

Multiplicativity of perverse filtration for Hilbert schemes of fibered surfaces

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Abstract of the Dissertation

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Let $S \rightarrow C$ be a smooth projective surface with numerically trivial canonical bundle fibered onto a curve. We prove the multiplicativity of the perverse filtration with respect to the cup product on $H^*(S^{[n]}, \mathbb{Q})$ for the natural morphism $S^{[n]} \rightarrow C^{(n)}$. We also prove the multiplicativity for five families of Hitchin systems obtained in a similar way and compute the perverse numbers of the Hitchin moduli spaces. We show that for small values of n the perverse numbers match the predictions of the numerical version of the de Cataldo-Hausel-Migliorini $P = W$ conjecture and of the conjecture by Hausel-Letellier-Villegas.

To my parents

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1 Introduction

1.1 Nonabelian Hodge theory and the $P = W$ conjecture

Given any smooth complex projective variety X and any algebraic reductive group G , there are two natural moduli spaces associated with them, the moduli of Higgs G -bundles M_D and the character variety M_B . In [17], Simpson proved that these moduli spaces are algebraic varieties, and that they are canonically diffeomorphic to each other. The diffeomorphism induces a canonical identification of the cohomology of these moduli spaces. The algebraicity endows the cohomology groups with mixed Hodge structures. Furthermore, the moduli of Higgs bundle carries a proper Hitchin map to an affine space, so that the rational cohomology $H^*(M_D)$ is endowed with the Leray filtration and with the perverse (Leray) filtration. Under the canonical isomorphism, one may compare the filtrations mentioned above. One remarkable result is in [6], where de Cataldo, Hausel and Migliorini considered the case when X is any curve with genus $g \geq 2$ and $G = GL(2, \mathbb{C})$. They proved that, via the non-abelian Hodge theorem, the perverse filtration P on $H^*(M_D)$ for the Hitchin map equals the mixed Hodge theoretic weight filtration W on $H^*(M_B)$; they proved that $P = W$. There are numerous and diverse Hitchin-type moduli spaces M_D that come with natural Hitchin-type maps $h : M_D \rightarrow \mathbb{A}^1$ and which have corresponding (twisted) character varieties M_B . It is implicitly conjectured in [5] and [6] that the $P = W$ phenomenon appears whenever the nonabelian Hodge theory holds.

Conjecture 1.1 (The $P = W$ conjecture). *Under the canonical isomorphism between the cohomology groups predicted by nonabelian Hodge theory, the perverse filtration and the weight filtration correspond to each other.*

It had known previously that the weight filtration in mixed Hodge structure is multiplicative with respect to the cup product, which means $\cup : W_k H^i \times W_l H^j \rightarrow W_{k+l} H^{i+j}$. The perverse filtration is not multiplicative in general, even for proper maps between smooth projective varieties. So one key step in [6] is to establish that the perverse filtration is also multiplicative.

This dissertation is devoted to prove that for five families of Hitchin moduli spaces, the perverse filtrations on the rational cohomology are multiplicative with respect to the cup product. By using the same method, we can also prove the multiplicativity of perverse filtration on the cohomology on Hilbert schemes of points on elliptic K3 surfaces defined by the natural elliptic fibration.

1.2 Perverse filtration

In this section, we define the perverse filtration on the cohomology group defined by maps between algebraic varieties. We always work with varieties over the field of complex numbers \mathbb{C} . All cohomology groups have rational coefficients.

Definition 1.2 ([6] 1.4.1). Let $f : X \rightarrow Y$ be a morphism between smooth quasi-projective varieties. Let $r(f) = \dim X \times_Y X - \dim X$ be the defect of semismallness. Define the geometric perverse filtration as

$$P_p H^d(X; \mathbb{Q}) := \text{Im} \left\{ \mathbb{H}^{d-\dim X+r(f)}(Y, {}^p\tau_{\leq p} Rf_* \mathbb{Q}_X[\dim X - r(f)]) \rightarrow H^d(X, \mathbb{Q}) \right\},$$

where the ${}^p\tau_{\leq p}$ is the truncation functor of the standard perverse t -structure. The filtration ranges from 0 to $2r(f)$, i.e. $\text{Gr}_p^P H^d(X) = 0$ if $p < 0$ or $p > 2r(f)$, where

$$\text{Gr}_p^P H^d(X; \mathbb{Q}) := P_p H^d(X; \mathbb{Q}) / P_{p-1} H^d(X; \mathbb{Q})$$

The perverse filtration is multiplicative with respect to the cup product if the following condition holds for any integer p, q, i, j .

$$P_p H^i(X; \mathbb{Q}) \cup P_q H^j(X; \mathbb{Q}) \rightarrow P_{p+q} H^{i+j}(X; \mathbb{Q}).$$

Definition 1.3. Let $f : X \rightarrow Y$ as before. Given a cohomology class $0 \neq \alpha \in H^*(X)$, define the perversity of α , denoted as $\mathbf{p}(\alpha)$, to be the integer such that $\alpha \in P_{\mathbf{p}(\alpha)} H^*(X)$ and $\alpha \notin P_{\mathbf{p}(\alpha)-1} H^*(X)$. By our choice of perversity, the function \mathbf{p} takes value in the interval $[0, 2r(f)]$. Define $\mathbf{p}(0) = -\infty$. Then the perverse filtration is multiplicative with respect to cup product if and only if for any two classes $\alpha, \beta \in H^*(X)$, one has $\mathbf{p}(\alpha \cup \beta) \leq \mathbf{p}(\alpha) + \mathbf{p}(\beta)$.

Definition 1.4. Let $f : X \rightarrow Y$ as before. A perverse decomposition for $Rf_* \mathbb{Q}_X$ is an isomorphism in the bounded derived category of constructible sheaves of complex vector spaces

$$Rf_* \mathbb{Q}_X[\dim X - r(f)] \cong \bigoplus_{i=0}^{2r(f)} \mathcal{P}_i[-i],$$

where \mathcal{P}_i are suitable perverse sheaves.

We will use the perversity function \mathbf{p} for a cohomology class $\alpha \in H^*(X)$ without mentioning the map $X \rightarrow Y$ when no confusion arises. We say “the perversity of α ” for $\mathbf{p}(\alpha)$. To simplify notation, we say the perverse decomposition for the map $f : X \rightarrow Y$ for the perverse decomposition for $Rf_* \mathbb{Q}_X$, the push-forward of the constant sheaf. We say the

perverse filtration for the map $f : X \rightarrow Y$ for the perverse filtration on the cohomology group $H^*(X; \mathbb{Q})$ defined by the map $f : X \rightarrow Y$.

Definition 1.5. Let $f : X \rightarrow Y$ be as before. A basis $\alpha_1, \dots, \alpha_k$ of cohomology group $H^*(X; \mathbb{Q})$ is filtered with respect to the perverse filtration if the following property holds for any $0 \leq p \leq 2r(f)$.

$$P_p H^*(X) = \text{Span} \{ \alpha_i \mid \mathbf{p}(\alpha_i) \leq p, 1 \leq i \leq k \}.$$

1.3 Dissertation work

In this dissertation, we study a beautiful and classical class of Hitchin systems $h : M_D \rightarrow \mathbb{A}^1$. They are five families of moduli spaces of parabolic Higgs bundles over \mathbb{P}^1 with marked points, labeled by the affine Dynkin diagrams $\widetilde{A}_0, \widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7$ and \widetilde{E}_8 . In this setting, Theorem 4.1 and 5.1 in [10] states that these M_D are the Hilbert schemes $S^{[n]}$ of n -points of five distinct smooth algebraic elliptically fibered surfaces $S \rightarrow \mathbb{A}^1$. There are, for each of the five surfaces and for each $n \geq 1$, Hitchin maps $h : M_D = S^{[n]} \rightarrow \mathbb{A}^n$, hence a perverse filtration P on the cohomology groups $H^*(M_D)$. The construction of Hitchin map h is analogous to the one that starts with an elliptic K3 surface $S \rightarrow \mathbb{P}^1$ and yields the natural map $h : S^{[n]} \rightarrow \mathbb{P}^n$. The Mmin result of the dissertation is the multiplicativity of these perverse filtrations.

Theorem 1.6 (Theorem 5.6). *For the five families of Hitchin systems, the perverse filtration on the rational cohomology $H^*(M_D; \mathbb{Q})$ defined by the map $h : M_D \rightarrow \mathbb{A}^1$ is multiplicative.*

In the proof, we develop a systematic framework to describe the perverse filtration on $H^*(M_D)$ in terms of the one on $H^*(S)$ defined by the map $S \rightarrow \mathbb{A}^1$. We use the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber [1] as our main tool to decompose $Rh_* \mathbb{Q}_S$, the push-forward of the constant sheaf by the Hitchin map. By using the explicit geometry, we pick a very special basis of $H^*(S)$ which is filtered respect to the perverse filtration, and use it to produce a filtered basis of $H^*(M_D)$. The key step is the determination of the precise perversity of the class of the small diagonals in the product S^n ; the general bounds for these perversities are too weak for the problem, and we have to improve upon them by using the special geometry. The description of Lehn in [14], of the cohomology ring of $H^*(S^{[n]})$ is a key ingredient in my approach. Since M_D is not compact, we prove that it is also valid for all the moduli spaces M_D we are working with.

By using similar techniques, we may also prove the multiplicativity of perverse filtration for Hilbert schemes of projective surfaces with numerically trivial canonical bundle. In fact,

one can start with any smooth quasi-projective surfaces $f : S \rightarrow C$ fibered over a curve and obtain a map $f^{[n]} : S^{[n]} \rightarrow C^{(n)}$, where $C^{(n)}$ is the n -th symmetric product of C . In this case, since we don't have explicit description of the map, we have to use the "relative" Hodge-Riemann bilinear relations, due to de Cataldo and Migliorini [3], to "calculate" the perverse filtration and to produce special basis for the cohomology groups which are adapted to our problem. We have the following result.

Theorem 1.7 (Theorem 4.17). *Let $f : S \rightarrow C$ be a surjective morphism from a smooth projective surface with numerically trivial canonical bundle to a smooth projective curve. Then the perverse filtration of $H^*(S^{[n]}; \mathbb{Q})$ defined by the morphism $f : S^{[n]} \rightarrow C^{(n)}$ is multiplicative, namely, we have*

$$P_p H^*(S^{[n]}; \mathbb{Q}) \cup P_{p'} H^*(S^{[n]}; \mathbb{Q}) \subset P_{p+p'} H^*(S^{[n]}; \mathbb{Q})$$

As a byproduct of our formalism, we can prove that if there is a pair of smooth surfaces S_P and S_W , such that the perverse filtration (defined by some proper map $h : S_P \rightarrow \mathbb{A}^1$) on the cohomology $H^*(S_P)$ corresponds naturally to the weight filtration in the mixed Hodge structure on the cohomology of $H^*(S_W)$, then this correspondence induces an identification between the perverse filtration on $H^*(S_P^{[n]})$ and the weight filtration on $H^*(S_W^{[n]})$. This generalizes [5] Theorem 4.1.1.

There is a numerical version of the $P = W$ conjecture, namely instead of requiring the filtrations to correspond via the nonabelian Hodge theorem, one only requires the dimensions of the graded pieces to be the same. Conjectures in [7] and [11] predict the perverse numbers and mixed Hodge numbers for the moduli of parabolic Higgs bundles over curves with marked points and the corresponding character varieties. In our five families of Hitchin systems, we compute the perverse filtration explicitly, and also the perverse numbers.

Theorem 1.8 (Theorem 5.11). *Let $f : S \rightarrow \mathbb{A}^1$ be the $n = 1$ case of the five families. Denote the perverse numbers by $p_n^{i,j} = \dim \text{Gr}_i^P H^j(S^{[n]})$. Let the perverse Poincaré polynomial be $P_n(q, t) = \sum_{i,j} p_n^{i,j} q^i t^j$. Then in the \widetilde{A}_0 case, the generating series is*

$$\sum_{n=0}^{\infty} s^n P_n(q, t) = \prod_{m=1}^{\infty} \frac{(1 + s^m q^m t^{2m-1})^2}{(1 - s^m q^{m-1} t^{2m-2})(1 - s^m q^{m+1} t^{2m})}.$$

In the other four cases \widetilde{D}_4 , \widetilde{E}_6 , \widetilde{E}_7 and \widetilde{E}_8 , the generating series are

$$\sum_{n=0}^{\infty} s^n P_n(q, t) = \prod_{m=1}^{\infty} \frac{1}{(1 - s^m q^{m-1} t^{2m-2})(1 - s^m q^m t^{2m})^k (1 - s^m q^{m+1} t^{2m})}$$

where k is an integer defined in Proposition 5.4.

Using the explicit description of the corresponding character varieties for $n = 1$ in [9], we prove the full version of the $P = W$ conjecture for $S \rightarrow \mathbb{A}^1$ in each of our five cases. For $n \geq 2$, little is known about the corresponding character varieties. However, there are conjectures concerning the shape of the filtration W on $H^*(M_B)$ in [11]. Mathematica computations show that for small n , the perverse numbers obtained in our theorem match the conjectural mixed Hodge numbers in [11].

2 Functoriality of the perverse filtrations

In this chapter we prove that external tensor products, symmetric products and alternating products of perverse sheaves are perverse. We also show we may describe the perverse filtrations for $f^n : X^n \rightarrow Y^n$ and $f^{(n)} : X^{(n)} \rightarrow Y^{(n)}$ in terms of the perverse filtration for $f : X \rightarrow Y$. We use terms “perversity”, “perverse decomposition”, “perverse filtration” under the convention defined in section 1.2.

2.1 External tensor product

Proposition 2.1. *Let $f_1 : X_1 \rightarrow Y_1$, $f_2 : X_2 \rightarrow Y_2$ be two proper morphisms between smooth quasi-projective varieties. Let $r(f)$ denote the defect of semismallness of f defined in section 1.2. Let \mathcal{F}_i and \mathcal{G}_j be suitable perverse sheaves and*

$$Rf_{1,*}\mathbb{Q}_{X_1}[\dim X_1 - r(f_1)] \cong \bigoplus_{i=0}^{2r(f_1)} \mathcal{F}_i[-i],$$

$$Rf_{2,*}\mathbb{Q}_{X_2}[\dim X_2 - r(f_2)] \cong \bigoplus_{j=0}^{2r(f_2)} \mathcal{G}_j[-j]$$

be the perverse decompositions for map f_1 , f_2 , respectively. Then

$$R(f_1 \times f_2)_*\mathbb{Q}_{X_1 \times X_2}[\dim X_1 \times X_2 - r(f_1 \times f_2)] \cong \bigoplus_{i,j} \mathcal{F}_i \boxtimes \mathcal{G}_j[-i-j]$$

is a perverse decomposition for the proper map $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$. In particular, for $\alpha_1 \in H^*(X_1)$, $\alpha_2 \in H^*(X_2)$, we have $\mathbf{p}(\alpha_1 \otimes \alpha_2) = \mathbf{p}(\alpha_1) + \mathbf{p}(\alpha_2)$, where $\alpha_1 \otimes \alpha_2$ is viewed as a cohomology class in $H^*(X_1 \times X_2)$, and the perverse filtration is defined by the map $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$.

