. Nonexistence of wandering intervals

In this section, we present a proof for the nonexistence of wandering intervals by using the expansion estimate (Proposition 13.61). This is a classical theorem [Guc79, dMvS88, dMvS89, Lyu89, BL89, MdMvS92]. The strategy of the proof in this article motivates the proof for the Hénon-like maps.

By the following proposition and the hyperbolicity of the renormalization operator, without lose of generality, we may start from a deep level of renormalization and assume that the map is close to the limiting map f_σ.

Proposition 13.62. *Assume that σ is an admissible unimodal permutation and $f \in \mathscr{U}^{\mu} \cup \mathscr{U}^{\sigma^2}$ where μ is the two-cycle. The unimodal map f has a wandering interval in $P_f(0)$ if and only if its renormalization Rf has a wandering interval in $P_{Rf}(0)$.*

Proof. The converse follows directly from Proposition 13.26.

For the period-doubling case, we may assume that f has a wandering interval J in P^1 by Proposition 13.32. Then $\phi \circ f(J)$ is a wandering interval of Rf in $P_{Rf}(0)$ by Proposition 13.26.

For the admissible permutation case, the unimodal map f has a wandering interval D_f by Proposition 13.32. Also, by Corollary 13.39 later, the map has a wandering interval in R_f. Finally, by Proposition 13.26, the renormalization Rf has a wandering interval in P_{RF}. $\square$

As an consequence, we obtain

Theorem 13.63. *Assume that the unimodal map f is infinitely remormalizable with stationary combinatorics but not period-doubling infinitely renormalizable. Then the map does not have a wandering interval.*

Proof. Prove by contradiction. Assume that $f \in \mathscr{U}^{\sigma^\infty}$ has a wandering interval. By the shifting trick and Proposition 13.62, we may assume that the map f is admissible.

We can assume that the unimodal map f is sufficient close to the hyperbolic fixed point f_σ by Proposition 13.62 and the hyperbolicity of the renormalization operator. That is, $\|f_n - f_\sigma\| < \overline{\varepsilon}$ for all $n \geq 0$ where $\overline{\varepsilon}$ is given by Proposition 13.61. Also, we may also assume that J is a wandering interval of f such that the J-closest approach is disjoint from the critical point in its associate renormalization level by selecting some forward iterates of a wandering interval and Corollary 13.8.

Proposition 13.61 shows that hyperbolic length of the J-closest approach diverges to infinity. However, this is impossible because the hyperbolic length of gaps and trapping intervals are uniformly bounded when the unimodal map is close enough to f_σ. Therefore, a wandering interval cannot exist. $\square$

14. Hénon-like Maps

An introduction to the theory of Hénon-like maps developed by the articles [dCLM05, LM11, Haz11] is given in this chapter. The tool for vertical graphs in Section 14.2 are new in addition to the papers which allows us to study the topology of the stable manifolds by the results from unimodal maps.

14.1. Class of Hénon-like maps

Definition 14.1 (Hénon-like map)**.** Assume that I^h and I^v are compact intervals with $I^v \supset I^h \Supset I$ and $\delta > 0$. A *Hénon-like map* on $I^h \times I^v$ is a map F of the form

$$F(x,y) = (h(x,y), x)$$

where $h(x,y)$ is an $\mathbb{R}$-symmetric holomorphic map defined on $I^h(\delta) \times I^v(\delta)$ and $f(x) \equiv h(x,x) \in \mathscr{U}_\delta(I^h)$. Define $\varepsilon(x,y) = f(x) - h(x,y)$ to be the perturbation. The reason to choose this convention is to ensure that the fixed points are preserved when varying ε. The Hénon-like map can be written as its standard form

$$F(x,y) = (f(x) - \varepsilon(x,y), x).$$

Denote the class of Hénon-like maps as $\mathscr{H}_\delta(I^h \times I^v)$.

We are interested in *strongly dissipative* Hénon-like maps, the case when the perturbation ε is small (h is close to a unimodal map). Assume that $\overline{\varepsilon} > 0$ is small. The class $\mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$ are the maps $F = (f - \varepsilon, \pi_x) \in \mathscr{H}_\delta(I^h \times I^v)$ such that

$$\|\varepsilon\|_{I^h(\delta) \times I^v(\delta)} \leq \overline{\varepsilon}.$$

Fix a unimodal map $g \in \mathscr{U}_\delta(I^h)$. The subset $\mathscr{H}_\delta(I^h \times I^v, g, \overline{\varepsilon})$ is the class of the Hénon-like maps $F = (f - \varepsilon, \pi_x) \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$ such that $f \in \mathscr{U}_\delta(I^h, g, \overline{\varepsilon})$. This is an open $\overline{\varepsilon}$-cylinder centered at the degenerate Hénon-like map $i(g)$.

Degenerate Hénon-like maps are important examples of Hénon-like maps. They are maps that has zero perturbation $\varepsilon = 0$. In this case, the x-component h is exactly the unimodal map f and the dynamics of the Hénon-like map is fully determined by the unimodal dynamics of f. Unimodal maps $\mathscr{U}_\delta(I^h)$ can be identified with degenerate Hénon-like maps $\mathscr{H}_\delta(I^h \times I^v, 0)$ by $i(f) = (f, \pi_x)$ for $f \in \mathscr{U}_\delta(I^h)$. The difference between degenerate and non-degenerate Hénon-like maps lead to the main difficulty of proving the absence of wandering domains.

For a Hénon-like map, the point $a = a(0) = (-1,-1)$ is a fixed point of F. When $\overline{\varepsilon}$ is small, the fixed point a is saddle. It has two multipliers λ_1 and λ_2 with $0 \leq |\lambda_1| < d$ and $\lambda_2 \geq 1 + \kappa - d > 1$ where d is a small number that has the order of $\overline{\varepsilon}$ and $\kappa > 0$ is the value defined for $\mathscr{U}_\kappa$. The sign of the contracting direction λ_1 is determined by the sign of Jacobian $\det DF = \frac{\partial \varepsilon}{\partial y}$ at the fixed point and the sign of λ_2 is positive.

14.2. Separators and Lipschitz curves

Separators and Lipschitz curves are introduced in this section to study the topology of a Hénon-like map. For a strongly dissipative Hénon-like map, a periodic point has a stable manifold of dimension one that is associated to the perturbation ε. Some connected components of the manifolds are separators. They are used to define a partition on the domain which is similar to the partition defined for unimodal maps.

Definition 14.2 (Separator)**.** Assume that I^h and I^v are intervals. Identify $I^h \times I^v$ as a subset in the $\mathbb{R}^2$ plane.

A *separator* ω is the vertical graph $\omega = \{(u(t),t); t \in I^v\}$ of a continuous curve $u : I^v \to I^h$. The separator ω is said to have Lipschitz constant L if the curve u is a Lipschitz function with constant L.

A simple (total) *order* $\preceq$ can be defined on a collection of disjoint separators $\mathscr{X}$ by the order of the intersection of separators with any horizontal line. The order does not depend on the choice of horizontal line because a separator intersect a horizontal line at a unique point and the separators are disjoint. A collection of disjoint separators has the induced order topology [Mun00, Section 14]. The notion of intervals on a ordered set from Section 2.1 is used here.

Definition 14.3 (Vertical Strip)**.** Assume that $\mathscr{X}$ is a collection of disjoint separators and $I \subset \mathscr{X}$ is an interval. A *vertical strip* S induced by I is the set consists of all points $(x_1, x_2) \in I^h \times I^v$ between any two separators α and β in I. If a vertical strip S is induced by an interval of the form $I = [\alpha, \beta]$, then the vertical strip S is called a *vertical strip with boundaries* and the separators α and β are called the *boundaries* of S. In this article, vertical strips will be identified with intervals of disjoint separators whenever there is no confusion.

Lipschitz curves are graph of Lipschitz functions which are transverse to separators. In Chapter 16 later, Lipschitz curves will be used to measure the hyperbolic size of a set. The proof of nonexistence of wandering domain relies on the study of the hyperbolic size for the orbit of a wandering domain.

Definition 14.4 (Lipschitz Curve)**.** A *Lipschitz curve* with constant $R \geq 0$ is the graph $\Gamma = \{(t, r(t)); t \in I^r\}$ of a Lipschitz function $r : I^r \to I^v$ with constant R defined on a subinterval $I^r \subset I^h$.

Lemma 14.5. *Assume that R and L are non-negative constants with* $0 \leq RL < 1$. *Then the intersection of a Lipschitz curve with constant R and a separator with Lipschitz constant L have at most one point.*

Proof. Prove by contradiction. Let r be a Lipschitz function with constant R and ω be the vertical graph of a Lipschitz curve u with constant L. Assume that the Lipschitz curve of r intersects the separator ω at two distinct points $(a, r(a))$ and $(b, r(b))$.

For the Lipschitz curve r, since r is a Lipschitz function with constant R, we have

$$|r(b) - r(a)| \leq R\,|b - a|\,. \tag{14.1}$$

For the separator ω, since u is a Lipschitz function with constant L, we have

$$|b - a| = |u(r(b)) - u(r(a))| \leq L\,|r(b) - r(a)|\,. \tag{14.2}$$

Combine (14.1) and (14.2) and the inequality $RL < 1$, we get

$$|b-a| < |b-a|$$

which is a contradiction. This proves the lemma. □

Next, we show that the order of a collection of disjoint separators is preserved on the intersection of the separators with a Lipschitz curve.

Proposition 14.6. *Assume that $\mathscr{X}$ is a collection of disjoint separators with Lipschitz constant L, $r : [a,b] \to I^v$ is a Lipschitz function with constant R, $0 \le RL < 1$, and $\alpha, \beta \in \mathscr{X}$.*

1. *If $\alpha \prec \beta$ and the graph of r intersects α and β at $(a, r(a))$ and $(b, r(b))$ respectively, then $a < b$.*
2. *In addition, if $\omega \in (\alpha, \beta)$, then the graph of r intersects ω at a unique point $(w, r(w))$ with $a < w < b$.*

Proof. For the first property, if $r(a) = r(b)$ then the intersection belongs to the same horizontal line $y = r(a)$. Thus, the property follows from the definition of $\alpha \prec \beta$.

Assume the case that $r(a) \neq r(b)$. Let $(c, r(b))$ be the intersection of the horizontal line $y = r(b)$ with α. By the definition of $\alpha \prec \beta$, we have $c < b$. Then

$$b - a > c - a \ge -|c-a|. \tag{14.3}$$

Since α is a vertical graph of a Lipschitz curve with constant L, we have

$$|c-a| \le L|r(b) - r(a)| \tag{14.4}$$

Also, since r is a Lipschitz function with constant R, we have

$$|r(b) - r(a)| \le R|b-a|. \tag{14.5}$$

Combine (14.3), (14.4), and (14.5), we obtain

$$a - b < L|r(b) - r(a)| \le RL|b-a| < |b-a|.$$

The only possibility to make this inequality holds is $a < b$. This proves the proposition.

The second property follows from connectivity. The separator ω splits the whole domain into two connected components: one contains α and the other contains β. The segment $\Gamma = \{(t, r(t)); t \in [v,w]\}$ must intersects ω because the two endpoints belong to different components. Therefore, the inequality follows from the first property and the uniqueness follows from Lemma 14.5. □

14.2.1. Induced unimodal map

Induced unimodal map is a tool to study the dynamics of a Hénon-like map on a partition formed by separators. Strongly disspative Hénon-like maps are similar to unimodal maps in the macroscopic scale. Induced unimodal map characterizes the similarities of the topology between the two types of maps. It will be used later to generalize the dynamical properties on the partition from unimodal maps to Hénon-like maps.

Compatible separators for Hénon-like maps are analogs of the periodic orbits for unimodal maps. An example will be given in the next section. It is defined by the following.

Definition 14.7 (Compatible Separators and Induced Map)**.** Given constants $\overline{\varepsilon} > 0$ and $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a Hénon-like map $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$. A pair of collection of disjoint separators $(\mathscr{X}, \mathscr{Y})$ for F is *compatible* if for all $\alpha \in \mathscr{X}$, there exists $\beta \in \mathscr{Y}$ such that $F(\alpha) \subset \beta$. The separator β is unique because separators in $\mathscr{Y}$ are disjoint. The *induced map* $f^s : \mathscr{X} \to \mathscr{Y}$ on the pair is defined to be $f^s(\xi) = \psi$ if $F(\xi) \subset \psi$ for $\xi \in \mathscr{X}$ and $\psi \in \mathscr{Y}$. A collection of separators $\mathscr{X}$ itself is compatible if $(\mathscr{X}, \mathscr{X})$ is compatible.

An induced map f^s is said to be *increasing* (resp. *decreasing*) on an interval $\mathscr{I} \subset \mathscr{X}$ if $f^s(\psi) \prec f^s(\xi)$ (resp. $f^s(\psi) \succ f^s(\xi)$) whenever $\alpha \prec \beta$ for $\alpha, \beta \in \mathscr{X}$. The induced map is said to be *monotone* on an interval $\mathscr{I} \subset \mathscr{X}$ if it is either increasing or decreasing on the interval. An induced map g is said to be *unimodal* if there exists a union $\mathscr{X} = \mathscr{X}_L \cup \mathscr{X}_R$ of nonempty disjoint intervals $\mathscr{X}_L$ and $\mathscr{X}_R$ with $\psi \prec \zeta$ for all $\psi \in \mathscr{X}_L$ and $\zeta \in \mathscr{X}_R$ such that

1. f^s is increasing on $\mathscr{X}_L$ and decreasing on $\mathscr{X}_R$ and
2. if $\psi \in \mathscr{X}_i$ for $i = L$ or R then $F^{-1}(f^s(\psi)) \cap S(\mathscr{X}_i) = \psi$ where $S(\mathscr{Y})$ is the vertical strip in $I^h \times I^v$ induced by the interval $\mathscr{X}_i$.

In the remaining part of the article, the function f^s will be the induced unimodal map of the Hénon-like map of interest and the domain of compatible separators will contain all local stable manifolds in interest.

The first proposition says that we can enlarge the domain of an induce map by taking the union of two smaller domains.

Proposition 14.8. *Given constants $\overline{\varepsilon} > 0$ and $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a Hénon-like map $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$. If $\mathscr{Y}$ and $\mathscr{Z}$ are two collections of compatible separators, then $\mathscr{X} = \mathscr{Y} \cup \mathscr{Z}$ is also a collection of compatible separators.*

In addition, if the two induced maps on $\mathscr{Y}$ and $\mathscr{Z}$ are both unimodal and the two collections of separators $\mathscr{Y}_L \cup \mathscr{Z}_L$ and $\mathscr{Y}_R \cup \mathscr{Z}_R$ are disjoint intervals in $\mathscr{X}$ where $\mathscr{Y} = \mathscr{Y}_L \cup \mathscr{Y}_R$ and $\mathscr{Z} = \mathscr{Z}_L \cup \mathscr{Z}_R$ are the decomposition of the increasing and decreasing intervals, then the induced map is also unimodal on $\mathscr{X}$.

Proof. The proposition follows from definition. □

The next two propositions are important applications of induced unimodal map. One can think of the induced maps are maps that record the combinatorics properties of a unimodal map or a Hénon-like map. The first proposition says that if the combinatorics properties of a unimodal map g and the induced map f^s are the same which is identified by the monotone map ι, then the induced unimodal map f^s acts like the unimodal map g on a monotone branch.

Proposition 14.9. *Let $g : I^h \to I^h$ be a unimodal map, the set $\mathscr{X}$ is a collection of compatible separators, and the induced map f^s on $\mathscr{X}$ is unimodal. Assume that there exists a monotone map $\iota : X \subset I^h \to \mathscr{X}$ such that $g(X) \subset X$, $\iota \circ g = f^s \circ \iota$ and maps points in a monotone branch of g to the associated monotone branch of f^s. If $\alpha_1, \beta_1, \alpha_2, \beta_2 \in X$, g is monotone on $[\alpha_1, \beta_1]$, and $g([\alpha_1, \beta_1]) = [\alpha_2, \beta_2]$, then $f^s([\iota(\alpha_1), \iota(\beta_1)]) = [\iota(\alpha_1), \iota(\beta_1)]$.*

Proof. Since g is monotone on $[\alpha_1, \beta_1]$, boundary points are mapped to boundary points. Without lose of generality, we assume that $g(\alpha_1) = \alpha_2$ and $g(\beta_1) = \beta_2$. Then $f^s \circ \imath(\alpha_1) = \alpha_2$ and $f^s \circ \imath(\beta_1) = \beta_2$ since f^s is conjugate to g. This shows that $f^s([\imath(\alpha_1), \imath(\beta_1)]) = [\imath(\alpha_1), \imath(\beta_1)]$ since f^s is monotone on $[\imath(\alpha_1), \imath(\beta_1)]$. □

The second proposition says that the Hénon-like map F acts like the induced unimodal map on a monotone branch. The two propositions together allow us to generalize the dynamical properties from unimodal maps to Hénon-like maps of the same combinatorics type.

Proposition 14.10. *Assume that $\mathscr{X}$ is a collection of compatible separators and the induced map f^s on $\mathscr{X}$ is unimodal. If $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathscr{X}$, f^s is monotone on $[\alpha_1, \beta_1]$, and $f^s([\alpha_1, \beta_1]) = [\alpha_2, \beta_2]$, then $F([\alpha_1, \beta_1]) \subset [\alpha_2, \beta_2]$.*

Proof. Let $\mathscr{X} = \mathscr{Y} \cup \mathscr{Z}$ be the decomposition into the increasing and decreasing intervals. Assume the case that $[\alpha_1, \beta_1] \subset \mathscr{Y}$. The other case $[\alpha_1, \beta_1] \subset \mathscr{Z}$ is similar.

Prove by contradiction. Assume that z_1 is a point in the vertical strip $[\alpha_1, \beta_1]$ and $z_2 = F(z_1)$ such that $z_2 \notin [\alpha_2, \beta_2]$.

Let Γ be the horizontal line segment connecting α_1 and β_1 that intersects z_1. Then $\Gamma \subset L$ and the image $F(\Gamma)$ is a curve connecting α_2 and β_2 that intersects z_2. Since $z_2 \notin [\alpha_2, \beta_2]$, the interior of the curve $F(\Gamma)$ intersects a boundary of the vertical strip $[\alpha_2, \beta_2]$ at some point by connectivity. Without lose of generality, assume the case that $F(\Gamma)$ intersects α_2 at $F(u)$ where $u \in \Gamma$. Thus, $u \in \alpha_1$ because $F^{-1}(\alpha_2) \cap S(\mathscr{Y}) = \alpha_1$ by the definition of induced unimodal map where $S(\mathscr{Y})$ is the vertical strip induced by $\mathscr{Y}$. This contradicts to the fact that a horizontal line have at most one intersection with a separator.

Therefore, $F([\alpha_1, \beta_1]) \subset [\alpha_2, \beta_2]$. □

Remark 14.11. In the statement of the proposition, the equality $f^s([\alpha_1, \sigma_1]) = [\alpha_2, \beta_2]$ means the interval $[\alpha_1, \beta_1] \subset \mathscr{X}$ is mapped to the interval $[\alpha_2, \beta_2] \subset \mathscr{X}$ by the induced unimodal map f^s. The equation $F([\alpha_1, \beta_1]) \subset [\alpha_2, \beta_2]$ means the vertical strip $[\alpha_1, \beta_1]$ is mapped to a subset of the vertical strip $[\alpha_2, \beta_2]$ by the Hénon-like map F. This type of identification between intervals and vertical strips will also be used later in this article. The reader should not be confused with this.

14.3. Cycle and Renormalizable

In this section, we begin to study the dynamics of a Hénon-like map by defining two vertical strips $P(0)$ and $P(1)$ on $I^h \times I^v$ that are invariant under some iterations. The set $P(0)$ turns the Hénon-like map into a self-map which enables us to study the dynamics. The other set $P(1)$ defines a first return map of a Hénon-like map which allows us to setup the notion of Hénon renormalizable. The vertical strips are similar to the intervals and the boundary separators are similar to the periodic orbits that are defined for unimodal maps. The same notation for the intervals and periodic orbits from unimodal maps will be adopted to Hénon-like maps in this section.

A Hénon-like map in interest has two saddle fixed points. First, consider the fixed point $a(0)$ with the expanding positive multiplier as follows.

Definition 14.12. Given $\bar{\varepsilon} > 0$ small, $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a Hénon-like map $F \in \mathscr{H}_\delta(I^h \times I^v, \bar{\varepsilon})$. The fixed point $a(0)$ is saddle. The expanding multiplier is positive and associated to the direction tangent to the graph of the unimodal part f. The contracting multiplier is associated to the strongly dissipative part ε. We focus on the stable manifold associated to the contracting multiplier.

1. If the connect component of the stable manifold that contains $a(0)$ is a separator, define $\alpha(0)$ to be the component.

2. If the preimage $F^{-1}(\alpha(0))$ has two components, one of the component is $\alpha(0)$ itself. Define $\overline{\alpha(0)}$ to be the other component if it is also a separator.

3. Define $P(0)$ to be the vertical strip $[\alpha(0), \overline{\alpha(0)}]$.

Then, consider the other fixed point $b(0)$ with the expanding negative multiplier as follows.

Definition 14.13. Given $\overline{\varepsilon} > 0$ small, $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a Hénon-like map $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$. Write $F = (f - \varepsilon, x)$. Assume that f has a unique fixed point b_x with negative multiplier that satisfies $\lambda^2 \geq 1 + \kappa$ and $\varepsilon \ll \kappa$. Then the Hénon-like map has a saddle fixed point $b = b(0) \equiv (b_x, b_x)$. The contracting multiplier is associated to the strongly dissipative part ε. And the expanding multiplier is negative. The separators $\beta(0)$ and $\overline{\beta(0)}$ are components of the stable manifold for $b(0)$ that are defined similar to the stable manifolds in Definition 14.12.

The domain $P(0)$ turns the Hénon-like map to be a self-map as follows.

Proposition 14.14. *Given $\delta > 0$ and intervals I^h and I^v with $I \Subset I^h \subset I^v$. There exist $\overline{\varepsilon} = \overline{\varepsilon}(\kappa) > 0$ and $c > 0$ such that for all $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$, the following properties hold:*

1. *The separators $\alpha(0)$ and $\overline{\alpha(0)}$ exist. They are separators with Lipschitz constant $c\,\|\varepsilon\|$.*

2. *The set $\{\alpha(0), \overline{\alpha(0)}\}$ forms a collection of compatible separators and f^s is an induced unimodal map on the separators.*

3. *$F(P(0)) \subset P(0)$.*

In addition, if f has a unique fixed point b_x that satisfies the properties in Definition 14.13, then

4. *The separators $\beta(0)$ and $\overline{\beta(0)}$ exist. They are also separators with Lipschitz constant $c\,\|\varepsilon\|$.*

5. *The set $\{\alpha(0), \overline{\alpha(0)}, \beta(0), \overline{\beta(0)}\}$ forms a collection of compatible separators and f^s is an induced unimodal map on the separators.*

Proof. The first and second properties follow from the graph transformation. See [LM11, Lemma 3.1, 3.2] for the period-doubling case.

The third property follows from the definition of the local stable manifolds when $\overline{\varepsilon} = \overline{\varepsilon}(\kappa) > 0$ is sufficiently small. □

Cyclic sets are analog of cyclic intervals for the unimodal maps in Definition 13.2. They are defined as follows.

Definition 14.15 (Cycle)**.** Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, a unimodal map $g \in \mathscr{U}^\sigma_\delta(I^h)$, and $\overline{\varepsilon} > 0$ small. Assume $F \in \mathscr{H}_\delta(I^h \times I^v, g, \overline{\varepsilon})$ is a Hénon-like map and the two separators $\beta(0)$ and $\overline{\beta(0)}$ exist. Let $p = |\sigma|$.

1. First, we assume that there exist a periodic orbit $\{a^t(1)\}_{t=0}^{p-1}$ and an orbit of separators $\{\alpha^t(1)\}_{t=0}^{p-1}$ which are all local stable manifolds containing the periodic points that satisfy the following properties. There are two different cases: $p=2$ and $p \geq 3$.
   For the case $p=2$, set $a(1)=a^0(1)=a^1(1)=b(0)$ and $\alpha(1)=\alpha^0(1)=\alpha^1(1)=\beta(0)$.
   For the case $p \geq 3$, when $\overline{\varepsilon}$ is small, assume the map has a saddle periodic orbit $\{a^t(1)\}_{t=0}^{p-1}$ on $P(0)$ with period p that is close to the periodic orbit of the degenerate map $i(g)$. The orbit has two multipliers. The contracting multiplier is associated to the strongly dissipative part ε. The expanding multiplier is positive and satisfies $\lambda_2 \geq 1+\kappa-d>1$. For each point z in the periodic orbit, assume that the connected component of the stable manifold that contains z is a separator. The p separators are disjoint and defines an order mentioned in Definition 14.2. Define $\alpha(1)$ to be the largest separator containing the periodic point $a(1)$. Also, define $a^t(1)=F^t(a(1))$ and $\alpha^t(1)$ be the separator that contains $a^t(1)$ for $t=0,\cdots,p-1$. The order of the separators coincide with the order of the periodic points of g and the induced map f^s acts like the unimodal permutation σ on the separators.

2. Assume the preimage $F^{-1}(\alpha^0(1))$ has two components. The component containing the periodic point $a^{p-1}(1)$ is $\alpha^{p-1}(1)$. Define $\overline{\alpha^{p-1}(1)}$ to be the other component if it is also a separator. The vertical strip $P^{p-1}=[\alpha^{p-1}(1),\overline{\alpha^{p-1}(1)}]$ is called the *center* cyclic set. The complement $P(0)\setminus \mathrm{int}(P^{p-1})$ contains two components L and R. One can think L and R are the orientation preserving and reversing parts of the Hénon-like map. The sets L and R will be extended to some larger sets that are monotone later in Definition 14.35.

3. Define $\{\overline{\alpha^t(1)}\}_{t=0}^{p-1}$ by induction in the reverse order on t. Assume that $\overline{\alpha^{t+1}(1)}$ exists. If $\alpha^t(1)\subset L$, define $\overline{\alpha^t(1)}=F^{-1}(\overline{\alpha^{t+1}(1)})\cap L$ if it is a separator. Similarly, if $\alpha^t(1)\subset R$, define $\overline{\alpha^t(1)}=F^{-1}(\overline{\alpha^{t+1}(1)})\cap R$ if it is a separator. Set $\overline{\alpha(1)}=\overline{\alpha^0(1)}$.

4. Define P^t be the vertical strip $[\alpha^t(1),\overline{\alpha^t(1)}]$ for $t=0,\cdots,p-1$.

5. Assume that the sets $\{P^t\}_{t=0}^{p-1}$ exists. The collection $\{P^t\}_{t=0}^{p-1}$ is called a *cycle* if the sets are disjoint and $F(P^{p-1})\subset P^0$. A set in cycle is called a *cyclic set*. Set $P(1)=P^0$.

See Table 14.1 for a summary of the notations.

Remark 14.16. By definition, the local stable manifolds form an orbit $\overline{\alpha^0(1)}\to\cdots\to\overline{\alpha^{p-1}(1)}\to \alpha^0(1)\to\cdots\to\alpha^{p-1}(1)\to\alpha^0(1)$ of the induced map f^s.

Remark 14.17. By identifying the periodic orbits of the unimodal map g with the orbits of the local manifolds, the unimodal map g is conjugated to the induced unimodal map f^s from the periodic orbits to the local stable manifolds because orbit points are mapped to orbit points. See Figure 14.1. The identification is monotone because the order is preserved by the definition of renormalizable. Thus, the topological results for the unimodal map g also holds for the induced unimodal map f^s by Proposition 14.9. Moreover, the topological properties for the induced unimodal map f^s (on a monotone branch) can be generalized to the Hénon-like map F by Proposition 14.10. This will be used later in the proof of the expansion argument to generalize the topological properties for unimodal maps to Hénon-like maps.

