Convergence of Manifolds and Metric Spaces with Boundary

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Abstract of the Dissertation

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We study sequences of oriented Riemannian manifolds with boundary and, more generally, integral current spaces and metric spaces with boundary. We prove theorems demonstrating when the Gromov-Hausdorff [GH] and Sormani-Wenger Intrinsic Flat [SWIF] limits of sequences of such metric spaces agree. Thus in particular the limit spaces are countably $\mathcal{H}^n$ rectifiable spaces. From these theorems we derive compactness theorems for sequences of Riemannian manifolds with boundary where both the GH and SWIF limits agree. For sequences of Riemannian manifolds with boundary we only require nonnegative Ricci curvature, upper bounds on volume, noncollapsing conditions on the interior of the manifold and diameter controls on the level sets near the boundary. In addition we survey prior results of the author concerning the SWIF limits of manifolds with boundary, prior work of the author with Sormani concerning glued limits of metric spaces with boundary, prior work of the author with Li concerning GH and SWIF limits agreeing for Alexandrov spaces without boundary and work of Kodani, Anderson-Katsuda-Kurylev-Lassas-Taylor, Wong, and Knox concerning limits of Riemannian manifolds with boundary.

Contents

List of Figures

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Chapter 1

Introduction

In the past few decades many important compactness theorems have been proven for families of compact Riemannian manifolds without boundary. Gromov introduced the notion of Gromov-Hausdorff (GH) convergence of Riemannian manifolds to metric spaces, (X,d). He proved that the family of manifolds with nonnegative Ricci curvature and uniformly bounded diameter are precompact in GH sense [14]. Cheeger-Colding have proven many properties of the GH limits of these manifolds including rectifiability [6]. The Sormani-Wenger Intrinsic Flat (SWIF) convergence of oriented Riemannian manifolds to countably $\mathcal{H}^n$ rectifiable metric spaces called integral current spaces, (X, d, T), was introduced in [31]. They proved that when the sequence of manifolds is noncollapsing and has nonnegative Ricci curvature the SWIF and GH limits agree [30]. In general GH and SWIF limits need not agree and GH limits need not be countably $\mathcal{H}^n$ rectifiable metric spaces (cf. the appendix of [30] by Schul and Wenger).

Here we prove GH and SWIF compactness theorems for oriented Riemannian manifolds with boundary. Note that there are sequences of flat manifolds with boundary of bounded diameter with volume bounded below which have no GH limit, see Example 5.1.2. Nevertheless, Wenger proved that a sequence of n-dimensional oriented Riemannian manifolds M_j with boundary that satisfy

$$\mathrm{Diam}(M_j) \le D, \;\; \mathrm{Vol}(M_j) \le V, \;\; \mathrm{Vol}(\partial M_j) \le A \tag{1.1}$$

has a SWIF convergent subsequence [33] (cf. [31]). Knox proved weak $L^{1,p}$ and $C^{1,\alpha}$ convergence of Riemannian manifolds with two sided bounds on the sectional curvature of the manifolds and of their boundaries, a lower bound on the volume of the boundaries and two sided bounds on the mean curvature of the boundary

[16]. Wong proved GH convergence of Riemannian manifolds with Ricci curvature bounded below and two sided bounds on the second fundamental form of the boundaries [34]. Under stronger conditions Anderson-Katsuda-Kurylev-Lassas-Taylor [2] and Kodani [17] have respectively proven $C^{1,\alpha}$ and Lipschitz compactness theorems.

We first prove compactness theorems for sequences of metric spaces and from them we derive compactness theorems for sequences of Riemannian manifolds with boundary where both the GH and SWIF limits agree. Thus we produce countably $\mathcal{H}^n$ rectifiable GH limit spaces. For sequences of Riemannian manifolds we only require Ricci curvature bounds, noncollapsing conditions on the interior of the manifold and additional controls on the boundary.

To precisely state our theorems we recall a few notions. Let (M, g) be a Riemannian manifold with boundary, ∂M. We denote by d the metric on M induced by g. For $\delta > 0$ we define the δ-inner region of M by

$$M^\delta = \{\, x \in M : d(x, \partial M) > \delta\}. \tag{1.2}$$

There are two metrics on M^δ. The restricted metric $d|_M$ (that we denote by d to simplify notation) and the length metric d_{M^δ} induced by g. The diameter of M^δ with respect to this metric is given by

$$\mathrm{Diam}(M^\delta, d_{M^\delta}) = \sup\left\{d_{M^\delta}(x, y) : x, y \in M^\delta\right\}. \tag{1.3}$$

In [25], the author and Sormani proved a GH compactness theorem for sequences of inner regions (M^δ_j, d_j) that have nonnegative Ricci curvature, upper bounds on volume and diameter as in (1.4) and a noncollapsing condition as in (1.5) (cf. Theorem 2.2.1 within). Now we add one additional condition on the boundary (1.6) to obtain GH convergence of the sequence of manifolds themselves:

1.0.1 Theorem. *Let $n \in \mathbb{N}$, $\delta, D_i, V, \theta > 0$ and $\{\delta_i\} \subset \mathbb{R}$ be a decreasing sequence that converges to zero. Let (M_j, g_j) be a sequence of compact oriented manifolds with boundary such that*

$$\mathrm{Ric}(M_j) \geq 0, \quad \mathrm{Vol}(M_j) \leq V, \quad \mathrm{Diam}(M_j^{\delta_i},\, d_{M_j^{\delta_i}}) \leq D_i, \tag{1.4}$$

$$\exists q \in M^\delta_j \text{ such that } \mathrm{Vol}(B(q, \delta)) \geq \theta\delta^n, \tag{1.5}$$

where $B(q, \delta)$ is the ball in M_j with center q and radius δ, and suppose that there is a compact metric space (X_∂, d_∂) such that

$$(\partial M_j, d_j) \xrightarrow{GH} (X_\partial, d_\partial). \tag{1.6}$$

Then a subsequence of $\{(M_j, d_j)\}_{j=1}^{\infty}$ *converges in GH sense.*

In Example 5.3.2 we define a sequence (M_j, d_j) with no GH converging subsequence that satisfies all the conditions of Theorem 1.0.1 except the Ricci lower bound. Note that, by Gromov's Embedding Theorem, if (M_j, d_j) converges in GH sense then a subsequence of $(\partial M_j, d_j)$ converges in GH sense. In Example 5.1.2 we define a sequence (M_j, d_j) with no GH converging subsequence that satisfies all the conditions of Theorem 1.0.1 except that $(\partial M_j, d_j)$ does not have any GH convergent subsequence. In Theorem 7.0.5 we obtain convergence of the boundary as in (1.6) by requiring uniform bounds on the second fundamental form of ∂M_j and its derivative in the normal direction.

Suppose that $\{(M_j, g_j)\}$ satisfies the hypotheses of Theorem 1.0.1 so that we have a subsequence such that $(M_j, d_j) \xrightarrow{\text{GH}} (X, d_X)$. Sormani-Wenger proved that for such a sequence, if for all j

$$\mathrm{Vol}(M_j) \leq V \text{ and } \mathrm{Vol}(\partial M_j) \leq A \tag{1.7}$$

then there exists a subsequence and an integral current space (Y, d_Y, T) such that

$$(M_{j_k}, d_{j_k}) \xrightarrow{\text{GH}} (X, d_X) \text{ and } (M_{j_k}, d_{j_k}, T_{j_k}) \xrightarrow{\mathcal{F}} (Y, d_Y, T), \tag{1.8}$$

where either $Y \subset X$ or (Y, d_Y, T) is the zero integral current space [31] (cf. Theorem 3.2.3). In [30] , they proved that for a sequence (M_j, g_j) of oriented compact n-dimensional Riemannian manifolds with no boundary, with nonnegative Ricci curvature and with

$$0 < v \leq \mathrm{Vol}(M_j) \leq V, \tag{1.9}$$

the GH and SWIF limits agree, $Y = X$ (cf. Theorem 3.3.1). They proved this by showing that the GH limit, X, is contained in a nonzero SWIF limit, Y, using work of Cheeger-Colding [6], Colding [7] and Perelman [26].

In this paper we prove the corresponding theorem for manifolds with boundary. We assume the same hypothesis as in Theorem 1.0.1 (1.10)-(1.12) and one additional area bound on the boundary (1.13):

1.0.2 Theorem. *Let* $n \in \mathbb{N}$, $\delta, D_i, V, \theta > 0$ *and* $\{\delta_i\} \subset \mathbb{R}$ *with* δ_i *decreasing to* 0. *Let* (M_j, g_j) *be a sequence of compact oriented manifolds with boundary such that*

$$\mathrm{Ric}(M_j) \geq 0, \ \ \mathrm{Vol}(M_j) \leq V, \ \ \mathrm{Diam}(M_j^{\delta_i}, d_{M_j^{\delta_i}}) \leq D_i, \tag{1.10}$$

$$\exists q \in M_j^{\delta} \textit{ such that } \mathrm{Vol}(B(q, \delta)) \geq \theta \delta^n, \tag{1.11}$$

and suppose that there is a compact metric space (X_∂, d_∂) such that

$$(\partial M_j, d_j) \xrightarrow{GH} (X_\partial, d_\partial). \tag{1.12}$$

In addition, if for all j we have

$$\mathrm{Vol}(\partial M_j) \leq A. \tag{1.13}$$

Then there is a subsequence that converges in SWIF sense to a non zero integral current space:

$$(M_{j_k}, d_{j_k}, T_{j_k}) \xrightarrow{\mathcal{F}} (Y \subset X, d_X, T). \tag{1.14}$$

such that

$$X \setminus X_\partial \subset Y. \tag{1.15}$$

If the GH limit of the boundaries is contained in the SWIF limit of the manifolds (or in the completion),

$$\partial M_j \xrightarrow{GH} X_\partial \subset Y, \tag{1.16}$$

then $X = Y$ ($X = \bar{Y}$). Thus, X is countably $\mathcal{H}^n$ rectifiable, where $\mathcal{H}^n$ denotes n-Hausdorff measure.

In Example 5.3.3 we construct a sequence of manifolds which have GH and SWIF limits that do not agree and that satisfies all the conditions of Theorem 1.0.2, except $X_\partial \subset Y$. In Example 5.3.2 we construct a sequence of manifolds which have GH and SWIF limits that do not agree and that satisfies all the conditions of Theorem 1.0.2, except the Ricci bound.

In [2], Anderson-Katsuda-Kurylev-Lassas-Taylor prove that a sequence of manifolds with bounds on various injectivity radii and on diameter as well as two sided Ricci curvature bounds on the manifolds and their boundaries, one has a subsequence converging in the $C^{1,\alpha}$ sense (cf. Theorem 4.2.1 within). Removing half of these hypothesis we can prove GH and SWIF convergence of the manifolds [Theorem 1.0.3]. Recall that the boundary injectivity radius of $p \in \partial M$ is defined by

$$i_\partial(p) = \inf\{t \,|\, \gamma_p \text{ stops minimizing at t}\}, \tag{1.17}$$

where γ_p is the geodesic in M such that $\gamma_p'(0)$ is the inward unitary normal tangent vector at p. The boundary injectivity radius of M is defined by

$$i_\partial(M) = \inf\{i_\partial(p) \,|\, p \in \partial M\}. \tag{1.18}$$

Inside we prove that a uniform bound on the boundary injectivity radius of the manifolds implies that the diameter bounds in (1.4) can be reduced to a single diameter bound. In addition, we prove GH convergence of the boundaries, (1.6), and $X_\partial \subset \bar{Y}$. Combining this with our theorems above we obtain:

1.0.3 Theorem. *Let $n \in \mathbb{N}$ and $\delta, D, V, \theta, \iota > 0$, $\iota > \delta$. Suppose that (M_j, g_j) is a sequence of n-dimensional compact oriented manifolds with boundary that satisfy*

$$\mathrm{Ric}(M_j) \geq 0, \ \mathrm{Vol}(M_j) \leq V, \ \mathrm{Diam}(M_j^\delta, d_{M_j^\delta}) \leq D, \tag{1.19}$$

$$\exists q \in M_j^\delta \text{ such that } \mathrm{Vol}(B(q,\delta)) \geq \theta\delta^n, \tag{1.20}$$

where $B(q,\delta)$ is the ball in M_j with center q and radius δ and

$$i_\partial(M_j) \geq \iota. \tag{1.21}$$

Then there is a subsequence such that

$$(M_{j_k}, d_{j_k}) \xrightarrow{GH} (X, d_X) \text{ and } (\partial M_{j_k}, d_{j_k}) \xrightarrow{GH} (X_\partial, d_X). \tag{1.22}$$

If in addition $\mathrm{Vol}(\partial M_j) \leq A$, then there is a subsequence and a non zero integral current space such that:

$$(M_{j_k}, d_{j_k}, T_{j_k}) \xrightarrow{\mathcal{F}} (Y \subset X, d_X, T), \tag{1.23}$$

$$X \setminus X_\partial \subset Y \ \text{ and } \ X = \bar{Y}. \tag{1.24}$$

In Example 5.3.3 we construct a sequence of manifolds which has a GH limit X and a SWIF limit Y such that $X \neq \bar{Y}$. The sequence satisfies all the conditions of Theorem 1.0.3, except that the boundary injectivity radii do not have a positive uniform lower bound. In Example 6.3.4 we construct a sequence of manifolds that satisfies all the conditions of the Theorem 1.0.3.

Our results concerning sequences of Riemannian manifolds with boundary are consequences of the next two theorems concerning sequences of metric spaces and integral current spaces. Let (X, d) be a metric space. In [25], the author and Sormani defined the boundary of a metric space to be

$$\partial X = \bar{X} \setminus X, \tag{1.25}$$

where $\bar{X}$ is the metric completion of X. This agrees with the notion of the boundary of a manifold with boundary if one takes X to be interior of the manifold.

For $\delta > 0$ let

$$X^\delta := \{ x \in X \mid d(x, \partial X) > \delta\} \tag{1.26}$$

be the δ-inner region of X.

In prior work of the author with Sormani [25], applying Gromov's Embedding Theorem [11], it was proven that given a sequence $\{\delta_i\}$ decreasing to zero and a sequence of compact metric spaces with boundary that converge in GH sense, $(X_j, d_j) \xrightarrow{\text{GH}} (X, d_X)$, there is a subsequence $\{j_k\}$ and compact subspaces

$$X(\delta_i) \subset X \text{ and } X_\partial \subset X \tag{1.27}$$

such that

$$(\overline{X^{\delta_i}_{j_k}}, d_{j_k}) \xrightarrow{\text{GH}} (X(\delta_i), d_X) \tag{1.28}$$

for all i and

$$(\partial X_{j_k}, d_{j_k}) \xrightarrow{\text{GH}} (X_\partial, d_X). \tag{1.29}$$

In Theorem 1.0.4 we prove the converse:

1.0.4 Theorem. *Let (X_j, d_j) be a sequence of precompact length metric spaces with precompact boundary. Suppose that there is a decreasing sequence $\{\delta_i\}_{i=1}^\infty \subset \mathbb{R}$ that converges to zero such that $X_j^{\delta_i} \neq \emptyset$ and $\{(\overline{X_j^{\delta_i}}, d_j)\}_{j=1}^\infty$ converges in GH sense for all i to some compact metric space $(X(\delta_i), d_{X(\delta_i)})$,*

$$(\overline{X_j^{\delta_i}}, d_j) \xrightarrow{GH} (X(\delta_i), d_{X(\delta_i)}). \tag{1.30}$$

Suppose that there is a compact metric space (X_∂, d_∂) such that

$$(\partial X_j, d_j) \xrightarrow{GH} (X_\partial, d_\partial). \tag{1.31}$$

Then a subsequence of $\{(\bar{X}_j, d_j)\}_{j=1}^\infty$ converges in GH sense.

The next theorem is the key ingredient to prove our theorems in which both GH and SWIF limits agree. In particular it is applied to prove Theorem 1.0.2, Theorem and 1.0.3.

1.0.5 Theorem. *Let (X_j, d_j, T_j) be precompact integral current spaces. Suppose that there exist a compact metric space (X, d) and a non zero integral current space $(Y \subset X, d, T)$ such that*

$$(\bar{X}_j, d_j) \xrightarrow{GH} (X, d) \text{ and } (X_j, d_j, T_j) \xrightarrow{\mathcal{F}} (Y, d, T). \tag{1.32}$$

and there is a subsequence such that

$$(\partial X_{j_k}, d_{j_k}) \xrightarrow{GH} (X_\partial, d) \tag{1.33}$$

where ∂X_{j_k} *is defined as in (1.25). Suppose in addition that*

$$X_\partial \subset \bar{Y}, \tag{1.34}$$

and there is a decreasing sequence $\delta_i \to 0$ *such that the inner regions converge*

$$(\overline{X^{\delta_i}_{j_k}}, d_{j_k}) \xrightarrow{GH} X(\delta_i) \tag{1.35}$$

with

$$X(\delta_i) \subset Y \qquad \forall i \in \mathbb{N}. \tag{1.36}$$

Then $X = \bar{Y}$.

In Example 5.3.2 we construct a sequence of manifolds which have GH and SWIF limits that do not agree that satisfies all the conditions of Theorem 1.0.5, except that $X^\delta \subset Y$. In Example 5.3.3 we describe a sequence of regions in Euclidean space with GH and SWIF limits that do not agree. This sequence satisfies all the conditions of Theorem 1.0.5, except $X_\partial \subset Y$.

We now provide an outline for the paper. We begin with two chapters reviewing key theorems needed to prove the results in this paper. In Chapter 2 we review GH convergence as defined in Gromov's book [14]. We state prior GH convergence results for manifolds with boundary proven by Sormani and the author [25]. Then we state Colding's volume estimate for balls GH close to balls in Euclidean space [7], and some of Cheeger-Colding's results about GH limits of non collapsed sequences of Riemannian manifolds with Ricci curvature bounded below [6].