Proof. Note that f_1, f_2 and $f_1 \times f_2$ are all proper, so $Rf_* = Rf_!$. By the Künneth formula (see exercise II.18 of [13]), we have

$$\begin{aligned}
& R(f_1 \times f_2)_* \mathbb{Q}_{X_1 \times X_2}[\dim X_1 + \dim X_2 - r(f_1) - r(f_2)] \\
&= R(f_1 \times f_2)_* \mathbb{Q}_{X_1} \boxtimes \mathbb{Q}_{X_2}[\dim X_1 + \dim X_2 - r(f_1) - r(f_2)] \\
&= Rf_{1,*} \mathbb{Q}_{X_1}[\dim X_1 - r(f_1)] \boxtimes Rf_{2,*} \mathbb{Q}_{X_2}[\dim X_2 - r(f_2)] \\
&\cong \bigoplus_{i,j} \mathcal{F}_i \boxtimes \mathcal{G}_j[-i-j]
\end{aligned}$$

By Proposition 10.3.6 (i)(ii) of [13], the external tensor product $\mathcal{F}_i \boxtimes \mathcal{G}_j$ is perverse. Therefore this gives a perverse decomposition. To check the perversity is additive with respect to tensor product is basically by definition as follows. Let $p_1 = \mathbf{p}(\alpha_1), p_2 = \mathbf{p}(\alpha_2)$. If $\alpha_1 = 0$ or $\alpha_2 = 0$, then there is nothing to prove. Suppose none of them are 0, recall that by definition of geometric perversity, we have

$$\alpha_1 \in \mathbb{H} \left(\bigoplus_{i \leq p_1} \mathcal{F}_i[-i] \right), \alpha_2 \in \mathbb{H} \left(\bigoplus_{j \leq p_2} \mathcal{G}_j[-j] \right)$$

So

$$\begin{aligned}
\alpha_1 \otimes \alpha_2 &\in \mathbb{H} \left(\bigoplus_{i \leq p_1, j \leq p_2} \mathcal{F}_i \boxtimes \mathcal{G}_j[-i-j] \right) \\
&\subset \mathbb{H} \left(\bigoplus_{i+j \leq p_1+p_2} \mathcal{F}_i \boxtimes \mathcal{G}_j[-i-j] \right) \\
&= P_{p_1+p_2} H^*(X_1 \times X_2)
\end{aligned}$$

This shows that $\mathbf{p}(\alpha_1 \otimes \alpha_2) \leq \mathbf{p}(\alpha_1) + \mathbf{p}(\alpha_2)$. On the other hand, $\mathbf{p}(\alpha_k) = p_k$ means that $\alpha_k \neq 0 \in \mathrm{Gr}_{p_k}^P H^*(X_k)$. So

$$0 \neq \alpha_1 \otimes \alpha_2 \in \mathrm{Gr}_{p_1}^P H^*(X_1) \otimes \mathrm{Gr}_{p_2}^P H^*(X_2) \subset \mathrm{Gr}_{p_1+p_2}^P H^*(X_1 \times X_2)$$

This shows that $\mathbf{p}(\alpha_1 \otimes \alpha_2) = \mathbf{p}(\alpha_1) + \mathbf{p}(\alpha_2)$. □

Corollary 2.2. *Let $f : X \rightarrow Y$ be a proper map between smooth quasi-projective varieties. Then the perverse filtration for the product map $f^n : X^n \rightarrow Y^n$ can be described as*

$$P_p H^*(X^n; \mathbb{Q}) = \mathrm{Span} \{ \alpha_1 \otimes \cdots \otimes \alpha_n \mid \mathbf{p}(\alpha_1) + \cdots + \mathbf{p}(\alpha_n) \leq p \},$$

where $\alpha_i \in H^*(X; \mathbb{Q})$ for $i = 1, \dots, n$.

2.2 Symmetric and alternating product

In this section, we give an explicit description of the \mathfrak{S}_n -action on the n -fold external tensor product of a bounded complex, where \mathfrak{S}_n is the symmetric group of n elements. We use the action to define the symmetric product in derived category of constructible sheaves, and show that the symmetric product of a perverse sheaf is still perverse. Our method is similar to the one in [16]. Let $X^{(n)} = X^n / \mathfrak{S}_n$ denote the n -th symmetric product of X .

Definition 2.3. Let K_i^\bullet be a bounded complex of constructible sheaves on a complex quasi-projective variety X_i , for $1 \leq i \leq n$. Then the n -fold external tensor product $\boxtimes_{i=1}^n K_i^\bullet$ on $\prod_{i=1}^n X_i$ is defined as follows.

1. The j -th component is $\bigoplus_{\sum k_i=j} \boxtimes_{i=1}^n K_i^{k_i}$.
2. The differential is $\sum_{i=1}^n (-1)^{k_1+\dots+k_{i-1}} d_i$ on the summand $\boxtimes_{i=1}^n K_i^{k_i}$, where d_i is induced by the differential of K_i .

Definition 2.4. Let K_i^\bullet and X_i as above. Then there is a natural \mathfrak{S}_n -action on $\boxtimes_{i=1}^n K_i^\bullet$ by:

$$\sigma^\# : \boxtimes_{i=1}^n K_i^\bullet \xrightarrow{\sim} \sigma_* (\boxtimes_{i=1}^n K_{\sigma(i)}^\bullet)$$

which is defined, for $m_i \in K_i^{p_i}$, by

$$\boxtimes_{i=1}^n m_i \mapsto (-1)^{\nu(\sigma, p)} \sigma_* (\boxtimes_{i=1}^n m_{\sigma(i)})$$

where $\nu(\sigma, p) = \sum_{i < j, \sigma(j) < \sigma(i)} p_i p_j$.

Definition 2.5. Let X be a complex quasi-projective variety. Let $q : X^n \rightarrow X^{(n)}$ be the quotient map. For a bounded complex of constructible sheaves K on X , we define the symmetric product and alternating product as

$$\begin{aligned} K^{(n)} &= (Rq_* K^{\boxtimes n})^{\mathfrak{S}_n} \\ K^{\{n\}} &= (Rq_* K^{\boxtimes n})^{\text{sign}-\mathfrak{S}_n} \end{aligned}$$

where

$$(-)^{\mathfrak{S}_n} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} Rq_*(\sigma^\#)$$

is the symmetrizing projector and

$$(-)^{\text{sign}-\mathfrak{S}_n} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\text{sign}(\sigma)} Rq_*(\sigma^\#)$$

is the alternating projector. Here we use the fact that \mathfrak{S}_n acts trivially on $X^{(n)}$.

Remark 2.6. By [15] (1.1), we have the following canonical isomorphisms.

$$H^*(X^{(n)}, K^{(n)}) = H^*(X^n, K^{\boxtimes n})^{\mathfrak{S}_n} = \bigoplus_{i+j=n} \text{Sym}^i H^{\text{even}}(X, K) \otimes \bigwedge^j H^{\text{odd}}(X, K)$$

$$H^*(X^{(n)}, K^{\{n\}}) = H^*(X^n, K^{\boxtimes n})^{\text{sign-}\mathfrak{S}_n} = \bigoplus_{i+j=n} \bigwedge^i H^{\text{even}}(X, K) \otimes \text{Sym}^j H^{\text{odd}}(X, K)$$

Furthermore, we have

$$(K[a])^{(n)} = \begin{cases} K^{(n)}[na] & \text{if } a \text{ is even.} \\ K^{\{n\}}[na] & \text{if } a \text{ is odd.} \end{cases}$$

Proposition 2.7. *Let \mathcal{P} be a perverse sheaf on X . Let $q : X^n \rightarrow X^{(n)}$ be the quotient map. Then $\mathcal{P}^{(n)}$ and $\mathcal{P}^{\{n\}}$ are perverse sheaves on $X^{(n)}$.*

Proof. By Proposition 2.1, $\mathcal{P}^{\boxtimes n}$ is perverse. Since the map $q : X^n \rightarrow X^{(n)}$ is finite, $Rq_*K^{\boxtimes n}$ is perverse due to Corollaire 2.2.6 (i) in [1]. It suffices to prove that the invariant part and the alternating part under the \mathfrak{S}_n -action are both perverse. By the definition of the projectors, we have

$$\begin{aligned} (Rq_*\mathcal{P}^{\boxtimes n})^{\mathfrak{S}_n} &\rightarrow Rq_*\mathcal{P}^{\boxtimes n} \rightarrow (Rq_*\mathcal{P}^{\boxtimes n})^{\mathfrak{S}_n} \\ (Rq_*\mathcal{P}^{\boxtimes n})^{\text{sign-}\mathfrak{S}_n} &\rightarrow Rq_*\mathcal{P}^{\boxtimes n} \rightarrow (Rq_*\mathcal{P}^{\boxtimes n})^{\text{sign-}\mathfrak{S}_n} \end{aligned}$$

where both compositions are the identity. This means that $(Rq_*\mathcal{P}^{\boxtimes n})^{\mathfrak{S}_n}$ and $(Rq_*\mathcal{P}^{\boxtimes n})^{\text{sign-}\mathfrak{S}_n}$ are both direct summands of $Rq_*\mathcal{P}^{\boxtimes n}$ in the bounded derived category of constructible sheaves. The proposition holds due to the following lemma. \square

Lemma 2.8. *Let \mathcal{P} be a perverse sheaf on X . Suppose that $\mathcal{P} = K \oplus K'$ holds in $D_c^b(X)$, the bounded derived category of constructible sheaves. Then K is perverse.*

Proof. The cohomology sheaf satisfies $\mathcal{H}^i\mathcal{P} = \mathcal{H}^iK \oplus \mathcal{H}^iK'$. Therefore $\text{Supp } \mathcal{H}^iK \subset \text{Supp } \mathcal{H}^i\mathcal{P}$, and thus $\dim \text{Supp } \mathcal{H}^iK \leq \dim \text{Supp } \mathcal{H}^i\mathcal{P} \leq -i$. This proves the support condition (4.0.1') in [1]. Note that $\mathcal{P}^\vee = K^\vee \oplus (K')^\vee$, the cosupport condition follows similarly. \square

2.3 Perverse filtration of symmetric products

In this section, we show that the perverse filtration for a symmetric product a morphism $f^{(n)} : X^{(n)} \rightarrow Y^{(n)}$ is compatible with the perverse filtration of the corresponding cartesian product $f^n : X^n \rightarrow Y^n$. We also use the symmetric product and the alternating product for perverse sheaves to give a perverse decomposition for the symmetric product of maps.

Lemma 2.9. *Let X be a smooth quasi projective variety. Let $q : X^n \rightarrow X^{(n)}$. Let $K_i \in D_c^b(X)$, $i = 1, \dots, n$. Then \mathfrak{S}_n acts on*

$$\tilde{K} = \bigoplus_{\sigma \in \mathfrak{S}_n} K_{\sigma(1)} \boxtimes \dots \boxtimes K_{\sigma(n)}$$

as an endomorphism. Furthermore, $(Rp_ \tilde{K})^{\mathfrak{S}_n} \cong Rp_*(K_1 \boxtimes \dots \boxtimes K_n)$. More generally, let $\mathbf{k} = (k_1, \dots, k_m)$ with $k_1 + \dots + k_m = n$. Let $\mathfrak{S}_{\mathbf{k}} = \{\sigma : [n] \rightarrow [m] : |f^{-1}(i)| = k_i\}$, where $[n]$ denotes the set $\{1, \dots, n\}$. Let $q_{\mathbf{k}} : X^{(k_1)} \times \dots \times X^{(k_m)} \rightarrow X^{(n)}$. Then \mathfrak{S}_n acts on*

$$\tilde{K}_{\mathbf{k}} := \bigoplus_{\sigma \in \mathfrak{S}_{\mathbf{k}}} K_{\sigma(1)} \boxtimes \dots \boxtimes K_{\sigma(n)},$$

and we have

$$(Rq_* \tilde{K}_{\mathbf{k}})^{\mathfrak{S}_n} \cong Rq_{\mathbf{k},*}(K_1^{(k_1)} \boxtimes \dots \boxtimes K_m^{(k_m)}).$$

Similarly, for the alternating part we have

$$(Rq_* \tilde{K}_{\mathbf{k}})^{\text{sign-}\mathfrak{S}_n} \cong Rq_{\mathbf{k},*}(K_1^{\{k_1\}} \boxtimes \dots \boxtimes K_m^{\{k_m\}}).$$

Proof. Note that \mathfrak{S}_n acts on \tilde{K} by permuting the direct summands (up to sign). The invariant part of the push-forward is determined by any one of its summands. \square

Proposition 2.10. *Let X be a smooth quasi-projective variety. Let $q : X^n \rightarrow X^{(n)}$. Let $K = \bigoplus_{i=1}^m K_i \in D_c^b(X)$. Then we have the expansion*

$$K^{(n)} \cong \bigoplus_{\mathbf{k}} Rq_{\mathbf{k},*}(K_1^{(k_1)} \boxtimes \dots \boxtimes K_n^{(k_n)}).$$

and

$$K^{\{n\}} \cong \bigoplus_{\mathbf{k}} Rq_{\mathbf{k},*}(K_1^{\{k_1\}} \boxtimes \dots \boxtimes K_n^{\{k_n\}}).$$

Proof. By Lemma 2.9, we have

$$\begin{aligned}
K^{(n)} &= (Rq_* K^{\boxtimes n})^{\mathfrak{S}_n} \\
&= \left(Rq_* \bigoplus_{1 \leq i_1, \dots, i_n \leq m} K_{i_1} \boxtimes \dots \boxtimes K_{i_n} \right)^{\mathfrak{S}_n} \\
&= \left(Rq_* \bigoplus_{\mathbf{k}} \tilde{K}_{\mathbf{k}} \right)^{\mathfrak{S}_n} \\
&= \bigoplus_{\mathbf{k}} \left(Rq_* \tilde{K}_{\mathbf{k}} \right)^{\mathfrak{S}_n} \\
&= \bigoplus_{\mathbf{k}} Rq_{\mathbf{k},*} \left(K_1^{(k_1)} \boxtimes \dots \boxtimes K_m^{(k_m)} \right)
\end{aligned}$$

□

Lemma 2.11. *Let $f : X \rightarrow Y$ be a proper morphism between smooth quasi-projective varieties. We have the following commutative diagram*

$$\begin{array}{ccc}
X^n & \xrightarrow{q} & X^{(n)} \\
\downarrow f^n & & \downarrow f^{(n)} \\
Y^n & \xrightarrow{q} & Y^{(n)}
\end{array}$$

Let $K \in D_c^b(X)$, then $Rf_*^{(n)} K^{(n)} \cong (Rf_* K)^{(n)}$.

