The Degree of Irrationality of Very General Hypersurfaces in Some Homogeneous Spaces

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Abstract of the Dissertation

The Degree of Irrationality of Very General Hypersurfaces in Some Homogeneous Spaces

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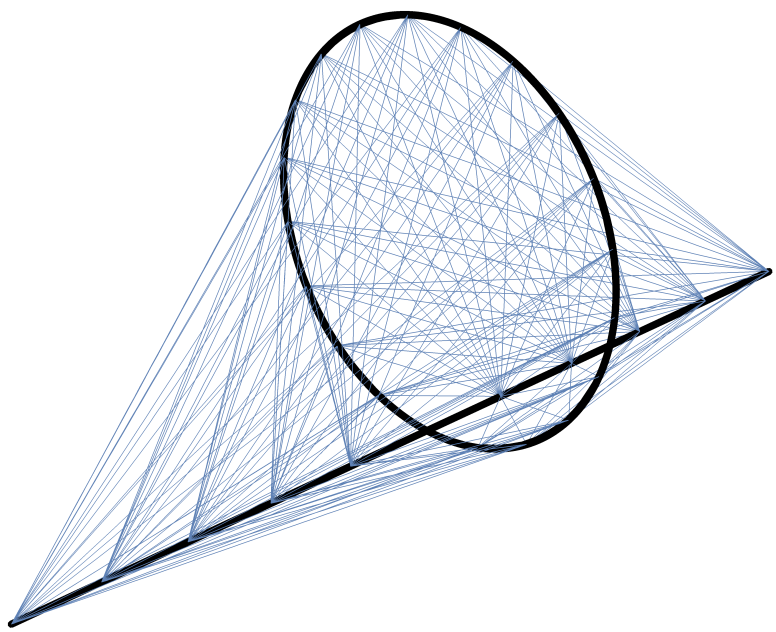
Abstract. The degree of irrationality of an n-dimensional algebraic variety X is the minimal degree of a rational map from X to $\mathbf{P}^n$. The degree of irrationality is a birational invariant with the purpose of measuring how far X is from being rational. For example the degree of irrationality of X is 1 if and only if X is rational. While the invariant has a very classical appearance, it has not attracted very much attention until very recently in [BDPE$^+$15] where it was shown that the degree of irrationality of a very general degree d hypersurface in $\mathbf{P}^{n+1}$ is $d-1$, if d is sufficiently large. The method of proof involves relating the geometry of a low degree map to projective space to the geometry of lines in projective space. In this dissertation we show that these methods can be extended to compute the degree of irrationality of hypersurfaces in other rational homogeneous spaces: quadrics, Grassmannians, and products of projective spaces. In particular, we relate the geometry of low degree maps from hypersurfaces in these rational homogeneous spaces to the geometry of lines inside these rational homegeneous spaces. These computations represent some of the first computations of the degree of irrationality for higher dimensional varieties.



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Chapter 1

Introduction

In this dissertation we compute the degree of irrationality of very general hypersurfaces in quadrics, Grassmannians, and products of projective spaces. This extends work of [BCDP14, BDPE$^+$15] who computed the degree of irrationality for very general hypersurfaces in projective space. This dissertation represents one piece of the candidate's research during graduate school. The candidate has also written two additional papers [Sta16], [BS16].

Let X be a smooth n-dimensional complex projective variety. Recall that X is *rational* if it is birational to $\mathbf{P}^n$. Historically there has been a great deal of interest in studying the behavior of rationality and in finding invariants that can distinguish between nonrational Fano varieties and $\mathbf{P}^n$ (e.g. [AM72, CG72, IM71, Kol95, Tot16, HPT16]). Here we are studying a slightly orthogonal question.

Question. *Given an arbitrary projective variety X, how can one measure how far X is from being rational?*

In the case X is 1-dimensional, i.e. X is a compact Riemann surface, then the *gonality of* X, denoted $\mathrm{gon}(X)$, provides the most natural answer to this question. Recall that the gonality of a Riemann surface X is the minimal degree of a branched covering $\pi\colon X \to \mathbf{P}^1$. We have

$$X \text{ is rational } \iff \mathrm{gon}(X) = 1.$$

When $\dim(X) \geq 2$ then one possible answer to this question is the *degree of irrationality of* X, denoted $\mathrm{irr}(X)$, which is the minimum degree of a dominant rational map

$$\varphi\colon X \dashrightarrow \mathbf{P}^n.$$

In [BDPE$^+$15] those authors prove that if $X = X_d \subset \mathbf{P}^{n+1}$ is a very general hypersurface of degree $d \geq 2n+1$ then $\mathrm{irr}(X) = d - 1$.

The main theorems from this dissertation are the computation of the degree of irrationality for very general high degree hypersurfaces in quadrics, Grassmannians, and products of projective spaces. These are some of the first computations of the degree of irrationality in higher dimensions.

Let $\mathbf{Q} \subset \mathbf{P}^{n+2}$ (with $n \geq 1$) be a smooth quadric in projective space.

Theorem 4.1. *Let*

$$X = X_d \subset \mathbf{Q}$$

be a very general hypersurface in $\mathbf{Q}$ *with* $X \in |\mathcal{O}_{\mathbf{Q}}(d)|$. *If* $d \geq 2n$, *then* $\mathrm{irr}(X) = d$.

Let $\mathbf{G} = \mathrm{Gr}(k,m) \subset \mathbf{P}^N$ be the Plücker embedding of the Grassmannian of k-planes in an m-dimensional vector space (with $k \neq 1, m-1$).

Theorem 4.2. *Let*

$$X = X_d \subset \mathbf{G}$$

be a very general hypersurface in $\mathbf{G}$ *with* $X \in |\mathcal{O}_{\mathbf{G}}(d)|$. *If* $d \geq 3m-5$ *then* $\mathrm{irr}(X) = d$.

Let $\mathbf{P} = \mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_k}$ be a product of $k \geq 2$ projective spaces.

Theorem 4.3. *Let*

$$X = X_{(a_1,\ldots,a_k)} \subset \mathbf{P}$$

be a very general divisor with $X \in |\mathcal{O}_{\mathbf{P}}(a_1,\ldots,a_k)|$, *and let* p *be the minimum of* $\{a_i - m_i - 1\}$. *If* $p \geq \max\{m_i\}$ *then* $\mathrm{irr}(X) = \min\{a_i\}$.

One of the main themes in this dissertation is the role that the positivity of K_X plays in bounding the degree of irrationality of X. As a guiding example, consider the case when X is a curve. Then we have

$$X \text{ is rational} \iff K_X \text{ is negative},$$

and a similar relationship holds for curves of higher gonality. To make such a statement precise we need to introduce a measurement of the positivity of K_X. Recall that a line bundle L on X is *p-very ample* if for every length $p+1$ subscheme, the map

$$H^0(X,L) \to H^0(X, L|_\xi)$$

is surjective. In particular, a 0-very ample line bundle is base point free, and a 1-very ample line bundle is very ample. Then we have the following relationship between the gonality of X and the positivity of K_X

$$\mathrm{gon}(X) \geq p+2 \iff K_X \text{ is } p\text{-very ample}.$$

Another important part of the dissertation is an extension of Ein and Voisin's work [Ein88, Voi96] on the nonexistence of rational curves in very general hypersurfaces of high degree to the nonexistence of curves with bounded gonality. For example, if $\mathbf{G} = \mathrm{Gr}(k,m) \subset \mathbf{P}^N$ is the Plücker embedding of the Grassmannian as before, we prove

Proposition 3.11. *Let* $X \in |\mathcal{O}_{\mathbf{G}}(d)|$ *be a very general divisor and set* $n = \dim(X)$. *Then there are no curves* $C \subset X$ *with*

$$\mathrm{gon}(C) \leq d - m - n + 2.$$

We have similar results for hypersurfaces in quadrics and hypersurfaces in products of projective space. This type of extension of the results of Ein and Voisin was first carried out in [BDPE+15] in the case of hypersurfaces in projective space.

Finally we prove some auxiliary results on the degree of irrationality of K3 surfaces, abelian surfaces, and complete intersections in projective space. In the case of polarized K3 surfaces (X, L) with $L^2 = 2d$ we are interested in the question

Question. *For (X, L) a very general polarized K3 surface with $L^2 = 2d$, does $\operatorname{irr}(X)$ go to infinity with d?*

We have the following partial answer to this question, which says $\operatorname{irr}(X)$ cannot grow too quickly.

Theorem 5.1. *There is a uniform constant C such that if (X, L) is a very general K3 surface, then*

$$\operatorname{irr}(X) \leq C\sqrt{L^2}.$$

We have a similar result for polarized abelian surfaces.

In the case of complete intersections we show

Theorem 5.4. *Let $Z = Z_{d,e} \subset \mathbf{P}^{n+2}$ be a very general complete intersection of type (d, e). Then*

$$\operatorname{irr}(Z) \geq \frac{e\lfloor \sqrt[n+1]{d} \rfloor}{n+1}.$$

This should be compared to the case of a complete intersection

$$C = C_{(a_1,\dots,a_k)} \subset \mathbf{P}^{k+1}$$

of hypersurfaces of degrees $a_1 \leq \cdots \leq a_k$. In this case it is known that $\operatorname{gon}(C) \geq (a_1 - 1)a_2 \cdots a_k$.

Throughout we work over the field $\mathbb{C}$. By a *variety* we mean a reduced and irreducible scheme of finite type. We say that a property holds *very generally* if it holds outside of the complement of a countable union of proper subvarieties.

In Chapter 2 we give the background to the dissertation, including the main definitions and some important results from the literature that we will need. In Chapter 3 we give our extensions of the theorems of Ein and Voisin to show the nonexistence of curves with bounded gonality in hypersurfaces of some homogeneous space of high degree. In Chapter 4 we prove the main results of the dissertation, which is the computation of the degree of irrationality of very general hypersurfaces in certain homegeneous spaces. In Chapter 5 we give our partial results about the degree of irrationality of K3 surfaces, abelian surfaces, and complete intersections.

Chapter 2

Background

In this chapter we recall the background and main definitions for this dissertation. This chapter is meant to be expository and we do not claim any originality. In §2.1 we introduce two measures of irrationality, the covering gonality and the degree of irrationality. In §2.2 we introduce a property called birational p-very ampleness, or $(\mathrm{BVA})_p$, which is meant to measure the birational positivity of a line bundle. In §2.3 we define a trace map which associates to any dominant, generically finite rational map between projective varieties

$$f\colon X \dashrightarrow Y$$

a linear map on global sections of canonical bundles

$$\mathrm{tr}_f\colon H^0(X,\omega_X)\to H^0(Y,\omega_Y).$$

In §2.4 we recall the Cayley-Bacharach condition on points in projective space. In §2.5 we give a simple intersection theoretic condition for a line in projective space to be contained in some subvariety of projective space. Finally, in §2.6 we recall some basic facts about homogeneous spaces including global generation of their tangent bundles, the Borel-Weil theorem, and we give some examples of Fano varieties of lines in homogeneous spaces.

2.1. Measures of Irrationality

Let X be a smooth projective variety over $\mathbb{C}$. In this section we consider two birational invariants of X, the *covering gonality* and the *degree of irrationality* which measure the distance X is from being rational. These definitions were systematically studied in [BDPE+15], although they have appeared earlier in the literature.

Definition 2.1. We say X is *swept out by curves of gonality* c if there is a smooth and connected family of proper curves

$$\pi\colon \mathcal{C}\to T$$

such that each fiber $C_t = \pi^{-1}(t)$ has gonality c, and there is a dominating map

$$\psi\colon \mathcal{C}\to X$$

such that for each $t\in T$, the restriction

$$\psi|_{C_t}\colon C_t\to X$$

is birational onto its image.

Definition 2.2 (see [BDPE⁺15, Def. 1.6])**.** The *covering gonality* of X, denoted $\mathrm{cov.gon}(X)$, is the minimum c such that X is swept out by curves with gonality c.

Remark 2.3. We could also define $\mathrm{cov.gon}(X)$ as

$$\mathrm{cov.gon}(X) = \min \left\{ c > 0 \;\middle|\; \begin{array}{c} \text{For general } x \in X, \text{ there} \\ \text{is } C \subset X \text{ an irred. curve} \\ \text{with } \mathrm{gon}(C) = c \text{ and } x \in C \end{array} \right\}.$$

Here the gonality of a curve C is by definition the gonality of its normalization. A standard argument using Hilbert schemes and the Baire category theorem implies these definitions are equivalent, but to make this argument one must use the fact that $\mathbb{C}$ is uncountable.

Example 2.4. If X is a curve then $\mathrm{cov.gon}(X) = \mathrm{gon}(X)$. Moreover, we have

$$\mathrm{gon}(X) = 1 \iff X \cong \mathbf{P}^1.$$

If $\dim(X) \geq 2$ then we have

$$\mathrm{cov.gon}(X) = 1 \iff X \text{ is uniruled.}$$

Remark 2.5. If $g \colon C \to D$ is a finite map of curves then $\mathrm{gon}(C) \geq \mathrm{gon}(D)$. Thus if

$$f \colon X \dashrightarrow Y$$

is a dominant and generically finite rational map then

$$\mathrm{cov.gon}(X) \geq \mathrm{cov.gon}(Y).$$

Example 2.6. If X is a K3 surface then by Bogomolov and Mumford's theorem [MM83, p. 351], X is swept out by genus 1 curves. Thus we have

$$\mathrm{cov.gon}(X) = 2.$$

Example 2.7. If A is a abelian surface then A is not uniruled so $\mathrm{cov.gon}(A) \geq 2$. Moreover, if A is general then A is isogenous to the Jacobian of a genus 2 curve C, $\mathrm{Jac}(C)$. Thus

$$C \subset \mathrm{Jac}(C)$$

is a hyperelliptic curve, and by translating C we have

$$\mathrm{cov.gon}(\mathrm{Jac}(C)) \leq 2.$$

Thus by Remark 2.5 we have $\mathrm{cov.gon}(A) = 2$.

Pirola showed in [Pir89] that a generic abelian variety A with $\dim(A) \geq 3$ contains no hyperelliptic curves. Thus,

$$\mathrm{cov.gon}(A) \geq 3.$$

Definition 2.8. Set $n = \dim(X)$. The *degree of irrationality of* X, denoted $\mathrm{irr}(X)$ is the minimal degree of a dominant rational map

$$\varphi \colon X \dashrightarrow \mathbf{P}^n.$$

Remark 2.9. If $\mathbb{C}(X)$ is the function field of X then $\mathbb{C}(X)$ has transcendence degree n thus it can be written as a finite extension of a purely transcendental extension of $\mathbb{C}$. The degree of irrationality is the minimal degree of such a finite extension,

$$\mathbb{C}(x_1, ..., x_n) \subset \mathbb{C}(X).$$

Example 2.10. We have

$$\mathrm{irr}(X) = 1 \iff X \simeq_{\mathrm{bir}} \mathbf{P}^n \iff X \text{ is rational.}$$

Remark 2.11. There is a straightforward inequality

$$\mathrm{irr}(X) \geq \mathrm{cov.gon}(X).$$

Remark 2.12. Historically there has been a great deal of interest in understanding the nature of rationality including finding invariants that can prove a Fano variety is not rational (see e.g. [AM72, CG72, Kol95, Tot16, HPT16]). In this dissertation our interests are somewhat orthogonal. We will focus on showing that certain general type varieties have a high degree of irrationality.

2.2. Birational positivity of line bundles and K_X

Let X be a projective variety over $\mathbb{C}$ with a line bundle L. Following [BDPE+15], we introduce a property $(\mathrm{BVA})_p$ which measures the birational positivity of L.

Definition 2.13. A line bundle L on X is *p-very ample* if for all subschemes $\xi \subset X$ of length $p+1$, the restriction map

$$H^0(X, L) \to H^0(X, L|_\xi)$$

is surjective.

Remark 2.14. The notion of p-very ampleness is supposed to measure the positivity of L. For example, we have

$$L \text{ is 0-very ample} \iff L \text{ is globally generated}$$

and

$$L \text{ is 1-very ample} \iff L \text{ is very ample.}$$

Example 2.15. The line bundle $\mathcal{O}_{\mathbf{P}^N}(p)$ on $\mathbb{P}^N$ is p-very ample. Moreover, if $Z \subset X$ is any subvariety and L is a p-very ample line bundle, then $L|_Z$ is also a p-very ample line bundle.

Remark 2.16. If X is a curve and $L = K_X$ then we have

$$K_X \text{ is } p\text{-very ample} \iff \mathrm{gon}(X) \geq p+2.$$

This illustrates the basic connection between the positivity of K_X and the degree of irrationality of X. In higher dimensions however, it becomes important to work with a measure of the birational positivity of K_X instead.