In Chapter 3 we go over Ambrosio-Kirchheim's results concerning integral currents [1]. We then review Sormani-Wenger's integral current spaces, SWIF distance and some of their theorems. We present a simplified proof of Sormani-Wenger's GH=SWIF theorem for manifolds with no boundary [30] (cf. Theorem 3.3.1). This simplified proof will be adapted later to prove Theorem 1.0.2. We end the Chapter discussing another GH=IF theorem for Alexandrov Spaces proven by Li with the author [19]. This proof uses results by Burago-Gromov-Perelman ([5][4]) for Alexandrov spaces rather than Colding and Perelman results for manifolds with boundary [7][26] which are used in Sormani-Wenger's proof.

In Chapter 4 we review prior convergence theorems concerning Riemannian manifolds with smooth boundary by Kodani [17], Wong [34], Knox [16] and Anderson-Katsuda-Kurylev-Lassas-Taylor [2]. A survey of this material by the author has been published before in [23].

In Section 4.5, we review prior work of the author with Sormani published in [25] in which the notion of a glued limit space is introduced. This limit is built from the GH limits of inner regions and it may exist even when there is no GH limit (cf. Example 5.1.2 and Example 5.1.3 within). If the GH limit exists then the glued limit space can be a proper subset of the GH limit. See Example 5.3.3. In that paper we also prove that for certain noncollapsing sequences of manifolds with boundary that have nonnegative Ricci, the glued limit space has Hausdorff dimension equal to n and positive $\mathcal{H}^n$ lower density (cf. Theorem 4.5.2).

In Section 4.6, we review prior work of the author alone concerning SWIF convergence of oriented Riemannian manifolds with boundary [24]. In Theorem 4.6.4 SWIF precompactness is proven for sequences of manifolds with nonnegative Ricci curvature, mean curvature bounded above and volume of the boundary bounded above and diameter bounded above. In Theorem 4.6.5, SWIF precompactness is obtained by replacing the volume and diameter on the manifolds with such bounds on the boundary alone but requiring the mean curvature is strictly negative. These theorems follow from volume and area estimates proven by the author (cf. Theorem 4.6.3) using a Laplace Comparison Theorem for manifolds with boundary (cf. Theorem 4.6.2). Once the volume estimates are obtained the result follows from Wenger's Compactness Theorem [33] (cf. Theorem 4.6.1).

In Chapter 5, we prove the new convergence theorems for metric spaces stated above: Theorem 1.0.4 and Theorem 1.0.5. We also present some important examples. There we see that the sequence of metric spaces described in Example 5.1.2 satisfies all the conditions of Theorem1.0.4, except the GH convergence of the boundaries. Meanwhile, the sequence of metric spaces described in Example 5.1.3 satisfies all the conditions of Theorem1.0.4, except that there is no GH convergent subsequence of inner regions (M_j^δ, d_j) for any δ small. In both examples the conclusion of Theorem1.0.4 does not hold. To prove the importance of our conditions in Theorem1.0.5, we present two examples. In Example 5.3.2 we describe a sequence for which the GH limit of the sequences of inner regions is not contained in the SWIF limit and in Example 5.3.2 we show a sequence for which the GH and SWIF limit do not agree since the GH limit of the boundaries is not contained in the SWIF limit.

In Chapter 5 we also prove Theorem 5.2.1 which deals with the case when the GH limit of (X_j, d_j) agrees with the GH limit of $(\partial X_j, d_j)$. In Example 5.2.3 we describe a sequence of 3-dimensional cylinders $X_j \subset \mathbb{R}^3$ such that $\{X_j\}$ and $\{\partial X_j\}$ GH converge to a segment. Notice that if the Hausdorff dimension of the GH limit drops then the GH limit cannot agree with the SWIF limit. In Lemma 5.3.4 we characterize the points of $X \setminus X_\partial$.

In Chapter 6 we prove our new theorems about limits of Riemannian manifolds with boundary (Theorem 1.0.1, Theorem 1.0.2 and Theorem 1.0.3) and present key examples related to these theorems. In Section 6.1 we prove Theorem 1.0.1 by applying Theorem 1.0.4. We note that Example 5.3.3 satisfies all the conditions of Theorem 1.0.1, hence it has a GH limit.

In Section 6.2 we prove Theorem 1.0.2 by adapting the simplified proof of Sormani-Wenger's GH=IF Theorem for manifolds with no boundary [31]. See Chapter 3.3 for details of their proof. We also see that the sequence described in Example 5.3.3 satisfies all the conditions of Theorem 1.0.2, except Equation (1.34) and so the GH limit does not coincide with the SWIF limit. Then we provide two examples, Example 6.2.1 and Example 6.2.2, in which the sequences satisfy all the conditions of Theorem 1.0.2. Additionaly, in these two examples the GH limit of the sequences of boundaries, X_∂, do not agree with the SWIF limit of the boundaries, $(Y_\partial, d, T_\partial)$, showing that we cannot replace Equation (1.34) by $X_\partial = Y_\partial$.

In Section 6.3 we prove Theorem 1.0.3. To prove it we will apply Theorem 1.0.2. To do so, we first prove uniform diameter bounds for sequences of inner regions, Lemma 6.3.1, and the GH convergence of the sequence of boundaries, Lemma 7.0.5. Then we show that $X_\partial \subset \bar{Y}$. From Example 5.3.3 we notice that a positive lower bound on the injectivity radii is not necessary for the GH convergence. In Example 6.3.4 we present a sequence that satisfies all the conditions of Theorem 1.0.3, except the posivite boundary injectivity radii bound. In this example the GH limit and the SWIF limit agree showing that the hypothesis in the boundary injectivity radii is stronger than necessary.

In Chapter 7 we prove GH convergence of sequences of boundaries, Theorem 7.0.5. In order to prove this theorem, in Proposition 7.0.6 we show that the GH convergence of $(\partial M_j, d_{\partial M_j})$ implies the convergence of $(\partial M_j, d_j)$ and in Proposition 7.0.7 we obtain a uniform Ricci curvature bound on the boundaries. Notice that in Theorem 1.0.1 and Theorem 1.0.2 one of the hypothesis is the convergence in GH sense of the sequence $(\partial M_j, d_j)$, Equation (1.6) and Equation (1.12). So Theorem 7.0.5 can be used in these cases.

Chapter 2

A Review of GH Limits

In this chapter we list GH convergence results that will be used in the next chapters including prior published results of the author with Sormani as well as work of Gromov and Cheeger-Colding. In Section 2.1 we define Gromov-Hausdorff distance and state Gromov's compactness theorem and its converse; Theorem 2.1.2 and 2.1.4. In Section 2.2 we review the GH compactness theorems for δ-inner regions of manifolds with boundary proven by the author and Sormani in [25], Theorem 2.2.1 and 2.2.3. In Section 2.3 we state Colding's theorem about the volume of balls being close to the volume of balls in $\mathbb{R}^n$, provided the balls are GH close and the Ricci curvature is bounded below [7], cf. Theorem 2.3.1. We also state Cheeger-Colding's theorem [6] cf. 2.3.7 about the singular set of the GH limit of a noncollapsing sequences of Riemannian manifolds with curvature bounded below having zero Hausdorff measure and the regular points having all tangent cones of the maximal dimension. These results will be aplied to prove that the GH limit agrees with the SWIF limit in Theorem 1.0.2.

2.1 Gromov-Hausdoff Convergence

Here we introduce Gromov-Hausdorff convergence. A detailed exposition see Burago-Burago-Ivanov [4] and Gromov [14].

The Hausdorff distance in a complete metric space Z, d_H^Z, between two subsets $A, B \subset Z$ is defined as

$$d_H^Z(A, B) = \inf\{\varepsilon > 0 : A \subset T_\varepsilon(B) \text{ and } B \subset T_\varepsilon(A)\}. \tag{2.1}$$

Here, $T_\varepsilon(A)$ denotes the ε neigborhood of A.

2.1.1 Definition (Gromov)**.** *Let* (X_i, d_{X_i})*,* $i = 1, 2$*, be two metric spaces. The Gromov-Hausdorff distance between them is defined as*

$$d_{GH}(X_1, X_2) = \inf\ d_H^Z(\varphi_1(X_1), \varphi_2(X_2)) \tag{2.2}$$

where Z *is a complete metric space and* $\varphi_i : X_i \to Z$ *are distance preserving maps.*

The above function is symmetric and satisfies the triangle inequality. It is a distance when considering compact metric spaces.

2.1.2 Theorem (Gromov)**.** *Let* $\left(X_j, d_j\right)$ *be a sequence of compact metric spaces. If there exist* D *and* $N : (0, \infty) \to \mathbb{N}$ *such that for all* j

$$\mathrm{Diam}(X_j) \leq D \tag{2.3}$$

and for all ε *there are* $N(\varepsilon)$ ε*-balls that cover* X_j*, then there exist a compact metric space* (X, d_X) *and a subsequence such that*

$$\left(X_{j_k}, d_{j_k}\right) \xrightarrow{GH} (X, d_X). \tag{2.4}$$

2.1.3 Definition. *We say that a family,* $\mathcal{F}$*, of compact metric spaces is equibounded if there exists a function* $N : (0, \infty) \to \mathbb{N}$ *as in theorem 2.1.2. For the purpose of clarity we will denote* N *by* $N(\,\cdot\,, \mathcal{F})$ *when working with different families.*

2.1.4 Theorem (Gromov)**.** *Let* $\{(X_j, d_j)\}$ *be a sequence of compact metric spaces that converges in GH sense. Then* $\{(X_j, d_j)\}$ *is equibounded and there is* $D > 0$ *such that* $\mathrm{Diam}(X_j) \leq D$ *for all* j*.*

2.1.5 Theorem (Gromov in [12])**.** *Let* $\{(X_j, d_j)\}$ *be a sequence of compact metric spaces that converges in GH sense to* (X_∞, d_∞)*. Then there is a compact metric space* (Z, d) *and isometric embeddings* $\varphi_j : X_j \to Z$ *such that a subsequence* $\{(\varphi_{j_k}(X_{j_k}), d)\}$ *converges in Hausdorff sense to* $(\varphi_\infty(X_\infty), d)$*.*

Whenever we have a GH converging sequence, we choose embeddings φ_j that satisfy the result of the previous theorem. Then we consider $\{(X_{j_k}, d_{j_k})\}$ to be our original sequence, $\{(X_j, d_j)\}$. We say that a sequence $x_j \in X_j$ converges to $x_\infty \in X_\infty$ if

$$\varphi_j(x_j) \to \varphi_\infty(x_\infty). \tag{2.5}$$

Moreover, using the following theorem we can say that a sequence $A_j \subset X_j$ GH converges to a set $A_\infty \subset X_\infty$.

2.1.6 Theorem. *[Blaschke] Let (Z, d) be a compact metric space and A_j be a sequence of closed subsets of Z. Then, there is a subsequence A_{j_k} that converges in Hausdorff sense.*

For Riemannian manifolds with no boundary the following compactness theorem holds.

2.1.7 Theorem (Gromov)**.** *Every sequence of n-dimensional compact Riemannian manifolds with diameter $\leq D$ and $\mathrm{Ric} \geq (n-1)k$ has a GH convergent subsequence.*

2.2 Prior GH Convergence Results of the Author with Sormani

In this section we review results published in [25].

Recall that for a Riemannian manifold with boundary (M, g) the δ-inner region of M is given by

$$M^\delta = \{\, x \in M : d(x, \partial M) > \delta\}. \tag{2.6}$$

The inner regions may be endowed with the induced length metric d_{M^δ}

$$d_{M^\delta}(x, y) := \inf\left\{L_g(C) : \; C : [0,1] \to M^\delta,\, C(0) = x,\, C(1) = y\right\} \tag{2.7}$$

(which is possibly infinite) or the restricted metric d

$$d(x, y) := \inf\left\{L_g(C) : \; C : [0,1] \to M,\, C(0) = x,\, C(1) = y\right\} \tag{2.8}$$

where

$$L_g(C) = \int_0^1 g(C'(t), C'(t))\, dt. \tag{2.9}$$

In the first theorem presented here, we proved GH subconvergence of the sequence M_j^δ with respect to the restricted metric since this provides more information about the original sequence of manifolds M_j. But note that the diameter bound we required is with respect to the induced length metric. In particular the inner regions were assumed to be path connected:

2.2.1 Theorem (P–Sormani)**.** *Given $n \in \mathbb{N}$ and $\delta, D, V, \theta > 0$ suppose that (M_j, g_j) is a sequence of compact oriented manifolds with boundary such that*

$$\mathrm{Ric}(M_j) \geq 0, \;\; \mathrm{Vol}(M_j) \leq V, \;\; \mathrm{Diam}(M_j^\delta,\, d_{M_j^\delta}) \leq D, \tag{2.10}$$

$$\exists q \in M_j^\delta \text{ such that } \operatorname{Vol}(B(q,\delta)) \geq \theta\delta^n, \tag{2.11}$$

where $B(q,\delta)$ is the ball in M_j with center q and radius δ. Then there is a subsequence $\{j_k\}$ and a compact metric space (X^δ, d_{X^δ}) such that

$$(\overline{M_{j_k}^\delta}, d_{M_{j_k}}) \xrightarrow{GH} (X(\delta), d_{X(\delta)}). \tag{2.12}$$

2.2.2 Remark. *In the proof of Theorem 2.2.1 it was shown that for all $p \in M^\delta$ and $\varepsilon < \delta/2$*

$$\operatorname{Vol}(B(p,\varepsilon)) \geq 2^{-nD/\varepsilon}\theta\varepsilon^n. \tag{2.13}$$

This estimate also works for $\varepsilon = \delta/2$. Choosing $\varepsilon = \delta/2$ and applying Bishop-Gromov Volume Comparison we get:

$$\operatorname{Vol}(B(p,r)) \geq 2^{-nD/(\delta/2)}\theta r^n. \tag{2.14}$$

for all $r \leq \delta/2$.

For a decreasing sequence of real numbers, $\delta_i \to 0$, we obtained simultaneous convergence of sequences of inner regions.

2.2.3 Theorem (P–Sormani)**.** *Take $n \in \mathbb{N}$, a decreasing sequence, $\delta_i \to 0$, $D_i > 0$, $i = 0, 1, 2...$, $V > 0$ and $\theta > 0$. Suppose that (M_j, g_j) is a sequence of compact n-dimensional Riemannian manifolds with boundary such that*

$$\operatorname{Ric}(M_j) \geq 0, \;\; \operatorname{Vol}(M_j) \leq V, \;\; \operatorname{Diam}(M_j^{\delta_i}, d_{M_j^{\delta_i}}) \leq D_i \;\; \forall i \tag{2.15}$$

and

$$\exists q \in M_j^\delta \text{ such that } \operatorname{Vol}(B(q,\delta)) \geq \theta\delta^n, \tag{2.16}$$

where $B(q,\delta)$ is the ball in M_j with center q and radius δ. Then there is a subsequence $\{j_k\}$ such that $(\overline{M_{j_k}^{\delta_i}}, d_{M_{j_k}})$ converges in Gromov-Hausdorff sense for all i.

In [25], the author and Sormani applied these theorems to construct "glued limit spaces" which are created by gluing together the GH limits of the inner regions in a canonical way. See Theorem 6.1 and Theorem 6.3 of [25] for an introduction to this notion of a glued limit space. These glued limit spaces may exist even when the sequence of manifolds with boundary has no GH limit. In fact they need not be compact. In Theorem 7.1 and Theorem 7.4 of [25] we constructed glued limit spaces under various curvature conditions. In Theorem 8.8 of [25] we proved nice properties for the glued limit spaces of noncollapsing sequences of manifolds with boundary that have nonnegative Ricci curvature. Numerous examples were constructed to demonstrate the importance of the various hypothesis. We do not review all these results here because they are not applied in this paper.

2.3 Cheeger-Colding Theorems

Here we review a result by Colding [7] and few of the many important theorems of Cheeger-Colding proven in [6] that we need to prove that the GH limit is inside the SWIF limit, see proof of Theorem 1.0.2.

The next theorem tells us that the volume of balls of manifolds are close to the volume of balls in Euclidean space when these balls are close in GH sense. This result is used in Sormani-Wenger [30] to prove that the GH limit of manifolds with no boundary coincides with the SWIF limit (cf Theorem 3.3.1 within). We will use this theorem as well to prove our new Theorem 1.0.2.

2.3.1 Theorem. *[Colding, Corollary 2.19 in [7]] For all $\varepsilon > 0$ and $n \in \mathbb{N}$ there exist $k(\varepsilon, n) > 0$ and $\delta(\varepsilon, n) > 0$ such that for any complete n-dimensional Riemannian manifold M that satisfies*

$$\mathrm{Ric}(M) \geq -(n-1)k \text{ and } d_{GH}(B(p,1), B(0,1)) < \delta, \tag{2.17}$$

the following holds:

$$|\,\mathrm{Vol}(B(p,1)) - \mathrm{Vol}(B(0,1))| < \varepsilon, \tag{2.18}$$

where $B(0,1)$ denotes the open ball of radius 1 and center 0 in the Euclidean space $\mathbb{R}^n$.

In the noncollapsing case the volume of the manifolds converge to the Hausdorff measure of the limit space.

2.3.2 Theorem (Cheeger-Colding [6])**.** *Let $k \in \mathbb{R}$, $v > 0$ and $\{M^n_j\}$ be a sequence of n-dimensional compact Riemannian manifolds such that*

$$\mathrm{Ric}(M_j) \geq (n-1)k, \;\; M_j \xrightarrow{GH} X \text{ and } \mathrm{Vol}(M_j) \geq v. \tag{2.19}$$

Then for all $r > 0$ and $x \in X$

$$\lim_{j\to\infty} \mathrm{Vol}(B(x_j, r)) = \mathcal{H}^n(B(x,r)), \tag{2.20}$$

where $x_j \in M_j$ such that $x_j \to x$ and $\mathcal{H}^n$ denotes n-Hausdorff measure. In particular,

$$\lim_{j\to\infty} \mathrm{Vol}(M_j) = \mathcal{H}^n(X). \tag{2.21}$$

2.3.3 Remark. *Since the theorem is proven locally, if M_j is a sequence of n-dimensional manifolds with boundary that satisfy*

$$\mathrm{Ric}(M_j) \geq (n-1)k, \;\; M_j \xrightarrow{GH} X \tag{2.22}$$

and for each $x \in M_j^{\delta}$

$$\mathrm{Vol}(B(x,r)) \geq v(\delta) > 0 \tag{2.23}$$

for $r \leq \delta/2$. Then

$$\lim_{j\to\infty} \mathrm{Vol}(B(x_j, r)) = \mathcal{H}^n(B(x,r)), \tag{2.24}$$

where $x \in X$, $x_j \in M_j^{\delta}$ such that $x_j \to x$ and $r \leq \delta/2$.

