Conical Kähler-Einstein metrics and Its Applications

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Chengjian Yao

We, the dissertation committe for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

Xiu-Xiong Chen - Dissertation Advisor
Professor, Department of Mathematics

Blaine Lawson - Chairperson of Defense
Professor, Department of Mathematics

Eric Bedford
Professor, Department of Mathematics

Dror Varolin
Professor, Department of Mathematics

Martin Rocek
Professor, C.N. Yang Institute for Theoretical Physics

This dissertation is accepted by the Graduate School

Charles Taber
Dean of the Graduate School

Abstract of the Dissertation

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In this dissertation we prove certain deformation behaviors of Kähler-Einstein metrics with singularities. Based on smooth approximation and *a priori* estimates, we first give a new proof to *Donaldson's Openness Theorem* of deforming the cone angle of conical Kähler-Einstein metric on smooth Fano manifold, and prove a direct generalization to conical Kähler-Einstein metrics along simple normal crossing pluri-anti-canonical divisors on smooth Fano manifold. Then we use continuity method to study the deformation of weak conical Kähler-Einstein metric on $\mathbb{Q}$-Fano variety. The idea of our new proof above is generalized to prove the openness part of the continuity method argument, while the closedness part directly follows from the weak compactness result of Chen-Donaldson-Sun.

To all who have taught me, both on mathematics and on life!

Contents

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List of Abbreviations

GH, *Gromov-Hausdorff*;

KE, *Kähler-Einstein*;

cscK, *constant scalar curvature Kähler*;

extK, *extremal Kähler*;

KLT, *Kawamata Log Terminal*;

PSH, *pluri-subharmonic*;

SNC, *simple normal crossing*;

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Contents

1 Background in Kähler Geometry

The interplay between Riemannian geometry, symplectic geometry and complex geometry makes Kähler geometry one of the most fruitful area of research in recent years. We first describe how Kähler geometry arises from the study of Riemannian geometry, and then we briefly outline several important conjectures that dominate the recent developments in past several decades.

1.1 Riemannian point of view

Start with a real vector bundle E with an inner product $\langle , \rangle$ over a differential manifold X. A connection on E is an operator

$$\nabla_A : \Gamma(E) \to \Omega^1(E)$$

such that

$$d\langle s, t\rangle = \langle \nabla_A s, t\rangle + \langle s, \nabla_A t\rangle$$

and

$$\nabla_A(fs) = df \otimes s + f\nabla_A s$$

for any sections s, t and any function f.

Coupled with the exterior differentiation operator on differential forms and extended via the Leibniz rule, ∇_A induces operators $d_A : \Omega^p(E) \to \Omega^{p+1}(E)$ for $p = 0, 1, \cdots ,$. The curvature operator of ∇_A is defined as

$$F_A(s) = d_A d_A s$$

i.e.

$$F_A(s)(u, v) = \nabla_{A,u}\nabla_{A,v}s - \nabla_{A,v}\nabla_{A,u}s - \nabla_{A,[u,v]}s$$

Let (X, g) be a $2n$ dimensional Riemannian manifold. For any affine connection ∇_A on TX, the *torsion* $\tau^A \in \Lambda^2 T^*X \otimes TX$ is defined as follows

$$\tau^A(u, v) = \nabla^A_u v - \nabla^A_v u - [u, v]$$

The tangent bundle TX equipped with the Riemannian metric admits a unique compatible torsion free connection ∇, called the *Levi-Civita connection.* The curvature form of ∇ is then the standard *Riemannian curvature tensor* R:

$$R(u,v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w$$

Suppose now X admits an *almost complex structure* J, i.e. a section of $End(TX) \cong TX \otimes T^*X$ such that $J^2 = -id$ on X. Then J induces a splitting of the complexified tangent bundle $T_{\mathbb{C}}X = T^{1,0}X \oplus X^{0,1}$, where $T^{1,0}X$ and $T^{0,1} = \overline{T}^{1,0}$ are the complex sub-bundle corresponding to the $\sqrt{-1}$ and $-\sqrt{-1}$ eigenspaces of J on $T_{\mathbb{C}}X$.

The tensor that is used to measure the defect of J from being *integrable*, i.e. coming from an complex structure, is called the *Nijenhuis Tensor* N_J and it is defined as

$$N_J(u,v) = [u, Jv] + [Ju, v] - J[u,v] + J[Ju, Jv]$$

The *Nijenhuis Tensor* could be represented in terms of $\nabla_A J$ and τ^A via the following formula:

$$\begin{aligned} N_J(u,v) = (\nabla_A J)(u,v) - (\nabla_A J)(Ju, Jv) - (\nabla_A J)(v,u) + (\nabla_A J)(Jv, Ju) \\ - \tau^A(u, Jv) - \tau^A(Ju, v) + J(\tau^A(u,v)) - J(\tau^A(Ju, Jv)) \end{aligned} \tag{1.1}$$

If J is compatible with the Riemannian metric g, i.e. $g(Ju, Jv) = g(u,v)$, then the tensor ω defined by $\omega(u,v) = g(Ju, v)$ is skew-symmetric, i.e. it is a two form. It is easily seen that it is nondegenerate, i.e. ω^n is nowhere zero on X. The exterior derivative is given by:

$$\begin{aligned} d\omega(u,v,w) = g((\nabla_A J)(u,v) + J\tau^A(u,v), w) + g((\nabla_A J)(v,w) \\ + J\tau^A(v,w), u) + g((\nabla_A J)(w,u) + J\tau^A(w,u), v) \end{aligned} \tag{1.2}$$

Now if we take ∇_A to be the Levi-Civita connection ∇, then $\tau^{\nabla} = 0$. There is an inverse formula to the above

$$2g((\nabla_u J)v, w) = d\omega(u,v,w) - d\omega(u, Jv, Jw) - g(u, N_J(v,w)) \tag{1.3}$$

By Equations (1.1, 1.2,1.3),

$$d\omega = 0, N_J = 0 \Longleftrightarrow \nabla J = 0$$

A Riemannian structure (X, g, J) is called a *Kähler structure* if one of the above equivalent conditions is satisfied. The *parallel* condition $\nabla J = 0$ implies that the splitting $T^{1,0}X$ and $T^{0,1}X$ are preserved under covariant derivatives. In this situation J is integrable, i.e. it comes from a complex structure, according to the famous theorem of Newlander-Nirenberg. Let $z_1, \cdots, z_n$ be a local complex coordinates, then ω is a nondegenerate closed real $(1,1)$-form, i.e.

$$\omega = \sqrt{-1} \sum g_{i\bar{j}} dz_i \wedge d\bar{z}_j$$

for $(g_{i\bar{j}})$ being positive Hermitian, and $\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}}$. By Poincaré's Lemma, locally

$$\omega = \sqrt{-1} \partial\bar{\partial}\phi$$

for a real valued smooth function, which is called *local Kähler potential*. This formulation is precisely Kähler's original description about a special kind of Riemannian metric on a complex manifold, whose metric tensor is locally given by the complex Hessian of a pluri-subharmonic (PSH) function. His motivation comes from the remarkable simplicity of the formulas of the various curvature tensors in this situation. Since $T^{1,0} = \mathrm{Span}\{\frac{\partial}{\partial z_i}\}$ and $T^{0,1} = \mathrm{Span}\ \{\frac{\partial}{\partial \bar{z}_j}\}$ are preserved under the covariant derivative of Levi-Civita connection, all other types of the Christoffel symbols except

$$\Gamma_{ij}^k, \Gamma_{\bar{i}\bar{j}}^{\bar{k}}$$

vanishes, and it is easy to derive that

$$\Gamma_{ij}^k = g^{k\bar{l}} \partial_i g_{j\bar{l}} \tag{1.4}$$

$$R_{i\bar{j}k}{}^l = -\partial_{\bar{j}} \Gamma_{ik}^l \tag{1.5}$$

$$R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} \log \det g_{k\bar{l}} \tag{1.6}$$

The globally defined closed real $(1,1)$-form

$$\mathrm{Ric}\ \omega = \sqrt{-1} \sum R_{i\bar{j}} dz_i \wedge d\bar{z}_j = -\sqrt{-1}\partial\bar{\partial} \log \omega^n \tag{1.7}$$

is called the *Ricci form* of the Kähler metric ω. On the other hand, ω^n could be viewed as a section of $K_X \otimes \overline{K}_X$, i.e. a Hermitian metric on the holomorphic line bundle K_X^{-1}, and the *Ricci form* is just the curvature form of the Chern connection corresponding to this Hermitian metric. For a different choice of Kähler metric ω',

$$\mathrm{Ric}\ \omega' - \mathrm{Ric}\ \omega = -\sqrt{-1}\partial\bar{\partial} \log \frac{\omega'^n}{\omega^n}$$

therefore the *Ricci form* of any Kähler metric on X defines the same de Rham cohomology class, which is $2\pi c_1(K_X^{-1}) = 2\pi c_1(X)$ according to Chern-Weil theory.

1.2 Calabi's conjectures

Riemannian metrics of constant Gaussian curvature on Riemann surfaces could be used to prove the *Uniformization Theorem*. It is E. Calabi who first consider *uniformization problems* for higher dimensional complex manifolds using "canonical metrics". He proposed several conjectures and laid the foundations for this very interesting subject of study, which is still very active after its inception for half a century.