Definition 2.17 ([BDPE+15, Def. 1.1])**.** A line bundle L on X *satisfies the property* $(\mathrm{BVA})_p$ if there exists a Zariski closed proper subset $Z = Z(L) \subsetneqq X$ depending on L such that

$$H^0(X, L) \to H^0(X, L|_\xi)$$

is surjective for every finite subscheme $\xi \subset X$ of length $p+1$ whose support is disjoint from Z.

Remark 2.18. In Definition 2.17, $(\mathrm{BVA})_p$ stands for *birationally p-very ample.* In particular, if L is p-very ample then L satisfies $(\mathrm{BVA})_p$ (in Definition 2.17, it suffices to take $Z(L) = \emptyset$). Moreover we have

$$\begin{array}{ccc} L \text{ satisfies } (\mathrm{BVA})_0 & \iff & L \text{ is effective,} \\ L \text{ satisfies } (\mathrm{BVA})_1 & \iff & \begin{array}{c} |L| \colon X \dashrightarrow \mathbf{P}^N \text{ is birational} \\ \text{onto its image,} \end{array} \end{array}$$

and

$$L \text{ satisfies } (\mathrm{BVA})_1 \implies L^{\otimes p} \text{ satisfies } (\mathrm{BVA})_p.$$

The property $(\mathrm{BVA})_p$ is birational in the following sense, if

$$\pi \colon X' \to X$$

is a birational map and L is a line bundle on X that satisfies $(\mathrm{BVA})_p$, then $L' = \pi^*(L)$ also satisfies $(\mathrm{BVA})_p$. In fact, if E is the exceptional locus of π then in the definition of $(\mathrm{BVA})_p$ it suffices to take

$$Z(L') = E \cup \pi^{-1}(Z(L)).$$

Remark 2.19. The following implications are clear,

$$\begin{array}{ccc} K_X \text{ satisfies } (\mathrm{BVA})_0 & \implies & X \text{ is not uniruled, and} \\ K_X \text{ satisfies } (\mathrm{BVA})_1 & \implies & X \text{ is of general type.} \end{array}$$

Example 2.20. If X is a regular surface and K_X satisfies BVA_0 (i.e. K_X is effective) then Mumford proved in [Mum68] that $\mathrm{CH}_0(X)$ is *infinite dimensional.* Moreover, if K_X does not satisfy $(\mathrm{BVA})_0$ then Bloch's conjecture predicts $\mathrm{CH}_0(X) \cong \mathbb{Z}$. Some other Chow-theoretic consequences of K_X satisfying $(\mathrm{BVA})_p$ are explored in [BDPE+15, §2].

Leveraging Remark 2.16, one can prove the following theorem.

Theorem 2.21 ([BDPE+15, Thm. 1.9])**.** *Let X be a smooth projective variety, and suppose there is an integer $p \geq 0$ such that its canonical bundle K_X satisfies* $(\mathrm{BVA})_p$. *Then*

$$\mathrm{cov.gon}(X) \geq p + 2.$$

Thus by Remark 2.11, if K_X satisfies $(\mathrm{BVA})_p$ then $\mathrm{irr}(X) \geq p + 2$.

2.3. Mumford's trace map

Let X and Y be smooth n-dimensional projective varieties, and let

$$f\colon X \dashrightarrow Y$$

be a dominant rational map. In [Mum68], Mumford defines a linear trace map

$$\mathrm{tr}_f\colon H^0(X, \omega_X) \to H^0(Y, \omega_Y)$$

which at a generic point of Y is defined by *summing over the fibers of f*. Mumford defines the trace in a more general setting than we need. Moreover, from a modern perspective the trace can be defined using duality theory. However, in this section we opt to give an elementary proof that this trace exist.

Definition 2.22 (Case 0: f regular, proper, and étale)**.** Assume

$$f\colon X \to Y$$

is a degree d proper, étale map between smooth (possibly nonproper) varieties. we can define the *trace of f*,

$$\mathrm{tr}_f\colon H^0(X, \omega_X) \to H^0(Y, \omega_Y)$$

as follows. If $B \subset Y$ is a simply connected open set, then the preimage is a disjoint union

$$f^{-1}(B) = B_1 \sqcup \cdots \sqcup B_d.$$

of opens sets B_i with $B_i \cong B$. For any $\eta \in H^0(X, \omega_X)$ define

$$\mathrm{tr}_f(\eta)|_B = \eta|_{B_1} + \cdots + \eta|_{B_d}.$$

Remark 2.23. For any dominant rational map

$$f\colon X \dashrightarrow Y$$

of projective n-dimensional varieties we can always find a nonempty open set $U \subset Y$ such that if $V = f^{-1}(U)$ then

$$f|_V\colon V \to U$$

is a regular, proper, and étale morphism. This gives rise to a diagram

$$\begin{tikzcd} H^0(X,\omega_X) \arrow[r, dashed, "?"] \arrow[d, "\text{restr.}"] & H^0(Y,\omega_Y). \arrow[d, "\text{restr.}"] \\ H^0(V,\omega_V) \arrow[r, "\mathrm{tr}_{f|_V}"] & H^0(U,\omega_U) \end{tikzcd}$$

So our aim is to show that we can fill in the dashed arrow. Equivalently we want to show that for any $\eta \in H^0(X,\omega_X)$, the a priori meromorphic n-form $\mathrm{tr}_{f|_V}(\eta|_V)$ has no poles in Y.

Proposition/Definition 2.24. *In the above setting, let $\eta \in H^0(X,\omega_X)$ be a holomorphic n-form. Then the holomorphic n-form*

$$\mathrm{tr}_{f|_V}(\eta|_V) \in H^0(U,\omega_U)$$

can be extended to a holomorphic n-form

$$\mathrm{tr}_f(\eta) \in H^0(Y,\omega_Y).$$

Proof. By applying Hartog's theorem it suffices to show that we can extend $\mathrm{tr}_{f|_V}(\eta|_V)$ to an open set $W \supset U$ such that $\mathrm{codim}(Y \setminus W) \geq 2$. Furthermore, by resolving f we can reduce to the case

$$f\colon X \to Y$$

is a regular, generically finite map. Here we need to use that any proper birational map of smooth projective varieties

$$\pi\colon X' \to X$$

induces an isomorphism

$$\pi^*\colon H^0(X,\omega_X) \to H^0(X',\omega_{X'}).$$

Having reduced to the case f is regular and proper, there exists an open subset $W \subset Y$ with $\mathrm{codim}(Y \setminus W) \geq 2$ such that for every point $p \in W$ and any point $q \in f^{-1}(p)$, there are local analytic coordinates $y_1,\dots,y_n$ around p and $x_1,\dots,x_n$ around q satisfying

$$f^*(y_1) = x_1^e$$
$$\text{and } f^*(y_i) = x_i \text{ if } i \neq 1.$$

That is, away from codimension 2 in Y, the map f is analytically locally a cyclic cover.

So it suffices to show that $\mathrm{tr}_{f|_V}(\eta|_V)$ has no poles in the case f is a cyclic cover branched over a smooth divisor. For points $p \in W$ and $q \in X$ as above and using the above coordinates, we can write

$$\eta = g(x_1,\dots,x_n)dx_1 \wedge \cdots \wedge dx_n$$

locally around q. If we expand g as a power series in x_1 we have

$$\eta = \sum_{j=0}^{\infty} g_j(x_2, \ldots, x_n) x_1^j dx_1 \wedge \cdots \wedge dx_n.$$

Then we can write the meromorphic n-form $\mathrm{tr}_{f|_V}(\eta|_V)$ near p as:

$$\begin{aligned}
\mathrm{tr}_{f|_V}(\eta|_V) &= \mathrm{tr}_{f|_V}\left(\sum_{j=0}^{\infty} g_j(x_2, \ldots, x_n) x_1^j \cdot dx_1 \wedge \cdots \wedge dx_n\right) \\
&= \sum_{i=1}^{e}\sum_{j=0}^{\infty} g_j(y_2, \ldots, y_n)(\omega^i \sqrt[e]{y_1})^j \cdot d(\omega^i \sqrt[e]{y_1}) \wedge \cdots \wedge dy_n \\
&\qquad\qquad \text{(where } \omega \text{ is a primitive } e\text{-th root of unity)} \\
&= \sum_{j=0}^{\infty}\left(\frac{1}{e}\sum_{i=1}^{e}\omega^{i(j+1)}\right) g_j(y_2, \ldots, y_n) y_1^{(j+1-e)/e} dy_1 \wedge \ldots \wedge dy_n.
\end{aligned}$$

Now, as ω is a primitive e-th root of unity we have:

$$\frac{1}{e}\sum_{i=1}^{e}\omega^{i(j+1)} = \begin{cases} 1 & \text{if } e|j+1 \\ 0 & \text{otherwise.} \end{cases}$$

So letting $k = (j+1)/e$ we get

$$\mathrm{tr}_{f|_V}(\eta|_V) = \sum_{k=1}^{\infty} y_1^{k-1} g_{ke-1}(y_2, \ldots, y_n) dy_1 \wedge \cdots \wedge dy_n.$$

In particular we see that the meromorphic n-form $\mathrm{tr}_{f|_V}(\eta|_V)$ does not have a pole at p, which completes the proof. □

For any correspondence Γ between X and Y, we can define a trace through Γ as follows. Assume Γ is a cycle on $X \times Y$ of the form

$$\Gamma = \sum n_i \Gamma_i,$$

with each Γ_i irreducible of dimension n. Let $\Gamma_i' \to \Gamma_i$ be a resolution of singularities of Γ_i and define

$$p_i \colon \Gamma_i' \to X \text{ and } q_i \colon \Gamma_i' \to Y$$

to be the natural maps.

Definition 2.25. In the above setting, define *the trace through* Γ to be the linear map

$$\mathrm{tr}_\Gamma = \sum_i n_i \mathrm{tr}_{q_i} \circ p_i^* \colon H^0(X, \omega_X) \to H^0(Y, \omega_Y).$$

2.4. The Cayley-Bacharach condition

In this section we introduce the Cayley-Bacharach condition and state some consequences for low degree maps to $\mathbf{P}^n$.

Definition 2.26. Let $\mathcal{S} = \{q_1, ..., q_d\} \subset \mathbf{P}^N$ be a set of d points in projective space. We say that $\mathcal{S}$ satisfies the *Cayley-Bacharach condition with respect to* $|mH|$ if every divisor $D \subset |mH|$ which contains at least $d-1$ points in $\mathcal{S}$, contains all of $\mathcal{S}$.

Lemma 2.27 ([BCDP14, Lemma 2.4])**.** *Let $N \geq 2$ and let*

$$\mathcal{S} = \{q_1, \ldots, q_d\} \subset \mathbf{P}^N$$

be a subset satisfying the Cayley-Bacharach condition with respect to $|mH|$ for some $m \geq 1$. Then $d \geq m+2$. Moreover, if $d \leq 2m+1$ then $\mathcal{S}$ is contained in a line.

Let $X \subset \mathbf{P}^N$ be a smooth n-dimensional projective variety, and assume that

$$K_X = mH|_X + E$$

for E some effective divisor.

Lemma 2.28 ([BCDP14])**.** *Let*

$$\Gamma \subset X \times \mathbf{P}^n$$

be an irreducible n-dimensional subvariety such that both projections

$$p_1 \colon \Gamma \to X \ \textit{and}\ p_2 \colon \Gamma \to \mathbf{P}^n,$$

are dominant maps. (For example, the closure of a graph of a rational map $\varphi \colon X \dashrightarrow \mathbf{P}^n$.) If $p \in \mathbf{P}^n$ is a general point then the subset

$$p_1(p_2^{-1}(p)) \subset \mathbf{P}^N$$

satisfies the Cayley-Bacharach condition with respect to $|mH|$.

The idea is the following, if we take the trace through Γ we get a map

$$\mathrm{tr}_\Gamma \colon H^0(X, \omega_X) \to H^0(\mathbf{P}^n, \omega_{\mathbf{P}^n}).$$

For a general point $p \in \mathbf{P}^n$, the projection p_2 is étale over p, and the map p_1 is étale around each point $q \in p_2^{-1}(p)$. Moreover we can choose p so that $p_1(p_2^{-1}(p)) \cap E$ is empty. Then coordinates $x_1, \ldots, x_n$ around p give rise to coordinates around each point $x \in p_1(p_2^{-1}(p))$ and there is a commuting diagram:

$$\begin{tikzcd} H^0(X,\omega_X) \arrow[r, "\mathrm{tr}_\Gamma"] \arrow[d, "\text{restr.}"] & H^0(\mathbf{P}^n,\omega_{\mathbf{P}^n}) = 0 \arrow[d, "\text{restr.}"] \\ \bigoplus_{x\in p_1(p_2^{-1}(p))} \mathbb{C}dx_1\wedge\cdots\wedge dx_n \arrow[r, "\sum"] & \mathbb{C}dx_1\wedge\cdots\wedge dx_n. \end{tikzcd}$$

The commutativity implies that every global n-form must restrict to $\ker(\sum)$. Suppose $D \in |mH|$ is a divisor that meets $p_1(p_2^{-1}(p))$ at all except possibly 1 point $x' \in p_1(p_2^{-1}(p))$. Then $D+E$ corresponds to a holomorphic n-form $\eta_{D+E} \in H^0(X,\omega_X)$ that vanishes at all points except possibly $x' \in p_1(p_2^{-1}(p))$. Finally we have

$$0 = \sum_{x\in p_1(p_2^{-1}(p))} \eta_{D+E}|_x = \eta_{D+E}|_{x'}.$$

So η_{D+E} vanishes at x' as well, and thus x' is contained in D.

2.5. A Bezout-type lemma

In this section we prove an elementary lemma which we will need at various stages of the dissertation. Let $X \subset \mathbf{P}^N$ be a subvariety of projective space. Assume that X is cut out by homogeneous polynomials $\{F_i\}$ satisfying $\deg(F_i) \le e$ for all i.

Lemma 2.29. *If $C \subset \mathbf{P}^N$ is a degree d integral curve such that*

$$\#(X\cap C) > de.$$

then $C \subset X$.

Proof. Set $Y_i = (F_i = 0) \subset \mathbf{P}^N$. For all i we have:

$$[Y_i]\cdot[C] \le de < \#(X\cap C) \le \#(Y_i \cap C).$$

So the intersection $Y_i \cap C$ is not proper. Thus $C \subset Y_i$ for all i, i.e. $C \subset X$ ☐

Corollary 2.30. *If $X \subset \mathbf{P}^N$ is cut out by quadrics and $\ell \subset \mathbf{P}^N$ is a line meeting X at 3 or more points, then $\ell \subset X$.*

2.6. Background on rational homogeneous spaces

In this section we recall some basic results about rational homogeneous spaces. In §2.6.1 we prove that the tangent bundle of a rational homogeneous space is globally generated. In §2.6.2 we recall the Borel-Weil theorem and some of its consequences. In §2.6.3 we give some examples of the Fano variety of lines of some homogeneous spaces.

Definition 2.31. Recall that a projective variety $\mathbf{Y}$ is a *rational homogeneous space* if there is a semisimple linear algebraic group G and a transitive algebraic group action

$$G \times \mathbf{Y} \to \mathbf{Y}.$$

Remark 2.32. A rational homogeneous space $\mathbf{Y}$ is necessarily smooth and rational. Moreover, we have

$$\mathbf{Y} \cong G/P$$

for $P \subset G$ some parabolic subgroup which is the stabilizer of a point $y \in \mathbf{Y}$.

We recall some of the terminology from Lie theory. If G is any semisimple linear algebraic group and we choose a maximal torus

$$H = (\mathbb{C}^*)^m \subset G$$

then the Lie algebra $\mathfrak{g}$ of G can be decomposed into the eigenvectors for the Adjoint representation of H on $\mathfrak{g}$. That is we can write

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \right).$$

Here, $\mathfrak{h}$ is the Lie algebra of H and the $\mathfrak{g}_\alpha$ are one dimensional weight spaces where H acts with the weight $\alpha \in R$. Thus the weights in R are some finite subset $R \subset \mathfrak{h}^*$ called the *roots of* $\mathfrak{g}$. Fixing a Borel subgroup $B \subset G$ that contains H allows us to decompose the roots into positive and negative roots

$$R = R^+ \sqcup R^-.$$

The positive roots, correspond to the eigenspaces of $\mathfrak{g}$ which are contained in the Lie algebra $\mathfrak{b}$ of B. Moreover, the *simple roots* in R^+ are the positive roots which are not the sum of 2 other positive roots. The simple roots form a natural basis for $\mathfrak{h}^*$.

2.6.1. **Global generation of $T_{\mathbf{Y}}$.** Let $\mathbf{Y} = G/P$ be a rational homogeneous space.