The next proposition and corollary show the existence of the objects and allow us to define a first return map for the Hénon-like map.

$$\begin{tikzcd}
\text{orbits of periodic points} \arrow[r, "\text{monotone}"] \arrow[d, "g"] & \text{objects } \alpha, \beta \arrow[d, "f^s"] & \text{orbits of local manifolds} \arrow[l] \arrow[d, "F"] \\
\text{orbits of periodic points} \arrow[r, "\text{monotone}"] & \text{objects } \alpha, \beta & \text{orbits of local manifolds} \arrow[l]
\end{tikzcd}$$

Figure 14.1.: The relation between the unimodal map, induced unimodal map, and the Hénon-like map.

Proposition 14.18. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and a unimodal map $g \in \mathscr{U}_\delta^\sigma(I^h)$. There exist $\overline{\varepsilon} = \overline{\varepsilon}(\sigma) > 0$ and $c > 0$ such that for all $F \in \mathscr{H}_\delta(I^h \times I^v, g, \overline{\varepsilon})$ the following properties hold:*

1. *The periodic orbit $a^t(1)$ and the separators $\alpha^t(1)$ and $\overline{\alpha^t(1)}$ exist for $t = 0, \cdots, p-1$. In fact, $\alpha^t(1)$ and $\overline{\alpha^t(1)}$ are separators with Lipschitz constant $c\,\|\varepsilon\|$.*

2. *The set $\{\alpha(0), \overline{\alpha(0)}, \alpha^0(1), \overline{\alpha^0(1)}, \cdots, \alpha^{p-1}(1), \overline{\alpha^{p-1}(1)}\}$ forms a collection of compatible separators and f^s is an induced unimodal map on the separators. The map f^s is increasing on $[\alpha(0), \alpha^{p-1,L}(1)]$ and decreasing on $[\alpha^{p-1,R}(1), \overline{\alpha(0)}]$ where $\alpha^{p-1,L}(1), \alpha^{p-1,R}(1) \in \{\alpha^{p-1}(1), \overline{\alpha^{p-1}(1)}\}$ with $\alpha^{p-1,L}(1) \prec \alpha^{p-1,R}(1)$.*

Proof. The first and second properties are proved by graph transformation. See [LM11, Lemma 3.1, 3.2] for the period-doubling case. □

Corollary 14.19. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and a unimodal map $g \in \mathscr{U}_\delta^\sigma(I^h)$. There exists $\overline{\varepsilon} = \overline{\varepsilon}(\sigma) > 0$ such that for all $F \in \mathscr{H}_\delta(I^h \times I^v, g, \overline{\varepsilon})$ the sets $\{P^t\}_{t=0}^{p-1}$ exists and $F(P^t) \subset P^{t+1}$ for $0 \le t < p-1$.*

Proof. The corollary follows from $f^s(P^t) = P^{t+1}$, f^s is monotone on P^t, and Propositions 14.10 and 14.18. □

Remark 14.20. The corollary does not say that the inclusion $F(P^{p-1}) \subset P^0$ holds for all Hénon-like maps close to the degenerate map g. In other words, it does not guarantee that the sets P^t form a cycle.

Finally, renormalizable is defined as follows.

Definition 14.21 (Renormalizable)**.** Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, a unimodal map $g \in \mathscr{U}_\delta^\sigma(I^h)$, and $\overline{\varepsilon} = \overline{\varepsilon}(\sigma) > 0$ small. A Hénon-like map $F \in \mathscr{H}_\delta(I^h \times I^v, g, \overline{\varepsilon})$ is said to be *renormalizable* of combinatorial type σ if

1. the map has a cycle $\{P^t\}_{t=0}^{p-1}$ and

2. the order of the separators $\alpha(0)$, $\overline{\alpha(0)}$, $\beta(0)$, $\overline{\beta(0)}$, $\alpha^0(1)$, $\cdots$, $\alpha^{p-1}(1)$, $\overline{\alpha^0(1)}$, $\cdots$, $\overline{\alpha^{p-1}(1)}$ coincide with the order of the corresponding periodic points for a renormalizable unimodal map g.

The class of renormaizable Hénon-like maps of combinatorial type σ is denoted as $\mathscr{H}^{\sigma}_{\delta}(I^h \times I^v, g, \overline{\varepsilon})$. Set $\mathscr{H}^{\sigma}_{\delta}(I^h \times I^v, \overline{\varepsilon}) = \cup_{g \in \mathscr{U}^{\sigma}_{\delta}(I^h)} \mathscr{H}^{\sigma}_{\delta}(I^h \times I^v, g, \overline{\varepsilon})$.

Remark 14.22. From the definition, the topological properties for unimodal maps can be generalized to induced unimodal maps on separators because the order of the separators is preserved for all renormalizable maps with the same combinatorics σ by definition. Proposition 14.10 can be used to generalize the dynamical properties from unimodal maps to Hénon-like maps.

14.4. Renormalization operator

When a Hénon-like map is renormalizable, the cyclic sets forms a periodic orbit

$$P(1) = P^0 \to \cdots \to P^{p-1} \to P^0.$$

Thus, a first return map can be defined on $P(1)$. However, it is no longer a Hénon-like map by direct computation. The article by [Haz11] generalized the renormalization operator to arbitrary stationary combinatorics. It introduced a nonlinear rescaling $H(x,y) \equiv (\pi_x \circ F^{p-1}(x,y), y)$ to turn the first return map into a Hénon-like map. The following proposition defines the renormalization operator.

Proposition 14.23 (Renormalization operator)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exist constants $\overline{\varepsilon} > 0$ and $c > 0$ such that for all $F \in \mathscr{H}^{\sigma}_{\delta}(I^h \times I^v, \overline{\varepsilon})$ there exists an $\mathbb{R}$-symmetric affine map $s = s(F)$ that depends continuously on F and its orientation depends only on σ so that the following properties hold:*
Let $\Lambda(x,y) = (s(x), s(y))$ and $\phi = \Lambda \circ H$.

1. *The renormalization $RF \equiv \phi \circ F^p \circ \phi^{-1}$ is an Hénon-like map in the class of $\mathscr{H}^{\sigma}_{\delta}(I^h \times I^v)$ for some $\delta_R > 0$ and intervals I^h_R and I^v_R. The intervals satisfy $I^v_R \supset I^h_R \Supset [-1,1]$ and $I^v_R = s(I^v)$.*
2. *The domain $I^h_R \times I^v_R$ contains $P_{RF}(0)$.*
3. *The rescaling map $\phi : P_F(1) \to P_{RF}(0)$ is a diffeomorphism on the restriction. It has a holomorphic extension on some complex neighborhood of $P_F(1)$ with image containing $I^h_R(\delta_R) \times I^v_R(\delta_R)$. It preserves the orientation on the x-direction.*
4. *The renormalization has the representation $RF = (f_R - \varepsilon_R, x)$ where $f_R \in \mathscr{U}_{\delta_R}(I^h_R)$. It satisfies the relations*
$$\|f_R - Rf\|_{I^h_R(\delta_R)} < c\,\|\varepsilon\|$$
and
$$\|\varepsilon_R\|_{I^h_R(\delta_R) \times I^v_R(\delta_R)} < c\,\|\varepsilon\|^p. \tag{14.6}$$

Proof. See [dCLM05, Section 3.5] for the period-doubling case and [Haz11, Section 3.2] for arbitrary stationary combinatorics. □

From the renormalization operator, we are able to define infinitely renormalizable maps for strongly dissipative Hénon-like maps.

Definition 14.24 (Infinitely Renormalizable)**.** Assume that $\overline{\varepsilon} > 0$ is sufficiently small, $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and σ is a unimodal permutation. Denote $\mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, \overline{\varepsilon}) \subset \mathscr{H}_\delta^{\sigma}(I^h \times I^v, \overline{\varepsilon})$ to be the class of *infinitely renormalizable* Hénon-like map with stationary combinatorics σ.

For an infinitely renormalizable map $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, \overline{\varepsilon})$, define $F_n = R^n F$. The subscript n is called the *renormalization level*. Subscript is used to indicate the associate renormalization level of an object. For example, the maps H_n, s_n, Λ_n, and ϕ_n with subscript n are the rescaling functions for F_n in Proposition 14.23. The vertical domain I_n^v satisfies $I_0^v = I^v$ and the recurrent relation $I_{n+1}^v = s_n(I_n^v)$ for all $n \geq 0$. The points $a_n(0)$ and $b_n(0)$ are the saddle fixed points of F_n. The separators $\alpha_n(0)$, $\overline{\alpha_n(0)}$, $\beta_n(0)$, $\overline{\beta_n(0)}$, $\alpha_n(1)$, and $\overline{\alpha_n(1)}$ are the local stable manifolds for F_n defined in Definitions 14.12 and 14.15. The sets $P_n(0)$ and $P_n(1)$ are the vertical strips for F_n defined in Definitions 14.12 and 14.15.

Also, define $\Phi_n^j = \phi_{n+j-1} \circ \cdots \circ \phi_n$ for $j \geq 1$, $\Phi_n^0 = \mathrm{id}$, and $\lambda_n = s_n'(x)$.

Recall from page 127 that $f_\sigma \in \mathscr{U}$ is the fixed point of the unimodal renormalization operator with combinatorics σ and λ_σ is its rescaling constant.

The renormalization operator is hyperbolic. The following proposition summarizes the properties of an infinitely renormalizable Hénon-like map.

Proposition 14.25 (Hyperbolicity of the Renormalization operator)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exist $\rho < 1$ (universal), $\overline{\varepsilon} > 0$, $c > 0$ such that for all $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, \overline{\varepsilon})$ there exist a constant δ_R with $0 < \delta_R < \delta$, an interval I_R^h with $I \Subset I_R^h \subset I^h$, and a constant $b = b(F) \in \mathbb{R}$ so that the following properties hold:*

Let $F_n = R^n F$ be the n-th renormalization of F. Then $F_n \in \mathscr{H}_{\delta_R}^{\sigma}(I_R^h \times I_n^v, \overline{\varepsilon})$ for all $n \geq 0$ and its representation $F_n = (f_n - \varepsilon_n, x)$ satisfies

1. $\|f_n - f_\sigma\|_{I_R^h(\delta_R)} < c\rho^n \|F - i(f_\sigma)\|_{I_R^h(\delta_R) \times I^v(\delta_R)}$,
2. $\|\varepsilon_{n+j}\|_{I_R^h(\delta_R) \times I_{n+j}^v(\delta_R)} < c \|\varepsilon_n\|_{I_R^h(\delta_R) \times I_n^v(\delta_R)}^{p^j}$,
3. $\varepsilon_n(x,y) = b^{p^n} a_\sigma(x)(y-x)(1 + O(\rho^n))$ *(universality)*

for all $n \geq 0$ where $a_\sigma(x)$ is a universal analytic positive function. The value δ_R in the estimates can be replaced by any smaller positive number.

Proof. See [dCLM05, Theorem 3.5, 4.1, 7.9, and Lemma 7.4] for the period-doubling case and [Haz11, Theorems 3.10, 3.11, and 6.1] for arbitrary stationary combinatorics. □

Remark 14.26. The value b is called the *average Jacobian* of F. See [dCLM05, Section 6] and [Haz11, Definition 3.19].

Abbreviate $\|\varepsilon_n\| = \|\varepsilon_n\|_{I_R^h(\delta_R) \times I_n^v(\delta_R)}$ when the context is clear.

To study wandering domain, it is enough to consider Hénon-like maps that are close to the hyperbolic fixed point $i(f_\sigma)$. By Proposition 18.12 later, an infinitely renormalizable Hénon-like map F has a wandering domain in $D(F)$ if and only if F_n has a wandering domain in $D(F_n)$ for all $n \geq 0$. Also, the n-th renormalized map converges to the fixed point $i(f_\sigma)$ as $n \to \infty$ by the hyperbolicity of the renormalization operator (Proposition 14.25). Thus, without lose of generality, we may focus on a small neighborhood of the fixed point $i(f_\sigma)$.

Definition 14.27. Given $\overline{\varepsilon} > 0$ small, $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. Define $\mathscr{I}_{\delta}^{\sigma}(I^h \times I^v, \overline{\varepsilon})$ to be the class of non-degenerate infinitely renormalizable Hénon-like maps $F \in \mathscr{H}_{\delta}^{\sigma^\infty}(I^h \times I^v, \overline{\varepsilon})$ that are sufficiently close to the hyperbolic fixed point $i(f_\sigma)$: $\|F_n - i(f_\sigma)\| < \overline{\varepsilon}$, the partial derivative $\frac{\partial^j \pi_x \circ F_n^t}{\partial x^j}$ is $\overline{\varepsilon}$-close to the limiting case $D^j(f_\sigma^t)$ for $t \in \{1, 2, \cdots, p\}$ and $j \in \{1, 2, 3\}$, $|\lambda_n - \lambda_\sigma| < \overline{\varepsilon}$, $\|s_n(x) - (-\lambda_\sigma)x\|_{I^h} < \overline{\varepsilon}$, and $\left\|\frac{\partial \varepsilon_n}{\partial y}\right\|_{I^h \times I_n^v} \geq \frac{c}{|I_n^v|}\|\varepsilon_n\|$ for all $n \geq 0$.

Remark 14.28. The technical conditions for the partial derivatives are required by Propositions 17.3, 17.26, and 17.29. These theorems are used to generalize the expansion of hyperbolic length under the iteration by unimodal maps with negative Schwarzian derivative to Hénon-like maps.

Remark 14.29. For any Hénon-like map $F \in \mathscr{H}_{\delta}^{\sigma^\infty}(I^h \times I^v, \overline{\varepsilon})$ with $\overline{\varepsilon} > 0$ sufficiently small, the condition $\left\|\frac{\partial \varepsilon_n}{\partial y}\right\|_{I^h \times I_n^v} \geq \frac{c}{|I_n^v|}\|\varepsilon_n\|$ from Definition 14.27 holds for all n large enough. The condition comes from the universality of the tip. See Lemma 5.31 for the proof of the period-doubling case. It controls the Jacobian of the map and is required by the area argument (Proposition 18.2) later.

14.5. Topological Structure of Infinitely Renormalizable Maps

Stable manifolds form the topology of a Hénon-like map. The macroscopic scale of the topology is characterized by the stable manifolds of the fixed points a and b. If the Hénon-like map is renormalizable, it means that the microscopic structure of the topology is controlled by the pull-back of a similar topology from some deeper level of renormalization using the rescaling map. Therefore, the condition infinitely renormalizable builds a self-similarity of the topology between the macroscopic and microscopic scale.

In this article, two types of local stable manifolds $\alpha(j)$ and $\beta(j)$ are used to study the topology. They are associated to the two types of fixed points a (with a positive expanding multiplier) and b (with a negative expanding multiplier) respectively. The value j means the associate scale (pulled back from j-deeper level) of the local stable manifold. The definitions are as follows.

Definition 14.30 (Periodic Points)**.** Given $\overline{\varepsilon} > 0$ sufficiently small, $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and $F \in \mathscr{H}_{\delta}^{\sigma^\infty}(I^h \times I^v, \overline{\varepsilon})$. See Table 14.1 for a summary of the notations.

Define $a_n(j) = \left(\Phi_n^j\right)^{-1}(a_{n+j})$ and $b_n(j) = \left(\Phi_n^j\right)^{-1}(b_{n+j})$ for integers $n \geq 0$ and $j \geq 0$.

Define $\alpha_n(j) = \left(\Phi_n^j\right)^{-1}(\alpha_{n+j})$, $\overline{\alpha_n(j)} = \left(\Phi_n^j\right)^{-1}(\overline{\alpha_{n+j}})$, $\beta_n(j) = \left(\Phi_n^j\right)^{-1}(\beta_{n+j})$, $\overline{\beta_n(j)} = \left(\Phi_n^j\right)^{-1}(\overline{\beta_{n+j}})$, and $P_n(j) = \left(\Phi_n^j\right)^{-1}(P_n(0))$ for integers $n \geq 0$ and $j \geq 0$.

Remark 14.31. The definition is consistent with the previous definition of $a(j)$, $\alpha(j)$, $\overline{\alpha(j)}$, and $P(j)$ for $j = 1$.

The next proposition summarizes the properties of the local stable manifolds for an infinitely renormalizable Hénon-like map with stationary combinatorics.

Proposition 14.32. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exist $\overline{\varepsilon} > 0$, $c > 0$, and $c' > 1$ such that for all $F \in \mathscr{H}_{\delta}^{\sigma^\infty}(I^h \times I^v, \overline{\varepsilon})$, the following properties hold for all $n \geq 0$:*

letter$_n^{t,(j)}(k)$		
letter	F	Hénon-like map
	ϕ	Rescaling map
	Φ_n^j	Rescaling from level n to level $n+j$
	a	Periodic or fixed point with a positive expanding multiplier
	b	Periodic or fixed point with a negative expanding multiplier
	t	Tip
	Greek	Separator (or local stable manifold): α, β, τ
	α	Local stable manifold of a
	β	Local stable manifold of b
	τ	Stable manifold of the tip t
	capital	Vertical strip: B, C, D, P, T, Q, R
	B	The vertical strip $[\beta^0(1), \overline{\alpha^0(1)}]$
	C	The vertical strip $[\overline{\alpha^1(1)}, \beta^1(1)]$
	D	Iteration set
	P	Cyclic set (Sets invariant under iterations)
	T	Trapping set
	Q	Prerescaling set / Prerescaling level
	R	Rescaling set / Rescaling level
n	Renormalization level. The object belongs to the domain of $F_n = R^n F$.	
t	Time. The superscript is neglected when $t = 0$. e.g. $F_n^t = F_n \circ \cdots \circ F_n$ (t times) , $\Phi_n^t = \phi_{n+t-1} \circ \cdots \circ \phi_{n+1} \circ \phi_n$ e.g. $P_n(1) = P_n^0$, $P_n^0 \xrightarrow{F_n} P_n^1 \xrightarrow{F_n} P_n^2 \xrightarrow{F_n} \cdots \xrightarrow{F_n} P_n^{p-1}$	
(j)	Row. See Definition 18.4.	
k	Rescaling level. The point or set that is pulled back from k levels deeper. e.g. $P_n(2) \xrightarrow{\phi_n} P_{n+1}(1) \xrightarrow{\phi_{n+1}} P_{n+2}(0)$, e.g. $F_n^{p^2}(P_n(2)) \subset P_n(2)$, $F_n^p(P_n(1)) \subset P_n(1)$, $F_n^{p^0}(P_n(0)) \subset P_n(0)$ e.g. $R_n(2) \xrightarrow{\phi_n} R_{n+1}(1) \xrightarrow{\phi_{n+1}} R_{n+2}(0) \subset D_{n+2} \cup B_{n+2}$	

Table 14.1.: Summary of notations.

1. *The points $a_n(j)$ and $b_n(j)$ are periodic points of F_n. When $p \neq 2$, the points $a_n(j)$ and $b_n(j)$ has period p^j for all $j \geq 0$. When $p = 2$, the points coincide $a_n(j+1) = b_n(j)$ and $b_n(j)$ has period p^j for all $j \geq 0$.*

2. *The sets $\alpha_n(j)$ and $\overline{\alpha_n(j)}$ are local stable manifolds of the periodic point $a_n(j)$ and the sets $\beta_n(j)$ and $\overline{\beta_n(j)}$ are local stable manifolds of the periodic point $b_n(j)$.*

3. *(analog of the critical value) The intersection contains exactly one point*

$$\{t_n\} = \cap_{j=0}^{\infty} \left(\Phi_n^j\right)^{-1} \left(P_{n+j}(0) \cap \left(I^h \times I^h\right)\right).$$

   The point t_n is called the tip of the Hénon-like map F_n.

4. *(geometric property) The local stable manifolds $\alpha_n(j)$, $\overline{\alpha_n(j)}$, $\beta_n(j)$, and $\overline{\beta_n(j)}$ are separators with Lipschitz constant $c\,\|\varepsilon_n\|$ for all $j \geq 0$. The local stable manifolds all converges to the "local stable manifold of the tip" τ_n as $j \to \infty$. On the horizontal slice that intersects the tip t_n, the manifolds satisfies the inequality*

$$\frac{1}{c'}\left(\frac{1}{\lambda_\sigma}\right)^{2j} < |z_n(j) - t_n| < c'\left(\frac{1}{\lambda_\sigma}\right)^{2j}$$

   where $z_n(j)$ is the intersection point of any one of the local stable manifolds $\alpha_n(j)$, $\overline{\alpha_n(j)}$, $\beta_n(j)$, and $\overline{\beta_n(j)}$ with the horizontal line that intersects the tip t_n.

5. *(self-similarity) The local stable manifolds satisfies the order*

$$\alpha_n(j) \prec \overline{\beta_n(j)} \prec \beta_n(j) \prec \alpha_n(j+1) \prec \overline{\alpha_n(j+1)} \prec \overline{\alpha_n(j)}$$

   for all $j \geq 0$.

Proof. The first two properties follow from the definition of the rescaling map ϕ_n and the definition of renormalization.

The third property comes from [dCLM05, Chapter 5] for the period-doubling case and [Haz11, Section 3.4 and Chapter 5] for the arbitrary stationary combinatorics case.

The proof of the fourth property is similar to [LM11, Proposition 3.5]. The next lemma, Lemma 14.34, proves that the local manifolds are separators with Lipschitz constant $c\,\|\varepsilon_n\|$. The estimation of the cross-section comes from [Haz11, Proposition 5.6]. They all converges to the same set τ_n as $j \to \infty$ because no open sets can be rescaled infinity many times since the horizontal domain is bounded.

The fifth property is true because it holds for $j = 0$ by the definition of renormalizable. The property also holds for other j values because the rescaling map ϕ_n preserves the orientation along the x-direction. □

Remark 14.33. One can also define the orbit $a^t(j) = F^t(a(j))$ of the periodic point $a(j)$ for $t = 0, \cdots, p^j - 1$. The reason to start from the index $t = 0$ here and in Definition 14.15 is because of the equality $a_{n+1}^t(j) = \phi_n(a_n^{pt}(j+1))$ for $t = 0, \cdots, p^j - 1$.

Lemma 14.34. *Given $\delta > 0$, $c > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exists $\overline{\varepsilon} > 0$ such that for all $F \in \mathscr{H}^{\sigma}_{\delta}(I^h \times I^v, \overline{\varepsilon})$ the following property holds for all $n \geq 0$:*

Assume that ω is a separator with Lipschitz constant $c\,\|\varepsilon_{RF}\|$ in $P_{RF}(0)$. Then $\chi = \phi^{-1}(\omega)$ is a separator with Lipschitz constant $c\,\|\varepsilon_F\|$ in $P_F(1)$.

Proof. The proof is similar to [LM11, Proposition 3.5]. □

14.6. Trapping Sets and Gaps

Trapping sets and gaps are introduced in this section. They are generalizations of the intervals from Definition 13.14. Extending the topological properties from unimodal maps to Hénon-like maps are routine but not trivial. It relies on the tools developed in Section 14.2 and the definition of Hénon-renormalization. Justifications will be left to the reader for properties that are similar to unimodal maps. To obtain the uniform expansion of the hyperbolic size, the dynamics of the Hénon-like map will be studied on a smaller set partitioned by trapping sets and gaps instead of the larger domain $P(0)$.

Definition 14.35 (Trapping Set and Gap)**.** Given $\overline{\varepsilon} > 0$ sufficiently small, $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and $F \in \mathscr{H}^{\sigma^\infty}_{\delta}(I^h \times I^v, \overline{\varepsilon})$.

Define $\beta^0(1) = \beta(1)$ and $\overline{\beta^0(1)} = \overline{\beta(1)}$. The sequence of local stable manifolds $\{\beta^t(1)\}_{t=0}^{p-1}$ and $\{\overline{\beta^t(1)}\}_{t=0}^{p-1}$ are defined similar to the local stable manifolds $\{\alpha^t(1)\}_{t=0}^{p-1}$ and $\{\overline{\alpha^t(1)}\}_{t=0}^{p-1}$. See Definition 14.15.

The trapping sets are defined to be $T^t = [\beta^t(1), \overline{\beta^t(1)}]$ for $t = 0, \cdots, p-1$. The set T^{p-1} is called the center trapping set.

A gap is a vertical strip between two neighboring trapping intervals. Precisely, it is a connected component of $[\overline{\beta^1(1)}, \overline{\beta^0(1)}] \setminus \cup_{t=2}^{p-1} \operatorname{int}(T^t)$.

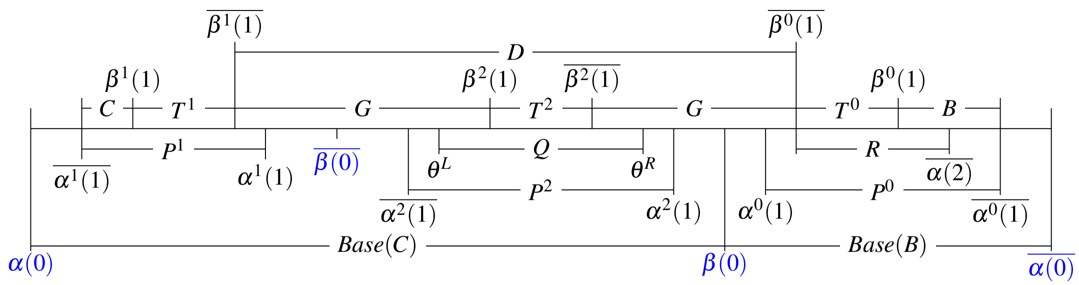
For the local stable manifold $\overline{\beta(1)}$, assume the preimage $F^{-1}(\overline{\beta(1)})$ contains two components and the two components are separators. Define θ^L and θ^R to be the left and right components of the preimage respectively.

Figure 14.1a shows an example for the period-tripling case.

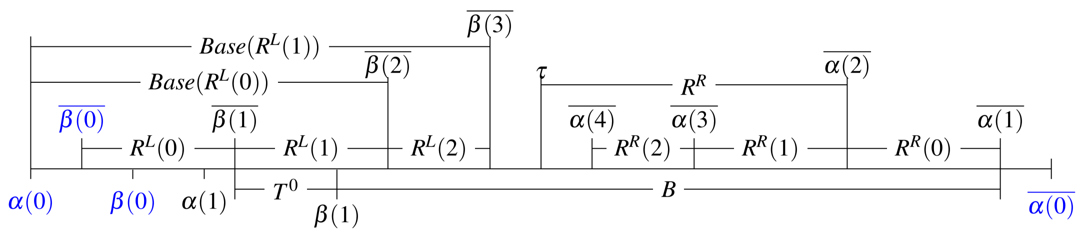
The following proposition summarizes the geometric properties of the local stable manifolds.

Proposition 14.36. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exist $\overline{\varepsilon} > 0$ and $c > 0$ such that for all $F \in \mathscr{H}^{\sigma^\infty}_{\delta}(I^h \times I^v, \overline{\varepsilon})$, the following properties hold:*

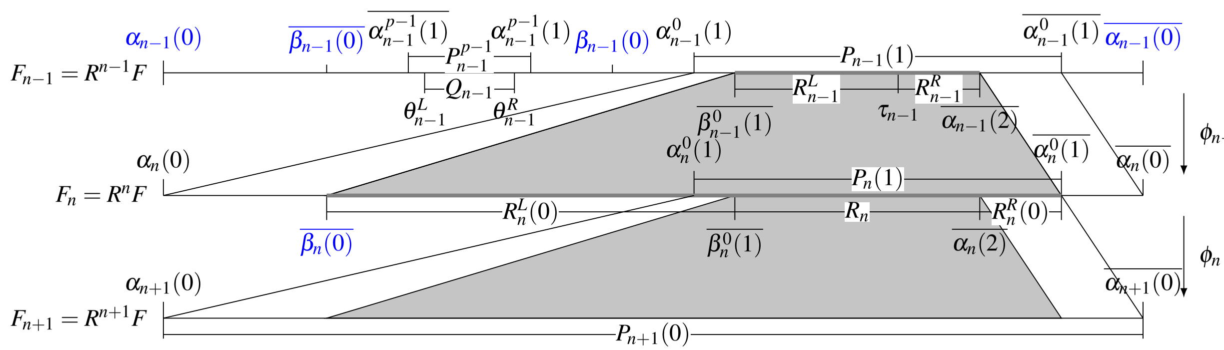
1. *The separators $\beta^t(1)$, $\overline{\beta^t(1)}$, θ^L, and θ^R exist for $t = 0, \cdots, p-1$. In fact, the local stable manifolds are separators with Lipschitz constant $c\,\|\varepsilon\|$.*
2. *The set $\{\alpha(0), \overline{\alpha(0)}, \beta(0), \overline{\beta(0)}, \alpha^0(1), \overline{\alpha^0(1)}, \beta^0(1), \overline{\beta^0(1)}, \cdots, \alpha^{p-1}(1), \overline{\alpha^{p-1}(1)}, \beta^{p-1}(1), \overline{\beta^{p-1}(1)}, \theta^L, \theta^R\}$ forms a collection of compatible separators and f^s is an induced unimodal map on the separators. The induced unimodal map f^s is increasing on $[\alpha(0), \beta^{p-1,L}(1)]$ and decreasing on $[\beta^{p-1,R}(1), \overline{\alpha(0)}]$ where $\beta^{p-1,L}(1), \beta^{p-1,R}(1) \in \{\beta^{p-1}(1), \overline{\beta^{p-1}(1)}\}$ with $\beta^{p-1,L}(1) \prec \beta^{p-1,R}(1)$.*



(a) The trapping sets, cyclic sets, gaps, and the extra strips B and C for the case $p = 3$.