Proof.

$$\begin{aligned}
(Rf_* K)^{(n)} &\cong (Rq_*(Rf_* K)^{\boxtimes n})^{\mathfrak{S}_n} \cong (Rq_* Rf_* K^{\boxtimes n})^{\mathfrak{S}_n} \\
&\cong \left(Rf_*^{(n)} Rq_* K^{\boxtimes n} \right)^{\mathfrak{S}_n} \cong Rf_*^{(n)} (Rq_* K^{\boxtimes n})^{\mathfrak{S}_n} \cong Rf_*^{(n)} K^{(n)}
\end{aligned}$$

□

Proposition 2.12. *Let $f : X \rightarrow Y$ be a proper morphism between smooth quasi-projective varieties. Let*

$$Rf_* \mathbb{Q}_X[\dim X - r(f)] \cong \bigoplus_{i=0}^{2r(f)} \mathcal{P}_i[-i]$$

be the perverse decomposition, where \mathcal{P}_i are perverse sheaves on Y . Then the perverse decomposition of the map $f^{(n)} : X^{(n)} \rightarrow Y^{(n)}$ is given as follows. The formula is slightly different

depending on the parity of $\dim X - r(f)$. When $\dim X - r(f)$ is even, then

$$\begin{aligned} & Rf_*^{(n)} \mathbb{Q}_{X^{(n)}}[n(\dim X - r(f))] \\ \cong & \bigoplus_{\mathbf{k}} Rq_{\mathbf{k},*} \left(\mathcal{P}_0^{(k_0)} \boxtimes (\mathcal{P}_1[-1])^{(k_1)} \boxtimes \cdots \boxtimes (\mathcal{P}_{2r(f)}[-2r(f)])^{(k_{2r(f)})} \right) \\ \cong & \bigoplus_{\mathbf{k}} Rq_{\mathbf{k},*} \left(\mathcal{P}_0^{(k_0)} \boxtimes \mathcal{P}_1^{\{k_1\}} \boxtimes \cdots \boxtimes \mathcal{P}_{2r(f)}^{(k_{2r(f)})} \right) \left[- \sum_{i=0}^{2r(f)} ik_i \right]. \end{aligned}$$

When $\dim X - r(f)$ is odd, then

$$\begin{aligned} & Rf_*^{(n)} \mathbb{Q}_{X^{(n)}}[n(\dim X - r(f))] \\ \cong & \bigoplus_{\mathbf{k}} Rq_{\mathbf{k},*} \left(\mathcal{P}_0^{\{k_0\}} \boxtimes (\mathcal{P}_1[-1])^{\{k_1\}} \boxtimes \cdots \boxtimes (\mathcal{P}_{2r(f)}[-2r(f)])^{\{k_{2r(f)}\}} \right) \\ \cong & \bigoplus_{\mathbf{k}} Rq_{\mathbf{k},*} \left(\mathcal{P}_0^{\{k_0\}} \boxtimes \mathcal{P}_1^{(k_1)} \boxtimes \cdots \boxtimes \mathcal{P}_{2r(f)}^{\{k_{2r(f)}\}} \right) \left[- \sum_{i=0}^{2r(f)} ik_i \right]. \end{aligned}$$

Proof. By the canonical isomorphism $(\mathbb{Q}_X)^{(n)} = \mathbb{Q}_{X^{(n)}}$, Remark 2.6 and Lemma 2.11, we have

$$Rf_*^{(n)} \mathbb{Q}_{X^{(n)}}[n(\dim X - r(f))] = \begin{cases} (Rf_* \mathbb{Q}_X[\dim X - r(f)])^{(n)} & \text{if } \dim X - r(f) \text{ is even.} \\ (Rf_* \mathbb{Q}_X[\dim X - r(f)])^{\{n\}} & \text{if } \dim X - r(f) \text{ is odd.} \end{cases}$$

Then we use Proposition 2.10 to obtain the isomorphism. Using Proposition 2.1, Proposition 2.7 and the fact that the projection $q_{\mathbf{k}}$ is finite, we know this isomorphism is indeed a perverse decomposition. \square

Although the perverse decomposition for the symmetric product is somewhat complicated, the perverse filtration is much simpler. It is compatible with the one for the cartesian product as one may expect. To see this, we need the following lemma.

Lemma 2.13. *Let $f : X \rightarrow Y$ as before. Then*

$${}^p\tau_{\leq p} \left((Rf_* \mathbb{Q}_X)^{(n)} \right) = \left(Rq_* \left({}^p\tau_{\leq p} (Rf_* \mathbb{Q}_X)^{\boxtimes n} \right) \right)^{\mathfrak{S}_n}$$

Proof. Note that the \mathfrak{S}_n -invariant part is a direct summand, so it commutes with the functor ${}^p\tau_{\leq p}$. Furthermore, the quotient map q is finite, hence Rq_* is t -exact. \square

Proposition 2.14. *Under the isomorphism*

$$H^*(X^{(n)}) = (H^*(X^n))^{\mathfrak{S}_n},$$

the perverse filtration can be identified as

$$P_p H^*(X^{(n)}) = (P_p H^*(X^n))^{\mathfrak{S}_n},$$

where the perversity in parentheses is taken with respect to $f^n : X^n \rightarrow Y^n$.

Proof. By Lemma 2.11 and Lemma 2.13, we have

$$\begin{aligned} {}^p\tau_{\leq p} Rf_*^{(n)} \mathbb{Q}_{X^{(n)}} &= {}^p\tau_{\leq p} (Rf_* \mathbb{Q}_X)^{(n)} \\ &= (Rq_* {}^p\tau_{\leq p} (Rf_* \mathbb{Q}_X)^{\boxtimes n})^{\mathfrak{S}_n} \end{aligned}$$

After taking cohomology, we have

$$\begin{aligned} P_p H^*(X^{(n)}) &= \mathbb{H} \left(Y^{(n)}, {}^p\tau_{\leq p} Rf_*^{(n)} \mathbb{Q}_{X^{(n)}} \right) \\ &= (P_p H^*(X^n))^{\mathfrak{S}_n} \end{aligned}$$

So the result follows. □

3 Perversity of the diagonal

We will prove a technical result about the diagonal embedding, which is true for any algebraic variety. The result is crucial in the proof of the multiplicativity of the perverse filtration for Hilbert schemes of surfaces. In fact, by using Lehn's description of ring structure of Hilbert scheme of surfaces with numerically trivial canonical bundle, the perversity estimation of the diagonals is equivalent to the multiplicativity of perversity filtration for the Hilbert schemes.

3.1 Filtered basis for cohomology groups

We choose and fix a basis for the cohomology group with the following properties, which is crucial in the perversity estimation of the diagonal embedding.

Proposition 3.1. *Let $f : X \rightarrow Y$ be a morphism between projective varieties. Let $k(p, d) = \dim \text{Gr}_p^P H^d(X)$. There exists an \mathbb{Q} -basis*

$$B = \{\beta_{p,i}^d \mid 0 \leq d \leq 2 \dim X, 0 \leq p \leq 2r(f), 1 \leq i \leq k(p, d)\} \subset H^*(X)$$

with the following properties:

1. $\beta_{p,i}^d \in P_p H^d(X)$. $\{\overline{\beta_{p,1}^d}, \dots, \overline{\beta_{p,k(p,d)}^d}\}$ is a basis of $\mathrm{Gr}_p^P H^d(X)$, where $\overline{\beta_{p,i}^d}$ is the image of $\beta_{p,i}^d$ under the natural quotient map $P_p H^d(X) \rightarrow \mathrm{Gr}_p^P H^d(X)$.
2. The basis B is signed orthonormal in the following sense.

$$\langle \beta_{p,i}^d, \beta_{p',j}^{d'} \rangle = \begin{cases} \pm 1 & d + d' = 2 \dim X, \ p + p' = 2r(f) \text{ and } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

where -1 can only appear when $d = d' = \dim X$, $p = p' = r(f)$ and $i = j$.

In particular, if $A = \{\alpha_{p,i}^d\}$ is the dual basis of B with respect to the Poincaré pairing, then we have

$$\mathfrak{p}(\alpha_{p,i}^d) + \mathfrak{p}(\beta_{p,i}^d) = 2r(f)$$

To prove Proposition 3.1, we need the following two results.

Lemma 3.2 ([3] Version 1, Lemma 2.9.1). *Let $f : X \rightarrow Y$ be a morphism of algebraic varieties. The Poincaré pairing*

$$P_p H^d(X) \times P_{p'} H^{2 \dim X - d}(X) \rightarrow \mathbb{Q}$$

is trivial for $p + p' < 2r(f)$.

Proof. Denote by $D_c^b(Y)$ the bounded derived category of constructible sheaf on Y . Let $\epsilon : Rf_* \mathbb{Q}_X[n] \rightarrow \mathcal{D}(Rf_* \mathbb{Q}_X[n])$ be the duality isomorphism. For every d , the map ϵ defines the non-degenerate Poincaré pairing

$$\int_X : H^d(X) \times H^{2 \dim X - d}(X) \rightarrow \mathbb{Q}.$$

So to prove the pairing is trivial, it suffices to prove that the following composition is 0:

$${}^p\tau_{\leq p-r(f)} Rf_* \mathbb{Q}_X[n] \rightarrow Rf_* \mathbb{Q}_X[n] \xrightarrow{\epsilon} \mathcal{D}(Rf_* \mathbb{Q}_X[n]) \rightarrow \mathcal{D}({}^p\tau_{\leq p'-r(f)} Rf_* \mathbb{Q}_X[n]).$$

By our choice of geometric perversity, the dualizing functor \mathcal{D} satisfies

$$\mathcal{D}\left(D_c^b(Y)^{\leq p'-r(f)}\right) \subset D_c^b(Y)^{\geq r(f)-p'}.$$

By the axioms of a t -structure, $\mathrm{Hom}(D_c^b(Y)^{\leq p-r(f)}, D_c^b(Y)^{\geq r(f)-p'}) = 0$, since $p - r(f) < r(f) - p'$. So the composition is 0. \square

Lemma 3.3. *The pairing induced by the Poincaré pairing*

$$\mathrm{Gr}_p^P H^d(X) \times \mathrm{Gr}_{2r(f)-p}^P H^{2n-d}(X) \rightarrow \mathbb{Q}$$

is non-degenerate.

Proof. By Lemma 3.2, the pairing is well-defined. The non-degeneracy is due to Theorem 2.1.4 and Corollary 2.1.8 of [3] \square

Proof of Proposition 3.1. Denote $B_p^d = \{\beta \in B \mid \mathfrak{p}(\beta) = p, \beta \in H^d(X)\}$. We construct B_p^d using a Gram-Schmidt type argument. We perform the construction inductively in the lexicographical order of pairs (p, d) .

Induction base: for $(p, d) \prec (r(f), \dim X)$, pick any basis of $\mathrm{Gr}_p^P H^d(X)$, and lift them to get B_p^d . For $(p, d) = (r(f), \dim X)$, by Lemma 3.3, the self-intersection form is nondegenerate, so we may pick a basis such that the intersection matrix is diagonal and has only ± 1 on the diagonal. Denote any lift of this basis by $B_{r(f)}^{\dim X}$.

We are now going to find $B_{p,d} = \{\beta_{p,1}^d, \dots, \beta_{p,k(p,d)}^d\}$, assuming that all cases below (p, d) are done. To simplify notation, we let $e = 2 \dim X - d$ and $q = 2r(f) - p$. Note that $(q, e) \prec (p, d)$. By Lemma 3.2, the pairing $\mathrm{Gr}_p^P H^d(X) \times \mathrm{Gr}_q^P H^e(X)$ is non-degenerate, so we may pick a basis $\widetilde{B}_p^d = \{\widetilde{\beta}_{p,1}^d, \dots, \widetilde{\beta}_{p,k(p,d)}^d\}$ such that the matrix of this bilinear pairing is the identity matrix with respect to the bases B_q^e and \widetilde{B}_p^d . Modify \widetilde{B}_p^d by setting

$$\begin{pmatrix} \beta_{p,1}^d \\ \dots \\ \beta_{p,k(p,d)}^d \end{pmatrix} = \begin{pmatrix} \widetilde{\beta}_{p,1}^d \\ \dots \\ \widetilde{\beta}_{p,k(p,d)}^d \end{pmatrix} + \sum_{i=q+1}^{p-1} A_i \begin{pmatrix} \beta_{i,1}^d \\ \dots \\ \beta_{i,k(i,d)}^d \end{pmatrix}, \quad (3.4)$$

where the A_i are $k(p, d) \times k(i, d)$ matrices of rational numbers to be determined. The condition that the A_i need to satisfy is slightly different when $d < \dim X$, $d > \dim X$ and $d = \dim X$.

1. $d < \dim X$. For degree reasons, it suffices to require the orthogonality between B_p^d and degree d basis which precedes (p, d) in the lexicographical order, namely B_1^e, \dots, B_{p-1}^e . If we denote the Poincaré pairing by regular multiplication, then the condition can be written in matrix notation as

$$\begin{pmatrix} \beta_{p,1}^d \\ \dots \\ \beta_{p,k(p,d)}^d \end{pmatrix} \begin{pmatrix} \beta_{j,1}^e & \dots & \beta_{j,k(j,e)}^e \end{pmatrix} = 0,$$

for $j = 0, \dots, \hat{q}, \dots, i - 1$, and

$$\begin{pmatrix} \beta_{p,1}^d \\ \dots \\ \beta_{p,k(p,d)}^d \end{pmatrix} \begin{pmatrix} \beta_{q,1}^e & \dots & \beta_{q,k(q,e)}^e \end{pmatrix} = I_{k(p,d)},$$

where I denotes the identity matrix. Plugging in (3.4), we have

$$\begin{pmatrix} \widetilde{\beta}_{p,1}^d \\ \dots \\ \widetilde{\beta}_{p,k(p,d)}^d \end{pmatrix} \begin{pmatrix} \beta_{j,1}^e & \dots & \beta_{j,k(j,e)}^e \end{pmatrix} + \sum_{i=q+1}^{p-1} A_i \begin{pmatrix} \beta_{i,1}^d \\ \dots \\ \beta_{i,k(i,d)}^d \end{pmatrix} \begin{pmatrix} \beta_{j,1}^e & \dots & \beta_{j,k(j,e)}^e \end{pmatrix} = 0$$

for $j = 0, \dots, \hat{q}, \dots, p - 1$, and

$$\begin{pmatrix} \widetilde{\beta}_{p,1}^d \\ \dots \\ \widetilde{\beta}_{p,k(p,d)}^d \end{pmatrix} \begin{pmatrix} \beta_{q,1}^e & \dots & \beta_{q,k(q,e)}^e \end{pmatrix} + \sum_{i=q+1}^{p-1} A_i \begin{pmatrix} \beta_{i,1}^d \\ \dots \\ \beta_{i,k(i,d)}^d \end{pmatrix} \begin{pmatrix} \beta_{q,1}^e & \dots & \beta_{q,k(q,e)}^e \end{pmatrix} = I$$

The second condition is always satisfied by $q + i < 2r(f)$ and by Lemma 3.2. The first condition is true when $j < q$ for the same reason. When $q \leq j \leq p - 1$, by induction hypothesis, the first condition is reduced to

$$\begin{pmatrix} \widetilde{\beta}_{p,1}^d \\ \dots \\ \widetilde{\beta}_{p,k(p,d)}^d \end{pmatrix} \begin{pmatrix} \beta_{j,1}^e & \dots & \beta_{j,k(j,e)}^e \end{pmatrix} + A_{2r(f)-j} = 0$$

This solves $A_{2r(f)-j}$. Note that $q + 1 \leq j \leq p - 1$, so $q + 1 \leq 2r(f) - j \leq p - 1$ (note that $p + q = 2r(f)$). That means that all A_i are determined.

2. $j > \dim X$. The only difference in this case is that B_p^e is already done, so we need one more condition to require B_p^d to be orthogonal to B_p^e . To make this work, the sum taken in (3.4) need to be from q to $p - 1$ instead of from $q + 1$ to $p - 1$. The computation is similar.
3. $j = \dim X$. In this case $\widetilde{B_p^{\dim X}}$ need to be modified to be orthogonal to itself. The condition to be satisfied is exactly the same as $j > \dim X$ case, the result is slightly different: the matrix A_q is different by a factor 2.

This completes the induction. In particular, the dual basis $\alpha_{p,i}^d = \pm\beta_{q,i}^e$, so $\mathbf{p}(\alpha_{p,i}^d) + \mathbf{p}(\beta_{q,i}^e) = 2r(f)$. \square

Remark 3.5. The assumption that X and Y are smooth varieties is not necessary. In fact the construction works for the intersection cohomology for singular varieties.