Proposition 2.33. *The tangent bundle $T_{\mathbf{Y}}$ is globally generated.*

Proof. As $\mathbf{Y} = G/P$ we have the map

$$\pi : G \to \mathbf{Y} \text{ by sending } g \mapsto [g \cdot P].$$

The tangent space at a point $[g \cdot P]$ canonically fits in the exact sequence

$$0 \to \mathrm{Ad}(g) \cdot \mathfrak{p} \to \mathfrak{g} \xrightarrow{d\pi_g} T_{\mathbf{Y}}|_{[g \cdot P]} \to 0$$

where Ad is the adjoint representation of G and $\mathfrak{p}$ is the Lie algebra of P. Note that it can happen that $[g \cdot P] = [h \cdot P]$ even when the transformations $\mathrm{Ad}(g)$ and $\mathrm{Ad}(h)$ are not the same. But the subspaces

$$\mathrm{Ad}(g) \cdot \mathfrak{p}, \mathrm{Ad}(h) \cdot \mathfrak{p} \subset \mathfrak{g}$$

will still be the same.

For any vector $X \in \mathfrak{g}$, we can define the global vector field $s_X \in H^0(Y, T_{\mathbf{Y}})$ fiber by fiber by setting

$$s_X|_{[g \cdot P]} = d\pi_g(\mathrm{Ad}(g) \cdot X).$$

By the surjectivity of $d\pi_g$ these vector fields generate $T_{\mathbf{Y}}$. $\square$

2.6.2. **The Borel-Weil theorem.** Now we recall the theorem of Borel and Weil on the cohomology of line bundles on homogeneous spaces (for a reference see [FH91, p. 393] or [Ser95]).

Let $\mathfrak{h}$ be the Lie algebra of our maximal torus $H \subset G$ and choose

$$\lambda \in \mathfrak{h}^*$$

an integral weight. Then the exponential of λ gives rise to a 1-dimensional representation of H, denote $\mathbb{C}_\lambda$, by

$$\chi_\lambda \colon H \to \mathbb{C}^*.$$

To construct a line bundle L_λ on G/B it suffices to construct a B-equivariant line bundle on G. That is we need an action of B on $G \times \mathbb{C}_\lambda$ which is linear on the fibers over G and such that the diagram

$$\begin{tikzcd} G \times \mathbb{C}_\lambda \times B \arrow[r] \arrow[d] & G \times \mathbb{C}_\lambda \arrow[d] \\ G \times B \arrow[r] & G \end{tikzcd}$$

commutes. To give the action we note that there is a quotient map

$$\psi : B \to H$$

(quotienting by the unipotent radical) such that $H \hookrightarrow B$ is a splitting of ψ. Thus we can extend the 1-dimensional representation $\mathbb{C}_\lambda$ to a representation of B by considering

$$\chi_\lambda \circ \psi \colon B \to \mathbb{C}^*.$$

The right action

$$G \times \mathbb{C}_\lambda \times B \longrightarrow G \times \mathbb{C}_\lambda$$

is given by

$$(b, g, v) \mapsto (g \cdot b, \chi_\lambda(\psi(b)) \cdot v).$$

Define L_λ to be the associated line bundle on G/B.

Remark 2.34. The global sections of L_λ naturally form a finite dimensional representation of G. Moreover, from the construction it is clear that for any two weights $\lambda, \mu \in \mathfrak{h}^*$ we have

$$L_{\lambda+\mu} = L_\lambda \otimes L_\mu.$$

Theorem 2.35 (Borel-Weil Theorem)**.** *Every line bundle on G/B is L_λ for some weight λ. If λ is a dominant weight then*

$$H^0(\mathbf{Y}, L_{-\lambda}) \cong V_\lambda^*$$

where V_λ is the irreducible representation with highest weight λ. If λ is not dominant then

$$H^0(G/B, L_{-\lambda}) = 0.$$

Remark 2.36. Let $\mathfrak{h}_\mathbb{Z}^*$ denote the set of integral weights of H and let $\mathfrak{h}_\mathbb{R}^* = \mathfrak{h}_\mathbb{Z}^* \otimes \mathbb{R}$. Together Remark 2.34 and Theorem 2.35 imply that there is an isomorphism

$$\mathrm{Pic}(\mathbf{Y}) \cong \mathfrak{h}_\mathbb{Z}^*.$$

This isomorphism sends the effective cone in $\mathrm{Pic}(G/B)\otimes\mathbb{R}$ to the cone spanned by weights $-\lambda \in \mathfrak{h}_\mathbb{R}^*$ such that λ is dominant, i.e. if

$$\mathcal{W} \subset \mathfrak{h}_\mathbb{R}^*$$

is the Weyl chamber spanned by dominant weights, then the effective cone corresponds to $-\mathcal{W}$. Moreover the effective cone is the same as the nef cone.

Remark 2.37 (Line bundles on arbitrary homogeneous spaces)**.** In practice we will be working with homogeneous spaces that are given by G/P where P is just a parabolic subgroup with $B \subset P \subset G$. Fortunately, it is still possible to apply the Borel-Weil Theorem in this setting.

To start there is a proper map of homogeneous spaces

$$\pi \colon G/B \to G/P$$

which is a P/B fiber bundle. Moreover the higher direct images vanish, i.e.

$$R^i\pi_*\mathcal{O}_{(G/B)} = 0 \text{ for } i > 0.$$

Thus for any line bundle L on G/P, we have $\pi^*L = L_\lambda$ for some weight λ and

$$H^0(G/P, L) = H^0(\mathbf{Y}, L_\lambda).$$

The latter is computable using the Borel-Weil theorem.

Remark 2.38. We can be slightly more precise. There is a correspondence

$$\{P \text{ parabolic} | B \subset P\} \overset{1:1}{\longleftrightarrow} \left\{ F \,\middle|\, \begin{array}{c} F \text{ is a face of the} \\ \text{Weyl chamber } \mathcal{W} \end{array} \right\}.$$

The map going from left to right can be defined by taking

$$\pi^*(\mathrm{Pic}(G/P) \otimes \mathbb{R}) \subset \mathrm{Pic}(G/B) \otimes \mathbb{R} \cong \mathfrak{h}^*_{\mathbb{R}}$$

and taking the face

$$F_P = \pi^*(\mathrm{Pic}(G/P) \otimes \mathbb{R}) \cap \mathcal{W}.$$

This implies that the Picard rank of G/P equals the dimension of the face F_P. Moreover, if L is ample on G/P then $\pi^*(L) = L_{-\lambda}$ is nef on (G/P) and λ corresponds to a dominant weight in the interior of F_P. In particular, every nef line bundle on G/B is the pullback of an ample line bundle from some G/P.

Proposition 2.39. *Let L be an ample line bundle on a homogeneous space G/P. Then L is very ample and the embedding given by the linear series of L*

$$\varphi_L \colon G/P \to \mathbf{P}^N$$

is projectively normal.

Proof. If L is ample, then $\pi^*L = L_{-\lambda}$ for $\lambda \in \mathcal{W}$ a dominant weight. Moreover, by the Borel-Weil theorem, the following diagram of G-representations commutes:

$$\begin{array}{ccc} \mathrm{Sym}^n(H^0(G/P, L)) & \longrightarrow & H^0(G/P, L^{\otimes n}). \\ \| & & \| \\ \mathrm{Sym}^n(V_\lambda^*) & \longrightarrow & V_{n\lambda}^* \end{array} \tag{2.40}$$

The horizontal arrows are nonzero, and therefore they must be surjective (else the image of $\mathrm{Sym}^n(V_\lambda^*)$ would give a nontrivial subrepresentation of $V_{n\lambda}^*$, which contradicts the irreducibility of $V_{n\lambda}^*$).

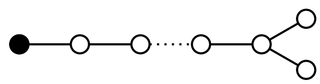
Assuming for contradiciton that every section of L vanishes at some point $x \in G/P$, then by diagram (2.40) so does every section of $L^{\otimes n}$ (for $n > 0$). This is absurd as for n large $L^{\otimes n}$ is very ample. Thus L is globally generated. A similar argument implies that L is very ample. Finally, the surjectivity of the arrows in diagram (2.40) implies that the image of the embedding $\varphi_L(G/P)$ is projectively normal. □

Finally, Lichtenstein proved

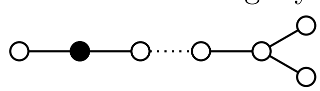
Theorem 2.41 ([Lic82]). *The image $\varphi_L(G/P)$ is cut out by quadrics.*

2.6.3. **Fano varieties of lines in homogeneous spaces.** Here we look at the Fano variety of lines of various homogeneous spaces under their natural embeddings.

Example 2.42. Let $\mathbf{Q} \subset \mathbf{P}^{2k-1}$ be a smooth even dimensional quadric. Then $\mathbf{Q}$ is a homogeneous space for the group $SO_{2k}(\mathbb{C})$. If $k \geq 3$ then $\mathbf{Q}$ has Picard rank 1 and corresponds to the ray in the Weyl chamber of $SO_{2k}(\mathbb{C})$ which is perpendicular to the unmarked simple roots in the following Dynkin diagram, D_k:

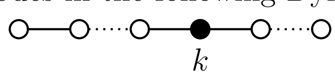


Moreover, $SO_{2k}(\mathbb{C})$ acts transitively on the set of set of lines in $\mathbf{Q}$. So we see that the Fano variety $\mathrm{Fano}(\mathbf{Q})$ is also a homogeneous space for $SO_{2k}(\mathbb{C})$. When $k = 3$, then $\mathbf{Q} \cong \mathrm{Gr}(2,4)$ and the Fano variety of lines is computed in the next example. If $k \geq 4$ then the Fano variety of lines in $\mathbf{Q}$ also has Picard rank 1. It corresponds to the ray in the Weyl chamber which is perpendicular to the unmarked simple roots in the following Dynkin diagram.



There is a similar picture for odd dimensional quadrics.

Example 2.43. Let $\mathbf{G} = \mathrm{Gr}(k,m) \subset \mathbf{P}(\wedge^k \mathbb{C}^m)$ be the Plücker embedding of the Grassmannian of k-planes in an m-dimensional vector space (with $k \neq 1, m-1$). Then $\mathbf{G}$ is a Picard rank 1 homogeneous space for $SL_m(\mathbb{C})$. It corresponds to the ray of the Weyl chamber of $SL_m(\mathbb{C})$ which is perpendicular to the unmarked simple nodes in the following Dynkin diagram, A_{m-1}:



Given a partial flag

$$[U \subset V \subset \mathbb{C}^m] \in \mathrm{Fl}(k-1, k+1, m)$$

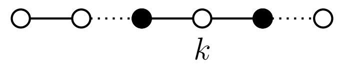
such that $\dim(U) = k-1$ and $\dim(V) = k+1$ we can construct a line $\ell = \ell_{U,V} \subset \mathbf{G}$ in the Plücker embedding by

$$\ell = \{[\Lambda] \in \mathbf{G} | U \subset \Lambda \subset V\}.$$

In fact all the lines in $\mathbf{G}$ correspond to such partial flags and we have

$$\mathrm{Fano}(\mathbf{G}) = \mathrm{Fl}(k-1, k+1, m).$$

Thus, $\mathrm{Fano}(\mathbf{G})$ is an $SL_m(\mathbb{C})$ homogeneous space of Picard rank 2. It corresponds to the face of the Weyl chamber which is perpendicular to the unmarked simple roots in the following Dynkin diagram.



Example 2.44. Let $\mathbf{P} = \mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_k}$ be a product of projective spaces. Then $\mathbf{P}$ is a homogeneous space for

$$G = SL_{m_1}(\mathbb{C}) \times \cdots \times SL_{m_k}(\mathbb{C}).$$

Let $\mathbf{P} \hookrightarrow \mathbf{P}^N$ be the Segre embedding. Any embedding $f \colon \mathbf{P}^1 \hookrightarrow \mathbf{P}$ comes from a set of k maps $f_i \colon \mathbf{P}^1 \to \mathbf{P}^{m_i}$. Moreover, the degree of f is computed by

$$\deg(f) = \sum \deg(f_i).$$

Therefore, the image of f is a line in the Segre embedding if and only if one of the factors f_i embeds $\mathbf{P}^1$ as a line in $\mathbf{P}^{m_i}$ and the rest of the factors have degree 0 (i.e. are constant maps). In particular, we can write the Fano variety of lines in $\mathbf{P}$ as a disjoint union

$$\mathrm{Fano}(\mathbf{P}) = \begin{array}{l} (\mathrm{Gr}(2, m_1+1) \times \mathbf{P}^{m_2} \times \cdots \times \mathbf{P}^{m_k}) \sqcup \cdots \\ \quad \cdots \sqcup (\mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_{k-1}} \times \mathrm{Gr}(2, m_k+1)) \,. \end{array}$$

Chapter 3

The gonality of curves in hypersurfaces in some homogeneous spaces

The goal of this chapter is extend the arguments of Ein and Voisin [Ein88, Voi96] on the nonexistence of rational curves in very general hypersurfaces of projective space to the situations which we are considering in this dissertation. In [BDPE$^+$15] these arguments were extended to give bounds on the gonality of curves appearing in very general hypersurfaces in projective spaces. For example, those authors proved:

Proposition (see [BDPE$^+$15, Prop. 3.7])**.** *Let $X \subset \mathbf{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n$. Then any irreducible curve $C \subset X$ satisfies*

$$\mathrm{gon}(C) \geq d - 2n + 1.$$

More generally, if X contains an irreducible subvariety of dimension $e > 0$ which is swept out by curves of gonality c, then

$$c \geq d - 2n + e.$$

In this chapter we prove similar propositions for very general hypersurfaces in quadrics, Grassmannians, and products of projective spaces. These propositions will be important in the computation of the degree of irrationality of these varieties, which we carry out in Chapter 4.

In §3.1 we outline the general argument for proving the nonexistence of families of curves with gonality c in a hypersurface in a homogeneous space. In §3.2 we prove a proposition similar to the above proposition in the case of hypersurfaces in quadrics. In §3.3 and §3.4 we prove similar propositions for hypersurfaces in Grassmannians and products of projective spaces respectively.

3.1. Outline of the general argument

Let $\mathbf{Y}$ be a rational homogeneous space. In this section we outline the general argument which gives bounds for the gonality of curves that can appear in very general hypersurfaces in $\mathbf{Y}$.

We start by defining a certain kernel bundle on $\mathbf{Y}$.

Definition 3.1. Let $L \in \mathrm{Pic}(\mathbf{Y})$ be a globally generated vector bundle, then the map

$$\mathrm{eval}\colon H^0(\mathbf{Y}, L) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{Y}} \to L$$

is surjective. Thus the kernel

$$\mathcal{M}_L := \ker(\mathrm{eval})$$

is a vector bundle which we call the *kernel bundle associated to L.*

From now on, assume that L and $\mathcal{O}(1) \in \mathrm{Pic}(\mathbf{Y})$ are very ample line bundles. Let

$$U \subset H^0(\mathbf{Y}, L)$$

be the open subset which parameterizes smooth divisors. Thus there is a universal divisor $\mathcal{X}$ over U

$$\begin{array}{ccc} \mathcal{X} \subset U \times \mathbf{Y} & \xrightarrow{p_2} & \mathbf{Y}. \\ \big\downarrow{\scriptstyle p_1} & & \\ U & & \end{array}$$

Here the fibers of $p_1|_{\mathcal{X}}$ are all smooth divisors $X \in |L|$. In particular, $\mathcal{X}$ is smooth. We have the following proposition, the argument here is due to Ein and Voisin.

Proposition 3.2. *Let $X \in |L|$ be a smooth divisor. Assume that*

$$H^1(X, \mathcal{M}_L(1)|_X) = 0$$

and that $\mathcal{M}_L(1)|_X$ is globally generated. Then $T_{\mathcal{X}}(1)|_X$ is globally generated.