2.3.4 Definition. *A sequence $\{(X_j, d_j, p_j)\}$, $p_j \in X_j$, converges in the pointed Gromov-Hausdorff sense to a metric space (X, d, p) if the following holds. For all $r > 0$ and $\varepsilon > 0$ there exists $N \in N$ and maps*

$$f_j : B(p_j, r) \to X \tag{2.25}$$

such that

$$f(p_j) = p, \quad d_{GH}(B(p_j, r), f_j(B(p_j, r))) < 2\delta \tag{2.26}$$

and

$$B(p, r-\varepsilon) \subset T_{\varepsilon} f_j(B(p_j, r)), \tag{2.27}$$

where $T_{\varepsilon} f_j(B(p_j, r))$ is the ε neighborhood of $f_j(B(p_j, r))$.

2.3.5 Definition. *Let (X, d) be a metric space. A tangent cone at $x \in X$ is a complete pointed GH limit (X, d_∞, x) of a sequence of the form $\{(X, r_j^{-1} d, x)\}$, where* $\lim_{j\to\infty} r_j = 0$.

2.3.6 Definition. *A point $x \in X$ is called regular if for some k every tangent cone of x is isometric to $\mathbb{R}^k$. A point is called non regular if it is not regular.*

2.3.7 Theorem (Cheeger-Colding, Theorem 2.1 and Theorem 5.9 [6])**.** *Let $k \in \mathbb{R}$, $v > 0$ and $\{M_j^n\}$ be a sequence of n-dimensional compact Riemannian manifolds such that*

$$\mathrm{Ric}(M_j) \geq (n-1)k, \;\; M_j \xrightarrow{GH} X \text{ and } \mathrm{Vol}(M_j) \geq v. \tag{2.28}$$

Then the set of nonregular points of X has zero n-Hausdorff measure and all the tangent cones of the regular points of X are isometric to $\mathbb{R}^n$.

2.3.8 Remark. *Since the theorem is proven locally, if M_j is a sequence of n dimensional manifolds with boundary that satisfy*

$$\mathrm{Ric}(M_j) \geq (n-1)k, \;\; M_j \stackrel{GH}{\longrightarrow} X \tag{2.29}$$

and for each $x \in M_j^\delta$

$$\mathrm{Vol}(B(x,r)) \geq v(\delta) > 0 \tag{2.30}$$

for $r < \delta/2$, then the set of nonregular points of X contained in

$$X(\delta) = \{x \in X \,|\, \exists\, x_j \in M_j^\delta \to x\} \tag{2.31}$$

has zero n-Hausdorff measure and all the tangent cones of the regular points of X contained in $X(\delta)$ are isometric to $\mathbb{R}^n$.

Chapter 3

A Review of Integral Current Spaces and SWIF Convergence

In Section 3.1 we review the notion and properties of integral currents that appear on Ambrosio-Kirchheim's paper "Currents in Metric Spaces" [1]. Here we see that an integral current, T, in a metric space is a current acting on a tuple of functions (rather than a differential form) that has integer valued Borel weight functions whose boundaries are also integer rectifiable currents. The set of the current, denoted set(T), is an oriented countably $\mathcal{H}^n$ rectifiable subset of the given metric space.

In Section 3.2 we see that Sormani-Wenger [31] defined integral rectifiable current spaces, (Y, d, T), where T is an integral current in $\bar{Y}$ and set(T) = Y. We also define the Sormani-Wenger intrinsic flat distance (SWIF distance) which was defined in imitation of Gromov's intrinsic Hausdorff distance (GH distance), except that the Hausdorff distance, d_H, in Definition 2.1.1 is replaced by Federer-Fleming's flat distance d_F [31]. We end Section 3.2 with Sormani-Wenger's Theorem that shows that under certain conditions the GH limit contains the SWIF limit from [31].

In Section 3.3 we explain Sormani-Wenger's GH=IF Theorem for manifolds with no boundary, [30] (cf. Theorem 3.3.1 within).

In Section 3.4 we review joint work of the author with Li appearing in a preprint entitled "On the Sormani-Wenger Intrinsic Flat Convergence of Alexandrov Spaces" [19].

3.1 Integral Currents

The aim of this section is to review Ambrosio-Kirchheim's notion of an integral current on a metric space (which extends the notion of Federer-Flemming) [1] [9]. To accomplish this we define currents, Definition 3.1.1, and integer currents, Definition 3.1.5. Then we mention two important properties of integer currents proven by Ambrosio-Kirchheim [1]. The characterization of the mass measure, Lemma 3.1.8, which is amply used in SWIF convergence and that the set is a countably rectifiable metric space, Lemma 3.1.9. The section finishes with the definition of an integral current, Definition 3.1.10.

For a metric space Z, denote by $\mathcal{D}^m(Z)$ the collection of $(m+1)$-tuples of Lipschitz functions where the first entry is a bounded function:

$$\mathcal{D}^m(Z) = \{(f,\pi) = (f,\pi_1...,\pi_m) \mid f,\pi_i : Z \to \mathbb{R} \text{ Lipschitz and } f \text{ is bounded}\}. \tag{3.1}$$

3.1.1 Definition (Ambrosio-Kirchheim)**.** *Let Z be a complete metric space. A multilinear functional $T : \mathcal{D}^m(Z) \to \mathbb{R}$ is called an m dimensional current if it satisfies:*

i) If there is an i such that π_i is constant on a neighborhood of $\{f \neq 0\}$ then $T(f,\pi) = 0$.

ii) T is continuous with respect to the pointwise convergence of the π_i for $\mathrm{Lip}(\pi_i) \leq 1$.

iii) There exists a finite Borel measure μ on Z such that for all $(f,\pi) \in \mathcal{D}^m(Z)$

$$|T(f,\pi)| \leq \prod_{i=1}^{m} \mathrm{Lip}(\pi_i) \int_Z |f|\, d\mu. \tag{3.2}$$

The collection of all m dimensional currents of Z is denoted by $\mathbf{M}_m(Z)$.

To each current we associate a measure and a mass:

3.1.2 Definition (Ambrosio-Kirchheim)**.** *Let $T : \mathcal{D}^m(Z) \to \mathbb{R}$ be an m-dimensional current. The mass measure of T is the smallest Borel measure $\|T\|$ such that (3.2) holds for all $(f,\pi) \in \mathcal{D}^m(Z)$.*

The mass of T is defined as

$$M(T) = \|T\|(Z) = \int_Z d\|T\|. \tag{3.3}$$

To give the definition of integer current, we first see how to get a current by pushing forward another one, Definition 3.1.3, and in Example 3.1.4 we define a current in Euclidean space, $\mathbb{R}^n$, that only requires an integer valued L^1 function.

3.1.3 Definition (Ambrosio-Kirchheim Defn 2.4)**.** *Let $T \in \mathbf{M}_m(Z)$ and $\varphi : Z \to Z'$ be a Lipschitz map. The* pushforward *of T to a current $\varphi_\# T \in \mathbf{M}_m(Z')$ is given by*

$$\varphi_\# T(f, \pi) = T(f \circ \varphi, \pi_1 \circ \varphi, ..., \pi_m \circ \varphi). \tag{3.4}$$

3.1.4 Example (Ambrosio-Kirchheim)**.** *Let $h : A \subset \mathbb{R}^m \to \mathbb{Z}$ be an L^1 function. Then $[\![h]\!] : \mathcal{D}^m(\mathbb{R}^m) \to \mathbb{R}$ given by*

$$[\![h]\!]\,(f, \pi) = \int_{A\subset\mathbb{R}^m} hf \det(\nabla\pi_i)\; d\mathcal{L}^m \tag{3.5}$$

in an m dimensional current, where $\nabla\pi_i$ are defined almost everywhere by Rademacher's Theorem.

Now we proceed to define integer currents:

3.1.5 Definition (Defn 4.2, Thm 4.5 in Ambrosio-Kirchheim [1])**.** *Let $T \in \mathbf{M}_m(Z)$. T is an integer rectifiable current if it has a parametrization of the form $(\{\varphi_i\}, \{\theta_i\})$, where*

i) $\varphi_i : A_i \subset \mathbb{R}^m \to Z$ is a countable collection of bilipschitz maps such that A_i are precompact Borel measurable with pairwise disjoint images,

ii) $\theta_i \in L^1(A_i, \mathbb{N})$ such that

$$T = \sum_{i=1}^{\infty} \varphi_{i\#}[\![\theta_i]\!] \quad \textit{and} \quad \mathbf{M}(T) = \sum_{i=1}^{\infty} \mathbf{M}\left(\varphi_{i\#}[\![\theta_i]\!]\right). \tag{3.6}$$

The mass measure is

$$\|T\| = \sum_{i=1}^{\infty} \|\varphi_{i\#}[\![\theta_i]\!]\|. \tag{3.7}$$

The space of m dimensional integer rectifiable currents on Z is denoted by $\mathcal{I}_m(Z)$.

In the next lemma we see that the mass measure of an integral current is concentrated in its set.

3.1.6 Definition (Ambrosio-Kirchheim)**.** *Let $T \in \mathbf{M}_m(Z)$, the canonical set of T, denoted* set(T)*, is*

$$\mathrm{set(T)} = \{\mathrm{p} \in \mathrm{Z} : \; \Theta_{*\mathrm{m}}\left(\|\mathrm{T}\|, \mathrm{p}\right) > 0\} \tag{3.8}$$

where
$$\Theta_{*m}(\|T\|, p) := \liminf_{r\to 0} \frac{\|T\|(B(p,r))}{\omega_m r^m}. \tag{3.9}$$
The function $\Theta_{*m}(\|T\|, p)$ is called the $\|T\|$ lower density of x and ω_m denotes the volume of the unit ball in $\mathbb{R}^m$.

3.1.7 Definition. *Given $T \in \mathbf{M}_m(Z)$ and $A \subset Z$ a Borel set, the restriction of T to A is a current, $T \llcorner A \in \mathbf{M}_m(Z)$, given by*

$$(T \llcorner A)(f, \pi) = T(\chi_A f, \pi). \tag{3.10}$$

where χ_A is the indicator funcion of A.

We note that the mass measure of $T \llcorner A$, $\|T \llcorner A\|$, equals $\|T\||_A$. Hence, $\|T \llcorner A\|(A) = |\, T \llcorner A\|(Z) = \|T\|(A)$. So
$$\Theta_{*m}(\|T\|, p) := \liminf_{r\to 0} \frac{\|T \llcorner B(p,r)\|(Z)}{\omega_m r^m}. \tag{3.11}$$

3.1.8 Lemma (Ambrosio-Kirchheim)**.** *Let $T \in \mathcal{I}_m(Z)$ with parametrization $(\{\varphi_i\}, \theta_i)$. Then there is a function*

$$\lambda : \mathrm{set(T)} \to [\mathrm{m}^{-\mathrm{m}/2}, 2^{\mathrm{m}}/\omega_{\mathrm{m}}] \tag{3.12}$$

such that
$$\Theta_{*m}(\|T\|, x) = \theta_T(x)\lambda(x) \tag{3.13}$$
for $\mathcal{H}^m$ almost every $x \in$ set(T) and

$$\|T\| = \theta_T \lambda \mathcal{H}^m \llcorner \mathrm{set(T)}, \tag{3.14}$$

where ω_m denotes the volume of an unitary ball in $\mathbb{R}^m$ and $\theta_T : Z \to \mathbb{N} \cup \{0\}$ is an L^1 function called weight given by

$$\theta_T = \sum_{i=1}^{\infty} \theta_i \circ \varphi_i^{-1} 1_{\varphi_i(A_i)}. \tag{3.15}$$

3.1.9 Lemma. *[Ambrosio-Kirchheim] If $T \in \mathcal{I}_m(Z)$, then* set(T) *is a countably $\mathcal{H}^m$ rectifiable metric space, ie. there exist a countable collection of biLipschitz charts*
$$\psi_i : A_i \subset \mathbb{R}^n \to U_i \subset Z \tag{3.16}$$
where A_i are Borel measurable sets and

$$\mathcal{H}^n(\,\mathrm{set(T)} \setminus \cup_{\mathrm{i}=1}^{\infty} \mathrm{U_i}\,) = 0. \tag{3.17}$$

Finally, we define integral currents.

3.1.10 Definition (Ambrosio-Kirchheim)**.** *An integral current is an integer rectifiable current, $T \in \mathcal{I}_m(Z)$, such that ∂T is also a current of finite mass where ∂T is defined by:*

$$\partial T\,(f, \pi_1, ..., \pi_{m-1}) = T\,(1, f, \pi_1, ..., \pi_{m-1}) \tag{3.18}$$

We denote the space of m dimensional integral currents on Z: $\mathbf{I}_m\,(Z)$.

3.2 Integral Current Spaces and SWIF Distance

In this section we define integral current spaces, Definition 3.2.1, the Sormani-Wenger intrinsic flat distance between these spaces, Definition 3.2.2 and state a theorem that shows that the SWIF limit is contained in the GH limit, Theorem 3.2.3.

3.2.1 Definition (Sormani-Wenger)**.** *Let (Y, d) be a metric space and $T \in \mathbf{I}_m(\bar{Y})$. If* $\mathrm{set}\,(\mathrm{T}) = \mathrm{Y}$ *then (Y, d, T) is called an m dimensional integral current space. T is called the integral current structure. X is called the canonical set.*

For technical reasons the zero integral current space is defined. It is denoted by **0** *and has current $T = 0$.*

We denote by $\mathcal{M}^m$ the space of m dimensional integral current spaces and by $\mathcal{M}^m_0$ the space of m dimensional integral current spaces whose canonical set is precompact.

Note that we can obtain an integral current space (M, d, T) from a compact oriented Riemannian manifold (M^n, g) (with or without boundary). In this case, d represents the metric induced by g and T is integration over M:

$$T(f, \pi_1, ..., \pi_n) = \int_M f d\pi_1 \wedge \cdots \wedge d\pi_n. \tag{3.19}$$

3.2.2 Definition (Sormani-Wenger)**.** *Let $(Y_i, d_i, T_i) \in \mathcal{M}^m$. Then the intrinsic flat distance between these two integral current spaces is defined by*

$$\begin{aligned} d_{\mathcal{F}}\,((Y_1, d_1, T_1), (Y_2, d_2, T_2)) &= \inf\{d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2)\} &(3.20)\\ &= \inf\{\mathbf{M}\,(U) + \mathbf{M}\,(V)\}, &(3.21)\end{aligned}$$

where the infimum is taken over all complete metric spaces, (Z, d)*, and all integral currents,* $U \in \mathbf{I}_m(Z), V \in \mathbf{I}_{m+1}(Z)$*, for which there exist isometric embeddings* $\varphi_i : \left(\bar{Y}_i, d_i\right) \to (Z, d)$ *with*

$$\varphi_{1\#}T_1 - \varphi_{2\#}T_2 = U + \partial V. \tag{3.22}$$

The **0** *m-dimensional integral current isometrically embeds into any Z with* $\varphi_\# 0 = 0 \in \mathbf{I}_m(Z)$*.*

It was proven in Theorem 3.27 of [31] that $d_{\mathcal{F}}$ is a distance on the class of precompact integral current spaces, $\mathcal{M}_0^m$.

We apply the following compactness theorem in all of our SWIF theorems. It is proven by Sormani-Wenger applying a combination of Gromov's Compactness Theorem and Ambrosio-Kirchheim's Compactness Theorem.

3.2.3 Theorem (Sormani-Wenger)**.** *Let* $\left(X_j, d_j, T_j\right)$ *be a sequence of m dimensional integral current spaces. If there exist D, M and* $N : (0, \infty) \to \mathbb{N}$ *such that for all j*

$$\mathrm{Diam}(X_j) \le D, \quad \mathbf{M}(T_j) + \mathbf{M}(\partial T_j) \le M \tag{3.23}$$

and, for all ε *there are* $N(\varepsilon)$ ε*-balls that cover* X_j*, then*

$$\left(X_{j_k}, d_{j_k}\right) \xrightarrow{GH} (X, d_X) \;\; \text{and} \;\; \left(X_{j_k}, d_{j_k}, T_{j_k}\right) \xrightarrow{\mathcal{F}} (Y, d, T), \tag{3.24}$$

where either (Y, d, T) *is an m dimensional integral current space with* $Y \subset X$ *or it is the* **0** *current space.*

3.2.4 Remark. *In a later theorem, Sormani-Wenger constructed a common compact metric space Z and isometric embeddings* $\varphi_j : X_j \to Z$ *and* $\varphi : X \to Z$ *such that*

$$\varphi_j(X_{j_k}) \xrightarrow{H} \varphi(X) \text{ and } \varphi_{j_k\#}(T_{j_k}) \xrightarrow{\mathcal{F}} \varphi_\#(T), \tag{3.25}$$

where $\mathrm{set}(\varphi_\#(\mathrm{T})) \subset \mathrm{X}$*. Here,* $\mathrm{set}(\varphi_\#(\mathrm{T}))$ *can be the empty set. Note that this is not proven using the common compact metric space constructed by Gromov in his work. In fact the Z constructed in [31] is a countably* $\mathcal{H}^{n+1}$ *rectifiable metric space.*

3.3 Sormani-Wenger: GH=SWIF when there is no boundary

For a sequence of compact oriented Riemannian manifolds, (M_j, g_j), with nonnegative Ricci curvature, $\partial M_j = \emptyset$ and two sided uniform volume bounds, Sormani-Wenger proved that the GH limit, X, of these type of sequences agree with the SWIF limit, Y, [30] (cf. Theorem 3.3.1 below). In [30], Sormani-Wenger prove a far more general theorem about a larger class of integral current spaces without boundary, and thus the proof is quite technically complicated. In this section we present an adapted and simplified version of their proof specialized to oriented Riemannian manifolds without boundary based upon Sormani's Geometry Festival presentation of the result. We refer to a review in a recent paper of Portegies-Sormani [27] that provides detailed proofs of some ideas presented there.