1.2.1 Volume Form conjecture

E. Calabi asked if every form $\alpha \in 2\pi c_1(X)$ could be realized as the Ricci form of some Kähler metrics. We might first restrict ourselves to a fixed *Kähler class* $[\omega_0]$, which consists of all Kähler metric cohomologous to ω_0.

$$[\omega_0] = \{\omega_\phi | \omega_\phi = \omega_0 + \sqrt{-}\partial\bar{\partial}\phi > 0\}$$

Let Ric $\omega_0 = \alpha_0$, suppose $\alpha = \alpha_0 - \sqrt{-1}\partial\bar{\partial}F$ for $F \in C^\infty(X)$, uniquely determined up to a constant. The equation needed to be solved is

$$\text{Ric } \omega_\phi = \alpha \tag{1.8}$$

By the formula 1.7, this equation is seen to be equivalent to

$$\omega_\phi^n = e^F \omega_0^n$$

Yau uses a continuity method to solve this equation:

$$\text{Ric } \omega_{\phi_t} = t\alpha + (1-t)\alpha_0 \tag{1.9}$$

or equivalently the family of Monge-Ampère equations

$$\omega_{\phi_t}^n = e^{tF+C_t}\omega_0^n \tag{1.10}$$

where C_t is some constant chosen to make the normalization condition

$$\int e^{tF+C_t}\omega_0^n = \int \omega_0^n$$

hold.

1.2.2 Kähler-Einstein problem

Calabi also proposed the question of finding "canonical metrics" inside a fixed Kähler class. If the Kähler class is proportional to the first Chern class, the natural candidates of those "canonical metrics" would be Kähler-Einstein metrics, i.e.

$$\text{Ric}\ \omega_\phi = \lambda\omega_\phi \tag{1.11}$$

for some constant λ.

By scaling the Kähler metric ω_ϕ, the constant could be assumed to be $-1, 0$ or 1. All three cases requires $c_1(X)$ to be definite, i.e. $c_1(K_X) > 0$, $c_1(K_X) = 0$[1] and $c_1(K_X^{-1}) > 0$ respectively. Similar to the continuity path (1.9) to attack the *Calabi's Volume Conjecture*, there is also natural continuity path (called *Aubin-Yau path*) to solve the Eq. 1.11:

$$\text{Ric}\ \omega_{\phi_t} = t\lambda\omega_{\phi_t} + (1-t)\alpha_0 \tag{1.12}$$

where $\alpha_0 = \text{Ric}\ \omega_0 = \lambda\omega_0 - \sqrt{-1}\partial\bar{\partial}h_{\omega_0}$. This is equivalent to a family of Monge-Ampère equations

$$\omega_{\phi_t}^n = e^{-\lambda t\phi_t + th_{\omega_0}}\omega_0^n \tag{1.13}$$

with parameter $t \in [0, 1]$.

The linearized of operator $\mathcal{L}_{\phi,t} = \Delta_\phi + \lambda t$ could be verified to be invertible on suitable Banach spaces, thus the *openness* part of the continuity method follows from a standard *implicit function theorem* in Banach spaces. The *closedness* part involves *a priori estimates* for solutions of the equation above. Yau derived the C^2 and C^1 bounds in terms of C^0 estimate, and Calabi did the calculation for C^3 estimate (in terms of the C^2 bound even earlier). Therefore, all works are reduced to obtain the C^0 estimate of the family ϕ_{t_j}. The case $\lambda = -1$ is the simplest one and very easy to get by *maximum principle* (cf. [Au, Yau].) The case $\lambda = 0$ is solved by Yau [Yau] (later called *Calabi-Yau manifolds*). The case $\lambda = 1$ (i.e. the Kähler-Einstein problem on a Fano manifold) turns out to be quite subtle and is only fully settled until very recently.

1.2.3 cscK/extK metrics

It is worth mentioning more general canonical metrics introduced by Calabi besides the Kähler-Einstein metrics. In analogy with the Yang-Mills functional, i.e. the L^2

[1] The Kähler class could be any Kähler class in the case $c_1(K_X) = 0$.

norm of the curvature F_A introduced in the first section, Calabi defined a functional, which is the L^2 norm of the Riemannian curvature of Kähler metrics, and studied the critical points of this functional inside a fixed Kähler class. Much different from the Riemannian case, the L^2 norms of the Riemannian curvature tensor, Ricci curvature tensor and scalar curvature function are essentially the same in a fixed Kähler class, in the sense that they differ from each other only by some topological constants depending on Chern classes and the Kähler class. The *space of Kähler potentials* is defined as

$$\mathcal{H}_{\omega_0} = \{\phi \in C^\infty(X, \mathbb{R}) | \omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi > 0\}$$

Calabi's functional Ca is defined as follows:

$$Ca : \mathcal{H}_{\omega_0} \to \mathbb{R}$$

$$\phi \mapsto \int (R_\phi - \underline{R}_\phi)^2 \omega_\phi^n$$

By Calabi's calculation, the variation is given by

$$\delta_\psi Ca(\phi) = \langle \psi, \mathcal{D}_\phi R_\phi \rangle_\phi$$

where $\mathcal{D}_\phi f = \sum_{\alpha,\beta} f_{,\alpha\beta}{}^{\alpha\beta}$ is a fourth order differential operator. The Euler-Lagrange equation is then

$$\mathcal{D}_\phi R_\phi = 0$$

On a compact Kähler manifold, integrating the above equation by parts, this Euler-Lagrange equation is reduced to the condition that $\nabla^{1,0} R_\phi = \sum_\alpha R_\phi{}^{,\alpha} \frac{\partial}{\partial z_\alpha}$ is a holomorphic vector field on X. A Kähler metric is called *extremal Kähler metric* (abbreviated as *extK metric*) if the $(1,0)$-gradient of the scalar curvature function is a holomorphic vector field on X. The special case for which $\nabla^{1,0} R_\phi = 0$, i.e. the scalar curvature is constant, is called *constant scalar curvature Kähler metrics* (abbreviated as *cscK metric*). It should be remarked that cscK metrics inside a class which is a multiple of the first Chern class are Kähler-Einstein metrics.

Based on the observation of Fujiki-Schumacher [FS] and Donaldson [Don97] that the scalar curvature of a Kähler metric could be viewed as the moment map for the action of the exact symplectomorphism group $\mathcal{G}$ on some infinite dimensional Kähler manifold. In this picture, *Calabi's functional* is just the "norm function" of the moment map, *cscK metrics* are zeros of the moment map, and $\mathcal{H}_{\omega_0}$ is the quotient of the *complexified orbit*. Thus, to find a metric of constant scalar curvature inside $\mathcal{H}_{\omega_0}$ is equivalent to find a zero of the moment map inside the corresponding *complexified* *orbit*. In finite dimensional picture, this correspondence is known as the Kempf-Ness Theorem, which relates the *Mumford stability* of a complex orbit and the existence of zeros of the moment map. This finite dimensional *Mumford stability* could be tested just along all the one parameter subgroups, known as the *Hilbert-Mumford criterion*. And it suggests a stability condition for our current picture, exploited as *K-stability* [SchYau, Tian97, Don02].

On the other hand, $\mathcal{H}_{\omega_0}$ could be equipped with a L^2 Reimannian metric

$$\langle \psi_1, \psi_2 \rangle_\phi = \int_X \psi_1 \psi_2 \omega_\phi^n$$

under which any two points ϕ_1, ϕ_2 could be joined by a unique $C^{1,1}$ geodesic [].

Mabuchi introduced a functional $\mathcal{M}_{\omega_0}$ on $\mathcal{H}_{\omega_0}$ whose critical points are constant scalar curvature Kähler (cscK) metrics inside $[\omega_0]$. Its variation along any direction $\psi \in T_\phi \mathcal{H}_{\omega_0} = C^\infty(X, \mathbb{R})$ is defined by the following formula:

$$\delta_\psi \mathcal{M}_{\omega_0}(\phi) = -\int_X \psi(R_\phi - \underline{R})\omega_\phi^n \tag{1.14}$$

where

$$\underline{R} = \frac{\int_X R_\phi \omega_\phi^n}{\int_X \omega_\phi^n} = \frac{2\pi n c_1(X) \cdot [\omega_0]^{n-1}}{[\omega_0]^n}$$

is the average of the scalar curvature and it is a topological constant. It is a simple calculation that $\mathcal{M}_{\omega_0}$ is convex along any smooth geodesics in $\mathcal{H}_{\omega_0}$ and therefore the existence of critical points would be expected to imply the *properness* (in terms of the geodesic distance) of this functional. Chen [Chen00] made the following important conjecture:

Conjecture 1 (Chen's Conjecture)**.** *There exists a cscK metric in* $[\omega_0]$ *iff* $\mathcal{M}_{\omega_0}$ *is proper in terms of the geodesic distance on* $\mathcal{H}_{\omega_0}$.

In the case when $[\omega_0] = 2\pi c_1(X)$, a cscK metric in $[\omega_0]$ would be necessarily a Kähler-Einstein metric. If $\mathrm{Aut}(X)$ is assumed to be discrete, there is one analytical criterion for the existence proved by Tian (cf. [Tian97]).

Theorem [Tian97] . *On a Fano manifold* X *without nontrivial holomorphic vector fields, there exists a Kähler-Einstein metric iff the Mabuchi functional* $\mathcal{M}_{\omega_0}$ *is proper.*

The Kähler-Einstein problem attracts the central attention in the area of complex differential geometry perhaps for two main reasons. On the one hand, the complex Monge-Ampère equation theory developed by Cafferalli-Spruck-Nirenberg-Kohn, serves as a big input to KE problem from the analytic perspective, and on the other hand, the powerful convergence theory for Riemannian manifolds with Ricci curvature bounded from below, thanks to the work of Anderson, Cheeger, Colding, Tian, serves as a powerful tool from the geometrical perspective.

In the recent years, the existence problem of smooth Kähler-Einstein metrics on Fano manifolds is fully settled by the seminal works of Donaldson-Sun [DS], Chen-Donaldson-Sun [CDS1, CDS2, CDS3]. The resolution of this problem utilizes a new "continuity method" of deforming the cone angles of *conical Kähler-Einstein metric* proposed by Donaldson [Don12]. This continuity path, especially the "openness" along the path, is the main focus point of this current dissertation.

