On Conformally Kähler Einstein-Maxwell Metrics

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Abstract of the Dissertation On Conformally Kähler Einstein-Maxwell Metrics

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We construct families of conformally Kahler Einstein-Maxwell metrics in arbitrary dimension. We also consider the rigidity of Yamabe metrics in the space of conformally Kahler Einstein-Maxwell metrics.

Contents

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1. Introduction

The purpose of this section is to familiarize the reader with the definitions and motivation needed to understand the remaining sections.

Let (M, g) be a compact, oriented, Riemannian 4-manifold. Suppose F is a 2-form obeying:

$$dF = 0$$

$$\star dF = 0$$

$$(ric_g + F \circ F)_0 = 0$$

where $F \circ F$ denotes the composition of F with itself, when thought of as an endomorphism of the tangent bundle of M, $((F \circ F)_{ij} = F_i^k F_{kj})$, and the 0 subscript denotes the trace-free part. Such a trio is called an Einstein-Maxwell metric. It arose originally in the physics literature, where g plays the role of gravitational field, and F the electromagnetic field. Notice that if M is compact, then F being both closed and co-closed implies that it's a harmonic 2-form. In this work, we will consider exclusively the case where M is compact. Furthermore, we always take our metrics to be of Riemannian, as opposed to Lorentzian signature. Whether there is any direct interest to physics, therefore, remains to be seen.

In [9], C. LeBrun showed that any such metric is necessarily of constant scalar curvature. Conversely, any constant scalar curvature Kähler surface, with F defined to be the 2-form

$$F = \omega + \frac{\rho_0}{2},$$

is an example of an Einstein-Maxwell manifold. Here, ω denotes the Kähler form, and ρ_0 denotes the primitive part of the Ricci form.

There are [9, 10] at least three interesting variational characterizations of the Einstein-Maxwell equations:

(1) They are the Euler-Lagrange equations for the functional

$$(g, F) \longmapsto \int_M (s_g + |F|_g^2)\mu_g$$

defined on the space $\{(g, F) | g$ is a metric of volume V, F is a 2-form in some fixed de Rham cohomology class$\}$.

(2) They are the Euler-Lagrange equations for the Einstein-Hilbert functional:

$$g \longmapsto \frac{\int_M s_g \mu_g}{\sqrt{\int_M \mu_g}},$$

with g varying in $G_{[\omega]}$, which, for a given element, $[\omega]$ of $H^2(M, \mathbb{R})$ with $[\omega]^2 > 0$ is defined to be the space of smooth metrics g, such that the harmonic representative of $[\omega]$ (with respect to g) is self-dual (also with respect to g).

(3) They imply the Euler-Lagrange equations for the Calabi functional

$$g \longmapsto \int_M s_g^2 \mu_g$$

as g varies in $G_{[\omega]}$.

The second characterization recasts the study of Einstein-Maxwell metrics as a sort of restricted Yamabe problem. The relationship between these two subjects was broached in [9] by C. LeBrun, and is further explored in section 4 of this paper. In particular, since $G_{[\omega]}$ contains the entire conformal class of $[\omega]$, we see that Einstein-Maxwell metrics are of constant scalar curvature. This connection bears further study, and will likely lead to much fruitful research.

In another direction, the third characterization above relates the theory to Calabi's theory of extremal Kähler metrics [4, 5]. Indeed, every constant scalar curvature Kähler manifold is extremal and, as mentioned above, they always provide examples of Einstein-Maxwell metrics. This connection should make the subject of Einstein-Maxwell metrics of particular interest to those studying Kähler geometry.

Definition 1. *If (M, J) denotes a complex surface, then a solution of the Einstein-Maxwell equations, (h, F), is called strongly Hermitian if both h and F are J-invariant.*

In [10], it was proven that, for any compact complex surface with a strongly Hermitian solution, (h, F), of the Einstein-Maxwell equations, there is always a Kähler metric g, and a holomorphy potential, $f > 0$, with $h = f^{-2}g$ and such that F^+, the self-dual part of the 2-form F, is a constant multiple of the Kähler form of g. Recall that a holomorphy potential is a (positive) real-valued function such that:

- the $(1,0)$-component of its gradient is a holomorphic vector field, or
- f has J-invariant Hessian, or
- $Jgrad(f)$ is a Killing field.

In [10, 11], concrete families of Einstein-Maxwell metrics are constructed (see also [8]). In particular, C. LeBrun found a family of Einstein-Maxwell metrics including the Einstein metric constructed by Page on $\mathbb{CP}^2\overline{\mathbb{CP}^2}$. In the next two sections of this paper, I extend these constructions to the following higher-dimensional generalization of Einstein-Maxwell metrics:

Apostolov and Maschler [1] then considered higher dimensional analogues of these strongly Hermitian Einstein-Maxwell metrics.

Definition 2. *Let h be a Hermitian metric on a complex manifold (M, J), and let u denote a smooth function such that $g := u^2h$ is a Kähler metric.*

Also suppose h has constant scalar curvature, and obeys

$$ric = J^* ric.$$

Then h is called a conformally Kähler Einstein-Maxwell metric.

Remark 1. *The condition on the Ricci curvature is equivalent to the condition that $Jgrad_g(u)$ is a Killing field for both metrics g and h.*

Remark 2. *If the Hermitian metric is Einstein, then the conditions on scalar and Ricci curvature are automatically met. Conformally Kähler Einstein metrics were studied by A. Derdzinski and G. Maschler in* [6].

Apostolov and Maschler also considered obstructions to the existence of conformally Kähler Einstein-Maxwell metrics by defining two invariants, similar in nature to the traditional Futaki invariants. In particular, they find obstructions to the problems of:

- finding, on a given compact symplectic manifold, and a fixed conformal factor (coming from a hamiltonian function with respect to the symplectic structure), a complex structure compatible with the symplectic structure, and such that the conformal change of this Kähler metric via the fixed conformal factor produces a conformally Kähler Einstein-Maxwell metric, and
- finding, in a fixed Kähler class, Ω, on a given compact complex manifold and with a fixed Killing field, K, to play the role of $Jgrad_g(u)$, and with some fixed positive real number a, a Kähler metric, ω, in Ω, such that $\frac{1}{f^2_{K,\omega,a}} g$ is conformally Kähler Einstein-Maxwell. Here, g is the metric corresponding to ω and $f_{K,\omega,a}$ denotes the hamiltonian function of K with respect to ω, normalized to have total integral a.

A very recent paper of A. Futaki and H. Ono [7] recasts the second Futaki-type invariant above in terms of the volume function on a suitable space of Killing fields. These results mirror the variational characterization of the 4-dimensional Einstein-Maxwell equations mentioned above (the second bullet point). They also construct the same higher-dimensional generalization of the LeBrun $\mathbb{CP}^1 \times \mathbb{CP}^1$ that I do in the following section of this paper, and so, there is some overlap between their results and my own.