Proof. To start we note that there is a map between the normal sequence of X in $\mathcal{X}$ and the normal sequence of X in $\mathbf{Y}$.

$$\begin{tikzcd} 0 \arrow[r] & T_X(1) \arrow[r] \arrow[d, equal] & T_{\mathcal{X}}(1)|_X \arrow[r] \arrow[d, "dp_2"] & H^0(\mathbf{Y}, L) \otimes_{\mathbb{C}} \mathcal{O}_X(1) \arrow[r] \arrow[d, "\mathrm{eval}"] & 0 \\ 0 \arrow[r] & T_X(1) \arrow[r] & T_{\mathbf{Y}}(1)|_X \arrow[r] & L(1)|_X \arrow[r] & 0 \end{tikzcd}$$

Then by the snake lemma, we have

$$\mathcal{M}_L(1)|_X = \ker(dp_2) = \ker(\mathrm{eval}).$$

To show $T_{\mathcal{X}}(1)|_X$ is globally generated consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{M}_L(1)|_X) \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r] \arrow[d, "\mathrm{eval}"] & H^0(T_{\mathcal{X}}(1)|_X) \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r] \arrow[d, "\mathrm{eval}"] & H^0(T_{\mathbf{Y}}(1)|_X) \otimes_{\mathbb{C}} \mathcal{O}_X \arrow[r] \arrow[d, "\mathrm{eval}"] & 0 \\ 0 \arrow[r] & \mathcal{M}_L(1)|_X \arrow[r] \arrow[d] & T_{\mathcal{X}}(1)|_X \arrow[r, "dp_2"] & T_{\mathbf{Y}}(1)|_X \arrow[r] \arrow[d] & 0. \\ & 0 & & 0 & \end{tikzcd}$$

The left and right evaluation maps are surjective - the left by the assumption that $\mathcal{M}_L(1)|_X$ is globally generated and the right because $\mathbf{Y}$ is a rational homogeneous space so $T_{\mathbf{Y}}$ is globally generated. Then by the snake lemma, the center evaluation map is surjective and thus $T_{\mathcal{X}}(1)$ is globally generated. $\square$

Now we assume that every general fiber X of $p_1\colon \mathcal{X}\to U$ contains an e-dimensional subvariety which is swept out by curves with gonality c. In this setting we have the following proposition:

Proposition 3.3. *Let X be a very general fiber with $n = \dim(X)$ and assume that $\omega_X(-n)$ is p-very ample. If $T_{\mathcal{X}}(1)|_X$ is globally generated then*

$$c \geq p + e + 2.$$

Proof. We have assumed that a very general X contains an e-dimensional subvariety $S' \subset X$ which is swept out by curves with gonality c. By a standard argument using Hilbert schemes and the Baire category theorem we can construct a family of such S'. I.e. there is a diagram

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[rr, bend left, "f"] \arrow[dr] & \mathcal{S}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "p_1"] \\ & V \arrow[r] & U \end{tikzcd}$$

where $V\to U$ is étale, $\mathcal{S}' \subset \mathcal{X} \times_U V$ is a family of e-dimensional subvarieties of $\mathcal{X} \times_U V$ which are swept out by curves with gonality c, and $\mathcal{S}\to\mathcal{S}'$ is a resolution of the total space of $\mathcal{S}'$. In particular, after shrinking V we can assume that $\mathcal{S}\to V$ is a smooth map of relative dimension e and that fiber by fiber $\mathcal{S}\to\mathcal{S}'$ is a resolution (in particular, the fibers of $\mathcal{S}$ and $\mathcal{S}'$ over V are birational).

Let $N = \dim(U)$. As the fibers of $\mathcal{S}$ and $\mathcal{S}'$ are birational, if $S_0 \subset \mathcal{S}$ is any fiber, then the exterior power of the differential:

$$\wedge^{e+N} df\colon f^*\left(\wedge^{e+N}\Omega_{\mathcal{X}}\right)|_{S_0}\to\omega_{\mathcal{S}}|_{S_0} \tag{3.4}$$

is not identically 0. The normal bundle of $S_0 \subset \mathcal{S}$ is trivial and hence

$$\omega_{\mathcal{S}}|_{S_0} \cong \omega_{S_0}.$$

Moreover, the exterior product gives rise to the isomorphism:

$$f^*\left(\wedge^{e+N}\Omega_{\mathcal{X}}\right)|_{S_0} \cong f^*\left(\wedge^{n-e}T_{\mathcal{X}}(1)\right) \otimes f^*(\omega_{\mathcal{X}}(e-n))|_{S_0}. \tag{3.5}$$

The triviality of the normal bundle of $X \subset \mathcal{X}$ implies $\omega_{\mathcal{X}}|_X = \omega_X$. So if we twist (3.4) by $(\omega_X(e-n))^{-1}$, use the isomorphism (3.5), and the assumption that $T_{\mathcal{X}}(1)|_X$ is globally generated, we see that the line bundle

$$\mathcal{O}_{S_0}(E) := \omega_{S_0} \otimes f^*(\omega_X(e-n))^{-1}|_{S_0}$$

is effective. Therefore we have

$$\omega_{S_0} = f^*(\omega_X(e-n))|_{S_0} \otimes \mathcal{O}_{S_0}(E)$$

is the tensor product of a line bundle which satisfies $(\mathrm{BVA})_{p+e}$ and an effective line bundle. Thus ω_{S_0} satisfies $(\mathrm{BVA})_{p+e}$ and we are done by Theorem 2.21. $\square$

Recall that by Proposition 2.39, the embedding of $\mathbf{Y}$ by the complete linear series $\mathcal{O}(1)$ is projectively normal. Thus we have the following proposition due to Ein that we will use in the setting L is a multiple of $\mathcal{O}(1)$.

Proposition 3.6 ([Ein88, Prop. 1.2(c)])**.** *If $L = \mathcal{O}(d)$ then $\mathcal{M}_L(1)|_X$ is globally generated.*

3.2. The gonality of curves on hypersurfaces in a quadric

Let $\mathbf{Q} \subset \mathbf{P}^{n+2}$ be a smooth $(n+1)$-dimensional quadric in projective space with $n \geq 2$.

Proposition 3.7. *Let $X \in |\mathcal{O}_{\mathbf{Q}}(d)|$ be very general. If $S \subset X$ is an e-dimensional subvariety swept out by curves with gonality c, then*

$$c \geq e + d - 2n + 1.$$

Proof. We have $\omega_X \cong \mathcal{O}_X(d-n-1)$. Therefore $\omega_X(-n)$ is p-very ample for $p = d - 2n - 1$. By the following lemma, we have that $T_{\mathcal{X}}(1)|_X$ is globally generated. Thus by Proposition 3.3 we are done. $\square$

Lemma 3.8. *The bundle $T_{\mathcal{X}}(1)|_X$ is globally generated.*

Proof. By Proposition 3.2 and Proposition 3.6 it suffices to show

$$H^1(X, \mathcal{M}_L(1)|_X) = 0$$

where $L = \mathcal{O}_{\mathbf{Q}}(d)$. Twisting the exact sequence that defines $\mathcal{M}_L$ by $\mathcal{O}_{\mathbf{Q}}(1)$ and restricting to X gives the following long exact sequence on cohomology.

$$\begin{aligned} \cdots \to H^0(\mathbf{Q}, \mathcal{O}_{\mathbf{Q}}(d)) \otimes H^0(X, \mathcal{O}_X(1)) &\xrightarrow{\text{eval}} H^0(X, \mathcal{O}_X(d+1)) \\ &\to H^1(X, \mathcal{M}_L(1)|_X) \to H^0(\mathbf{Q}, \mathcal{O}_{\mathbf{Q}}(d)) \otimes H^1(X, \mathcal{O}_X(1)) \to \cdots \end{aligned}$$

Now the evaluation map is surjective because $X \subset \mathbf{P}^{n+2}$ is projectively normal, and $H^1(X, \mathcal{O}_X(1)) = 0$ is an easy computation. Therefore, $H^1(X, \mathcal{M}_L(1)|_X)$ vanishes. $\square$

It is interesting to ask to what extent the bounds in Proposition 3.7 are optimal.

Example 3.9. When $(d, n) = (3, 2)$ then X is a K3 surface then by [MM83] X is covered by genus 1 curves and always contains rational curves. Therefore the inequality in Proposition 3.7 (which becomes $c \geq e$) is sharp.

Example 3.10. Consider the case $n = e = 3$, i.e. we are looking at the covering gonality of a hypersurface in a quadric fourfold. The inequality in Proposition 3.7 reads

$$\operatorname{cov.gon}(X) \geq d - 2.$$

Now for each point $x \in \mathbf{Q}$, the family of 2-planes through x

$$F_x := \left\{ [P] \in \operatorname{Gr}(3,6) \middle| P \subset \mathbf{Q} \subset \mathbf{P}^5 \text{ with } x \in P \right\} \subset \operatorname{Gr}(3,6)$$

is a disjoint union of 2 $\mathbf{P}^1$s. Fixing $x \in X$ then for every plane in P_x the intersection $C = P \cap X$ is a degree d plane curve containing x. If $d > 1$ then by varying the plane $[P] \in F_x$ in its 1-dimensional family it is always possible to find a singular intersection $C = P \cap X$, which necessarily has $\operatorname{gon}(C) \leq d-2$. Therefore

$$\operatorname{cov.gon}(X) = d - 2.$$

3.3. The gonality of curves on hypersurfaces in a Grassmannian

Let $\mathbf{G} = \operatorname{Gr}(k, m) \subset \mathbf{P}^N$ be the Plücker embedding of the Grassmannian of k-planes in an m-dimensional vector space.

Proposition 3.11. *Let $X \in |\mathcal{O}_{\mathbf{G}}(d)|$ be very general and set*

$$n = \dim(X).$$

If $S \subset X$ is an e-dimensional subvariety swept out by curves with gonality c, then

$$c \geq e + d - m - n + 2.$$

Proof. We have $\omega_X \cong \mathcal{O}_X(d - m)$. Thus $\omega_X(-n)$ is p-very ample for $p = d - m - n$. As in Proposition 3.7, we need to show that $T_{\mathcal{X}}(1)|_X$ is globally generated, which we prove in the following lemma. Then we are done by Proposition 3.3. □

Lemma 3.12. *The bundle $T_{\mathcal{X}}(1)|_X$ is globally generated.*

Proof. The proof is identical to the proof of Lemma 3.8 □

3.4. The gonality of curves on hypersurfaces in a product of two or more projective spaces

For simplicity we start out by looking at a product of two projective spaces: $\mathbf{P} = \mathbf{P}^a \times \mathbf{P}^b$.

Proposition 3.13. *Assume at least one of $a, b > 1$. Let $X \in |\mathcal{O}_{\mathbf{P}}(k, \ell)|$ be a very general divisor with $k, \ell > 1$ and set $n = \dim(X)$. If $S \subset X$ is an e-dimensional subvariety swept out by curves with gonality c, then*

$$c \geq e + \min\{k - a, l - b\} - n + 1.$$

Proof. By adjunction we have $\omega_X \cong \mathcal{O}_X(k-a-1, \ell-b-1)$. Thus $\omega_X(-n)$ is p-very ample for $p = \min\{k-a, \ell-b\}-1$. Thus, as in Proposition 3.7 and 3.11 we need to show that $T_{\mathcal{X}}(1,1)$ is globally generated, which we do in the following two lemmas. Once we do this we are done by Proposition 3.3. $\square$

In this setting we are not able to immediately apply Proposition 3.6 to deduce that $\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(k,\ell)}(1,1)|_X$ is globally generated, as $\mathcal{O}_{\mathbf{P}}(k,\ell)$ is not necessarily a tensor power of $\mathcal{O}_{\mathbf{P}}(1,1)$. So it requires a bit more work to prove that $T_{\mathcal{X}}(1,1)|_X$ is globally generated, although the proof is not hard.

Lemma 3.14. *The vector bundle $\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(k,\ell)}(1,1)|_X$ is globally generated.*

Proof. The idea is to understand the kernel bundle of $\mathcal{O}_{\mathbf{P}}(k,\ell)$ in terms of the kernel bundles of $\mathcal{O}_{\mathbf{P}}(k,0)$ and $\mathcal{O}_{\mathbf{P}}(0,\ell)$. The defining exact sequence of the kernel bundle of $\mathcal{O}_{\mathbf{P}}(k,0)$ is

$$0 \to \mathcal{M}_{\mathcal{O}_{\mathbf{P}}(k,0)} \to H^0(\mathcal{O}_{\mathbf{P}^a}(k)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{P}} \xrightarrow{\text{eval}} \mathcal{O}_{\mathbf{P}}(k,0) \to 0. \tag{3.15}$$

Here we are using the natural isomorphism

$$H^0(\mathbf{P}^a, \mathcal{O}_{\mathbf{P}^a}(k)) \cong H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(k,0)).$$

Likewise we have the exact sequence for the kernel bundle of $\mathcal{O}_{\mathbf{P}}(0,\ell)$

$$0 \to \mathcal{M}_{\mathcal{O}_{\mathbf{P}}(0,\ell)} \to H^0(\mathcal{O}_{\mathbf{P}^b}(\ell)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{P}} \xrightarrow{\text{eval}} \mathcal{O}_{\mathbf{P}}(0,\ell) \to 0. \tag{3.16}$$

These two sequences are the pullback of the defining sequences of the kernel bundles of $\mathcal{O}_{\mathbf{P}^a}(k)$ and $\mathcal{O}_{\mathbf{P}^b}(\ell)$ respectively. Then Proposition 3.6 implies that the bundles $\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(k,0)}(1,0)$ and $\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(0,\ell)}(0,1)$ both are globally generated. Now we take the tensor product of the sequence (3.15) with the sequence (3.16) to get the following 9-term commuting diagram

$$\begin{tikzcd}
\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(k,0)} \otimes \mathcal{M}_{\mathcal{O}_{\mathbf{P}}(0,\ell)} \arrow[r] \arrow[d] & H^0(\mathcal{O}_{\mathbf{P}^a}(k)) \otimes_{\mathbb{C}} \mathcal{M}_{\mathcal{O}_{\mathbf{P}}(0,\ell)} \arrow[r] \arrow[d] & \mathcal{M}_{\mathcal{O}_{\mathbf{P}}(0,\ell)}(k,0) \arrow[d] \\
\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(k,0)} \otimes_{\mathbb{C}} H^0(\mathcal{O}_{\mathbf{P}^b}(\ell)) \arrow[r] \arrow[d] & H^0(\mathcal{O}_{\mathbf{P}}(k,\ell)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{P}} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbf{P}}(k,0) \otimes_{\mathbb{C}} H^0(\mathcal{O}_{\mathbf{P}^b}(\ell)) \arrow[d] \\
\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(k,0)}(0,\ell) \arrow[r] & H^0(\mathcal{O}_{\mathbf{P}^a}(k)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{P}}(0,\ell) \arrow[r] & \mathcal{O}_{\mathbf{P}}(k,\ell).
\end{tikzcd}$$

All the rows and columns are short exact sequences. A diagram chase yields a surjection

$$(H^0(\mathcal{O}_{\mathbf{P}^a}(k)) \otimes_{\mathbb{C}} \mathcal{M}_{\mathcal{O}_{\mathbf{P}}(0,\ell)}) \oplus (\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(k,0)} \otimes_{\mathbb{C}} H^0(\mathcal{O}_{\mathbf{P}^b}(\ell))) \to \mathcal{M}_{\mathcal{O}_{\mathbf{P}}(k,\ell)}.$$

After twisting by $\mathcal{O}_X(1,1)$, the term on the left is globally generated, and hence $\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(k,\ell)}(1,1)|_X$ is globally generated. $\square$

Lemma 3.17. *The bundle $T_{\mathcal{X}}(1,1)|_X$ is globally generated.*

Proof. By Proposition 3.2 and Lemma 3.14, it suffices to show that

$$H^1(X, \mathcal{M}_{\mathcal{O}_{\mathbf{P}}(k,\ell)}(1,1)) = 0.$$

As in Lemma 3.8 it suffices to show that

$$\mathrm{eval}\colon H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(k,\ell)) \otimes H^0(X, \mathcal{O}_X(1,1)) \to H^0(X, \mathcal{O}_X(k+1, \ell+1))$$

is surjective and that

$$H^1(X, \mathcal{O}_X(1,1)) = 0.$$

The surjectivity of the evaluation map follows from the surjectivity of

$$H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(k+1, \ell+1)) \to H^0(X, \mathcal{O}_X(k+1, \ell+1)).$$

By the ideal sequence of X the vanishing of $H^1(X, \mathcal{O}_X(1,1))$ is equivalent to the vanishing of $H^2(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(1-k, 1-\ell))$. We can decompose the latter group using using Künneth to write

$$H^2(\mathcal{O}_{\mathbf{P}}(1-k, 1-\ell)) \cong \bigoplus_{p+q=2} H^p(\mathcal{O}_{\mathbf{P}^a}(1-k)) \otimes H^q(\mathcal{O}_{\mathbf{P}^b}(1-\ell)).$$

The terms with H^0 vanish by the assumption that $k, \ell > 1$. The term

$$H^1(\mathcal{O}_{\mathbf{P}^a}(1-k)) \otimes H^1(\mathcal{O}_{\mathbf{P}^b}(1-\ell))$$

vanishes by the assumption that at least one of $a, b > 1$. $\square$

Example 3.18. Let $a = b = 2$, so $X \subset \mathbf{P}^2 \times \mathbf{P}^2$ is a smooth threefold. Assume $k = \ell$. In this case Proposition 3.13 says that if X is very general in $|\mathcal{O}_{\mathbf{P}}(k,k)|$ then X does not contain any curves C with $\mathrm{gon}(C) \leq k - 4$. For example, when $k = 5$ this implies X has no rational curves. Similarly we see that when $k = 4$ there are no ruled surfaces (with arbitrary singularities) inside X.

Now we consider products of more than two projective spaces

$$\mathbf{P} = \mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_k}$$

with $k \geq 3$.