(b) The structure of the rescaling set.



(c) The dynamics of rescaling. The set Q_{n-1} is mapped to $R_{n-1} \cup R^R_{n-1}(0)$. In addition, if the Hénon-like map is close to the renormalization fixed point, then Q_{n-1} is mapped into R_{n-1}. Then R_{n-1} is rescaled to $D_n \cup R_n \cup R^R_n(0)$. Finally, the set $R^R_n(0) \subset B_n$ is mapped to C_n then to D_n.

Figure 14.1.: The partition of an infinitely period-tripling renormalizable Hénon-like map. The figure illustrates the order of the local stable manifolds. Note that $\beta(1) \equiv \beta^0(1)$ and $\overline{\beta(1)} \equiv \overline{\beta^0(1)}$ by definition.

Proof. The first and second properties are proved by graph transformation. See [LM11, Lemma 3.1, 3.2] for the period-doubling case. □

Corollary 14.37. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exists $\overline{\varepsilon} > 0$ such that for all $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, \overline{\varepsilon})$, we have*

1. $F(T^t) \subset T^{t+1}$ *for* $0 \leq t < p-1$ *and*
2. $F(T^{p-1}) \subset [\beta^0(1), \overline{\alpha^0(1)}]$.

Proof. The first property follows from $g(T^t) = T^{t+1}$, Propositions 14.10, and 14.36.

For the second property, we have $F(T^{p-1}) \subset [\alpha^0(1), \overline{\alpha^0(1)}] = P^0$ by the definition of a renormalizable map. The image $F(T^{p-1})$ is disjoint from the interior of the vertical strip $[\alpha^0(1), \overline{\beta^0(1)}]$ because that the interior of the preimage $F^{-1}([\alpha^0(1), \beta^0(1)]) = [\overline{\alpha^{p-1}(1)}, \beta^{p-1}(1)] \cup [\overline{\beta^{p-1}(1)}, \alpha^{p-1}(1)]$ is disjoint from T^{p-1}. □

14.7. Dynamics of Wandering Domain

The dynamics of a wandering domain is studied in this section. Recall the definition of a wandering domain.

Definition 14.38 (Wandering domain)**.** Given $\overline{\varepsilon} > 0$ sufficiently small, $\delta > 0$, and intervals I^h and I^v with $I \Subset I^h \subset I^v$. Assume that $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$, $P(0)$ exists, and F is an open map (diffeomorphism from $P(0)$ to its image). A nonempty connected open set $J \subset P(0)$ is a *wandering domain* of F if the orbit elements $\{F^n(J)\}_{n \geq 0}$ are disjoint from the stable manifolds of any periodic point.

A wandering domain can be constructed from a wandering domain by iteration and rescaling.

Proposition 14.39. *Assume that $\overline{\varepsilon} > 0$ is sufficiently small, $\delta > 0$, I^h and I^v are intervals with $I \Subset I^h \subset I^v$, σ is a unimodal permutation, and $F \in \mathscr{H}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$.*

1. *If $J \subset P(0)$ is a wandering domain of F, then $F(J) \subset P(0)$ is also a wandering domain of F.*
2. *If $J \subset P_F(1)$ is a wandering domain of F, then $\phi(J) \subset P_{RF}(0)$ is a wandering domain of RF.*

A wandering domain will be iterated or rescaled based on the dynamics on the iteration set and the rescaling set. The sets are similar to the intervals defined for the unimodal case with some adjustments due to the fact that $F(P^{p-1}) \not\subset [\alpha^0(1), \tau]$ for the non-degenerate case where τ is the local stable manifold of the tip t defined in Proposition 14.32. The sets are defined as follows.

Definition 14.40. Given $\overline{\varepsilon} > 0$ sufficiently small, $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, \overline{\varepsilon})$.

The *iteration set* is defined to be $D = [\overline{\beta^1(1)}, \overline{\beta^0(1)}]$.

The *rescaling set* is defined to be $R = [\overline{\beta^0(1)}, \overline{\alpha(2)}]$. The *rescaling level j* in R is defined to be $R(j) = R^L(j) \cup R^R(j)$ where $R^L(j) = [\overline{\beta(j)}, \overline{\beta(j+1)}]$ and $R^R(j) = [\overline{\alpha(j+2)}, \overline{\alpha(j+1)}]$ for $j \geq 0$. Let $R^L = \cup_{j=1}^\infty R^L(j)$ and $R^R = \cup_{j=1}^\infty R^R(j)$.

The *prerescaling set* is defined to be $Q = [\theta^L, \theta^R]$. The *rescaling level* j in Q is defined to be $Q(j) = F^{-1}(R(j))$ for $j \geq 1$.

Define $B = [\beta^0(1), \overline{\alpha^0(1)}]$ and $C = [\overline{\alpha^1(1)}, \beta^1(1)]$.

See Figure 14.1 for an example of the period-tripling case.

Remark 14.41. The rescaling levels in Q cannot be defined by boundary separators. This is because that the preimage of the local stable manifolds may not be separator in general. The levels where this happen are the locations where the expansion argument breaks down. See Figure 6.1, Figure 9.1, and Proposition 15.2 later.

The rescaling level of a set on the rescaling set R or the prerescaling set Q is defined as follows.

Definition 14.42 (Rescaling level)**.** Assume that $U \subset R$ is a connected set that does not intersect any stable manifolds. The *rescaling level* of U in the rescaling set R is the positive integer $k(U)$ such that $U \subset R(k(U))$.

Similarly, if $U \subset Q$ is a connected set that does not intersect any stable manifolds, the *rescaling level* of U in the prerescaling set Q is the positive integer $k(U)$ such that $U \subset Q(k(U))$. For convenience, set $k(U) = 0$ if $U \subset B \cup C \cup (D \setminus Q)$.

We will consider wandering domains that belongs to B, C, D, or R. A wandering domain will be iterated or rescaled by the following rules:

1. If the wandering domain is in B, C, or D, then it is iterated by F.
2. If the wandering domain is in R, then it is rescaled by ϕ.

The next goal is to shows that the rescale orbit of a wandering domain that follows the rule always stays inside the sets B, C, D, and R. The vertical strips B and C, which are addition to the unimodal case, come from the case when the orbit of a wandering domain enters the bad region which will be defined later in Chapter 15. The following proposition generalizes Proposition 13.30 to Hénon-like maps.

Proposition 14.43. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exists $\overline{\varepsilon} > 0$ such that for all $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, \overline{\varepsilon})$ and $n \geq 0$, the iteration set and rescaling set satisfy the following properties:*

1. $F_n(Q_n) \subset T^0 \cup B = R_n \cup R_n^R(0)$.
2. $F_n(D_n) \subset D_n \cup R_n \cup R_n^R(0)$.
3. $F_n(B_n) \subset C_n$.
4. $F_n(C_n) \subset G_n \subset D_n$ *if $p \geq 3$ where G_n is the gap left to T_n^2.*
5. $R_n = \cup_{j=1}^{\infty} R_n(j)$. *For $j \geq 1$,* $\phi_n(R_n(j)) = R_{n+1}(j-1)$, $\phi_n(R_n^L(j)) = R_{n+1}^L(j-1)$, *and* $\phi_n(R_n^R(j)) = R_{n+1}^R(j-1)$.
6. $\phi_n(R_n) = R_{n+1}(0) \cup R_{n+1}$ *and* $R_n^R(0) \subset B_n$. *If σ is admissible, then* $R_n^L(0) \subset D_n$.

In addition, when the map F is close to the fixed point $i(f_\sigma)$, i.e. $F \in \mathscr{I}^{\sigma}_{\delta}(I^h \times I^v, \overline{\varepsilon})$ for some small $\overline{\varepsilon} > 0$, the first two conditions can be improved as

1. $F_n(Q_n) \subset R_n$ *and*
2. $F_n(D_n) \subset D_n \cup R_n$.

Proof. The first property is true because $F_n(T_n^{p-1}) \subset [\beta_n^0(1), \overline{\alpha_n^0(1)}]$ by Corollary 14.37 and $f_n^s([\theta_n^L, \beta_n^{p-1,L}(1)]) = f_n^s([\beta_n^{p-1,R}(1), \theta_n^R]) = T_n^0$ where $\beta_n^{p-1,L}(1), \beta_n^{p-1,R}(1) \in \{\beta_n^{p-1}(1), \overline{\beta_n^{p-1}(1)}\}$ and $\beta_n^{p-1,L}(1) \prec \beta_n^{p-1,R}(1)$. Thus, $F_n(Q_n) \subset R_n \cup [\overline{\alpha_n(2)}, \overline{\alpha_n(1)}]$ by Proposition 14.10.

To prove the second property, the iteration set D_n can be separated into three parts $[\overline{\beta_n^1(1)}, \theta_n^L]$, $[\theta_n^R, \overline{\beta_n^0(1)}]$, and Q_n. The iteration of the first two components are

$$f_n^s([\overline{\beta_n^1(1)}, \theta_n^L]) = [\overline{\beta_n^2(1)}, \overline{\beta_n^0(1)}] \subset D_n$$

and

$$f_n^s([\theta_n^R, \overline{\beta_n^0(1)}]) = [\overline{\beta_n^1(1)}, \overline{\beta_n^0(1)}] \subset D_n.$$

Thus, the second property follows from Proposition 14.10.

The third property follows from $f_n^s(B_n) = C_n$ and Proposition 14.10.

The fourth property follows from $f_n^s(C_n) = [\overline{\alpha^2(1)}, \beta^2(1)]$, Proposition 14.10, and the vertical strip $[\overline{\alpha^2(1)}, \beta^2(1)]$ is inside the gap left to T_n^2 when $p \geq 3$.

The fifth property follows from definition of the boundary local stable manifolds and Proposition 14.32.

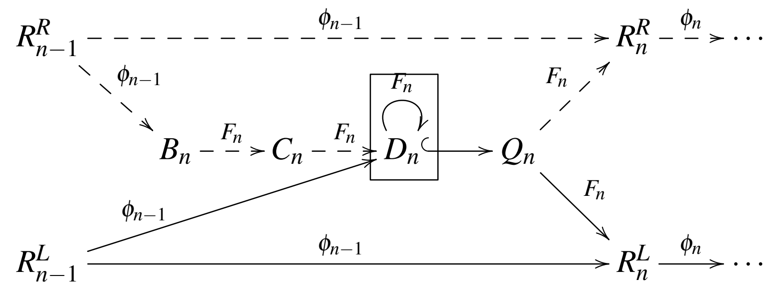
For the last property, $\phi_n(R_n) = R_{n+1}(0) \cup R_{n+1}$ follows from the fifth property. The property $R_n^R(0) \subset B_n$ follows from the fifth property of Proposition 14.32. See Figure 14.1b. If σ is admissible, then the property $R_n^L(0) \subset D_n$ holds because Hénon-like maps preserves the order of the period orbits from unimodal maps. See the second property of Proposition 13.30 for the proof. Note that the results for admissible permutations also apply to Hénon-like maps.

When the map is close to the fixed point $i(f_\sigma)$, we have $F_n(P_n(0)) \cap R_n^R(0) = \phi$ by Proposition 15.2 (first property of the good region) later. This gives the improvement of the first two conditions. □

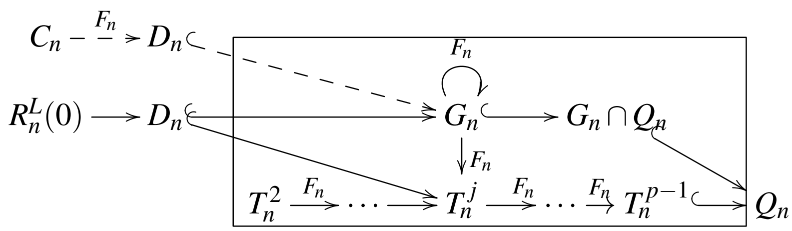
It is sufficient to study the problem of wandering domain in the iteration set because of the following proposition.

Proposition 14.44. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exists $\overline{\varepsilon} > 0$ such that if $F \in \mathscr{H}^{\sigma}_{\delta}(I^h \times I^v, \overline{\varepsilon})$ has a wandering domain in $P(0)$, then it also has a wandering domain in the vertical strip $[\overline{\beta(0)}, \beta(0)]$. In particular, if σ is an admissible permutation and $F \in \mathscr{H}^{\sigma^2}_{\delta}(I^h \times I^v, \overline{\varepsilon})$, then F has a wandering domain in D.*

Proof. The proof is similar to the unimodal case Proposition 13.32. The vertical strip $P(0)$ can be partitioned into three components: $[\alpha(0), \overline{\beta(0)}]$, $[\overline{\beta(0)}, \beta(0)]$, and $[\beta(0), \overline{\alpha(0)}]$. The components satisfy $f^s([\alpha(0), \overline{\beta(0)}]) = f^s([\beta(0), \overline{\alpha(0)}]) = [\alpha(0), \overline{\beta(0)}] \cup [\overline{\beta(0)}, \beta(0)]$. Thus, we only need to consider the orbit of a wandering domain on the vertical strip $[\alpha(0), \overline{\beta(0)}]$. When a wandering domain is close to the saddle fixed point $a(0)$, its orbit follows the unstable manifold and moves away from the point. When a wandering domain is far from the saddle fixed point, the x-component



(a) Brief diagram of the dynamics in different levels.



(b) Details of the step $D_n \to D_n$ are illustrated in the rectangle.

Figure 14.1.: The dynamics of a wandering domain on the partition when the Hénon-like map is close to the hyperbolic fixed point of the renormalization operator. The arrow $\xrightarrow{F_n}$ means iterated by F_n, the arrow $\xrightarrow{\phi_n}$ means rescaled by ϕ_n, and the arrow $\hookrightarrow$ means belongs to the target interval without iteration and rescaling. The dashed arrow $\dashrightarrow$ emphasizes the paths that are addition to the unimodal case (compare Figure 13.1).

of its orbit is increasing when $\overline{\varepsilon} = \overline{\varepsilon}(\kappa) > 0$ is sufficiently small. Therefore, the proposition follows by Proposition 14.39. $\square$

The dynamics of a wandering domain will be studied on the iteration set and the rescaling set. Figure 14.1a describes the dynamics on the iteration set and the rescaling set from Proposition 14.43 when the Hénon-like map is close to the hyperbolic fixed point $i(f_\sigma)$ of the renormalization operator. Similar to the unimodal case, the iteration set can be partitioned by trapping sets and gaps. A wandering domain is contained fully inside a partition element because it cannot intersect any stable manifolds. Thus, the dynamics of a wandering domain follows the dynamics of trapping sets and gaps. A more detail diagram for the dynamics on the partition elements is illustrated in Figure 14.1b. The details are left to the reader.

Finally, if a Hénon-like map has a wandering domain, a subsequence of a rescaled orbit of the wandering domain, called the closest approach, can be constructed to study the dynamics of the wandering domain. The construction follow the rules for iteration set and rescaling set. The sequence is defined as follows.

Definition 14.45 (Closest approach)**.** Assume that $\overline{\varepsilon} > 0$ is sufficiently small, $\delta > 0$, I^h and I^v are intervals with $I \Subset I^h \subset I^v$, an admissible unimodal permutation, and $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, \overline{\varepsilon})$.

Given a set $J \subset B \cup C \cup D$ that does not intersect any of the stable sets. Define a sequence of sets $\{J_n\}_{n=0}^\infty$ and the associate renormalization level $\{r(n)\}_{n=0}^\infty$ by induction such that $J_n \subset B_{r(n)} \cup C_{r(n)} \cup D_{r(n)}$ for all $n \geq 0$ as follows.

1. Set $J_0 = J$ and $r(0) = 0$.
2. Abbreviate the rescaling level $k_n = k(J_n)$ whenever J_n is defined.
3. If $J_n \subset B_{r(n)} \cup C_{r(n)} \cup (D_{r(n)} \backslash Q_{r(n)})$, set $J_{n+1} = F_{r(n)}(J_n)$ and $r(n+1) = r(n)$.
4. If $J_n \subset Q_{r(n)}$, set $J_{n+1} = \Phi_{r(n)}^{k_n} \circ F_{r(n)}(J_n)$ and $r(n+1) = r(n) + k_n$.

The transition between two constitutive sequence element $J_n \to J_{n+1}$, one iteration together plus some number of rescaling if possible, is called *one step*. The sequence $\{J_n\}_{n=0}^{\infty}$ is called the rescaled iterations of J that closest approaches to the tip, or J-*closest approach* for short.

15. The Good Region and the Bad Region

In this chapter, the rescaling levels $\{R_n(j)\}_{j=1}^{\infty}$ and prerescaling levels $\{Q_n(j)\}_{j=1}^{\infty}$ will be grouped into two regions, called the good region and the bad region.

The regions were introduced in Chapter 9 to prove the nonexistence of wandering domain for the period-doubling case. The good region is an area when the rescaling levels j are small. In the good region, the topology and the dynamics of a Hénon-like map behave similar to a unimodal map. In particular, a prerescaling level is the union of two vertical strips (Figure 15.1a) and the expansion argument holds. On the contrary, the bad region is an area when rescaling levels j are large. It is a special feature in higher dimension: a degenerate Hénon-like map does not have bad region. In the bad region, the topology and the dynamics behave different from a unimodal map. In particular, a prerescaling level has only one component which looks like an arc (Figure 15.1b) and the expansion argument fails.

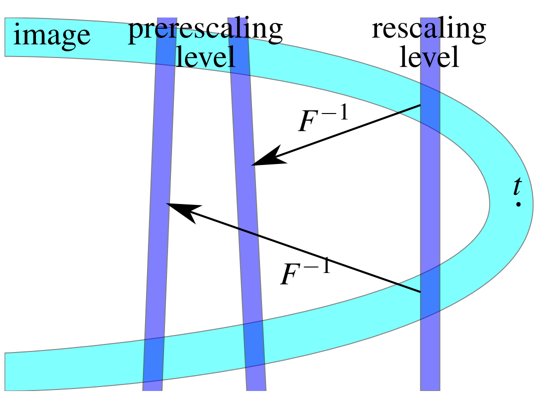
The concept of the regions can be generalized from the period-doubling case to arbitrary stationary combinatorics directly. Recall the definition from the period-doubling case (Definition 9.1).

Definition 15.1 (The Good Region and the Bad Region)**.** Fix a constant $b > 0$. Assume that $\overline{\varepsilon} > 0$ is sufficiently small and $F \in \mathscr{I}_{\delta}^{\sigma}(I^h \times I^v, \overline{\varepsilon})$ so that geometric properties of the local stable manifolds $\alpha_n(j)$, $\overline{\alpha_n(j)}$, $\beta_n(j)$, and $\overline{\beta_n(j)}$ hold (Proposition 14.32). For each $n \geq 0$, define $K_n = K_n(b)$ to be the largest positive integer such that

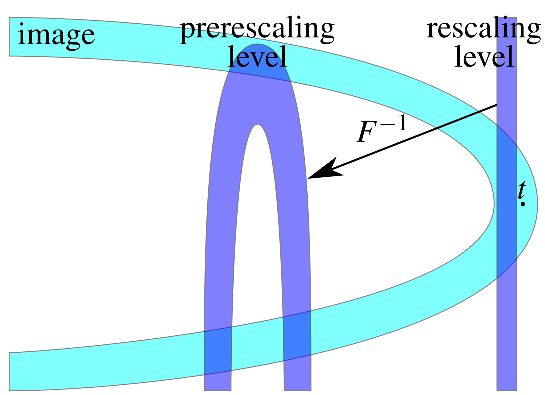
$$\left|\pi_x z_n^{(0)}(K_n) - \pi_x t_n\right| > b\,\|\varepsilon_n\|$$

where $z_n^{(0)}(K_n)$ is the intersection point of $\alpha_n(K_n)$ with the horizontal line through the tip t_n.

The rescaling level $R_n(j)$ (resp. prerescaling level $Q_n(j)$) is said to be in the *good region* if



(a) Good region



(b) Bad region

Figure 15.1.: Topology of the good region and the bad region

$j \le K_n$; in the *bad region* if $j > K_n$. The sequence K_n is called the *boundary* for the good region and the bad region.

The next proposition summarizes the properties of the good region and the bad region.

Proposition 15.2 (Geometric properties for the good region and the bad region)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exist $\overline{\varepsilon} > 0$, $b > 0$, and $c > 1$ such that for all $F \in \mathscr{I}^{\sigma}_{\delta}(I^h \times I^v, \overline{\varepsilon})$ the following properties hold for all $n \ge 0$:*

The boundary $K_n = K_n(b)$ satisfies the estimation

$$\frac{1}{c}\frac{1}{\sqrt{\|\varepsilon_n\|}} \le \lambda_{\sigma}^{K_n} \le c\frac{1}{\sqrt{\|\varepsilon_n\|}}. \tag{15.1}$$

In the good region, we have

1. *$R^R_n(j) \cap F_n(P_n(0)) = \phi$ for $0 \le j \le K_n + 1$,*
2. *$|\pi_x z - \pi_x t_n| > \frac{1}{c}\|\varepsilon_n\|$ for all $z \in R_n(j) \cap F_n(P_n(0))$ and $1 \le j \le K_n + 1$,*
3. *$\left|\pi_x z - \pi_y t_n\right| > \frac{1}{c}\sqrt{\|\varepsilon_n\|}$ for all $z \in Q_n(j)$ and $1 \le j \le K_n + 1$,*
4. *$\left|\frac{\partial h_n}{\partial x}(z)\right| > \frac{1}{c}\sqrt{\|\varepsilon_n\|}$ for all $z \in Q_n(j)$ and $1 \le j \le K_n + 1$,*
5. *The preimage $F_n^{-1}(\overline{\beta_n(j)})$ contains exactly two components for $1 \le j \le K_n + 2$. The two components are both separators with Lipschitz constant $c\sqrt{\|\varepsilon_n\|}$. Denote the left and right components as $\theta^L_n(j)$ and $\theta^R_n(j)$ respectively. The prerescaling level $Q_n(j)$ is the union of two disjoint vertical strips $Q^L_n(j)$ and $Q^R_n(j)$ for $1 \le j \le K_n + 1$ where $Q^L_n(j) = [\theta^L_n(j), \theta^L_n(j+1)]$ and $Q^R_n(j) = [\theta^R_n(j+1), \theta^R_n(j)]$.*

In the bad region $j > K_n$, we have

1. *$|\pi_x z - \pi_x t_n| < c\|\varepsilon_n\|$ for all $z \in R_n(j) \cap F_n(P_n(0))$ and*
2. *$\left|\pi_x z - \pi_x \circ F_n^{-1}(t_n)\right| = \left|\pi_x z - \pi_y t_n\right| < c\sqrt{\|\varepsilon_n\|}$ for all $z \in Q_n(j)$.*

Proof. The fourth property of the good region follows from the third property and the proof of Lemma 6.21.

The sixth property of the good region follows from the graph transformation and the fourth property.

Other properties are similar to the period-doubling case. The proof depends only on the perturbation ε and the geometric structure of the local stable manifolds and the tip which are the same as the period-doubling case. See Proposition 6.27 and Proposition 9.4 for the proof. □

Remark 15.3. By definition, $\theta^L_n(1) = \theta^L_n$ and $\theta^R_n(1) = \theta^R_n$.

In the remaining part of this article, the parameter b will be a fixed large value that makes the proposition hold so the boundaries K_n are also constants that depends only on the Hénon-like map.

16. Hyperbolic Size

Hyperbolic size is introduced in this chapter. It is a generalization of the hyperbolic length from the dimension one setting. It measures the relative horizontal displacement of a set inside a vertical strip. The expansion of hyperbolic size for the sequence elements in a closest approach will be studied later to prove the nonexistence of wandering domain.

Regular curves are used to measure the hyperbolic size of a set inside a vertical strip. It is defined as follows.

Definition 16.1 (Regular curve)**.** Given a parameter $R > 0$. The graph of a C^3 function $r : [a,b] \subset I^h \to I^v$ is said to be *R-regular* (with respect to the Hénon-like map F) if

1. $\|r'\| < \frac{R}{\|\varepsilon\|^{1/4}}$,
2. $\|r''\| < \frac{R}{\|\varepsilon\|}$,
3. $\|r'''\| < \frac{R}{\|\varepsilon\|}$, and
4. $\|r'\|\,\|r''\| < \frac{R}{\|\varepsilon\|}$.

Remark 16.2. These are the conditions that make the restriction of a Hénon-like map to a regular curve preserves the property of negative Schwarzian derivative. This produces the expansion of hyperbolic size for a set under iteration. See the proof of Lemma 17.1 and Remark 17.2 later. Chapter 10 has a geometric explanation for the first condition.

An R-regular curve is also a Lipschitz curve with constant $\frac{R}{\|\varepsilon\|^{1/4}}$. The tools for Lipschitz curves in Section 14.2 applies to regular curves.

If a set is inside a vertical strip, then we can measure the hyperbolic size of the set by R-regular curves.

Definition 16.3 (Hyperbolic Size)**.** Assume that $\overline{\varepsilon} > 0$, $\delta > 0$, I^h and I^v are intervals with $I \Subset I^h \subset I^v$, $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$ is a Hénon-like map, R and L are positive constants with $RL < 1$, and $\mathscr{X}$ is a collection of disjoint separators with Lipschitz constant $L\|\varepsilon\|^{1/4}$. Given a vertical strip $S = [\alpha, \beta]$ and a set $J \subset S$ where $\alpha, \beta \in \mathscr{X}$.

The *hyperbolic size* of J in S is defined to be

$$|J|_S \equiv \sup_{(r,c,d)} |[c,d]|_{[a,b]}$$

where the supremum is evaluated over all R-regular curves $r : [a,b] \to I^v$ and all constants $c, d \in (a,b)$ that satisfies

1. the two ends $(a, r(a))$ and $(b, r(b))$ of the graph are attached to the two boundaries α and β of the strip S respectively and

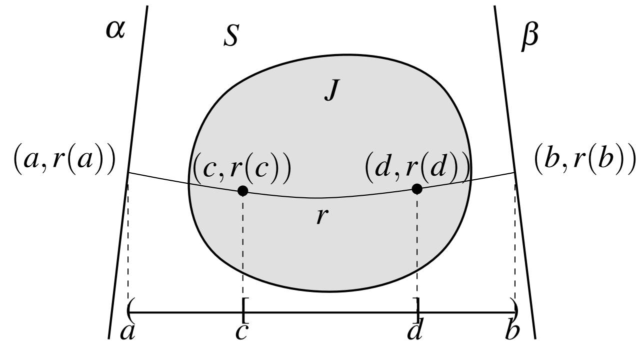


Figure 16.1.: Hyperbolic size measured by regular curves.

2. the two points $(c,r(c))$ and $(d,r(d))$ on the graph belong to the set J.

See Figure 16.1 for illustration.