The outline of this work is as follows.

In section 2, we construct conformally Kähler Einstein-Maxwell metrics on $\mathbb{CP}^1 \times X$, where X denotes a constant scalar curvature Kähler manifold.

In section 3, we construct conformally Kähler Einstein-Maxwell metrics on $\mathbb{CP}^1$-bundles over $\mathbb{CP}^m - 1$. These metrics, in particular, include a family of Einstein metrics first constructed by Berard-Bergery [2], and later studied by Page and Pope [13].

In section 4, we consider the question of rigidity of conformally Kähler Einstein-Maxwell metrics. In the case that the Kähler metric is Yamabe, and assuming some other minor condition, we show that there can be no family of Einstein-Maxwell metrics inhabiting the same Kähler class, passing through this solution.

2. Einstein-Maxwell metrics on $\mathbb{CP}^1 \times X$

Let (X, g_2) be a Kähler manifold of complex dimension $m-1$, and constant scalar curvature c, and let g_1 denote the metric

$$\frac{1}{\Psi(t)} dt^2 + \Psi(t)\,\sigma^2, \tag{2.1}$$

on $(\alpha, \beta) \times S^1$. Here σ denotes a closed 1-form on S^1. Our goal in this section is to put a conformally Kähler Einstein-Maxwell metric on the product of these manifolds.

Lemma 1. *There exists a complex structure on $(\alpha, \beta) \times S^1 \times X$ with respect to which the product metric on $(\alpha, \beta) \times S^1 \times X$ is Kähler.*

Proof. Since X is already Kähler, we need only show that the metric g_1 on $(\alpha, \beta) \times S^1$ is a Kähler metric. Following [11], we define an almost complex structure J on our product manifold by having J send dt to $-\Psi(t)\sigma$. In particular,

$$\frac{dt}{\Psi(t)} + i\sigma$$

is a $(1,0)$-form and

$$d\left(\frac{dt}{\Psi(t)} + i\sigma\right) = i d\sigma = 0.$$

Since the $(1,0)$-forms span a closed differential ideal, J is integrable. Notice that the metric g_1 is Hermitian with respect to this complex structure, and, since the complex dimension is 1, it must be Kähler. $\square$

The associated $(1,1)$-form to g_1, is given by

$$\omega_1 = \sigma \wedge dt.$$

In particular, if η denotes the vector field generating rotation in the S^1-component, so that

$$dt(\eta) = 0$$

$$\sigma(\eta) \equiv 1,$$

then η is a Killing, Hamiltonian vector field for ω_1, with t as its Hamiltonian function. Notice, too, that $J\nabla t = J(\Psi(t)\partial_t) = \eta$ is Killing, so that t is a holomorphy potential. (Also notice that the S^1-action lifts to $(\alpha, \beta) \times S^1 \times X$.)

Theorem 1. *Let g denote the product metric of g_1 and g_2 on $(\alpha, \beta) \times S^1 \times X$, and let $h = t^{-2}g$, where t denotes the real variable on (α, β). There exists a choice of $\Psi(t)$ so that h is an Einstein-Maxwell metric on $\mathbb{CP}^1 \times X$.*

Since we've already seen that t is a holomorphy potential, we need only arrange for the scalar curvature of g to be constant. The scalar curvature of g is given by $s_g = c - \Psi''(t)$, and, so, the scalar curvature of h is given by

$$s_h = d = 2\frac{2m-1}{m-1} t^{m+1} \Delta_g \left(\frac{1}{t^{m-1}} \right) + t^2 \left(c - \Psi''(t) \right). \tag{2.2}$$

Here, d denotes the intended (constant) scalar curvature of s_h. In order to determine how Δ_g acts on functions of t, notice that if $\eta_1, ..., \eta_{2m-2}$ denotes an orthonormal coframe for X, then $\frac{1}{\sqrt{\Psi(t)}}dt, \sqrt{\Psi(t)}\sigma, \eta_1, ..., \eta_{2m-2}$ forms an orthonormal coframe for g. Then

$$\left(\frac{1}{\sqrt{\Psi}} dt \right) \wedge * \left(\frac{1}{\sqrt{\Psi}} dt \right) = dt \wedge \sigma \wedge \eta_1 ... \wedge \eta_{2m-2}.$$

This implies that $*dt = \Psi\sigma \wedge \eta_1 ... \wedge \eta_{2m-2}$. In particular, if we let $\varphi(t)$ denote any function of t, then

$$\Delta_g(\varphi) = - * d * d\varphi = - * d * (\varphi' dt) = - * d(\Psi\varphi')\sigma \wedge \eta_1 ... \wedge \eta_{2m-2} =$$

$$- * (\Psi\varphi')' dt \wedge \sigma \wedge \eta_1 ... \wedge \eta_{2m-2} = -(\Psi\varphi')'.$$

The case we're interested in is $\varphi = t^{1-m}$, which gives $\Delta_g\left(t^{1-m}\right) = -\left(1-m\right)\left(\Psi t^{-m}\right)' = \left(m-1\right)\left(\Psi' t^{-m} - m\Psi t^{-m-1}\right)$, so plugging into our formula for s_h gives

$$s_h = d = 2\frac{2m-1}{m-1}t^{m+1}\left(m-1\right)\left(\Psi' t^{-m} - m\Psi t^{-m-1}\right) + t^2\left(c - \Psi''\left(t\right)\right) \tag{2.3}$$

$$\Rightarrow d - ct^2 = \left(-2m\left(2m-1\right)\right)\Psi + \left(2\left(2m-1\right)t\right)\Psi' - t^2\Psi''.$$

The right hand side acts as a differential operator on powers of t by sending

$$t^L \mapsto [-2m\left(2m-1\right) + 2\left(2m-1\right)L - L\left(L-1\right)]t^L = [-L^2 + \left(4m-1\right)L + 2m - 4m^2]t^L.$$

In particular, t^{2m-1} and t^{2m} are sent to 0, and $\frac{d}{2m(1-2m)}, \frac{ct^2}{2(m-1)(2m-3)}$ are sent to $d, -ct^2$ respectively. The general solution is then

$$\Psi = \frac{d}{2m\left(1-2m\right)} + \frac{ct^2}{2\left(m-1\right)\left(2m-3\right)} + Bt^{2m-1} + At^{2m} = \tag{2.4}$$

$$M + Nt^2 + Bt^{2m-1} + At^{2m}$$

where A, B are arbitrary constants and $M = \frac{d}{2m(1-2m)}, N = \frac{c}{2(m-1)(2m-3)}$.

We have, thus far, focused on the local character of the problem. Now we consider the global question of compactifying our manifolds.