2 Outline and Definitions

The organization of this dissertation is as follows: In this section, the necessary concepts, definitions, notations are introduced and defined, together with the statement of the theorems proved. In Sect. 3, we firstly present a new proof to *Donaldson's Openness Theorem* ([Don12, Theorem 2]) by smoothing out the current equation of *weak conical KE metrics* (cf. Theorem 2.1), and then we present a generalization to simple normal crossing pluri-anti-canonical divisors on smooth Fano manifolds (cf. Theorem 2.2). The subsection 3.1 is setting up and solving two two-parameter continuity paths (the first used to approximate *weak conical KE metric* by smooth Kähler metrics with Ricci curvature bounded from below by the same number, while the second is used to deform the cone angles) for *weak conical KE metrics* on a pair $(X, (1-\beta)D)$ where X is smooth Fano and D is a smooth (pluri-)anti-canonical divisor. The subsection 3.2 is devoting to prove that the smooth approximate solutions are actually quasi-isometric to local model metrics (this enhances the *rough C^2 bound* in [CDS1, Theorem 2.2] to a *uniform C^2 bound*), which enables us to show that the limiting metrics are quasi-isometric to the model conical Kähler metrics (cf. Theorem 3.11). Combining with the regularity results of [GP, CW], the deformed *weak conical KE metrics* are actually *conical* in Donaldson's sense. In subsection 3.3, we uses the same idea of smoothing out conical KE metrics to prove the openness of deforming the cone angles along SNC pluri-anti-canonical divisors on a smooth Fano manifold.

In Sect. 4, we are going to study the deformation behavior of weak conical KE metrics on a $\mathbb{Q}$-Fano variety, and prove Theorem 2.13 about deforming the cone angles of weak conical KE metrics on $\mathbb{Q}$-Gorenstein smoothable $\mathbb{Q}$-Fano varieties. This is an important step in the author's joint work with Spotti and Sun [SSY], where existence of weak KE metric on K-polystable $\mathbb{Q}$-Gorenstein smoothable $\mathbb{Q}$-Fano varieties is established.

2.1 local model conical metrics

Let us first look at a model metric $\omega_{(\beta)}$ on $\mathbb{C}^n$ (the Euclidean space equipped with this metric is then usually written as $\mathbb{C}_\beta \times \mathbb{C}^{n-1}$), and an obvious family of smooth Kähler metrics on $\mathbb{C}^n$ approximating $\omega_{(\beta)}$,

$$\omega_{(\beta)} = \sqrt{-1}\partial\bar{\partial}(|z|^{2\beta} + \sum_{i=2}^{n} |z_i|^2) = \sqrt{-1}\{\beta^2|z|^{2\beta-2}dz \wedge d\bar{z} + \sum_{i=2}^{n} dz_i \wedge d\bar{z}_i\} \quad (2.1)$$

$$\omega_{(\beta,\epsilon)} = \sqrt{-1}\{\beta^2(|z|^2 + \epsilon)^{\beta-1}dz \wedge d\bar{z} + \sum_{i=2}^{n} dz_i \wedge d\bar{z}_i\} \quad (2.2)$$

As $\omega_{(\beta,\epsilon)}$ approaches $\omega_{(\beta)}$, the Ricci form

$$\text{Ric } \omega_{(\beta,\epsilon)} = \sqrt{-1}\frac{\epsilon(1-\beta)}{(|z|^2+\epsilon)^2}dz \wedge d\bar{z}$$

approaches the delta function $2\pi(1-\beta)[z=0]$ in the distributional sense. It is by this reason that $\omega_{(\beta)}$ could be viewed as a Kähler metric on $\mathbb{C}^n$ with Ric $\omega_{(\beta)} = 2\pi(1-\beta)[z=0]$.

The recent development of Kähler-Einstein problem on Fano manifold makes use Kähler metrics with cone singularities along a smooth divisor (called *conical Kähler metric*). Roughly speaking, conical Kähler metric on a Kähler manifold could be viewed as just usual Kähler metric with Ricci curvature concentrated as delta function near the divisor. Donaldson proposed an analogous continuity path of the *Aubin-Yau continuity path* by replacing the smooth $(1,1)$-form α_0 in Eq. 1.12 as the *integration current* along a divisor D, taking the shape (cf. [Don12, Eq. 27]):

$$\text{Ric } \omega_\beta = \beta\omega_\beta + 2\pi(1-\beta)[D] \quad (2.3)$$

The parameter β in this path corresponds to the angle of the cone which is $2\pi\beta$. The idea of Donaldson's new continuity method is to find ω_{β_0} with very small cone angle $2\pi\beta_0 > 0$ first, and then to "open up" the cone, i.e. deforming the cone angles, to 2π.

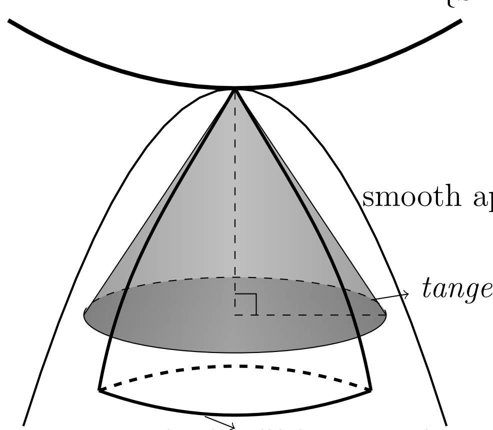

Figure 2.1: conical Kähler metric ω_β; tangent cone $\omega_{(\beta)}$ and smooth approximation ω_{ϕ_ϵ}

The Eq. 2.3 should be understood in the distributional sense. Let Ric $\omega_0 = \omega_0 + \sqrt{-1}\partial\bar\partial h_{\omega_0}$, and let $\omega_\beta = \omega_0 + \sqrt{-1}\partial\bar\partial\phi_\beta$. Then for any smooth test $(n-1, n-1)$-form η on X,

$$\int_X (h_{\omega_0} - \log\frac{\omega_\beta^n}{\omega_0^n}) \wedge \sqrt{-1}\partial\bar\partial\eta$$
$$= \int_X \beta\phi_\beta \wedge \sqrt{-1}\partial\bar\partial\eta + (1-\beta)(2\pi\int_D \eta - \int_X \omega_0 \wedge \eta)$$

The condition that the above integration equality makes sense is that $\log\frac{\omega_\beta^n}{\omega_0^n}$ and ϕ_β are integrable, which could be satisfied fairly easily. However, for the KE problem, the condition is much easier to satisfy. In order to spell those out, let us first write down the PDE for a conical Kähler-Einstein metric. Write L_D for the holomorphic line bundle associated with the divisor D, and let S be its defining section, and let h be the Hermitian metric on L_D with curvature form ω_0. Using the *Poincaré-Lelong's formula* $\sqrt{-1}\partial\bar\partial \log|S|^2 = 2\pi[D]$, the Eq. 2.3 could be translated as a Monge-Ampère equation with mildly singular right hand side:

$$\omega_\beta^n = e^{-\beta\phi_\beta + h_{\omega_0}}|S|_h^{2\beta-2}\omega_0^n \tag{2.4}$$

By the general theory developed by Bedford-Taylor [BT], ω_ϕ^n makes sense as a non-pluripolar measure, called *Monge-Ampère measure*, for any continuous ω_0-PSH function ϕ on X. Kolodziej even proved that if the Monge-Ampère measure ω_ϕ^n has L^p density for some $p > 1$, then the potential function ϕ should be Hölder continuous for some $\alpha \in (0, 1)$. Therefore, it is natural to require ϕ_β to be Hölder continuous on X and satisfy Eq. 2.4 in the smooth sense on $X \backslash D$. A Kähler metric ω_β with this described regularity is called *weak conical Kähler-Einstein metric.* The distributional equality above makes sense.

On the other hand, based on more geometrical consideration, Donaldson requires the singular Kähler metric ω_ϕ to have $C^{2,\alpha,\beta}$ regularity, which means $\phi, \partial\phi, \partial\bar{\partial}\phi$ are all Hölder continuous in the real coordinates adapted to the presence of D (see more precise Definition 2.7). This regularity requirement implies the tangent cone of a conical Kähler metric ω_ϕ at any point on D is the standard model metric cone $\mathbb{C}_\beta \times \mathbb{C}^{n-1}$ (see Figure 2.1 for the illustration). The conical Kähler metric ω_β in Donaldson's sense, which also satisfies the smooth Monge-Ampère equation 2.4 on $X \backslash D$ is called a *conical Kähler-Einstein metric* with angle $2\pi\beta$ along D, or sometimes as conical Kähler-Einstein metric on the pair $(X, (1-\beta)D)$. The distributional equality is easier to see to make sense for conical Kähler metrics.

Similar to the smooth case, conical KE metric corresponds to the critical point of *log-Mabuchi-functional* $\mathcal{M}_{\omega_0,(1-\beta)D}$, which is the usual *Mabuchi functional* modified with an extra term coming from the current term $2\pi(1-\beta)[D]$, i.e.

$$\mathcal{M}_{\omega_0,(1-\beta)D} = \mathcal{M}_{\omega_0} + (1-\beta)J_{2\pi[D]} \tag{2.5}$$

where

$$\delta_\psi J_{2\pi[D]}(\phi) = n \int_X (2\pi[D] - \omega_\phi) \wedge \omega_\phi^{n-1} \tag{2.6}$$

Using a variational approach, the existence of *weak conical KE metric* with small cone angles is established by Berman [Berm1]. Actually, Berman showed the properness of $\mathcal{M}_{\omega_0,(1-\beta_0)D}$ for small $\beta_0 > 0$ (he actually finds an explicit estimate of the range for β_0 in terms of Alpha-invariant introduced by Tian).

In this dissertation, we first prove the deformation property of *weak conical KE metrics* on smooth Fano manifold, which is *a priori* with weaker regularity than $C^{2,\alpha,\beta}$.