The following proposition is an analog of Corollary 2.15. It says that the hyperbolic size is larger when measuring the hyperbolic size inside a smaller base set.

Proposition 16.4. *Assume that $\overline{\varepsilon} > 0$, $\delta > 0$, I^h and I^v are intervals with $I \Subset I^h \subset I^v$, $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$ is a Hénon-like map, R and L are positive constants with $RL < 1$, and $\mathscr{X}$ is a collection of disjoint separators with Lipschitz constant $L\|\varepsilon\|^{1/4}$.*

If S_1 and S_2 are vertical strips with boundaries in $\mathscr{X}$ such that $S_2 \subset S_1$, then

$$|J|_{S_2} \geq |J|_{S_1}$$

for all subsets J in the vertical strip S_2 where the hyperbolic size is measured by R-regular curves.

Proof. Assume that $S_1 = [\alpha_1, \beta_1]$ and $S_2 = [\alpha_2, \beta_2]$ with $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$. Then $\alpha_1 \preceq \alpha_2 \prec \beta_2 \preceq \beta_1$.

Let $r : [a_1, b_1] \to I^v$ be an R-regular curve that intersects J at $(c, r(c))$ and $(d, r(d))$ with $c < d$ and the two endpoints $(a_1, r(a_1))$ and $(b_1, r(b_1))$ are attached to α_1 and β_1 respectively. By Proposition 14.6, the curve intersects α_2 and β_2 at $(a_2, r(a_2))$ and $(b_2, r(b_2))$ respectively and $a_1 \leq a_2 < c < d < b_2 \leq b_1$. Then

$$|J|_{S_2} \geq |[c,d]|_{[a_2,b_2]} \geq |[c,d]|_{[a_1,b_1]}$$

by Corollary 2.15 and the definition of hyperbolic size. Therefore, the proposition follows because r, c, and d are arbitrary chosen. $\square$

The next proposition is an analog of Proposition 2.17. It is an important property that allows us to quantify the expansion of hyperbolic size when a wandering domain is embedded from a larger vertical strip to a smaller vertical strip. The expansion depends on the two spacing between the larger vertical strip and the smaller vertical strip. In Chapter 17 later, the proposition will be used to estimate the size of the expansion given by the expansion from the topology.

Define the *minimal displacement* of two separators α and β as

$$\mathrm{SDisp}(\alpha, \beta) = \inf\{|a_x - b_x| ; (a_x, a_y) \in \alpha, (b_x, b_y) \in \beta\} \tag{16.1}$$

and the *maximal displacement* as

$$\mathrm{LDisp}(\alpha,\beta) = \sup\{|a_x - b_x|\,;(a_x,a_y)\in\alpha,(b_x,b_y)\in\beta\}.$$

Proposition 16.5. *Assume that $\overline{\varepsilon} > 0$, $\delta > 0$, I^h and I^v are intervals with $I \Subset I^h \subset I^v$, $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$ is a Hénon-like map, R and L are positive constants with $RL < 1$, and $\mathscr{X}$ is a collection of disjoint separators with Lipschitz constant $L\|\varepsilon\|^{1/4}$.*

If $S_1 = [\alpha_1,\beta_1]$ and $S_2 = [\alpha_2,\beta_2]$ are vertical strips with $\alpha_1 \prec \alpha_2 \prec \beta_2 \prec \beta_1$, $\frac{SDisp(\alpha_1,\alpha_2)}{LDisp(\alpha_1,\beta_1)} > M$, and $\frac{SDisp(\beta_1,\beta_2)}{LDisp(\alpha_1,\beta_1)} > M$ for some constant $M > 0$, then

$$|J|_{S_2} \geq \frac{1}{1-M}\,|J|_{S_1}$$

for all subsets J in the vertical strip S_2 where the hyperbolic size is measured by R-regular curves.

Proof. The proof is similar to Proposition 16.4.

Let $r : [a_1,b_1] \to I^v$ be an R-regular curve that intersects J at $(c,r(c))$ and $(d,r(d))$ with $c < d$ and the two endpoints $(a_1,r(a_1))$ and $(b_1,r(b_1))$ are attached to α_1 and β_1 respectively. By Lemma 14.6, the curve intersects α_2 and β_2 at $(a_2,r(a_2))$ and $(b_2,r(b_2))$ respectively and $a_1 < a_2 < c < d < b_2 < b_1$. Then

$$\frac{|a_2-a_1|}{|b_1-a_1|} \geq \frac{M}{|I^h|} \text{ and } \frac{|b_1-b_2|}{|b_1-a_1|} \geq \frac{M}{|I^h|}.$$

By Proposition 2.17 and the definition of hyperbolic size, we get

$$|J|_{S_2} \geq |[c,d]|_{[v_2,w_2]} > \frac{1}{1-M/\left|I^h\right|}\,|[c,d]|_{[v_1,w_1]}\,.$$

Therefore, the proposition follows because r, c, and d are arbitrary chosen. □

The hyperbolic size can be compared with the Euclidean size. The first proposition says that if the two sides have definite size, then the hyperbolic size is bounded above by the horizontal Euclidean displacement.

Proposition 16.6. *Assume that $\overline{\varepsilon} > 0$, $\delta > 0$, I^h and I^v are intervals with $I \Subset I^h \subset I^v$, $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$ is a Hénon-like map, R and L are positive constants with $RL < 1$, and $\mathscr{X}$ is a collection of disjoint separators with Lipschitz constant $L\|\varepsilon\|^{1/4}$.*

If $S_1 = [\alpha_1,\beta_1]$ and $S_2 = [\alpha_2,\beta_2]$ are vertical strips with $\alpha_1 \prec \alpha_2 \prec \beta_2 \prec \beta_1$, $SDisp(\alpha_1,\alpha_2) \geq M$, and $SDisp(\beta_1,\beta_2) \geq M$ for some constant $M > 0$, then there exists $c = c(M) > 0$ such that

$$|J|_{S_1} \leq \frac{2}{M}\cdot \sup\{|x_2-x_1| : (x_1,y_1),(x_2,y_2)\in J\}$$

for all subsets J in the vertical strip S_2 where the hyperbolic size is measured by R-regular curves.

Proof. Assume that r is an R-regular curve such that its graph is attached to α_1 and β_1 at $(a,r(a))$ and $(b,r(b))$ respectively and intersects the set J at $(c,r(c))$ and $(d,r(d))$ from left to right. Also, let $(a',r(a'))$ and $(b',r(b'))$ be the intersection of the graph with α_2 and β_2 respectively. Then

$$|[c,d]|_{[a,b]} \leq \left(\frac{1}{c-a} + \frac{1}{b-d}\right)(d-c) \leq \frac{2}{M} \cdot \sup\{|x_2 - x_1| : (x_1,y_1),(x_2,y_2) \in J\} \tag{16.2}$$

by Proposition 2.13. Therefore, the proposition follows because the regular curve r is arbitrary chosen. □

The next proposition says that the hyperbolic size is bounded below by the size of the horizontal cross-section.

Proposition 16.7. *Assume that $\overline{\varepsilon} > 0$, $\delta > 0$, I^h and I^v are intervals with $I \Subset I^h \subset I^v$, $F \in \mathscr{H}_\delta(I^h \times I^v, \overline{\varepsilon})$ is a Hénon-like map, R and L are positive constants with $RL < 1$, and $\mathscr{X}$ is a collection of disjoint separators with Lipschitz constant $L\|\varepsilon\|^{1/4}$.*

If $S = [\alpha, \beta]$ is a vertical strips with $\alpha \prec \beta$, then there exists $c > 0$ such that

$$|J|_{S_1} \geq \frac{c}{LDisp(\alpha,\beta)} \cdot \sup\{|x_2 - x_1| : (x_1,y),(x_2,y) \in J\}$$

for all subsets J in the vertical strip S where the hyperbolic size is measured by R-regular curves.

Proof. We may assume that J is compact. Let $x \to (x, y_0)$ be a horizontal line that intersects J and $u_2 - u_1 = \sup\{|x_2 - x_1| : (x_1,y),(x_2,y) \in J\}$ for some $(u_1,y_0),(u_2,y_0) \in J$. Also let $r(x) = y_0$ be the constant function. It is clear that r is R-regular. The curve intersects the boundaries α and β at points a and b respectively. Then

$$l(J) \geq |[u_1,u_2]|_{[a,b]}\,.$$

By Proposition 2.12, there exists a constant $c > 0$ such that

$$|[u_1,u_2]|_{[a,b]} \geq c\frac{u_2 - u_1}{b-a} \geq \frac{c}{\mathrm{LDisp}(\alpha,\beta)} \cdot \sup\{|x_2 - x_1| : (x_1,y),(x_2,y) \in J\}.$$

□

17. Expansion of Hyperbolic Size in the Good Region

This chapter generalize the expansion argument from unimodal maps to Hénon-like maps. The proof follows the work from Chapter 13 with three necessary adjustments:

1. Expansion from iteration by generalizing the measurements from hyperbolic length to hyperbolic size. (Propositions 17.3, 17.26, and 17.29)
2. Adjustments of the base sets for the prerescaling set and the rescaling set to avoid regular curves intersecting the bad region. (Definition 17.16)
3. Additional steps that comes from the Hénon rescaling. ($B \xrightarrow{F} C$ and $C \overset{F}{\Rightarrow} D$)

Figure 17.1 summarizes all the expansion estimates. The goal of this chapter is to prove Proposition 17.32, the hyperbolic size of the elements in a closest approach expands uniformly.

17.1. Expansion from iteration

This section will generalize the expansion argument from unimodal maps to Hénon-like maps for the steps containing only iteration without rescaling. Our first goal is to prove Proposition 17.3, a generalization of Proposition 2.10. It states that the hyperbolic size of a set expands under iteration when the set is away from the center trapping set.

To prove the proposition, we first show that the tools for negative Schwarzian diffeomorphisms apply to a Hénon-like map when the map is close to a unimodal map with negative Schwarzian derivative. The first condition says that the restriction of the Hénon-like map to an R-regular curve is also a map with negative Schwarizan derivative. The second condition says that the class of R-regular curves is invariant under iterations. This allows us to generalize the expansion estimates from unimodal maps to Hénon-like maps.

$$\begin{tikzcd}
B_0 \arrow[r, dashed, "F_0", "\text{Lem 17.10}"'] & C_0 \arrow[r, Rightarrow, dashed, "F_0", "\text{Lem 17.11}"'] & G_0 \arrow[r, Rightarrow, "F_0"] & \cdots_{\text{Lem 17.12\&17.13}} \arrow[r, Rightarrow, "F_0"] & G_0 \arrow[rrr, hook, "\text{Lem 17.19}"'] \arrow[d, Rightarrow, "F_0", "\text{Lem 17.14}"'] & & & Q_0(j)\ \text{(good)} \\
& D_0 \arrow[ur, hook] \arrow[rrr, hook] & & & T_0^j \arrow[r, "F_0"] & \cdots_{\text{Lem 17.15}} \arrow[r, "F_0"] & T_0^{p-1} \arrow[ur, hook, "\text{Lem 17.20}"'] \arrow[r, hook, dashed, "\text{Lem 17.31}"'] & Q_0(j)\ \text{(bad)}
\end{tikzcd}$$

$$Q_0(j)\ \text{(good)} \underset{\text{Cor 17.27}}{\xrightarrow{\phi_0 \circ F_0}} R_1^L(j-1) \xrightarrow{\phi_1} \underset{\text{Cor 17.30}}{\cdots} \xrightarrow{\phi_{j-1}} R_j^L(0) \underset{\text{Lem 17.31}}{\hookrightarrow\!\!\Rightarrow} D_j \longrightarrow \cdots$$

Figure 17.1.: A summary of all expansion estimates. The arrow $\to$ represents expansion and the arrow $\Rightarrow$ represents uniform expansion. The dash arrows are the paths addition to the unimodal case (compare Figure 13.1).

Lemma 17.1. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and a unimodal map $g \in \mathscr{U}_{\delta}^{\sigma^\infty}(I^h)$. For all $R > 0$ sufficiently small (depending on g) and $\overline{\varepsilon} > 0$ sufficiently small (depending on g and R)*[1]*, there exists $c = c(g) > 0$ such that the following properties hold for all $F \in \mathscr{H}_{\delta}^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$:*

Assume that $r_1 : [a_1, b_1] \to I^v$ is an R-regular curve (associated to F). Let $\hat{f}(x) = \pi_x \circ F(x, r_1(x)) = h(x, r_1(x))$. If the graph of r_1 is disjoint from the center trapping set T^{p-1}, then

1. *the map $\hat{f}$ is diffeomorphic to its image and has negative Schwarzian derivative and*
2. *the image of the graph of r_1 (under the iteration of F) is the graph of an R-regular curve $r_2 : [a_2, b_2] \to I^v$. In fact, the derivatives of the curve r_2 are uniformly bounded $\|r_2'\|, \|r_2''\|, \|r_2'''\| < c$.*

Proof. Proposition 2.7 is used to prove the first property. To estimate the C^2 norm of $\hat{f}'$, compute

$$\begin{aligned} \left|\hat{f}'(x) - g'(x)\right| &= \left|(f' - g') + \frac{\partial \varepsilon}{\partial x} + \frac{\partial \varepsilon}{\partial y} \cdot (r_1')\right| \\ &\leq c_1 \|f - g\| + c_1 \|\varepsilon\| + c_1 R \|\varepsilon\|^{3/4}, \end{aligned} \tag{17.1}$$

$$\begin{aligned} \left|\hat{f}''(x) - g''(x)\right| =& \left|(f'' - g'') + \frac{\partial^2 \varepsilon}{\partial x^2} + 2\frac{\partial^2 \varepsilon}{\partial x \partial y} \cdot (r_1') + \frac{\partial^2 \varepsilon}{\partial y^2} \cdot (r_1')^2 + \frac{\partial \varepsilon}{\partial y} \cdot (r_1'')\right| \\ \leq& c_1 \|f - g\| + c_1 \|\varepsilon\| + c_1 R \|\varepsilon\|^{3/4} + c_1 R^2 \|\varepsilon\|^{1/2} + c_1 R, \end{aligned} \tag{17.2}$$

and

$$\begin{aligned} \left|\hat{f}'''(x) - g'''(x)\right| =& \Big|(f''' - g''') + \frac{\partial^3 \varepsilon}{\partial x^3} + 3\frac{\partial^3 \varepsilon}{\partial x^2 \partial y} \cdot (r_1') + 3\frac{\partial^3 \varepsilon}{\partial x \partial y^2} \cdot (r_1')^2 \\ &+ \frac{\partial^3 \varepsilon}{\partial y^3} \cdot (r_1')^3 + 3\frac{\partial^2 \varepsilon}{\partial x \partial y} \cdot (r_1'') + 3\frac{\partial^2 \varepsilon}{\partial y^2} \cdot (r_1')(r_1'') + \frac{\partial \varepsilon}{\partial y} \cdot (r_1''')\Big| \\ \leq& c_1 \|f - g\| + c_1 \|\varepsilon\| + 3c_1 R \|\varepsilon\|^{3/4} + 3c_1 R^2 \|\varepsilon\|^{1/2} + c_1 R^3 \|\varepsilon\|^{1/4} + 7c_1 R. \end{aligned} \tag{17.3}$$

In the equations, the partial derivatives are evaluated at $(x, r_1(x))$. Also, the derivatives are estimated by the C^0 norm using Lemma 2.1 and c_1 is the positive constant. The inequalities show that the map $\hat{f}'$ is C^2 close to g'. Since g is a map with negative Schwarzian derivative, the map $\hat{f}$ also has negative Schwarzian derivative by Proposition 2.7 when $\overline{\varepsilon}$ and R are sufficiently small (depending on g).

Next, we show that $\hat{f}$ is diffeomorphic to its image. Note that the curve r_2 is exactly the inverse $\hat{f}^{-1}$ because the two curves r_1 and r_2 satisfy the relation

$$(\hat{f}(t), t) = F(t, r_1(t)) = (x, r_2(x)).$$

[1] This means that there exists $\hat{R} > 0$ depending on g. And for all R with $0 < R < \hat{R}$, there exists $\hat{\varepsilon} > 0$ depending on R. Then the properties hold for all $\overline{\varepsilon}$ with $0 < \overline{\varepsilon} < \hat{\varepsilon}$.

Let $c_2 > 0$ be a constant such that $\left|\frac{\partial h}{\partial x}(x,y)\right| \geq c_2$ whenever $(x,y) \notin T^{p-1}$. The constant exists because the points are away from the critical locus, and the constant is chosen so that the estimate holds for all Hénon maps F close to the degenerate map $i(g)$ ($\overline{\varepsilon}$ is sufficiently small). Then

$$\begin{aligned}\left|\hat{f}'(x)\right| &\geq \left|\frac{\partial h}{\partial x}(x,r_1(x))\right| - \left|\frac{\partial \varepsilon}{\partial y}(x,r_1(x))\right| \left|r_1'(x)\right| \geq c_2 - \frac{1}{\delta}\|\varepsilon\| \left\|r_1'\right\| \\ &\geq c_2 - \frac{1}{\delta} R \|\varepsilon\|^{3/4} \geq \frac{c_2}{2}\end{aligned}$$

when $\overline{\varepsilon}$ and R are small. By the inverse function theorem (Lemma A.1), the curve $r_2 = \hat{f}^{-1}$ exists and is C^3.

It remains to prove that r_2 is R-regular. The derivatives of $\hat{f}$ are uniformly bounded on I^h because they are close to derivatives of g by (17.1), (17.2), and (17.3) and I^h is compact. By computing the derivatives of the inverse function, we have

$$\left|r_2' \circ \hat{f}(x)\right| = \frac{1}{\left|\hat{f}'(x)\right|} \leq 2/c_2 \leq c_3,$$

$$\left|r_2'' \circ \hat{f}(x)\right| = \frac{\left|\hat{f}''(x)\right|}{\left|\hat{f}'(x)\right|^2} \leq \frac{4}{c_2^2}\left|\hat{f}''(x)\right| \leq c_3,$$

and

$$\begin{aligned}r_2''' \circ \hat{f}(x) &= -\frac{1}{\left[\hat{f}'(x)\right]^5}\left\{\hat{f}'(x)\hat{f}'''(x) - 3\left[\hat{f}''(x)\right]^2\right\} \\ \left|r_2''' \circ \hat{f}(x)\right| &\leq \left(\frac{2}{c_2}\right)^5 \left\{\left|\hat{f}'(x)\hat{f}'''(x)\right| + 3\left|\hat{f}''(x)\right|^2\right\} \leq c_3\end{aligned}$$

for some constant $c_3 > 0$. We get

$$\left\|r_2'\right\| \leq c_3 < \frac{R}{\|\varepsilon\|^{1/4}},$$

$$\left\|r_2''\right\| \leq c_3 < \frac{R}{\|\varepsilon\|},$$

$$\left\|r_2'''\right\| \leq c_3 < \frac{R}{\|\varepsilon\|},$$

and

$$\left\|r_2'\right\| \left\|r_2''\right\| \leq c_3^2 < \frac{R}{\|\varepsilon\|}$$

whenever $\overline{\varepsilon}$ is sufficiently small (depending on R). Therefore, the curve r_2 is R-regular. □

Remark 17.2. The condition regular is defined to ensure the inequalities (17.1), (17.2), and (17.3) are small.

Finally, the proposition generalizes Proposition 2.10 to Hénon-like maps. It says that when a

Hénon-like map is close to a unimodal map with negative Schwarzian derivative, the hyperbolic size of a set away from the center expands under iteration.

Proposition 17.3. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and a unimodal map $g \in \mathscr{U}_\delta^{\sigma^\infty}(I^h)$. For all $R > 0$ sufficiently small (depending on g) and $\overline{\varepsilon} > 0$ sufficiently small (depending on g and R), the following property holds for all $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$:*

Assume that $f^s : \mathscr{X} \to \mathscr{Y}$ is an induced unimodal map on the pair of compatible separators $\mathscr{X}$ and $\mathscr{Y}$ with Lipschitz constant $L\|\varepsilon\|^{1/4}$ and $RL < 1$. If S_1 and S_2 are vertical strips of $\mathscr{X}$ and $\mathscr{Y}$ respectively, S_1 is disjoint from the center trapping region T^{p-1}, and $f^s(S_1) = S_2$, then the hyperbolic size expands under iteration:

$$|F(J)|_{S_2} \geq |J|_{S_1}$$

for all $J \subset S_1$. The hyperbolic size is measured by R-regular curves.

Proof. Let R and $\overline{\varepsilon}$ be the constants given by Lemma 17.1. Given an R-regular curve $r_1 : [a_1, b_1] \to I_n^v$ such that the two endpoints $(a_1, r_1(a_1))$ and $(b_1, r_1(b_1))$ of its graph are attached to the two boundaries of S_1 and the two points $(c_1, r_1(c_1))$ and $(d_1, r_1(d_1))$ on the graph belong to J.

By Lemma 17.1, the image of the graph of r_1 is the graph of an R-regular curve $r_2 : [a_2, b_2] \to I_n^v$. The endpoints $(a_2, r_2(a_2))$ and $(b_2, r_2(b_2))$ of its graph are attached to the two boundaries of S_2 because boundaries of S_1 maps to boundaries of S_2. Also, the two points $(c_2, r_2(c_2))$ and $(d_2, r_2(d_2))$ belong to $F(J)$ where $c_2 = \hat{f}(c_1)$, $d_2 = \hat{f}(d_1)$, and $\hat{f}(x) = \pi_x \circ F(x, r_1(x))$. Hence, r_2 is an R-regular curve that satisfies the conditions for measuring the hyperbolic size. We get

$$|F(J)|_{S_2} \geq |[c_2, d_2]|_{[a_2, b_2]}. \tag{17.4}$$

Moreover, the map $\hat{f}$ has negative Schwarizan derivative by Lemma 17.1. This yields the expansion of hyperbolic length

$$|[c_2, d_2]|_{[a_2, b_2]} = \left|[\hat{f}(c_1), \hat{f}(d_1)]\right|_{[\hat{f}(a_1), \hat{f}(b_1)]} \geq |[c_1, d_1]|_{[a_1, b_1]} \tag{17.5}$$

by Proposition 2.10. Combine (17.4) and (17.5), we get

$$|F(J)|_{S_2} \geq |[c_1, d_1]|_{[a_1, b_1]}.$$

The inequality holds for all R-regular curves r_1 that satisfy the conditions for measuring the hyperbolic size. Therefore, the proposition is proved. □

The proposition does not guarantee that the expansion of hyperbolic size is uniform. Proposition 16.5 is the tool that allows us to estimate the size of expansion from the expansion of the topology. The topological results from Section 13.6.1 showing the expansion of a unimodal map's topology also apply to this context with the help of the induced unimodal map f^s. This produces the expansion of topology for a Hénon map in a bounded number of iterations and ensures definite expansion on the hyperbolic size.

Fix the induced unimodal map f^s defined by Proposition 14.36. Given $R > 0$ sufficiently small by Proposition 17.3. The local stable manifolds are separators with Lipschitz constant

$L\|\varepsilon\| = (L\|\varepsilon\|^{3/4})\,\|\varepsilon\|^{1/4}$ by Propositions 14.18 and 14.36. Thus, the separators satisfy the condition $R(L\|\varepsilon\|^{3/4}) < 1$ when the perturbation $\overline{\varepsilon}$ is sufficiently small. Therefore, Propositions 16.4, 16.5, and 17.3 apply to the vertical strips defined by the separators $\alpha(0), \overline{\alpha(0)}, \beta(0), \overline{\beta(0)}, \alpha^0(1), \overline{\alpha^0(1)}, \beta^0(1), \overline{\beta^0(1)}, \cdots, \alpha^{p-1}(1), \overline{\alpha^{p-1}(1)}, \beta^{p-1}(1), \overline{\beta^{p-1}(1)}, \theta^L, \theta^R$.

To measure the hyperbolic size of a wandering domain, a base set is assigned to each partition element: B, C, trapping set, and gap. The hyperbolic size of a wandering domain will be measured inside the base set of the partition element that contains the wandering domain instead of the partition element itself. It is defined as follows.

Definition 17.4 (Base set).
1. If T^j is a trapping set with $2 \le j \le p-1$, define its *base set* as $Base(T^j) = P^j$.
2. If G is a gap, define its *base set* as $Base(G) = T_L \cup G \cup T_R$ where T_L and T_R are the two adjacent trapping sets of G.
3. The *base set* of B is $Base(B) = [\beta(0), \overline{\alpha(0)}]$.
4. The *base set* of C is $Base(C) = [\alpha(0), \beta(0)]$.

If $J \subset B \cup C \cup D$ is a wandering domain, its base set is defined to be $Base(J) = Base(S)$ where S is one of the vertical strips above that contains J. Denote the hyperbolic size of the wandering domain as

$$l(J) = |J|_{Base(J)}\,.$$

Remark 17.5. The third and fourth definitions are new in the Hénon setting. See Proposition 14.43 and Figure 14.1.

The final goal of this section is to prove the following two propositions. The first proposition shows that the hyperbolic size of a set expands under iteration.

Proposition 17.6 (Expansion for one iteration). *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and a unimodal map $g \in \mathscr{U}_\delta^{\sigma^\infty}(I^h)$. For all $R > 0$ sufficiently small (depending on g) and $\overline{\varepsilon} > 0$ sufficiently small (depending on g and R), the following property holds for all $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$:*

Assume that J is a wandering domain of F. If $J \subset D$ and $F(J) \subset D$ then

$$l(F(J)) \ge l(J)$$

where the hyperbolic size is measured by R-regular curves.

Proof. Assume the expansion estimates in the later subsections hold. The proposition follows from Lemma 17.12 ($G \xrightarrow{F} G$), Lemma 17.14 ($G \overset{F}{\Rightarrow} T^j$), and Lemma 17.15 ($T^j \xrightarrow{F} T^{j+1}$) later. □

The next proposition shows that a definite amount of expansion can be obtained in a bounded number of iterations.

Proposition 17.7 (Uniform expansion for iterations). *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and a unimodal map $g \in \mathscr{U}_\delta^{\sigma^\infty}(I^h)$. For all $R > 0$ sufficiently small (depending on g) and $\overline{\varepsilon} > 0$ sufficiently small (depending on g and R), the following property holds for all $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$:*

Assume that J is a wandering domain of F and $\min_{0\le t\le p-1} SDisp(\beta^t(1),\overline{\beta^t(1)}) \ge K$ *for some constant* $K>0$*. If the sets* $J, F(J), \cdots, F^p(J)$ *are all in the iteration set* D*, then*

$$l(F^p(J)) \ge E \cdot l(J)$$

for some constant $E>1$ *that depends only on* $K/\left|I^h\right|$*. The hyperbolic size is measured by R-regular curves.*

Proof. Assume the expansion estimates in the later subsections hold. If the sets $J, F(J), \cdots, F^p(J)$ all belong to gaps, then

$$l(F^p(J)) \ge E \cdot l(J)$$

by Lemma 17.13 ($G \overset{F^p}{\Rightarrow} G$).

If $F^t(J) \subset T^j$ for some $0 \le t \le p-1$, let t be the smallest integer. The integer $t \ne 0$ because T^0 and T^1 are disjoint from D. Then

$$l(F^t(J)) \ge E \cdot l(F^{t-1}(J)) \tag{17.6}$$

by Lemma 17.14 ($G \overset{F}{\Rightarrow} T^j$). The sets $F^t(J), \cdots, F^p(J)$ belong to trapping intervals implies that

$$l(F^p(J)) \ge \cdots \ge l(F^t(J)) \tag{17.7}$$

by Lemma 17.15 ($T^j \xrightarrow{F} T^{j+1}$). Also, the sets $J, \cdots, F^{t-1}(J)$ belong to gaps implies that

$$l(F^{t-1}(J)) \ge \cdots \ge l(J) \tag{17.8}$$

by Lemma 17.12 ($G \xrightarrow{F} G$). After combining (17.6), (17.7), and (17.8), we obtain

$$l(F^p(J)) \ge E \cdot l(J).$$

□

An immediate consequence is the orbit of an wandering domain cannot stay in the iteration set forever. The orbit must eventually enters the prerescaling set and the rescaling set.