Remark 3.6. We point out an easy but important fact about B . The basis B is filtered in the sense that

$$P_p H^*(X) = \text{Span} \{ \beta \in B \mid \mathbf{p}(\beta) \leq p \}.$$

By the additivity of perversities with respect to tensor products, we have the following.

Corollary 3.7. *Let $f : X \rightarrow Y$ be a morphism between smooth projective varieties. Let $B = \{ \beta_1, \dots, \beta_k \}$ be the basis of $H^*(X)$ in Proposition 3.1. Then the set B^n defined by*

$$B^n := \{ \beta_{i_1} \otimes \dots \otimes \beta_{i_n} \mid 1 \leq i_1, \dots, i_n \leq k \}$$

is a basis of $H^(X^n)$. Furthermore, this basis is filtered with respect to the perverse filtration induced by map $f^n : X^n \rightarrow Y^n$.*

3.2 Perversity estimation of small diagonals

In this section, we study the perversity of the small diagonal of the cartesian self-product. This estimation is crucial to prove the multiplicativity of perverse filtration of Hilbert schemes.

Proposition 3.8. *Let $f : X \rightarrow Y$ be any morphism between smooth projective varieties. Suppose the perverse filtration for $f : X \rightarrow Y$ is multiplicative, i.e. for any two classes $\alpha_1, \alpha_2 \in H^*(X)$, we have $\mathbf{p}(\alpha_1 \cup \alpha_2) \leq \mathbf{p}(\alpha_1) + \mathbf{p}(\alpha_2)$. The small diagonal embedding $\Delta_n : X \rightarrow X^n$ induces a Gysin push-forward of cohomology*

$$\Delta_{n,*} : H^*(X) \rightarrow H^{*+2(n-1)\dim X}(X)$$

Then for any $\gamma \in H^(X)$, we have that*

$$\mathbf{p}(\Delta_{n,*}(\gamma)) \leq \mathbf{p}(\gamma) + 2(n-1)r(f),$$

where the perversity on the left side is defined by the map $f^n : X^n \rightarrow Y^n$ and the one on the right side is defined by $f : X \rightarrow Y$.

We need an easy fact to prove the proposition.

Lemma 3.9. *Let X be a compact smooth manifold. Let β_1, \dots, β_k be an additive \mathbb{Q} -basis of $H^*(X)$. Let $\alpha_1, \dots, \alpha_k$ be the dual basis with respect to the Poincaré pairing, namely $\langle \alpha_i, \beta_j \rangle = \delta_{ij}$. Then*

$$\Delta_{2,*}(\gamma) = \sum_{i=1}^k \alpha_i \otimes (\beta_i \cup \gamma)$$

Proof. Let $\text{pr}_1, \text{pr}_2 : X \times X \rightarrow X$ be the projection maps to the two factors. Any cohomology class $\Phi \in H^*(X \times X)$ induces a correspondence

$$\begin{aligned} [\Phi] : H^*(X) &\rightarrow H^*(X) \\ \xi &\mapsto \text{pr}_{2,*}(\text{pr}_1^*(\xi) \cup \Phi) \end{aligned}$$

Now the correspondence induced by left hand side is

$$\begin{aligned} [\Delta_{2,*}(\gamma)](\xi) &= \text{pr}_{2,*}(\text{pr}_1^*(\xi) \cup \Delta_{2,*}(\gamma)) \\ &= \text{pr}_{2,*}(\xi \otimes 1 \cup \Delta_{2,*}(\gamma)) \\ &= \text{pr}_{2,*} \Delta_{2,*}(\Delta_2^*(\xi \otimes 1) \cup \gamma) \\ &= \Delta_2^*(\xi \otimes 1) \cup \gamma \\ &= \xi \cup \gamma, \end{aligned}$$

where the second equality is due to the projection formula, and the third equality uses $\Delta_2 \circ \text{pr}_2 = \text{id}$. The correspondence on right hand side computes as

$$\begin{aligned} \left[\sum_{i=1}^k \alpha_i \otimes \beta_i \cup \gamma \right] (\beta_j) &= \sum_{i=1}^k \text{pr}_{2,*}(\beta_j \cup \alpha_i \otimes \beta_i \cup \gamma) \\ &= \text{pr}_{2,*}(\beta_j \cup \alpha_j \otimes \beta_j \cup \gamma) \\ &= \beta_j \cup \gamma \end{aligned}$$

Here we use the fact that the nontrivial push-forward takes place only when $\beta_j \cup \alpha_i$ is a cohomology class of top degree, and hence in this case $\beta_j \cup \alpha_i = \langle \beta_j, \alpha_i \rangle = \delta_{ij}$ by our choice of $\{\alpha_i\}$ and $\{\beta_i\}$. So there is only one non-zero pairing left in the summation. Now extending by linearity, we have

$$\left[\sum_{i=1}^k \alpha_i \otimes \beta_i \cup \gamma \right] (\xi) = \xi \cup \gamma.$$

So the lemma follows. □

Proof of Proposition 3.8. We use induction on n to prove the statement. Since $n = 1$ is trivial, we prove for $n = 2$ as induction basis.

Let $\{\beta_{p,i}^d\}, \{\alpha_{p,i}^d\}$ be the basis in Proposition 3.1. Then by Lemma 3.9 we have

$$\Delta_{2,*}(\gamma) = \sum_{p,d,i} \alpha_{p,i}^d \otimes (\beta_{p,i}^d \cup \gamma)$$

Now by Proposition 2.1, Proposition 3.1, and the hypothesis that the perverse filtration for $f : X \rightarrow Y$ is multiplicative, we have

$$\begin{aligned} \mathfrak{p}(\Delta_{2,*}(\alpha)) &\leq \max_{p,d,i} \mathfrak{p}(\alpha_{p,i}^d \otimes (\beta_{p,i}^d \cup \gamma)) \\ &\leq \max_{p,d,i} (\mathfrak{p}(\alpha_{p,i}^d) + \mathfrak{p}(\beta_{p,i}^d) + \mathfrak{p}(\gamma)) \\ &\leq 2r(f) + \mathfrak{p}(\gamma) \end{aligned}$$

For general n , Δ_n can be decomposed into the following two diagonal maps.

$$X \xrightarrow{\Delta_{n-1}} X^{n-1} \xrightarrow{\Delta_2 \times \text{Id}^{n-2}} X^n$$

Then by induction hypothesis, we have

$$\begin{aligned} \mathfrak{p}(\Delta_{n,*}(\gamma)) &\leq \mathfrak{p}(\Delta_{n-1,*}\gamma) + 2r(f) \\ &\leq \mathfrak{p}(\gamma) + 2(n-2)r(f) + 2r(f) \\ &= \mathfrak{p}(\gamma) + 2(n-1)r(f) \end{aligned}$$

□

4 Hilbert scheme of points on surfaces

In this section we produce a perverse decomposition for the Hilbert schemes of points on smooth surfaces in terms of a perverse decomposition for the fibered surface. We use Lehn's description of the ring structure of the cohomology of Hilbert schemes and the perversity estimation of the diagonal in self-cartesian product to prove the multiplicativity of the perverse filtration for Hilbert schemes.

4.1 Ring structure of Hilbert scheme of K3 surfaces

In this section we recall the notation, definition and results on the cup product on the cohomology ring of the Hilbert scheme of points for K3 surfaces. All results in this section are due to Lehn's paper [14].

Let S be a projective surface with numerically trivial canonical bundle. Let $A = H^*(S; \mathbb{Q})$

be the cohomology with \mathbb{Q} coefficients. Let $[n]$ denote the set $\{1, \dots, n\}$.

Definition 4.1 ([14] 2.1). Let I be a finite set of cardinality n . Define

$$A^I = \left(\bigoplus_{f: [n] \xrightarrow{\sim} I} A_{f(1)} \otimes \cdots \otimes A_{f(n)} \right) / \mathfrak{S}_n$$

Remark 4.2. In fact, A^I is isomorphic to $A^{|I|}$. This isomorphism is canonical once an isomorphism $\varphi : [n] \rightarrow I$ is fixed.

Definition 4.3 ([14] 2.1). Let $\varphi : I \rightarrow J$ be a surjective map between sets. Then φ induces a morphism $\varphi : S^J \rightarrow S^I$ by sending $(x_1, \dots, x_{|J|})$ to $(x_{\varphi(1)}, \dots, x_{\varphi(|I|)})$. Define φ_* and φ^* to be the push-forward and pull-back map associated with φ between the cohomology groups $H^*(S^I)$ and $H^*(S^J)$.

Remark 4.4. The pull-back map can be described explicitly as follows. First note that φ is a product of diagonal embedding map: the j -th copy of S in S^J is embedded diagonally in $S^{f^{-1}(j)}$. Pulling-back along the diagonal embedding is exactly the definition of cup product. So if we fix isomorphism $f : [n] \xrightarrow{\sim} I$, $g : [m] \xrightarrow{\sim} J$, we will have

$$\tilde{\varphi} : [n] \rightarrow I \rightarrow J \rightarrow [m]$$

Therefore

$$\begin{aligned} \tilde{\varphi}^* : A^n &\rightarrow A^m \\ a_1 \otimes \cdots \otimes a_n &\mapsto \bigotimes_{j=1}^m \bigcup_{i \in \tilde{\varphi}^{-1}(j)} a_i \end{aligned}$$

and

$$\varphi^* : A^I \rightarrow A^n \xrightarrow{\tilde{\varphi}^*} A^m \rightarrow A^J$$

It is easy to check that this is independent of choice of f and g .

Now we define the wreath product of A and \mathfrak{S}_n , which is used to describe the cohomology of Hilbert scheme of points on smooth surfaces. For a permutation $\sigma \in \mathfrak{S}_n$ and a partition $\nu = 1^{a_1} \cdots n^{a_n}$ of n , we say σ is of type ν if σ has exactly a_i i -cycles. For K a subgroup of \mathfrak{S}_n , and for a K -stable subset $E \subset [n]$, let $K \backslash E$ denote the set of orbits for the induced action of K on E .

Definition 4.5. For $\sigma, \tau \in \mathfrak{S}_n$, the graph defect $g(\sigma, \tau) : \langle \sigma, \tau \rangle \setminus [n] \rightarrow \mathbb{Q}$ is defined by

$$g(\sigma, \tau)(E) = \frac{1}{2}(|E| + 2 - |\langle \sigma \rangle \setminus E| - |\langle \tau \rangle \setminus E| - |\langle \sigma\tau \rangle \setminus E|).$$

In fact, $g(\sigma, \tau)$ is always non-negative integer.

Definition 4.6 ([14] 2.8). The wreath product of A and symmetric group \mathfrak{S}_n as follows.

$$A\{\mathfrak{S}_n\} := \bigoplus_{\sigma \in \mathfrak{S}_n} A^{\otimes \langle \sigma \rangle \setminus [n]}[-2|\sigma|] \cdot \sigma$$

\mathfrak{S}_n acts on $A\{\mathfrak{S}_n\}$ as follows: the action of $\tau \in \mathfrak{S}_n$ on $[n]$ induces a bijection

$$\begin{aligned} \sigma : \langle \sigma \rangle \setminus [n] &\rightarrow \langle \tau\sigma\tau^{-1} \rangle \setminus [n] \\ x &\mapsto \tau x \end{aligned}$$

for each σ and hence an isomorphism

$$\begin{aligned} \tilde{\tau} : A\{\mathfrak{S}_n\} &\rightarrow A\{\mathfrak{S}_n\} \\ a\sigma &\mapsto \tau^*(a)\tau\sigma\tau^{-1} \end{aligned}$$

Let

$$A^{[n]} := (A\{\mathfrak{S}_n\})^{\mathfrak{S}_n}$$

be the subspace of invariants.

Any inclusion $H \subset K$ of subgroups of \mathfrak{S}_n induces a surjection $H \setminus [n] \twoheadrightarrow K \setminus [n]$ of set of orbits and hence induces a pull-back map

$$f^{H,K} : A^{\otimes H \setminus [n]} \rightarrow A^{\otimes K \setminus [n]}$$

and a push-forward map

$$f_{K,H} : A^{\otimes K \setminus [n]} \rightarrow A^{\otimes H \setminus [n]}$$

Definition 4.7 ([14] 2.12). For $\sigma, \tau \in \mathfrak{S}_n$, define

$$\begin{aligned} m_{\sigma,\tau} : A^{\otimes \langle \sigma \rangle \setminus [n]} \otimes A^{\otimes \langle \tau \rangle \setminus [n]} &\rightarrow A^{\otimes \langle \sigma\tau \rangle \setminus [n]} \\ a \otimes b &\mapsto f_{\langle \sigma,\tau \rangle, \langle \sigma\tau \rangle}(f^{\langle \sigma \rangle, \langle \sigma,\tau \rangle}(a) \cdot f^{\langle \tau \rangle, \langle \sigma,\tau \rangle}(b) \cdot e^{g(\sigma,\tau)}) \end{aligned}$$

where e is the Euler class of S .

Proposition 4.8 ([14] Proposition 2.13). *The product $A\{\mathfrak{S}_n\} \times A\{\mathfrak{S}_n\} \rightarrow A\{\mathfrak{S}_n\}$ defined by*

$$a\sigma \cdot b\tau := m_{\sigma,\tau}(a \otimes b)\sigma\tau$$

is associative and \mathfrak{S}_n -equivariant. So it descends to a product on $A^{[n]}$.

Theorem 4.9 ([14] Theorem 3.2). *Let S be a smooth projective surface with numerically trivial canonical divisor. Then there is a canonical isomorphism of graded rings*

$$H^*(S; \mathbb{Q})^{[n]} \xrightarrow{\cong} H^*(S^{[n]}; \mathbb{Q})$$

4.2 Perverse filtration for Hilbert schemes of fibered surfaces

Partitions of n are denoted as $\nu = 1^{a_1} \cdots n^{a_n}$, where $\sum_{i=1}^n ia_i = n$. The length of a partition is denoted by $l(\nu) = \sum_{i=1}^n a_i$. Set $\mathfrak{S}_\nu := \mathfrak{S}_{a_1} \times \cdots \times \mathfrak{S}_{a_n}$. For a quasi-projective variety X , set $X^{(\nu)} := X^{l(\nu)}/\mathfrak{S}_\nu = X^{(a_1)} \times \cdots \times X^{(a_n)}$. For $K \in D_c^b(X)$, denote the multi-symmetric and multi-alternating external tensor product by

$$K^{(\nu)} = \boxtimes_{i=1}^n K^{(a_i)} \in D_c^b(X^{(\nu)}),$$

$$K^{\{\nu\}} = \boxtimes_{i=1}^n K^{\{a_i\}} \in D_c^b(X^{(\nu)}).$$

We still have

$$(K[a])^{(\nu)} = \begin{cases} K^{(\nu)}[l(\nu)a], & a \text{ is even,} \\ K^{\{\nu\}}[l(\nu)a], & a \text{ is odd.} \end{cases}$$

In fact, since the external tensor product is compatible with push-forward and perversity, all result in section 2.3 can be generalized to the multi-symmetric or multi-alternating context.