Proposition 3.19. *Let $X \in |\mathcal{O}_{\mathbf{P}}(a_1, \ldots, a_k)|$ be a very general divisor with all the $a_i > 1$ and set $n = \dim(X)$. If $S \subset X$ is an e-dimensional subvariety swept out by curves with gonality c then*

$$c \geq e + \min\{a_1 - m_1, \ldots, a_k - m_k\} - n + 1.$$

Proof. We have

$$\omega_X(-n, \ldots, -n) \cong \mathcal{O}_X(a_1 - m_1 - n - 1, \ldots a_k - m_k - n - 1)$$

is p-very ample for

$$p = \min\{a_1 - m_1, \ldots, a_k - m_k\} - n - 1.$$

In the following Lemma we prove that $T_{\mathcal{X}}(1, \ldots, 1)|_X$ is globally generated. Then by applying Proposition 3.3 we are done. $\square$

Lemma 3.20. *The bundle $T_{\mathcal{X}}(1,\ldots,1)|_X$ is globally generated.*

Proof. The proof of Lemma 3.20 follows the method of proof of Lemma 3.14 and Lemma 3.17. To start we prove global generation of

$$\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(a_1,\ldots,a_k)}(1,\ldots,1).$$

As in Lemma 3.14 we see there is a surjection

$$\bigoplus_{i=1}^{k}\left(\left(\bigotimes_{j\neq i} H^0(\mathcal{O}_{\mathbf{P}^{m_j}}(a_j))\right)\otimes_{\mathbb{C}}\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(0,\ldots,a_i,\ldots,0)}\right)\to\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(a_1,\ldots,a_k)}.$$

If we twist this map by $\mathcal{O}_{\mathbf{P}}(1,\ldots,1)$ then the left hand side is globally generated by Proposition 3.6. Therefore $\mathcal{M}_{\mathcal{O}_{\mathbf{P}}(a_1,\ldots,a_k)}(1,\ldots,1)$ is globally generated.

The rest of the proof that $T_{\mathcal{X}}(1,\ldots,1)$ is globally generated is the same as Lemma 3.17. We just remark that every term in the Künneth decomposition of $H^2(\mathcal{O}_{\mathbf{P}}(1-a_1,\ldots,1-a_k))$ contains a tensor factor of $H^0(\mathcal{O}_{\mathbf{P}^{m_i}}(1-a_i))$ for some i, and all these terms vanish because $1-a_i$ is negative. $\square$

Example 3.21. Consider $X\subset(\mathbf{P}^1)^4$ such that $X\in|kH|$ is very general - i.e. X is a smooth threefold. In this case Proposition 3.14 says that X contains no curves C with $\mathrm{gon}(C)\leq k-3$. In particular, when $k=4$ we see that X contains no rational curves, and when $k=3$ we see that X contains no ruled surfaces. In this setting, this is the best possible bound on k such that X contains no ruled surfaces. When $k=2$ (X is a Calabi-Yau threefold) we get that the projection:

$$p_1\colon X\to\mathbf{P}^1$$

is generically fibered in smooth K3 surfaces. By the following lemma, (based on the theorem of Bogomolov and Mumford [MM83]) any threefold which is fibered in K3 surfaces necessarily contains a ruled surface.

Lemma 3.22. *Let X be a projective threefold and C a smooth curve. If there exists a map*

$$\pi\colon X\to C$$

which is generically fibered in K3 surfaces, then X contains a ruled surface.

Sketch of Proof. As a general fiber $X_p=\pi^{-1}(p)$ is a K3 surface [MM83, p. 351] implies that X_p contains a rational curve. This implies that the following locus in the Hilbert scheme of X

$$H=\left\{[R\subset X]\;\middle|\;\begin{array}{c}R\text{ is integral, }R\simeq_{\mathrm{bir}}\mathbf{P}^1,\\ \text{and }\pi(R)\text{ is a point}\end{array}\right\}\subset\mathrm{Hilb}(X)$$

dominates C. Thus, by the Baire category theorem there must be an integral curve $D\subset H$. Then the universal subscheme over D

$$\mathcal{R}=\{[R]\times x|x\in R\}\subset D\times X$$

projects onto a ruled surface in X. $\square$

Remark 3.23. Conjecturally, every K3 surface contains infinitely many rational curves. This is known for very general K3 surfaces. If this were true then one could show that X contains countably many ruled surfaces.

Chapter 4

The degree of irrationality of hypersurfaces in some rational homogeneous spaces

In this chapter we compute the degree of irrationality of hypersurfaces in quadrics, Grassmannians, and products of projective spaces. These are the main results from this dissertation and they represent some of the first computations of the degree of irrationality of higher dimensional varieties. The arguments here are extensions of the arguments in [BDPE+15] where those authors proved

Theorem ([BDPE+15, Thm. C])**.** *Let $X \subset \mathbf{P}^{n+1}$ be a very general smooth hypersurface of degree $d \geq 2n+1$. Then*

$$\operatorname{irr}(X) = d - 1.$$

Furthermore, if $d \geq 2n+2$ then any rational mapping:

$$\varphi\colon X \dashrightarrow \mathbf{P}^n \text{ with } \deg(f) = d - 1$$

is given by projection from a point of X.

This theorem resolved a conjecture [BCDP14, Conj. 1.5] where the conjecture was established in the cases X is a surface or a threefold.

The idea of the proof is that assuming for contradiction there is a rational map

$$\varphi\colon X \dashrightarrow \mathbf{P}^n$$

with $\delta = \deg(\varphi) \leq d-2$, then the Cayley-Bacharach condition on the fibers of φ (Lemma 2.27) implies the fibers of φ lie on lines $\ell \subset \mathbf{P}^{n+1}$. Thus if $y \in \mathbf{P}^n$ is a general point then

$$\ell \cap X = \varphi^{-1}(y) \cup \{x_1(y), ..., x_{d-\delta}(y)\}.$$

However, Theorem 2.21 implies that $d-\delta \leq n$. So if we let y vary in a rational curve $\mathbf{P}^1 \subset \mathbf{P}^n$ then the residual points:

$$\{x_1(y), ..., x_{d-\delta}(y) | y \in \mathbf{P}^1\}$$

trace out a curve with gonality $\leq n$. This is a contradiction, as the existence of such a curve is ruled out by Propositions similar to those proved in Chapter 3 (following ideas of Ein and Voisin [Ein88, Voi96]).

Here we extend these arguments to other rational homogeneous spaces. For example, let $\mathbf{Q} \subset \mathbf{P}^{n+2}$ be a smooth quadric in $\mathbf{P}^{n+2}$.

Theorem 4.1. *Let*

$$X = X_d \subset \mathbf{Q} \subset \mathbf{P}^{n+2}$$

be a very general hypersurface in $\mathbf{Q}$ with $X \in |\mathcal{O}_{\mathbf{Q}}(d)|$. If $d \geq 2n$, then $\operatorname{irr}(X) = d$.

Similarly, let $\mathbf{G} = \mathrm{Gr}(k, m) \subset \mathbf{P}^N$ be the Plücker embedding of the Grassmannian of k-planes in $\mathbb{C}^m$.

Theorem 4.2. *Let*

$$X = X_d \subset \mathbf{G}$$

be a very general hypersurface with $X \equiv_{\mathrm{lin}} dH$. *If* $d \geq 3m - 5$ *then* $\mathrm{irr}(X) = d$.

Finally, let $\mathbf{P} = \mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_k}$ be a product of k projective spaces with $k \geq 2$.

Theorem 4.3. *Let*

$$X = X_{(a_1,\ldots,a_k)} \subset \mathbf{P}$$

be a very general divisor with $X \in |\mathcal{O}_{\mathbf{P}}(a_1, \ldots, a_k)|$ *and let* p *be the minimum of* $\{a_i - m_i - 1\}$. *If* $p \geq \max\{m_i\}$ *(for example, if* $\min\{a_i\} \geq 2 \cdot \max\{m_i\} + 1$*) then* $\mathrm{irr}(X) = \min\{a_i\}$.

We prove Theorem 4.1 in §4.1, Theorem 4.2 in §4.2 and Theorem 4.3 in §4.3.

4.1. The degree of irrationality of a hypersurface in a quadric

Let $\mathbf{Q} \subset \mathbf{P}^{n+2}$ be a smooth quadric hypersurface in projective space with $\dim(\mathbf{Q}) \geq 2$. The goal of this section is to prove Theorem 4.1, that is if $d \geq 2n$ and

$$X = X_d \subset \mathbf{Q}$$

is a very general hypersurface with $X \in |\mathcal{O}_{\mathbf{Q}}(d)|$ then $\mathrm{irr}(X) = d$. In §4.1.1 we construct a degree d rational map:

$$\varphi_0 \colon X \dashrightarrow \mathbf{P}^n,$$

which implies $\mathrm{irr}(X) \leq d$. In §4.1.2 we assume for contradiction that there is a rational map

$$\varphi \colon X \dashrightarrow \mathbf{P}^n$$

with $\deg(\varphi) < d$. Using the Cayley-Bacharach condition (Lemma 2.27) we show that the fibers of φ lie on lines $\ell \subset \mathbf{P}^{n+2}$ which are contained in $\mathbf{Q}$. Finally in §4.1.3 we give an argument along the lines of [BDPE$^+$15, Pf. of Thm. C], which is outlined at the beginning of this chapter. Finally we use §3.2 to arrive at a contradiction.

4.1.1. **Construction of a degree d rational map.** We start by proving.

Proposition 4.4. *If* $X = X_d \subset \mathbf{Q}$ *is any element in* $|\mathcal{O}_{\mathbf{Q}}(d)|$ *then there exists*

$$\varphi_0 \colon X \dashrightarrow \mathbf{P}^n$$

of degree d.

Proof. As $\dim(\mathbf{Q}) \geq 2$, $\mathbf{Q}$ is covered by lines $\ell \subset \mathbf{Q} \subset \mathbf{P}^{n+2}$. Linear projection from one such ℓ defines a rational map:

$$p_\ell \colon \mathbf{P}^{n+2} \dashrightarrow \mathbf{P}^n.$$

The map p_ℓ is defined away from ℓ, and the fibers of p_ℓ (or rather the closure of the fibers) consist of the planes $P \subset \mathbf{P}^{n+2}$ which contain ℓ.

For a general choice of $\ell \subset \mathbf{Q}$ set $\varphi_0 = p_\ell|_X$. Now let $y \in \mathbf{P}^n$ be a general point and P_y the plane which is the closure of the fiber $p_\ell^{-1}(y)$. Then we have

$$\varphi_0^{-1}(y) \subset P_y \cap X \subset P_y \cap \mathbf{Q}.$$

The last intersection consists of ℓ and a residual line ℓ_y, i.e.

$$P_y \cap \mathbf{Q} = \ell \cup \ell_y.$$

As $\ell \subset \mathbf{Q}$ is general and y is general, it is easy to see that in fact

$$\ell_y \cap X = \varphi_0^{-1}(y),$$

thus $\#\varphi_0^{-1}(y) = d$. $\square$

Example 4.5. Theorem 4.1 says that if $X = X_d$ is very general with $d \geq 2n$ then $\mathrm{irr}(X) = d$. However, if $n \geq 2$ then there are special smooth hypersurfaces $X \subset \mathbf{Q}$ which contain a line $\ell \subset X$. In this case, the projection:

$$p_\ell|_X \colon X \dashrightarrow \mathbf{P}^n$$

is dominant of degree $d-1$. Thus there are special smooth hypersurfaces which satisfy $\mathrm{irr}(X) < d$.

Example 4.6 (Another example of a degree d map when n is odd)**.** Suppose n is odd, equal to $2k-1$. In this case there exist nonintersecting linear subvarieties of dimension k

$$\mathbf{P}(V), \mathbf{P}(W) \subset \mathbf{Q}$$

(for example when $n = 1$ take 2 lines in the same ruling of $\mathbf{Q}$). We can write $\mathbf{P}(V \oplus W) = \mathbf{P}^{n+2}$. Now we consider the linear projections away from $\mathbf{P}(V)$ and $\mathbf{P}(W)$ respectively

$$p_V \colon \mathbf{Q} \dashrightarrow \mathbf{P}(W) \text{ and } p_W \colon \mathbf{Q} \dashrightarrow \mathbf{P}(V),$$

and set

$$p_{V,W} = p_V \times p_W \colon \mathbf{Q} \dashrightarrow \mathbf{P}(V) \times \mathbf{P}(W).$$

For a general point $x \in \mathbf{Q}$ we can write $x = [v \oplus w]$ with $v \in V$ and $w \in W$ both nonzero. Thus there is a line $\ell_x \subset \mathbf{P}^{n+2}$ which connects the three points $\{x, [v \oplus 0], [0 \oplus w]\} \subset \mathbf{Q}$. So by Lemma 2.27 we know $\ell_x \subset \mathbf{Q}$. Further, it is easy to see that ℓ_x is actually the closure of the fiber of $p_{V,W}$ containing x.

Let

$$B = \overline{\mathrm{Image}(p_{V,W} \colon \mathbf{Q} \dashrightarrow \mathbf{P}(V) \times \mathbf{P}(W))}.$$

be the closure of the image of $\mathbf{Q}$. It is straightforward to show that $B \subset \mathbf{P}(V) \times \mathbf{P}(W)$ is a type (1,1) divisor. In particular, B is rational. Thus if $X = X_d \subset \mathbf{Q}$ is a general hypersurface with $X \in |\mathcal{O}_{\mathbf{Q}}(d)|$ then

$$p_{V,W}|_X : X \dashrightarrow B \simeq_{\mathrm{bir}} \mathbf{P}^n$$

has degree equal to $\ell_x \cdot X = d$. Therefore we see in a different way that $\mathrm{irr}(X) \leq d$.

4.1.2. **Low degree maps give rise to congruences on Q.** From now on, in order to reach a contradiction, we make the following assumption

Assumption 4.7. *There exists $\varphi \colon X \dashrightarrow \mathbf{P}^n$ with $\delta = \deg(\varphi) < d$.*

We start by proving that all the fibers of φ must lie on lines $\ell \subset \mathbf{P}^{n+2}$ which are contained inside $\mathbf{Q}$.

Lemma 4.8. *If $d \geq 2n$ and $(d, n) \neq (2, 1)$ then a general fiber of φ lies on a line $\ell \subset \mathbf{P}^{n+2}$ which is contained in $\mathbf{Q}$.*

Proof. The canonical bundle K_X is $\mathcal{O}_X(d-n-1)$. A general fiber $\phi^{-1}(y)$ satisfies Cayley-Bacharach with respect to $|K_X|$. The degree assumption implies that

$$\delta \leq 2(d-n-1)+1.$$

Thus, Lemma 2.27 implies that $\phi^{-1}(y)$ lies on a line $\ell \subset \mathbf{P}^{n+2}$.

By Theorem 2.21 (using the assumption that $(d, n) \neq (2, 1)$):

$$\#\varphi^{-1}(y) \geq d-n+1 \geq 3.$$

Thus the line ℓ meets $\mathbf{Q}$ at a minimum of

$$\#\varphi^{-1}(y) = \delta \geq 3$$

points, Corollary 2.30 implies that $\ell \subset \mathbf{Q}$. $\square$

Having proved Lemma 4.8 we can think of a general point $y \in \mathbf{P}^n$ as parameterizing a line $\ell_y \subset \mathbf{Q}$. That is, the rational map φ induces a rational map $\mathbf{P}^n \dashrightarrow \mathrm{Fano}(\mathbf{Q})$, where $\mathrm{Fano}(\mathbf{Q})$ is the Fano variety of lines in $\mathbf{Q}$ (the so-called *orthogonal Grassmannian*). Resolving this rational map, gives rise to a regular map

$$f \colon B \to \mathrm{Fano}(\mathbf{Q})$$

where B is a smooth rational projective variety. The map f gives rise to the following fundamental diagram

$$
\begin{tikzcd}
X & X' \arrow[l] \\
\mathbf{Q} \arrow[from=1-1, hook] & F \arrow[l, "\pi"'] \arrow[from=1-2, hook] \arrow[d, "\psi"] \\
& B \arrow[r, "f"] & \mathrm{Fano}(\mathbf{Q}).
\end{tikzcd}
\tag{4.9}
$$

Here $\psi\colon F\to B$ is the $\mathbf{P}^1$-bundle defined as the pullback of the natural $\mathbf{P}^1$-bundle over $\mathrm{Fano}(\mathbf{Q})$. Thus F comes with a natural projection $\pi\colon F\to\mathbf{Q}$. Note that X is not uniruled implies π is generically finite. To define X' consider the rational map:

$$\mathrm{id}_X \times \varphi\colon X \dashrightarrow X \times B \subset \mathbf{Q} \times B$$

which is the graph of φ. The image of $\mathrm{id}_X \times \varphi$ is contained in F. Set

$$X' := \overline{\mathrm{Image}(\mathrm{id}_X \times \varphi)},$$

i.e. let X' be the closure of the image of the graph of φ.

Lemma 4.10. *If $d \geq 2n$ and $(d,n) \neq (2,1)$ then in (4.9) the map π is birational. In particular*

$$f\colon B\to\mathrm{Fano}(\mathbf{Q})$$

determines a "congruence of order one" on $\mathbf{Q}$.