Theorem 2.1. *If there exists a weak conical Kähler-Einstein metric ω_{φ_β} with angle $2\pi\beta$ $(0 < \beta < 1)$ along a smooth anti-canonical divisor D on a smooth Fano manifold X, then there exists $\delta > 0$, such that for all $\beta' \in (\beta - \delta, \beta + \delta)$, there exists a weak conical Kähler-Einstein metric $\omega_{\varphi_{\beta'}}$ with angle $2\pi\beta'$ along the divisor D.*

The method is a combination of two two-parameter continuity paths, the first one (the one that is used in [CDS1]) is used to approximate ω_{φ_β} by smooth Kähler metrics with Ricci curvature bounded below by β, and the second one is used to deform the angle parameter β to nearby β', thus approximating *Donaldson's continuity path.* The potentials of the smooth approximating Kähler metrics are shown to have uniform L^∞ bound, which enables us to pass to the higher order bounds outside the divisor. The weak conical KE metric with angle $2\pi\beta'$ is obtained by taking the limit of these smooth Kähler metrics.

By the global regularity result of Guenancia-Păun [GP, Theorem A], and also independently the local version of Chen-Wang [CW, Theorem 1.2], the deformed *weak conical KE metrics* are actually Hölder continuous, i.e. $\omega_{\varphi_{\beta'}} \in C^{2,\alpha,\beta'}$. Therefore, together with this regularity result, Theorem 2.1 gives a new proof of *Donaldson's Openness Theorem* ([Don12, Theorem 2]) for conical KE metrics.

Since the essential tool used in the study of continuity path, the *log-Mabuchi-functional* (cf. Definition 3.1), requires relatively less regularity, we bypass the *implicit function theorem* for singular Kähler metrics. This observation enables us to generalize this openness property to some other settings, where the Banach spaces theory seems difficult or subtle to set up. For instance, the case of *simple normal crossing divisor (pluri-anticanonical)* on a smooth Fano manifold:

Theorem 2.2. *On a smooth Fano manifold X, suppose there is a* weak conical KE metric *with angle $2\pi\beta_i$ (for $\beta_i \in (0,1)$) along D_i, where $D_i \in |-\lambda_i K_X|$ with $\lambda_i > 0$ and $\cup_i D_i$ being simple normal crossing, and also assume there is no holomorphic vector field tangential to any D_i. Then there exists $\delta > 0$ small enough such that for all $\beta_i' \in (\beta_i - \delta, \beta_i + \delta)$, there exists a weak conical KE metric with angle $2\pi\beta_i'$ along D_i for $i = 1, \cdots, k$.*

Actually, the method exploited here shows that in this situation the *weak conical KE metric* is the limit of a family of smooth Kähler metrics with Ricci curvature bounded from below. And it could be shown that the metrics spaces defined by those smooth approximate Kähler metrics converges to the metric space defined by the *weak conical KE metric* in GH sense. Therefore, Theorem 2.2 generalizes the approximation result of [CDS1, Theorem 1.1] to the case of SNC pluri-anticanonical divisors.

It should be remarked that by [GP, Theorem A], weak conical KE metrics along simple normal crossing divisors is also quasi-isometric to the standard model conical Kähler metrics. The deformation result above, Theorem 2.2, is thus a generalization of *Donaldson's Openness Theorem* to the case of simple normal crossing pluri-anticanonical divisors on a smooth Fano manifold. We hope this new result is useful in some future works.

"Approximation" is a very general principle in almost all branches of mathematics. We use smooth objects to approximate the singular objects and then take limit to derive the properties of the singular objects from the corresponding properties of the approximate smooth objects. The *deformation property* we proved here also falls into this general framework, however the key new input here is the "gap" type argument that enables us to carry out a contradiction. This bypasses lots of technical difficulties which are inevitable via the direct method.

As was already remarked above, the usual *implicit function theorem* requires us to prove the linearized operator is invertible in some Banach space, which is $C^{2,\alpha,\beta}$ in the setting of conical KE metrics. The Schauder-type estimate, i.e. the $C^{2,\alpha,\beta}$ theory, for conical Kähler metrics on a smooth Kähler manifold is achieved by a beautiful elementary and classical construction of Donaldson [Don12]. In general, the related Schauder-type estimates for metrics with even weaker regularities, for instance Kähler current on a possibly singular projective variety, seems to be problematic to achieve/set up at this moment. In this direction, we are able to use the idea developed here to prove the *openness* about the cone angles for *weak conical KE metrics* on a $\mathbb{Q}$-Gorenstein smoothable $\mathbb{Q}$-Fano variety. Before we state our result (Theorem 2.13) rigorously, we need to make some definitions and notations.

2.2 Definitions/Notations

A complex line bundle L over an complex analytic space X is called *holomorphic* on X if $L|_{U\cap X^{reg}}$ is isomorphic to the pull back under j of some holomorphic line bundle on $\mathbb{C}^N$ for any local holomorphic embedding $j : U(\subset X) \to \mathbb{C}^N$. The transition functions of the holomorphic line bundle over X are understood through local embeddings into affine spaces. Therefore, we also have the notion of holomorphic sections. The holomorphic line bundle L is called *ample* if the space of holomorphic sections $H^0(X, L^k)$ gives an embedding $X \to \mathbb{P}^M$ for some integer $k \geq 1$.

By saying $-K_X$ is a $\mathbb{Q}$-Cartier divisor, we mean that for some integer $m \geq 1$, the holomorphic line bundle $K^{-m}_{X^{reg}}$ could be extended to a holomorphic line bundle on X. A singular variety X is called with (at worst) *log terminal* singularity if there exists a resolution $\mu : \tilde{X} \to X$ such that the exceptional divisors $\{E_i\}$ are simple normal crossing and

$$K_{\tilde{X}} \equiv_{\mathbb{Q}} \mu^* K_X + \sum_i a_i E_i$$

with the discrepancy $a_i > -1$ for any any exceptional divisor E_i, where "$\equiv_{\mathbb{Q}}$" means the numerical equivalence as $\mathbb{Q}$-Cartier divisors. Similarly, we could define the notion of *Kawamata Log Terminal* (abbreviated as KLT in the following) for a pair (X, D) (cf. [BBEGZ, Sect. 3.1]). We remark that two dimensional log terminal singularities are precisely the quotient singularity, i.e. locally modeled as $\mathbb{C}^2/\Gamma$ for some finite subgroup Γ of $GL(2, \mathbb{C})$.

Definition 2.3 ($\mathbb{Q}$-Fano variety)**.** A normal projective variety X over $\mathbb{C}$ with at worst log-terminal singularities is called $\mathbb{Q}$-*Fano* if $K_{X^{reg}}^{-k}$ extends to a ample holomorphic line bundle over X for some positive integer k. If X is smooth, then X is called a Fano manifold.

Typical examples of Fano manifolds are $\mathbb{P}^n$, low degree complete intersections in $\mathbb{P}^n$, Mukai 3-folds and so on. Cubic surfaces in $\mathbb{P}^3$ with quotient singularities, for instance the *Cayley's nodal cubic surface* defined as

$$\{wxy + xyz + yzw + zwx = 0\},$$

are typical examples of two dimensional $\mathbb{Q}$-Fano varieties (often called as *del Pezzo surfaces* in literature).

Definition 2.4 (KE metric)**.** A Kähler manifold (X, ω) is said to be Kähler-Einstein if

$$\mathrm{Ric}\ \omega = c\,\omega,$$

or equivalent, an Einstein manifold (X, g) with a compatible complex structure.

Definition 2.5 (weak KE metric, [EGZ])**.** A *weak KE metric* on a $\mathbb{Q}$-Fano variety X is a Kähler current in $2\pi c_1(X)$ with continuous local potentials and that is a smooth KE metric on the smooth part X^{reg}.

Let X be a $\mathbb{Q}$-Fano variety and D be a Cartier divisor in the linear system $|-\lambda K_X|$ for some integer $\lambda \geq 1$ with defining section S. Fix a smooth Kähler metric $\omega \in 2\pi c_1(X)$ which is the curvature form of a Hermitian metric h on K_X^{-1} (defined up to a constant multiple). A choice of h can also be viewed as a choice of a smooth volume form vol_h on X, or equivalently, a choice of the *Ricci potential* h_ω of ω (that is to say $\mathrm{Ric}\ \omega = \omega + \sqrt{-1}\partial\bar{\partial} h_\omega$ by the relation $\mathrm{vol}_h = e^{h_\omega}\frac{\omega^n}{n!}$).

Denote by $PSH(X, \omega)$ the space of ω-plurisubharmonic functions on X. For $\beta \in (0, 1]$ we define $r(\beta) = 1 - (1 - \beta)\lambda$ and denote $V = (2\pi)^n \frac{<c_1(X)^n, [X]>}{n!}$ (this is a fixed topological quantity all through this paper).

Definition 2.6 (weak conical KE metric, [BBEGZ])**.** Fix a smooth Kähler metric ω on X, let ϕ be smooth on $X^{reg}\backslash D$ and locally continuous near $X^{sing}\cup D$ such that $\omega_\phi=\omega+\sqrt{-1}\partial\bar\partial\phi$ is a Kähler current on X. It is called *weak conical KE* if it satisfies the current equation:

$$\text{Ric }\omega_\phi=r(\beta)\omega_\phi+2\pi(1-\beta)[D] \tag{2.7}$$

in a suitable sense or, equivalently, the complex Monge-Ampère equation

$$\omega_\phi^n=n!V\frac{e^{-r(\beta)\phi}|S|_h^{2\beta-2}\text{vol}_h}{\int_X e^{-r(\beta)\phi}|S|_h^{2\beta-2}\text{vol}_h}. \tag{2.8}$$

where S is the defining section of D and h on L_D is the natural Hermitian metric induced from the one on K_X^{-1}.