Corollary 17.8. *Given* $\delta>0$*, intervals* I^h *and* I^v *with* $I \Subset I^h \subset I^v$*, a unimodal permutation* σ*, and a unimodal map* $g \in \mathscr{U}_\delta^{\sigma^\infty}(I^h)$*. For all* $\overline{\varepsilon}>0$ *sufficiently small (depending on g), the following property holds for all* $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$*:*

If the map F has a wandering domain in D, then it also has a wandering domain in the rescaling set R.

Proof. Let σ be a unimodal permutation, $g \in \mathscr{U}_\delta^{\sigma}(I^h)$ be a unimodal map, $R>0$ and $\overline{\varepsilon}>0$ be two constants sufficiently small by Proposition 17.7, $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$, and $K = \min_{0\le t\le p-1}$ $\mathrm{SDisp}(\beta^t(1),\overline{\beta^t(1)})>0$. Assume that J is a wandering domain of F.

By Proposition 17.7, the hyperbolic size of the orbit of J diverges to infinity if the orbit stays in the iteration set D forever. This cannot happen because the hyperbolic size of gaps and trapping sets are bounded inside their base sets. Thus, $F^t(J) \subset R$ for some $t \ge 0$. By Proposition 14.39, $F^t(J)$ is a wandering domain in R. □

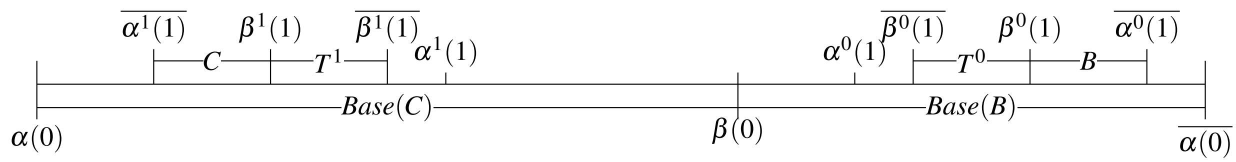


Figure 17.1.: The iteration from B to C.

Another immediate consequence is the expansion constant E can be chosen to be uniform when the Hénon-like map F it is sufficiently close to the hyperbolic fixed point $i(f_\sigma)$ of the renormalization operator.

Corollary 17.9. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exists a constant $E > 1$ such that for all $R > 0$ sufficiently small and $\overline{\varepsilon} > 0$ sufficiently small (depending on R), the following property holds for all $F \in \mathscr{I}^{\sigma}_{\delta}(I^h \times I^v, \overline{\varepsilon})$:*

Assume that J is a wandering domain of F. If $J, F(J), \cdots, F^p(J) \subset D$, then

$$l(F^p(J)) \geq E \cdot l(J)$$

where the hyperbolic size is measured by R-regular curves.

Proof. Set $g = f_\sigma$ in Proposition 17.7. The constant $K > 0$ can be chosen to be uniform for all Hénon-like maps close enough to $i(f_\sigma)$. Thus, the expansion constant E is uniform. □

The remaining part of this section generalizes the expansion argument in the iteration set by using Proposition 17.3. For the unimodal map, the expansion of the hyperbolic length is fully determined by the expansion of the topology under iteration. For Hénon-like maps, the topology of a Hénon-like maps inside the good region behaves like unimodal maps under iteration. Precisely, the order of the local stable manifolds for a renormalizable Hénon-like map are the same as the order of the associate periodic points for a renormalizable unimodal map with the same combinatoric type. Thus, the expansion argument for unimodal maps can be fully adopted to Hénon-like maps by using the induced unimodal map f^s. The following subsections will study the expansion case by case according to Figure 14.1. The three cases $G \xrightarrow{F} G$ (Subsection 17.1.3), $G \overset{F}{\Rightarrow} T^j$ (Subsection 17.1.4), and $T^j \xrightarrow{F} T^{j+1}$ (Subsection 17.1.5) are similar to Section 13.6.1. The two cases $B \xrightarrow{F} C$ (Subsection 17.1.1) and $C \overset{F}{\Rightarrow} G$ (Subsection 17.1.2) are addition in the Hénon settings.

17.1.1. $B \xrightarrow{F} C$

Lemma 17.10 (Expansion for $B \xrightarrow{F} C$)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and a unimodal map $g \in \mathscr{U}^{\sigma^\infty}_{\delta}(I^h)$. For all $R > 0$ sufficiently small (depending on g) and $\overline{\varepsilon} > 0$ sufficiently small (depending on g and R), the following property holds for all $F \in \mathscr{H}^{\sigma^\infty}_{\delta}(I^h \times I^v, g, \overline{\varepsilon})$:*

If $J \subset B$, then

$$|F(J)|_{Base(C)} \geq |J|_{Base(B)}$$

where the hyperbolic size is measured by R-regular curves.

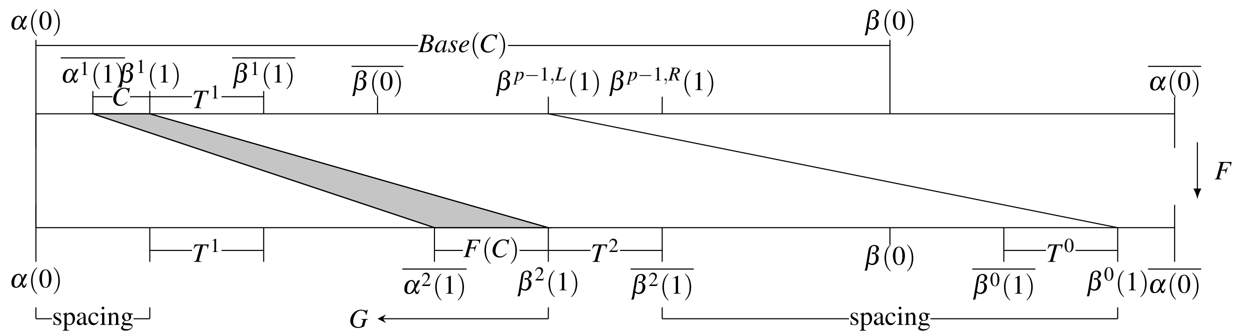


Figure 17.2.: The iteration from C to G.

Proof. By definition, the vertical strip $Base(B)$ is disjoint from the center trapping set T^{p-1} and $f^s(Base(B)) = Base(C)$. See Figure 17.1. The lemma follows from Proposition 17.3. $\square$

17.1.2. $C \overset{F}{\Rightarrow} G$

Lemma 17.11 (Expansion for $C \overset{F}{\Rightarrow} G$). *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, $p \geq 3$, and a unimodal map $g \in \mathscr{U}_{\delta}^{\sigma^\infty}(I^h)$. For all $R > 0$ sufficiently small (depending on g) and $\overline{\varepsilon} > 0$ sufficiently small (depending on g and R), the following property holds for all $F \in \mathscr{H}_{\delta}^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$:*

Assume that $SDisp(\beta^0(1), \overline{\beta^0(1)}) \geq K$ and $SDisp\left(\alpha(0), \beta^1(1)\right) \geq K$ for some constant $K > 0$. If $J \subset C$ and $F(J) \subset G$ for some gap G, then

$$|F(J)|_{Base(G)} \geq E\,|J|_{Base(C)}$$

for some constant $E > 1$ that depends only on $K/\left|I^h\right|$. The hyperbolic size is measured by R-regular curves.

Proof. See Figure 17.2 for illustration. Assume that $\beta^{p-1,L}(1)$ be the left separator in $\{\beta^{p-1}(1), \overline{\beta^{p-1}(1)}\}$. Then $C \subset [\alpha(0), \beta^{p-1,L}(1)] \subset Base(C)$. We get

$$|J|_{[\alpha(0),\beta^{p-1,L}(1)]} \geq |J|_{Base(C)} \tag{17.9}$$

by Proposition 16.4.

Then we iterate the sets J and $[\alpha(0), \beta^{p-1,L}(1)]$. The vertical strip $[\alpha(0), \beta^{p-1,L}(1)]$ is disjoint from the center trapping set T^{p-1} and $f^s([\alpha(0), \omega]) = [\alpha(0), \beta^0(1)]$. By Proposition 16.4, we get

$$|F(J)|_{[\alpha(0),\beta^0(1)]} \geq |J|_{[\alpha(0),\beta^{p-1,L}(1)]}\,. \tag{17.10}$$

Consider the images $F(J)$ and $[\alpha(0), \beta^0(1)]$. The set $[\alpha(0), \beta^0(1)] \backslash Base(G)$ contains a component on each side of the vertical strip $Base(G)$. The left component contains $[\alpha(0), \beta^1(1)]$ and the right component contains $[\overline{\beta^0(1)}, \beta^0(1)]$ because $p \geq 3$. By Proposition 16.5, we obtain

$$|F(J)|_{Base(G)} \geq E\,|F(J)|_{[\alpha(0),\beta^0(1)]}\,. \tag{17.11}$$

for some constant $E > 1$ determined by $K/\left|I^h\right|$. The lemma follows by combining (17.9), (17.10), and (17.11). $\square$

17.1.3. $G \xrightarrow{F} G$

Lemma 17.12 (Expansion for $G \xrightarrow{F} G$)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and a unimodal map $g \in \mathscr{U}_\delta^{\sigma^\infty}(I^h)$. For all $R > 0$ sufficiently small (depending on g) and $\overline{\varepsilon} > 0$ sufficiently small (depending on g and R), the following property holds for all $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$:*

If $J \subset G^1$ and $F(J) \subset G^2$ for some gaps G^1 and G^2 in D, then

$$|F(J)|_{Base(G^2)} \geq |J|_{Base(G^1)}$$

where the hyperbolic size is measured by R-regular curves.

Proof. The proof is similar to Corollary 13.41.

The result of Lemma 13.40 can be generalized to Hénon-like maps. There exists a vertical strip I that is disjoint from the center trapping set T^{p-1} such that $Base(G^1) \supset I$ and $f^s(I) \supset Base(G^2)$. By Proposition 16.4 and Proposition 17.3, we get

$$|F_n(J)|_{Base(G^2)} \geq |F_n(J)|_{f^s(I)} \geq |J|_I \geq |J|_{Base(G^1)}.$$

$\square$

Lemma 17.13 (Uniform expansion for $G \overset{F^p}{\Rightarrow} G$)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and a unimodal map $g \in \mathscr{U}_\delta^{\sigma^\infty}(I^h)$. For all $R > 0$ sufficiently small (depending on g) and $\overline{\varepsilon} > 0$ sufficiently small (depending on g and R), the following property holds for all $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$:*

Assume that $\min_{0 \leq t \leq p-1} SDist(\beta^t(1), \overline{\beta^t(1)}) \geq K$ for some constant $K > 0$. If $F^j(J) \subset G^j$ for $0 \leq j \leq p$ where G^j are gaps in D, then

$$|F^p(J)|_{Base(G^p)} \geq E\,|J|_{Base(G^0)}$$

for some constant $E > 1$ that depends only on $K/\left|I^h\right|$. The hyperbolic size is measured by R-regular curves.

Proof. The proof is similar to Corollary 13.43.

The result of Lemma 13.42 can be generalized to Hénon-like maps. Let $\left\{I^j\right\}_{j=0}^p$ be vertical strips similar to the intervals in the lemma. The vertical strips satisfy the properties:

1. the vertical strips I^j are disjoint from the center trapping set T^{p-1},
2. $f^s(I^j) \supset I^{j+1}$ for $j = 0, \cdots, p-1$,
3. $I^0 = G^0$, $I^j \supset G^j$ for $j = 1, \cdots, p-1$, and $I^p = Base(G^p)$.

The two components of $Base(G^0)\backslash I^0$ are both trapping sets. By Proposition 16.5, we have

$$|J|_{I^0} \geq E\,|J|_{Base(G^0)} \tag{17.12}$$

for some constant $E > 1$ determined by $K/\left|I^h\right|$.

Since $f^s(I^j) \supset I^{j+1}$ for $j = 0, \cdots, p-1$ and the vertical strips I^j are disjoint from the center trapping set T^{p-1}, we get

$$|F^p(J)|_{Base(G^p)} \geq |F^p(J)|_{f^s(I^{p-1})} \geq \left|F^{p-1}(J)\right|_{I^{p-1}} \geq \left|F^{p-1}(J)\right|_{f^s(I^{p-2})} \geq \cdots \geq |J|_{I^0} \tag{17.13}$$

by Proposition 16.4 and Proposition 17.3. The lemma follows by combining (17.12) and (17.13). □

17.1.4. $G \overset{F}{\Rightarrow} T^j$ with $j \neq 0, 1$

Lemma 17.14 (Uniform expansion for $G \overset{F}{\Rightarrow} T^j$)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and a unimodal map $g \in \mathscr{U}_\delta^{\sigma^\infty}(I^h)$. For all $R > 0$ sufficiently small (depending on g) and $\overline{\varepsilon} > 0$ sufficiently small (depending on g and R), the following property holds for all $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$:*

Assume that $\min_{0 \leq t \leq p-1} SDist(\beta^t(1), \overline{\beta^t(1)}) \geq K$ for some constant $K > 0$. If $J \subset G$ for some gap G, $F(J) \subset T^j$ with $j \neq 0$, then

$$|F(J)|_{Base(T^j)} \geq E\,|J|_{Base(G)}$$

for some constant $E > 1$ that depends only on $K/\left|I^h\right|$. The hyperbolic size is measured by R-regular curves.

Proof. The proof is similar to Corollary 13.45.

The set $Base(G)\backslash G$ has a component on each side of G. Both components are trapping sets. By Proposition 16.5, the hyperbolic size has definite expansion

$$|J|_G \geq E\,|J|_{Base(G)} \tag{17.14}$$

for some constant $E > 1$ determined by $K/\left|I^h\right|$.

Moreover, by Proposition 17.3, the hyperbolic size expands under iteration

$$|F(J)|_{f^s(G)} \geq |J|_G\,. \tag{17.15}$$

The result from Lemma 13.44 can be generalized to Hénon-like maps: $f^s(G) \supset P^j$. Hence,

$$|F(J)|_{Base(T^j)} \geq |F(J)|_{f^s(G)} \tag{17.16}$$

by Proposition 16.4.

The lemma follows by combining (17.14), (17.15), and (17.16). □

17.1.5. $T^j \xrightarrow{F} T^{j+1}$ with $j \leq p-2$

Lemma 17.15 (Expansion for $T \to T$)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and a unimodal map $g \in \mathscr{U}_{\delta}^{\sigma^\infty}(I^h)$. There exist $R > 0$ sufficiently small (depending on g) and $\overline{\varepsilon} > 0$ sufficiently small (depending on g and R) such that the following property holds for all $F \in \mathscr{H}_{\delta}^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$:*

If $J \subset T^j$ where $0 \leq j \leq p-2$, then

$$|F(J)|_{Base(T^{j+1})} \geq |J|_{Base(T^j)}$$

where the hyperbolic size is measured by R-regular curves.

Proof. By the definition of the trapping sets, we have $F(J) \subset T^{j+1}$ and $f^s(P^j) = P^{j+1}$. The inequality follows from Proposition 17.3. □

17.2. Expansion from rescaling

In this section, we study the expansion of hyperbolic size for the case when a step in a closest approach contains rescaling.

Recall from Proposition 15.2, the constant K_n is the boundary for the good region and the bad region of F_n and $\theta_n^L(j)$ and $\theta_n^R(j)$ are the two separators such that $F_n^{-1}(\overline{\beta_n(j)}) = \theta_n^L(j) \cup \theta_n^R(j)$ and $\theta_n^L(j) \prec \theta_n^R(j)$ for $j = 1, \cdots, K_n+2$. Fix the induced unimodal map $f_n^s : \mathscr{X}_n \to \mathscr{Y}_n$ where $\mathscr{X}_n = \{\alpha_n^{p-1}(1), \overline{\alpha_n^{p-1}(1)}, \theta_n^L(1), \theta_n^R(1), \theta_n^L(2), \theta_n^R(2), \cdots, \theta_n^L(K_n+2), \theta_n^R(K_n+2)\}$ and $\mathscr{Y}_n = \{\alpha_n^0(1), \beta_n(1), \beta_n(2), \cdots, \beta_n(K_n+2)\}$.

To measure the hyperbolic size in the rescaling set and the prerescaling set, a base set is assigned to each level as follows.

Definition 17.16 (Base set)**.** Assume that $\overline{\varepsilon} > 0$ is sufficient small such that Proposition 15.2 holds, $F \in \mathscr{I}_{\delta}^{\sigma}(I^h \times I^v, \overline{\varepsilon})$, and $1 \leq j \leq K_n$.

1. The *base se*t of $R_n^L(j) = [\overline{\beta_n(j)}, \overline{\beta_n(j+1)}]$ is defined to be

$$Base(R_n^L(j)) = [\alpha_n(0), \overline{\beta_n(j+2)}] = [\alpha_n(0), \overline{\beta_n(j)}] \cup R_n^L(j) \cup R_n^L(j+1)$$

   for all $j \geq 0$. See Figure 14.1b for an illustration of the base sets of R_n^L.

2. The *base set* of $Q_n^i(j) = [\theta_n^i(j), \theta_n^i(j+1)]$ is defined to be

$$Base(Q_n^i(j)) = [\alpha_n^{p-1,i}(1), \theta_n^i(j+2)] = [\alpha_n^{p-1,i}(1), \theta_n^i(j)] \cup Q_n^i(j) \cup Q_n^i(j+1)$$

   for $i = L$ or R where $\alpha_n^{p-1,L}(1), \alpha_n^{p-1,R}(1) \in \{\alpha_n^{p-1}(1), \overline{\alpha_n^{p-1}(1)}\}$ with $\alpha_n^{p-1,L}(1) \prec \alpha_n^{p-1,R}(1)$.

Remark 17.17. Compare to the unimodal case, the definition for the base sets of the prerescaling set and the rescaling set here are adjusted to avoid an R-regular curve intersecting the bad region. Compare with Definition 13.48.

The hyperbolic size for the step containing rescaling will be studied in four separated parts:

1. Conversion of the hyperbolic size from the iteration set to the prerescaling set (Subsection 17.2.1 $D \hookrightarrow Q$).

2. Expansion of the hyperbolic size under one iteration plus one rescaling (Subsection 17.2.2 $Q_n(j) \xrightarrow{F_n} R_n^L(j) \xrightarrow{\phi_n} R_n^L(j-1)$).

3. Expansion of the hyperbolic size under the remaining rescalings (Subsection 17.2.3 $R_n^L(j) \xrightarrow{\phi_n} R_{n+1}^L(j-1)$).

4. Conversion of the hyperbolic size from the rescaling set back to the iteration set (Subsection 17.2.4 $R_n^L(0) \Rightarrow D_n$).

The final goal of this section is to prove the following proposition.

Proposition 17.18 (Uniform expansion for rescaling)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and an admissible unimodal permutation σ. For all $R > 0$ sufficiently small and $\overline{\varepsilon} > 0$ sufficiently small (depending on R), there exists $E > 1$ such that the following property holds for all $F \in \mathscr{I}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$ and all $n \geq 0$:*

Assume that J is a wandering domain of F_n. If $J \subset Q_n$ is in the good region, then

$$l(\Phi_n^{k(J)} \circ F_n(J)) \geq E \cdot l(J)$$

where the hyperbolic size is measured by R-regular curves.

Proof. Assume the expansion estimates in the later subsections. Fix the cylindrical neighborhood to be centered at the fixed point $g = f_\sigma$ in Lemmas 17.19, 17.20, and 17.31. Also, select a constant K for Lemma 17.31 ($R_n^L(0) \to D_n$) such that $\operatorname{SDisp}\left(\alpha_F(0), \beta_F^1(1)\right) \geq K$ and $\operatorname{SDisp}\left(\beta_F(1), \overline{\beta_F(2)}\right) \geq K$ for all $F \in \mathscr{I}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$. The constant K can be chosen to be positive when the Hénon-like maps are sufficiently close to the hyperbolic fixed point $i(f_\sigma)$, i.e. $\overline{\varepsilon} > 0$ is sufficiently small. Thus, the expansion constant E from Lemma 17.31 ($R_n^L(0) \to D_n$) is uniform on $\mathscr{I}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$.

By Lemma 17.19 ($G \hookrightarrow Q$) and Lemma 17.20 ($T \hookrightarrow Q$), we have

$$|J|_{Base(Q_n^i(k(J)))} \geq l(J) \tag{17.17}$$

where $i = L$ or R such that $J \subset Q_n^i(k(J))$. Also, by Corollary 17.27 ($Q_n(j) \xrightarrow{F_n} R_n^L(j) \xrightarrow{\phi_n} R_{n+1}^L(j-1)$), we have

$$|\phi_n \circ F_n(J)|_{Base(R_{n+1}^L(k(J)-1))} \geq |J|_{Base(Q_n^i(k(J)))}\,. \tag{17.18}$$

Finally, by Corollary 17.30 ($R_n^L(j) \xrightarrow{\phi_n} R_{n+1}^L(j-1)$) and Lemma 17.31 ($R_n^L(0) \Rightarrow D_n$), we get

$$l(\Phi_n^{k(J)} \circ F_n(J)) \geq E \cdot |\phi_n \circ F_n(J)|_{Base(R_{n+1}^L(k(J)-1))} \tag{17.19}$$

where the constant $E > 1$ is obtained by Lemma 17.31 ($R_n^L(0) \Rightarrow D_n$).

The lemma follows by combining (17.17), (17.18), and (17.19). □

17.2.1. $D_n \hookrightarrow Q_n$ good

When a wandering domain is in the prerescaling set Q, it can be in either the center trapping set T^{p-1} or the gap adjacent to the center trapping set. The first lemma studies the case of gap.

Lemma 17.19 (Expansion for $G \hookrightarrow Q(1)$)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and a unimodal map $g \in \mathscr{U}_{\delta}^{\sigma^\infty}(I^h)$. For all $R > 0$ sufficiently small (depending on g) and $\overline{\varepsilon} > 0$ sufficiently small (depending on g and R), the following property holds for all $F \in \mathscr{H}_{\delta}^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$:*

If $J \subset G \cap Q^i(1)$ for some gap G and $i = L$ or R, then

$$|J|_{Base(Q^i(1))} \geq |J|_{Base(G)}$$

where the hyperbolic size is measured by R-regular curves.

Proof. If $G \cap Q^i(1) \neq \phi$, then T^{p-1} is one of the adjacent trapping set of G. So $Base(G) \supset Base(Q^i(1))$ by definition. The lemma follows from Proposition 16.4. □

The next lemma studies the case in the center trapping set.

Lemma 17.20 (Expansion for $T^{p-1} \hookrightarrow Q$)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and a unimodal map $g \in \mathscr{U}_{\delta}^{\sigma^\infty}(I^h)$. For all $R > 0$ sufficiently small (depending on g) and $\overline{\varepsilon} > 0$ sufficiently small (depending on g and R), the following property holds for all $F \in \mathscr{H}_{\delta}^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$:*

If $J \subset T^{p-1} \cap Q^i(k(J))$, $i = L$ or R, and $1 \leq k(J) \leq K_F$, then

$$|J|_{Base(Q^i(k(J)))} \geq |J|_{Base(T^{p-1})}$$

where the hyperbolic size is measured by R-regular curves.

Proof. By definition, $Base(T^{p-1}) = P^{p-1} \supset Base(Q^i(j))$. The lemma follows from Proposition 16.4. □

17.2.2. $Q_n(j) \xrightarrow{F_n} R_n^L(j) \xrightarrow{\phi_n} R_n^L(j-1)$ where $1 \leq j \leq K_n$

The goal of this section is to prove Proposition 17.26, the hyperbolic size of a wandering domain expands under one iteration plus one rescaling. Unlike the unimodal case, the combination of iteration and rescaling cannot be separated into two parts. This is because the class of R-regular curves is not invariant under one iteration when the curves are close to the bad region. The trick is to apply one additional rescaling to make the curves to be R-regular in the next level of renormalization because the size of the perturbation term ε is contracted by taking the power of p whenever the map is renormalized by Proposition 14.23.

Similar to Proposition 17.3, we first prove that the property of negative Schwarzian derivative is preserved when restricting the combination of iteration and rescaling to an R-regular curve and the class of regular curves is invariant under the composed map.

Lemma 17.21. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and $p \geq 3$. For all $R > 0$ sufficiently small and $\overline{\varepsilon} > 0$ sufficiently small (depending on R), the following properties hold for all $F \in \mathscr{I}_{\delta}^{\sigma}(I^h \times I^v, \overline{\varepsilon})$ and $n \geq 0$:*

Assume that $r_1 : [a_1, b_1] \to I_n^v$ *is an R-regular curve (associated to* F_n*). If the graph of* r_1 *lies in the vertical strip* $[\alpha_n^{p-1,L}(1), \theta_n^L(K_n+2)]$ *or* $[\theta_n^R(K_n+2), \alpha_n^{p-1,R}(1)]$ *where* $\alpha_n^{p-1,L}(1), \alpha_n^{p-1,R}(1) \in \{\alpha_n^{p-1}(1), \overline{\alpha_n^{p-1}(1)}\}$ *with* $\alpha_n^{p-1,L}(1) \prec \alpha_n^{p-1,R}(1)$*, then*

1. *the map* $x \to \pi_x \circ \phi_n \circ F_n(x, r_1(x))$ *is injective with negative Schwarzian derivative and*
2. *the image of the graph of* r_1 *under* $\phi_n \circ F_n$ *is the graph of an R-regular (associated to* F_{n+1}*) curve* $r_2 : [a_2, b_2] \to I_{n+1}^v$.

Remark 17.22. The two vertical strips $[\alpha_n^{p-1,L}(1), \theta_n^L(K_n+2)]$ and $[\theta_n^R(K_n+2), \alpha_n^{p-1,R}(1)]$ are exactly the union of all base sets of Q in the good region.

To prove the lemma, we need the estimates for the partial derivatives of $\pi_x \circ H_n^{p-1}$. The estimates will also be used to prove Lemma 17.28 later.

Lemma 17.23. *Given* $\delta > 0$*, intervals* I^h *and* I^v *with* $I \Subset I^h \subset I^v$*, a unimodal permutation* σ*, and an integer* $t \in \{1, 2, \cdots, p\}$*. For all* $\overline{\varepsilon} > 0$ *sufficiently small (independent of* t*), there exists* $c = c(t) > 0$ *such that for all* $F \in \mathscr{I}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$ *and* $n \geq 0$ *the inequalities hold*

$$\left|\frac{\partial \pi_x \circ F_n^t}{\partial y}\right|, \left|\frac{\partial^2 \pi_x \circ F_n^t}{\partial x \partial y}\right|, \left|\frac{\partial^2 \pi_x \circ F_n^t}{\partial y^2}\right|, \left|\frac{\partial^3 \pi_x \circ F_n^t}{\partial x^2 \partial y}\right|, \left|\frac{\partial^3 \pi_x \circ F_n^t}{\partial x \partial y^2}\right|, \left|\frac{\partial^3 \pi_x \circ F_n^t}{\partial y^3}\right| < c\,\|\varepsilon_n\|$$

for all points in $P_n(0)$.

Proof. By Lemma 2.1, there exists a constant $c_1 = c_1(\delta) > 0$ such that

$$\left|\frac{\partial \varepsilon_n}{\partial y}\right|, \left|\frac{\partial^2 \varepsilon_n}{\partial x \partial y}\right|, \left|\frac{\partial^2 \varepsilon_n}{\partial y^2}\right|, \left|\frac{\partial^3 \varepsilon_n}{\partial x^2 \partial y}\right|, \left|\frac{\partial^3 \varepsilon_n}{\partial x \partial y^2}\right|, \left|\frac{\partial^3 \varepsilon_n}{\partial y^3}\right| < c_1\,\|\varepsilon_n\|$$

for all points in $P_n(0)$.

Prove by induction on t.

For the case $t = 1$, we have $\frac{\partial \pi_x \circ F_n}{\partial y} = \frac{\partial \varepsilon_n}{\partial y}$. Also, the partial derivatives can be estimated by the C^0 norm $\|\varepsilon_n\|$ using Lemma 2.1. Hence, the lemma holds for $t = 1$.