Proof. This proof follows the proof of [BCDP14, Thm. 4.3]. Note that

$$\pi_*(\pi^*([X])) = \deg(\pi)\cdot[X].$$

Thus to prove Lemma 4.10 it suffices to show that $\pi_*(\pi^*([X])) = [X]$. Write

$$\pi^*(X) = cX' + \sum a_i E_i$$

where the E_i are irreducible and $c, a_i \in \mathbb{Z}^{>0}$. Thus it suffices to show that $c = 1$ and $\pi_* E_i = 0$.

Consider a general fiber $\ell = \psi^{-1}(b)$. Since $X \in |\mathcal{O}_{\mathbf{Q}}(d)|$ we have

$$\pi^*(X)\cdot[\ell] = d.$$

Moreover, as ℓ is a general fiber of ψ we have that ℓ meets E_i and X' transversely. Thus $E_i\cdot[\ell] \geq 0$ and $X'\cdot[\ell] = \delta(= \deg(\varphi))$.

To prove $c = 1$ it suffices to show that $\delta > d/2$ since

$$d = \pi^{-1}(X)\cdot[\ell] \geq X'\cdot[\ell] = c\cdot\delta.$$

Theorem 2.21 implies that $\delta \geq d - n + 1$, and the assumption that $d \geq 2n$ implies that $d - n + 1 > d/2$. Thus $c = 1$.

Proving that $\pi_*(E_i) = 0$ uses a similar analysis. Assume for contradiction that $E = E_i$ satisfies $\pi_*(E) \neq 0$. Thus the image $\pi(E)$ is all of X. This also implies that $\psi(E) = B$ as otherwise E would be covered by the fibers of ψ which would then imply that X is uniruled, a contradiction. So E gives a correspondence between X and B. By Lemma 2.28 we see that for a general point $y \in B$, the image $\pi(\psi|_E^{-1}(y))$ satisfies the Cayley-Bacharach condition with respect to K_X. Therefore, as in the previous paragraph, $\#\pi(\psi|_E^{-1}(y)) > d/2$. So we have

$$d = \pi^{-1}X \cdot [\ell] \geq (X' + E) \cdot [\ell] > d/2 + d/2,$$

which is a contradiction. □

4.1.3. **Proof of Theorem 4.1.** Having proved the map π in (4.9) is birational, we are now ready to prove Theorem 4.1. The following argument is almost identical to [BDPE+15, Pf. of Thm. C].

Proof of Theorem 4.1. First, consider the case $(d, n) = (2, 1)$. I.e. X is a $(2, 2)$ complete intersection genus 1 curve in $\mathbf{P}^3$. In this case X is hyperelliptic, so $\operatorname{irr}(X) = \operatorname{gon}(X) = 2$. Thus Theorem 4.1 holds in this case.

Now assume that $(d, n) \neq (2, 1)$ and assume for contradiction that there is a map:

$$\varphi\colon X \dashrightarrow \mathbf{P}^n$$

with $\deg(\varphi) = \delta < d$. We want to study:

$$\pi^* X = X' + \sum a_i E_i,$$

where $a_i > 0$. As above let ℓ be a general fiber of ψ. By the proof of Lemma 4.10 we know that $\pi_* E_i = 0$ and $E_i \cdot [\ell] \geq 0$. We also know that

$$X' \cdot [\ell] = \delta \text{ and } \pi^* X \cdot [\ell] = d.$$

Now Theorem 2.21 implies $\delta \geq d - n + 1$. Thus there exists $E = E_i$ with

$$0 < c = \deg(\psi|_E) = E \cdot [\ell] \leq d - \delta \leq n - 1.$$

If $n = 1$ then we are done. So assume $n \geq 2$. Now the image $\pi(E)$ satisfies $e := \dim(\pi(E)) \geq 1$ as every point $x \in \mathbf{Q}$ is connected to $\pi(E)$ via a line inside $\mathbf{Q}$, and the dimension of lines through a single point in $\mathbf{Q}$ is $n-1$. Thus the image $\pi(E)$ has $\operatorname{cov.gon}(\pi(E)) \leq n - 1$, and Proposition 3.7 implies

$$c \geq e + d - 2n + 1. \tag{4.11}$$

On the other hand, there is another inequality relating e and c by looking at the contribution of E to $K_{F/\mathbf{Q}}$. In particular, in the appendix to [BDPE+15] it is shown that

$$\operatorname{ord}_E(K_{F/\mathbf{Q}}) \geq n - e.$$

For a fiber ℓ of ψ we have

$$-2 = K_F \cdot [\ell] = (K_{F/\mathbf{Q}} + \pi^* K_{\mathbf{Q}}) \cdot \ell = K_{F/\mathbf{Q}} \cdot [\ell] - n - 1.$$

Thus

$$n - 1 = K_{F/\mathbf{Q}} \cdot [\ell] \geq \mathrm{ord}_E(K_{F/\mathbf{Q}})E \cdot [\ell] \geq (n-e)c.$$

Now if we divide the above inequality by $(n-e)$ and combine it with equation (4.11) we get

$$\frac{n-1}{n-e} \geq e + d - 2n + 1.$$

Rearranging, and using that $\frac{n-1}{e(n-e)} \leq 1$ for $1 \leq e \leq n-2$, we get

$$2n - 1 \geq 2n + e\left(\frac{n-1}{e(n-e)} - 1\right) - 1 \geq d,$$

which contradicts the assumption that $d \geq 2n$. □

4.2. The degree of irrationality of a hypersurface in a Grassmannian

Now we turn to hypersurfaces in a Grassmannian. Let $\mathbf{G} = \mathrm{Gr}(k, m) \subset \mathbf{P}(\wedge^k \mathbb{C}^m)$ be the Plücker embedding of the Grassmannian of k-planes in $\mathbb{C}^m$ (with $k \neq 1, m-1$). The purpose of this section is to prove Theorem 4.2, that is if $d \geq 3m - 5$ and

$$X = X_d \subset \mathbf{G}$$

is a very general hypersurface with $X \in |\mathcal{O}_{\mathbf{G}}(d)|$ then $\mathrm{irr}(X) = d$. The structure in this section is very similar to §4.1. In §4.2.1 we construct a degree d rational map

$$\varphi_0 \colon X \dashrightarrow \mathbf{P}^n \text{ (where } n := \dim(X)).$$

In §4.2.2 we assume for contradiction that there is a rational map

$$\varphi : X \dashrightarrow \mathbf{P}^n$$

with $\deg(\varphi) < d$ and conclude as in §4.1.2 that the fibers of φ lie on lines $\ell \subset \mathbf{P}(\wedge^k \mathbb{C}^m)$ contained in $\mathbf{G}$. Finally, in §4.2.3 we prove Theorem 4.2.

4.2.1. Construction of a degree d map to a flag variety.

To start we show

Proposition 4.12. *If $X = X_d \subset \mathbf{G}$ is any element in $|\mathcal{O}_{\mathbf{G}}(d)|$ then there exists*

$$\varphi_0 \colon X \dashrightarrow \mathbf{P}^n$$

of degree d.

Proof. In the following construction we actually construct a degree d rational map to the partial flag variety

$$\mathbf{F} = \mathrm{Fl}(k-1,k,m-1) = \left\{ [U \subset V \subset (\mathbb{C}^m/L)] \,\middle|\, \begin{array}{l} \dim(U) = k-1 \\ \text{and } \dim(V) = k \end{array} \right\},$$

but as flag varieties are rational this suffices for our purposes.

To start, fix subspaces $L, W \subset \mathbb{C}^m$ such that $L \cap W = 0$, $\dim(L) = 1$, and $\dim(W) = m-1$. Define $T\colon \mathbb{C}^m \to (\mathbb{C}^m/L)$ to be the natural quotient map. In this setting we define a rational map:

$$p_{L,W}\colon \mathbf{G} \dashrightarrow \mathbf{F}$$

by sending

$$[\Lambda \subset \mathbb{C}^m] \mapsto [T(\Lambda \cap W) \subset T(\Lambda) \subset (\mathbb{C}^m/L)].$$

The base locus of the map is the set of k-planes $[\Lambda \subset \mathbb{C}^m]$ where $\ell \subset \Lambda$ or $\Lambda \subset W$. Choosing any point $y = [U \subset V \subset (\mathbb{C}^m/L)]$, the fiber $p_{L,W}^{-1}(y)$ consists of k-planes $\Lambda \subset \mathbb{C}^m$ which satisfy

$$T^{-1}(U) \cap W \subset \Lambda \subset T^{-1}(V).$$

The subspace $T^{-1}(U) \cap W$ is $k-1$-dimensional and the subspace $T^{-1}(V)$ is $(k+1)$-dimensional. So there are always k-dimensional subspaces Λ between these two spaces, i.e. the map is surjective. Moreover, by Example 2.43 we see that the closure of the fiber $p_{L,W}^{-1}(y)$ is a line.

Fixing a general choice of $L, W \subset \mathbb{C}^m$, we define

$$\varphi_0 := (p_{L,W})|_X \colon X \dashrightarrow \mathbf{F}.$$

So we want to show that φ_0 is dominant of degree d. To prove φ_0 is dominant it suffices to find a fiber of φ_0 which is finite but nonempty. We can reverse-engineer this by choosing a line in $\mathbf{G}$ which we want to arise as the closure of a fiber of $p_{L,W}$. Following Example 2.43 choose a line $\ell \subset \mathbf{G}$ that arises from a two step flag $A \subset B \subset \mathbb{C}^m$ such that $\ell \cap X$ is a proper intersection (in particular $\mathrm{length}(\ell \cap X) = d$).

Now we want to choose L and W so that $\mathbf{P}^1$ is the closure of the fiber of $p_{L,W}$ over some point. This is possible if we choose any L such that $L \subset B$ and $L \not\subset A$ and any W such that $W \supset A$ and $W \not\supset B$. In this setup the locus in ℓ where $p_{L,W}$ is not defined consists of only the two k-planes: $[B + L \subset \mathbb{C}^m]$, $[W \cap A \subset \mathbb{C}^m] \in \mathbf{G}$. If we further require that these two points do not meet $\ell \cap X$ we see that $\deg(\varphi_0) = \mathrm{length}(\ell \cap X) = d$. □

Remark 4.13. When $\mathbf{G} = \mathrm{Gr}(2,4)$ then $\mathbf{G}$ is well known to be a quadric in $\mathbf{P}(\wedge^2\mathbb{C}^4) = \mathbf{P}^5$. So it is natural to compare the map $p_{L,W}$ to the maps constructed in §4.1.1. Indeed, the map φ_0 is the same as the map constructed in Example 4.6.

4.2.2. **Low degree maps give rise to congruences on $\mathbf{G}$.** From now on, in order to arrive at a contradiction we make the assumption

Assumption 4.14. *There exists* $\varphi : X = X_d \dashrightarrow \mathbf{P}^n$ *with* $\delta = \deg(\varphi) < d$.

Lemma 4.15. *If* $d \geq 3m - 5$ *then a general fiber of* φ *lies on a line* $\ell \subset \mathbf{P}(\wedge^k \mathbb{C}^m)$ *which is contained in* $\mathbf{G}$.

Proof. The argument is the same as the proof of Lemma 4.9. We just note that by Theorem 2.41, $\mathbf{G}$ is cut out by quadrics in $\mathbf{P}(\wedge^k \mathbb{C}^m)$. □

So given a map φ with $\delta < d$ we can think of a general point in $\mathbf{P}^n$ as parametrizing a line $\ell \subset \mathbf{G}$. According to Example 2.43 the Fano variety of lines in $\mathbf{G} \subset \mathbf{P}(\wedge^k \mathbb{C}^m)$ is given by

$$\mathrm{Fano}(\mathbf{G}) = \mathrm{Fl}(k-1, k+1, m).$$

Thus as in §4.1.2, φ induces a rational map

$$\mathbf{P}^n \dashrightarrow \mathrm{Fano}(\mathbf{G})$$

and resolving this map gives rise to a regular map

$$f : B \rightarrow \mathrm{Fano}(\mathbf{G})$$

where B is a smooth rational variety. Then as in §4.1.2 we have the fundamental diagram

$$\begin{tikzcd} X & X' \arrow[l] \arrow[d, hook] \\ \mathbf{G} \arrow[u, hookleftarrow] & F \arrow[l, "\pi"'] \arrow[d, "\psi"] \\ & B \arrow[r, "f"] & \mathrm{Fano}(\mathbf{G}). \end{tikzcd} \tag{4.16}$$

The map $\psi\colon F \rightarrow B$ is the $\mathbf{P}^1$ bundle given by pulling back the tautological $\mathbf{P}^1$-bundle over $\mathrm{Fano}(\mathbf{G})$. The tautological $\mathbf{P}^1$-bundle over $\mathrm{Fano}(\mathbf{G})$ lies inside the product $\mathrm{Fano}(\mathbf{G}) \times \mathbf{G}$ so there is a natural projection $\psi\colon F \rightarrow \mathbf{G}$. Finally, the rational map $\varphi\colon X \dashrightarrow \mathbf{P}^n \simeq_{\mathrm{bir}} B$ gives rise to a rational section $X \dashrightarrow F$ given by the graph of ϕ. The closure of the image of X is by definition X'.

Similar to Lemma 4.10 we have the following

Lemma 4.17. *If* $d \geq 3m - 5$ *then the map* π *is birational, i.e.*

$$B \rightarrow \mathrm{Fano}(\mathbf{G})$$

is a "congruence of order 1" on $\mathbf{G}$.

Proof. The proof is identical to the proof of Lemma 4.10, so we omit it here. □

4.2.3. **Proof of Theorem 4.2.** Before completing the proof of Theorem 4.2 we compute the dimension of lines through a point $x \in \mathbf{G}$.

Lemma 4.18. *Let $S_x \subset \mathrm{Fano}(\mathbf{G})$ be the set of lines in $\mathbf{G}$ through a point $x \in \mathbf{G}$. Then*
$$\dim(S_x) = m - 2$$

Proof. Assume that $x \in \mathbf{G}$ represents the subspace $W \subset \mathbb{C}^m$. It easily follows from Example 2.43 that S_x is equivalent to:
$$\left\{ \begin{array}{c} [\ell] \in \mathrm{Fano}(\mathbf{G}) \\ \text{such that } x \in \ell \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} [V \subset U \subset \mathbb{C}^m] \in \mathrm{Fl}(k-1, k+1, m) \\ \text{such that } V \subset W \subset U \end{array} \right\}.$$
Thus we need the dimension of $(k-1)$-planes inside W and the $(k+1)$-planes containing W. The former is the dual projective space $\mathbf{P}(W^*)$ and the latter is the projective space $\mathbf{P}(\mathbb{C}^m/W)$. Thus
$$\dim(S_x) = (k-1) + (m-k-1) = m-2,$$
which completes the dimension count. $\square$

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. Let $X \in |\mathcal{O}_{\mathbf{G}}(d)|$ be very general with $d \geq 3m-5$. Suppose for contradiction that there exists $\varphi \colon X \dashrightarrow \mathbf{P}^n$ a rational map with $\deg(\varphi) = \delta < d$. Now following the fundamental diagram (4.16) we look at
$$\pi^* X = X' + \sum a_i E_i$$
where the E_i are divisors which are contracted by π and $a_i \geq 0$. As X is an element of $|\mathcal{O}_{\mathbf{G}}(d)|$ we have that if $\ell \subset F$ is a fiber of ψ then
$$d = \pi^* X \cdot [\ell] = X' \cdot [\ell] + \sum a_i E_i \cdot [\ell] = \delta + \sum a_i E_i \cdot [\ell].$$
Thus as we assumed that $\delta < d$ we see that at least one component $E = E_i$ must dominate B. Set
$$c = \deg(\psi|_E).$$
Thus $\pi(E)$ satisfies $\mathrm{cov.gon}(\pi(E)) \leq c$. As $\delta \geq d - m + 2$ we have
$$m - 2 \geq c.$$
Let $e = \dim(\pi(E))$. Then applying Proposition 3.11 we have that
$$m - 2 \geq c \geq e + d - n - m + 2.$$

As E dominates B and the map F is birational, we must have that every point in $\mathbf{G}$ lies on a line that goes through $\pi(E)$. From this we can deduce a lower bound on the dimension of $\pi(E)$. In particular, applying Lemma 4.22 we must have
$$e + m - 1 \geq \dim(\mathbf{G}) = n + 1.$$

Combining these inequalities we get:

$$m - 2 \geq d - 2m + 4,$$

or

$$3m - 6 \geq d.$$

This contradicts our degree assumption. $\square$

4.3. The degree of irrationality of hypersurfaces in products of projective space

Let $\mathbf{P} = \mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_k}$ be a product of projective spaces with at least 2 factors. The goal of this section is to prove Theorem 4.3, i.e. to show that if

$$X = X_{(a_1,\ldots,a_k)} \subset \mathbf{P}$$

is a very general divisor with $X \in |\mathcal{O}_{\mathbf{P}}(a_1, \ldots, a_k)|$, and if

$$\min\{a_i - m_i - 1\} \geq \max\{m_i\} + 1$$

then

$$\operatorname{irr}(X) = \min\{a_i\}.$$

Throughout the section we define

$a := \min\{a_i\}$,
$m := \max\{m_i\}$,
$p := \min\{a_i - m_i - 1\}$, and
$n := \dim(X)$.