To compactify this solution, we let α, β be two consecutive zeroes of Ψ. Then we require $\Psi'\left(\alpha\right) = 2 = -\Psi'\left(\beta\right)$ and, of course, we need Ψ to be positive on the interval $\left(\alpha, \beta\right)$. We may therefore suppose

$$\Psi = \left(t-\alpha\right)\left(t-\beta\right)P\left(t\right) \tag{2.5}$$

for some polynomial, P, of degree $2m-2$. Since $\Psi' = \left(t-\beta\right)P\left(t\right) + \left(t-\alpha\right)P\left(t\right) + \left(t-\alpha\right)\left(t-\beta\right)P'\left(t\right)$, the conditions that $\Psi\left(0\right) = M, \Psi'\left(0\right) = 0, \Psi'\left(\alpha\right) = 2 = -\Psi'\left(\beta\right)$ imply the following facts about P:

$$P\left(0\right) = \frac{M}{\alpha\beta}$$

$$P'(0) = \frac{M(\alpha+\beta)}{(\alpha\beta)^2} \tag{2.6}$$

$$P(\alpha) = P(\beta) = \frac{2}{\alpha-\beta}.$$

We get further conditions on P if we notice that Ψ only has terms of order $0, 2, 2m-1, 2m$. Since

$$\Psi^{(n)} = \left(n^2 - n\right) P^{(n-2)} + n(2t-\alpha-\beta) P^{(n-1)} + (t-\alpha)(t-\beta) P^{(n)}, \tag{2.7}$$

these conditions on Ψ imply for $(3 < n < 2m-1)$

$$\Psi^{(n)}(0) = 0 \Rightarrow \tag{2.8}$$

$$\frac{P^{(n)}(0)}{n!} = \frac{\alpha+\beta}{\alpha\beta}\frac{P^{(n-1)}(0)}{(n-1)!} - \frac{1}{\alpha\beta}\frac{P^{(n-2)}(0)}{(n-2)!}.$$

We must also take into account the non-zero terms of Ψ:

$$\Psi''(0) = 2N \Rightarrow \frac{P''(0)}{2} = \left(\frac{N}{\alpha\beta} - \frac{M}{(\alpha\beta)^2} + \frac{M(\alpha+\beta)^2}{(\alpha\beta)^3}\right)$$

$$\Psi^{(2m-1)}(0) = (2m-1)!B \tag{2.9}$$

$$\Psi^{(2m)}(0) = (2m)!A.$$

For clarity, we make the following definitions:

$$u = \frac{1}{\alpha\beta}, v = \frac{(\alpha+\beta)}{\alpha\beta}$$

$$P_0 = \frac{M}{\alpha\beta} = Mu$$

$$P_1 = \frac{M(\alpha+\beta)}{\alpha^2\beta^2} = Muv \tag{2.10}$$

$$P_2 = \frac{N}{\alpha\beta} - M\left(\frac{1}{\alpha^2\beta^2} + \frac{(\alpha+\beta)^2}{\alpha^3\beta^3}\right) = Nu - Mu(u+v).$$

In general,

$$P_j = \frac{P^{(j)}(0)}{j!}.$$

Then our inductive formula for the coefficients of P is now

$$P_j = vP_{j-1} - uP_{j-2}. \tag{2.11}$$

By inductively expanding the right hand side of this equation, we can reduce it to a linear combination of P_1 and P_2. So define q_j and r_j, for $j > 1$ so that $P_j = q_j P_2 + r_j P_1$. The q_j, r_j are polynomials in u and v and each obey similar inductive relations to P_j. Indeed, one has that

$$q_j = \sum_{l=1}^{\lfloor j/2 \rfloor} (-1)^{l+1} \binom{j-l-1}{l-1} u^{l-1} v^{j-2l} \tag{2.12}$$

$$r_j = \sum_{l=1}^{\lceil j/2-1 \rceil} (-1)^l \binom{j-l-2}{l-1} u^l v^{j-2l-1} = -uq_{j-1}.$$

(Since we have $u < \frac{v^2}{4}$each $q_j > \frac{j-1}{2^{j-2}} v^{j-2}$ is positive, so that each $r_j = -uq_{j-1}$ is negative.) Now, we can write

$$\frac{2}{\alpha - \beta} = P(\alpha) = P_0 + P_1\alpha + ... + P_{2m-2}\alpha^{2m-2} =$$

$$Mu + Muv\alpha + Nu\alpha^2 - Mu\left(u+v\right)\alpha^2 + \sum_{j=3}^{2m-2} [q_j P_2 + r_j P_1]\alpha^j =$$

(2.13)
$$M\left[u + uv\alpha - u\left(u+v\right)\alpha^2 + \sum_{j=3}^{2m-2} \left(-u\left(u+v\right)q_j + r_j uv\right)\alpha^j\right] + N\left[u\alpha^2 + \sum_{j=3}^{2m-2} \left(uq_j\right)\alpha^j\right] =$$

$$MS_\alpha + NR_\alpha.$$

Here

$$S_\alpha := \left[u + uv\alpha - u\left(u+v\right)\alpha^2 + \sum_{j=3}^{2m-2} \left(-u\left(u+v\right)q_j + r_j uv\right)\alpha^j\right]$$

and

$$R_\alpha := \left[u\alpha^2 + \sum_{j=3}^{2m-2} (uq_j)\,\alpha^j \right].$$

Similarly,

$$\frac{2}{\alpha - \beta} = MS_\beta + NR_\beta.$$

This system of equations can be solved for M, N:

$$M = \frac{2\,(R_\beta - R_\alpha)}{(\beta - \alpha)\,(S_\beta R_\alpha - S_\alpha R_\beta)} \tag{2.14}$$

$$N = \frac{-2\,(S_\beta - S_\alpha)}{(\beta - \alpha)\,(S_\beta R_\alpha - S_\alpha R_\beta)}.$$

In order to better understand these formulas, it is helpful to define:

$$\Omega_\alpha = \sum_{j=3}^{2m-2} q_j \alpha^j$$

$$\Lambda_\alpha = \sum_{j=3}^{2m-2} r_j \alpha^j,$$

so that we can now write

$$S_\alpha = u + uv\alpha - u\,(u+v)\,\Omega_\alpha + uv\Lambda_\alpha$$

$$R_\alpha = u\Omega_\alpha$$

$$R_\beta - R_\alpha = u\,(\Omega_\beta - \Omega_\alpha)$$

$$S_\beta - S_\alpha = uv[\beta - \alpha + \Lambda_\beta - \Lambda_\alpha] - u\,(u+v)\,[\Omega_\beta - \Omega_\alpha] \tag{2.15}$$

$$S_\beta R_\alpha - S_\alpha R_\beta = -u^2[\Omega_\beta - \Omega_\alpha] - u^2 v[\alpha\Omega_\beta - \beta\Omega_\alpha] - u^2 v[\Omega_\beta\Lambda_\alpha - \Omega_\alpha\Lambda_\beta].$$