Assume the induction hypothesis for t. For the case $t+1$, apply the chain rule. We get

$$\begin{aligned}\frac{\partial \pi_x \circ F_n^{t+1}}{\partial y} &= \frac{\partial \pi_x \circ F_n^t}{\partial x} \circ F_n \cdot \frac{\partial \varepsilon_n}{\partial y},\\ \left|\frac{\partial \pi_x \circ F_n^{t+1}}{\partial y}\right| &\leq c_1 \left|\frac{\partial \pi_x \circ F_n^t}{\partial x} \circ F_n\right| \|\varepsilon_n\|,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \pi_x \circ F_n^{t+1}}{\partial x \partial y} &= \frac{\partial^2 \pi_x \circ F_n^t}{\partial x^2} \circ F_n \cdot \frac{\partial h_n}{\partial x} \cdot \frac{\partial \varepsilon_n}{\partial y} + \frac{\partial^2 \pi_x \circ F_n^t}{\partial x \partial y} \circ F_n \cdot \frac{\partial \varepsilon_n}{\partial y} + \frac{\partial \pi_x \circ F_n^t}{\partial x} \circ F_n \cdot \frac{\partial^2 \varepsilon_n}{\partial x \partial y},\\ \left|\frac{\partial^2 \pi_x \circ F_n^{t+1}}{\partial x \partial y}\right| &\leq c_1 c_2^2 \|\varepsilon_n\| + c(t) c_1 \|\varepsilon_n\|^2 + c_1 c_2 \|\varepsilon_n\|,\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \pi_x \circ F_n^{t+1}}{\partial y^2} =& \frac{\partial^2 \pi_x \circ F_n^t}{\partial x^2} \circ F_n \cdot \left(\frac{\partial \varepsilon_n}{\partial y}\right)^2 + \frac{\partial \pi_x \circ F_n^t}{\partial x} \circ F_n \cdot \frac{\partial^2 \varepsilon_n}{\partial y^2}, \\
\left|\frac{\partial^2 \pi_x \circ F_n^{t+1}}{\partial y^2}\right| \le& c_1^2 c_2 \|\varepsilon_n\|^2 + c_1 c_2 \|\varepsilon_n\|,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 \pi_x \circ F_n^{t+1}}{\partial x^2 \partial y} =& \frac{\partial^3 \pi_x \circ F_n^t}{\partial x^3} \circ F_n \cdot \left(\frac{\partial h_n}{\partial x}\right)^2 \cdot \frac{\partial \varepsilon_n}{\partial y} + 2\frac{\partial^3 \pi_x \circ F_n^t}{\partial x^2 \partial y} \circ F_n \cdot \frac{\partial h_n}{\partial x} \cdot \frac{\partial \varepsilon_n}{\partial y} \\
&+ \frac{\partial^3 \pi_x \circ F_n^t}{\partial x \partial y^2} \circ F_n \cdot \frac{\partial \varepsilon_n}{\partial y} + \frac{\partial^2 \pi_x \circ F_n^t}{\partial x^2} \circ F_n \cdot \left(\frac{\partial^2 h_n}{\partial x^2} \cdot \frac{\partial \varepsilon_n}{\partial y} + 2\frac{\partial h_n}{\partial x} \cdot \frac{\partial^2 \varepsilon_n}{\partial x \partial y}\right) \\
&+ 2\frac{\partial^2 \pi_x \circ F_n^t}{\partial x \partial y} \circ F_n \cdot \frac{\partial^2 \varepsilon_n}{\partial x \partial y} + \frac{\partial \pi_x \circ F_n^t}{\partial x} \circ F_n \cdot \frac{\partial^3 \varepsilon_n}{\partial x^2 \partial y}, \\
\left|\frac{\partial^3 \pi_x \circ F_n^{t+1}}{\partial x^2 \partial y}\right| \le& c_1 c_2^3 \|\varepsilon_n\| + 2c(t) c_1 c_2 \|\varepsilon_n\|^2 + c(t) c_1 \|\varepsilon_n\|^2 + c_2 \left(c_1 c_2 \|\varepsilon_n\| + 2 c_1 c_2 \|\varepsilon_n\|\right) \\
&+ 2c(t) c_1 \|\varepsilon_n\|^2 + c_1 c_2 \|\varepsilon_n\|,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 \pi_x \circ F_n^{t+1}}{\partial x \partial y^2} =& \frac{\partial^3 \pi_x \circ F_n^t}{\partial x^3} \circ F_n \cdot \frac{\partial h_n}{\partial x} \cdot \left(\frac{\partial \varepsilon_n}{\partial y}\right)^2 + \frac{\partial^3 \pi_x \circ F_n^t}{\partial x^2 \partial y} \circ F_n \cdot \left(\frac{\partial \varepsilon_n}{\partial y}\right)^2 \\
&+ \frac{\partial^2 \pi_x \circ F_n^t}{\partial x^2} \circ F_n \cdot \left(2\frac{\partial \varepsilon_n}{\partial y} \cdot \frac{\partial^2 \varepsilon_n}{\partial x \partial y} + \frac{\partial h_n}{\partial x} \cdot \frac{\partial^2 \varepsilon_n}{\partial y^2}\right) + \frac{\partial \pi_x \circ F_n^t}{\partial x} \circ F_n \cdot \frac{\partial^3 \varepsilon_n}{\partial x \partial y^2}, \\
\left|\frac{\partial^3 \pi_x \circ F_n^{t+1}}{\partial x \partial y^2}\right| \le& c_1^2 c_2^2 \|\varepsilon_n\|^2 + c(t) c_1^2 \|\varepsilon_n\|^3 + c_2 \left(2c_1^2 \|\varepsilon_n\|^2 + c_1 c_2 \|\varepsilon_n\|\right) + c_1 c_2 \|\varepsilon_n\|,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^3 \pi_x \circ F_n^{t+1}}{\partial y^3} =& \frac{\partial^3 \pi_x \circ F_n^t}{\partial x^3} \circ F_n \cdot \left(\frac{\partial \varepsilon_n}{\partial y}\right)^3 + 3\frac{\partial^3 \pi_x \circ F_n^t}{\partial x^2 \partial y} \circ F_n \cdot \frac{\partial \varepsilon_n}{\partial y} \cdot \frac{\partial^2 \varepsilon_n}{\partial y^2} + \frac{\partial^3 \pi_x \circ F_n^t}{\partial x \partial y^2} \circ F_n \cdot \frac{\partial^3 \varepsilon_n}{\partial y^3}, \\
\left|\frac{\partial^3 \pi_x \circ F_n^{t+1}}{\partial y^3}\right| \le& c_1^3 c_2 \|\varepsilon_n\|^3 + 3c(t) c_1^2 \|\varepsilon_n\|^3 + c(t) c_1 \|\varepsilon_n\|^2.
\end{aligned}$$

Note that the partial derivatives $\frac{\partial^j \pi_x \circ F_n^t}{\partial x^j}$ and $\frac{\partial^j h_n}{\partial x^j}$ for $j \in \{1,2,3\}$ are uniform bounded above by the constant c_2 because the map F_n is close to the hyperbolic fixed point $i(f_\sigma)$ by the definition of $\mathscr{I}$. This proves the induction step $t+1$. □

Remark 17.24. Lemma 2.1 does not apply to the partial derivatives of $\frac{\partial \pi_x \circ F_n^t}{\partial y}$ directly because the lemma requires $\frac{\partial \pi_x \circ F_n^t}{\partial y}$ to be defined on a complex neighborhood of $P_n(0)$ and the constant depends on the size of the neighborhood. It is not easy to find a neighborhood such that all maps are defined on the same neighborhood.

Finally, we prove the lemma for the iteration and rescaling of a regular curve.

Proof of Lemma 17.21. Let $\hat{F} = H_n \circ F_n$ and $\hat{f}(x) = \pi_x \circ \hat{F}(x, r_1(x)) = \pi_x \circ F^p(x, r_1(x))$. For the first property, it is sufficient to prove that $\hat{f}$ has negative Schwarzian derivative because $s_n \circ \hat{f}(x) =$

$\pi_x \circ \phi_n \circ F_n(x, r_1(x))$ and s_n is an affine map. Let $\overline{\varepsilon} > 0$ be small such that the estimates in Lemma 17.23 hold.

Proposition 2.7 is used to prove the first property. To estimate the C^2 norm of $\hat{f}'$, we apply Lemma 17.23 and the definition of $\mathscr{I}$. Compute

$$\begin{aligned} \hat{f}'(x) - \left(f_\sigma^p\right)'(x) &= \left[\frac{\partial \pi_x \circ F^p}{\partial x} - \left(f_\sigma^p\right)'\right] + \frac{\partial \pi_x \circ F^p}{\partial y} \cdot \left(r_1'\right), \\ \left|\hat{f}'(x) - \left(f_\sigma^p\right)'(x)\right| &\leq \overline{\varepsilon} + c_1 R \left\|\varepsilon_n\right\|^{3/4}, \end{aligned} \tag{17.20}$$

$$\begin{aligned} \hat{f}''(x) - \left(f_\sigma^p\right)''(x) &= \left[\frac{\partial^2 \pi_x \circ F^p}{\partial x^2} - \left(f_\sigma^p\right)''\right] + 2\frac{\partial^2 \pi_x \circ F^p}{\partial x \partial y} \cdot \left(r_1'\right) + \frac{\partial^2 \pi_x \circ F^p}{\partial y^2} \cdot \left(r_1'\right)^2 \\ &\quad + \frac{\partial \pi_x \circ F^p}{\partial y} \cdot \left(r_1''\right), \\ \left|\hat{f}''(x) - \left(f_\sigma^p\right)''(x)\right| &\leq \overline{\varepsilon} + 2c_1 R \left\|\varepsilon_n\right\|^{3/4} + c_1 R^2 \left\|\varepsilon_n\right\|^{1/2} + c_1 R, \end{aligned} \tag{17.21}$$

and

$$\begin{aligned} \hat{f}'''(x) - \left(f_\sigma^p\right)'''(x) &= \left[\frac{\partial^3 \pi_x \circ F^p}{\partial x^3} - \left(f_\sigma^p\right)'''\right] + 3\frac{\partial^3 \pi_x \circ F^p}{\partial x^2 \partial y} \cdot \left(r_1'\right) + 3\frac{\partial^3 \pi_x \circ F^p}{\partial x \partial y^2} \cdot \left(r_1'\right)^2 \\ &\quad + \frac{\partial^3 \pi_x \circ F^p}{\partial y^3} \cdot \left(r_1'\right)^3 + 3\frac{\partial^2 \pi_x \circ F^p}{\partial x \partial y} \cdot \left(r_1''\right) + 3\frac{\partial^2 \pi_x \circ F^p}{\partial y^2} \cdot \left(r_1'\right)\left(r_1''\right) \\ &\quad + \frac{\partial \pi_x \circ F^p}{\partial y} \cdot \left(r_1'''\right), \\ \left|\hat{f}'''(x) - \left(f_\sigma^p\right)'''(x)\right| &\leq \overline{\varepsilon} + 3c_1 R \left\|\varepsilon_n\right\|^{3/4} + 3c_1 R^2 \left\|\varepsilon_n\right\|^{1/2} + c_1 R^3 \left\|\varepsilon_n\right\|^{1/4} + 7c_1 R \end{aligned} \tag{17.22}$$

for some constant $c_1 > 0$ given by Lemma 17.23. In the equations, the partial derivatives are evaluated at the point $(x, r_1(x))$. Also, the derivatives are estimated by the C^0 norm using Lemma 2.1. The inequalities show that the map $\hat{f}'$ is C^2 close to $f_\sigma^{p\prime}$ where f_σ^p is a map with negative Schwarzian derivative. Thus, $\hat{f}$ has negative Schwarzian derivative by Proposition 2.7 when $\overline{\varepsilon}$ and R are small (depending on g).

Next we show that the map $x \to \pi_x \circ \phi_n \circ F_n(x, r_1(x))$ is diffeomorphic to its image and prove the existence of the curve by finding the inverse function $\hat{f}^{-1}$ using the inverse function theorem. Observe that if the graph of u is the image of the graph of r_1 under $\hat{F}$, then $r_2 = s_n \circ u \circ s_n^{-1}$ and $u = \hat{f}^{-1}$ because the two curves r_1 and u satisfy the relation

$$(\hat{f}(t), t) = \hat{F}(t, r_1(t)) = (x, u(x)).$$

By the third property of the good region from Proposition 15.2, there exists a constant $c_2 > 0$ such that $\left|\frac{\partial h_n}{\partial x}(x, y)\right| > c_2 \sqrt{\|\varepsilon_n\|}$ whenever the point (x, y) belongs to one of the vertical strips

$[\alpha_n^{p-1,L}(1), \theta_n^L(K_n+2)]$ or $[\theta_n^R(K_n+2), \alpha_n^{p-1,R}(1)]$. Compute

$$\begin{aligned}
\hat{f}'(x) =& \frac{\partial \pi_x \circ F_n^{p-1}}{\partial x} \circ F(x, r_1(x)) \cdot \frac{\partial h_n}{\partial x}(x, r_1(x)) + \frac{\partial \pi_x \circ F_n^{p-1}}{\partial y} \circ F(x, r_1(x)) \\
&+ \frac{\partial \pi_x \circ F^p}{\partial y}(x, r_1(x)) \cdot r_1'(x) \\
\left|\hat{f}'(x)\right| \geq & c_2 c_3 \|\varepsilon_n\|^{1/2} - c_1 \|\varepsilon_n\| - c_1 \|\varepsilon_n\| \left\|r_1'\right\| \\
\geq & c_2 c_3 \|\varepsilon_n\|^{1/2} - c_1 \|\varepsilon_n\| - c_1 R \|\varepsilon_n\|^{3/4} \geq \frac{c_2 c_3}{2} \|\varepsilon_n\|^{1/2}
\end{aligned}$$

whenever $\overline{\varepsilon}$ and R are sufficiently small. For the first term, the partial derivative $\left|\frac{\partial \pi_x \circ F_n^{p-1}}{\partial x}\right|$ is bounded below by a constant c_3 on $(x,y) \in P_n(1)$ because $P_n(0)$ is away from the critical locus. The constant c_3 can be chosen to be independent of the Hénon-like map F_n when it is close to the fixed point $i(f_\sigma)$. The estimates for the remaining terms come from Lemma 17.23. Consequently, the curve $u = \hat{f}^{-1}$ exists and is C^3 by the inverse function theorem (Lemma A.1).

It remains to prove that r_2 is R-regular. The derivatives of $\hat{f}$ are uniformly bounded on I^h because they are close to derivatives of the limiting map f_σ^p by (17.20), (17.21), and (17.22). By computing the derivatives of the inverse function, we get

$$\begin{aligned}
u' \circ \hat{f}(x) &= \frac{1}{\hat{f}'(x)}, \\
\left|u' \circ \hat{f}(x)\right| &\leq \frac{2}{c_2 c_3} \|\varepsilon_n\|^{-1/2} \leq c_4 \|\varepsilon_n\|^{-1/2},
\end{aligned}$$

$$\begin{aligned}
u'' \circ \hat{f}(x) &= -\frac{\hat{f}''(x)}{\left[\hat{f}'(x)\right]^2}, \\
\left|u'' \circ \hat{f}(x)\right| &\leq \left|\hat{f}''(x)\right| \left(\frac{2}{c_2 c_3}\right)^2 \|\varepsilon_n\|^{-1} \leq c_4 \|\varepsilon_n\|^{-1},
\end{aligned}$$

and

$$\begin{aligned}
u''' \circ \hat{f}(x) =& -\frac{1}{\left[\hat{f}'(x)\right]^5} \left\{\hat{f}'(x) \hat{f}'''(x) - 3\left[\hat{f}''(x)\right]^2\right\}, \\
\left|u''' \circ \hat{f}(x)\right| \leq & \left\{\left|\hat{f}'(x) \hat{f}'''(x)\right| + 3\left|\hat{f}''(x)\right|^2\right\} \left(\frac{2}{c_2 c_3} \|\varepsilon_n\|^{-1/2}\right)^5 \leq c_4 \|\varepsilon_n\|^{-5/2}
\end{aligned}$$

for some constant $c_4 > 0$ when $\overline{\varepsilon}$ and R are sufficiently small.

By the definition of $\mathscr{I}$, the constant λ_n is close to λ_σ. We have $|\lambda_n| \geq c_5$ for some constant $c_5 > 0$ when $\overline{\varepsilon}$ is sufficiently small. By (14.6), we get

$$\left\|r_2'\right\| = \left\|u'\right\| \leq c_4 \|\varepsilon_n\|^{-1/2} \leq c_4 c_6 \|\varepsilon_{n+1}\|^{-1/2p} < \frac{R}{\|\varepsilon_{n+1}\|^{1/4}}, \tag{17.23}$$

$$\|r_2''\| = \frac{1}{|s_n'|}\|u''\| \le \frac{c_4}{|\lambda_n|}\|\varepsilon_n\|^{-1} \le \frac{c_4 c_6}{c_5}\|\varepsilon_{n+1}\|^{-1/p} < \frac{R}{\|\varepsilon_{n+1}\|},$$

$$\|r_2'''\| = \frac{1}{|s_n'|^2}\|u''\| \le \frac{c_4}{|\lambda_n|^2}\|\varepsilon_n\|^{-5/2} \le \frac{c_4 c_6}{c_5^2}\|\varepsilon_{n+1}\|^{-5/2p} < \frac{R}{\|\varepsilon_{n+1}\|} \tag{17.24}$$

and

$$\|r_2'\|\,\|r_2''\| \le \frac{c_4^2 c_6^2}{c_5}\|\varepsilon_{n+1}\|^{-3/2p} < \frac{R}{\|\varepsilon_{n+1}\|}$$

for some constant $c_6 > 0$ given by (14.6) whenever $p \ge 3$ and $\overline{\varepsilon}$ is sufficiently small (depending on R). Therefore, the curve r_2 is R-regular. □

Remark 17.25. The estimates (17.23) and (17.24) are the inequalities that do not work for $p = 2$.

The proposition generalizes Proposition 2.10 to the step in the good region of Q_n.

Proposition 17.26. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and $p \ge 3$. For all $R > 0$ sufficiently small and $\overline{\varepsilon} > 0$ sufficiently small (depending on R), the following property holds for all $F \in \mathscr{I}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$ and $n \ge 0$:*

Assume that $\mathscr{X}_n$ is a collection of separators on $I^h \times I_n^v$ with Lipschitz constant $L\|\varepsilon_n\|^{1/4}$, $\mathscr{Z}_{n+1}$ is a collection of separators on $I^h \times I_{n+1}^v$ with Lipschitz constant $L\|\varepsilon_{n+1}\|^{1/4}$, and $RL < 1$. If S_1 and S_2 are vertical strips of $\mathscr{X}_n$ and $\mathscr{Z}_n$ respectively, S_1 is in the base set of the good region for the prerescaling set Base$(Q_n^i(j))$ where $i = L$ or R, and the boundaries of S_1 are mapped to the boundaries of S_2 by $\phi_n \circ F_n$, then

$$|\phi_n \circ F_n(J)|_{S_2} \ge |J|_{S_1}$$

for all $J \subset S_1$. Here, the hyperbolic size of the sets are measured by R-regular curves on the sets' associate renormalization level.

Proof. Let R and $\overline{\varepsilon}$ be the constants given by Lemma 17.21. Given an R-regular curve $r_1 : [a_1, b_1] \to I_n^v$ associate to F_n such that the two endpoints $(a_1, r_1(a_1))$ and $(b_1, r_1(b_1))$ of its graph belong on the two boundaries of S_1 and the two points $(c_1, r_1(c_1))$ and $(d_1, r_1(d_1))$ on the graph belong to J.

By Lemma 17.21, the image of the graph of r_1 is the graph of an R-regular curve $r_2 : [a_2, b_2] \to I_{n+1}^v$ associate to F_{n+1}. The boundaries $(a_2, r_2(a_2))$ and $(b_2, r_2(b_2))$ of its graph belong to the two boundaries of S_2 because boundaries of S_1 maps to boundaries of S_2. Also, the two points $(c_2, r_2(c_2))$ and $(d_2, r_2(d_2))$ belong to $\phi_n \circ F_n(J)$ where $c_2 = \hat{f}(c_1)$, $d_2 = \hat{f}(d_1)$, and $\hat{f}(x) = \pi_x \circ \phi_n \circ F_n(x, r_1(x))$. Hence, r_2 is an R-regular curve that satisfies the conditions for computing the hyperbolic size. We get

$$|\phi_n \circ F_n(J)|_{S_2} \ge |[c_2, d_2]|_{[a_2,b_2]}. \tag{17.25}$$

Moreover, the map $\hat{f}$ has negative Schwarizan derivative by Lemma 17.21. It implies the expansion of hyperbolic length

$$|[c_2, d_2]|_{[a_2,b_2]} = \left|[\hat{f}(c_1), \hat{f}(d_1)]\right|_{[\hat{f}(a_1),\hat{f}(b_1)]} \ge |[c_1, d_1]|_{[a_1,b_1]} \tag{17.26}$$

by Proposition 2.10. Combine (17.25) and (17.26), we get

$$|\phi_n \circ F_n(J)|_{S_2} \ge |[c_1, d_1]|_{[a_1,b_1]}.$$

This inequality holds for all R-regular curves r_1 that satisfy the conditions for measuring the hyperbolic size. Therefore, the proposition is proved. □

Finally, we conclude the expansion of hyperbolic size in the base sets from one iteration plus one rescaling.

Corollary 17.27 (Expansion for $Q_n(j) \xrightarrow{F_n} R_n^L(j) \xrightarrow{\phi_n} R_n^L(j-1)$). *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ, and $p \geq 3$. For all $R > 0$ sufficiently small and $\overline{\varepsilon} > 0$ sufficiently small (depending on R), the following property holds for all $F \in \mathscr{I}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$ and $n \geq 0$:*

If $J \subset Q_n^i(j)$ for some $1 \leq j \leq K_n$ and $i = L$ or R, then

$$|\phi_n \circ F_n(J)|_{Base(R_{n+1}^L(j-1))} \geq |J|_{Base(Q_n^i(j))}$$

where the hyperbolic size is measured by R-regular curves in the sets renormalization levels.

Proof. By definition, the boundary separators of $Base(Q_n^i(j))$ are mapped to the boundaries of $Base(R_{n+1}^L(j-1))$ by $\phi_n \circ F_n$. The boundary separators of $Base(Q_n^i(j))$ have Lipschitz constant $L\|\varepsilon_n\|^{1/2} = \left(L\|\varepsilon_n\|^{1/4}\right)\|\varepsilon_n\|^{1/4}$ by the fifth property of the good region from Proposition 15.2 and the boundary separators of $Base(R_{n+1}^L(j-1))$ have Lipschitz constant $L\|\varepsilon_{n+1}\| = \left(L\|\varepsilon_{n+1}\|^{3/4}\right) \cdot \|\varepsilon_{n+1}\|^{1/4}$ by Proposition 14.32. Also, the inequalities $RL\|\varepsilon_n\|^{1/4} < 1$ and $RL\|\varepsilon_{n+1}\|^{3/4} < 1$ hold when $\overline{\varepsilon}$ is small enough. Therefore, Proposition 17.26 applies to this corollary. □

17.2.3. $R_n^L(j) \xrightarrow{\phi_n} R_{n+1}^L(j-1)$

When a wandering domain enters the rescaling set, it is then rescalied by ϕ_n according to the procedure of defining a closest approach. This section will study the expansion of hyperbolic size during the step of rescaling.

Again, we repeat the work done in the proof of Propositions 17.3 and 17.26. We first prove that the property of negative Schwarzian derivative is preserved under rescaling when the rescaling map is restricted to an R-regular curve.

Lemma 17.28. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. For all $R > 0$ sufficiently small and $\overline{\varepsilon} > 0$ sufficiently small (depending on R), the following properties hold for all $F \in \mathscr{I}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$ and $n \geq 0$:*

Assume that $r_1 : [a_1, b_1] \to I_n^v$ is an R-regular curve (associated to F_n). If the graph of r_1 is in $P_n(1)$, then

1. *the map $x \to \pi_x \circ \phi_n(x, r_1(x))$ is diffeomorphic to its image and has negative Schwarzian derivative and*

2. *the rescaling ϕ_n of the graph of r_1 is the graph of an R-regular curve $r_2 : [a_2, b_2] \to I_{n+1}^v$ associated to F_{n+1}.*

Proof. Proposition 2.7 is used to prove the first property. Let $\hat{\phi}(x) = \pi_x \circ \phi_n(x, r_1(x))$ and $\hat{f}(x) = \pi_x \circ H_n(x, r_1(x)) = \pi_x \circ F_n^{p-1}(x, r_1(x))$. Then $\hat{\phi}(x) = s_n \circ \hat{f}(x)$. It is sufficient to estimate the C^2 norm of $\hat{f}'$ because s_n is an affine map. By Lemma 17.23 and the definition of $\mathscr{I}$, compute

$$\hat{f}'(x) - \left(f_\sigma^{p-1}\right)'(x) = \left[\frac{\partial \pi_x \circ F_n^{p-1}}{\partial x} - \left(f_\sigma^{p-1}\right)'\right] + \frac{\partial \pi_x \circ F_n^{p-1}}{\partial y} \cdot \left(r_1'\right),$$

$$\left|\hat{f}'(x) - \left(f_\sigma^{p-1}\right)'(x)\right| \leq \overline{\varepsilon} + c_1 R \left\|\varepsilon_n\right\|^{3/4}, \tag{17.27}$$

$$\begin{aligned}\hat{f}''(x) - \left(f_\sigma^{p-1}\right)''(x) = &\left[\frac{\partial^2 \pi_x \circ F_n^{p-1}}{\partial x^2} - \left(f_\sigma^{p-1}\right)''\right] + 2\frac{\partial^2 \pi_x \circ F_n^{p-1}}{\partial x \partial y} \cdot \left(r_1'\right) \\ &+ \frac{\partial^2 \pi_x \circ F_n^{p-1}}{\partial y^2} \cdot \left(r_1'\right)^2 + \frac{\partial \pi_x \circ F_n^{p-1}}{\partial y} \cdot \left(r_1''\right),\end{aligned}$$

$$\left|\hat{f}''(x) - \left(f_\sigma^{p-1}\right)''(x)\right| \leq \overline{\varepsilon} + 2c_1 R \left\|\varepsilon_n\right\|^{3/4} + c_1 R^2 \left\|\varepsilon_n\right\|^{1/2} + c_1 R, \tag{17.28}$$

and

$$\begin{aligned}\hat{f}'''(x) - \left(f_\sigma^{p-1}\right)'''(x) = &\left[\frac{\partial^3 \pi_x \circ F_n^{p-1}}{\partial x^3} - \left(f_\sigma^{p-1}\right)'''\right] + 3\frac{\partial^3 \pi_x \circ F_n^{p-1}}{\partial x^2 \partial y} \cdot \left(r_1'\right) + 3\frac{\partial^3 \pi_x \circ F_n^{p-1}}{\partial x \partial y^2} \cdot \left(r_1'\right)^2 \\ &+ \frac{\partial^3 \pi_x \circ F_n^{p-1}}{\partial y^3} \cdot \left(r_1'\right)^3 + 3\frac{\partial^2 \pi_x \circ F_n^{p-1}}{\partial x \partial y} \cdot \left(r_1''\right) + 3\frac{\partial^2 \pi_x \circ F_n^{p-1}}{\partial y^2} \cdot \left(r_1'\right)\left(r_1''\right) \\ &+ \frac{\partial \pi_x \circ F_n^{p-1}}{\partial y} \cdot \left(r_1'''\right),\end{aligned}$$

$$\left|\hat{f}'''(x) - \left(f_\sigma^{p-1}\right)'''(x)\right| \leq \overline{\varepsilon} + 3c_1 R \left\|\varepsilon_n\right\|^{3/4} + 3c_1 R^2 \left\|\varepsilon_n\right\|^{1/2} + c_1 R^3 \left\|\varepsilon_n\right\|^{1/4} + 3c_1 R + 3c_1 R + c_1 R \tag{17.29}$$

for some constant $c_1 > 0$ from Lemma 17.23. The derivatives in the equations are evaluated at the point $(x, r_1(x))$. Thus, $\hat{f}$ has negative Schwarzian derivative when $\overline{\varepsilon}$ and R are sufficiently small by Proposition 2.7.