3.3.1 Theorem. *[Sormani-Wenger, Theorem 7.1 in [30]] Let (M_j, g_j) be a sequence of n dimensional oriented compact Riemannian manifolds with no boundary that satisfy the following*

$$\mathrm{Ric}(M_j) \geq 0, \ \ \mathrm{Diam}(M_j) \leq D \textit{ and } \mathrm{Vol}(M_j) \geq v \tag{3.26}$$

for some constants $v, D > 0$. Then there exist subsequence and an n-integral current space (X, d, T) such that

$$(M_{j_k}, d_{j_k}) \xrightarrow{GH} (X, d) \tag{3.27}$$

and

$$(M_{j_k}, d_{j_k}, T_{j_k}) \xrightarrow{\mathcal{F}} (X, d, T), \tag{3.28}$$

where T_j is integration of top forms over M_j.

From (3.26) by Gromov's Compactness theorem [14], Sormani-Wenger obtain a subsequence converging to a metric space (X, d) in GH sense, (3.27). Then applying Sormani-Wenger's theorem [31] (cf. Theorem 3.2.3 above), they get a further subsequence converging in SWIF sense to an integral current space (Y, d_Y, T). Note that by Remark 3.2.4 we can suppose that (M_{j_k}, d_{j_k}), (X, d), $(M_{j_k}, d_{j_k}, d_{j_k})$, (Y, d_Y, T) lie in a common metric space.

Recall that by the definition of an integral current space $x \in Y$ if and only if

$$\liminf_{r \to 0} \frac{||T||(B(x, r))}{\omega_n r^n} > 0. \tag{3.29}$$

Then, to prove that the GH limit coincides with the SWIF limit they estimate $\|T\|(B(x,r))$ for all $x \in X$.

Sormani-Wenger show that for each $x \in \mathcal{R}(X)$, where $\mathcal{R}(X)$ denotes the regular points in Cheeger-Colding sense of X (see Definition 2.3.6) there is $C(x) > 0$ and $r_0(x) > 0$ such that

$$\|T\|(B(x,r)) > C(x)r^n \qquad \forall r \leq r_0 \tag{3.30}$$

We now review how they prove this. Since the mass measure is only lower semi-continuous with respect to SWIF convergence [1], they use the notion of filling volume of a current [30]. The filling volume by definition is smaller than the mass and is continuous with respect to SWIF convergence:

3.3.2 Definition. *(c.f. [27]) Given an n-integral current space $N = (Y, d, T)$, $n \geq 1$, define the filling volume of ∂N by*

$$\mathrm{FillVol}(\partial N) = \inf\{\mathbf{M}(S)\,|S \textit{ is an } n+1 \textit{ integral current space such that } \partial S = \partial N\}. \tag{3.31}$$

That is, there is a current preserving isometry $\varphi : \partial S \to \partial N$ such that $\varphi_\sharp \partial S = \partial N$.

Thus, from the definition of filling volume and mass it follows that

$$\|T\|(N) = \mathbf{M}(N) \geq \mathrm{FillVol}(\partial N). \tag{3.32}$$

The continuity of the filling volume with respect to SWIF convergence theorem follows from the following theorem. This fact was first observed by Sormani-Wenger [30] building upon work by Wenger on flat convergence of integral currents in metric spaces [32]. The precise statement given here is Theorem 2.48 in work of Portegies-Sormani [27]:

3.3.3 Theorem (cf. Portegies-Sormani [27])**.** *For any pair of integral current spaces, M_i, we have*

$$\mathrm{FillVol}(\partial M_1) \leq \mathrm{FillVol}(\partial M_2) + d_{\mathcal{F}}(M_1, M_2). \tag{3.33}$$

We note that the notion of filling volume given in Definition 3.3.2 is not exactly the same notion as the Gromov Filling Volume [13], however many similar properties hold. Gromov's Filling volume is defined using chains rather than integral current spaces and the notion of volume used by Gromov is not the same as Ambrosio-Kirchheim's mass.

Now, in order to use the notion of filling volume and its continuity under SWIF convergence to estimate $\|T\|(B(x,r))$ for $x \in \mathcal{R}(X)$, we state Portegies-Sormani [27], Lemma 3.3.4 which allows us to view a ball as an integral current space and Sormani [29], Theorem 3.3.5, which allows us to take the limits of balls.

3.3.4 Lemma (cf. Lemma 3.1 in [27]). *Let M be a Riemannian manifold, $p \in M$. For almost every $r > 0$, the ball $B(p, r)$ with the current restricted from the current structure of the Riemannian manifold, T,*

$$(\mathrm{set}(T \llcorner B(p, r)), d, T \llcorner B(p, r)), \tag{3.34}$$

is an integral current space itself.

3.3.5 Theorem (Sormani [29]). *Let $\left(X_j, d_j, T_j\right)$ be a sequence of integral current spaces such that*

$$\left(X_j, d_j, T_j\right) \xrightarrow{\mathcal{F}} (X_\infty, d_\infty, T_\infty) \tag{3.35}$$

and $x_j \in X_j$ a Cauchy sequence. Then there is a subsequence such that for almost all $r > 0$

$$(B(x_j, r), d_j, T_j \llcorner B(x_j, r)) \tag{3.36}$$

are integral current spaces and

$$(B(x_j, r), d_j, T_j \llcorner B(x_j, r)) \xrightarrow{\mathcal{F}} (B(x_\infty, r), d_\infty, T_\infty \llcorner B(x_\infty, r)). \tag{3.37}$$

From Lemma 3.3.4 and Theorem 3.3.6 and applying Theorem thm-fillvolCont to a sequence of converging balls we see that for $x \in \mathcal{R}(X)$ we have

$$\begin{aligned} \|T\|(B(x, r)) &\geq \mathrm{FillVol}(\partial(B(x, r), d, T \llcorner B(x, r))) && (3.38) \\ &= \lim_{j \to \infty} \mathrm{FillVol}(\partial(B(x_j, r), d_j, T_j \llcorner B(x_j, r))). && (3.39) \end{aligned}$$

Thus to get a lower estimate of $\mathrm{FillVol}(\partial(B(x, r), d, T \llcorner B(x, r)))$ it is enough to estimate

$$\mathrm{FillVol}(\partial(B(x_j, r), d_j, T_j \llcorner B(x_j, r))) \geq Cr^n \text{ for } x_j \to x. \tag{3.40}$$

Sormani-Wenger proved the following filling volume estimate in [30]. It holds in a more general setting and was proven using work of Gromov [13] and Ambrosio-Kirchheim [1]. In the case of Riemannian manifolds the proof is quite similar to Greene-Petersen's proof of the existence of lower bounds on volume of balls of manifolds with a local geometric contractibility function [10].

3.3.6 Theorem (Sormani-Wenger [30], cf. Theorem 3.19 in [27]). *Let (M, g) be a compact n-dimensional Riemannian manifold (with or without boundary) and $p \in M$. If there exist $r_0 > 0$ and $k \geq 1$ such that $B(p, kr_0) \cap \mathrm{set}(\partial M) = \emptyset$, and every*

$B(z,r) \subset B(p,r_0)$ is contractible within $B(z,kr)$ for all $r \le 2^{-(n+6)}k^{-(n+1)}r$. Then there is $C_k > 0$ such that

$$\mathrm{FillVol}(\partial(\overline{B(p,r)}, d, T \llcorner B(p,r))) \ge C_k r^n \tag{3.41}$$

for all $r \le 2^{-(n+6)}k^{-(n+1)}r$.

With the hypotheses of Theorem 3.3.1, Sormani-Wenger obtain r_0 and k that only depend on x for x_j where j is large. They use the fact that if x is a regular point of X then all its tangent cones are isometric to $\mathbb{R}^n$ as in Cheeger-Colding [6] (cf. Theorem 2.3.7). They then apply Colding's Volume Estimate [8] (cf. 2.3.1) and the GH convergence of the balls to show that the volume of small balls contained in $B(x_j, r_0)$ satisfy inequality (3.42) of Perelman's Main Lemma in [26]:

3.3.7 Theorem (Perelman, Main Lemma and remark [26])**.** *For any $c_2 > c_1 > 1$ and integer $k > 0$ there is $\delta(k, c_1, c_2) > 0$ with the following property:*

Let M be an n-dimensional Riemannian manifold with $\mathrm{Ric}(M) \ge 0$. Suppose that $p \in M$ and that

$$\mathrm{Vol}(B(q,\rho)) \ge (1-\delta)\omega_n \rho^n \tag{3.42}$$

for every ball $B(q,\rho) \subset B(p, c_2 R)$. Then,

- *any continuous function $f : S^k \to B(p,R)$ can be continuously extended to a function*
$$\bar{f} : B^{k+1} \to B(p, c_1 R). \tag{3.43}$$
- *any continuous function $f : S^k \to M \setminus B(p, c_1 R)$ can be continuously deformed to a function*
$$\bar{f} : S^k \to M \setminus B(p, c_2 R). \tag{3.44}$$

This theorem allows them to obtain the contractibility of balls so that they can apply Theorem 3.3.6 to obtain (3.40) which implies (3.38). Thus every $x \in \mathcal{R}(X)$ is in the SWIF limit $Y = \mathrm{set}(T)$.

To prove that the non regular points of X are contained in the SWIF limit, Sormani-Wenger use the fact that the singular set of the GH limit has zero measure [6] (cf. Theorem 2.3.7) and Ambrosio-Kircheim's characterization of the mass measure [1] (cf. Lemma 3.1.8).

3.4 Li-Perales: GH=SWIF for Alexandrov Spaces

In recent work of the the author with Nan Li appearing on the arxiv [19], an approach different to using Colding [7] and Perelman [26] is used to prove that GH and SWIF limits of sequences of Alexandrov spaces agree. There they prove:

3.4.1 Theorem. *[Li-] Let (Y_j, d_j, T_j) be n-dimensional integral current spaces with weight equal to 1 (cf. Lemma 3.1.8) and no boundary. Suppose that (Y_j, d_j) are Alexandrov spaces with nonnegative curvature and $\mathrm{Diam}(Y_j) \leq D$. Then either*

$$(Y_j, d_j, T_j) \xrightarrow{\mathcal{F}} \mathbf{0} \tag{3.45}$$

or a subsequence converges in GH and SWIF sense to the same space:

$$(Y_{j_k}, d_{j_k}) \xrightarrow{GH} (X, d), \tag{3.46}$$

$$(Y_{j_k}, d_{j_k}, T_{j_k}) \xrightarrow{\mathcal{F}} (Y, d, T) \tag{3.47}$$

and $X = Y$.

Now we describe the proof and state the theorems applied. Notice that the proof follows very closely the proof of Sormani-Wenger [30] (cf Theorem 3.3.1 within) except that we apply Alexandrov space theorems of Burago-Gromov-Perelman in [5] and Otsu-Shioya [**?**] to prove the regular points of the GH limit are contained in the SWIF limit.

3.4.2 Theorem (Burago-Gromov-Perelman 8.5 in [5], Corollary 10.10.11 in [4])**.** *Let Y_j be a sequence of n-dimensional Alexandrov spaces that satisfy*

$$\sec(Y_j) \geq \kappa \text{ and } \mathrm{Diam}(Y_j). \tag{3.48}$$

Then there is a subsequence Y_{j_k} and an Alexandrov space X with $\sec(X) \geq \kappa$, $\mathrm{Diam}(X) \leq D$ and Hausdorff dimension $\leq n$ such that

$$Y_{j_k} \xrightarrow{GH} X. \tag{3.49}$$

Moreover, $\lim_{k\to\infty} \mathcal{H}^n(Y_{j_k}) = 0$ if and only if the Hausdorff dimension of X is less than n.

This compactness theorem for Alexandrov spaces gives us a GH convergent subsequence. To get a SWIF limit we find an upper bound of $\|T_j\|(Y_j)$. In the case of Riemannian manifolds no calculation of this type is needed since there $\|T_j\|(M_j) = \mathrm{Vol}(M_j)$ which is bounded by hypotheses. To find the bound we use Ambrosio-Kirhheim characterization of the mass measure [1], cf. Lemma 3.1.8, the Hausdorff measure bound obtained from the Bishop volume comparison for Alexandrov spaces and the uniform diameter bound that we have by hypotheses.

3.4.3 Theorem (Theorem 10.6.6 in [4])**.** *Let Y be an n-dimensional Alexandrov space with* $\sec(Y) \geq \kappa$*. Let $V_n^\kappa(r)$ denote the volume of a ball of radius r in the simply connected space with constant sectional curvature equal to κ. Then for all $y \in Y$ the function:*

$$r \quad \longmapsto \quad \frac{\mathcal{H}^n(B(y,r))}{V_n^\kappa(r)} \tag{3.50}$$

is non-increasing.

Once we have bounded $\|T_j\|(Y_j)$, by Sormani-Wenger [30], cf. Theorem 3.2.3, we obtain a SWIF limit space, (Y, d, T) that is either zero integral current space or it is contained in X. In the first case we get (3.45). In the second case, we have to prove that $X \subset Y$. To do that we divide the proof in two parts. First we show that the regular points of X are contained in Y. Secondly, we show that the non regular points of X are contained in Y.

If $x \in X$ is regular point then we consider a sequence converging to that point, $y_j \to x$. By fixing δ and ε, from Burago-Gromov-Perelman's result below we obtain a sequence of bi-Lipschitz maps $f_j : B(y_j, \delta r) \to W_j \subset \mathbb{R}^n$ with Lipschitz constants as close to one as we choose ε.

3.4.4 Theorem (See Theorem 9.4 [5], Theorem 10.8.4 [4])**.** *For any $\varepsilon > 0$ there is $\delta = \delta(n, \kappa, \varepsilon) > 0$ such that for any n-dimensional Alexandrov space Y with* $\sec(Y) \geq \kappa$ *and* $\mathrm{Diam}(Y) \leq D$*, $y \in Y$ and $r > 0$ if*

$$d_{GH}(B(y,r), B(0,r)) < \delta r, \tag{3.51}$$

where $B(0,r)$ denotes the ball of radius r contained in $\mathbb{R}^n$ centered at 0. Then there exists a bi-Lipschitz map

$$f : B(y, \delta r) \to W \subset \mathbb{R}^n \tag{3.52}$$

with $\mathrm{Lip}(f)$*,* $\mathrm{Lip}(f^{-1}) \leq 1 + \varepsilon$*. In particular,*

$$B(f(y), (1-\epsilon)r) \subset f(B(y,r)) \subset B(f(y), (1+\epsilon)r). \tag{3.53}$$

The existence of these maps provide an estimate on the intrinsic flat distance between the currents defined on $B(y_j, \delta r)$ and W_j:

3.4.5 Lemma (Sormani-Wenger [31]). *Let (Y_i, d_i) be complete metric spaces and $\varphi : Y_1 \to Y_2$ be a $\lambda > 1$ biLipschitz map. If $T \in \mathbf{I}_n(Y_1)$, then*

$$d_{\mathcal{F}}(M_1, M_2) \le k_{\lambda,n} \max\{\mathrm{Diam}(\operatorname{spt} T), \mathrm{Diam}(\varphi(\operatorname{spt} T))\}\mathbf{M}(T) \tag{3.54}$$

where $M_1 = (\mathrm{set(T)}, \mathrm{d}_1, \mathrm{T})$*,* $M_2 = (\mathrm{set}(\varphi_\# \mathrm{T}), \mathrm{d}_2, \varphi_\# \mathrm{T})$ *and* $k_{\lambda,n} = \frac{1}{2}(n+1)\lambda^{n-1}(\lambda-1)$.

By applying the triangle inequality and noticing that W_j has the standard current structure given of $\mathbb{R}^n$ for $\varepsilon < 1$ we obtain

$$d_{\mathcal{F}}((B(y_j, \delta r), d_j, T_j \llcorner B(y_j, \delta r)), \mathbf{0}) \ge C(n, r)(\delta r)^n \tag{3.55}$$

such that for r sufficiently small $C(n, r)$ is positive and $\lim_{r\to 0} C(n, r) > 0$. Now we just combine this estimate with the inequiality

$$\|T\|(B(x, \delta r)) \ge d_{\mathcal{F}}((B(x, \delta r), d, T \llcorner B(x, \delta r)), \mathbf{0}) \tag{3.56}$$

and Sormani's convergence theorem [29], cf. Theorem 3.3.5, to show that $\|T\|(B(x, \delta r)) \ge C(n, r)(\delta r)^n$. This shows that $x \in Y$. Hence, the regular points of X are contained in Y.

To prove that the nonregular points are contained in Y we apply Otsu-Shioya's result about the density of the regular set and the Hausdorff dimension of the singular set of X, Ambrosio-Kirchheim's characterization of the mass measure, the Bishop-Gromov Volume Comparison Theorem for Alexandrov spaces and the fact that the Hausdorff dimension of X is n, see Theorem 3.4.2, otherwise we would be in the first case of Theorem 3.4.1, to show that

$$\|T\|(B(x, r)) \ge n^{-n/2}\mathcal{H}^n(X)/D^n \, r^n$$

Thus, $\liminf_{r\to 0} \|T\|(B(x, r))/r^n > 0$. This shows that $X \setminus \mathcal{R}(X) \subset Y$.

3.4.6 Theorem (Otsu-Shioya, Theorem A in [22]). *Let Y be an n-dimensional Alexandrov space. Then $Y \setminus \mathcal{R}(Y)$ has Hausdorff dimension less or equal than $n-1$. In particular, $\mathcal{R}(Y)$ is dense in Y.*

Chapter 4

A Survey on Convergence of Manifolds with Boundary

In this chapter, we review precompactness theorems for sequences of manifolds and metric spaces with boundary. We begin with a survey of the work of Kodani [17], Anderson-Katsuda-Kurylev-Lassas-Taylor [2], Wong [34], and Knox [16] on the limits of Riemannian manifolds with boundary. We then present published joint work of the author with Sormani concerning glued limit spaces of metric spaces with boundary [25]. Finally we present SWIF precompactness theorems of the author building upon Wenger's Compactness Theorem that have appeared on the arxiv in [24].