Let us recall Donaldson's notion of conical Kähler metrics [Don12] for a smooth pair $(X,(1-\beta)D)$. Let U be a local chart X with coordinate system $\{z,z_2,\cdots,z_n\}$ and $D\cap U=\{z=0\}$, we could define a new (non-holomorphic)coordinate system

$$\{\zeta=|z|^{\beta-1}z,z_2,\cdots,z_n\}.$$

A function φ on X is said to be in $C^{,\alpha,\beta}$ if φ is in the usual Hölder space C^α under the new coordinate system. φ is said to be in $C^{2,\alpha,\beta}$ if the coefficients of $\varphi,\partial\varphi,\partial\bar\partial\varphi$ under the new coordinate systems are all in the usual Hölder space C^α.

Definition 2.7 (conical KE metric [Don12])**.** A weak conical KE metric ω_{φ_β} on a smooth pair $(X,(1-\beta)D)$ is called *conical KE* if ω_{φ_β} is quasi-isometric to the standard local model conical Kähler metric $\omega_{(\beta)}=\sqrt{-1}(\beta^2|z|^{2\beta-2}\text{d}z\wedge\text{d}\bar z+\sum_{i=2}^n\text{d}z_i\wedge\text{d}\bar z_i)$ around each point $p\in D$, and $\varphi_\beta\in C^{2,\alpha,\beta}$.

Then, we refer the read to Berman [Berm2] for the definition of K-polystability and log-K-polystability. He indeed proved the existence of weak (conical) Kähler-Einstein metrics implies (log-)K-polystability.

Next, we need to introduce several important functionals in Kähler geometry that are needed in this paper. Fix ω_0 to be a smooth Kähler metric on the smooth Kähler manifold X. The *space of Kähler potentials* is

$$\mathcal{H}_{\omega_0}=\{\phi\in C^\infty(X)|\omega_0+\sqrt{-1}\partial\bar\partial\phi>0\}$$

Let α be a smooth closed $(1,1)$-form on X (not necessarily in $2\pi c_1(X)$, not necessarily positive either), and let D be a smooth divisor on X. Denote two constants,

$$\underline{\alpha} = \frac{n[\alpha]\cdot[\omega_0]^{n-1}}{[\omega_0]^n}, \qquad \underline{D} = \frac{2\pi n[D]\cdot[\omega_0]^{n-1}}{[\omega_0]^n}$$

which is the "average" of the corresponding quantities, depending only on the cohomology/homology classes. For any $\phi \in \mathcal{H}_{\omega_0}$, take ϕ_t with $t \in [0,1]$ to be a smooth path in $\mathcal{H}_{\omega_0}$ connecting 0 and ϕ. All the functionals described below are defined by integration along the path ϕ_t (it can be verified that the values do not depend on the particular choice of the path ϕ_t).

Definition 2.8. [J-Functional [Chen00]]

$$J_\alpha(\phi) := \int_0^1 \mathrm{d}t \int_X \dot{\phi}_t(n\ \alpha - \underline{\alpha}\ \omega_{\phi_t}) \wedge \omega_{\phi_t}^{n-1} \tag{2.9}$$

$$J_{2\pi[D]}(\phi) = 2\pi n \int_0^1 \mathrm{d}t \int_D \dot{\phi}_t \omega_{\phi_t}^{n-1} - \underline{D} \int_0^1 \mathrm{d}t \int_X \dot{\phi}_t \omega_{\phi_t}^{n} \tag{2.10}$$

By choosing the standard linear path $\phi_t = t\phi$ and integrating the above formulas out, we could write down well-known more explicit formulas which we include a proof for convenience:

$$J_\alpha(\phi) = \sum_{k=0}^{n-1} \int_X \phi\alpha \wedge \omega_0^k \wedge \omega_\phi^{n-1-k} - \frac{\underline{\alpha}}{n+1} \sum_{k=0}^{n} \int_X \phi\omega_0^k \wedge \omega_\phi^{n-k} \tag{2.11}$$

$$J_{2\pi[D]}(\phi) = \sum_{k=0}^{n-1} 2\pi \int_D \phi\omega_0^k \wedge \omega_\phi^{n-1-k} - \frac{\underline{D}}{n+1} \sum_{k=0}^{n} \int_X \phi\omega_0^k \wedge \omega_\phi^{n-k} \tag{2.12}$$

Proof. The proof for the first formula is given below, and the proof for the divisor case is similar.

$$\begin{aligned} J_\alpha(\phi) &= \int_0^1 \mathrm{d}t \int_X \phi n\alpha \wedge \big((1-t)\omega_0 + t\omega_\phi\big)^{n-1} - \underline{\alpha}\phi\big((1-t)\omega_0 + t\omega_\phi\big)^n \\ &= \sum_{k=0}^{n-1} \int_0^1 n\binom{n-1}{k}(1-t)^k t^{n-1-k}\mathrm{d}t \int_X \phi\alpha \wedge \omega_0^k \wedge \omega_\phi^{n-1-k} \\ &\quad - \sum_{k=0}^{n} \int_0^1 \underline{\alpha}\binom{n}{k}(1-t)^k t^{n-k}\mathrm{d}t \int_X \phi\omega_0^k \wedge \omega_\phi^{n-k} \\ &= \sum_{k=0}^{n-1} \int_X \phi\alpha \wedge \omega_0^k \wedge \omega_\phi^{n-1-k} - \frac{\underline{\alpha}}{n+1} \sum_{k=0}^{n} \int_X \phi\omega_0^k \wedge \omega_\phi^{n-k} \end{aligned}$$

□

The functional J_α was introduced by Chen [Chen00] to study the lower bound of Mabuchi functional (cf. Definition 2.9) on $\mathcal{H}_{\omega_0}$, and later was used by Székelyhidi to study the Aubin-Yau continuity path [Sz] initiated from different $(1,1)$ forms in $2\pi c_1(X)$.

If $\alpha \in 2\pi c_1(X)$, then this J_α functional is familiar in earlier work of Aubin[Au], Bando and Mabuchi [BM]. In particular, take $\alpha = \omega_0$, then $\underline{\alpha} = n$ and

$$J_{\omega_0}(\phi) = \frac{1}{n+1}\sum_{i=0}^{n}\int_X \phi\omega_\phi^i \wedge \omega_0^{n-i} - \int_X \phi\omega_\phi^n = (I-J)(\phi) \tag{2.13}$$

where I and J are the classical functionals defined by Aubin [Au],

$$I(\phi) := \int_X \phi\omega_0^n - \int_X \phi\omega_\phi^n \tag{2.14}$$

$$J(\phi) := \int_X \phi\omega_0^n - \frac{1}{n+1}\sum_{i=0}^{n}\int_X \phi\omega_\phi^i \wedge \omega_0^{n-i} \tag{2.15}$$

There is an easy comparison according to [Au]:

$$\frac{1}{n+1}I(\phi) \le J_{\omega_0}(\phi) \le I(\phi) \tag{2.16}$$

which could be seen from the following calculation:

$$\begin{aligned}
(n+1)J_{\omega_0}(\phi) - I(\phi) &= \sum_{i=0}^{n}\int_X \phi\omega_0^i \wedge \omega_\phi^{n-i} - \int_X \phi\omega_0^n - n\int_X \phi\omega_\phi^n \\
&= \sum_{i=1}^{n-1}\int_X \phi\omega_0^i \wedge \omega_\phi^{n-i} - \phi\omega_\phi^n \\
&= \sum_{i=1}^{n-1}\int_X -\phi\sqrt{-1}\partial\bar{\partial}\phi \wedge (\omega_0^{i-1} + \omega_0^{i-2}\omega_\phi + \cdots + \omega_\phi^{i-1}) \wedge \omega_\phi^{n-i} \\
&= \sum_{i=1}^{n-1}\int_X \sqrt{-1}\partial\phi \wedge \bar{\partial}\phi \wedge (\omega_0^{i-1} + \omega_0^{i-2}\omega_\phi + \cdots + \omega_\phi^{i-1}) \wedge \omega_\phi^{n-i} \\
&\ge 0
\end{aligned}$$

Definition 2.9 (Mabuchi functional $\mathcal{M}_{\omega_0}$, [Ma])**.** The variation of $\mathcal{M}_{\omega_0}$ at $\phi \in \mathcal{H}_{\omega_0}$ along a tangent direction ψ is defined as

$$\delta_\psi \mathcal{M}_{\omega_0}(\phi) = -\int_X \psi(R_\phi - \underline{R})\omega_\phi^n. \tag{2.17}$$

By integrating along the path $\phi_t = t\phi$, Chen [Chen00] derived an explicit *Decomposition Formula* for the Mabuchi functional, the sum of an *entropy* term and J_{Ric} term.

$$\mathcal{M}_{\omega_0}(\phi) = \int_X \log \frac{\omega_\phi^n}{\omega_0^n}\omega_\phi^n - J_{\mathrm{Ric}\,\omega_0}(\phi) \tag{2.18}$$

Let us look at some consequence of this decomposition formula. The *entropy* term is always bounded from below, and moreover it is proper in terms of the I functional by Tian [Tian97]. If $c_1(X) = 0$, thanks to Yau's solution on Calabi's Volume Conjecture, we could pick ω_0 to be the unique Ricci flat Kähler metric in its Kähler class. It follows that $\mathcal{M}_{\omega_0}$ is proper in any Kähler class on a Calabi-Yau manifold. If $c_1(X) < 0$, by Aubin-Yau's theorem on Kähler-Einstein metrics, we could pick ω_0 to be the unique Kähler metric in $-2\pi c_1(X)$ with $\mathrm{Ric}\ \omega_0 = -\omega_0$. Therefore, $\mathcal{M}_{\omega_0}$ is proper since $\mathcal{M}_{\omega_0}(\phi) \geq -C + J_{\omega_0}(\phi)$.