In particular, the degree assumption on X can be restated as $p \geq m$.

The method of proof of Theorem 4.3 is similar to the proof of Theorem 4.1 and Theorem 4.2. In §4.3.1 we give an example of a rational map

$$\varphi_0 \colon X \dashrightarrow \mathbf{P}^n$$

such that $\deg(\varphi_0) = a$. In §4.3.2 we show that any rational map

$$\varphi \colon X \dashrightarrow \mathbf{P}^n$$

with $\deg(\varphi) < a$ gives rise to a congruence of lines of order 1 on $\mathbf{P}$. Finally in §4.3.3 we prove Theorem 4.3 by a reduction to [BDPE+15, Thm. C].

4.3.1. **Construction of a degree a map.**

In this subsection we give a degree a map

$$\varphi_0 \colon X \dashrightarrow \mathbf{P}^n.$$

Example 4.19. Assume for simplicity that $a_1 = a$. Choosing a general point $z \in \mathbf{P}^{m_1}$ gives rise to a constant section of the projection

$$\begin{tikzcd} \mathbf{P} \arrow[r, "p_2\times\cdots\times p_k"] & \mathbf{P}^{m_2}\times\cdots\times\mathbf{P}^{m_k} \arrow[l, bend left, "\sigma_z"] \end{tikzcd}$$

by

$$\sigma_z(x) = z \times x.$$

Fiber by fiber linear projection from the image $\sigma_z(\mathbf{P}^{m_2}\times\cdots\times\mathbf{P}^{m_k})$ gives rise to a rational map

$$p_z\colon \mathbf{P} \dashrightarrow \mathbf{P}^{m_1-1}\times\mathbf{P}^{m_2}\times\cdots\times\mathbf{P}^{m_k}.$$

Now for a general choice of $z \in \mathbf{P}^{m_1}$ set

$$\varphi_0 := p_z|_X\colon X \dashrightarrow \mathbf{P}^{m_1-1}\times\mathbf{P}^{m_2}\times\cdots\times\mathbf{P}^{m_k}.$$

Then it is easy to see that the degree of φ_0 is a. In particular, $\operatorname{irr}(X) \le a$.

Remark 4.20. There are examples of low degree k-tuples $(a_1,\dots,a_k)$ such that $\operatorname{irr}(X) < a$ for every $X = X_{(a_1,\dots,a_k)} \in |\mathcal{O}_{\mathbf{P}}(a_1,\dots,a_k)|$. Indeed there are values $(a_1,\dots,a_k)$ such that every hypersurface $X \in |\mathcal{O}_{\mathbf{P}}(a_1,\dots,a_k)|$ contains the image of a section

$$\sigma\colon \mathbf{P}^{m_2}\times\cdots\times\mathbf{P}^{m_k} \to \mathbf{P}.$$

When this happens, the relative linear projection from the image of σ gives rise to a degree $a_1 - 1$ map

$$X \dashrightarrow \mathbf{P}^n.$$

In Examples 4.21 and 4.22 we give some concrete examples of $(a_1,\dots,a_k)$ where this occurs.

Example 4.21. Let $X \subset \mathbf{P} = \mathbf{P}^n \times \mathbf{P}^1$ be a smooth ample divisor of type (d,e) such that $d \le e$ and $e+1 \le n$. Then a general fiber of X over a point $y \in \mathbf{P}^1$ is a hypersurface $X_y \subset \mathbf{P}^n$ of degree d. This gives rise to a map

$$f_X : \mathbf{P}^1 \to \mathbf{P}(H^0(\mathbf{P}^n, \mathcal{O}(d))).$$

As this map is defined by equations of degree e, the image has linear span satisfying

$$\dim(\operatorname{Span}(f_X(\mathbf{P}^1)) \subset \mathbf{P}^n) \le e.$$

This implies that the intersection of all the hypersurfaces parametrized by $\mathbf{P}^1$ is nonempty, i.e.

$$\bigcap_{y\in\mathbf{P}^1} X_y \ne \emptyset.$$

If $z \in \cap X_y$ is any point then σ_z is a section of $p_2|_X\colon X \to \mathbf{P}^1$ and thus

$$p_z\colon X \dashrightarrow \mathbf{P}^n$$

has degree $d-1$.

Note that in Example 4.21, d is smaller than $n-1$ so the total space X is not of general type. This example admits obvious generalizations.

Example 4.22. Let $X \subset \mathbf{P}^{m_1} \times \mathbf{P}^{m_2}$ be a smooth ample divisor of type (d, e) with $d \leq e$ and $\binom{e+m_2}{e} \leq m_1$, then a similar argument to Example 4.21 shows that the projection $p_2|_X \colon X \to \mathbf{P}^{m_2}$ admits a constant section and therefore $\operatorname{irr}(X) \leq d-1$.

4.3.2. **Low degree maps give rise to congruences on P.** In order to reach a contradiction we make the following assumption

Assumption 4.23. *There exists $\varphi \colon X \dashrightarrow \mathbf{P}^n$ with $\delta = \deg(\varphi) < a$.*

We start by proving:

Lemma 4.24. *If $p \geq m$ then a general fiber of φ lies on a line $\ell \subset \mathbf{P}^N$ which is contained in $\mathbf{P}$.*

Proof. We have

$$\omega_X \cong \mathcal{O}_{\mathbf{P}}(a_1 - m_1 - 1, \ldots, a_k - m_k - 1) \cong \mathcal{O}_{\mathbf{P}}(p, \ldots, p) \otimes \mathcal{O}_{\mathbf{P}}(E),$$

where E is linearly equivalent to an effective divisor which misses any finite set of points in X. Thus if $\varphi^{-1}(y)$ is a general fiber then we can choose a representative of E which does not meet $\varphi^{-1}(y)$. Then $\varphi^{-1}(y)$ satisfies Cayley-Bacharach with respect to $|\mathcal{O}_{\mathbf{P}}(p, \ldots, p)|$. Our degree assumption implies $3 \leq \delta \leq 2p+1$. Thus by Lemma 2.27, $\varphi^{-1}(y)$ lies on a line $\ell \subset \mathbf{P}^N$ which meets $\mathbf{P}$ at a minimum of $\delta \geq 3$ points. Using the fact that $\mathbf{P}$ is cut out by quadrics and applying Corollary 2.30, we have $\ell \subset \mathbf{P}$. $\square$

As in §4.1.2 and §4.2.2, we have that a general point $y \in \mathbf{P}^n$ determines a line in $\mathbf{P}$. If $\operatorname{Fano}(\mathbf{P})$ is the Fano variety of lines in $\mathbf{P}^N$ which are contained in $\mathbf{P}$ then Lemma 4.24 implies there is a rational map

$$\mathbf{P}^n \dashrightarrow \operatorname{Fano}(\mathbf{P}),$$

and resolving this map we get a regular map

$$f : B \to \operatorname{Fano}(\mathbf{P})$$

from a smooth variety $B \simeq_{\mathrm{bir}} \mathbf{P}^n$.

Remark 4.25. Recall that in Example 2.44 we showed that $\operatorname{Fano}(\mathbf{P})$ is a disjoint union

$$\operatorname{Fano}(\mathbf{P}) = \begin{array}{c} (\operatorname{Gr}(2, m_1+1) \times \mathbf{P}^{m_2} \times \cdots \times \mathbf{P}^{m_k}) \sqcup \cdots \\ \cdots \sqcup (\mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_{k-1}} \times \operatorname{Gr}(2, m_k+1)) \,. \end{array}$$

I.e. every line in $\mathbf{P}$ is necessarily constant in all factors of $\mathbf{P}$ except one. Thus the map $f\colon B \to \mathrm{Fano}(\mathbf{P})$ lands in a single component. For simplicity we assume

$$f(B) \subset \mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_{k-1}} \times \mathrm{Gr}(2, m_k + 1).$$

To streamline notation we define

$$\mathbf{P}_0 := \mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_{k-1}}, \text{ and}$$
$$\mathrm{Fano}(\mathbf{P})_0 := \mathbf{P}_0 \times \mathrm{Gr}(2, m_k + 1).$$

Then following §4.1 and §4.2 we arrive at the fundamental diagram

$$\begin{tikzcd} X & X' \arrow[l] \arrow[d, hook] & \\ \mathbf{P} & F \arrow[l, "\pi"] \arrow[d, "\psi"] & \\ & B \arrow[r, "f"] & \mathrm{Fano}(\mathbf{P})_0. \end{tikzcd} \tag{4.26}$$

As usual, the map $\psi\colon F \to B$ is the pullback of the tautological $\mathbf{P}^1$-bundle along the map $f\colon B \to \mathrm{Fano}(\mathbf{P})$. The map $\pi\colon F \to \mathbf{P}$ is the natural projection. Finally $X' \subset F$ is given by taking the closure of the graph of $\varphi : X \dashrightarrow B$.

Note that all the varieties in the (4.26) admit a map to $\mathbf{P}_0$ and all the maps in (4.26) are compatible with these maps to $\mathbf{P}_0$.

Lemma 4.27. *If $p \geq m$ then the map π is birational, i.e.*

$$f\colon B \to \mathrm{Fano}(\mathbf{P})_0$$

is a "congruence of order 1" on $\mathbf{P}$.

Proof. The proof is similar to the proof of Lemma 4.10 but is more delicate. Again the goal is to show that $\pi_*(\pi^*([X])) = [X]$ and again we start by writing:

$$\pi^*(X) = cX' + \sum a_i E_i$$

where $c, a_i > 0$. Thus it suffices to show that $c = 1$ and $\pi_* E_i = 0$.

To prove that $c = 1$ we again note that for a line $\ell \subset F$ which is a fiber of ψ, we have $\pi^*(X) \cdot [\ell] = a_k$, and $X' \cdot [\ell] = \delta < a$. In particular, as in Lemma 4.10 it suffices to show that $\delta > a_k/2$ and $\deg(\psi|_{E_i}) > a_k/2$.

If $x \in B$ is a general point then we know that $\#\varphi^{-1}(x) = \delta$ and $\varphi^{-1}(x) \subset \mathbf{P}^N$ satisfies Cayley-Bacharach with respect to $|\omega_X|$. As $\varphi^{-1}(x)$ actually lies in a linear subspace $y \times \mathbf{P}^{m_k}$ for some $y \in \mathbf{P}_0$ we can conclude that $\varphi^{-1}(x)$ satisfies Cayley-Bacharach with respect to the restriction of the linear series:

$$|(\omega_X|_{y \times \mathbf{P}^{m_k}})| = |\mathcal{O}_{\mathbf{P}^{m_k}}(a_k - m_k - 1)|$$

Thus applying Lemma 2.27 and our degree assumption we have that $\delta > a_k/2$ (and as in Lemma 4.10 we also have $\deg(\psi|_{E_i}) > a_k/2$). $\square$

4.3.3. **Proof of Theorem 4.3.**

Proof of Theorem 4.3. Assume for contradiction that there is a rational map $\varphi\colon X \dashrightarrow \mathbf{P}^n$ with $\delta < a$. As in §4.3.2, assume for simplicity that the lines spanned by the fiber of φ are constant in all factors except $\mathbf{P}^{m_k}$.

First we deal with the case $m_k = 1$. In this case $\mathrm{Gr}(2, m_k + 1)$ is a single point, so

$$\mathrm{Fano}(\mathbf{P})_0 = \mathbf{P}_0.$$

As $\pi\colon F \to \mathbf{P}$ is birational, the only possibility is that φ is projection onto $\mathbf{P}_0$ and thus $\delta = a_k \geq a$, a contradiction.

Now assume $m_k \geq 2$. All varieties in (4.26) admit a map to $\mathbf{P}_0$, so the base change of diagram (4.26) to a fiber over a general point $y \in \mathbf{P}_0$ (denoted by subscript y) gives rise to the following diagram

$$\begin{tikzcd} X_y \arrow[d, hook] & X'_y \arrow[l] \arrow[d, hook] & \\ \mathbf{P}^{m_k} & F_y \arrow[l, "\pi_y"'] \arrow[d, "\psi_y"] & \\ & B_p \arrow[r, "f_y"] & \mathrm{Gr}(2, m_k + 1). \end{tikzcd} \tag{4.28}$$

Because y is general, every variety in (4.28) is integral. Moreover, π_y and $\pi_y|_{X'_y}$ are birational maps, and the degree of $\psi|_{X'_y}$ is δ.

The variety X is very general so if $y \in \mathbf{P}_0$ is a very general point then X_y is a very general degree a_k hypersurface in $\mathbf{P}^{m_k}$ with a degree $\delta < a \leq a_k$ rational map

$$\varphi_p : X_p \dashrightarrow B_p.$$

As F_p is rational and is a $\mathbf{P}^1$-bundle over B_p this implies B_p is rationally connected. The proof of [BDPE$^+$15, Thm. C] works for dominant rational maps to any rationally connected base. Thus as $\delta < a_k$ and X_p is a very general hypersurface in $\mathbf{P}^{m_k}$ (with $m_k \geq 2$) then [BDPE$^+$15, Thm. C] implies that $\delta = a_k - 1$, φ_p is given by projection from a point, and B_p is in fact rational.

Now looking at the whole diagram (4.26) the previous paragraph implies there is a component E in $\pi^* X$ distinct from X' such that $\psi|_E\colon E \to B$ has degree 1 and $\pi(E)$ dominates $\mathbf{P}_0$. Then $\pi(E)$ is a uniruled subvariety of X of dimension $n+1-m_k$. So by Propositions 3.13 and 3.19 we see that

$$m_k \geq p+3 > p,$$

which contradicts the degree assumption. $\square$

Chapter 5

Nonlinear inequalities for the degree of irrationality

In this chapter we prove two auxiliary results. The first is about the degree of irrationality of K3 and abelian surfaces and the second about the degree of irrationality of complete intersections.

A pair (X, L) is a polarized K3 surface if X is a K3 surface and L is a primitive ample line bundle on X. There are countably many irreducible families of polarized K3 surfaces, which are distinguished by the degree of the polarization, L^2. A priori there is no reason that there should be a universal upper bound for the degree of irrationality of all K3 surfaces. Indeed it is natural to ask

Question. *Can the degree of irrationality a K3 surface be arbitrarily large?*

Similarly, polarized abelian surfaces (A, L) come in countably many families where the components of these families depend on L^2 and other topological data. E.g. principally polarized abelian varieties form one such family. One can similarly ask in this setting

Question. *Can the degree of irrationality of an abelian surface be arbitrarily large?*

More greedily one might ask

Question. *Does the degree of irrationality of a very general polarized K3 surface (X, L) or a very general polarized abelian surface (A, L) grow linearly with L^2?*

In this chapter we show the answer to this last question is no.

Theorem 5.1. *There exists a uniform constant C such that if (X, L) is a very general polarized K3 surface then*

$$\mathrm{irr}(X) \leq C \cdot \sqrt{L^2}.$$

Theorem 5.2. *There exists a uniform constant D such that if (A, L) is a very general polarized abelian surface then*

$$\mathrm{irr}(A) \leq D \cdot \sqrt{L^2}.$$

Remark 5.3. In Theorem 5.1, it suffices to take $C = 3\sqrt{2}$. In Theorem 5.2 it suffices to take $D = 6\sqrt{2}$. Presumably these values of C and D are not optimal.

The last result from this chapter is about the degree of irrationality of a complete intersection. Recall that for a curve C we have

$$\mathrm{gon}(C) = c \iff c = \max\{p + 2 | K_C \text{ is } p\text{-very ample}\}.$$

Let C be a complete intersection curve

$$C = C_{(a_1,\ldots,a_k)} \subset \mathbf{P}^{k+1}$$

of type $(a_1, \ldots, a_k)$ (with $a_1 \leq \cdots \leq a_k$). Adjunction for K_C implies that

$$K_C = (a_1 + \cdots + a_k - k - 2)H.$$

Therefore, we have the naive lower bound

$$\mathrm{gon}(C) \geq a_1 + \cdots + a_k.$$

So naively we have that the gonality grows at least additively in the a_is. However, it is known that the gonality is multiplicative in the a_i. More precisely, [Laz97, p. 185] gives the lower bound

$$\mathrm{gon}(C) \geq (a_1 - 1) \cdot a_2 \cdots a_k.$$

There is a similar picture in the case

$$Z = Z_{(a_1,\ldots,a_k)} \subset \mathbf{P}^{n+k}$$

is a smooth n-dimensional complete intersection of type $(a_1, \ldots, a_k)$. That is there is a naive bound

$$\mathrm{irr}(Z) \geq a_1 + \cdots + a_k - n - k + 1$$

coming from adjunction which is additive in the a_is. But extrapolating from the one-dimensional case one might expect that the degree of irrationality should actually grow multiplicatively in the a_is. In this chapter we prove that the degree of irrationality of a complete intersection of two hypersurfaces in $\mathbf{P}^{n+2}$ grows superlinearly, but the results we give are not optimal.