Next, we prove that the map $x \to \pi_x \circ \phi_n(x, r_1(x))$ is diffeomorphic to its image and the curve r_2 exists by finding the inverse function of $\hat{f}$. Observe that the two curves r_1 and r_2 satisfy the relation

$$\left(s_n \circ \hat{f}(t), s_n \circ r_1(t)\right) = \phi_n(t, r_1(t)) = (x, r_2(x)).$$

Thus, $r_2 = s_n \circ r_1 \circ \hat{f}^{-1} \circ s_n^{-1}$.

To prove the existence of the inverse function $\hat{f}^{-1}$, compute

$$\left|\hat{f}'(x)\right| \geq \left|\frac{\partial \pi_x \circ F_n^{p-1}}{\partial x}(x, r_1(x))\right| - \left|\frac{\partial \pi_x \circ F_n^{p-1}}{\partial y}(x, r_1(x)) r_1'(x)\right|$$

$$\geq c_2 - c_1 \|\varepsilon_n\| \left\|r_1'\right\| \geq c_2 - c_1 R \|\varepsilon_n\|^{3/4} \geq c_2/2$$

when $\overline{\varepsilon}$ and R are small. The first term is bounded below by the constant $c_2 > 0$ because $P_n(0)$ is away from the critical locus. The constant c_2 can be chosen to be uniform for all maps in a neighborhood of $i(f_\sigma)$. The second term is estimated by Lemma 17.23 and the definition of R-regular curves. Consequently, the inverse function $\hat{f}^{-1}$ exists and is C^3 by the inverse function theorem (Lemma A.1).

It remains to prove that r_2 is R-regular. The derivatives of $\hat{f}$ are uniformly bounded on I^h because they are close to derivatives of the limiting map f_σ^{p-1} by (17.27), (17.28), (17.29) and I^h is compact. By evaluating the derivatives of the inverse function, we have

$$\left|(\hat{f}^{-1})' \circ \hat{f}(x)\right| = \frac{1}{\left|\hat{f}'(x)\right|} \leq \frac{2}{c_2} \leq c_5,$$

$$(\hat{f}^{-1})'' \circ \hat{f}(x) = -\frac{\hat{f}''(x)}{\left[\hat{f}'(x)\right]^2},$$

$$\left|(\hat{f}^{-1})'' \circ \hat{f}(x)\right| \leq \left(\frac{2}{c_2}\right)^2 \left|\hat{f}''(x)\right| \leq c_3$$

and

$$(\hat{f}^{-1})''' \circ \hat{f}(x) = -\frac{1}{\left[\hat{f}'(x)\right]^5} \left\{\hat{f}'(x)\hat{f}'''(x) - 3\left[\hat{f}''(x)\right]^2\right\},$$

$$\left|(\hat{f}^{-1})''' \circ \hat{f}(x)\right| \leq \left(\frac{2}{c_2}\right)^3 \left\{\left|\hat{f}'(x)\hat{f}'''(x)\right| + 3\left|\hat{f}''(x)\right|^2\right\} \leq c_3$$

for some constant $c_3 > 0$. By (14.6), we get

$$r_2' = \left[r_1' \circ \hat{f}^{-1} \circ s_n^{-1}\right] \left[(\hat{f}^{-1})' \circ s_n^{-1}\right],$$

$$\left|r_2'\right| \leq c_3 R \|\varepsilon_n\|^{-1/4} \leq c_3 c_4 R \|\varepsilon_{n+1}\|^{-1/4p} \leq R \|\varepsilon_{n+1}\|^{-1/4},$$

$$r_2'' = \frac{1}{\lambda_n} \left\{\left(r_1'' \circ \hat{f}^{-1} \circ s_n^{-1}\right) \left[(\hat{f}^{-1})' \circ s_n^{-1}\right]^2 + \left(r_1' \circ \hat{f}^{-1} \circ s_n^{-1}\right) \left[(\hat{f}^{-1})'' \circ s_n^{-1}\right]\right\},$$

$$\begin{aligned}\left|r_2''\right| \leq &\frac{1}{c_5} \left[c_3^2 R \|\varepsilon_n\|^{-1} + c_3 R \|\varepsilon_n\|^{-1/4}\right] \leq \frac{c_4}{c_5} \left[c_3^2 \|\varepsilon_{n+1}\|^{(p-1)/p} + c_3 \|\varepsilon_{n+1}\|^{(4p-1)/4p}\right] R \|\varepsilon_{n+1}\|^{-1} \\ &\leq R \|\varepsilon_{n+1}\|^{-1},\end{aligned}$$

$$\begin{aligned}r_2''' = &\frac{1}{\lambda_n^2} \left\{\left(r_1''' \circ \hat{f}^{-1} \circ s_n^{-1}\right) \left[(\hat{f}^{-1})' \circ s_n^{-1}\right]^3 + 3\left(r_1'' \circ \hat{f}^{-1}\right) \left[(\hat{f}^{-1})' \circ s_n^{-1}\right] \left[(\hat{f}^{-1})'' \circ s_n^{-1}\right]\right. \\ &\left. + \left(r_1' \circ \hat{f}^{-1}\right) \left[(\hat{f}^{-1})''' \circ s_n^{-1}\right]\right\},\end{aligned}$$

$$\begin{aligned}
\left|r_2'''\right| \leq & \frac{1}{c_5^2}\left[c_3^3 R\left\|\varepsilon_n\right\|^{-1}+3 c_3^2 R\left\|\varepsilon_n\right\|^{-1}+c_3 R\left\|\varepsilon_n\right\|^{-1/4}\right] \\
\leq & \frac{c_4}{c_5^2}\left[c_3^3\left\|\varepsilon_{n+1}\right\|^{(p-1)/p}+3 c_3^2\left\|\varepsilon_{n+1}\right\|^{(p-1)/p}+c_3\left\|\varepsilon_{n+1}\right\|^{(4p-1)/4p}\right] R\left\|\varepsilon_{n+1}\right\|^{-1} \\
\leq & R\left\|\varepsilon_{n+1}\right\|^{-1},
\end{aligned}$$

and

$$\left|r_2'\right|\left|r_2''\right| \leq \frac{c_3 c_4^2}{c_5}\left[c_3^2 R\left\|\varepsilon_{n+1}\right\|^{(4p-5)/4p}+c_3 R\left\|\varepsilon_{n+1}\right\|^{(2p-1)/2p}\right] R\left\|\varepsilon_{n+1}\right\|^{-1}<R\left\|\varepsilon_{n+1}\right\|^{-1}$$

for some constant $c_4 > 0$ given by (14.6) whenever $\overline{\varepsilon}$ and R are sufficiently small. The constant λ_n also has a bound $|\lambda_n| \geq c_5$ for some constant $c_4 > 0$ when $\overline{\varepsilon}$ is sufficiently small because λ_n is close to λ_σ by the definition of $\mathscr{I}$. Therefore, the curve r_2 is R-regular associated to F_{n+1}. □

Proposition 17.29. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. For all $R > 0$ sufficiently small and $\overline{\varepsilon} > 0$ sufficiently small (depending on R), the following property holds for all $F \in \mathscr{I}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$ and $n \geq 0$:*

Assume that $\mathscr{Y}_n$ is a collection of separators on $I^h \times I_n^v$ with Lipschitz constant $L\|\varepsilon_n\|^{1/4}$, $\mathscr{Z}_{n+1}$ is a collection of separators on $I^h \times I_{n+1}^v$ with Lipschitz constant $L\|\varepsilon_{n+1}\|^{1/4}$, and $RL < 1$. If S_1 and S_2 are vertical strips of $\mathscr{Y}_n$ and $\mathscr{Z}_n$ respectively, S_1 is in $P_n(1)$, and the boundaries of S_1 are mapped to the boundaries of S_2 by ϕ_n, then

$$|\phi_n(J)|_{S_2} \geq |J|_{S_1}$$

for all $J \subset S_1$. Here, the hyperbolic size of the sets are measured by R-regular curves in their renormalization level.

Proof. Let R and $\overline{\varepsilon}$ be the constants given by Lemma 17.28. Given an R-regular curve $r_1 : [a_1, b_1] \to I_n^v$ associate to F_n such that the two endpoints $(a_1, r_1(a_1))$ and $(b_1, r_1(b_1))$ of its graph belong on the two boundaries of S_1 and the two points $(c_1, r_1(c_1))$ and $(d_1, r_1(d_1))$ on the graph belong to J.

By Lemma 17.28, the image of the graph of r_1 is the graph of an R-regular curve $r_2 : [a_2, b_2] \to I_{n+1}^v$ associate to F_{n+1}. The boundaries $(a_2, r_2(a_2))$ and $(b_2, r_2(b_2))$ of its graph belong to the two boundaries of S_2 because boundaries of S_1 maps to boundaries of S_2. Also, the two points $(c_2, r_2(c_2))$ and $(d_2, r_2(d_2))$ belong to $\phi_n(J)$ where $c_2 = \hat{\phi}(c_1)$, $d_2 = \hat{\phi}(d_1)$, and $\hat{\phi}(x) = \pi_x \circ \phi_n(x, r_1(x))$. Hence, r_2 is an R-regular curve that satisfies the conditions for measuring the hyperbolic size. We get

$$|\phi_n(J)|_{S_2} \geq |[c_2, d_2]|_{[a_2, b_2]}. \tag{17.30}$$

Moreover, the map $\hat{\phi}$ has negative Schwarizan derivative by Lemma 17.28. It implies the expansion of hyperbolic length

$$|[c_2, d_2]|_{[a_2, b_2]} = \left|[\hat{\phi}(c_1), \hat{\phi}(d_1)]\right|_{[\hat{\phi}(a_1), \hat{\phi}(b_1)]} \geq |[c_1, d_1]|_{[a_1, b_1]} \tag{17.31}$$

by Proposition 2.10. Combine (17.30) and (17.31), we get

$$|\phi_n(J)|_{S_2} \geq |[c_1, d_1]|_{[a_1, b_1]}.$$

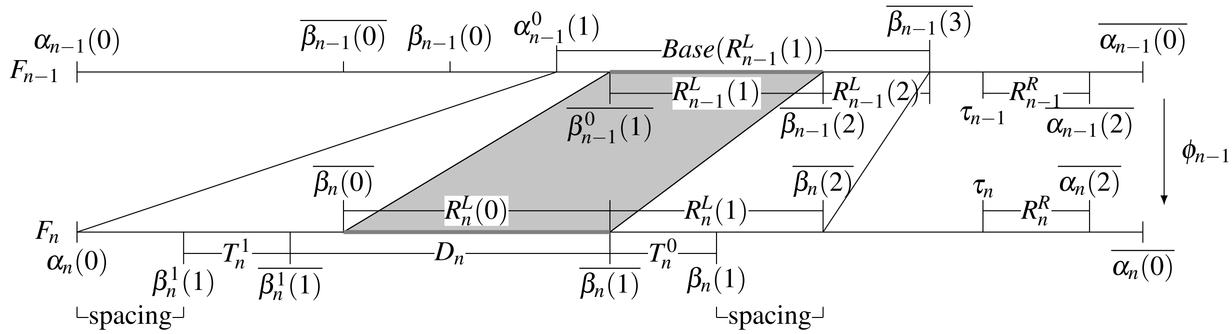


Figure 17.1.: The rescaling of the last step from R to D.

This inequality holds for all R-regular curves that satisfy the conditions for measuring the hyperbolic size. Therefore, the proposition is proved. □

Finally, we obtain the expansion of hyperbolic size in the base sets from rescaling.

Corollary 17.30 (Expansion for $R^L_n(j) \xrightarrow{\phi_n} R^L_n(j-1)$)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. For all $R > 0$ sufficiently small and $\overline{\varepsilon} > 0$ sufficiently small (depending on R), the following property holds for all $F \in \mathscr{I}^{\sigma}_{\delta}(I^h \times I^v, \overline{\varepsilon})$ and $n \geq 0$:*

If $J \subset R^L_n(j)$ and $j \geq 1$, then

$$|\phi_n(J)|_{Base(R^L_{n+1}(j-1))} \geq |J|_{Base(R^L_n(j))}$$

where the hyperbolic size of the sets are measured by R-regular curves in their renormalization level.

Proof. First, $[\alpha_n(1), \overline{\beta_n(j+2)}] \subset [\alpha_n(0), \overline{\beta_n(j+2)}] = Base(R^L_n(j))$. By Proposition 16.4, we have

$$|J|_{[\alpha_n(1),\overline{\beta_n(j+2)}]} \geq |J|_{Base(R^L_n(j))} \tag{17.32}$$

The vertical strip $[\alpha_n(1), \overline{\beta_n(j+2)}]$ is in $P_n(1)$ and its boundaries are mapped to the boundaries of $Base(R^L_{n+1}(j-1))$ by the rescaling map ϕ_n. Also, the separators $\alpha_n(1)$ and $\overline{\beta_n(j+2)}$ have Lipschitz constant $L\|\varepsilon_n\| = \left(L\|\varepsilon_n\|^{3/4}\right)\|\varepsilon_n\|^{1/4}$ and the boundary separators of $Base(R^L_{n+1}(j-1))$ have Lipschitz constant $L\|\varepsilon_{n+1}\| = \left(L\|\varepsilon_{n+1}\|^{3/4}\right)\|\varepsilon_{n+1}\|^{1/4}$. Then $RL\|\varepsilon_n\|^{3/4} < 1$ and $RL\|\varepsilon_{n+1}\|^{3/4} < 1$ hold when $\overline{\varepsilon} > 0$ is sufficiently small. Thus, Proposition 17.29 applies to the vertical strips and the rescaling map. We get

$$|\phi_n(J)|_{Base(R^L_{n+1}(j-1))} \geq |J|_{[\alpha_n(1),\overline{\beta_n(j+2)}]}. \tag{17.33}$$

Therefore, the corollary follows by combining (17.32) and (17.33). □

17.2.4. $R^L_n(0) \Rightarrow D_n$ (G or T^j_n)

Lemma 17.31 (Expansion for $R^L_n(0) \Rightarrow D_n$)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, an admissible unimodal permutation σ, and a unimodal map $g \in \mathscr{U}^{\sigma^\infty}_{\delta}(I^h)$. For all $R > 0$ sufficiently* *small (depending on g) and $\overline{\varepsilon} > 0$ sufficiently small (depending on g and R), the following property holds for all $F \in \mathscr{H}_{\delta}^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$:*

Assume that SDisp $\left(\alpha_F(0), \beta_F^1(1)\right) \geq K$ and SDisp $\left(\beta_F(1), \overline{\beta_F(2)}\right) \geq K$ for some constant $K > 0$. If $J \subset S$ where S is a trapping set or gap in D_n, then

$$|J|_{Base(S)} \geq E\,|J|_{Base(R_n^L(0))}$$

for some constant $E > 1$ that depends only on $K/\left|I^h\right|$. The hyperbolic size is measured by R-regular curves.

Proof. See Figure 17.1. The set $Base(R_n^L(0)) \backslash Base(S)$ contains a component on each side of $Base(S)$. The left component contains $[\alpha_n(0), \beta_n^1(1)]$ and the right component contains $[\beta_n(1), \overline{\beta_n(2)}]$. By Proposition 16.5, we get

$$|J|_{Base(S)} \geq E\,|J|_{Base(R_n^L(0))} \tag{17.34}$$

for some constant $E > 1$ determined by $K/\left|I^h\right|$. □

17.3. Uniform expansion in the good region

This section summarize the uniform expansion of the hyperbolic length for a closest approach by the following proposition.

Proposition 17.32 (Uniform expansion in the good region)**.** *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and an admissible unimodal permutation σ. For all $R > 0$ sufficiently small and $\overline{\varepsilon} > 0$ sufficiently small (depending on R), the following property holds for all $F \in \mathscr{I}_{\delta}^{\sigma}(I^h \times I^v, \overline{\varepsilon})$:*

Assume that $J \subset B \cup C \cup D$ is a wandering domain of F and $\{J_n\}_{n=0}^{\infty}$ is the J-closest approach. If $J_0, \cdots, J_{n-1}$ belong to the good region, that is, $0 \leq k_{r(m)}(J_m) \leq K_{r(m)}$ for all $0 \leq m < n$, then

$$l_n \geq E^{n-p} \cdot l_0$$

for some constant $E > 1$ where the hyperbolic length is measured by R-regular curves.

Sketch of the proof. The proposition follows by Lemma 17.10 ($B_n \overset{F_n}{\to} C_n$), Lemma 17.11 ($C_n \overset{F_n}{\Rightarrow} D_n$), Proposition 17.6 ($D_n \overset{F_n}{\to} D_n$), Proposition 17.7 ($D_n \overset{F_n^p}{\Rightarrow} D_n$), and Proposition 17.18 ($D_n \hookrightarrow Q_n \overset{\Phi_n^j \circ F_n}{\Rightarrow} D_{n+j}$).

In any p steps $J_t \to \cdots \to J_{t+p}$, if $C_n \overset{F_n}{\Rightarrow} D_n$ or the rescaling $D_n \hookrightarrow Q_n \overset{\Phi_n^j \circ F_n}{\Rightarrow} D_{n+j}$ occurs, then uniform expansion happens due to Lemma 17.11 ($C_n \overset{F_n}{\Rightarrow} D_n$) and Proposition 17.18 ($D_n \hookrightarrow Q_n \overset{\Phi_n^j \circ F_n}{\Rightarrow} D_{n+j}$). Otherwise, the wandering domain are all in D_n for some n. In the later case, expansion of definite size is provided by Proposition 17.7 ($D_n \overset{F_n^p}{\Rightarrow} D_n$) for every p steps inside D_n. □

18. Bad region and Thickness

When an element J_n from a closest approach enters the bad region, the expansion estimate breaks down and the hyperbolic sizes have a strong contraction when the element is iterated and rescaled from J_n to J_{n+1}. This leads to the main difficulty of showing that the horizontal sizes approach infinity. Our goal is to prove that the closest approach have at most finite entries to the bad regions to show that the total amount of contraction is bounded.

To estimate the size of contraction, we first introduce the quantity "thickness" (Definition 18.1) . Thickness gives a good estimation for the lower bound of the hyperbolic size when the expansion estimate breaks down. To study the number of entries to the bad regions, we define a sequence with two indices, called a double sequence (Definition 18.4). The sequence consists of rows. Each row is associated to one entry to the bad region. Then we study the relationships between the hyperbolic sizes and the thicknesses of the elements in a double sequence. From Propositions 18.5, 18.6, and 18.7, we have a full control over the hyperbolic sizes and the thicknesses of all elements in a double sequence.

Finally, we prove that the number of rows in a double sequence is bounded to show that the amount of contraction is bounded (Proposition 18.11). Of course, the reader can replicate the proof from the period-doubling case. We proved a similar version of the expansion estimate for hyperbolic size and contraction estimate for thickness for the case of other stationary combinatorics. However, here we present a different but shorter proof.

As a result, if a wandering domain exists, then we study the sizes of the elements in a closest approach. We showed that the hyperbolic sizes expand at a definite rate while the elements are in the good regions (Chapter 17), and the sizes have a strong contraction during every entries to the bad regions which is estimated by the thickness (Section 18.1). The total amount of contraction is bounded because the elements have at most finite entries to the bad regions (Proposition 18.11). This shows that the hyperbolic sizes approach infinity. However, the hyperbolic size of the gaps and trapping sets are uniform bounded, and the sizes of the elements are bounded by the sizes of the gaps and trapping sets. This is a contradiction. Therefore, a wandering domain cannot exist.

18.1. Thickness and largest square subset

Thickness is a quantity to estimate the size of the contraction when the expansion argument breaks down. It was first introduced in Chapter 11 to prove the nonexistence of wandering domain for the period-doubling case. Whenever an element J_n in a closest approach enters the bad region, the hyperbolic size of the next sequence element J_{n+1} is determined by the set's horizontal cross-section. Thickness is defined to estimate the size of the horizontal cross-section in terms of area, roughly speaking. Moreover, the element J_{n+1} is very thin because the area is contracted by the size of its Jacobian which is very small for a strongly dissipative Hénon-like map. Thus, a strong contraction on the hyperbolic length applies to the step $J_n \to J_{n+1}$.

The universality for the tip [dCLM05, Section 7.3] is the key property that allows us to estimate the contraction of thickness in the period-doubling case. It gives a lower bound for estimating the Jacobian $\frac{\partial \varepsilon}{\partial y}$ in terms of the size of perturbation ε. For the arbitrary stationary combinatorics case, there is also a version for the universality [Haz11, Section 6.1]. Therefore, the techniques for thickness and largest square subset can be generalized to this context without making any adjustment.

This section gives a brief review of thickness and largest square subset from Section 11.1. Most of the properties will be stated without proof.

Definition 18.1 (Square, Largest square subset, and Thickness)**.** A set $I \subset \mathbb{R}^2$ is a *square* if $I = [x_1, x_2] \times [y_1, y_2]$ with $x_2 - x_1 = y_2 - y_1$. This means that I is a closed square with horizontal and vertical sides.

Given a set $J \subset \mathbb{R}^2$. Define the *thickness* of J to be the quantity $w(J) = \sup_I \{|I|\}$ where the supremum is evaluated over all square subsets $I \subset J$. A subset $I \subset J$ is a *largest square subset* of J if I is a square such that $|I| = w(J)$. A largest square subset of a compact set always exists. See Figure 11.1 for illustration.

For a closest approach $\{J_n\}_{n=0}^{\infty}$, write $w_n = w(J_n)$. The contraction rate of the thickness is estimated by

Proposition 18.2. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exist $\overline{\varepsilon} > 0$ sufficiently small and $c > 0$ such that the following property holds for all $F \in \mathscr{I}_{\delta}^{\sigma}(I^h \times I^v, \overline{\varepsilon})$:*

Assume that $J \subset B \cup C \cup D$ is a wandering domain of F and $\{J_n\}_{n=0}^{\infty}$ is the J-closest approach. Then

$$w_{n+1} \geq c \frac{\left\|\varepsilon_{r(n)}\right\|}{\left|I^v_{r(n)}\right|} w_n$$

for all $n \geq 0$.

Proof. The proof is similar to Proposition 11.7. It depends on the universality of Hénon-like maps in Proposition 14.25. □

Since $\|\varepsilon_n\|$ decreases super-exponentially and $|I^v_n|$ increases exponentially, we can simplify

Corollary 18.3. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exist $\overline{\varepsilon} > 0$ sufficiently small and $c > 0$ such that the following property holds for all $F \in \mathscr{I}_{\delta}^{\sigma}(I^h \times I^v, \overline{\varepsilon})$:*

Assume that $J \subset B \cup C \cup D$ is a compact subset of a wandering domain of F and $\{J_n\}_{n=0}^{\infty}$ is the J-closest approach. Then

$$w_{n+1} \geq c \left\|\varepsilon_{r(n)}\right\|^{3/2} w_n$$

for all $n \geq 0$.

18.2. Double sequence

Next, we study the number of times that a closest approach enters the bad region by defining a double sequence of sets. The definition is the same as in the period-doubling case Definition

11.10. A double sequence is a sequence with two indices, one index represents the rows and the other represents the columns. Each row in a double sequence is associated to entering the bad region once. The total number of rows is the number of times that a closest approach enters the bad region.

Recall the definition.

Definition 18.4 (Double sequence, Row, and Time span in good region)**.** Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. Assume that $\overline{\varepsilon} > 0$ be sufficiently small so that Proposition 15.2 holds and $F \in \mathscr{I}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$ is a non-degenerate open map.

Given a square subset $J \subset B \cup C \cup D$ of a wandering domain for F. Define sets $\left\{J_n^{(j)}\right\}_{n\geq 0, 0\leq j\leq \overline{j}}$, Hénon-like maps $\left\{F_n^{(j)} = (f_n^{(j)} - \varepsilon_n^{(j)}, x)\right\}_{n\geq 0, 0\leq j\leq \overline{j}}$, and non-negative integers $\left\{n^{(j)}\right\}_{0\leq j\leq \overline{j}}$ for some $\overline{j} \in \mathbb{N} \cup \{0, \infty\}$[1] by induction on j such that the following properties hold.

Base For $j = 0$, set $J_0^{(0)} = J$ and $F_0^{(0)} = F$.

Row The super-script j is called *row*. The first set $J_0^{(j)}$ of a row j is a square subset of a wandering domain of $F_0^{(j)}$ in $B(F_0^{(j)}) \cup C(F_0^{(j)}) \cup D(F_0^{(j)})$. Each row j is a $J_0^{(j)}$-closest approach. Precisely, if $J_0^{(j)}$ and $F_0^{(j)}$ are defined, set $F_n^{(j)} = R^n F_0^{(j)}$ and $K_n^{(j)}$ be the boundary for the good region and the bad region of $F_n^{(j)}$. Let $\left\{J_n^{(j)}\right\}_{n=0}^\infty$ and $\left\{r^{(j)}(n)\right\}_{n=0}^\infty$ be the $J_0^{(j)}$-closest approach. See Definition 14.45 and Definition 15.1.

Induction step For a row j, if an element in the row enters the bad region, i.e. $k_n^{(j)} > K_{r^{(j)}(n)}^{(j)}$ for some $n \geq 0$, set $J_{n^{(j)}}^{(j)}$ to be the first element. The nonnegative integer $n^{(j)}$ is called the *time span in good region* of row j. Define the first element $J_0^{(j+1)}$ of the next row $j+1$ to be a largest square subset of the next element $J_{n^{(j)}+1}^{(j)}$ and set $F_0^{(j+1)} = F_{r^{(j)}(n^{(j)}+1)}^{(j)}$. If the row never enters the bad region, then the construction stops, set $\overline{j} = j$ and $n^{(j)} = \infty$. If the procedure never stops, set $\overline{j} = \infty$.

The two dimensional sequence $\left\{J_n^{(j)}\right\}_{n\geq 0, 0\leq j\leq \overline{j}}$ is called a *double sequence* generated by J or a J-double sequence. The integer $\overline{j}$ is called the *number of rows* in the double sequence. It means the double sequence enters the bad region $\overline{j}$ times.

Figure 11.1 illustrates the construction.

To be consistent and avoid confusion, the superscript is assigned for the row and the subscript is assigned for the renormalization level or the index of sequence element in the closest approach. For example, abbreviate $D_n^{(j)} = D(F_n^{(j)})$, $R_n^{(j)} = B(F_n^{(j)})$, $T_n^{i,(j)} = T^i(F_n^{(j)})$, $P_n^{i,(j)} = P^i(F_n^{(j)})$, $l_n^{(j)} = l(J_n^{(j)})$, $w_n^{(j)} = w(J_n^{(j)})$, and $k_n^{(j)} = k(J_n^{(j)})$ as before.

[1] For the case $\overline{j} = \infty$, this means that the sequence is defined for all finite positive integers j.

In the following, write $r^{(j)}(n) = r(n)$ when the context is clear, for example $F^{(j)}_{r(n^{(j)}+1)} = F^{(j)}_{r^{(j)}(n^{(j)}+1)}$. Also, set $\varepsilon^{(j)} = \varepsilon^{(j)}_{r(n^{(j)})}$, $K^{(j)} = K^{(j)}_{r(n^{(j)})}$, and $k^{(j)} = k^{(j)}_{n^{(j)}}$. The quantities $\varepsilon^{(j)}$, $K^{(j)}$, and $k^{(j)}$ are the representative quantities of those objects in the row. For convenience, let $m^{(j)} = n^{(j)} + 1$.

First, we study the relations of hyperbolic size and thickness in a double sequence. The expansion of hyperbolic size in a row was proved by Proposition 17.32. The result is rephrased in terms of double sequence by the next proposition.