$$\begin{aligned}
\mathcal{M}_{\omega_0}(\phi) =& -n\int_0^1 \mathrm{d}t \int_X \phi \mathrm{Ric}\ \omega_{\phi_t} \wedge \omega_{\phi_t}^{n-1} + \underline{R}\int_0^1 \mathrm{d}t\int_X \phi\omega_{\phi_t}^n \\
=& -n\int_0^1 \mathrm{d}t \int_X \phi(\mathrm{Ric}\ \omega_{\phi_t} - \mathrm{Ric}\ \omega_0) \wedge \omega_{\phi_t}^{n-1} \\
& \qquad - \int_0^1 \mathrm{d}t \int_X \phi(n\ \mathrm{Ric}\ \omega_0 - \underline{R}\ \omega_{\phi_t})\omega_{\phi_t}^{n-1} \\
=& n\int_0^1 \mathrm{d}t \int_X \phi\sqrt{-1}\partial\bar\partial \log\frac{\omega_{\phi_t}^n}{\omega_0^n} \wedge \omega_{\phi_t}^{n-1} - J_{\mathrm{Ric}\,\omega_0}(\phi) \\
=& n\int_0^1 \mathrm{d}t \int_X \log\frac{\omega_{\phi_t}^n}{\omega_0^n}\sqrt{-1}\partial\bar\partial\phi \wedge \omega_{\phi_t}^{n-1} - J_{\mathrm{Ric}\,\omega_0}(\phi) \\
=& \int_0^1 \mathrm{d}t \int_X (n\sqrt{-1}\partial\bar\partial\phi + \log\frac{\omega_{\phi_t}^n}{\omega_0^n} n\sqrt{-1}\partial\bar\partial\phi) \wedge \omega_{\phi_t}^{n-1} - J_{\mathrm{Ric}\,\omega_0}(\phi) \\
=& \int_0^1 \mathrm{d}t \int_X \frac{d}{dt}(\log\frac{\omega_{\phi_t}^n}{\omega_0^n}\omega_{\phi_t}^n) - J_{\mathrm{Ric}\,\omega_0}(\phi) \\
=& \int_X \log\frac{\omega_\phi^n}{\omega_0^n}\omega_\phi^n - J_{\mathrm{Ric}\,\omega_0}(\phi)
\end{aligned}$$

Firstly, define $C^{1,1}(X) := \bigcup_{\beta\in(0,1)} C^{1,1}_\beta(X)$, where $C^{1,1}_\beta(X)$ denotes the space of all functions ϕ which are C^2 on $X^{reg}\backslash D$ with $\omega + i\partial\bar\partial\phi \geq 0$ and, locally around each point p in D, we have $C^{-1}\omega_{(\beta)} \leq \omega + \sqrt{-1}\partial\bar\partial\phi \leq C\omega_{(\beta)}$ for some $C > 0$, where $\omega_{(\beta)}$ is a standard model conical Kähler metric in a neighborhood of p (cf. the beginning of this section).

Definition 2.10 (log-Mabuchi-functional $\mathcal{M}_{\omega,(1-\beta)D}$)**.** Suppose (X, D) is smooth and ω be a smooth Kähler metric on X, define the log-Mabuchi-functional on $\mathcal{H}_\omega$ to be

$$\mathcal{M}_{\omega,(1-\beta)D} = \mathcal{M}_\omega + (1-\beta)J_{2\pi[D]}$$

In the particular case when X is a Fano manifold and $D \in |-\lambda K_X|$ is an pluri-anticanonical divisor, let $\omega \in 2\pi c_1(X)$ have Ricci potential h_ω, then we could write

$$\text{Ric } \omega = r(\beta)\omega + 2\pi(1-\beta)[D] + \sqrt{-1}\partial\bar\partial H_{\omega,(1-\beta)D},$$

where $H_{\omega,(1-\beta)D} = h_\omega - \log|S|_h^{2-2\beta}$. For $\phi \in PSH(X,\omega)$ which is in $C^{1,1}(X)$, it follows from the *Decomposition Formula* 2.18 that,

$$\mathcal{M}_{\omega,(1-\beta)D}(\phi) := \int_X \log\frac{\omega_\phi^n}{e^{H_{\omega,(1-\beta)D}}\omega^n}\omega_\phi^n - r(\beta)J_\omega(\phi) + \int_X H_{\omega,(1-\beta)D}\omega^n.$$

In the particular case $\beta = 1$, the *log-Mabuchi-functional* reduces to the usual *Mabuchi-functional*. The *log-Mabuchi-functional* was introduced to study *conical KE metrics*, see [Berm1, BBEGZ, SW, LS, CDS1].

On a KLT pair $(X, (1-\beta)D)$, we could define the following functional which only requires the potentials to be bounded.

Definition 2.11 (log-Ding-functional)**.** For $\phi \in L^\infty(X) \cap PSH(X,\omega)$, we define

$$\mathcal{F}_{\omega,(1-\beta)D}(\phi) := F^0_\omega(\phi) + F^1_\omega(\phi),$$

where

$$F^0_\omega(\phi) := -\frac{1}{n+1}\sum_{i=0}^n \int_X \phi(\omega + \sqrt{-1}\partial\bar\partial\phi)^i \wedge \omega^{n-i},$$

$$F^1_\omega(\phi) := -\frac{V}{r(\beta)}\log\frac{1}{V}\int_X e^{-r(\beta)\phi}|S|_h^{2\beta-2}\text{vol}_h.$$

The first term, essentially the second term appears in the formula of J_α functional, is well-defined by the pluri-potential theory [BBEGZ] and the second one makes sense when $(X, (1-\beta)D)$ is KLT. The Euler-Lagrange equation for *log-Ding-functional* is precisely the above Eq. 2.8.

In the second half of the paper, we would like to study the existence of weak KE metric on a $\mathbb{Q}$-Fano variety X via Donaldson's continuity method of deforming the cone angle of weak conical KE metric on X. The idea of this part comes from the joint work with Spotti and Sun [SSY], therefore we will be very sketchy and refer the reader to [SSY] for more detail. We only illustrate how to prove the *openness* of Donaldson's continuity path on this type of variety without using implicit function theorem.

As already remarked above, to overcome the difficult of proving the *openness* part, we need to use some kind of smooth approximations. Since the essential difference between weak conical KE metrics on smooth Fano manifolds and $\mathbb{Q}$-Fano variety is caused by the presence of singularities of the variety, rather than the singular behavior of the metrics, we could not simply smooth out the *current of integration* $[D]$ to χ_ϵ. The "approximation" is instead achieved by utilizing the smoothing of the variety (one dimensional $\mathbb{Q}$-Gorenstein smoothing). We use (weak) conical KE metrics on the nearby smooth fibers in $\mathcal{X}$ to approximate the weak conical KE metrics on X_0.

Definition 2.12 ($\mathbb{Q}$-Gorenstein smoothing)**.** A $\mathbb{Q}$-Fano variety X is called $\mathbb{Q}$-*Gorenstein smoothable* if there is a flat family

$$\pi : \mathcal{X} \to \Delta,$$

over a disc Δ in $\mathbb{C}$ such that $X \cong X_0$, X_t is smooth for $t \neq 0$ and $\mathcal{X}$ admits a relatively $\mathbb{Q}$-Cartier anti-canonical divisor $-K_{\mathcal{X}/\Delta}$ (in this case $\pi : \mathcal{X} \to \Delta$ is called a $\mathbb{Q}$-*Gorenstein smoothing* of X_0).

By possibly shrinking Δ, we can assume X_t is a Fano manifold for $t \neq 0$ and there exists an integer $\lambda > 0$ such that $K_{X_t}^{-\lambda}$ are very ample line bundles with vanishing higher cohomology for all $t \in \Delta$. Moreover, the dimension of the corresponding linear systems $|-\lambda K_{X_t}|$ is constant in t, denoted as $N(\lambda)$. Therefore, we could assume that the family $\mathcal{X}$ is relatively very ample, i.e. there is a smooth embedding $i : \mathcal{X} \hookrightarrow \mathbb{P}^{N(\lambda)} \times \mathbb{C}$ such that $i_t = i|_{X_t} : X_t \hookrightarrow \mathbb{P}^{N(\lambda)} \times \{t\}$ pulls the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^{N(\lambda)}$ back to $K_{X_t}^{-\lambda}$. It follows from purely algebraic geometry argument that there exists a divisor $\mathcal{D} \in |-\lambda K_{\mathcal{X}/\Delta}|$ such that $D_t = \mathcal{D}|_{X_t}$ is smooth and $(X_t, (1-\beta)D_t)$ is KLT pair for any $\beta \in (0, 1]$ (cf. appendix of [SSY]).

Theorem 2.13. *If there exists a weak conical Kähler-Einstein metric $\omega_{\varphi_{\beta_*}}$ on the KLT pair $(X_0, (1-\beta_*)D_0)$ (for $\beta_* \in (1-\lambda^{-1}, 1)$) introduced above, then there exists $\delta > 0$, such that for all $\beta \in (\beta_* - \delta, \beta_* + \delta)$, there exists a weak conical Kähler-Einstein metric ω_{φ_β} on the KLT pair $(X_0, (1-\beta)D_0)$.*

Several well-known properties needed in this paper is listed in the following proposition.