Since $\beta > \alpha \Rightarrow \Omega_\beta > \Omega_\alpha$, we see that $R_\beta - R_\alpha > 0$. $S_\beta - S_\alpha$ and $S_\beta R_\alpha - S_\alpha R_\beta$, on the other hand, are negative. This will follow from the inequality:

$$\alpha + \Lambda_\alpha > \beta + \Lambda_\beta \Leftrightarrow 1 < \frac{\Lambda_\alpha - \Lambda_\beta}{\beta - \alpha} = \sum_{j=3}^{2m-2} r_j \frac{\alpha^j - \beta^j}{\beta - \alpha} = u \sum_{j=2}^{2m-1} q_j \frac{\beta^{j+1} - \alpha^{j+1}}{\beta - \alpha}.$$

That $S_\beta - S_\alpha$ is negative is now clear. To see that $S_\beta R_\alpha - S_\alpha R_\beta$ is also negative, we need only note that

$$(\alpha + \Lambda_\alpha)\,\Omega_\beta > (\beta + \Lambda_\beta)\,\Omega_\alpha. \tag{2.16}$$

In particular, we see that N and, therefore, c must be positive.

3. Generalized Berard-Bergery Metrics

Our goal in this section is to construct examples of conformally Kähler Einstein-Maxwell metric on $\mathbb{CP}^1$-bundles over complex projective spaces of arbitrary dimension. These metrics include, in particular, examples of Einstein metrics originally constructed by Berard-Bergery [2] and studied by Page and Pope [13]. Our discussion begins with local considerations, then we compactify our local solutions to finish the construction.

Let us consider a metric g on an S^1-bundle over $(\alpha, \beta) \times \mathbb{CP}^{m-1}$ given by

$$\frac{1}{2t\Phi(t)}dt^2 + 2t\Phi(t)\,\sigma^2 + 2t\pi^* g_{FS} \tag{3.1}$$

where σ denotes a 1-form with $d\sigma = \pi^*\omega_{FS}$, π denotes the projection to the $\mathbb{CP}^{m-1}$ factor and g_{FS}, ω_{FS} denote the Fubini-Study metric and Kähler form respectively. The scalar curvature formula tells us that:

$$-s_g = \frac{-4m(m-1)}{2t} + \frac{2m(m-1)}{t}\Phi + 4m\Phi' + 2t\Phi''. \tag{3.2}$$

An argument entirely analogous to 1 ensures that this metric is Kähler, and that t is a holomorphy potential. In fact, since one can certainly add any constant to a holomorphy potential to get another (corresponding to the same Killing field), we may take our conformal factor to be $(t-\xi)^{-2}$ for some constant ξ. This added degree of freedom will be essential in the construction.

Now, we seek to conformally change our Kähler metric into one of constant scalar curvature M, by multiplying by the function $(t-\xi)^2$. It will also simplify notation if we define a new function Ψ by:

$$\Psi = t^m \Phi. \tag{3.3}$$

Then the equation for scalar curvature becomes:

(3.4)

$$t^{m-1}\left[M + \frac{-4m(m-1)}{2t}(t-\xi)^2\right] = -2(t-\xi)^2\Psi'' + (8m-4)(t-\xi)\Psi' - m(8m-4)\Psi.$$

Define a differential operator D by:

$$D\Psi = -2(t-\xi)^2\Psi'' + (8m-4)(t-\xi)\Psi' - m(8m-4)\Psi. \tag{3.5}$$

Then notice

$$D\left(t-\xi\right)^L = [-2L(L-1) + (8m-4)L - m(8m-4)](t-\xi)^L = (3.6)$$

$$[-2L^2 + (8m-2)L - 8m^2 + 4m](t-\xi)^L,$$

so, in particular, $D(t-\xi)^{2m} = D(t-\xi)^{2m-1} = 0$. An arbitrary linear combination of these two terms will therefore solve the corresponding homogeneous equation for our constant scalar curvature equation. In order to find a specific solution, we will write the left hand side of 3.4 as a polynomial in $(t-\xi)$.

$$t^{m-1}\left[M + \frac{-4m(m-1)}{2t}(t-\xi)^2\right] = \tag{3.7}$$

$$M\sum_{j=0}^{m-1}\binom{m-1}{j}\xi^{m-1-j}(t-\xi)^j + \frac{-4m(m-1)}{2}\sum_{j=0}^{m-2}\binom{m-2}{j}\xi^{m-2-j}(t-\xi)^{j+2} =$$

$$M\xi^{m-1} + M(m-1)\xi^{m-2}(t-\xi) +$$

$$[\sum_{j=0}^{m-3}\left[M\binom{m-1}{j+2}\xi^{m-3-j} + \frac{-4m(m-1)}{2}\binom{m-2}{j}\xi^{m-2-j}\right](t-\xi)^{j+2}] + \frac{k(m-1)}{2}(t-\xi)^m.$$

So a solution to the scalar curvature equation has the general form:

$$\Psi = A(t-\xi)^{2m} + B(t-\xi)^{2m-1} + \sum_{j=0}^{m} c_j (t-\xi)^j \tag{3.8}$$

for some constants A, B. Here,

$$c_j = \frac{M\binom{m-1}{j}\xi^{m-1-j} - 2m(m-1)\binom{m-2}{j-2}\xi^{m-j}}{-2(j-2m)(j-2m+1)} = M\xi^{m-j-1}a_j + \xi^{m-j}b_j \tag{3.9}$$

for $j < m$, $c_m = 1$, and $c_j = 0$ for $j > m$. Here,

$$a_j := \frac{\binom{m-1}{j}}{-2(j-2m)(j-2m+1)},$$

and

$$b_j := \frac{-2m(m-1)\binom{m-2}{j-2}}{-2(j-2m)(j-2m+1)}.$$

We have thus far considered only the local picture of our manifolds. We would now like to impose certain boundary conditions to compactify our solutions. The $SU(m)$-invariance of our metrics guarantees that our manifolds, thus far, are open dense sets in some $\mathbb{P}(\mathcal{O}(p) \oplus \mathcal{O})$ (see [11]), and the proper boundary conditions will allow us to extend our metrics to the total space, smoothly.