Let $f : S \rightarrow C$ be a proper map from a smooth quasi-projective surface to a smooth quasi-projective curve. Let $S^{[n]}$ denote the Hilbert scheme of n points on the surface S . Then we have the following diagram.

$$\begin{array}{ccccc} & & & & S^{[n]} \\ & & & & \downarrow \pi \\ S^{l(\nu)} & \xrightarrow{/\mathfrak{S}_\nu} & S^{(\nu)} & \xrightarrow{r_S^{(\nu)}} & S^{(n)} \\ f^{l(\nu)} \downarrow & & f^{(\nu)} \downarrow & & f^{(n)} \downarrow \\ C^{l(\nu)} & \xrightarrow{/\mathfrak{S}_\nu} & C^{(\nu)} & \xrightarrow{r_C^{(\nu)}} & C^{(n)} \end{array} \quad \left. \vphantom{\begin{array}{ccccc} S^{l(\nu)} & \xrightarrow{/\mathfrak{S}_\nu} & S^{(\nu)} & \xrightarrow{r_S^{(\nu)}} & S^{(n)} \\ f^{l(\nu)} \downarrow & & f^{(\nu)} \downarrow & & f^{(n)} \downarrow \\ C^{l(\nu)} & \xrightarrow{/\mathfrak{S}_\nu} & C^{(\nu)} & \xrightarrow{r_C^{(\nu)}} & C^{(n)} \end{array}} \right) h$$

To obtain a perverse decomposition for the map h , we need the following result:

Theorem 4.10 ([2] Theorem 4.1.1). *Let S be a quasi-projective algebraic surface. Then we have the decomposition theorem for the Hilbert-Chow morphism.*

$$R\pi_*\mathbb{Q}_{S^{[n]}}[2n] \cong \bigoplus_{\nu} Rr_{S,*}^{(\nu)}\mathbb{Q}_{S^{(\nu)}}[2l(\nu)]$$

The main result of this section is the following.

Proposition 4.11. *Let $f : S \rightarrow C$ be a proper map from a smooth algebraic surface to a smooth algebraic curve. Let*

$$Rf_*\mathbb{Q}_S[1] = \mathcal{P}_0 \oplus \mathcal{P}_1[-1] \oplus \mathcal{P}_2[-2]$$

be a perverse decomposition, where $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2$ are perverse sheaves on C . Then a perverse decomposition of the morphism $h : S^{[n]} \rightarrow C^{(n)}$ is given by:

$$\begin{aligned} Rh_*\mathbb{Q}_{S^{[n]}}[n] &\cong \bigoplus_{\nu=1^{a_1}\dots n^{a_n}} Rr_{C,*}^{(\nu)} \left((\mathcal{P}_0[0] \oplus \mathcal{P}_1[-1] \oplus \mathcal{P}_2[-2])^{l(\nu)} \right) [l(\nu) - n] \\ &\cong \bigoplus_{\nu=1^{a_1}\dots n^{a_n}} Rr_{C,*}^{(\nu)} \left(\bigoplus_{\mathbf{r}+\mathbf{s}+\mathbf{t}=\mathbf{a}} Rq_{\mathbf{r},\mathbf{s},\mathbf{t},*} \mathcal{P}_0^{\{\mathbf{r}\}} \boxtimes \mathcal{P}_1^{\{\mathbf{s}\}} \boxtimes \mathcal{P}_2^{\{\mathbf{t}\}} \right) [-C(\mathbf{r}, \mathbf{s}, \mathbf{t})] \end{aligned}$$

where $C(\mathbf{r}, \mathbf{s}, \mathbf{t}) = n - l(\nu) + \sum_{i=1}^n (s_i + 2t_i)$, $\mathbf{r} = (r_1, \dots, r_n)$, and similarly for \mathbf{s} and \mathbf{t} . Here $\mathbf{r} + \mathbf{s} + \mathbf{t} = \mathbf{a}$ means that $r_i + s_i + t_i = a_i$ holds for any i , and

$$q_{\mathbf{r},\mathbf{s},\mathbf{t}} : \prod_{i=1}^{l(\nu)} C^{(r_i)} \times C^{(s_i)} \times C^{(t_i)} \rightarrow \prod_{i=1}^{l(\nu)} C^{(a_i)}$$

is induced by the natural map $C^{(r_i)} \times C^{(s_i)} \times C^{(t_i)} \rightarrow C^{(a_i)}$.

Proof. The proof is formal.

$$\begin{aligned}
Rh_*\mathbb{Q}_{S^{[n]}}[n] &\cong Rf_*^{(n)}R\pi_*\mathbb{Q}_{S^{[n]}}[n] \\
&\cong Rf_*^{(n)}\left(\bigoplus_{\nu} Rr_{S,*}^{(\nu)}\mathbb{Q}_{S^{(\nu)}}[2l(\nu) - n]\right) \\
&\cong \bigoplus_{\nu} Rf_*^{(n)}Rr_{S,*}^{(\nu)}\mathbb{Q}_{S^{(\nu)}}[2l(\nu) - n] \\
&\cong \bigoplus_{\nu} Rr_{C,*}^{(\nu)}Rf_*^{(\nu)}\mathbb{Q}_{S^{(\nu)}}[2l(\nu) - n] \\
&\cong \bigoplus_{\nu} Rr_{C,*}^{(\nu)}(Rf_*\mathbb{Q}_S)^{(\nu)}[2l(\nu) - n] \\
&\cong \bigoplus_{\nu} Rr_{C,*}^{(\nu)}(Rf_*\mathbb{Q}_S[1])^{\{\nu\}}[l(\nu) - n] \\
&\cong \bigoplus_{\nu} Rr_{C,*}^{(\nu)}(\mathcal{P}_0[0] \oplus \mathcal{P}_1[-1] \oplus \mathcal{P}_2[-2])^{\{\nu\}}[l(\nu) - n] \\
&\cong \bigoplus_{\nu=1^{a_1}\dots n^{a_n}} Rr_{C,*}^{(\nu)}\left(\bigoplus_{\mathbf{r}+\mathbf{s}+\mathbf{t}=\mathbf{a}} Rq_{\mathbf{r},\mathbf{s},\mathbf{t},*}\mathcal{P}_0^{\{\mathbf{r}\}} \boxtimes \mathcal{P}_1^{\{\mathbf{s}\}} \boxtimes \mathcal{P}_2^{\{\mathbf{t}\}}\right)[-C(\mathbf{r}, \mathbf{s}, \mathbf{t})]
\end{aligned}$$

Here we use Lemma 2.11 and Proposition 2.12 and the fact that $\dim S - r(h) = 1$ is odd. By Proposition 2.1 and Proposition 2.7 and the fact that the map $q_{\mathbf{r},\mathbf{s},\mathbf{t}}$ is finite, all terms in the parenthesis are perverse. Note that $r_{C,*}^{(\nu)}$ is a closed embedding, so $Rr_{C,*}^{(\nu)}$ is t -exact, it preserves perversity. Therefore the above gives a perverse decomposition. \square

Remark 4.12. In the Proposition 4.11, we have symmetric product for odd perversity term and alternating products for even perversity terms. This counter-intuitive result is due to the fact that $\mathcal{P}_i[-i]$ are direct summands of $Rf_*\mathbb{Q}_S[1]$ rather than $Rf_*\mathbb{Q}_S$. See Proposition 2.12.

Corollary 4.13. *Under the isomorphism*

$$H^*(S^{[n]}) = \bigoplus_{\nu} (H^*(S^{l(\nu)}))^{\mathfrak{S}_{\nu}}[2l(\nu) - 2n],$$

the perverse filtration can be identified as

$$P_p H^*(S^{[n]}) = \bigoplus_{\nu} (P_{p+l(\nu)-n} H^*(S^{l(\nu)}))^{\mathfrak{S}_{\nu}}[2l(\nu) - 2n],$$

where the perversity in the parenthesis is taken with respect to $f^{l(\nu)} : S^{l(\nu)} \rightarrow C^{l(\nu)}$.

Proof. By t -exactness of $Rr_{C,*}^{(\nu)}$ and Proposition 4.11, we have

$$\begin{aligned} {}^p\tau_{\leq p}Rh_*\mathbb{Q}_{S^{[n]}}[n] &= \bigoplus_{\nu=1^{a_1}\dots n^{a_n}} Rr_{C,*}^{(\nu)} \left({}^p\tau_{\leq p+l(\nu)-n}(Rf_*\mathbb{Q}_S[1])^{\{l(\nu)\}} \right) [l(\nu) - n] \\ &= \bigoplus_{\nu=1^{a_1}\dots n^{a_n}} Rr_{C,*}^{(\nu)} \left(Rq_* {}^p\tau_{\leq p+l(\nu)-n}(Rf_*\mathbb{Q}_S[1])^{\boxtimes l(\nu)} \right)^{\text{sign-}\mathfrak{S}_n} [l(\nu) - n] \end{aligned}$$

where q is the quotient map denoted as $/\mathfrak{S}_\nu$ in the previous diagram and the last isomorphism is due to Corollary 2.13. After taking the cohomology, we have

$$\begin{aligned} P_p H^*(S^{[n]})[n] &= \mathbb{H} \left(C^{(n)}, {}^p\tau_{\leq p}Rh_*\mathbb{Q}_{S^{[n]}}[n] \right) \\ &= \bigoplus_{\nu=1^{a_1}\dots n^{a_n}} \mathbb{H} \left(C^{(\nu)}, \left(Rq_* {}^p\tau_{\leq p+l(\nu)-n}(Rf_*\mathbb{Q}_S[1])^{\boxtimes l(\nu)} \right)^{\text{sign-}\mathfrak{S}_n} \right) [l(\nu) - n] \\ &= \bigoplus_{\nu=1^{a_1}\dots n^{a_n}} \mathbb{H} \left(C^{l(\nu)}, {}^p\tau_{\leq p+l(\nu)-n}(Rf_*\mathbb{Q}_S[1])^{\boxtimes l(\nu)} \right)^{\text{sign-}\mathfrak{S}_n} [l(\nu) - n] \\ &= \bigoplus_{\nu=1^{a_1}\dots n^{a_n}} \left(P_{p+l(\nu)-n} H^* \left(C^{l(\nu)}, (Rf_*\mathbb{Q}_S[1])^{\boxtimes l(\nu)} \right) \right)^{\text{sign-}\mathfrak{S}_n} [l(\nu) - n] \\ &= \bigoplus_{\nu=1^{a_1}\dots n^{a_n}} \left(P_{p+l(\nu)-n} H^* \left(S^{l(\nu)}, (\mathbb{Q}_S[1])^{\boxtimes l(\nu)} \right) \right)^{\text{sign-}\mathfrak{S}_n} [l(\nu) - n] \\ &= \bigoplus_{\nu=1^{a_1}\dots n^{a_n}} \left(P_{p+l(\nu)-n} H^* \left(S^{l(\nu)}, \mathbb{Q}_S^{\boxtimes l(\nu)} \right) \right)^{\mathfrak{S}_n} [2l(\nu) - n] \\ &= \bigoplus_{\nu=1^{a_1}\dots n^{a_n}} \left(P_{p+l(\nu)-n} H^* (S^{l(\nu)}) \right)^{\mathfrak{S}_n} [2l(\nu) - n] \end{aligned}$$

So the result follows. \square

In fact, we may define an abstract perversity function on the wreath product $H^*(S)\{\mathfrak{S}_n\}$ which is easier to handle with. We will show that after restrict to the \mathfrak{S}_n -invariant part, it is the same as the one defined by the map $S^{[n]} \rightarrow C^{(n)}$.

Definition 4.14. Let $f : S \rightarrow C$ be a morphism from a smooth algebraic surface to a smooth algebraic curve. Let B be the basis of $H^*(S)$ obtained in Proposition 3.1. Then we have that

$$B\{\mathfrak{S}_n\} := \left\{ \bigotimes_{i=1}^n \bigotimes_{j=1}^{a_i} \alpha_{ij} \cdot \sigma \mid \alpha_{ij} \in B, \sigma \text{ is of type } 1^{a_1} \dots n^{a_n} \right\}$$

is a basis of $H^*(X)\{\mathfrak{S}_n\}$. We define an abstract perversity on $B\{\mathfrak{S}_n\}$ as

$$\mathfrak{p}_{abs} \left(\bigotimes_{i=1}^n \bigotimes_{j=1}^{a_i} \alpha_{ij} \cdot \sigma \right) = \sum_{i=1}^n \sum_{j=1}^{a_i} \mathfrak{p}(\alpha_{ij}) + \sum_{i=1}^n (i-1)a_i,$$

and extend by linearity in the sense that

$$P_p H^*(S)\{\mathfrak{S}_n\} = \text{Span}\{\beta \in B\{\mathfrak{S}_n\} \mid \mathfrak{p}_{abs}(\beta) \leq p\}.$$

In particular, the basis is filtered with respect to the abstract perverse filtration by definition.

Proposition 4.15. *The abstract perversity is invariant under the \mathfrak{S}_n -action. Furthermore, after restriction to $H^*(S^{[n]})$, it is the same as the perversity given by the morphism $h : S^{[n]} \rightarrow C^{(n)}$ in Corollary 4.13.*

Proof. Note that \mathfrak{S}_n acts on cohomology by permuting the factors, so the abstract perversity is invariant under \mathfrak{S}_n -action. By definition, we have

$$n - l(\nu) = \sum_{i=1}^n (i-1)a_i$$

and

$$\mathfrak{p}_{abs} \left(\bigotimes_{i=1}^n \bigotimes_{j=1}^{a_i} \alpha_{ij} \cdot \sigma \right) = p \text{ if and only if } \mathfrak{p}_{abs} \left(\bigotimes_{i=1}^n \bigotimes_{j=1}^{a_i} \alpha_{ij} \right) = p + n - l(\nu).$$

where $\nu = 1^{a_1} \cdots n^{a_n}$ be any partition of n . Comparing with Corollary 4.13, the abstract perversity is the same as the geometric perversity induced by the morphism h on the basis. Note that on both sides the bases are filtered with respect to perverse filtration, so the abstract perverse filtration coincides with the geometric perverse filtration. \square

4.3 Multiplicativity of the perverse filtration

Proposition 4.16. *Let $f : S \rightarrow C$ be a proper surjective morphism from a smooth algebraic surface to a smooth algebraic curve. Then the perverse filtration on $H^*(S)$ is multiplicative.*

Proof. The perverse decomposition of a proper map from a surface to curve can be found in [4] Theorem 3.2.3. Let $f : S \rightarrow C$ be a proper surjective map from a smooth surface to a smooth curve. Let $\hat{f} : \hat{S} \rightarrow \hat{C}$ be its smooth part. Let $j : \hat{C} \rightarrow C$ be the open embedding. Let $\hat{R}^i = R^i \hat{f}_* \mathbb{Q}_{\hat{S}}$. Then one has a non-canonical perverse decomposition of map f :

$$Rf_* \mathbb{Q}_S[1] \cong \left\{ j_* \hat{R}^0[1] \right\} \oplus \left\{ j_* \hat{R}^1[1] \oplus \bigoplus_{p \in C \setminus \hat{C}} \mathbb{Q}_p^{n_p-1} \right\} [-1] \oplus \left\{ j_* \hat{R}^2[1] \right\} [-2]$$

where n_p is the number of irreducible components of fiber over p . Note that the ordinary Leray filtration is multiplicative, and the only difference between ordinary Leray filtration

and the perverse filtration is that the classes corresponding to $\bigoplus_{p \in C \setminus \hat{C}} \mathbb{Q}_p^{n_p-1}$ are shifted from $R^2 f_* \mathbb{Q}_S[-2]$ to $\mathcal{P}_1[-1]$. Therefore, it suffices to check cup products with these classes. Furthermore, these classes are in perversity 1, the only possibility to violate the multiplicativity is that their cup with perversity 0 classes have perversity 2. However, they are fundamental classes of irreducible components of special fibers, and perversity 0 classes are pull-backs of classes on the curve. They don't meet if the pull-back class is not the surface itself, and cupping with fundamental class of the surface is the identity map. So in both cases, the multiplicativity is preserved. \square

Theorem 4.17. *Let $f : S \rightarrow C$ be a surjective morphism from a smooth projective surface with numerically trivial canonical bundle to a smooth projective curve. Then the perverse filtration of $H^*(S^{[n]}; \mathbb{Q})$ with respect to the morphism $h : S^{[n]} \rightarrow C^{(n)}$ is multiplicative, namely, we have*

$$P_p H^*(S^{[n]}; \mathbb{Q}) \cup P_{p'} H^*(S^{[n]}; \mathbb{Q}) \subset P_{p+p'} H^*(S^{[n]}; \mathbb{Q})$$

To prove this theorem, we need two lemmata. We use the notation introduced in section 4.1.