Theorem 5.4. *Let $Z = Z_{d,e} \subset \mathbf{P}^{n+2}$ be a very general complete intersection of type (d, e). Then*

$$\mathrm{irr}(Z) \geq \frac{e\lfloor \sqrt[n+1]{d} \rfloor}{n+1}.$$

Moreover precisely we show that if

$$\frac{e\lfloor \sqrt[n+1]{d} \rfloor}{n+1} \geq p + 1$$

then K_Z separates p general points on Z.

We prove Theorem 5.1 and Theorem 5.2 in §5.1 and Theorem 5.4 in §5.2.

5.1. A sublinear upper bound on the degree of irrationality of a K3 or abelian surface

In this section we give upper bounds on the degree of irrationality of a very general, polarized K3 or abelian surface. These bounds are sublinear in the degree of the polarization.

5.1.1. **Bounding the degree of irrationality of a K3 surface.** Let (X, L) be a polarized K3 surface with $L^2 = 2d$ with and assume that $d \geq 2$ and that L generates $\mathrm{Pic}(X)$. Suppose we choose sections

$$s_0, s_1, s_2 \in H^0(X, L)$$

which give a dominant map to $\mathbf{P}^2$ by

$$\begin{aligned} \varphi \colon X &\dashrightarrow \mathbf{P}^2 \\ x &\mapsto [s_0(x) : s_1(x) : s_2(x)] \end{aligned}$$

then the degree of φ can be computed in terms of L^2 and the Hilbert-Samuel multiplicity $e(I_\varphi)$ of the base ideal. Recall,

Definition 5.5. The *base ideal of* φ, denoted I_φ, is defined to be the image

$$I_\varphi := \mathrm{Image}((L^{-1})^{\oplus 3} \xrightarrow{(s_0, s_1, s_2)} \mathcal{O}_X).$$

The *base locus of* φ, denote $\mathrm{Bs}(\varphi)$, is the closed subscheme of X associated to I_φ.

Remark 5.6. The assumption that L generated $\mathrm{Pic}(X)$ implies that there are no curves in $\mathrm{Bs}(\varphi)$.

Remark 5.7. If $x \in \mathrm{Bs}(\varphi) \subset X$ is a point in the base locus, then the Hilbert-Samuel multiplicity of I_φ at x, denoted by $e(I_\varphi, x)$ can be computed by

$$e(I_\varphi, x) = \mathrm{length}(\mathcal{O}_{X,x}/(f, g)),$$

where $f, g \in I_\varphi$ are generic linear combinations of the generators $\{s_{0,x}, s_{1,x}, s_{2,x}\}$ of I_φ at x.

Moreover, it is easily verified that

$$\deg(\varphi) = L^2 - e(I_\varphi) = 2d - \sum_{x \in \mathrm{Bs}(\varphi)} e(I_\varphi, x). \tag{5.8}$$

Thus if we want to give a small upper bound on $\mathrm{irr}(X)$ then we want to find a base ideal I_φ with high Hilbert-Samuel multiplicity. Our strategy is to fix a point $x \in X$ and to take sections of L which vanish to multiplicity at least k at x. That is we want to choose 3 independent sections in $H^0(X, L \otimes m_x^k)$ for some k.

Lemma 5.9. *If* $d + 2 - \binom{k+1}{2} \geq 3$ *then* $\dim(H^0(X, L \otimes m_x^k)) \geq 3$.

Proof. The L-twisted ideal sequence

$$o \to L \otimes m_x^k \to L \to L \otimes \mathcal{O}_X/m_x^k \to 0$$

give rise to the exact sequence

$$0 \to H^0(X, L \otimes m_x^k) \to H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_X/m_x^k).$$

The right most term $L \otimes \mathcal{O}_X/m_x^k$ is a skyscraper sheaf of length $\binom{k+1}{2}$ and $\dim(H^0(X,L)) = d+2$ by Riemann-Roch. □

Lemma 5.10. *Any 3 independent sections*

$$s_0, s_1, s_2 \in H^0(L)$$

give rise to a dominant rational map

$$\varphi\colon X \dashrightarrow \mathbf{P}^2.$$

Proof. Assume for contradiction that φ is not dominant. Then (as we assumed the s_i are independent) the image of φ is a curve $C \subset \mathbf{P}^2$ with $\deg(C) \geq 2$. Thus a general member of the linear series spanned by the s_i is in correspondence with $\varphi^{-1}(\ell \cap C)$ where $\ell \subset \mathbf{P}^2$ is a general line. But the curve $\varphi^{-1}(\ell \cap C)$ is necessarily reducible if ℓ is general, and this contradicts the assumption that $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot L$. □

Proof of Theorem 5.1. Set

$$k = \max\left\{ k_0 \middle| d + 2 - \binom{k_0+1}{2} \geq 3 \right\}.$$

Then one easily checks that

$$k = \left\lfloor \frac{-1+\sqrt{8d-7}}{2} \right\rfloor.$$

By Lemma 5.9 we can choose 3 general independent

$$s_0, s_1, s_2 \in H^0(X, L \otimes m_x^k).$$

By Lemma 5.10 we know these sections give rise to a dominant rational map

$$\varphi : X \dashrightarrow \mathbf{P}^2.$$

Now, as each element of the linear series $H^0(X, L \otimes m_x^k)$ has multiplicity at least k at x, we can give a lower bound

$$e(I_\varphi, x) \geq k^2.$$

So by equation (5.8) we have

$$\begin{aligned} \mathrm{irr}(X) \ \leq \ \deg(\varphi) &\leq 2d - k^2 \\ &= 2d - \left(\left\lfloor \tfrac{-1+\sqrt{8d-7}}{2} \right\rfloor\right)^2 \\ &\leq 2d - \left(\tfrac{-1+\sqrt{8d-7}}{2} - 1\right)^2 \\ &= 2d - 2d - \tfrac{3}{2} + \tfrac{3}{2}\sqrt{8d-7} \ \leq \ (3\sqrt{2})\sqrt{d}, \end{aligned}$$

which completes the proof. □

5.1.2. **Bounding the degree of irrationality of an abelian surface.** The results in this section are similar to the results in §5.1.1. Let (A, L) be a very general polarized abelian surface with $d \geq 4$ and $L^2 = 2d$. Suppose that the Néron-Severi group of A is generated by L, i.e.

$$\mathrm{NS}(A) = \mathrm{Pic}(A)/\mathrm{Pic}^0(A) \cong \mathbb{Z} \cdot L.$$

Let $x \in A$ be any point and m_x the corresponding maximal ideal. As in §5.1.1 we have the following lemma.

Lemma 5.11. *If* $d - \binom{k+1}{2} \geq 3$ *then* $\dim(H^0(A, L \otimes m_x^k)) \geq 3$.

Proof. See the proof of Lemma 5.9. In the present setting, Riemann-Roch implies $\dim(H^0(A, L)) = L^2/2 = d$. □

Lemma 5.12. *Any 3 independent sections*

$$s_0, s_1, s_2 \in H^0(A, L)$$

give rise to a dominant rational map

$$\varphi \colon X \dashrightarrow \mathbf{P}^2.$$

Proof. See the proof of Lemma 5.7. □

Proof of Theorem 5.2. As in the proof of Theorem 5.1 set

$$k = \max\left\{ k_0 \,\middle|\, d + 2 - \binom{k_0 + 1}{2} \geq 3 \right\}.$$

Thus

$$k = \left\lfloor \frac{-1 + \sqrt{8d - 23}}{2} \right\rfloor.$$

By Lemma 5.11 we can choose 3 independent sections in $H^0(A, L \otimes m_x^k)$, and by Lemma 5.12 these sections define a dominant map

$$\varphi \colon A \dashrightarrow \mathbf{P}^2.$$

As before we know that $e(I_\varphi, x) \geq k^2$, so we have the inequalities

$$\begin{aligned} \mathrm{irr}(A) \;\leq\; \deg(\varphi) &\leq 2d - k^2 \\ &= 2d - \left(\left\lfloor \tfrac{-1+\sqrt{8d-23}}{2} \right\rfloor\right)^2 \\ &\leq 2d - \left(\tfrac{-1+\sqrt{8d-23}}{2} - 1\right)^2 \\ &= 2d - 2d + \tfrac{7}{2} + \tfrac{3}{2}\sqrt{8d-7} \;\leq\; (6\sqrt{2})\sqrt{d}, \end{aligned}$$

which proves Theorem 5.2. □

5.2. A multiplicative lower bound on the degree of irrationality of a complete intersection

Let $Z \subset \mathbf{P}^{n+2}$ be a very general complete intersection of type (d, e). The purpose of this section is to prove Theorem 5.4, i.e. to give a lower bound on $\mathrm{irr}(Z)$ which is multiplicative in d and e. To do this we want to show that the positivity of K_Z grows multiplicatively with d and e. However, to make this precise we will need a slightly weaker notion of positivity than $(\mathrm{BVA})_p$ which is only concerned with separating points rather than all possible subschemes. To prove that the positivity of K_Z grows multiplicatively in d and e we use a theorem of Ito on the Seshadri constants of very general hypersurfaces in projective space.

Definition 5.13. A line bundle L on a variety X *satisfies property* $(\mathrm{BSP})_p$ if there exists a Zariski closed subset $Z = Z(L) \subset X$ depending on L such that

$$H^0(X, L) \to H^0(X, L|_\xi)$$

is surjective for every finite set of $p+1$ points $\xi = \{x_0, \ldots, x_p\}$ such that $\xi \cap Z$ is empty.

Remark 5.14. If X is smooth and projective and K_X satisfies $(\mathrm{BSP})_p$ then $\mathrm{irr}(X) \geq p + 2$. The proof of Theorem 2.21 works equally well for $(\mathrm{BSP})_p$ instead of $(\mathrm{BVA})_p$.

We recall the definition of the Seshadri constant of a hnef divisor L at a point $x \in X$.

Definition 5.15. Let X be an irreducible projective variety with $\dim(X) \geq 2$. Let L be a nef Cartier divisor on X. Fix a point $x \in X$ and let

$$\mu : X' = \mathrm{Bl}_x(X) \to X$$

be the blowing up of X, with exceptional divisor $E \subset X'$. The *Seshadri constant*

$$\epsilon(X, L; x) = \epsilon(L, x)$$

of L *at* x is the non-negative real number

$$\epsilon(L, x) = \max\{\epsilon \geq 0 | \mu^* L - \epsilon E \text{ is nef}\}.$$

Remark 5.16. The idea for the following proposition comes from Lemma 4.1 in [EKL95] where the authors prove the same result but only in the case $p = 1$. The proof of Proposition 5.17 is essentially the same but we include it here for completeness.

Proposition 5.17 (compare to [EKL95, Lemma 4.1])**.** *Let X be a smooth projective variety with a nef divisor L and $n = \dim(X) \geq 2$. Suppose there exists*

$$\mathcal{V} = \bigcup_i V_i \subsetneq X$$

a countable union of proper subvarieties such that

$$\epsilon(L, x) \geq (p+1)n$$

for all $x \in X \setminus \mathcal{V}$. Then the adjoint divisor $K_X + L$ satisfies $(\mathrm{BSP})_p$.

Proof. Choose $p+1$ points $x_0, \ldots, x_p \in X \setminus \mathcal{V}$ and blow up these points to get

$$\mu\colon X' = \mathrm{Bl}_{x_0,\ldots,x_p} X \to X.$$

Then by the assumption on the Seshadri constants we get that

$$\frac{1}{p+1}\mu^*(L) - nE_i$$

is nef for each i. Thus

$$\mu^*(L) - \sum_{i=0}^{p} nE_i$$

is also nef.

We want to show that the higher cohomology of

$$K_{X'} + \mu^*(L) - \sum_{i=0}^{p} nE_i$$

vanishes. So it is enough to show that $\mu^*(L) - \sum nE_i$ is big. Equivalently, we need to show

$$(\mu^*(L) - \sum_{i=0}^{p} nE_i)^n = L^n - (p+1)n^n > 0.$$

By Remark 1.8 of [EKL95] we know that

$$(\mu^* L)^n \geq \epsilon(L, x_i)^n \geq (p+1)^n n^n > (p+1)n^n.$$

Therefore, $\mu^*(L) - \sum nE_i$ is big and nef and we can apply vanishing for big and nef divisors to get

$$\begin{aligned} 0 &= H^0(X', K_{X'} + \mu^*(L) - \textstyle\sum nE_i) \\ &= H^1(X', \mu^*(K_X + L) + K_{X'/X} - \textstyle\sum nE_i) \\ &= H^1(X', \mu^*(K_X + L) - \textstyle\sum E_i) \\ &= H^1(X, (K_X + L) \otimes \mathcal{I}_{x_0,\ldots,x_p}). \end{aligned}$$

This implies that for any subset $\xi = \{x_0, \dots, x_p\} \subset X \setminus \mathcal{V}$, the sections of $K_X + L$ separate ξ.

As vanishing of cohomology is an open condition, we see there is actually an open subset

$$U \subset X^{p+1} \setminus \Delta$$

such that the sections of K_X+L separates any $(p+1)$-tuple of points in U. Let Z' be the complement of this open set. Then for any irreducible component $Z'_0 \subset Z'$ and any point $z = (z_0, \dots, z_p) \in Z'_0$ there is a factor $z_k \in \mathcal{V}$. This implies we can write Z'_0 as a countable union of the preimages of the V_i under the different projections. Thus by the Baire category theorem, there must be some projection

$$\pi_k \colon X^{p+1} \setminus \Delta \to X$$

and some V_i such that $\pi_k(Z'_0) \subset V_i$. If we set Z equal to the union of these finitely many V_is then we see that $K_X + L$ separates any $(p+1)$-tuple $\xi \subset X$ such that $\xi \cap Z = \emptyset$. $\square$

If $X \subset \mathbf{P}^{n+2}$ is a very general degree d hypersurface then Ito [Ito14] gives bounds on the Seshadri constants of $\mathcal{O}_X(1)$.

Theorem 5.18 ([Ito14, Theorem 4.9]). *Let $X \subset \mathbf{P}^{n+2}$ be a very general degree d hypersurface. If $x \in X$ is a very general point then*

$$\lfloor \sqrt[n+1]{d} \rfloor \leq \epsilon(X, \mathcal{O}_X(1), x) \leq \sqrt[n+1]{d}.$$

Remark 5.19. In fact all we need is the lower bound

$$\lfloor \sqrt[n+1]{d} \rfloor \leq \epsilon(X, \mathcal{O}_X(1), x).$$

In [Ito14, Remark 4.7], Ito shows that the lower bound can be given by degenerating to a rational hypersurface $X_0 \subset \mathbf{P}^{n+2}$. E.g. if $d = c^{n+1}$ then Ito takes $n+3$ general sections

$$s_0, \dots, s_{n+2} \in H^0(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(c))$$

to give a map

$$\pi \colon \mathbf{P}^{n+1} \to \mathbf{P}^{n+2}$$

and Ito sets $X_0 = \pi(\mathbf{P}^{n+1})$. Then degeneration to X_0 and the computation of the Seshadri constant on $\mathbf{P}^{n+1}$ gives the desired lower bound.

Using Proposition 5.17 and Theorem 5.18, the proof of Theorem 5.4 follows easily.

Proof of Theorem 5.4. Let $X = X_d$, $Y = Y_e \subset \mathbf{P}^{n+2}$ be very general hypersurfaces of degrees d and e respectively and let

$$Z = X \cap Y \subset \mathbf{P}^{n+2}$$

be their intersection. Thus by Theorem 5.18 we have that the divisor $Z \subset X$ has Seshadri constant at least

$$e\lfloor \sqrt[n+1]{d} \rfloor \leq \epsilon(X, \mathcal{O}_X(Z), x)$$

for a very general point $x \in X$.

Now applying Proposition 5.17 we get that if

$$e\lfloor \sqrt[n+1]{d} \rfloor \geq (p+1)(n+1), \text{ i.e. if } \frac{e\lfloor \sqrt[n+1]{d} \rfloor}{n+1} - 1 \geq p,$$

then the adjoint series $K_X + Z$ satisfies $(\mathrm{BSP})_p$. So if $Z \in \mathcal{O}_X(e)$ is very general then we also get that K_Z satisfies $(\mathrm{BSP})_p$. Therefore, by Remark 5.14 we have

$$\mathrm{irr}(Z) \geq \frac{e\lfloor \sqrt[n+1]{d} \rfloor}{n+1},$$

which finishes the proof. □

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