To express the required boundary conditions for the zeroes of Ψ as easily as possible, we begin by defining $x = t - \xi$, and let

$$\Psi(x) = (x-\alpha)(x-\beta)Q(x), \tag{3.10}$$

for some polynomial Q of degree $2m-2$, and for some $\alpha < \beta$. Define Q_j to be the coefficients of Q:

$$Q(x) = \sum_{j=0}^{2m-2} Q_j x^j. \tag{3.11}$$

Define

$$u = \frac{1}{\alpha\beta}$$

$$v = \frac{\alpha+\beta}{\alpha\beta},$$

and notice that, since Ψ has no terms of order n for $m < n < 2m-1$,

$$0 = \frac{\Psi^{(n)}(0)}{n!} = \frac{Q^{(n-2)}(0)}{(n-2)!} - (\alpha+\beta)\frac{Q^{(n-1)}(0)}{(n-1)!} + \alpha\beta\frac{Q^{(n)}(0)}{(n)!} \tag{3.12}$$

$$\Rightarrow Q_n = vQ_{n-1} - uQ_{n-2}.$$

Then we have that

$$\Psi(x) = \left(x^2 - (\alpha+\beta)x + \alpha\beta\right)\sum_{j=0}^{2m-2} Q_j x^j = \sum_{j=2}^{2m} Q_{j-2}x^j - (\alpha+\beta)\sum_{j=1}^{2m-1} Q_{j-1}x^j$$

$$+\alpha\beta\sum_{j=0}^{2m-2} Q_j x^j = \alpha\beta Q_0 + [\alpha\beta Q_1 - (\alpha+\beta)Q_0]x$$

$$+\sum_{j=2}^{2m-2}[Q_{j-2} - (\alpha+\beta)Q_{j-1} + \alpha\beta Q_j]x^j$$

$$+[-(\alpha+\beta)Q_{2m-2} + Q_{2m-3}]x^{2m-1} + Q_{2m-2}x^{2m} = \alpha\beta Q_0 + [\alpha\beta Q_1 - (\alpha+\beta)Q_0]x$$

$$+\sum_{j=2}^{m}[Q_{j-2} - (\alpha+\beta)Q_{j-1} + \alpha\beta Q_j]x^j$$

$$+\sum_{j=m+1}^{2m-2}[Q_{j-2} - (\alpha+\beta)Q_{j-1} + \alpha\beta Q_j]x^j + [-(\alpha+\beta)Q_{2m-2} + Q_{2m-3}]x^{2m-1} + Q_{2m-2}x^{2m}.$$

Setting this equal to our general formula for Ψ gives:

$$c_0 = \alpha\beta Q_0$$

$$c_1 = \alpha\beta Q_1 - (\alpha+\beta)Q_0$$

$$c_j = Q_{j-2} - (\alpha+\beta)Q_{j-1} + \alpha\beta Q_j \; (j > 2)$$

$$B = -(\alpha+\beta)Q_{2m-2} + Q_{2m-3}$$

$$A = Q_{2m-2}.$$

For $j < 2m-1$, it follows that

$$Q_j = \sum_{k=0}^{j} q_{j-k}c_k \tag{3.13}$$

where

$$q_j = \sum_{l=1}^{\lfloor \frac{j}{2}+1 \rfloor} (-1)^{l+1} \binom{j-l+1}{l-1} u^l v^{j+2-2l}.$$

For our boundary conditions at α, β to be met, we require (for some positive integer p) that

$$\Psi'(\alpha) = (\alpha - \beta)\, Q\,(\alpha) = p\,(\alpha + \xi) \tag{3.14}$$

$$\Psi'\,(\beta) = (\beta - \alpha)\, Q(\beta) = -p(\beta + \xi)$$

$$\frac{Q(\alpha)}{\alpha + \xi} = \frac{-p}{\beta - \alpha} = \frac{Q(\beta)}{\beta + \xi} \tag{3.15}$$

Define $R_\alpha, S_\alpha, R_\beta, S_\beta$ so that

$$Q(\alpha) = \sum_{j=0}^{2m-2} \sum_{k=0}^{j} M[a_k q_{j-k} \xi^{m-k-1} \alpha^j] + [b_k q_{j-k} \xi^{m-k} \alpha^j] = MR_\alpha + S_\alpha \tag{3.16}$$

and similarly for $Q(\beta)$. Then the boundary conditions read

$$\frac{Q(\alpha)}{\alpha + \xi} = \frac{MR_\alpha + S_\alpha}{\alpha + \xi} = \frac{MR_\beta + S_\beta}{\beta + \xi} = \frac{Q(\beta)}{\beta + \xi} \tag{3.17}$$

$$\Rightarrow M = \frac{S_\beta(\alpha + \xi) - S_\alpha(\beta + \xi)}{R_\alpha(\beta + \xi) - R_\beta(\alpha + \xi)}. \tag{3.18}$$

Plugging this back in for M gives

$$F(\xi) := [S_\beta R_\alpha - S_\alpha R_\beta] + \frac{p}{\beta - \alpha}[R_\alpha(\beta + \xi) - R_\beta(\alpha + \xi)] = 0. \tag{3.19}$$

This is a polynomial equation in ξ. To show there exists a solution for some ξ, suppose $\alpha = 1$, and define ρ_j^α and σ_j^α by

$$R_\alpha = \sum_{j=0}^{m-1} \Big(\sum_{k=m-1-j}^{2m-2} q_{k-m+j+1} \alpha^k \Big) a_{m-1-j} \xi^j = \sum_{j=0}^{m-1} \rho_j^\alpha a_{m-1-j} \xi^j \tag{3.20}$$

$$S_\alpha = \sum_{j=0}^{m} \Big(\sum_{k=m-j}^{2m-2} q_{k-m+j} \alpha^k \Big) b_{m-j} \xi^j = \sum_{j=0}^{m} \sigma_j^\alpha b_{m-j} \xi^j \tag{3.21}$$

$$\sigma_j^\alpha = \rho_{j-1}^\alpha. \tag{3.22}$$

We now claim that, for any $p > 0$, a large enough choice of $\beta > \alpha$ implies $F(0) > 0$, and that the highest coefficient of F is negative, from which it immediately follows that F has a (positive) root ξ.

Indeed,

$$F(0) = [\sigma_0^\beta \rho_0^\alpha - \sigma_0^\alpha \rho_0^\beta] b_m a_{m-1} + \frac{p}{\beta - \alpha} a_{m-1} [\beta \rho_0^\alpha - \alpha \rho_0^\beta] \tag{3.23}$$

where

$$a_{m-1} = \frac{-1}{2m(m+1)}$$
$$b_m = 1$$

and, when $\alpha = 1$

$$\sigma_0^\alpha = \frac{1}{\beta^{m-1}}(1 + 2\beta + 3\beta^2 + \ldots + (m-1)\beta^{m-2})$$

$$\rho_0^\alpha = \frac{1}{\beta^m}(1 + 2\beta + 3\beta^2 + \ldots + m\beta^{m-1})$$

$$\sigma_0^\beta = \beta^{m-1}(\beta^{m-2} + 2\beta^{m-3} + \ldots + (m-1))$$

$$\rho_0^\beta = \beta^{m-2}(\beta^{m-1} + 2\beta^{m-2} + \ldots + m).$$

From which it follows

$$F(0) = \frac{-\beta^{-m-2}}{2m(m+1)(\beta-1)^3}(\beta^3 - p\beta^{m+1} - (m+1)\beta^{m+3} + m\beta^{m+4} + m(mp-1)\beta^{2m}$$
$$+(1 + m + 2p - 2pm^2)\beta^{2m+1} + pm^2\beta^{2m+2} - (1+p)\beta^{3m+1})$$