Lemma 4.18. *For surjective map between sets $\varphi : I \rightarrow J$, the pullback map $\varphi^* : H^*(S)^I \rightarrow H^*(S)^J$ does not increase perversity.*

Proof. By Corollary 3.7, the basis $B^{|I|}$ is filtered, so it suffices to compute the perversity of the pull-back of the elements in $B^{|I|}$. Pick an element $\alpha_1 \otimes \cdots \otimes \alpha_{|I|} \in B^{|I|}$, then

$$\varphi^*(\alpha_1 \otimes \cdots \otimes \alpha_{|I|}) = \bigotimes_{j=1}^{|J|} \bigcup_{i \in \varphi^{-1}(j)} \alpha_i$$

Note that by Proposition 2.1

$$\mathfrak{p}(\alpha_1 \otimes \cdots \otimes \alpha_{|I|}) = \sum_{j=1}^{|I|} \mathfrak{p}(\alpha_j)$$

where perversity on the left side is defined by map $S^{|I|} \rightarrow C^{|I|}$, and the perversity on the right side is defined by $S \rightarrow C$. Now by Proposition 2.1 and Lemma 4.16, we have that

$$\mathfrak{p} \left(\bigotimes_{j=1}^{|J|} \bigcup_{i \in \varphi^{-1}(j)} \alpha_i \right) \leq \sum_{j=1}^{|J|} \sum_{i \in \varphi^{-1}(j)} \mathfrak{p}(\alpha_i) = \sum_{j=1}^{|I|} \mathfrak{p}(\alpha_j),$$

where the perversity on the left side is defined by the map $S^{|J|} \rightarrow C^{|J|}$. \square

Lemma 4.19. *For any surjective map $\varphi : I \twoheadrightarrow J$, the push-forward map*

$$\varphi_* : H^*(S)^J \rightarrow H^*(S)^I$$

increases the perversity at most by $2(|I| - |J|)$.

Proof. Again, it suffices to prove for a basis element $\alpha_1 \otimes \cdots \otimes \alpha_{|J|} \in B^{|J|}$. Let $b_j = |\varphi^{-1}(j)|$. By definition,

$$\varphi_*(\alpha_1 \otimes \cdots \otimes \alpha_{|J|}) = \pm \bigotimes_{j=1}^{|J|} \Delta_{b_j, *}(\alpha_j).$$

By Proposition 3.8, we have

$$\begin{aligned} \mathfrak{p} \left(\pm \bigotimes_{j=1}^{|J|} \Delta_{b_j, *}(\alpha_j) \right) &\leq \sum_{j=1}^{|J|} \mathfrak{p} (\Delta_{b_j, *}(\alpha_j)) \\ &= \sum_{j=1}^{|J|} \mathfrak{p}(\alpha_j) + 2(b_j - 1) \\ &= \sum_{j=1}^{|J|} \mathfrak{p}(\alpha_j) + 2 \sum_{j=1}^{|J|} b_j - 2 \sum_{j=1}^{|J|} 1 \\ &= \sum_{j=1}^{|J|} \mathfrak{p}(\alpha_j) + 2(|I| - |J|), \end{aligned}$$

where the perversity on the left side is defined by the map $S^{|I|} \rightarrow C^{|I|}$, the perversities on the right side of the first line is defined by the map $S^{b_j} \rightarrow C^{b_j}$. \square

Proof of Theorem 4.17. By Proposition 4.15, it suffices to prove that the abstract perverse filtration defined on $H^*(S; \mathbb{Q})\{\mathfrak{S}_n\}$ is multiplicative. Furthermore, it suffices to prove the result for our basis, namely

$$\begin{aligned} &\mathfrak{p} \left(\bigotimes_{i=1}^n \bigotimes_{j=1}^{a_i} \alpha_{ij} \cdot \sigma \cup \bigotimes_{i=1}^n \bigotimes_{j=1}^{a'_i} \alpha'_{ij} \cdot \tau \right) \\ &\leq \sum_{i=1}^n \sum_{j=1}^{a_i} \mathfrak{p}(\alpha_{ij}) + \sum_{i=1}^n (i-1)a_i \\ &+ \sum_{i=1}^n \sum_{j=1}^{a'_i} \mathfrak{p}(\alpha'_{ij}) + \sum_{i=1}^n (i-1)a'_i, \end{aligned}$$

where α_{ij} and α'_{ij} run over basis B obtained in Proposition 3.1. Note that the cup product formula computes independently on each orbit of $\langle \sigma, \tau \rangle$ -action on $[n]$ individually. Let E be

an orbit of the action $\langle \sigma, \tau \rangle$ on $[n]$, i.e. $|\langle \sigma, \tau \rangle \setminus E| = 1$. The product is computed by

$$\begin{aligned} A^{\otimes \langle \sigma \rangle \setminus E} \cdot \sigma|_E \otimes A^{\otimes \langle \tau \rangle \setminus E} \cdot \tau|_E &\rightarrow A^{\otimes \langle \sigma \tau \rangle \setminus E} \cdot \sigma \tau|_E \\ a \cdot \sigma|_E \otimes a' \cdot \tau|_E &\mapsto f_{\langle \sigma, \tau \rangle, \langle \sigma \tau \rangle}(f^{\langle \sigma \rangle, \langle \sigma, \tau \rangle}(a) \cdot f^{\langle \tau \rangle, \langle \sigma, \tau \rangle}(a') \cdot e^{g(\sigma, \tau)}) \cdot \sigma \tau|_E \end{aligned}$$

for every E . Note that the Euler class e is of top degree, hence $e^g = 0$ for $g \geq 2$, so that it suffices to consider the following two cases.

1. $g(\sigma, \tau) = 0$. By Lemma 4.18 and Lemma 4.19, we have

$$\begin{aligned} &\mathbf{p} \left(f_{\langle \sigma, \tau \rangle, \langle \sigma \tau \rangle}(f^{\langle \sigma \rangle, \langle \sigma, \tau \rangle}(a) \cdot f^{\langle \tau \rangle, \langle \sigma, \tau \rangle}(a') \cdot e^{g(\sigma, \tau)}) \cdot \sigma \tau|_E \right) \\ &= \mathbf{p} \left(f_{\langle \sigma, \tau \rangle, \langle \sigma \tau \rangle}(f^{\langle \sigma \rangle, \langle \sigma, \tau \rangle}(a) \cdot f^{\langle \tau \rangle, \langle \sigma, \tau \rangle}(a')) \right) + |B| - |\langle \sigma \tau \rangle \setminus E| \\ &= \mathbf{p} \left(f^{\langle \sigma \rangle, \langle \sigma, \tau \rangle}(a) \cdot f^{\langle \tau \rangle, \langle \sigma, \tau \rangle}(a') \right) + 2(|\langle \sigma \tau \rangle \setminus E| - 1) + |E| - |\langle \sigma \tau \rangle \setminus E| \\ &= \mathbf{p}(a) + \mathbf{p}(a') + |E| + |\langle \sigma \tau \rangle \setminus E| - 2 \\ &= \mathbf{p}(a \cdot \sigma) - (|E| - |\langle \sigma \rangle \setminus E|) + \mathbf{p}(a' \cdot \tau) - (|E| - |\langle \tau \rangle \setminus E|) + |E| + |\langle \sigma \tau \rangle \setminus E| - 2 \\ &= \mathbf{p}(a \cdot \sigma) + \mathbf{p}(a' \cdot \tau) - 2g(\sigma, \tau) \\ &= \mathbf{p}(a \cdot \sigma) + \mathbf{p}(a' \cdot \tau) \end{aligned}$$

2. $g(\sigma, \tau) = 1$. Since e itself is already in top degree, so the only nonzero case arise for $a = a' = 1$. Then

$$\begin{aligned} &\mathbf{p} \left(f_{\langle \sigma, \tau \rangle, \langle \sigma \tau \rangle}(f^{\langle \sigma \rangle, \langle \sigma, \tau \rangle}(1) \cdot f^{\langle \tau \rangle, \langle \sigma, \tau \rangle}(1) \cdot e^{g(\sigma, \tau)}) \cdot \sigma \tau|_E \right) \\ &= \mathbf{p} \left(f_{\langle \sigma, \tau \rangle, \langle \sigma \tau \rangle}(e) \right) + |E| - |\langle \sigma \tau \rangle \setminus E| \\ &= 2 + 2(|\langle \sigma \tau \rangle \setminus E| - 1) + |E| - |\langle \sigma \tau \rangle \setminus E| \\ &= |E| + |\langle \sigma \tau \rangle \setminus E| \\ &= |E| - |\langle \sigma \rangle \setminus E| + |E| - |\langle \tau \rangle \setminus E| \\ &= \mathbf{p}(1 \cdot \sigma) + \mathbf{p}(1 \cdot \tau) \end{aligned}$$

The last but one equality is due to $g(\sigma, \tau) = 1$, which means $|E| = |\langle \sigma \rangle \setminus E| + |\langle \tau \rangle \setminus E| + |\langle \sigma \tau \rangle \setminus E|$.

□

An application of the theorem is the multiplicativity of perverse filtration for the elliptic fibration of Hilbert schemes of points on K3 surfaces.

Theorem 4.20. *Let S be an elliptic K3 surface and $f : S \rightarrow \mathbb{P}^1$ be the elliptic fibration. Then the perverse filtration on $H^*(S^{[n]})$ defined by the natural map $h : S^{[n]} \rightarrow \mathbb{P}^n$ is multiplicative.*

4.4 The Hilbert schemes of a surface $P = W$ package

Definition 4.21. A $P = W$ package is a 5-tuple (X_P, X_W, h, A, Ξ) where

1. X_P, X_W, A are smooth quasi-projective varieties. $h : X_P \rightarrow A$ is proper morphism. $\Xi : X_P \rightarrow X_W$ is a diffeomorphism.
2. $P_k H^*(X_P) = W_{2k} H^*(X_W) = W_{2k+1} H^*(X_W)$ for any k . Here perverse filtration is defined for map f , and the identity is viewed as pull-back via Ξ .

A homological $P = W$ package (X_P, X_W, h, A, Φ) is the same as a $P = W$ package except that the diffeomorphism Ξ is replaced by an isomorphism $\Phi : H^*(X_W) \xrightarrow{\sim} H^*(X_P)$.

Theorem 4.22. *If S_P and S_W are smooth surfaces and $(S_P, S_W, h, \mathbb{A}^1, \Phi)$ is a homological $P = W$ package. Then the Cartesian product $(S_P^n, S_W^n, h^n, \mathbb{A}^n, \Phi^n)$, the symmetric product $(S_P^{(n)}, S_W^{(n)}, h^{(n)}, \mathbb{A}^n, \Phi^{(n)})$ and the Hilbert scheme $(S_P^{[n]}, S_W^{[n]}, h^{[n]}, \mathbb{A}^n, \Phi^{[n]})$ are also homological $P = W$ packages, where $h^{[n]} : X_P^{[n]} \rightarrow X_P^{(n)} \rightarrow \mathbb{A}^n$ and*

$$\begin{aligned} \Phi^{[n]} : H^* \left(X_W^{[n]}; \mathbb{Q} \right) &= \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} \bigotimes_{i=1}^n H^* \left(X_W^{(a_i)}; \mathbb{Q} \right) [2n - 2l(\nu)] \\ &\xrightarrow{\oplus \Phi^{(\nu)}} \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} \bigotimes_{i=1}^n H^* \left(X_P^{(a_i)}; \mathbb{Q} \right) [2n - 2l(\nu)] = H^* \left(X_P^{[n]}; \mathbb{Q} \right) \end{aligned}$$

Proof. The proof is obtained by comparing the functoriality of the weight filtration for the mixed Hodge structure and the one for the perverse filtration.

Step 1. On one hand, by the Künneth formula for mixed Hodge structures,

$$W_w H^*(S_W^n; \mathbb{Q}) = \text{Span} \{ \alpha_1 \otimes \dots \otimes \alpha_n \mid \mathfrak{w}(\alpha_1) + \dots + \mathfrak{w}(\alpha_n) \leq w \},$$

where $\alpha_i \in H^*(S_W)$, and the function \mathfrak{w} denotes the weight of a cohomology class. On the other hand, by Corollary 2.2,

$$P_p H^*(S_P^n; \mathbb{Q}) = \text{Span} \{ \alpha'_1 \otimes \dots \otimes \alpha'_n \mid \mathfrak{p}(\alpha'_1) + \dots + \mathfrak{p}(\alpha'_n) \leq p \},$$

where $\alpha'_i \in H^*(S_P)$. By the hypothesis that $(S_P, S_W, h, \mathbb{A}^1, \Phi)$ is a homological $P = W$ package, $2\mathfrak{p}(\Phi\alpha_i) = \mathfrak{w}(\alpha_i)$. This implies that

$$W_{2k} H^*(S_W^n; \mathbb{Q}) = W_{2k+1} H^*(S_W^n; \mathbb{Q}) = P_k H^*(S_P^n; \mathbb{Q}).$$

So $(S_P^n, S_W^n, h^n, \mathbb{A}^n, \Phi^n)$ is a homological $P = W$ package.

Step 2. On one hand, the mixed Hodge structure is functorial with respect to finite group

quotient. So we have

$$W_w H^*(S_W^{(n)}; \mathbb{Q}) = (W_w H^*(S_W^n; \mathbb{Q}))^{\mathfrak{S}_n}.$$

On the other hand, by Proposition 2.14, we have

$$P_p H^*(S_P^{(n)}; \mathbb{Q}) = (P_p H^*(S_P^n; \mathbb{Q}))^{\mathfrak{S}_n}.$$

Then result in step 1 immediately implies

$$W_{2k} H^*(S_W^{(n)}; \mathbb{Q}) = W_{2k+1} H^*(S_W^{(n)}; \mathbb{Q}) = P_k H^*(S_P^{(n)}; \mathbb{Q}).$$

So $(S_P^{(n)}, S_W^{(n)}, h^{(n)}, \mathbb{A}^n, \Phi^{(n)})$ is a homological $P = W$ package.