This chapter is based in part on "A survey on the Convergence of Manifolds with Boundary" written by the author [23]. It was a survey article solicited by Taller de Vinculación Matemáticos Mexicanos Jóvenes en el Mundo as part of the proceedings volume of a conference at CIMAT in Guanajuato. It will appear in the collection *Contemporary Mathematics of the American Mathematical Society* in cooperation with the Mexican Mathematical Society.

4.1 Kodani's GH Precompactness Theorem

In this section we state the convergence theorems that Kodani proved in [17]. First we recall the definition of Lipschitz distance between metric spaces and interior injectivity radius of a manifold.

4.1.1 Definition. *The Lipschitz distance between two metric spaces* (X, d_X) *and* (Y, d_Y) *is defined by*

$$d_L(X,Y) = \inf_{f:X\to Y} \log(\max\{dil(f), dil(f^{-1})\})$$

where the infimum is taken over all bi-Lipschitz homeomorphisms $f : X \to Y$.

4.1.2 Definition. *If* M *is a Riemannian manifold with boundary, for* $p \in M \setminus \partial M$ *define the interior injectivity radius of* p, $i_{int}(p)$, *to be the supremum over all* $r > 0$ *such that all unitary geodesics* $\gamma : [0, t_\gamma] \to M$ *that start at* $\gamma(0) = p$ *are minimizing from* 0 *to* $\min\{t_\gamma, r\}$, *where* t_γ *is the first time* γ *intersects* ∂M. *The interior injectivity radius of* M *is defined as*

$$i_{int}(M) = \inf\{i_{int}(p) | p \in M \setminus \partial M\}. \tag{4.1}$$

Kodani proved that in the class defined below GH convergence implies Lipschitz convergence.

4.1.3 Theorem (Kodani)**.** *Let* $\mathcal{M}(n, K, \lambda, i)$ *be the class of connected n-dimensional Riemannian manifolds, M, with boundary that satisfy*

$$|\mathrm{sec(M)}| \le \mathrm{K}, \quad |\mathrm{II}_{\partial\mathrm{M}}| \le \lambda, \quad \mathrm{i_{int}(M)} \ge \mathrm{i}, \text{ and } \mathrm{i}_\partial\mathrm{(M)} \ge \mathrm{i}, \tag{4.2}$$

where $II_{\partial M}$ *is the second fundamental form of* ∂M, i_{int} *the interior injectivity radius and* i_∂ *the boundary injectivity radius. Then for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $M, N \in \mathcal{M}(n, K, \lambda, i)$ *and* $d_{GH}(M, N) < \delta$ *implies* $d_L(M, N) < \varepsilon$. *Thus sequences in* $\mathcal{M}(n, K, \lambda, i)$ *that converge in GH sense also converge in Lipschitz sense.*

Hence, by proving GH convergence of a subclass of $\mathcal{M}(n, K, \lambda, i)$, Kodani manages to prove Lipschitz convergence as well.

4.1.4 Theorem (Kodani)**.** *Let* $n \in \mathbb{N}$, $K, \lambda, v > 0$. *Denote by* $\mathcal{M}(n, K, \lambda, v)$ *the class of connected n-dimensional Riemannian manifolds with boundary that satisfy*

$$|\mathrm{sec(M)}| \le \mathrm{K}, \quad 0 \le \mathrm{II}_{\partial\mathrm{M}} \le \lambda, \quad \textit{and} \quad \mathrm{Vol(M)} \ge \mathrm{v}. \tag{4.3}$$

Then

$$\mathcal{M}(n, K, \lambda, v) \subset \mathcal{M}(n, K, \lambda, i). \tag{4.4}$$

Also, every sequence of manifolds in $\mathcal{M}(n, K, \lambda, v)$ *has a GH convergent subsequence. Hence, a Lipschitz convergent subsequence.*

To prove that $\mathcal{M}(n, K, \lambda, v) \subset \mathcal{M}(n, K, \lambda, i)$ Kodani estimates i_∂ and i_{int} by studying appropiate geodesics. To prove GH convergence he obtains uniform upper volume bounds for the manifolds and uniform lower bounds for their balls. Then it is possible to apply Gromov's Compactness Theorem (cf. Theorem 2.1.2)..

4.2 Anderson-Katsuda-Kurylev-Lassas-Taylor's Precompactness Theorem

Here we present Anderson-Katsuda-Kurylev-Lassas-Taylor's $C^{1,\alpha}$ precompactness theorem (cf. Theorem 4.2.1). This theorem appears in [2] and extends the techniques used to prove Theorem 1.1 of [3] of Anderson to manifolds with boundary.

4.2.1 Theorem (Anderson-Katsuda-Kurylev-Lassas-Taylor)**.** *Let $\mathcal{M}(n, R, i, H_0, D)$ be the class of compact, connected Riemannian n-manifolds with boundary M satisfying*

$$\begin{array}{lll} |\operatorname{Ric}(M)| \leq R, & |\operatorname{Ric}(\partial M)| \leq R & \\ injrad(M) \geq i, & i_{int}(M) \geq i, & i_\partial(M) \geq 2i \\ \operatorname{Diam}(M) \leq D, & |H_{\partial M}|_{Lip(\partial M)} \leq H_0 & \end{array}$$

where $i_\partial(M)$ denotes the boundary injectivity radius of M, and $H_{\partial M}$ is the mean curvature of ∂M in M. Then $\mathcal{M}(n, R, i, H_0, D)$ is precompact in the $C^{1,\alpha}$ topology.

4.3 Wong's GH Precompactness Theorems

Here we review the two GH compactness theorems proved by Wong in [34]. The first theorem that we present only requires a lower bound on sectional curvature and allows the second fundamental form to be negative unlike Kodani's compactness theorem (cf. Theorem 4.1.4). Moreover, sequences of manifolds in $\mathcal{M}(n, K-, \lambda^\pm, D)$ can collapse. Nonetheless, for the noncollapsing case Kodani obtains finitely many diffeomorphism types while Wong obtaines finetely many homeomorphism types.

4.3.1 Theorem (Wong [34])**.** *The class $\mathcal{M}(n, K-, \lambda^\pm, D)$ of n-dimensional Riemannian manifolds with boundary with*

$$\sec(M) \geq K-, \quad \lambda^- \leq II_{\partial M} \leq \lambda^+, \quad \textit{and} \quad \operatorname{Diam}(M) \leq D, \tag{4.5}$$

where $II_{\partial M}$ denotes the second fundamental form of ∂M, is precompact in the GH topology. If $\mathcal{M}(n, r-, \lambda^{\pm}, D)$ is restricted to manifolds with $\mathrm{Vol}(M) \geq v$*, then it has finitely many homeomorphism classes.*

Wong shows that each $M \in \mathcal{M}(n, K^{-}, \lambda^{\pm}, D)$ has an isometric extension $\tilde{M}$ that is an Alexandrov space with uniform sectional and diameter bounds. Thus, using the theory for Alexandrov spaces he can show that $M \in \mathcal{M}(n, K^{-}, \lambda^{\pm}, D)$ is equibounded.

In the following theorem, Wong requires a uniform lower bound on the Ricci curvature of the manifolds. Recall that in order to show $C^{1,\alpha}$ convergence [2] (cf. Theorem 4.2.1), Anderson et. al. assumed several injectivity radii bounds and a bound on the Ricci curvature of the boundary.

4.3.2 Theorem (Wong [34])**.** *The class $\mathcal{M}(n, r-, \lambda^{\pm}, D)$ of n-dimensional Riemannian manifolds with boundary with*

$$\mathrm{Ric}(M) \geq r-, \quad \lambda^{-} \leq II_{\partial M} \leq \lambda^{+}, \quad \textit{and} \quad \mathrm{Diam}(M) \leq D, \tag{4.6}$$

where $II_{\partial M}$ denotes the second fundamental form of ∂M, is precompact in the GH topology.

The proof of Theorem 4.3.2 is very similar to the proof of Theorem 4.3.1. Here, Wong extends each $M \in \mathcal{M}(n, r-, \lambda^{\pm}, D)$ to a C^2 manifold with boundary that has a diameter bound and a Ricci curvature lower bound that only depends on the class.

4.4 Knox's C^α and $L^{1,p}$ Precompactness Theorem

In this section we go over Knox's C^α and weak $L^{1,p}$ precompactness Theorem [16], cf. Theorem 4.4.1. The proof of this theorem builds upon work of Anderson in [3] and [2].

4.4.1 Theorem (Knox)**.** *Let $\mathcal{M}(n, K, H_0, D, v_\partial)$ be the class of compact connected Riemannian n-manifolds with connected boundary satisfying*

$$|\sec(M)| \leq K, \ |\sec(\partial M)| \leq K$$
$$0 < 1/H_0 < H_{\partial M} < H_0$$
$$\mathrm{Diam}(M) \leq D, \ \mathrm{Vol}(\partial M) \geq v_\partial,$$

where $H_{\partial M}$ is the mean curvature of the boundary. Then $\mathcal{M}(n, K, H_0, D, v_\partial)$ is precompact in the C^α, $0 < \alpha < 1$, and weak $L^{1,p}$, $p < \infty$, topologies.

From the bounds on the sectional curvature of the boundaries and their mean curvature we see that the class defined by knox, $\mathcal{M}(n, K, H_0, D, v_\partial)$, is contained in one of Wong's classes, $\mathcal{M}(n, K-, \lambda^\pm, D)$. Hence, Knox's class is GH precompact by Wong.

Knox's proof relies on calculating the harmonic radius, r_h of the Riemannian manifolds within the class he defined. For points in the interior by looking at the volume of cylinders whose base is in ∂M, he concludes that there is a $c > 0$ that only depends on the class such that

$$r_h(x) \geq cd(x, \partial M) \tag{4.7}$$

for all $x \in M \setminus \partial M$, where $(M, g) \in \mathcal{M}(n, K, H_0, D, v_\partial)$. For points in the boundary, he shows that there is r that only depends on $\mathcal{M}(n, K, H_0, D, v_\partial)$ for which

$$r_h(x) \geq r \tag{4.8}$$

for all $x \in \partial M$.

4.5 Glued Limits of Sormani and the Author

In this section we go over the definition of glued limit space of a sequence of open Riemannian manifolds (M_j, d_j) as defined by Sormani and the author in[25]. This glued limit space may exist even when (M_j, d_j) has no GH limit. See Example 5.1.2 and Example 5.1.3. If (M_j, d_j) has a GH limit then the glued limit space can be a proper subset of the GH limit. See Example 5.3.3.

We recall that in [25] the authors consider open Riemannian manifolds (M^n, g) and define ∂M as in (1.25). There they prove GH compactness theorems for sequences of inner regions, cf. Theorem 2.2.1 and Theorem 2.2.3. Moreover, they define glued limit space, cf.Theorem 4.5.1 and prove that under the conditions of Theorem 2.2.3 the glued limit space has positive lower density, cf. Theorem 4.5.2.

4.5.1 Theorem (P–Sormani)**.** *Let (M_j, g_j) be a sequence of open manifolds with boundary and $\delta_i \to 0$ a decreasing sequence. If*

$$(\bar{M}_j^{\delta_i}, d_j) \xrightarrow{GH} (Y(\delta_i), d_{Y(\delta_i)}) \tag{4.9}$$

for all i. Then there exists a metric space (Y, d_Y) with the following properties. For each $\delta \in (0, \delta_0]$ there is a subsequence and an isometric embedding:

$$(\bar{M}_{j_k}^{\delta}, d_{j_k}) \xrightarrow{GH} (Y(\delta), d_{Y(\delta)}) \text{ and } F_\delta : Y(\delta) \to Y \tag{4.10}$$

such that

$$F_{\delta_i}(Y(\delta_i)) \subset F_{\delta_{i+1}}(Y(\delta_{i+1}))\ \forall i, \tag{4.11}$$

$$Y = \bigcup_{i=1}^{\infty} F_{\beta_j}(Y(\beta_j)) \ \textit{for any decreasing}\ \beta_j \to 0. \tag{4.12}$$

The theorem was proven by constructing isometric embeddings between the limit spaces, $\varphi_{\delta_{i+1},\delta_i} : Y(\delta_i) \to Y(\delta_{i+1})$, and defining

$$Y := Y(\delta_0) \sqcup \left(\sqcup_{i=1}^{\infty} \left(Y(\delta_{i+1}) \setminus \varphi_{\delta_{i+1},\delta_i}\left(Y(\delta_i)\right)\right)\right), \tag{4.13}$$

where $\sqcup$ denotes disjoint union. This set was then endowed with the metric d_Y given by

$$d_Y(x,y) = d_{Y(\delta_k)}(x,y), \tag{4.14}$$

where for k large x and y are seen as elements in $Y(\delta_k)$ via compositions of the form

$$\varphi_{\delta_{i+j},\delta_i} = \varphi_{\delta_{i+j},\delta_{i+j-1}} \circ \cdots \circ \varphi_{\delta_{i+1},\delta_i}. \tag{4.15}$$

Applying the results of Cheeger-Colding [6] reviewed in Section 2.3, Sormani and the author proved the following.

4.5.2 Theorem (P–Sormani)**.** *Let $n \in \mathbb{N}$, $\delta_i \to 0$ be a decreasing sequence, $D_i > 0$, $i = 0, 1, 2...$, $V > 0$ and $\theta > 0$. Suppose that (M_j, g_j) is a sequence of open Riemannian manifolds with boundary that satisfy*

$$\mathrm{Ric}(M_j) \geq 0, \ \ \mathrm{Vol}(M_j) \leq V, \ \ \mathrm{Diam}(M_j^{\delta_i}, d_{M_j^{\delta_i}}) \leq D_i \tag{4.16}$$

and

$$\exists q \in M_j^{\delta} \ \textit{such that}\ \ \mathrm{Vol}(B(q,\delta)) \geq \theta\delta^n, \tag{4.17}$$

where $B(q,\delta)$ is the ball in M_j with center q and radius δ. Then there is a subsequence $(M_{j_k}^{\delta_i}, d_{j_k})$ that has a glued limit space Y such that Y has Hausdorff dimension n, $\mathcal{H}^n(Y) \leq V$ and every point in Y has positive lower density.

4.6 Prior SWIF Precompactness Theorems of the Author

In this section we review SWIF compactness theorems for sequences of manifolds with boundary proven by the author and posted in the arxiv in [24]: Theorem 4.6.4, Theorem 4.6.5 and Theorem 4.6.6. Since these theorems build upon

Wenger's Compactness Theorem [33], we begin by stating his theorem. Although his theorem is proven for sequences of integral current spaces, we state it here for oriented compact manifolds with boundary (cf. [30]). Recall that an oriented manifold with boundary M can be viewed as an integral current space, (M, d, T), taking d to be the induced length metric and $T(f, \pi_1, ..., \pi_m) = \int_M f\, d\pi_1 \wedge \cdots \wedge d\pi_m$.

4.6.1 Theorem. *[Wenger [33]] Let $n, V, D > 0$ and (M_j, g_j) be a sequence of oriented compact n-dimensional Riemannian manifolds (possibly with boundary) that satisfy the following:*

$$\mathrm{Vol}(M_j) \leq V, \ \ \mathrm{Vol}(\partial M_j) \leq A \tag{4.18}$$

and

$$\mathrm{Diam}(M_j) \leq D. \tag{4.19}$$

Then there exists a subsequence that converges in SWIF sense.

To apply Wenger's Compactness Theorem the author first obtained a Laplacian comparison theorem and then calculated volume and area bounds.

4.6.2 Theorem ([24])**.** *Let $n \geq 2$ and M^n be an n-dimensional connected Riemannian manifold with boundary with* $\mathrm{Ric}(M) \geq 0$ *and (M, d) complete. Let $r : M \to \mathbb{R}$ be the function*

$$r \ = \ d(\,\cdot\,, \partial M). \tag{4.20}$$

Then for all $p \in M$

$$\Delta r(p) \leq \frac{(n-1)H_{\partial M}(q)}{H_{\partial M}(q) r(p) + n - 1} \tag{4.21}$$

holds in the barrier sense, where $q \in \partial M$ such that $r(p) = d(p, q)$, Δr is the Laplacian and $H_{\partial M} : \partial M \to \mathbb{R}$ the mean curvature of ∂M with respect to the the normal inward pointing direction.

4.6.3 Theorem ([24])**.** *Let $n \geq 2$ and M^n be an n-dimensional and connected Riemannian manifold with smooth boundary such that (M, d) is complete,* $\mathrm{Ric}(M) \geq 0$ *and $H_{\partial M} \leq H$. Furthermore, let $A_{n,H} : [0, \infty) \to \mathbb{R}$ be the function $A_{n,H}(t) =$* $\max\{(1 + Ht/(n-1))^{n-1}, 0\}$.

If $\delta_1 \geq \delta_2 \geq 0$ then

$$\mathrm{Vol}(M^{\delta_2} \setminus M^{\delta_1}) \leq \mathrm{Vol}(\partial M) \int_{\delta_2}^{\delta_1} A_{n,H}(t) dt \tag{4.22}$$

and the area of the boundary of M^δ (as a metric subspace of M) satisfies $\mathcal{L}^1$-almost everywhere

$$\mathrm{Vol}(\partial M^\delta) \le \mathrm{Vol}(\partial M) A_{n,H}(\delta). \tag{4.23}$$

Notice that in the above theorems if $H_{\partial M} = 0$ then we get $\Delta r(p) \le 0$. For $H_{\partial M} = (n-1)/H$ we get $\Delta r(p) \le (n-1)/(r(p)+H)$. Also that the equality of both, volume and area, estimates is achieved by standard balls in $\mathbb{R}^n$. We mention that Sakurai proved a Laplacian comparison theorem for the function r that works only where it is smooth [28] and that different volume estimates have been obtained using a different approach by Heintz and Karcher in [15].

4.6.4 Theorem. *Let $D, A > 0$, $H \in \mathbb{R}$ and (M_j^n, g_j) be a sequence of n-dimensional oriented connected Riemannian manifolds with smooth boundary. Suppose that for all j the spaces (M_j, d_j) are complete as metric spaces,*

$$\mathrm{Ric}(M_j) \ge 0,\ \ H_{\partial M_j} \le H,\ \ \mathrm{Vol}(\partial M_j) \le A \tag{4.24}$$

and

$$\mathrm{Diam}(M_j) \le D. \tag{4.25}$$

Then there is an n-integral current space (W, d, T) and a subsequence that converges in SWIF sense:

$$(M_{j_k}, d_{j_k}, T_{j_k}) \xrightarrow{\mathcal{F}} (W, d, T). \tag{4.26}$$

If (M^n, g) is a complete connected Riemannian manifold with smooth boundary such that $\mathrm{Ric}(M) \ge 0$ and $H_{\partial M} \le H < 0$ then $r \le -(n-1)/H$ [18] [20]. Thus, $\mathrm{Diam}(M) \le \mathrm{Diam}(\partial M) - 2(n-1)/H$.