Proposition 18.5. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and an admissible unimodal permutation σ. There exist constants $E > 1$, $R > 0$, and $\overline{\varepsilon} > 0$ sufficiently small such that the following property holds for all non-degenerate open maps $F \in \mathscr{I}^{\sigma}_{\delta}(I^h \times I^v, \overline{\varepsilon})$:*

Let $J \subset B \cup C \cup D$ be a square subset of a wandering domain of F and $\left\{J^{(j)}_n\right\}_{n \geq 0, 0 \leq j \leq \overline{j}}$ be a J-double sequence. Then

$$l^{(j)}_n \geq E^{n-p} l^{(j)}_0$$

for all $n \leq n^{(j)}$ and $0 \leq j \leq \overline{j}$. The hyperbolic size is measured by R-regular curves.

The relation of sizes between two consecutive rows are connected by the thickness. By definition, the first element $J^{(j+1)}_0$ in row $j+1$ is a largest square subset of the sequence element $J^{(j)}_{n^{(j)}+1}$ in row j. This yields the relation $w^{(j+1)}_0 = w^{(j)}_{n^{(j)}+1}$. The hyperbolic size of the elements in row $j+1$ cannot be obtained from the hyperbolic size of the elements in row j because the hyperbolic size of $J^{(j)}_{n^{(j)}}$ fails to expand under iteration. The next proposition summarizes the relation of thickness between two rows.

Proposition 18.6. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exists $\overline{\varepsilon} > 0$ sufficiently small such that the following property holds for all non-degenerate open maps $F \in \mathscr{I}^{\sigma}_{\delta}(I^h \times I^v, \overline{\varepsilon})$:*

Let $J \subset B \cup C \cup D$ be a square subset of a wandering domain of F and $\left\{J^{(j)}_n\right\}_{n \geq 0, 0 \leq j \leq \overline{j}}$ be a J-double sequence. Then

$$\ln w^{(j+1)}_0 \geq 2m^{(j)} \ln \left\| \varepsilon^{(j)} \right\| + \ln w^{(j)}_0$$

for all $0 \leq j \leq \overline{j} - 1$.

Proof. The proof is similar to the proof of Proposition 11.13. See also Corollary 18.3. The details are left to the reader. □

The next proposition allows us to relate the horizontal size with the thickness for the first element in a row.

Proposition 18.7. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exist $\overline{\varepsilon} > 0$ sufficiently small and $c = c(I^h) > 0$ such that the following property holds for all non-degenerate open maps $F \in \mathscr{I}^{\sigma}_{\delta}(I^h \times I^v, \overline{\varepsilon})$:*

Let $J \subset B \cup C \cup D$ be a square subset of a wandering domain of F and $\left\{J^{(j)}_n\right\}_{n \geq 0, 0 \leq j \leq \overline{j}}$ be a J-double sequence. Then

$$l^{(j)}_0 \geq c w^{(j)}_0$$

for all $0 \le j \le \bar{j}$.

Proof. The proposition follows from Proposition 16.7 and the set $J_0^{(j)}$ is a square by definition. $\square$

Next, we relate the perturbation ε between two rows as follows.

Proposition 18.8. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exist $\bar{\varepsilon} > 0$ sufficiently small and $\alpha > 0$ such that the following property holds for all non-degenerate open maps $F \in \mathscr{I}_\delta^\sigma(I^h \times I^v, \bar{\varepsilon})$:*

Let $J \subset B \cup C \cup D$ be a square subset of a wandering domain of F and $\left\{J_n^{(j)}\right\}_{n \ge 0, 0 \le j \le \bar{j}}$ be a J-double sequence. Then

$$\left\|\varepsilon^{(j+1)}\right\| \le \left\|\varepsilon^{(j)}\right\|^{\left\|\varepsilon^{(j)}\right\|^{-\alpha}} \tag{18.1}$$

for all $0 \le j \le \bar{j} - 1$.

Proof. The proof is similar to Proposition 11.14.

By the definition of $\varepsilon^{(j)}$ and Proposition 14.25, we have

$$\left\|\varepsilon^{(j+1)}\right\| = \left\|\varepsilon^{(j+1)}_{r(n^{(j+1)})}\right\| \le \left\|\varepsilon_0^{(j+1)}\right\| = \left\|\varepsilon^{(j)}_{r(n^{(j)}+1)}\right\| \le c \left\|\varepsilon^{(j)}_{r(n^{(j)})}\right\|^{p^{k(j)}} = c \left\|\varepsilon^{(j)}\right\|^{p^{k(j)}}$$

for some constant $c > 0$. Here we assume that $\bar{\varepsilon} > 0$ is sufficiently small so that the size of the perturbation ε is decreasing in each row. Apply logarithm to both sides, we get

$$\ln\left\|\varepsilon^{(j+1)}\right\| \le p^{k(j)} \ln\left\|\varepsilon^{(j)}\right\| + \ln c \le \frac{1}{2} p^{k(j)} \ln\left\|\varepsilon^{(j)}\right\|. \tag{18.2}$$

Here we assume that $\bar{\varepsilon} > 0$ is small enough such that

$$\ln c < -\frac{1}{2} p^{k(j)} \ln\left\|\varepsilon^{(j)}\right\|$$

for all $j \ge 0$.

The element $J^{(j)}_{n^{(j)}}$ enters the bad region, we have $k^{(j)} > K^{(j)}$. By Proposition 15.2 and the change base formula, we get

$$p^{k(j)} > p^{K(j)} = \left(\lambda^{K(j)}\right)^{\frac{\ln p}{\ln \lambda}} \ge c' \left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{\frac{\ln p}{2\ln \lambda}} \tag{18.3}$$

for some constant $c' > 0$. Let $\alpha = \frac{\ln p}{4 \ln \lambda} > 0$. Combine (18.2) and (18.3), we obtain

$$\ln\left\|\varepsilon^{(j+1)}\right\| \le \frac{c'}{2} \left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{2\alpha} \ln\left\|\varepsilon^{(j)}\right\| < \left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{\alpha} \ln\left\|\varepsilon^{(j)}\right\|.$$

Here we also assume that $\bar{\varepsilon}$ is small enough such that

$$\frac{c'}{2} \left(\frac{1}{\left\|\varepsilon^{(j)}\right\|}\right)^{\alpha} > 1$$

for all $j \geq 0$. This proves the proposition. $\square$

18.3. Closest approach cannot enter the bad region infinitely many times

From the vertical line argument, a strong contraction applies to the horizontal size whenever an element in a closest approach enters the bad region. This conflicts our final goal of showing that the horizontal sizes approach infinity. In this section, we prove that the total amount of contraction is bounded. This is done by showing that a double sequence has at most finite number of rows (Proposition 18.11).

When an element in a closest approach enters the bad region, a restriction also applies to the element: the size of the element cannot exceed the size of the bad region. This is the key condition that is used to prove Proposition 18.11. The next lemma estimates an upper bound of the hyperbolic size if an element is in the bad region.

Lemma 18.9. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and a unimodal permutation σ. There exist $\overline{\varepsilon} > 0$ sufficiently small and $c > 0$ such that the following property holds for all non-degenerate open maps $F \in \mathscr{I}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$ and $n \geq 0$:*

If the set $J \subset Q_n$ is in the bad region, then

$$l(J) \leq c\sqrt{\|\varepsilon_n\|}.$$

Proof. The set J is in the center trapping set T_n^{p-1} because J lands in the bad region. Since the both sides of T_n^{p-1} has definite size, the hyperbolic size of J is bounded by

$$l(J) \leq c_1 \cdot \sup\{|x_2 - x_1| : (x_1, y_1), (x_2, y_2) \in J\}$$

for some constant $c_1 > 0$ by Proposition 16.6. Also, the size of the bad region bounds the Euclidean size of the set J by

$$\sup\{|x_2 - x_1| : (x_1, y_1), (x_2, y_2) \in J\} \leq c_2\sqrt{\|\varepsilon_n\|}$$

for some constant $c_2 > 0$ by Proposition 15.2 (second property of the bad region). Therefore, the lemma follows. $\square$

Now, we had prepared all of the ingredients, Proposition 18.5, Proposition 18.6, Proposition 18.8, Proposition 18.7, and Lemma 18.9, in order to prove Proposition 18.11. Of course, the reader can follow the arguments from the period doubling case (Section 11.3) to obtain the same result. Here, we present a different proof.

First, we derive a recurrence relation for the thickness.

Lemma 18.10. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and an admissible unimodal permutation σ. There exist $\overline{\varepsilon} > 0$ sufficiently small and a constant $c > 0$ such that the following property holds for all non-degenerate open maps $F \in \mathscr{I}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$ and $n \geq 0$:*

Let $J \subset B \cup C \cup D$ be a square subset of a wandering domain of F and $\left\{J_n^{(j)}\right\}_{n \geq 0, 0 \leq j \leq \overline{j}}$ be a

J-double sequence. Then the recurrence relation

$$\ln w_0^{(j+1)} \geq c a_j^2 + a_j \ln w_0^{(j)} \tag{18.4}$$

holds for all j with $0 \leq j \leq \overline{j} - 1$ where $a_j = -\frac{4}{\ln E} \ln \left\| \varepsilon^{(j)} \right\| > 0$.

Proof. The element $J_{n^{(j)}}^{(j)}$ from the j-th row enters the bad region. By Lemma 18.9, the hyperbolic size is bounded by

$$l_{n^{(j)}}^{(j)} \leq c_1 \sqrt{\left\| \varepsilon^{(j)} \right\|} \tag{18.5}$$

for some constant $c_1 > 0$. From Proposition 18.5 and Proposition 18.7, we have

$$l_{n^{(j)}}^{(j)} \geq E^{n^{(j)}} l_0^{(j)} \geq c_2 E^{n^{(j)}} w_0^{(j)} \tag{18.6}$$

for some constants $c_2 > 0$ and $E > 1$. Combine (18.5) and (18.6), we get

$$\begin{aligned} n^{(j)} \leq & \frac{1}{\ln E} \left[\ln c_1 - \ln c_2 + \frac{1}{2} \ln \left\| \varepsilon^{(j)} \right\| - \ln w_0^{(j)} \right] \\ \leq & \frac{1}{4 \ln E} \ln \left\| \varepsilon^{(j)} \right\| - \frac{1}{\ln E} \ln w_0^{(j)} \end{aligned} \tag{18.7}$$

when $\overline{\varepsilon}$ is small.

By Proposition 18.6 and (18.7), we have

$$\begin{aligned} \ln w_0^{(j+1)} \geq & 2 \left[\frac{1}{\ln E} \left(\ln c_1 - \ln c_2 + 1 + \frac{1}{2} \ln \left\| \varepsilon^{(j)} \right\| - \ln w_0^{(j)} \right) \right] \ln \left\| \varepsilon^{(j)} \right\| + \ln w_0^{(j)} \\ = & \frac{1}{\ln E} \left[2 \left(\ln c_1 - \ln c_2 + 1 \right) + \ln \left\| \varepsilon^{(j)} \right\| \right] \ln \left\| \varepsilon^{(j)} \right\| + \left(1 - \frac{2}{\ln E} \ln \left\| \varepsilon^{(j)} \right\| \right) \ln w_0^{(j)} \\ \geq & \frac{1}{2 \ln E} \left(\ln \left\| \varepsilon^{(j)} \right\| \right)^2 + \frac{4}{\ln E} \left(- \ln \left\| \varepsilon^{(j)} \right\| \right) \ln w_0^{(j)} \end{aligned}$$

when $\overline{\varepsilon} > 0$ is small since $\ln \left\| \varepsilon^{(j)} \right\| < 0$. □

Then, we use the recurrence relation to conclude that a double sequence has finite number of rows.

Proposition 18.11. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and an admissible unimodal permutation σ. There exists a constant $\overline{\varepsilon} > 0$ sufficiently small such that the following property holds for all non-degenerate open maps $F \in \mathscr{I}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$ and $n \geq 0$:*

Let $J \subset B \cup C \cup D$ be a square subset of a wandering domain of F and $\left\{ J_n^{(j)} \right\}_{n \geq 0, 0 \leq j \leq \overline{j}}$ be J-double sequence. Then the number of rows $\overline{j}$ is finite.

Proof. Prove by contradiction. If the double sequence have infinite number of rows, then (18.4) holds for all $j \geq 0$. We get

$$\begin{aligned}\ln w_0^{(j+1)} &\geq c a_j^2 + c a_j a_{j-1}^2 + \cdots + c\left(\prod_{k=1}^{j} a_k\right) a_0^2 + \left(\prod_{k=0}^{j} a_k\right) \ln w_0^{(0)} \\ &\geq \left(\prod_{k=0}^{j} a_k\right)\left(c\frac{a_j^2}{\prod_{k=0}^{j} a_k} + \ln w_0^{(0)}\right)\end{aligned}$$

for some constant $c > 0$.

In order to estimate the size of the lower bound, we estimate the ratio a_{j+1}/a_j. By Proposition 18.8, we have

$$\frac{a_{j+1}}{a_j} = \frac{\ln\left\|\varepsilon^{(j+1)}\right\|}{\ln\left\|\varepsilon^{(j)}\right\|} \geq \left\|\varepsilon^{(j)}\right\|^{-\alpha}$$

for some constant $\alpha > 0$ since $\ln\left\|\varepsilon^{(j)}\right\| < 0$. Apply the inequality $x > \ln x$ to $x = \left\|\varepsilon^{(j)}\right\|^{-\alpha/2}$, we get

$$\frac{a_{j+1}}{a_j} \geq c'\left\|\varepsilon^{(j)}\right\|^{-\alpha/2} a_j \tag{18.8}$$

for some constant $c' > 0$.

Finally, we estimate the size of the term $a_j^2/\prod_{k=0}^{j} a_k$. By (18.8), we have

$$\frac{a_j^2}{\prod_{k=0}^{j} a_k} \geq \prod_{k=0}^{j-1}\left(c'\left\|\varepsilon^{(k)}\right\|^{-\alpha/2}\right) a_0.$$

However, this implies that $\lim_{j\to\infty} \ln w_0^{(j+1)} = \infty$ which is a contradiction. Therefore, the number of rows $\overline{j}$ is finite. □

18.4. Nonexistence of wandering domains

Finally, we put all of the ingredients together to prove the main theorem, an infinitely renormalizable Hénon-like map with arbitrary stationary combinatorics does not have a wandering domain. The proof assumes the contradictory, there exists a wandering domain. By the next proposition, we may assume without lose of generality that the combinatorics is admissible by the shifting trick (Example 13.24) and the Hénon-like map is close to the hyperbolic fixed point of the renormalization operator with the same combinatorics (Propositions 14.25, 14.44, and 18.12).

Proposition 18.12. *Given $\delta > 0$ and intervals I^h and I^v with $I \Subset I^h \subset I^v$. Assume that σ is an admissible unimodal permutation and μ is a two-cycle and $g \in \mathscr{U}_\delta^{\mu}(I^h) \cup \mathscr{U}_\delta^{\sigma^\infty}(I^h)$. Set $\mathscr{H} = \mathscr{H}_\delta^{\mu}(I^h \times I^v, g, \overline{\varepsilon})$ if $g \in \mathscr{U}_\delta^{\mu}(I^h)$ and $\mathscr{H} = \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$ if $g \in \mathscr{U}_\delta^{\sigma^\infty}(I^h)$. There exists $\overline{\varepsilon} > 0$ (depending on g) such that the following property holds for all Hénon-like maps $F \in \mathscr{H}$:*

The Hénon-like map F has a wandering domain in $P_F(0)$ if and only if its renormalization RF has a wandering domain in $P_{RF}(0)$.

Proof. The proof is similar to the unimodal case Proposition 13.62. The proposition follows from Proposition 14.44 and Corollary 17.8. □

Finally, we prove the main theorem.

Theorem 18.13. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, a unimodal permutation σ that is not the two cycle, and $g \in \mathscr{U}_\delta^{\sigma^\infty}(I^h)$. There exists $\overline{\varepsilon} > 0$ (depending on g) such that every non-degenerate open Hénon-like map $F \in \mathscr{H}_\delta^{\sigma^\infty}(I^h \times I^v, g, \overline{\varepsilon})$ does not have a wandering domain.*

Proof. Prove by contradiction. Assume that F has a wandering domain. Without lose of generality, we may assume that the combinatorics σ is admissible and the Hénon-like map F is close to the fixed point $i(f_\sigma)$ by the shifting trick, Proposition 18.12, and Proposition 14.25. We may also assume that the wandering domain is in D_F by 14.44.

Let J be a square subset of a wandering domain in D_F and $\left\{J_n^{(j)}\right\}_{n\geq 0, 0\leq j\leq \overline{j}}$ be a J-double sequence. By Proposition 18.11, the number of rows $\overline{j}$ is finite. This implies that the amount of contraction is bounded and

$$\lim_{n\to\infty} l_n^{(\overline{j})} = \infty$$

by Proposition 18.5. However, this is impossible because the hyperbolic size of the trapping sets and gaps are uniform bounded in their base sets when a Hénon-like map is close enough to the hyperbolic fixed point $i(f_\sigma)$. Therefore, a wandering domain cannot exist. □

As a consequence, the absence of wandering domain shows the size of the strange attractor is small in the topological sense.

Corollary 18.14. *Given $\delta > 0$, intervals I^h and I^v with $I \Subset I^h \subset I^v$, and an admissible unimodal permutation σ. There exists $\overline{\varepsilon} > 0$ such that for every non-degenerate open Hénon-like map $F \in \mathscr{I}_\delta^\sigma(I^h \times I^v, \overline{\varepsilon})$, the union of the stable manifolds of the period points is dense in the domain $P_F(0)$.*

Nomenclature

Notation	Description	Page
a	Fixed point with a positive multiplier associate to its noncontracting direction for a Hénon-like map	149
$a^t(1)$	Periodic orbit of period p with a positive multiplier for a Hénon-like map	155
$a(j)$	The family of periodic points with a positive multiplier for an infinite renormalizable Hénon-like Map	159
$\alpha(0)$	(unimodal) Fixed point with positive multiplier	124
$\alpha(0)$	(Hénon) The connected component of the stable manifold of $a(0)$ that contains the point	154
$\overline{\alpha(0)}$	(unimodal) The preimage of the fixed point $\alpha(0)$	124
$\overline{\alpha(0)}$	(Hénon) The connected component of the preimage of $\alpha(0)$ that does not contain the point	154
$\alpha^t(1)$	(unimodal) Periodic orbit of period p with positive multiplier	124
$\alpha^t(1)$	(Hénon) The connected component of the stable manifold of $a^t(1)$ that contains the point	155
$\overline{\alpha^t(1)}$	(unimodal) Preimages of the periodic orbit $\alpha^t(1)$, page 125	125
$\overline{\alpha^t(1)}$	(Hénon) Preimages of $\alpha(1)$ that forms an orbit of local stable manifolds	155
$\alpha(j)$	(unimodal) The family of periodic points with positive multiplier	127
$\alpha(j)$	(Hénon) A component of the stable manifolds containing the periodic point $a(j)$	159
B	The vertical strip $[\beta^0(1), \overline{\alpha^0(1)}]$	165
b	Fixed point with a negative multiplier associate to its noncontracting direction for a Hénon-like map	154
$b(j)$	The family of periodic points with a negative multiplier for an infinite renormalizable Hénon-like Map	159
$\beta(0)$	(unimodal) Fixed point of the unimodal map with negative multiplier	124
$\beta(0)$	(Hénon) The connected component of the stable manifold of $b(0)$ that contains $b(0)$	154
$\overline{\beta(0)}$	(unimodal) The preimage of the fixed point $\beta(0)$	124
$\overline{\beta(0)}$	(Hénon) The connected component of the preimage of $\beta(0)$ that does not contain $b(0)$	154
$\beta^t(1)$	(unimodal) Periodic orbit of period p with negative multiplier	128
$\beta^t(1)$	(Hénon) The connected component of the stable manifold of $b^t(1)$ that contains the point	162
$\overline{\beta^t(1)}$	(unimodal) Preimages of the fixed point $\beta(1)$, page 129	128

Notation	Description	Page
$\overline{\beta^t(1)}$	(Hénon) Preimages of $\beta(1)$ that forms an orbit of local stable manifolds	162
$\beta(j)$	(unimodal) The family of periodic points with negative multiplier	127
$\beta(j)$	(Hénon) A component of the stable manifolds containing the periodic point $b(j)$	159
C	The vertical strip $[\overline{\alpha^1(1)}, \beta^1(1)]$	165
D	(unimodal) Iteration interval	133
D	(Hénon) Iteration set	164
ε	Perturbation part of a Hénon-like map	149
F	Hénon-like map	149
f	Unimodal map	123
f^s	Induced unimodal map on separators	152
f_σ	The fixed point of the unimodal renormalization operatior with combinatorial type σ	126
G	Gap	128, 162
H	Nonlinear part of the Hénon rescaling	157
h	The x-component of the Hénon-like map	149
$\mathscr{H}_\delta^\sigma$	Class of renormaizable Hénon-like maps of combinatorial type σ	157
$\mathscr{H}_\delta$	Class of Hénon-like maps with holomorphic extension on a δ-neighborhood	149
$i(f)$	Degenerate Hénon-like map	149
I^h	Horizontal domain for a Hénon-like map	149
I^v	Vertical domain for a Hénon-like map	149
$\mathscr{I}^\sigma$	Class of infinite renormalizable Hénon-like maps with stationary combinatorics σ	158
$\mathscr{I}_\delta^\sigma$	The class of infinite renormalizable Hénon-like maps with combinatoric type σ that are close to the fixed point $i(f_\sigma)$	159
$\bar{j}$	Number of rows in a double sequence	201
J_n	The J-closest approach	135, 167
$J_n^{(j)}$	A J-double sequence	201
k	Rescaling level	134, 165
K_n	Boundary for good region and bad region	169
l	(unimodal) Hyperbolic length	136
l	(Hénon) Hyperbolic size	179
Λ	Affine part of the Hénon rescaling	157
λ_n	The scaler s_n'	158
λ_σ	The rescaling factor for the fixed point f_σ	126
$n^{(j)}$	Time span in the good region for row j in a double sequence of wandering domain	201

Notation	**Description**	**Page**
$P(0)$	(unimodal)The interval $[\alpha(0), \overline{\alpha(0)}]$. The unimodal map is a self-map on the interval	124
$P(0)$	(Hénon) The vertical strip of $[\alpha(0), \overline{\alpha(0)}]$ that makes the Hénon-like map to be a self-map	154
P^t	(unimodal) Cyclic intervals	125
P^t	(Hénon) Cyclic sets	155
ϕ	Hénon rescaling	126, 157
Φ_n^j	Nonlinear rescaling from renormalization level n to $n+j$	158
Q	(unimodal) Prerescaling interval	133
Q	(Hénon) Prerescaling set	165
R	Renormalization operator	126, 157
$r(n)$	Renormalization scale of the closest approach J_n	135, 167
R	Parameter for regular curves	171
R	(unimodal) Rescaling interval	133
R	(Hénon) Rescaling set	164
s	(unimodal) Affine rescaling for the unimodal renormalization	126
s	(Hénon) Affine part of the Hénon rescaling	157
σ	Unimodal permutation	124
t	Tip of a Hénon-like map	161
T^t	(unimodal) Trapping intervals	128
T^t	(Hénon) Trapping sets	162
θ	Preimage of $\overline{\beta(1)}$	128, 162
$\mathscr{U}$	Class of unimodal maps	123
$\mathscr{U}_\delta$	Class of unimodal maps with holomorphic extension on a δ-neighborhood	123
$\mathscr{U}^\sigma$	Class of renormaizable unimodal maps of combinatorial type σ	125
w	Thickness	200

A. Tools

Lemma A.1 (Inverse function theorem)**.** *Assume that $f : [a,b] \to [c,d]$ is C^3 onto. If $f'(x) \neq 0$ for all $a \in [a,b]$, then f has a inverse function $s : [c,d] \to [a,b]$ that is C^3 and*

1. $s' \circ f(x) = \frac{1}{f'(x)}$,
2. $s'' \circ f(x) = -\frac{f''(x)}{[f'(x)]^2}$, *and*
3. $s''' \circ f(x) = -\frac{1}{[f'(x)]^5}\left\{f'(x)f'''(x) - 3\,[f''(x)]^2\right\}$.

Proof. The lemma follows directly from the inverse function theorem and chain rule. See for example [Rob99, P.140]. □

Lemma A.2. *Assume that $d > 0$ and f, g are continuous function on $J = [p-d, p+d]$. If f is decreasing and has a fixed point at p and $\|f-g\|_J < d$, then g has a fixed point in $(p-d, p+d)$.*

Proof. Consider the function $x \to x - g(x)$. Compute

$$\begin{aligned} p+d-g(p+d) &= (p+d-p)-(f(p+d)-p)+f(p+d)-g(p+d) \\ &\geq d - \|f-g\|_J > 0 \end{aligned}$$

and

$$\begin{aligned} p-d-g(p-d) &= (p-d-p)-(f(p-d)-p)+f(p-d)-g(p-d) \\ &\leq -d + \|f-g\|_J < 0. \end{aligned}$$

Therefore, g has a fixed point in $(p-d, p+d)$ by the intermediate value theorem. □

The following technical lemma estimates the change of the root when a function is perturbed.

Lemma A.3. *Let $J = [c,d]$, $m > 0$, and $\varepsilon > 0$ with $d - c > \frac{2\varepsilon}{m}$. Assume that $f, g \in C^1(J)$ such that $\|f-g\|_J < \varepsilon$ and $|f'|, |g'| > m$. If $f(u) = g(v)$ for some $u, v \in [c + \frac{\varepsilon}{m}, d - \frac{\varepsilon}{m}]$, then*

$$|u - v| < \frac{\varepsilon}{m}.$$

Proof. Without lose of generality, we may assume that $f' > m > 0$ by multiplying -1 to f. By the mean value theorem, there exist $\xi \in (u - \frac{\varepsilon}{m}, u)$ and $\eta \in (u, u + \frac{\varepsilon}{m})$ such that

$$f(u) - f(u - \frac{\varepsilon}{m}) = f'(\xi)\frac{\varepsilon}{m} > \varepsilon$$

and

$$f(u + \frac{\varepsilon}{m}) - f(u) = f'(\eta)\frac{\varepsilon}{m} > \varepsilon.$$

Since $\left|f(u-\frac{\varepsilon}{m})-g(u-\frac{\varepsilon}{m})\right|<\varepsilon$ and $\left|f(u+\frac{\varepsilon}{m})-g(u+\frac{\varepsilon}{m})\right|<\varepsilon$, we get $g(u-\frac{\varepsilon}{m})<g(v)$ and $g(u+\frac{\varepsilon}{m})>g(v)$. It follows by the intermediate value theorem, the unique solution v lies in the interval $\left(u-\frac{\varepsilon}{m},u+\frac{\varepsilon}{m}\right)$. That is, $|u-v|<\frac{\varepsilon}{m}$. □

We also need

Lemma A.4. *For all $\alpha>0$, we have $x^\alpha \ln x<0$ for all $0<x<1$ and*

$$\lim_{x\to 0+} x^\alpha \ln x = 0.$$

Proof. The limit follow directly from the L'Hôpital's rule. □

B. Comparison of the Notations

In Part I and Part II, we used two different systems of notations to express periodic orbits, local stable manifolds, and vertical strips. Here, we give a conversion table to relate the notations. Assume that the Hénon-like map is infinitely period-doubling renormalizable.

- Point:

Part I		Part II
$p(-1)$	=	$a(0)$
$p(j)$	=	$a(j+1) = b(j)$
τ	=	t

- Local stable manifold:

Part I		Part II
$W^0(-1)$	=	$\alpha(0)$
$W^2(-1)$	=	$\overline{\alpha(0)}$
$W^0(0)$	=	$\alpha(1) = \beta(0)$
$W^1(0)$	=	$\overline{\alpha^1(1)} = \overline{\beta(0)}$
$W^2(0)$	=	$\overline{\alpha^0(1)}$
$W^0(j)$	=	$\alpha(j+1) = \beta(j)$
$W^2(j)$	=	$\overline{\alpha(j+1)}$

- Vertical Strip:

Part I		Part II
B	=	P^1
C	=	$P^0 = P(1)$
D	=	$P(0)$

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