This is positive for large β. The highest two coefficients of F, thought of as a polynomial in ξ vanish, and the third is negative, given by:

$$\frac{-1}{8(2m-3)(2m-1)\beta^{2m-2}}(\sum_{j=0}^{2m-4}(2m-3-j)\beta^j)(\sum_{j=0}^{2m-2}(j+1)\beta^j).$$

This proves

Theorem 2. *The metric obtained above can be extended to the compactification of the S^1-bundle over $(\alpha, \beta) \times \mathbb{CP}^{m-1}$ obtained by adding copies of $\mathbb{CP}^{m-1}$ to each end. Namely, there exist conformally Kähler Einstein-Maxwell metrics on $\mathbb{P}(\mathcal{O}(p) \oplus \mathcal{O})$ for all positive integers p.*

Finally, we remark that the Einstein metrics constructed by Berard-Bergery [2], and also studied by Page/Pope [13] are specific examples of the metrics we've constructed here. Indeed, Page and Pope defined their metric as

$$\frac{(1-r^2)^{m-1}}{P(r)}dr^2 + c^2\frac{P(r)}{(1-r^2)^{m-1}}\sigma^2 + c(1-r^2)ds^2,$$

for some polynomial P, a constant c and Einstein-Kähler metric ds^2 on the base. (Their r variable is related to our x variable according to $x = \frac{-2\xi}{1+r}$.)

4. Rigidity in the Yamabe Case

Apostolov and Maschler [1] considered the question: which Kähler classes of a compact complex m-manifold (M, J) admit a representative which is conformal to a conformally Kähler Einstein-Maxwell metric? More precisely, they begin by fixing a compact subgroup G in the reduced automorphism group of (M, J), with Lie algebra $\mathfrak{g}$, and a Kähler class Ω. If K is any element of $\mathfrak{g}$, and $\omega \in \Omega$ is any G-invariant Kähler metric, then the Hamiltonian function $f_{K,\omega,a}$ is defined by the pair of equations:

$$\iota_K \omega = -df_{K,\omega,a}$$

$$\int_M f_{K,\omega,a} \mu_\omega = a,$$

where a is a positive real number, and μ_ω is the volume element. The square of this function serves as the conformal factor acting on ω. Apostolov and Maschler then define a Futaki-like invariant $\mathfrak{F}^G_{\Omega,K,a}$ which must vanish if there exists an $\omega \in \Omega$ with $h(X, Y) = \frac{1}{f^2_{K,\omega,a}} \omega(X, JY)$ defines an Einstein-Maxwell metric.

In this section, we address the question: to what extent is this metric, should it exist, locally rigid? That is, can we vary ω in Ω and K in $\mathfrak{g}$ along a curve of Einstein-Maxwell solutions? In the special case of Yamabe metrics, the answer is, in general, no.

Suppose we fix a Kahler class Ω on (M, J), and suppose we also fix a compact subgroup of reduced automorphisms G. Let (K, a, ω) be a triple in $\mathfrak{g} \times \mathbb{R}_{>0} \times \Omega$ such that ω is G-invariant, and $\frac{1}{f^2_{K,\omega,a}} \omega\left(X, JY\right)$ an Einstein-Maxwell metric. Consider a curve in the space $\mathfrak{g} \times \mathbb{R}_{>0} \times \Omega$ of the form $\left(K + tL, a + tb, \omega + ti\partial\overline{\partial}\varphi\right)$. To simplify matters later on, one can choose b to arrange that we are varying our holomorphy potential f in a direction L^2-orthogonal to the space of constant functions. Consider the scalar curvature of the conformally changed metric as a function

$$\widetilde{s} : \mathfrak{g} \times \mathbb{R}_{>0} \times \Omega \to C^\infty(M).$$

Our goal is to calculate the first variation of this function along the curve. To this end, define

$$f_t = f_{K+tL, a+tb, \omega+ti\partial\overline{\partial}\varphi},$$

$$s_t = scal(\omega + ti\partial\overline{\partial}\varphi),$$

$$\Delta_t = \Delta_{\omega+ti\partial\overline{\partial}\varphi},$$

$$\widetilde{s}_t = 2\frac{2m-1}{m-1} f_t^{1+m} \Delta_t \left(f_t^{1-m}\right) + s_t f_t^2.$$

To simplify notation, we write f for f_0, s for s_0, and Δ for Δ_0. Then the general formula for the first variation of conformally changed scalar curvature is

$$\dot{\widetilde{s}}_0 = 2\frac{2m-1}{m-1}\left[(m+1)f^m \dot{f}_0 \Delta\left(f^{1-m}\right) + f^{1+m}\dot{\Delta}_0\left(f^{1-m}\right) - (m-1)f^{1+m}\Delta\left(f^{-m}\dot{f}_0\right)\right] + \dot{s}_0 f^2 + 2sf\dot{f}_0.$$

In the case that our Kahler metric is of constant scalar curvature, these formulas simplify considerably. In particular, we may take the conformal factor $f = f_{K,a,\omega}$ to be identically equal to 1. This corresponds to $K = 0 \in \mathfrak{g}$.

Furthermore, one can show [12] that the Lichnerowicz fourth-order operator

$$\mathfrak{L} = \left(\overline{\partial}\partial^\#\right)^* \overline{\partial}\partial^\# = \frac{1}{4}(\Delta^2 + 2ric \cdot \nabla\nabla)$$

when acting on smooth complex-valued functions. Here, ric denotes the Ricci curvature of g, and $\partial^{\#}$ denotes the $(1,0)$-projection of the gradient of the function. In particular, notice that $\mathfrak{L}$ is self-adjoint with respect to the L^2-inner product on functions.

In this constant scalar curvature case, the first variation of the scalar curvature term simplifies ([12]) to

$$\dot{s}_0 = -4\mathfrak{L}(\varphi).$$

With these simplifications, we have

$$\dot{\tilde{s}}_0 = -2(2m-1)\Delta\left(\dot{f}_0\right) + \dot{s}_0 + 2s\dot{f}_0.$$

Assume our Kähler metric g is a Yamabe metric. In particular, it has constant scalar curvature s_g, so the above simplifications apply.

If we take the L^2-inner product of $\dot{\tilde{s}}_0$ with $\dot{f}_0$, we can arrange our original choice of b to get 0. This implies

$$0 = \int_M -2(2m-1)\dot{f}_0\Delta\left(\dot{f}_0\right) + 2s\dot{f}_0^2\mu_g = \int_M -2(2m-1)|\nabla\dot{f}_0|^2 + 2s\dot{f}_0^2\mu_g,$$

where we've used the fact that $\mathfrak{L}(\varphi) \in \text{image}(\left(\overline{\partial}\partial^{\#}\right)^* \overline{\partial}\partial^{\#}) = (\ker(\left(\overline{\partial}\partial^{\#}\right)^* \overline{\partial}\partial^{\#}))^{\perp}$, and so is L^2-orthogonal to the holomorphy potential $\dot{f}_0 = f_{K,a,i\partial\overline{\partial}\varphi} + f_{L,b,\omega}$.