4.6.5 Theorem. *Let $D', A > 0$ and (M_j^n, g_j) be a sequence of n-dimensional oriented connected Riemannian manifolds with smooth boundary. Suppose that for all j the spaces (M_j, d_j) are complete as metric spaces,*

$$\mathrm{Ric}(M_j) \ge 0,\ \ H_{\partial M_j} \le H < 0,\ \ \mathrm{Vol}(\partial M_j) \le A \tag{4.27}$$

$$\mathrm{Diam}(\partial M_j) \le D'. \tag{4.28}$$

Then there is a subsequence and an n-integral current space (W, d, T) such that

$$(M_{j_k}, d_{j_k}, T_{j_k}) \xrightarrow{\mathcal{F}} (W, d, T). \tag{4.29}$$

In the theorem below we see that if a sequence converges in SWIF sense to a current space (X, d, T) then the SWIF limits of its sequences of inner regions converge to (X, d, T). This does not happen for GH limits. See Example 5.3.3 and Example 4.10 of [25].

4.6.6 Theorem. *Let $D, A > 0$, $H \in \mathbb{R}$ and (M_j^n, g_j) be a sequence of n-dimensional complete oriented connected Riemannian manifolds with smooth boundary. Suppose that for all j the spaces (M_j, d_j) are complete metric spaces that satisfy*

$$\mathrm{Ric}(M_j) \geq 0, \;\; H_{\partial M_j} \leq H, \;\; \mathrm{Vol}(\partial M_j) \leq A \tag{4.30}$$

and

$$\mathrm{Diam}(M_j) \leq D. \tag{4.31}$$

Suppose that there exist an integral current space (W, d, T), a non increasing sequence $\delta_i \to 0$ and integral current spaces $(W_{\delta_i}, d_{W_{\delta_i}}, T_{\delta_i})$ such that

$$(M_j, d_j, T_j) \xrightarrow{\mathcal{F}} (W, d, T) \tag{4.32}$$

and for all i

$$(M_{j_k}^{\delta_i}, d_{j_k}, T_{j_k}^{\delta_i}) \xrightarrow{\mathcal{F}} (W_{\delta_i}, d_{W_{\delta_i}}, T_{\delta_i}). \tag{4.33}$$

Then we have

$$(W_{\delta_i}, d_{W_{\delta_i}}, T_{\delta_i}) \xrightarrow{\mathcal{F}} (W, d, T). \tag{4.34}$$

Chapter 5

Convergence of Metric Spaces with Boundary

In this chapter we prove our new GH compactness theorems for sequences of metric spaces (X_j, d_j): Theorem 1.0.4, Theorem 5.2.1 and Theorem 1.0.5. Theorem 1.0.4 is applied in all our GH convergence theorems, except Theorem 5.2.1, stated in this paper. The theorem relies on the GH convergence of inner regions (X_j^δ, d_j) and the boundaries, $(\partial X_j, d_j)$. Theorem 5.2.1 deals with the collapsed case when the GH limit of (X_j, d_j) agrees with the GH limit of $(\partial X_j, d_j)$. If the Hausdorff dimension of the GH limit drops then the GH limit cannot agree with the SWIF limit. Under the conditions of Theorem 1.0.5 we prove that GH and SWIF limits agree. We provide examples showing the necessity of our hypotheses.

5.1 GH convergence: Theorem 1.0.4

We first prove the following useful lemma and then prove Theorem 1.0.4. Recall that if (X, d) is a metric space with non empty boundary, where the boundary is defined as $\partial X = \bar{X} \setminus X$, then $X^\delta = \{x \in X \,|\, d(x, \partial M) > \delta\}$.

5.1.1 Lemma. *Let (X, d) be a precompact metric space with boundary ∂X defined as in 1.25. If $\{B(x_k, \delta)\}_{k=1}^N$ is a cover of $(\partial X, d)$, then $\{B(x_k, 2\delta)\}_{k=1}^N$ is a cover of $(\bar{X} \setminus X^\delta, d)$.*

Proof. We have to show that for all $x \in \bar{X} \setminus X^\delta$ there is k such that $x \in B(x_k, 2\delta)$. Let $x \in \bar{X} \setminus X^\delta$, then $d(x, \partial X) \leq \delta$. Since ∂X is precompact there is $x' \in \overline{\partial X}$ such that

$$d(x, x') = d(x, \partial X) \leq \delta. \tag{5.1}$$

If $\{B(x_k, \delta)\}$ is a cover of $(\partial X, d)$, then it is a cover of $(\overline{\partial X}, d)$ and $x' \in \partial X$ then there is k such that

$$d(x', x_k) < \delta. \tag{5.2}$$

Hence,

$$d(x, x_k) < 2\delta. \tag{5.3}$$

This proves that $x \in B(x_k, 2\delta)$. The result follows. □

We now apply this lemma to prove our GH convergence theorem for metric spaces.

Proof of Theorem 1.0.4. In order to prove that $(\bar{X}_j, d_j)$ converges in GH sense we will construct a function

$$N : (0, \infty) \to \mathbb{N} \tag{5.4}$$

for $\{(\bar{X}_j, d_j)\}$ as in Definition 2.1.3 and then apply Gromov's Compactness Theorem (cf. Theorem 2.1.2).

Since $\overline{X_j^{\delta_i}}$ and $\overline{\partial X_j}$ converge in GH sense, by the converse of Gromov Compactness Theorem, cf. Theorem 2.1.4, there exist functions $N(\,\cdot\,, \{\overline{X_j^{\delta_i}}\})$ and $N(\,\cdot\,, \{\overline{\partial X_j}\})$, respectively, that uniformly bound the number of balls needed to cover each element of the sequences. Using these functions we first define $N : \{2\delta_i\} \to \mathbb{N}$. A bound on the number of $2\delta_i$-balls needed to cover $\bar{X}_j$ can be obtained by adding the number of $2\delta_i$-balls needed to cover $\overline{X_j^{\delta_i}}$ to the number of $2\delta_i$-balls needed to cover $\bar{X}_j \setminus X_j^{\delta_i}$. With the notation of Definition 2.1.3 and applyng Lemma 5.1.1, define

$$N(2\delta_i) = N(\delta_i, \left\{\overline{X_j^{\delta_i}}\right\}) + N(\delta_i, \{\overline{\partial X_j}\}). \tag{5.5}$$

The domain of N is extended to $(0, \infty)$ by defining

$$N(\varepsilon) = N(2\delta_i) \text{ where } 2\delta_{i+1} \leq \varepsilon < 2\delta_i \tag{5.6}$$

and $N(\varepsilon) = N(2\delta_1)$ for $\varepsilon > 2\delta_1$.

Since X_j is a length metric space and can be covered with $N(\delta_1)$ balls with radius δ_1 then

$$\mathrm{Diam}(\bar{X}_j) \leq 2\delta_1 N(\delta_1). \tag{5.7}$$

The result follows from Gromov's Compactness Theorem (cf. Theorem 2.1.2). □

Figure 5.1: $\bar{X}_1, \bar{X}_2, \bar{X}_3$...

We now present an example demonstrating that the conclusion of Theorem 1.0.4 does not hold if the sequence of boundaries does not converge.

5.1.2 Example. *Let X_j be contained in $\mathbb{R}^n$, $n \geq 2$, endowed with the length metric that comes from the standard metric defined on $\mathbb{R}^n$. See Figure 5.1 above. Each X_j consists of an open ball of radius r with j increasingly thin splines of constant length, L and width $w_j \to 0$.*

Observe that each X_j is precompact and that its boundary, ∂X_j, is compact. For each $\delta > 0$, there is N such that X_j^δ does not contain any spline for $j \geq N$. Actually, the sequence $(\overline{X_j^\delta}, d_j)$ converges in GH sense to a closed ball,

$$(\overline{X_j^\delta}, d_j) \xrightarrow{GH} \overline{B(0, r-\delta)}. \tag{5.8}$$

Due to the increasing number of splines of constant length, the sequence $\{(\partial X_j, d_j)\}_{j=1}^\infty$ is not equibounded. Thus, by the converse of Gromov Compactness Theorem, (cf. Theorem 2.1.4), $\{(\partial X_j, d_j)\}_{j=1}^\infty$ does not have any GH convergent subsequences. Note that $\{(X_j, d_j)\}_{j=1}^\infty$ does not have GH convergent subsequences for the same reason.

The next example is modeled after the pictures depicted in Frank Morgan's book [21]. It shows that in Theorem 1.0.4 the GH convergence of sequences of δ_i-inner regions is necessary.

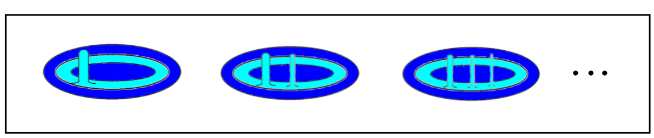

Figure 5.2: X_1, X_2, X_3...

5.1.3 Example. *Let* $0 < c < C < 1$ *be constant numbers. Consider the sequence* (X_j, d_j) *of precompact surfaces in 3-dimensional Euclidean space,* $\mathbb{E}^3$ *as depicted in Figure 5.2 above. More precisely,*

$$X_j = \{(x, y, 0) \in \mathbb{R}^3 | x^2 + y^2 \leq c\} \cup A_j \cup \{(x, y, 0) \in \mathbb{R}^3 | C \leq x^2 + y^2 < 1\}, \quad (5.9)$$

where A_j *(depicted in light blue in Figure 5.2) has* j *increasingly thin splines of constant height located in such a way that for all* $(x, y, 0) \in \partial X_j$*:*

$$d_j((x, y, 0), (0, 0, 0)) \leq 1 - c + \alpha_j, \quad (5.10)$$

where $\alpha_j \to 0$ *and the* d_j*'s are length metrics induced by the Euclidean metric* $d_{\mathbb{E}^3}$.

By the condition on the metrics, Equation (5.10), it follows that

$$(\partial X_j, d_j) \xrightarrow{GH} (\{(x, y, 0) \in \mathbb{R}^3 | x^2 + y^2 = 1\}, d_{\mathbb{E}^3}). \quad (5.11)$$

For $\delta \in [0, 1-c)$*, the sequence* $\{(\overline{X_j^\delta}, d_j)\}_{j=1}^\infty$ *is not equibounded due to the splines in* A_j*. Then, by the converse of Gromov's compactness theorem (cf. Theorem 2.1.4),* $\{(\overline{X_j^\delta}, d_j)\}_{j=1}^\infty$ *does not have any GH convergent subsequence, nor the sequence* $\{\bar{X}_j\}$.

5.2 Collapsing to the Boundary or Not

In this section we prove the following collapsing theorem, Theorem 5.2.1. Then we describe a sequence of cylinders that collapses to a segment, Example 5.2.3.

5.2.1 Theorem. *Let* (X_j, d_j) *be a sequence of precompact length metric spaces with compact boundary. Suppose that there is a compact metric space* (X_∂, d_∂) *such that*

$$(\partial X_j, d_j) \xrightarrow{GH} (X_\partial, d_\partial). \quad (5.12)$$

Then either:

1. *there is* $\delta > 0$ *such that* $X_j^\delta \neq \emptyset$ *for infinitely many* j *or*
2. $(\bar{X}_j, d_j) \xrightarrow{GH} (X_\partial, d_\partial)$.

5.2.2 Remark. *When the sequence of boundaries converge and (1) in Theorem 5.2.1 is satisfied we cannot conclude anything. Example 5.1.3 shows a sequence that satisfies these two conditions but* (X_j, d_j) *does not have GH convergent subsequence. Meanwhile, the sequence from Example 5.3.3 satisfies both conditions and GH converges.*

In the next example we describe a sequence of length metric spaces that illustrates the case in which the GH limit of X_j equals the GH limit of ∂X_j.

5.2.3 Example. *Let X_j be a sequence of increasingly thin cylinders in $\mathbb{R}^n$,*

$$X_j = \{(x_1, ..., x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_{n-1}^2 < r_j,\ -1 < x_n < 1\}, \tag{5.13}$$

$r_j = 1/j$, with the restricted standard metric of $\mathbb{R}^n$. With this metric each X_j is a precompact length metric space and it is clear that each ∂X_j is non empty and precompact. Let

$$X_\infty = \{0\} \times \{0\} \times \cdots \times \{0\} \times [-1, 1] \subset R^n \tag{5.14}$$

Then,

$$d_H^{\mathbb{R}^n}(\bar{X}_j, X_\infty) < 2/j. \tag{5.15}$$

Also,

$$d_H^{\mathbb{R}^n}(\partial X_j, X_\infty) < 2/j. \tag{5.16}$$

Thus,

$$\partial X_j \xrightarrow{GH} X_\infty \text{ and } \bar{X}_j \xrightarrow{GH} X_\infty. \tag{5.17}$$

Proof of Theorem 5.2.1. Suppose that there is no $\delta > 0$ such that $X_j^\delta \neq \emptyset$ for infinitely many j. Fix $\delta > 0$, then $X_j^\delta = \emptyset$ for finitely many j. Thus, for except a finite number of j's, $\bar{X}_j = \bar{X}_j \setminus X_j^\delta$. Hence, we define a function N that counts the number of ε-balls needed to cover $\bar{X}_j$ by

$$N(2\delta) := N(\delta, \{\partial X_j\}), \tag{5.18}$$

where we are using the notation of Definition 2.1.3 and applying Lemma 5.1.1

Since each $\bar{X}_j$ is a length space and can be covered by $N(\varepsilon_0)$ ε_0-balls we get $\mathrm{Diam}(\bar{X}_j) \leq 2\varepsilon_0 N(\varepsilon_0)$. The result follows from Gromov's compactness theorem, Theorem 2.1.2. □

5.3 GH=SWIF: Theorem 1.0.5

In this section we prove Theorem 1.0.4. Theorem 1.0.4 assures that a the GH and SWIF limits agree for sequences of integral currents that converge in GH and SWIF sense that satisfy condition (1.36) and (1.34), namely:

$$X(\delta_i) \subset Y \text{ and } X \setminus X_\partial \subset Y. \tag{5.19}$$

In Example 5.3.2 we present a sequence that has GH and SWIF limits, that satisfies (1.34) but does not satisfy (1.36). Then, in Example 5.3.3 we describe a sequence that has GH and SWIF limits, that satisfy (1.36) but does not satisfy (1.34). In both cases, the conclusion of the theorem does not hold. At the end of the section we prove Lemma 5.3.4 that characterizes the points in $X \setminus X_\partial$. With this lemma Theorem 1.0.5 can be rewritten as Corollary **??**. We also use this lemma to prove Theorem 1.0.2.

Proof of Theorem 1.0.5. By hypothesis we know that the SWIF limit is contained in the GH limit, $Y \subset X$. We only have to show that $X \subset \bar{Y}$.

Since the sequence $\{\bar{X}_j\}$ converges in GH sense, there is a sequence $x_j \in \bar{X}_j$ that converges to x. If there is $i > 0$ such that $x_j \in X_j^{\delta_i}$ for infinite many j, then by the GH convergence of the sequence $\{X_j^{\delta_i}\}$, $x \in X(\delta_i)$. Thus, by hypothesis (1.36), $x \in Y$.

Otherwise, for each i there is j_k such that $x_{j_k} \notin X_{j_k}^{\delta_i}$. Thus, $d(x_{j_k}, \partial X_{j_k}) \leq \delta_i$. Since each boundary, ∂X_j, is precompact we can choose $y_{j_k} \in \overline{\partial X_{j_k}}$ such that

$$d_{j_k}(x_{j_k}, y_{j_k}) = d(x_{j_k}, \overline{\partial X_{j_k}}) \leq \delta_i. \tag{5.20}$$

Then by the triangle inequality,

$$d_{j_k}(y_{j_k}, x) \to 0 \; as \; k \to \infty. \tag{5.21}$$

Thus, $x \in X_\partial$. From hypothesis (1.34) follows that $x \in \bar{Y}$. Hence, $X \subset \bar{Y}$. This together with $Y \subset X$ implies that $X = \bar{Y}$. □

5.3.1 Remark. *From Example 5.1.3 it follows that in Theorem 1.0.4 the convergence of the sequences of inner regions is necessary.*

In the next example we describe a sequence that satisfies all the conditions of Theorem 1.0.4, except condition (1.36). The conclusion of the theorem does not hold.

5.3.2 Example. *Just as in Example 5.1.3, let* $0 < c < C < 1$ *be constant numbers. Consider the sequence* (X_j, d_j) *of 2-dimensional precompact length metric spaces in 3-dimensional Euclidean space as depicted in the figure above given by:*

$$X_j = \{(x, y, 0) \in \mathbb{R}^3 | x^2 + y^2 \leq c\} \cup A_j \cup \{(x, y, 0) \in \mathbb{R}^3 | C \leq x^2 + y^2 \leq 1\}, \tag{5.22}$$

where A_j *has only one increasingly thin spline of constant height located in such a way that*

$$d_j((x, y, 0), (0, 0, 0)) \leq 1 - c + \alpha_j \tag{5.23}$$

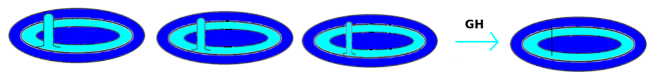

Figure 5.3: $X_1, X_2, X_3...$

for all $(x, y, 0) \in \partial X_j$ *and* $\alpha_j \to 0$.