Step 3. On one hand, Theorem 5.3.1 in [2] asserts that

$$H^*(S_W^{[n]}; \mathbb{Q})(n) \cong \bigoplus_{\nu} H^*(S_W^{(\nu)}; \mathbb{Q})[2l(\nu) - 2n](l(\nu))$$

is an isomorphism of mixed Hodge structures, so we have

$$W_w H^*(S_W^{[n]}; \mathbb{Q}) = \bigoplus_{\nu} W_{w+2l(\nu)-2n} H^*(S_W^{(\nu)}; \mathbb{Q})[2l(\nu) - 2n].$$

On the other hand, by Corollary 4.13, the perverse filtration for the map $h^{[n]} : S_P^{[n]} \rightarrow \mathbb{A}^n$ can be expressed as

$$\begin{aligned} P_p H^*(S_P^{[n]}; \mathbb{Q}) &= \bigoplus_{\nu} \left(P_{pl(\nu)-n} H^*(S_P^{l(\nu)}; \mathbb{Q}) \right)^{\mathfrak{S}_{\nu}} [2l(\nu) - 2n] \\ &= \bigoplus_{\nu} P_{p+l(\nu)-n} H^*(S_P^{(\nu)}; \mathbb{Q})[2l(\nu) - 2n] \end{aligned}$$

where the perversities in the parentheses are taken with respect to $h^{l(\nu)} : S_P^{l(\nu)} \rightarrow \mathbb{A}^{l(\nu)}$ and $h^{(\nu)} : S_P^{(\nu)} \rightarrow \mathbb{A}^{l(\nu)}$ respectively. The result in step 2 implies that

$$W_{2k} H^*(S_W^{[n]}; \mathbb{Q}) = W_{2k+1} H^*(S_W^{[n]}; \mathbb{Q}) = P_k H^*(S_P^{[n]}; \mathbb{Q}).$$

So $(S_P^{[n]}, S_W^{[n]}, h^{[n]}, \mathbb{A}^n, \Phi^{[n]})$ is a homological $P = W$ package. □

5 Applications to the $P = W$ conjecture

In this chapter, we will consider five families of Hitchin systems and the corresponding character varieties. We will prove the multiplicativity of the perverse filtration for the Hitchin map, compute perverse numbers and prove the full version of $P = W$ for the $n = 1$ case.

5.1 Five families of Hitchin systems

We first define the five families of Hitchin systems we consider.

1. Type $\widetilde{A}_0(n)$. Consider the moduli space of degree 0 rank n parabolic Higgs bundles over an elliptic curve $(E, 0)$, whose Higgs field can have at worst a first order pole at 0 and the residue of the Higgs field at 0 is nilpotent with respect to a multi-dimension $\{n, 1, 0\}$ flag.
2. Type $\widetilde{D}_4(n)$. Consider the moduli space of degree 0 rank $2n$ parabolic Higgs bundles over a weighted curve $(\mathbb{P}^1, p_1, p_2, p_3, p_4)$, whose Higgs field can have at worst a first order pole at marked points and the residues of the Higgs field are nilpotent with respect to a multi-dimension $\{2n, n, 0\}$ flag for p_1, p_2, p_3 , and a multi-dimension $\{2n, n, 1, 0\}$ flag for p_4 .
3. Type $\widetilde{E}_6(n)$. Consider the moduli space of degree 0 rank $3n$ parabolic Higgs bundles over a weighted curve $(\mathbb{P}^1, p_1, p_2, p_3)$, whose Higgs field can have at worst a first order pole at marked points and the residues of the Higgs field is nilpotent with respect to a multi-dimension $\{3n, 2n, n, 0\}$ flag for p_1, p_2 and a multi-dimension $\{3n, 2n, n, 1, 0\}$ flag for p_3 .
4. Type $\widetilde{E}_7(n)$. Consider the moduli space of degree 0 rank $4n$ parabolic Higgs bundles over a weighted curve $(\mathbb{P}^1, p_1, p_2, p_3)$, whose Higgs field can have at worst a first order pole at marked points and the residues of the Higgs field are nilpotent with respect to a multi-dimension $\{4n, 2n, 0\}$ flag for p_1 , a multi-dimension $\{4n, 3n, 2n, n, 0\}$ flag for p_2 and a multi-dimension $\{4n, 3n, 2n, n, 1, 0\}$ flag for p_3 .
5. Type $\widetilde{E}_8(n)$. Consider the moduli space of degree 0 rank $6n$ parabolic Higgs bundles over a weighted curve $(\mathbb{P}^1, p_1, p_2, p_3)$, whose Higgs field can have at worst a first order pole at marked points and the residues of the Higgs field are nilpotent with respect to a multi-dimension $\{6n, 3n, 0\}$ flag for p_1 , a multi-dimension $\{6n, 4n, 2n, 0\}$ flag for p_2 and a multi-dimension $\{6n, 5n, 4n, 3n, 2n, n, 1, 0\}$ flag for p_3 .

The geometry of the above moduli spaces of parabolic Higgs bundles are described explicitly by the following theorem in [10] due to Gröchenig.

Theorem 5.1 ([10] Theorem 4.1). *We consider the moduli of parabolic Higgs bundles in $n = 1$ case for any of the five families. Let $\Gamma =: \{0\}, \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}$ respectively. Let M_D denote the moduli of parabolic Higgs bundles. Then M_D is isomorphic to Γ -equivariant Hilbert scheme on T^*E , which is the crepant resolution of the quotient T^*E/Γ .*

Theorem 5.2 ([10] Theorem 5.1). *Let $M_D(n)$ denote the moduli space of parabolic Higgs bundle in any of the five families, and $M_D(1)$ is abbreviated to M_D . Then we have*

$$M_D^{[n]} \cong M_D(n)$$

The Hitchin map $M_D^{[n]} \rightarrow \mathbb{A}^n$ factors through the Hilbert-Chow map

$$M_D^{[n]} \rightarrow M_D^{(n)} \rightarrow (\mathbb{A}^1)^{(n)} = \mathbb{A}^n,$$

where $M_D^{(n)} \rightarrow (\mathbb{A}^1)^{(n)}$ is induced by $M_D^n \rightarrow (\mathbb{A}^1)^n$.

In parabolic non-abelian Hodge theory, the moduli of parabolic Higgs bundle is canonically diffeomorphic to the corresponding character variety. The $P = W$ Conjecture 1.1 asserts that under this canonical diffeomorphism, the weight filtration in mixed Hodge structure on the cohomology of character variety corresponds to the perverse filtration on the cohomology of the Higgs moduli space with respect to the Hitchin map. By the Simpson's table on page 720 in [17], we may find the character varieties corresponding to our five families of moduli of parabolic Higgs bundles. The Conjecture 1.2.1 in [11] predicts that all cohomology class are of Hodge-Tate type and the mixed Hodge numbers depend on the multiplicities of eigenvalues of the monodromy action around the punctures rather than the eigenvalues themselves. So for our purpose, we list the corresponding moduli description of character varieties for our five families of Hitchin systems without mentioning the specific eigenvalues for the monodromy action.

1. Let E be any elliptic curve. Consider $GL(n, \mathbb{C})$ -representations of $\pi_1(E \setminus p)$ such that the image of small loops around punctures are in a prescribed conjugacy class whose multiplicities of eigenvalue are of type

$$(n - 1, 1).$$

2. Let $C = \mathbb{P}^1 \setminus \{p_1, \dots, p_4\}$. Consider $GL(2n, \mathbb{C})$ -representations of $\pi_1(C)$ such that the image of small loops around punctures are in a prescribed conjugacy classes whose

multiplicities of eigenvalue are of type

$$(n, n)(n, n)(n, n)(n, n - 1, 1).$$

3. Let $C = \mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$. Consider $GL(3n, \mathbb{C})$ -representations of $\pi_1(C)$ such that the image of small loops around punctures are in a prescribed conjugacy classes whose multiplicities of eigenvalue are of type

$$(n, n, n)(n, n, n)(n, n, n - 1, 1).$$

4. Let $C = \mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$. Consider $GL(4n, \mathbb{C})$ -representations of $\pi_1(C)$ such that the image of small loops around punctures are in a prescribed conjugacy classes whose multiplicities of eigenvalue are of type

$$(2n, 2n)(n, n, n, n)(n, n, n, n - 1, 1).$$

5. Let $C = \mathbb{P}^1 \setminus \{p_1, p_2, p_3\}$. Consider $GL(6n, \mathbb{C})$ -representations of $\pi_1(C)$ such that the image of small loops around punctures are in a prescribed conjugacy classes whose multiplicities of eigenvalue are of type

$$(3n, 3n)(2n, 2n, 2n)(n, n, n, n, n, n - 1, 1).$$

We have the following explicit description for these character varieties for $n = 1$ cases.

Theorem 5.3 ([9] Theorem 6.14 and 6.19). *The character varieties $M_B(1)$ above can be described explicitly as follows.*

1. Type \widetilde{A}_0 . $\mathbb{C}^* \times \mathbb{C}^*$.
2. Type \widetilde{D}_4 . Degree 3 del Pezzo surface with a triangle removed.
3. Type \widetilde{E}_6 . Degree 3 del Pezzo surface with a nodal \mathbb{P}^1 removed.
4. Type \widetilde{E}_7 . Degree 2 del Pezzo surface with a nodal \mathbb{P}^1 removed.
5. Type \widetilde{E}_8 . Degree 1 del Pezzo surface with a nodal \mathbb{P}^1 removed.

Furthermore, these del Pezzo surfaces can be expressed by an explicit formula in weighted projective space away from the singularities, and the removed triangle or nodal \mathbb{P}^1 are cut out by a hyperplane section.

Contrary to the moduli of parabolic Higgs bundle case, we don't know much about character varieties for $n > 1$. Nevertheless, there are conjectures in [11] which predict the behavior of the mixed Hodge numbers of character varieties. We will go back to this point in section 5.4.

5.2 Multiplicativity of perverse filtrations for Hitchin systems

We will use the technique we developed in previous chapters to prove the multiplicativity of the five families of Hitchin systems.

Proposition 5.4. *Let $h : M_D \rightarrow \mathbb{C}$ be $n = 1$ cases for five families. Then M_D has trivial canonical bundle. The dual graph of irreducible components of the fiber over 0 is affine Dynkin diagram $\widetilde{A}_0, \widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, respectively. Let $\hat{h} : h^{-1}\mathbb{C}^* \rightarrow \mathbb{C}^*$ be the smooth part of the map, let $j : \mathbb{C}^* \rightarrow \mathbb{C}$, let $\hat{R}^1 = R^1\hat{h}_*\mathbb{Q}$. Then a perverse decomposition of $h : M_D \rightarrow \mathbb{C}$ can be written as follows.*

$$Rh_*\mathbb{Q}_{M_D}[1] \cong \{\mathbb{Q}_{\mathbb{C}}[1]\} \bigoplus \left\{ j_*\hat{R}^1 \oplus \mathbb{Q}_0^k \right\} [-1] \bigoplus \{\mathbb{Q}_{\mathbb{C}}[1]\}[-2]$$

where

$$k = \begin{cases} 0 & \widetilde{A}_0 \text{ case} \\ 4 & \widetilde{D}_4 \text{ case} \\ 6 & \widetilde{E}_6 \text{ case} \\ 7 & \widetilde{E}_7 \text{ case} \\ 8 & \widetilde{E}_8 \text{ case} \end{cases}$$

In particular, the dimension of the perverse filtration is given by

$$\dim \text{Gr}_p H^d(M_D, \mathbb{Q}) = \begin{cases} 1 & p = d = 0 \\ 1 & p = d = 2 \\ 2 & p = d = 1, \widetilde{A}_0 \text{ case} \\ k & p = 1, d = 2, \widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8 \text{ case} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The quotient of T^*E by Γ is computed using elementary methods. We list the type of singularities in our five cases.

Case	Singularities
\widetilde{A}_0	none
\widetilde{D}_4	4 A_1
\widetilde{E}_6	3 A_2
\widetilde{E}_7	1 A_1 , 2 A_3
\widetilde{E}_8	1 A_1 , 1 A_2 , 1 A_5

Note that all singularities take place in the fiber over 0, so the dual graph of irreducible components match the affine Dynkin diagram. The action of Γ on T^*E preserves the canonical form, so the trivial canonical bundle descends to the quotient. The minimal resolution of type A singularities is crepant, so M_D has trivial canonical bundle. The perverse decomposition is again due to Theorem 3.2.2 of [4]. Here the map $h : M_D \rightarrow \mathbb{C}$ has connected fibers, so $j_*\hat{R}^0 = j_*\hat{R}^2 = \mathbb{Q}_{\mathbb{C}}$. The dimension of the perverse filtration will follow if we show $\mathbb{H}^*(j_*\hat{R}^1) = 0$ in $\widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ cases. In fact, the local systems \hat{R}^1 can be described explicitly. They are rank 2 representation of $\mathbb{Z} = \pi_1(C^*)$ with monodromy $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. A simple Čech cohomology argument shows that all cohomology group of \hat{R}^1 vanishes, so does $j_*\hat{R}^1$. \square

When M_D is smooth and non-compact, the small diagonal embedding $\Delta_{n,*} : M_D \rightarrow M_D^n$ is still proper. So we have the push-forward in Borel-Moore homology. We may still define

$$\Delta_{*,n} : H^*(M_D) \cong H_{4-*}^{BM}(M_D) \rightarrow H_{4-*}^{BM}(M_D^n) \cong H^{*+4(n-1)}(M_D^n)$$

The following proposition is a counterpart of Proposition 3.8.

Proposition 5.5. *Let $f : M_D \rightarrow \mathbb{C}$ be as in Proposition 5.4. In \widetilde{A}_0 case, the Gysin push-forward by the small diagonal embedding $\Delta_{n,*}(\gamma) = 0$ for any $\gamma \in H^*(M_D)$ and $n > 1$. In the other four cases, let E_i be exceptional divisors of the resolution, then*

$$\Delta_{2,*}(1) = - \sum_{i=1}^k [E_i] \otimes [E_i]$$

and $\Delta_{n,*}(\gamma) = 0$ for any $n > 2$ or $n = 2$, $\gamma \neq 1$. In particular, the perversity estimation of diagonal in Proposition 3.8

$$\mathfrak{p}(\Delta_{n,*}(\gamma)) \leq \mathfrak{p}(\gamma) + 2(n-1)$$

is still true.

Proof. Note that $\Delta_{n,*}$ increases the degree by $4(n-1)$, however in our cases, the top nontrivial degree for $H^*(M_D^n)$ is $2n$. So when $n \geq 3$, the push-forward is automatically 0. When $n = 2$, the only possible nonzero term is $\Delta_{2,*}(1)$. In the \widetilde{A}_0 case, $H^4(M_D \times M_D)$ is one dimensional, generated by the class $[\mathbb{C}] \otimes [\mathbb{C}]$ and $H_4(M_D \times M_D)$ is generated by $E \otimes E$. $\langle \Delta_{2,*}(1), E \otimes E \rangle_{M_D \times M_D} = \langle E, E \rangle_{M_D} = 0$, so we have $\Delta_{2,*}(1) = 0$. For other four cases, according to the decomposition, we pick a basis $[E_1], \dots, [E_k], \Sigma \in H^2(M_D)$, where Σ is a generic section of map $f : M_D \rightarrow \mathbb{C}$ whose perversity is 2. To write $\Delta_{2,*}(1)$ in terms of the basis, it suffices to intersect it with the dual basis. The dual basis in $H_2(M_D)$ is $\{E_1, \dots, E_k, F\}$, where F denote the cycle class of general fiber. Since

$$\langle \Delta_{2,*}(1), E_i \otimes E_j \rangle_{M_D \times M_D} = \langle E_i, E_j \rangle_{M_D} = -\delta_{ij},$$

$$\langle \Delta_{2,*}(1), E_i \otimes F \rangle_{M_D \times M_D} = \langle E_i, F \rangle_{M_D} = 0,$$

$$\langle \Delta_{2,*}(1), F \otimes F \rangle_{M_D \times M_D} = \langle F, F \rangle_{M_D} = 0.$$

We conclude that

$$\Delta_{2,*}(1) = - \sum_{i=1}^k [E_i] \otimes [E_i]$$

□

Theorem 5.6. *Let $f : M_D \rightarrow \mathbb{C}$ be as in Proposition 5.4. Then the perverse filtration defined by the map $h : M_D^{[n]} \rightarrow \mathbb{C}^n$ is multiplicative .*

Proof. First, we show that Theorem 4.9 works in our five cases of non-compact surfaces. In fact, for all of our five cases, M_D can be compactified to $\overline{M_D}$ such that the restriction map $H^*(\overline{M_D}) \rightarrow H^*(M_D)$ is surjective, so does $H^*(\overline{M_D}^{[n]}) \rightarrow H^*(M_D^{[n]})$. To compute the cup product, we may pull-back to the compact cases, use the result for nontrivial canonical bundle case and then restrict. Since the extra terms involving the canonical bundle vanish after restricting, the cup product formula in Theorem 4.9 still hold for our five cases.