By the Rayleigh quotient interpretation of eigenvalues, the first (non-zero) eigenvalue of the Laplacian is equal to

$$\lambda = \inf \frac{\int_M |\nabla f|^2\mu_g}{\int_M f^2\mu_g},$$

where the inf is taken over all functions L^2-orthogonal to the constants.

But this implies

$$0 = \int_M -2(2m-1)|\nabla \dot{f}_0|^2 + 2s\dot{f}_0^2 \mu_g \leq \int_M \left[-2(2m-1)\lambda + 2s\right] \dot{f}_0^2 \mu_g,$$

from which it follows that

$$\lambda \leq \frac{s}{2m-1}. \tag{4.1}$$

However, we also have the following

Theorem 3. *If g is a Yamabe metric on a compact manifold of dimension n, s denotes its (constant) scalar curvature, and λ denotes the first non-zero eigenvalue of its Laplacian, then*

$$\lambda \geq \frac{s}{n-1}.$$

Furthermore, if $\lambda = \frac{s}{n-1}$, then

$$\int_M v^3 \mu_g = 0,$$

where v denotes any eigenfunction of the Laplacian with eigenvalue λ. In this case, we also have that

$$\frac{\left(\int_M \mu_g\right)\left(\int_M v^4 \mu_g\right)}{\left(\int_M v^2 \mu_g\right)^2} \leq 3\frac{n+2}{6-n}.$$

Proof. Since g is Yamabe, it minimizes the Einstein-Hilbert functional within its conformal class. So, if $\hat{g} = u^{\frac{4}{n-2}} g = u^{p-2} g$ for some function u and $p = \frac{2n}{n-2}$, then

$$\frac{\int_M \hat{s}\mu_{\hat{g}}}{\left[\int_M \mu_{\hat{g}}\right]^{2/p}} \geq \frac{\int_M s\mu_g}{\left[\int_M \mu_g\right]^{2/p}} = s\left[\int_M \mu_g\right]^{\frac{p-2}{p}}. \tag{4.2}$$

Suppose v is an eigenfunction of the Laplacian, Δ, of g, corresponding to the eigenvalue λ. That is,

$$\Delta(v) = \lambda v.$$

By the Rayleigh quotient interpretation of eigenvalues, we have that

$$\lambda = \frac{\int_M |\nabla v|^2 \mu_g}{\int_M v^2 \mu_g} = \inf \frac{\int_M |\nabla w|^2 \mu_g}{\int_M w^2 \mu_g},$$

where the inf is taken over all functions L^2-orthogonal to the constants. For our conformal factor, take

$$u = 1 + \epsilon v.$$

Then, using that [3]

$$\mu_{\hat{g}} = u^p \mu_g = (1 + \epsilon v)^p \mu_g,$$

and

$$\hat{s} = (p+2) u^{1-p} \Delta(u) + s u^{2-p} = (p+2)(1+\epsilon v)^{1-p} \Delta(1 + \epsilon v) + s(1+\epsilon v)^{2-p},$$

we have

(4.3)
$$\frac{\int_M \hat{s} \mu_{\hat{g}}}{\left[\int_M \mu_{\hat{g}}\right]^{2/p}} = \frac{(p+2) \int (1+\epsilon v)^{1-p} \epsilon \lambda v (1+\epsilon v)^p \mu_g + s \int_M (1+\epsilon v)^{2-p} (1+\epsilon v)^p \mu_g}{[\int_M (1+\epsilon v)^p \mu_g]^{2/p}}.$$

We'd like to expand this expression as series in ϵ (at least to 4th order). To that end, define

$$r(\epsilon) = \sum_{k=0}^{p} \left(\binom{p}{k} \int_M v^k \mu_g \right) \epsilon^k$$

and

$$f(\epsilon) = \left[\int_M (1 + \epsilon v)^p \mu_g \right]^{-2/p} = [r(\epsilon)]^{-2/p}.$$

So that $f(\epsilon)$ is the denominator of the above expression.

Since eigenfunctions corresponding to different eigenvalues of Δ must be orthogonal in the L^2 inner product, we have that $\int_M 1 \cdot v\mu_g = 0$, which implies $r'(0) = 0$. A simple calculation then gives us the following:

$$f'(\epsilon) = \frac{-2}{p} r(0)^{\frac{-2-p}{p}} r'(0),$$

$$f''(\epsilon) = \frac{-2}{p} r(0)^{\frac{-2-p}{p}} r''(0),$$

$$f'''(\epsilon) = \frac{-2}{p} r(0)^{\frac{-2-p}{p}} r'''(0),$$

$$f^{(4)}(\epsilon) = \frac{6(2+p)}{p^2} r(0)^{\frac{-2-2p}{p}} (r''(0))^2 + \frac{-2}{p} r(0)^{\frac{-2-p}{p}} r^{(4)}(0).$$

This allows us to write down the Taylor expansion of f (up to order ϵ^5) as

$$\begin{aligned}(4.4)\quad f(\epsilon) &= \left[\textstyle\int_M \mu_g\right]^{-2/p} \\ &- \epsilon^2 \left[\frac{2}{p}\binom{p}{2} \int_M v^2\mu_g(\int_M \mu_g)^{\frac{-2-p}{p}}\right] - \epsilon^3 \left[\frac{2}{p}\binom{p}{3} \int_M v^3\mu_g(\int_M \mu_g)^{\frac{-2-p}{p}}\right] \\ &+\epsilon^4 \left[\frac{2+p}{p^2}\binom{p}{2}^2 (\int_M v^2\mu_g)^2(\int_M \mu_g)^{\frac{-2-2p}{p}} - \frac{2}{p}\binom{p}{4} \int_M v^4\mu_g(\int_M \mu_g)^{\frac{-2-p}{p}}\right] + O(\epsilon^5).\end{aligned}$$

As for the numerator of 4.3, we have

$$s\left(\int_M \mu_g\right) + \epsilon^2 \left[(p+2)\lambda + s\right]\left(\int_M v^2\mu_g\right)$$

so that 4.3 takes the form

$$s\left(\int_M \mu_g\right)^{\frac{p-2}{p}} + \epsilon^2\left[\left((p+2)\lambda + s\left(1-\tfrac{2}{p}\tbinom{p}{2}\right)\right)\left(\int_M v^2\mu_g\right)\left(\int_M \mu_g\right)^{-2/p}\right] - \epsilon^3\left[s\frac{2}{p}\binom{p}{3}\left(\int_M v^3\mu_g\right)\left(\int_M \mu_g\right)^{-2/p}\right] + \epsilon^4\left[\left(\frac{(2+p)s}{p^2}\binom{p}{2}^2 - \frac{2}{p}\binom{p}{2}((p+2)\lambda+s)\right)\left(\int_M v^2\mu_g\right)^2\left(\int_M \mu_g\right)^{\frac{-2-p}{p}} - \frac{2s}{p}\binom{p}{4}\int_M v^4\mu_g\left(\int_M \mu_g\right)^{-2/p}\right] + O(\epsilon^5). \tag{4.5}$$