The spline in A_j *converges to a segment. Thus, the GH limit of* $(\bar{X}_j, d_j)$ *is a disc with the segment attached to it and its SWIF limit,* Y*, equals the disc.*

The sequences $(\overline{X_j^\delta}, d_j)$ *GH converge for all* $\delta < 1$*:*

$$(\overline{X_j^\delta}, d_j) \xrightarrow{GH} (X(\delta), d_\delta). \tag{5.24}$$

But $(X(\delta), d_\delta) \not\subset \bar{Y}$ *for* $\delta \in [0, 1 - c)$ *since each* $X(\delta)$ *contains a segment that* Y *does not. Hence, this sequence does not satisfy condition 1.36. It satisfies* $X_\partial \subset Y$ *since from Equation (5.10) follows that*

$$(\partial X_j, d_j) \xrightarrow{GH} (\{(x, y, 0) \in \mathbb{R}^3 | x^2 + y^2 = 1\}, d_{\mathbb{E}^3}). \tag{5.25}$$

In the next example we show a sequence that satisfies all the conditions of Theorem 1.0.5, except $X_\partial \subset Y$, for which the conclusion of the theorem does not hold.

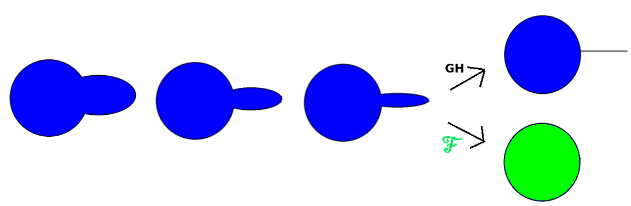

Figure 5.4: $X_1, X_2, X_3...$

5.3.3 Example. *Let* M_j *be a sequence of n-dimensional manifolds diffeomorphic to a closed ball in* R^n*,* $n \geq 2$*, consisting of a ball with a single increasingly thin spline as depicted in Figure 5.4 for* $n = 2$*. Let* $X_j = M_j \setminus \partial X_j$*. Then, each* X_j *is a precompact length metric space with compact boundary.*

The GH limit, X, of the sequence is a closed ball of radius r, $\overline{B(0,r)}$, with a segment attached. The SWIF limit of the manifolds is $Y = \overline{B(0,r)}$.

For each $\delta > 0$, there is N such that X_j^δ does not contain any spline for $j \geq N$. Actually, the sequence $(\overline{X_j^\delta}, d_j)$ converges in GH sense to a closed ball,

$$(\overline{X_j^\delta}, d_j) \xrightarrow{GH} \overline{B(0, r-\delta)}. \tag{5.26}$$

Hence, $\overline{B(0, r-\delta)} \subset Y$

The GH limit of the boundaries is a sphere with a segment attached. Thus, $X_\partial \not\subset Y$.

Now we characterize the points in $X \setminus X_\partial$. This characterization will help us to prove GH=SWIF in Chapter 6.

5.3.4 Lemma. *Let $\{(X_j, d_j)\}_{j=1}^\infty$ be a sequence of precompact metric spaces with compact boundary that converge in GH sense to a compact metric space (X, d):*

$$(\bar{X}_j, d_j) \xrightarrow{GH} (X, d). \tag{5.27}$$

Denote by X_∂ the GH limit of the boundaries:

$$(\partial X_j, d_j) \xrightarrow{GH} (X_\partial, d). \tag{5.28}$$

Let $x \in X$. Then $x \in X \setminus X_\partial$ if and only if there is $\delta > 0$ and a sequence $x_j \in X_j^\delta$ such that

$$\lim_{j\to\infty} x_j = x. \tag{5.29}$$

Proof. Suppose that there is a sequence $x_j \in X_j^\delta$ that converges to x and that x is contained in X_∂. Then, by the GH convergence of the boundaries, there is a sequence $y_j \in \partial X_j$ that converges to x as well. By the triangle inequality:

$$d_j(x_j, y_j) \leq d_j(x_j, x) + d_j(x, y_j). \tag{5.30}$$

Since $d_j(x_j, x), d_j(x, y_j) \to 0$ as $j \to \infty$, for j big enough, we have

$$d_j(x_j, y_j) < \delta, \tag{5.31}$$

which is a contradiction since $x_j \in X_j^\delta$.

Now, suppose that there is no δ and no sequence as in the statement of the lemma. Since $x \in X$, there is $x_j \in M_j$ that converges to x. By our supposition, for each k it is possible to choose $k \in \mathbb{N}$ such that

$$d(x_{j_k}, \partial X_{j_k}) \leq 1/k. \tag{5.32}$$

Since ∂X_{j_k} is compact there exists $y_{j_k} \in \partial X_{j_k}$ such that $d(x_{j_k}, \partial X_{j_k}) = d(x_{j_k}, y_{j_k})$. Hence, there is a subsequence $\{y_{j_k}\} \subset \partial X_{j_k}$ that converges to x. Hence, $x \in X_\partial$. $\square$

Chapter 6

Convergence of Riemannian Manifolds with Boundary

In this chapter we prove the compactness theorems for manifolds stated in the introduction: Theorem 1.0.1, Theorem 1.0.2 and Theorem 1.0.3. These theorems are consequence of the general cases that hold for metric spaces: Theorem 1.0.4 and Theorem 1.0.5. Theorem 1.0.1 about GH convergence is proven in Section 6.1. Then in Section 6.2 we show that the GH limit is contained in the SWIF limit, Theorem 1.0.2. This is done adapting Sormani-Wenger's GH=SWIF theorem for manifolds with no boundary [30]. In Section 6.3 we prove that the GH limit agrees with the completion of the SWIF limit when having a uniform lower bound on the boundary injectivity radii, Theorem 1.0.3. To do so we first obtain diameter bounds of δ-inner regions, Lemma 6.3.1, and then prove that the sequence of boundaries converges in GH sense, Lemma 6.3.2.

6.1 GH Convergence: Theorem 1.0.1

In this section we prove a GH convergence theorem for manifolds with boundary, Theorem 1.0.1.

Proof of Theorem 1.0.1. By hypotheses, we have a sequence of manifolds with nonnegative Ricci curvature, M_j, satisfying (1.4)-(1.5). These hypotheses are the same as the ones in prior work of the author with Sormani [25] (cf. Theorem 2.2.3). Thus, applying that theorem we obtain a subsequence $\{j_k\}$ such that the sequence of inner regions converges for all i:

$$(\bar{M}^{\delta_i}_{j_k}, d_{M^{\delta_i}_{j_k}}) \xrightarrow{\text{GH}} (X(\delta_i), d_{X(\delta_i)}). \tag{6.1}$$

By hypothesis $(\partial M_j, d_{M_j})$ also converges in GH sense. Thus the hypotheses of our GH convergence theorem for metric spaces, Theorem 1.0.4, proven in Chapter 5 above, are satisfied. By applying Theorem 1.0.4 we obtain a further subsequence, also denoted j_k, such that the sequence $\{(M_{j_k}, d_{M_{j_k}})\}$ converges in the Gromov-Hausdorff sense. □

The following remark shows a sequence that satisfies all the conditions of Theorem 1.0.1. So the sequence has a GH convergent subsequence.

6.1.1 Remark. *In Example 5.3.3 we describe a sequence M^n_j of n-dimensional compact Riemannian manifolds with boundary that satisfy all the conditions of Theorem 1.0.1:*

$$\mathrm{Ric}(M_j) = 0,\ \mathrm{Vol}(M_j) \le \omega_n (r+L)^n,\ \mathrm{Diam}(M^{\delta}_j) \le 2(r+L) \tag{6.2}$$

and the center q_j of the closed ball to which the spline is atacched in each M_j satisfies

$$\mathrm{Vol}(B(q_j, \delta)) = \omega_n \delta^n, \tag{6.3}$$

where ω_n is the volume of a ball in $\mathbb{R}^n$ of radius 1. *Finally, the sequence of boundaries, ∂M_j, also converges in GH sense.*

Recall first the notion of a "glued limit" as defined by the author and Sormani [25] [Theorem 7.4] (cf. Theorem 4.5.1). Theorem 1.0.1 can be restated as follows.

6.1.2 Theorem. *If (M_j, d_j) has a glued limit and $(\partial M_j, d_j)$ converges in GH sense, then there is a subsequence of (M_j, d_j) that converges in GH sense.*

Notice that the completion of the glued limit might not equal the GH limit. Cannot say GH=Completed Glued Limit here. In Example 5.3.3 the completion of the glued limit is a disc while the GH limit of the sequence is a disc with a segment attached.

Proof. The hypotheses for a sequence (M_j, d_j) to have a glued limit space coincide with the hypotheses of Theorem 1.0.1, except for the convergence of the sequence of boundaries $(\partial M_j, d_j)$. Hence, by adding this last condition all the hypotheses of Theorem 1.0.1 are satisfied. Thus, the theorem follows. □

6.1.3 Remark. *In Theorem 7.0.5 we will prove that the assumption on the GH convergence of the boundaries can be replaced by an assumption on* $\mathrm{Diam}(\partial M_j, d_{\partial M_j})$ *and the second fundamental form of the boundaries,* ∂M_j*, and their derivatives, (7.1).*

6.2 Proving GH=SWIF Theorem 1.0.2 and Examples

In this section we prove Theorem 1.0.2. The key step is to show that $X \setminus X_\partial \subset Y$. A sequence of compact, oriented, n-dimensional Riemannian manifolds with nonnegative Ricci curvature and positive uniform upper and lower bounds on the diameter and volume, respectively, have subsequences that converge in GH and SWIF sense, cf. Theorem 2.1.7 and Theorem 4.6.1. Sormani-Wenger proved that when the manifolds have no boundary a single subsequence can be chosen in such a way that both limits coincide [30], cf. Theorem 3.3.1. By avoiding the boundaries of the manifolds and provided with uniform lower bounds on the volume of the balls, Equation (2.14), Sormani-Wenger's proof can be adapted to show that $X \setminus X_\partial \subset Y$.

In Example 6.2.1 and Example we describe sequences of manifolds that satisfy all the conditions stated in Theorem 1.0.2. What is interesting to see is that the GH limit and SWIF limit of $\{\partial M_j\}$ do not need to coincide in order to get that the GH and the SWIF limits of $\{M_j\}$ agree.

Proof of Theorem 1.0.2. By Theorem 1.0.1 there exist a compact metric space (X, d_X) such that

$$(M_{j_k}, d_{j_k}) \xrightarrow{\text{GH}} (X, d_X). \tag{6.4}$$

By Sormani-Wenger [31], cf. Theorem 3.2.3 , there exist an n-integral current space (Y, d_Y, T) and a further subsequence that we denote in the same way such that

$$(M_{j_k}, d_{j_k}, T_{j_k}) \xrightarrow{\mathcal{F}} (Y, d_Y, T) \tag{6.5}$$

where either (Y, d_Y, T) is the zero integral current space or $Y \subset X$ and $d_Y = d_X|_Y$. With no loss of generality our convergent subsequences will be indexed by $\{j\}$.

Cheeger and Colding classified the points of a GH limit space into regular and nonregular according to their tangent cones [6], cf. Definition 2.3.6. Based on this, we divide the proof of the theorem in the following three claims.

Recall that from the definition of integral current space, Definition 3.2.1, $x \in Y$ if and only if

$$\liminf_{r\to 0} \frac{\|T\|(B(x,r))}{r^n} > 0 \tag{6.6}$$

holds.

Claim 1: *Let $x \in \mathcal{R}(X(\delta_i))$ and $x_j \in M_j^{\delta_i}$ such that $x_j \to x$. Then there is $r_0(x) > 0$ and $k(x) \geq 1$ such that $B(x_j, r_0) \cap \mathrm{set}(\partial \mathrm{T_j}) = \emptyset$ and every $B(z,r) \subset B(x_j, r_0(x))$ is contractible within $B(z,kr)$.*

Proof of Claim 1: Since x is a regular point contained in $X(\delta_i)$ by Cheeger-Colding [6] (cf. Theorem 2.3.7 and Remark 2.3.8), every tangent cone of x is isometric to $\mathbb{R}^n$. Thus, for any $\alpha > 0$ there is $r(\alpha) \leq \delta_i$ such that

$$d_{GH}(B(x,r), B(0,r)) < \alpha r/2, \tag{6.7}$$

where $B(0,r) \subset \mathbb{R}^n$ denotes the Euclidean ball in $\mathbb{R}^n$ with radius r and center 0. Now $x_j \to x$ and $M_j \stackrel{\mathrm{GH}}{\longrightarrow} X$ imply that for large j

$$d_{GH}(B(x_j,r), B(x,r)) < \alpha r/2. \tag{6.8}$$

Using the triangle inequality we obtain that

$$d_{GH}(B(x_j,r), B(0,r)) < \alpha r \tag{6.9}$$

for large j.

For $\varepsilon > 0$, by Colding's Volume Convergence Theorem [7] (cf. Theorem 2.3.1) there is $\alpha(\varepsilon, n) > 0$ such that

$$\mathrm{Vol}(B(x_j,r)) \geq (1-\varepsilon)\omega_n r^n \tag{6.10}$$

holds if (6.9) is satisfied. By what we proved in the previous paragraph (6.10) holds for large j taking $\alpha = \alpha(\varepsilon, n)$ and $r(\alpha) < \delta_i$.

Now, for any k, by Perelman's Contractibility Theorem [26] (cf. Theorem 3.3.7 within), there is ε such that if (6.10) holds then each $B(z,r) \subset B(x_j, r_0(x))$ is contractible within $B(z,kr)$. This finishes the proof of Claim 1.

Claim 2: *$\mathcal{R}(X(\delta_i)) \subset Y$. That is, (6.6) holds for all the regular points of $X(\delta_i)$.*

Proof of Claim 2: Since the result of Claim 1 holds then by Sormani-Wenger Filling Volume Theorem [30] which uses work by Gromov [13] and was also proven in [10], cf. Theorem 3.3.6, there exists $C_{k(x)} > 0$ such that

$$\mathrm{FillVol}(\partial(\overline{B(x_j,r)}, d_j, T_j \llcorner B(x_j,r))) \geq C_{k(x)} r^n \tag{6.11}$$

for sufficiently large j and all $r \leq 2^{-(n+6)}k^{-(n+1)}r_0(x)$.

Thus, by the continuity of the filling volume under SWIF convergence [27], cf. Theorem 3.3.3, we have

$$\mathrm{FillVol}(\partial(\overline{B(x,r)}, d, T \llcorner B(x,r))) \geq C_{k(x)}r^n \tag{6.12}$$

for all $r \leq 2^{-(n+6)}k^{-(n+1)}r_0(x)$. Since

$$||T||(S) \geq \mathrm{FillVol}(\partial S) \tag{6.13}$$

holds for all integral current spaces S. Then,

$$||T||(B(x,r)) \geq C_{k(x)}r^n. \tag{6.14}$$

Thus, (6.6) holds which proves that $x \in Y$. Hence, $\mathcal{R}(X(\delta_i)) \subset Y$ and (Y, d, T) is not the zero integral current space.

Claim 3: *If $\mathcal{R}(X(\delta_i)) \subset Y$ for all $\delta_i \leq \delta$, then $\mathcal{S}(X(\delta_i)) \subset Y$, ie. (6.6) holds for all the nonregular points of $X(\delta_i)$.*

Proof of Claim 3: Let $x \in \mathcal{S}(X(\delta_i))$. From the characterization of the mass measure given by Ambrosio-Kirchheim [1] (cf. Lemma 3.1.8), for $r > 0$

$$||T||(B(x,r)) \geq \lambda_n \mathcal{H}^n(B(x,r) \cap Y), \tag{6.15}$$

where $\lambda_n > 0$ only depends on n.

Now we bound $\mathcal{H}^n(B(x,r) \cap Y)$. First we note that

$$\mathcal{H}^n(B(x,r) \cap Y) \geq \mathcal{H}^n(B(x,r) \cap X(\delta_k)) \tag{6.16}$$

since in Claim 2 we proved that $\mathcal{R}(X(\delta_k)) \subset Y$ for all k and by Cheeger-Colding [6] (cf. Theorem 2.3.7 and Remark 2.3.8) $\mathcal{H}^n(\mathcal{R}(X(\delta_k)) = \mathcal{H}^n(X(\delta_k))$.

Now we estimate $\mathcal{H}^n(B(x,r) \cap X(\delta_k))$. Actually, if $r \leq \delta_i - \delta_{i+1}$ we have

$$B(x,r) \subset X(\delta_{i+1}). \tag{6.17}$$

Thus, we only need to estimate $\mathcal{H}^n(B(x,r))$. By the volume estimate calculated by the author and Sormani [25], cf. Remark 2.2.2,

$$\mathrm{Vol}(B(x_j,r)) \geq v(\delta_i)r^n, \tag{6.18}$$

where $x_j \in M_j^{\delta_i} \to x$ and $r \leq \delta_i/2$. Applying Colding's Volume Convergence [7] (cf. Therem 2.3.1) we have

$$\mathcal{H}^n(B(x,r)) \geq v(\delta_i)r^n \tag{6.19}$$

for $r \leq \delta_i/2$.

Thus, for $r < \min\{\delta_i/2, \delta_i - \delta_{i+1}\}$ from (6.15), (6.16) and (6.19), we see that

$$\|T\|(B(x,r)) \geq \lambda_n \mathcal{H}^n(B(x,r) \cap Y) \geq \lambda_n v(\delta_i) r^n. \tag{6.20}$$

From this, Equation (6.6) holds. Thus, $x \in Y$. This proves $S(X(\delta_i)) \subset Y$.

From Claim 2 and Claim 3 we conclude that $X(\delta_i) \subset Y$ for all i. By hypotheses we know that $X_\partial \subset Y$. Then we get by Theorem 1.0.4 that $X = Y$ and by Lemma 5.3.4 that $X \setminus X_\partial \subset Y$. □

In Example 5.3.3 we described a sequence that satisfies all the conditions of Theorem 1.0.2, except $X_\partial \subset Y$. See Remark 6.1.1. There the conclusion of the theorem does not hold.

From Gromov's compactness theorem, cf. Theorem 2.1.2, we know that if (M_j, d_j) converges in GH sense then a further subsequence of $(\partial M_j, d_j)$ also converge in GH sense. As in Federer-Fleming, if (M_j, d_j, T_j) converges in SWIF sense then $(\partial M_j, d_j, \partial T_j)$ converges in SWIF sense [9] [1] [31].