The pull-back Lemma 4.18 holds regardless of compactness. The push-forward Lemma 4.19 holds once we have Proposition 5.5. In the non-compact case, the Euler class $e = 0$, so case (2) in proof of Theorem 4.17 is trivial, and case (1) follows line by line. □

Combining Theorem 5.6 and Theorem 5.2, we have

Theorem 5.7. *For the five families of moduli space of parabolic Higgs bundles described in Theorem 5.2, the perverse filtration defined by the Hitchin map is multiplicative.*

5.3 Full version of $P = W$ for $n = 1$

In this section, we prove the full version of $P = W$ conjecture in the $n = 1$ cases by using the explicit geometry of the Hitchin map.

Lemma 5.8. *Let X be any of the del Pezzo surface in Theorem 5.3. Let $i : T \rightarrow X$ be the closed embedding of the removed curve in Theorem 5.3 and let $j : U \hookrightarrow X$ be its complement. Then*

$$W_2H^2(U) \cong \text{Im} (H_c^2(U) \rightarrow H^2(U)).$$

Proof. We have a diagram where the row and the column are distinguished triangles

$$\begin{array}{ccccc} & & i_*i^!\mathbb{Q}_X & & \\ & & \downarrow & & \\ Rj_!\mathbb{Q}_U & \longrightarrow & \mathbb{Q}_X & \longrightarrow & i_*\mathbb{Q}_T \\ & & \downarrow & & \\ & & Rj_*\mathbb{Q}_U & & \end{array}$$

Taking cohomology in degree 2, we have

$$\begin{array}{ccccc} & & H_T^2(X) & & \\ & & \downarrow i_* & \searrow \psi & \\ H_c^2(U) & \xrightarrow{j_!} & H^2(X) & \xrightarrow{i^*} & H^2(T) \\ & \searrow \phi & \downarrow j^* & & \\ & & H^2(U) & & \end{array}$$

By [8], Corollaire 3.2.17, the image of j^* is precisely $W_2H^2(U)$. So it suffices to prove that $\text{Im } j_! + \ker j^* = H^2(X)$. By exactness, this is equivalent to proving that $\text{Im } i_* + \ker i^* = H^2(X)$. Therefore, it suffices to prove that $\psi = i^*i_*$ is an isomorphism. In fact, the morphism ψ maps $\xi \in H_T^2(X) \cong H_2(T)$ to $\xi^\dagger : H_2(T) \rightarrow \mathbb{Q}$, where $\xi^\dagger(\gamma) = \int_X \xi \cup i_*(\gamma)$. This defines a symmetric bilinear form on $H_2(T)$ defined by the intersection number of components of T viewed in X . To show that ψ is an isomorphism, it suffices to show that this bilinear form is nondegenerate. In the case where T is a triangle, then the intersection matrix is

$$\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

In the case when T is a nodal \mathbb{P}^1 , since it is cut out by hyperplane section away from the singularities of the weighted projective space, so it is an ample divisor, therefore the self intersection of T is nonzero. \square

Theorem 5.9. *The perverse filtration for the map $M_D(1) \rightarrow \mathbb{C}$ and the mixed Hodge filtration on $M_B(1)$ correspond.*

Proof. By the spectral sequence of weight filtration of mixed Hodge structure, dimensions of graded pieces of weight filtration is easily computed.

$$\dim \mathrm{Gr}_w^W H^d(M_B) = \begin{cases} 1 & w = d = 0 \\ 1 & w = 4, d = 2 \\ 2 & w = 2, d = 1, \widetilde{A}_0 \text{ case} \\ k & w = 2, d = 2, \widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8 \text{ case} \\ 0 & \text{otherwise.} \end{cases}$$

Compare with Proposition 5.4, numerical $P = W$ holds for $n = 1$ in our five cases. To prove the full version of the $P = W$ conjecture, it suffices to prove that $P_1 H^2(M_D) = W_2 H^2(M_B)$ in $\widetilde{D}_4, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ cases. (\widetilde{A}_0 cases is trivially true.) By Proposition 5.4, $P_1 H^2(M_D)$ in four cases are all spanned by the fundamental classes of exceptional curves of the minimal resolutions, and $H_c^2(M_D)$ is generated by the fundamental classes of exceptional curves and a generic fiber of the morphism $M_D \rightarrow \mathbb{C}$. Note that the fiber class is 0 in $H^2(M_D)$, so we have

$$P_1 H^2(M_D) = \mathrm{Im} (H_c^2(M_D) \rightarrow H^2(M_D)).$$

By Lemma 5.8, we have $W_2 H^2(M_B) = \mathrm{Im}(H_c^2(M_B) \rightarrow H^2(M_B))$. This completes the proof. \square

Remark 5.10. In [6], Theorem 3.2.1 asserts that the forgetful map $H_c^{6g-6}(M_D) \rightarrow H^{6g-6}(M_D)$ is the zero map, where $6g - 6$ is the complex dimension of the moduli space in the context. In fact, they consider moduli space of degree 1 Higgs bundles and twisted representations, so their result does not contradict ours.

5.4 Perverse numbers and numerical $P = W$

In this section, we give some partial numerical evidence for $P = W$ Conjecture 1.1 for our five families of Hitchin fibrations. We use Proposition 4.11 and Proposition 5.4 to compute the perverse numbers $\mathrm{Gr}_p H^d(M_D^{[n]})$. Conjecture 1.2.1 in [11] which predicts that the mixed

Hodge numbers of character varieties can be computed by a combinatorial formula. We made a conjecture that in our five families of Hitchin systems, the perverse numbers equal the conjectural mix Hodge numbers of the corresponding character varieties. We have verified our conjecture for small n .

Theorem 5.11. *Let $f : M_D \rightarrow \mathbb{C}$ be $n = 1$ case of the five families. Denote perverse numbers by $p^{i,j} = \dim \text{Gr}_i H^j(M_D^{[n]})$. Let perverse Poincaré polynomial be $P_n(q, t) = \sum_{i,j} p^{i,j} q^i t^j$. Then for the \widetilde{A}_0 case, the generating series is*

$$\sum_{n=0}^{\infty} s^n P_n(q, t) = \prod_{m=1}^{\infty} \frac{(1 + s^m q^m t^{2m-1})^2}{(1 - s^m q^{m-1} t^{2m-2})(1 - s^m q^{m+1} t^{2m})}.$$

For the other four cases, the generating series are

$$\sum_{n=0}^{\infty} s^n P_n(q, t) = \prod_{m=1}^{\infty} \frac{1}{(1 - s^m q^{m-1} t^{2m-2})(1 - s^m q^m t^{2m})^k (1 - s^m q^{m+1} t^{2m})}$$

where k is defined in Proposition 5.4.

Proof. We prove the equalities by expanding both hand sides and identifying the corresponding terms. Since all cases are similar, we prove the \widetilde{D}_4 case as an illustration of the calculations. By Künneth formula and MacDonal theorem, we have

$$\begin{aligned} & H^* \left(M_D^{[n]} \right) [n] \\ &= \mathbb{H} \left(\bigoplus_{\nu=1^{a_1} \dots n^{a_n}} Rr_{C,*}^{(\nu)} \left(\bigoplus_{\mathbf{r}+\mathbf{s}+\mathbf{t}=\mathbf{a}} \mathcal{P}_0^{\{\mathbf{r}\}} \boxtimes \mathcal{P}_1^{\{\mathbf{s}\}} \boxtimes \mathcal{P}_2^{\{\mathbf{t}\}} \right) [-C(\mathbf{r}, \mathbf{s}, \mathbf{t})] \right) \\ &= \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} \bigoplus_{\mathbf{r}+\mathbf{s}+\mathbf{t}=\mathbf{a}} \mathbb{H} \left(\mathcal{P}_0^{\{\mathbf{r}\}} \boxtimes \mathcal{P}_1^{\{\mathbf{s}\}} \boxtimes \mathcal{P}_2^{\{\mathbf{t}\}} \right) [-C(\mathbf{r}, \mathbf{s}, \mathbf{t})] \\ &= \bigoplus_{\nu=1^{a_1} \dots n^{a_n}} \bigoplus_{\mathbf{r}+\mathbf{s}+\mathbf{t}=\mathbf{a}} \mathbb{H} \left(\mathcal{P}_0^{\{\mathbf{r}\}} \right) \otimes \mathbb{H} \left(\mathcal{P}_1^{\{\mathbf{s}\}} \right) \otimes \mathbb{H} \left(\mathcal{P}_2^{\{\mathbf{t}\}} \right) [-C(\mathbf{r}, \mathbf{s}, \mathbf{t})] \end{aligned}$$

By Proposition 5.4, we have $\mathcal{P}_0 = \mathbb{Q}_{\mathbb{C}}[1]$, $\mathcal{P}_1 = j_* \hat{R}^1[1] \oplus \mathbb{Q}_0^4$, $\mathcal{P}_2 = \mathbb{Q}_{\mathbb{C}}[1]$. Note that $j_* \hat{R}^1$ has no cohomology so that we have

	$\mathbb{H}(\mathcal{P}_0)$	$\mathbb{H}(\mathcal{P}_1)$	$\mathbb{H}(\mathcal{P}_2)$
dimension	1	4	1
degree	-1	0	-1

Together with Proposition 2.1 and Remark 2.6, the table shows that the summand

$$\mathbb{H} \left(\mathcal{P}_0^{\{\mathbf{r}\}} \right) \otimes \mathbb{H} \left(\mathcal{P}_1^{\{\mathbf{s}\}} \right) \otimes \mathbb{H} \left(\mathcal{P}_2^{\{\mathbf{t}\}} \right) [-C(\mathbf{r}, \mathbf{s}, \mathbf{t})]$$

is of dimension $\prod_{i=1}^n \binom{s_i + 3}{3}$ as a vector space, and all cohomology classes in it are of degree $\sum_{i=1}^n -r_i + s_i + t_i$ and perversity $\sum_{i=1}^n s_i + 2t_i$. Here we use the fact that $\mathbb{H}(\mathcal{P}_0)$ and $\mathbb{H}(\mathcal{P}_2)$ are in odd degree, so that

$$\mathbb{H}(\mathcal{P}_0^{\{r\}}) = \bigotimes_{i=1}^n \text{Sym}^{r_i} \mathbb{H}^{-1}(\mathcal{P}_0) = \mathbb{C},$$

and similarly for $\mathbb{H}(\mathcal{P}_2^{\{t\}})$. So

$$\begin{aligned} P_n(q, t) &= \sum_{\nu=1^{a_1} \dots n^{a_n}} q^{n-l(\nu)} t^{2n-l(\nu)} \prod_{i=1}^n \sum_{r_i+s_i+t_i=a_i} \binom{s_i + 3}{3} q^{s_i+2t_i} t^{-r_i+s_i+t_i} \\ &= \sum_{\nu=1^{a_1} \dots n^{a_n}} q^{n-l(\nu)} t^{2n-l(\nu)} \prod_{i=1}^n t^{-a_i} \sum_{r_i+s_i+t_i=a_i} \binom{s_i + 3}{3} q^{s_i+2t_i} t^{2s_i+2t_i} \\ &= \sum_{\nu=1^{a_1} \dots n^{a_n}} q^{n-l(\nu)} t^{2n-2l(\nu)} \prod_{i=1}^n \sum_{r_i+s_i+t_i=a_i} \binom{s_i + 3}{3} q^{s_i+2t_i} t^{2s_i+2t_i} \end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} P_n(q, t) = \sum_{n=0}^{\infty} \sum_{\nu=1^{a_1} \dots n^{a_n}} q^{n-l(\nu)} t^{2n-2l(\nu)} \prod_{i=1}^n \sum_{r_i+s_i+t_i=a_i} \binom{s_i + 3}{3} q^{s_i+2t_i} t^{2s_i+2t_i}$$

On the other hand, to get a term in the product generating series with factor s^n is equivalent to the following data: (1) A partition $\nu = 1^{a_1} \dots n^{a_n}$ of n , such that the factor $m = i$ contributes $(s^i)^{a_i}$, (2) a triple (r_i, s_i, t_i) for each i satisfying $r_i + s_i + t_i = a_i$, such that the expansions of three parenthesis contribute $(s^i)^{r_i}$, $(s^i)^{s_i}$ and $(s^i)^{t_i}$, respectively. So the term obtained in this way is

$$\begin{aligned} & \prod_{i=1}^n \binom{s_i + 3}{3} s^{i(r_i+s_i+t_i)} q^{(i-1)r_i+is_i+(i+1)t_i} t^{(2i-2)r_i+2is_i+2it_i} \\ &= \prod_{i=1}^n \binom{s_i + 3}{3} s^{ia_i} q^{(i-1)a_i+s_i+2t_i} t^{(2i-2)a_i+2s_i+2t_i} \\ &= s^n q^{n-l(\nu)} t^{2n-2l(\nu)} \prod_{i=1}^n \binom{s_i + 3}{3} q^{s_i+2t_i} t^{2s_i+2t_i}. \end{aligned}$$

Here we use the fact that $a_i = r_i + s_i + t_i$, $n = \sum_{i=1}^n ia_i$ and $l(\nu) = \sum_{i=1}^n a_i$. By comparing with the expansion of the additive generating series, the theorem follows. \square

If we believe $P = W$, the perverse numbers of Hitchin system should equal the mixed

Hodge numbers of corresponding character varieties. In fact, the Conjecture 1.2.1 in [11] and Theorem 5.11 suggest the following conjecture.

Conjecture 5.12. *Let μ be multi-partition which encodes the parabolic data of $M_B(n)$. Let k be defined as in Proposition 5.4. Let \mathbb{H}_μ be defined as in section 1.1 of [11]. Then for $\widetilde{A}_0(n)$ case, we have*

$$\prod_{m=1}^{\infty} \frac{(1 + s^m q^m t^{2m-1})^2}{(1 - s^m q^{m-1} t^{2m-2})(1 - s^m q^{m+1} t^{2m})} = \sum_{n=0}^{\infty} (st^2 q)^n \mathbb{H}_\mu(-\sqrt{q}, \frac{\sqrt{q}}{t})$$

For the other four cases, we have

$$\prod_{m=1}^{\infty} \frac{1}{(1 - s^m q^{m-1} t^{2m-2})(1 - s^m q^m t^{2m})^k (1 - s^m q^{m+1} t^{2m})} = \sum_{n=0}^{\infty} (st^2 q)^n \mathbb{H}_\mu(-\sqrt{q}, \frac{\sqrt{q}}{t})$$

Remark 5.13. In fact, the conjecture for \widetilde{A}_0 case is the cohomological version of Conjecture 4.2.1 in [12], and the other four cases are new. With the help of Mathematica, we prove for $n \leq 6$ for \widetilde{D}_4 case, $n \leq 4$ for \widetilde{E}_6 case, $n \leq 3$ for \widetilde{E}_7 and $n \leq 2$ for \widetilde{E}_8 case. Efforts to prove for general n so far all ended with combinatorial difficulties. More understanding on q, t -Kostka numbers would be helpful.

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