Proposition 2.14. *1. ([Sz]) If* $\alpha' = \alpha + \sqrt{-1}\partial\bar{\partial}\psi$*, then*

$$J_{\alpha'}(\phi) - J_{\alpha}(\phi) = \int_X \psi(\omega_\phi^n - \omega_0^n)$$

2. ([Tian00, Berm1])

$$\mathcal{M}_{\omega,(1-\beta)D}(\phi) \geq r(\beta)\mathcal{F}_{\omega,(1-\beta)D}(\phi) + \int_X H_{\omega,(1-\beta)D}\frac{\omega^n}{n!}$$

3.([Tian97]) Let ω_0 *and* $\omega_\phi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi$ *be two smooth Kähler metrics on* X*, and let* C_P, C_S *be the corresponding Poincaré and Sobolev constants. Then for some* $\delta_n > 0$*, which is a dimensional constant, the following estimate holds:*

$$Osc\ \phi \leq \{C_S(\omega_0)^{\delta_n}C_P(\omega_0) + C_S(\omega_\phi)^{\delta_n}C_P(\omega_\phi)\}J_{\omega_0}(\phi) + C_S(\omega_0)^{\delta_n} + C_S(\omega_\phi)^{\delta_n}$$

4.[Chern, Lu] Suppose ω *and* η *are two Kähler metrics on a compact Kähler manifold, if* $Ric\ \omega \geq C_1\omega - C_2\eta$ *and the holomorphic bisectional curvature* $R^{\eta}_{i\bar{j}k\bar{l}} \leq C_3(h_{i\bar{j}}h_{k\bar{l}} + h_{i\bar{l}}h_{k\bar{j}})$*, then*

$$\Delta_\omega \log tr_\omega\eta \geq C_1 - (C_2 + 2C_3)tr_\omega\eta.$$

The final definition in this section is the *Gromov-Hausdorff distance* and *Gromov-Hausdorff convergence* for compact metric spaces.

Definition 2.15 (GH distance, GH convergence)**.** For two compact metric spaces (X_1, d_1) and (X_2, d_2), define the *Gromov-Hausdorff distance* (abbreviated as GH distance) between them to be:

$$d_{GH}(X_1, X_2) = \inf\{\eta > 0 | \exists Y \text{ such that } X_1, X_2 \text{ are isometrically embedded in } Y \text{ and } \eta \text{ close to each other}\}$$

A sequence of compact metric spaces (X_i, d_i) is said to converge to (X_∞, d_∞) in *Gromov-Hausdorff sense*(abbreviated as GH sense) if the GH distance between X_i and X_∞ converges to zero as i goes to infinity.

3 Deformation of conical KE metric on smooth Fano Manifold

3.1 Smoothing Continuity paths

3.1.1 Two two-parameter Continuity paths

Let X be a smooth Fano manifold, fix a smooth background Kähler metric ω_0 in $2\pi c_1(X)$, and write h_{ω_0} as its Ricci potential (with the normalization $\sup_X h_{\omega_0} = 0$). Let S be a holomorphic defining section of a smooth divisor $D \in |-K_X|$ and let h be a smooth Hermitian metric on the line bundle L_D defined by D with curvature ω_0. Let $[D]$ denote the *current of integration* along D. By Poincaré-Lelong's formula, $\omega_0 + \sqrt{-1}\partial\bar{\partial}\log|S|_h^2 = 2\pi[D]$ as currents. We define a smooth approximation $\chi_\epsilon = \omega_0 + \sqrt{-1}\partial\bar{\partial}\log(|S|_h^2 + \epsilon)$ for $\epsilon \in (0, 1]$. Let ω_{φ_β} be a *weak conical KE metric*, i.e.

$$\omega_{\varphi_\beta}^n = e^{-\beta\varphi_\beta + h_{\omega_0}}|S|_h^{2\beta-2}\omega_0^n \tag{3.1}$$

The authors of [CDS1] used a smooth continuity path depending on two parameters to construct a family of smooth Kähler metrics (with Ricci curvature bounded below by β) to approximate ω_{φ_β} in the GH sense. We adopt the same two-parameter continuity path. Let us first recall the construction. Since the volume form $\omega_{\varphi_\beta}^n \in L^p(X, \omega_0^n)$ for $p \in [1, \frac{1}{1-\beta})$, we can choose a family of smooth volume forms η_ϵ with $\epsilon \in (0, 1]$ approximating it in L^p, and then solve the Monge-Ampère equation

$$\omega_{\varphi_\epsilon}^n = \eta_\epsilon \tag{3.2}$$

by using the solution in [Yau]. Kołodziej's L^p estimate [Ko08] asserts that $||\varphi_\epsilon||_{C^\gamma}$ is uniformly bounded for some $\gamma \in (0, 1)$, and consequently φ_ϵ (normalized to have 0 as supremum) subsequentially converges to some φ_0. By a general uniqueness theorem for bounded solutions of complex Monge-Ampère equations, $\varphi_0 = \varphi_\beta + C$, and therefore we can assume φ_ϵ converges to φ_β by passing to a subsequence.

Again from Yau's solution [Yau], we have smooth Kähler potentials $\psi_{\epsilon,\beta}$ such that

$$\text{Ric}\ \omega_{\psi_{\epsilon,\beta}} = \beta\omega_{\varphi_\epsilon} + (1 - \beta)\chi_\epsilon \tag{3.3}$$

or equivalently satisfying Monge-Ampère equation

$$\omega_{\psi_{\epsilon,\beta}}^n = e^{-\beta\varphi_\epsilon + h_{\omega_0}}\frac{\omega_0^n}{(|S|_h^2 + \epsilon)^{1-\beta}} \tag{3.4}$$

It follows from Kołodziej's L^p estimate [Ko08] that $||\psi_{\epsilon,\beta}||_{L^\infty(X)}$ is uniformly bounded.

Following [CDS1, Eq. 3.4], to deform the Kähler metrics $\omega_{\psi_{\epsilon,\beta}}$ to possess more positive Ricci curvature, we use the two-parameter continuity path[2] $\star^{\beta}_{\epsilon,t}$ with $\epsilon \in (0,1]$ and $t \in [0,\beta]$(cf. Figure 3.1 for illustration) :

$$\star^{\beta}_{\epsilon,t} : \begin{cases} \text{Ric } \omega_{\phi^{\beta}_{\epsilon,t}} & = t\, \omega_{\phi^{\beta}_{\epsilon,t}} + (\beta - t)\omega_{\varphi_\epsilon} + (1-\beta)\chi_\epsilon \\ \phi^{\beta}_{\epsilon,0} & = \psi_{\epsilon,\beta} \end{cases}$$

In [CDS1], the authors first use *Donaldson's Openness Theorem* to deform the cone angles of conical KE metrics to show that the *log-Mabuchi-functional* is coercive, which enables them to solve the above path for $t \in [0,\beta]$. However, in our situation, we can not use *Donaldson's Openness Theorem* to deform the cone angle and thus we lack of the "coercivity" of the *log-Mabuchi-functional*. Instead, we use the *modified log-Mabuchi-functional* (Definition 3.1) to establish the solvability for $t \in [0,\beta']$ for any parameter $\beta' < \beta$. Then we will argue by contradiction to solve $\star^{\beta}_{\epsilon,t}$ for $t \in [0,\beta]$ with a uniform L^∞ bound on $\phi^{\beta}_{\epsilon,\beta}$. The next step is to start from $\phi^{\beta}_{\epsilon,\beta}$ and use the standard openness property for another smooth continuity path [3] $\star_{\epsilon,t}$ with $\epsilon \in (0,1]$ and t near β (cf. Figure 3.2 for illustration):

$$\star_{\epsilon,t} : \begin{cases} \text{Ric } \omega_{u_{\epsilon,t}} & = t\, \omega_{u_{\epsilon,t}} + (1-t)\chi_\epsilon \\ u_{\epsilon,\beta} & = \phi^{\beta}_{\epsilon,\beta} \end{cases}$$

to deform the parameter t to nearby β' with uniform L^∞ bound on $u_{\epsilon,\beta'}$.

In [CDS1], a smooth approximation (with parameter $\epsilon \in (0,1]$) of the *log-Mabuchi-functional* was introduced to study the corresponding smooth continuity path:

$$\mathcal{M}_{\omega_0,(1-\beta)\chi_\epsilon}(\phi) := \mathcal{M}_{\omega_0}(\phi) + (1-\beta)J_{\chi_\epsilon}(\phi) \tag{3.5}$$

where $J_{\chi_\epsilon}(\phi)$ is the J functionals in Definition 2.8.

As explained in the introduction, in order to solve $\star^{\beta}_{\epsilon,t}$, we need to introduce the *modified log-Mabuchi-functional* by adding an extra term to achieve *coercivity*.

Definition 3.1. [modified log-Mabuchi-functional] For any $\beta' < \beta$ and $\epsilon \in (0,1]$, define

$$\widetilde{\mathcal{M}}_{\epsilon,\beta'} := \mathcal{M}_{\omega_0,(1-\beta)\chi_\epsilon} + (\beta - \beta')J_{\omega_{\varphi_\epsilon}}$$

[2] This two-parameter family is designed in [CDS1] to approximate *conical KE metric* by smooth Kähler metrics with Ricci curvature bounded below by the same number.

[3] This two-parameter family is one smoothing of *Donaldson's continuity path*.

where φ_ϵ is the solution to Eq. 3.2.

Notice that this modified functional depends on the choice of φ_ϵ and later we will see that its precise form is made such that it is decreasing along the continuity path $\star^{\beta}_{\epsilon,t}$ for $t \in [0, \beta']$. The following lemma shows that this *modified log-Mabuchi-functional* is coercive.