In the following two examples we present sequences $\{M_j\}$ of manifolds that satisfy all the conditions of Theorem 1.0.2. But for which the GH limit and SWIF limits of $\{\partial M_j\}$ do not agree. Hence, the hypothesis $X_\partial \subset Y$ in Theorem 1.0.2 cannot be replaced to requiring that both limits of $\{\partial M_j\}$ coincide.

6.2.1 Example. *Let $M_j = S^n \setminus B(p, 1/j)$, $n \geq 2$, be the standard n-dimensional unit sphere minus a ball of radius $1/j$ with center the north pole, p. Each M_j^n is a compact oriented Riemannian manifold with boundary that satisfy*

$$\mathrm{Ric}(M_j) = n-1, \mathrm{Vol}(M_j) \leq \mathrm{Vol}(S^n), \mathrm{Diam}(M_j^\delta) \leq \mathrm{Diam}(S^{n-1}) \tag{6.21}$$

and the south pole $q \in S^n$ satisfies

$$\mathrm{Vol}(B(q,\delta)) = \frac{\mathrm{Vol}(S^n)}{\pi^n} \delta^n \tag{6.22}$$

for small δ.

The SWIF limit of M_j is S^n and the north pole is the GH limit of ∂M_j. Thus, $\{p\} \subset Y$. These shows that this example satisfies all the hypotheses of Theorem 1.0.2. Hence, the GH limit of M_j is S^n. But the SWIF limit of ∂M_j is the zero current. This example shows that both limits of M_j can agree even though the limits of ∂M_j do not.

6.2.2 Example. *Let*

$$I^3 = [-1,1]\times[-1,1]\times[-1,1]\subset\mathbb{R}^3. \tag{6.23}$$

For $j \geq 2$, let

$$M_j = I^3 \setminus (-1/j,1/j)\times(-1/j,1/j)\times(0,1] \tag{6.24}$$

with the induced flat metric. Notice that the elements of this sequence are not manifolds but they can be smoothened. This sequence converges in both GH and SWIF sense to the cube

$$X = Y = I^3. \tag{6.25}$$

The boundaries, however, have different limits.

$$\partial M_j \xrightarrow{GH} \partial I^3 \ \cup\ \{0\}\times\{0\}\times[0,1] \tag{6.26}$$

However the SWIF limit of boundaries is the boundary of the limits:

$$\partial M_j \xrightarrow{\mathcal{F}} \partial I^3. \tag{6.27}$$

This shows that both limits of M_j can agree even though the limits of ∂M_j do not agree.

6.3 GH=SWIF when $i_\partial(M_j)$ is bounded above: Theorem 1.0.3

In this section we prove Theorem 1.0.3 that the closure of the SWIF limit coincides with the GH limit when having a uniform positive lower bound on the boundary injectivity radii. To prove GH convergence we first prove Lemma 6.3.1 that gives a uniform bound on the diameters of sequences of inner regions. Then we prove Lemma 6.3.2 that shows that the sequence of boundaries, $\{(\partial M_j, d_{M_j})\}$, converges in GH sense. Then we apply these lemmas combined with our GH convergence theorem for manifolds with boundary, Theorem 1.0.4. To show that $X = \bar{Y}$ we give an argument that shows that $X_\partial \subset \bar{Y}$ and then we apply our GH=SWIF theorem for manifolds with boundary, Theorem 1.0.2.

First we set some notation. Let (M,g) be a Riemannian manifold with boundary such that the boundary injectivity radius of M satisfies $i_\partial(M) \geq \iota$, then let

$$\gamma : \partial M \times [0,\iota] \to M \tag{6.28}$$

denote the function that assigns to each $(p,t) \in \partial M \times [0,\iota]$ the point at time t of the unitary normal geodesic that starts at p. Note that γ is well defined and a bijection onto its image by the definition of boundary injectivity radius. Finally, let

$$\pi : M \to \partial M \tag{6.29}$$

the function that assigns to each $p \in M$ a point $\pi(p) \in \partial M$ that satisfies $d(p,\pi(p)) = d(p,\partial M)$. Notice that by the boundary injectivity radius bound this point is unique for all $p \in \gamma(\partial M \times [0,\iota])$.

6.3.1 Lemma. *Let (M_j, g_j) be a sequence of Riemannian manifolds with boundary such that*

$$i_\partial(M_j) \geq \iota > \delta_0 \tag{6.30}$$

and

$$\mathrm{Diam}(M_j^{\delta_0}, d_{M_j^{\delta_0}}) \leq D_0 \tag{6.31}$$

for all j. Then for all $\delta \in [0,\delta_0]$ and all j the following holds

$$\mathrm{Diam}(M_j^{\delta},\ d_{M_j^{\delta}}) < D(\delta), \tag{6.32}$$

where $D(\delta) = D_0 + 2(\delta_0 - \delta)$.

Proof. Let $\delta < \delta_0$. Let's estimate $\mathrm{Diam}(M_j^{\delta},\ d_{M_j^{\delta}})$. Let $p_1, p_2 \in M_j^{\delta}$. Since $\delta < \delta_0$ and from the definion of inner region we know that $M_j^{\delta_0} \subset M_j^{\delta}$. If $p_i \notin M_j^{\delta_0}$, define

$$p_i' := \gamma(\pi(p_i), \delta_0) \in M_j^{\delta_0}. \tag{6.33}$$

where γ is defined in (6.28). This point is well defined since $i_\partial(M_j) > \delta_0 > \delta$. Notice that

$$d_{M_j}(p_i, p_i') = \delta_0 - \delta. \tag{6.34}$$

If $p_i \in M_j^{\delta_0}$, set $p_i' = p_i$. To end the proof we apply the triangle inequality:

$$d_{M_j^{\delta}}(p_1,p_2) \leq d_{M^{\delta_0}}(p_1', p_2') + 2(\delta_0 - \delta) \leq D_0 + 2(\delta_0 - \delta). \tag{6.35}$$

□

6.3.2 Lemma. *Let (M_j^n, g_j) be a sequence of Riemannian manifolds with boundary such that*

$$i_\partial(M_j) \geq \iota > 0. \tag{6.36}$$

Suppose that there is a decreasing sequence $\{\delta_i\} \subset \mathbb{R}$ that converges to zero and the inner regions, $(\overline{M_j^{\delta_i}}, d_{M_j})$, converge in GH sense for all i.

Then a subsequence of $\{(\partial M_j, d_{M_j})\}$ converges in GH sense.

Proof. By Gromov's compactness theorem, cf. Theorem 2.1.2, it is enough to show that $\{(\partial M_{j_k}, d_{j_k})\}$ is equibounded and has a uniform diameter bound.

Let $\delta < \iota$. We claim that if $\{B(x_l, \delta)\}$ is a δ cover of $(\partial M_j^\delta, d_j)$ then $\{B(\pi(x_l), 3\delta)\}$ is a 3δ cover of $(\partial M_j, d_j)$. Let $x \in \partial M_j$. Since $\gamma(x, \delta) \in \partial M_j^\delta$ and $\{B(x_l, \delta)\}$ is a cover of ∂M_j^δ, there is l such that

$$d_j(\gamma(x, \delta), x_l) < \delta. \tag{6.37}$$

Then, by the triangle inequality we get

$$d_j(x, \pi(x_l)) \leq d_j(x, \gamma(x, \delta)) + d_j(\gamma(x, \delta), x_l) + d_j(x_l, \pi(x_l)) < 3\delta. \tag{6.38}$$

This proves the claim.

Now, since $(\bar{M}_j^{\delta_i}, d_j)$ converges in GH sense, it follows from Gromov's compactness theorem and its converse (cf. Theorem 2.1.2 and Theorem 2.1.4), that there is a GH convergent subsequence, $(\partial M_{j_k}^{\delta_i}, d_{j_k})$ and there exists a function $N(\,\cdot\,, \{\partial M_{j_k}^{\delta_i}\})$ as in Definition 2.1.3.

Without any loss of generality suppose that $\delta_i < \iota$ for all i. We define

$$N(3\delta_i) := N(\delta_i, \{\partial M_{j_k}^{\delta_i}\}) \tag{6.39}$$

and extend the domain of N by defining $N(\varepsilon) = N(3\delta_i)$, where $3\delta_i \leq \varepsilon < 3\delta_{i-1}$. Thus $\{(\partial M_{j_k}, d_{M_{j_k}})\}$ is equibounded.

Finally, by Lemma 6.3.1

$$\mathrm{Diam}(\partial M_{j_k}, d_{\partial M_{j_k}}) \leq \mathrm{Diam}(M_{j_k}) \leq D(0). \tag{6.40}$$

Thus, by Gromov's Compactness Theorem (cf. Theorem 2.1.2), there is a subsequence of $\{(\partial M_{j_k}, d_{M_{j_k}})\}$ that converges in GH sense. □

Proof of Theorem 1.0.3. To prove this theorem we only need to show that the hypotheses of Theorem 1.0.2 are satisfied. Namely, we first have to prove the existence of diameter bounds, GH convergence of a subsequence of $(\partial M_j, d_j)$.

Choose a decreasing sequence $\{\delta_i\} \subset \mathbb{R}$ that converges to zero such that $\iota > \delta_i$ for all i. Using Lemma 6.3.1 we obtain the diameter bounds required in Theorem 1.0.2. Now we need to show that there is a GH convergent subsequence of $(\partial M_j, d_j)$. By the hypotheses and the diameter bounds that we obtained in Lemma 6.3.1 we can apply a result by the author and Sormani [25], cf. Theorem 2.2.3, to obtain GH convergent subsequences of inner regions:

$$(\bar{M}_{j_k}^{\delta_i}, d_{j_k}) \xrightarrow{\text{GH}} (X(\delta_i), d_{X(\delta_i)}) \ \forall i. \tag{6.41}$$

By Lemma 6.3.2, there is a further subsequence and a metric space $(X_\partial, d_{X_\partial})$ such that

$$\{(\partial M_{j_k}, d_{M_{j_k}})\} \xrightarrow{\text{GH}} (X_\partial, d_\partial). \tag{6.42}$$

Then, by Theorem 1.0.2 we have a further GH and SWIF convergent subsequence:

$$(M_{j_k}, d_{j_k}) \xrightarrow{\text{GH}} (X, d_X) \tag{6.43}$$

and

$$(M_{j_k}, d_{j_k}, T_{j_k}) \xrightarrow{\text{GH}} (Y \subset X, d_X, T) \tag{6.44}$$

such that $X \setminus X_\partial \subset Y$.

To prove that $X = \bar{Y}$ it remains to prove that $X_\partial \subset \bar{Y}$. With no loss of generality we suppose that (M_j, d_j) converges in GH sense. Let $x \in X_\partial$ and $x_j \in \partial M_j$ be a sequence that converges to x. For all i, by the GH convergence of (M_j, d_j), there is a subsequence j_k such that

$$\gamma_{j_k}(x_{j_k}, \delta_i) \to y(\delta_i) \text{ as } k \to 0, \tag{6.45}$$

where γ_{j_k} denotes the normal exponential function defined on $\partial M_{j_k} \times [0, \iota]$, see 6.28. From (6.45) and the GH conververgence of $(M_j^{\delta_i}, d_j)$ we know that $y(\delta_i) \subset X(\delta_i) \subset Y$.

Using the triangle inequality we get $d_X(y(\delta_i), x) = \delta_i$. Hence,

$$y(\delta_i) \in Y \to x \text{ as } i \to 0. \tag{6.46}$$

This proves that $x \in \bar{Y}$. Thus, $X_\partial \subset \bar{Y}$. This finishes the proof. □

6.3.3 Remark. *If all the hypotheses of Theorem 1.0.3 hold except the uniform positive lower bound on the boundary injectivity radii then the conclusion of the theorem might not hold. See Example 5.3.3 where* $\lim_{j\to\infty} i_\partial(M_j) = 0$ *due to the increasingly thin splines.*

Questions: 1. Is it possible to get an example that satisfies the hypotheses of Theorem 1.0.3, except the area bound?. Does the volume remain bounded?. Partial Answer: Innami proved that it is not possible for flat manifolds in Euclidean space. 2. Is the completion of the glued limit equal to the GH limit when having the hypotheses of Theorem 1.0.3.

The next example shows that although the injectivity radius of the boundary is a popular assumption in theorems about convergence of manifolds with boundary, it is not a necessary condition.

6.3.4 Example. *Let* $M_j = \overline{B(0,r)} \cup A_j$ *be the sequence in Euclidean space that consists of a closed ball of radius r with an increasingly thin and short spline* A_j *attached to the ball such that* M_j *converges in GH and SWIF sense to* $\overline{B(0,r)}$*. See Example 5.3.3 and Figure 5.4. There the splines have constant length.*

Chapter 7

GH Convergence of $(\partial M_j, d_{M_j})$

Let (M, g) be a Riemannian manifold with smooth boundary. We denote by d_M the metric given by g. Since M has smooth boundary, ∂M can be endowed with two different metrics, d_M which is the restriction of d_M to ∂M and $d_{\partial M}$ which is the metric given by the Riemannian metric of ∂M.

Some of our GH compactness theorems require GH convergence of the sequence $(\partial M_j, d_{M_j})$. Observe that GH convergence of $(\partial M_j, d_{\partial M_j})$ implies GH convergence of $(\partial M_j, d_{M_j})$ provided each $(\partial M_j, d_{\partial M_j})$ is connected or have a bounded number of connected components (cf. Proposition 7.0.6). Thus, by uniformly bounding the Ricci curvature of ∂M_j we will prove a GH compactness theorem, Theorem 7.0.5, for $(\partial M_j, d_{\partial M_j})$ and $(\partial M_j, d_{\partial M_j})$.

7.0.5 Theorem. *Let $\{(M_j^n, g_j)\}$ be a sequence of Riemannian manifolds with smooth boundary such that* $\mathrm{Ric}(M_j) \geq 0$*,*

$$\mathrm{Diam}(\partial M_j, d_{\partial M_j}) \leq D_\partial, \;\; (\mathrm{R}(e_n, X)X, e_n) \leq \gamma \textit{ and } \; \alpha \leq B(X, X) \leq \beta, \tag{7.1}$$

where e_n denotes the normal unitary vector field, B is the second fundamental form of ∂M and X is a vector field in $T\partial M$ such that $\nabla_{e_n} X = 0$. Then, there is a subsequence $\{j_k\}$ such that both $(\partial M_{j_k}, d_{\partial M_{j_k}})$ and $(\partial M_{j_k}, d_{j_k})$ converge in GH sense.

We now prove two propositions which we will apply to prove this theorem.

7.0.6 Proposition. *Let (M, g) be a Riemannian manifold with boundary. If $\{B_{d_{\partial M}}(x_i, \varepsilon)\}$ is a cover of $(\partial M_j, d_{\partial M_j})$ then $\{B_{d_M}(x_i, \varepsilon)\}$ is a cover of $(\partial M_j, d_{M_j})$.*

Proof. It is enough to show that for all $x \in \partial M$ the following holds

$$B_{d_{\partial M}}(x,\varepsilon) \subset B_{d_M}(x,\varepsilon). \tag{7.2}$$

By the definition of d_M and $d_{\partial M}$ we know that

$$d_M(x,x') \leq d_{\partial M}(x,x') \text{ for all } x,x' \in \partial M. \tag{7.3}$$

Thus,

$$d_{\partial M}(x,x') < \varepsilon \text{ implies } d_M(x,x') < \varepsilon. \tag{7.4}$$

Hence,

$$B_{d_{\partial M}}(x,\varepsilon) \subset B_{d_M}(x,\varepsilon). \tag{7.5}$$

□

7.0.7 Proposition. *Let (M,g) be a Riemannian manifold with boundary. If*

$$\mathrm{Ric}(M) \geq 0,\ \ (\mathrm{R}(e_n,X)X,e_n) \leq \gamma \ \textit{and}\ \ \alpha \leq B(X,X) \leq \beta. \tag{7.6}$$

Then,

$$\mathrm{Ric}_\partial(X,X) \geq c(n,\gamma,\alpha,\beta) = -\gamma + (n-1)(\alpha^2 - \beta^2). \tag{7.7}$$

Proof. Let $p \in \partial M$. Choose an orthonormal basis e_i of T_pM such that e_n is perpendicular to $T_p\partial M$ and $\nabla_{e_n} e_i = 0$ for $1 \leq i \leq n-1$. Using Gauss formula we get

$$(\mathrm{R}_\partial(e_i,e_j)e_j,e_i) = (\mathrm{R}(e_i,e_j)e_j,e_i) + B(e_j,e_j)B(e_i,e_i) - B^2(e_j,e_i), \tag{7.8}$$

for $1 \leq i,j \leq n-1$. Adding over i and adding and substracting $(\mathrm{R}(e_n,e_j)e_j,e_n)$ we obtain

$$\mathrm{Ric}_\partial(e_j,e_j) = \mathrm{Ric}(e_j,e_j) - (\mathrm{R}(e_n,e_j)e_j,e_n) \tag{7.9}$$

$$+\ B(e_j,e_j)\sum_{i=1}^{n-1} B(e_i,e_i) - \sum_{i=1}^{n-1} B^2(e_i,e_j). \tag{7.10}$$

□

Proof of Theorem 7.0.5. We know that

$$\mathrm{Diam}(\partial M_j, d_{\partial M_j}) \leq D_\partial. \tag{7.11}$$

From Proposition 7.0.7 we get

$$\mathrm{Ric}_\partial(X, X) \geq -\gamma + (n-1)(\alpha^2 - \beta^2). \tag{7.12}$$

Thus, by Gromov's Ricci Compactness Theorem (cf. Theorem 2.1.7) there is a GH convergent subsequence $(\partial M_{j_k}, d_{\partial M_{j_k}})$ and this subsequence is equibounded (cf. Theorem 2.1.4). Then $(\partial M_{j_k}, d_{j_k})$ is equibounded by Proposition 7.0.6. Moreover, by the definition of the restricted metric and the induced length metric,

$$\mathrm{Diam}(\partial M_{j_k}, d_{j_k}) \leq \mathrm{Diam}(\partial M_{j_k}, d_{\partial M_{j_k}}) \leq D_\partial. \tag{7.13}$$

Then by Gromov's Compactness theorem there is a subsequence of $(\partial M_{j_k}, d_{j_k})$ that converges. in GH sense. □

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