Recall that this expression is just the left hand side of 4.2, and notice that the first term of this expression coincides with the right hand side of 4.2. So we must have

$$0 \le \epsilon^2\left[\left((p+2)\lambda + s\left(1-\tfrac{2}{p}\tbinom{p}{2}\right)\right)\left(\int_M v^2\mu_g\right)\left(\int_M \mu_g\right)^{-2/p}\right] - \epsilon^3\left[s\frac{2}{p}\binom{p}{3}\left(\int_M v^3\mu_g\right)\left(\int_M \mu_g\right)^{-2/p}\right] + \epsilon^4\left[\left(\frac{(2+p)s}{p^2}\binom{p}{2}^2 - \frac{2}{p}\binom{p}{2}((p+2)\lambda+s)\right)\left(\int_M v^2\mu_g\right)^2\left(\int_M \mu_g\right)^{\frac{-2-p}{p}} - \frac{2s}{p}\binom{p}{4}\int_M v^4\mu_g\left(\int_M \mu_g\right)^{-2/p}\right] + O(\epsilon^5). \tag{4.6}$$

Define $P(\epsilon)$ to be the right hand side of this inequality. Notice that $P : \mathbb{R} \to \mathbb{R}$ is a real analytic function with no 0th or 1st order terms.

Define Q by $P(\epsilon) = \epsilon^2 Q(\epsilon)$, and define a_n so that $Q(\epsilon) = \sum_{n=0}^{\infty} a_n\epsilon^n$. Our inequality tells us that $P(\epsilon)$ must always be non-negative. Since

$$\lim_{\epsilon\to 0} g(\epsilon) = a_0,$$

a_0 negative would imply $g(\epsilon) < 0$ for small, but non-zero, ϵ. This would make P negative for small ϵ as well, contradicting the inequality. So

$$a_0 = \left[\left((p+2)\lambda + s\left(1 - \frac{2}{p}\binom{p}{2}\right)\right) \left(\int_M v^2 \mu_g\right) \left(\int_M \mu_g\right)^{-2/p} \right] \geq 0,$$

which establishes the first claim of the theorem.

If $a_0 = 0$, then $P(\epsilon) = \epsilon^3 \sum_{n=0}^{\infty} a_{n+1}\epsilon^n$. If $a_1 > 0$, then $\sum_{n=0}^{\infty} a_{n+1}\epsilon^n \to a_1$ as $\epsilon \to 0$, so $\sum_{n=0}^{\infty} a_{n+1}\epsilon^n$ is positive for small, non-zero ϵ. But for $\epsilon < 0$, this would make $P(\epsilon) < 0$ as well. So a_1 can't be positive. Similar reasoning shows that it can't be negative, so

$$a_1 = \left[s\frac{2}{p}\binom{p}{3} \left(\int_M v^3 \mu_g\right) \left(\int_M \mu_g\right)^{-2/p} \right] = 0 \Rightarrow \left(\int_M v^3 \mu_g\right) = 0,$$

establishing the second claim in the theorem.

We have thus shown that, should the second order term vanish, the third order term must also vanish. In this case, the positivity of $P(\epsilon)$ rests on the fourth order term. Here, reasoning similar to the second order term shows that the coefficient of ϵ^4 in $P(\epsilon)$ must be non-negative. That is,

$$0 \leq \left[\left(\tfrac{(2+p)s}{p^2}\tbinom{p}{2}^2 - \tfrac{2}{p}\tbinom{p}{2}\left((p+2)\lambda + s\right)\right) \left(\textstyle\int_M v^2\mu_g\right)^2 \left(\textstyle\int_M \mu_g\right)^{\frac{-2-p}{p}} \right.$$
$$\left. - \frac{2s}{p}\binom{p}{4} \int_M v^4 \mu_g \left(\int_M \mu_g\right)^{-2/p} \right]$$

Under the assumption that $\lambda = \frac{s}{n-1} = \frac{p-2}{p+2}s$, this simplifies to

$$s\left[\frac{[(p-2)(p-1)(p+3)]}{4} - (p-1)(p-2)\right] \frac{\left[\int_M v^2 \mu_g\right]}{\int_M \mu_g} \geq \frac{(p-1)(p-2)(p-3)}{12}\left[\int_M v^4 \mu_g\right].$$

The final claim in the theorem follows from this. $\square$

Remark 3. *As an example, consider the standard product of two spheres with different radii,* $S^2_1 \times S^2_r$*, for* $r < 1$*. The scalar curavature is* $s = 2 + \frac{2}{r^2}$ *and, since the eigenvalues of a product are sums of eigenvalues from either component,* $\lambda = 2$*. In this dimension, we are relating* λ *to* $\frac{s}{3}$*, and we find that, if the metric is Yamabe, then* $r \leq \frac{1}{\sqrt{2}}$*. In the equality case, one can use that the eigenfunctions for the first non-zero eigenvalue of the laplacian on* S^2 *are just projections to lines through the origin (in particular, one can take projection to the* z*-axis) to verify that the other conditions in the theorem hold, so we can not rule out the possibility of a* $\lambda = \frac{s}{3}$ *Yamabe metric in this way.*

Definition 3. *A Yamabe metric of dimension* n *will be called strict, if* $\lambda > \frac{s}{n-1}$*.*

Finally, we combine the two results in this section.

Theorem 4. *Let* (M, J) *be a compact complex manifold of (complex) dimension* m*. Let* G *be a compact subgroup of its reduced automorphism group, and let* $\mathfrak{g}$ *denote its Lie algebra. Suppose* g *denotes a Kähler metric on* (M, J)*, that is* G*-invariant, and a strict Yamabe metric. Let* ω *denote its Kähler form, and let* Ω *denote the Kähler class containing* ω*. Then, considering* g *as an Einstein-Maxwell metric (with conformal factor identically equal to* 1*), it is rigid with respect to perturbations in* $\mathfrak{g} \times \mathbb{R}_{>0} \times \Omega$*.*

Proof. Taking $f \equiv 1$ amounts to taking $K = 0 \in \mathfrak{g}$ (and $a > 0$ can be taken to be any positive real number). In the case when our Yamabe metric obeys the strict version of this inequality, we have a contradiction to 4.1, and so there can be no curve of Einstein-Maxwell solutions passing through our metric, in $\mathfrak{g} \times \mathbb{R}_{>0} \times \Omega$. $\square$

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