Proposition 3.2. *For any fixed $\beta' < \beta$, there exists $C = C_{\beta'}$ independent of ϵ such that for all $\epsilon \in (0, 1]$:*

$$\widetilde{\mathcal{M}}_{\epsilon,\beta'} \geq (\beta - \beta')J_{\omega_0} - C$$

Proof. Rewrite the functional as following:

$$\widetilde{\mathcal{M}}_{\epsilon,\beta'} = \mathcal{M}_{\omega_0} + (1-\beta)J_{\chi_\epsilon} + (\beta - \beta')J_{\omega_{\varphi_\epsilon}}$$

$$= \mathcal{M}_{\omega_0,(1-\beta)D} + (1-\beta)(J_{\chi_\epsilon} - J_{2\pi[D]}) + (\beta - \beta')(J_{\omega_{\varphi_\epsilon}} - J_{\omega_0}) + (\beta - \beta')J_{\omega_0}.$$

- The first term $\mathcal{M}_{\omega_0,(1-\beta)D}$ is the *log-Mabuchi-functional*. It follows from [BBEGZ, Theorem 4.8] that $\mathcal{M}_{\omega_0,(1-\beta)D}$ is bounded from below under the assumption that there exists a *weak conical KE metric* ω_{φ_β};

- The first error term $J_{\chi_\epsilon} - J_{2\pi[D]}$ is bounded from below by the following computation (see [CDS1] for more detailed calculation):

$$\begin{aligned}(J_{\chi_\epsilon} - J_{2\pi[D]})(\phi) &= \int_X \{\log(|S|_h^2 + \epsilon) - \log|S|_h^2\}(\omega_\phi^n - \omega_0^n) \\ &\geq \int_X \{\log|S|_h^2 - \log(|S|_h^2 + \epsilon)\}\omega_0^n \\ &\geq -\sup_X \log(|S|_h^2 + 1) + \int_X \log|S|_h^2 \omega_0^n\end{aligned}$$

- The second error term $J_{\omega_{\varphi_\epsilon}} - J_{\omega_0}$, whose formula appears in the first part of Proposition 2.14, is bounded from below:

$$\begin{aligned}(J_{\omega_{\varphi_\epsilon}} - J_{\omega_0})(\phi) &= \int_X \varphi_\epsilon(\omega_\phi^n - \omega_0^n) \\ &\geq -2||\varphi_\epsilon||_{L^\infty(X)}\end{aligned}$$

Therefore, if C is chosen to be

$$C = 2(\beta - \beta')||\varphi_\epsilon||_{L^\infty(X)} - (1-\beta)\int_X \log|S|_h^2 \omega_0^n + (1-\beta)\sup_X \log(|S|_h^2 + 1),$$

the coercivity of $\widetilde{M}_{\epsilon,\beta'}$ in the lemma holds. □

In the next section, we will try to solve $\star_{\epsilon,t}^\beta$ for $t \in [0, \beta'], \epsilon \in (0,1]$ with a uniform L^∞ bound on $\phi_{\epsilon,t}^\beta$.

3.1.2 Solution for $\star_{\epsilon,t}^\beta$ with $t \in [0, \beta']$ and $\epsilon \in (0,1]$

We need two lemmas which enable us to obtain uniform estimates of $\phi_{\epsilon,t}^\beta$ along the continuity path $\star_{\epsilon,t}^\beta$. The first lemma appeared in the work of [JMR] whose proof is based on the Chern-Lu inequality and is given here for completeness.

Lemma 3.3. *[JMR] There exists $C = C_A$ such that if $\star_{\epsilon,t}^\beta$ have solution $\phi_{\epsilon,t}^\beta$ with Osc $\phi_{\epsilon,t}^\beta \leq A$, then*

$$\omega_{\phi_{\epsilon,t}^\beta} \geq C^{-1}\omega_0$$

Proof. By comparing the Kähler metric $\omega_{\phi_{\epsilon,t}^\beta}$ (denoted by ω_{ϕ_ϵ} for simplifying notation) which has Ricci curvature bounded below by 0 and the fixed smooth Kähler metric ω_0 which has a fixed upper bound Λ on the bisectional curvature, the Chern-Lu Inequality (see part 4 of Proposition 2.14) tells that

$$\Delta_{\omega_{\phi_\epsilon}} \log \mathrm{tr}_{\omega_{\phi_\epsilon}}\omega_0 \geq -2\Lambda \mathrm{tr}_{\omega_{\phi_\epsilon}}\omega_0.$$

By using the fact that $\Delta_{\omega_{\phi_\epsilon}}\phi_\epsilon = \mathrm{tr}_{\omega_{\phi_\epsilon}}(\omega_{\phi_\epsilon} - \omega_0) = n - \mathrm{tr}_{\omega_{\phi_\epsilon}}\omega_0$, we get the inequality

$$\Delta_{\omega_{\phi_\epsilon}}\{\log \mathrm{tr}_{\omega_{\phi_\epsilon}}\omega_0 - (2\Lambda+1)\phi_\epsilon\} \geq \mathrm{tr}_{\omega_{\phi_\epsilon}}\omega_0 - n(2\Lambda+1).$$

The maximum principle on X tells us $\mathrm{tr}_{\omega_{\phi_\epsilon}}\omega_0 \leq \{n(2\Lambda+1)\}e^{(2\Lambda+1)\mathrm{Osc}\ \phi_\epsilon} \leq C$. Thus the lower bound of ω_{ϕ_ϵ} claimed in this lemma is obtained. □

The second lemma is an application of Evans-Krylov's theorem [Ev, Kr] for $C^{2,\alpha}$ bound and higher order bounds.

Lemma 3.4. *For any subset $K \subset\subset X\backslash D$, and $k \in \mathbb{N}$, there exists $C = C_{A,K,k}$ such that if $\star_{\epsilon,t}^\beta$ have solutions $\phi_{\epsilon,t}^\beta$ with Osc $\phi_{\epsilon,t}^\beta \geq A$, then*

$$||\phi_{\epsilon,t}^\beta||_{C^k(K)} \leq C$$

Proof. First we have the equation:

$$\omega^n_{\phi^\beta_{\epsilon,t}} = e^{-t\phi^\beta_{\epsilon,t}-(\beta-t)\varphi_\epsilon+h_{\omega_0}} \frac{\omega_0^n}{(|S|_h^2+\epsilon)^{1-\beta}} \tag{3.6}$$

By Lemma 3.3 above, on any Euclidean ball U inside K, the above equation reads as an equation of the type

$$\omega_\phi^n = F\omega_0^n$$

where $||F||_{C^\alpha}$ and $\Delta_{\omega_0}\phi$ are uniformly bounded. By the standard Evans-Krylov theorem [Ev, Kr],

$$||\phi||_{C^{2,\alpha}} \leq C$$

and the higher order bound follows from a standard bootstrapping argument. □

Proposition 3.5. *For any fixed $\beta' < \beta$, $\star^\beta_{\epsilon,t}$ is solvable for $t \in [0,\beta'], \epsilon \in (0,1]$ and there exists $C = C_{\beta'}$ independent of ϵ such that*

$$||\phi^\beta_{\epsilon,\beta'}||_{L^\infty(X)} \leq C$$

Proof. The proof follows a standard line, which is similar to the argument in [CDS1] if the *modified log-Mabuchi-functional* is used instead of the usual *log-Mabuchi-functional*. For the readers' convenience, the calculation is included below.

On the one hand, along the interval $t \in [0,\beta')$ on the continuity path $\star^\beta_{\epsilon,t}$, the functional $\widetilde{\mathcal{M}}_{\epsilon,\beta'}$ is decreasing by a direct calculation:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{M}}_{\epsilon,\beta'}(\phi^\beta_{\epsilon,t}) =& -n\int_X \dot\phi^\beta_{\epsilon,t}\{\mathrm{Ric}\ \omega_{\phi^\beta_{\epsilon,t}} - \omega_{\phi^\beta_{\epsilon,t}}\}\wedge\omega^{n-1}_{\phi^\beta_{\epsilon,t}} + n(\beta-\beta')\int_X \dot\phi^\beta_{\epsilon,t}\{\omega_{\varphi_\epsilon} - \omega_{\phi^\beta_{\epsilon,t}}\}\wedge\omega^{n-1}_{\phi^\beta_{\epsilon,t}} \\
&+ n(1-\beta)\int_X \dot\phi^\beta_{\epsilon,t}\{\chi_\epsilon - \omega_{\phi^\beta_{\epsilon,t}}\}\wedge\omega^{n-1}_{\phi^\beta_{\epsilon,t}} \\
=& n(\beta'-t)\int_X \dot\phi^\beta_{\epsilon,t}(\omega_{\phi^\beta_{\epsilon,t}} - \omega_{\varphi_\epsilon})\wedge\omega^{n-1}_{\phi^\beta_{\epsilon,t}} \\
=& (\beta'-t)\int_X (\phi^\beta_{\epsilon,t}-\varphi_\epsilon)\Delta_{\omega_{\phi^\beta_{\epsilon,t}}}\dot\phi^\beta_{\epsilon,t}\omega^n_{\phi^\beta_{\epsilon,t}} \\
=& (\beta'-t)\int_X (\phi^\beta_{\epsilon,t}-\varphi_\epsilon)\{-(\phi^\beta_{\epsilon,t}-\varphi_\epsilon) - t\dot\phi^\beta_{\epsilon,t}\}\omega^n_{\phi^\beta_{\epsilon,t}} \\
=& -(\beta'-t)\int_X (\phi^\beta_{\epsilon,t}-\varphi_\epsilon)^2\omega^n_{\phi^\beta_{\epsilon,t}} + t(\beta'-t)\int_X \dot\phi^\beta_{\epsilon,t}\{\Delta_{\omega_{\phi^\beta_{\epsilon,t}}}+t\}\dot\phi^\beta_{\epsilon,t}\omega^n_{\phi^\beta_{\epsilon,t}} \\
\leq& 0
\end{aligned}$$

where in the last inequality the second term is nonpositive by Lichnerowicz's estimate of the first eigenvalue of Laplacian operator on a manifold with a positive lower bound of the Ricci curvature.

On the other hand, since the initial potentials $\psi_{\epsilon,\beta}$ satisfes Eq. 3.4, by using the explicit *Decomposition Formula* in Definition 2.10 for the *log-Mabuchi-functional*, we have

$$\widetilde{\mathcal{M}}_{\epsilon,\beta'}(\psi_{\epsilon,\beta}) = \int_X \{-\beta\varphi_\epsilon - (1-\beta)\log(|S|_h^2+\epsilon)\}e^{-\beta\varphi_\epsilon + h_{\omega_0}} \frac{\omega_0^n}{(|S|_h^2+\epsilon)^{1-\beta}} - J_{\omega_0}(\psi_{\epsilon,\beta}) + \int_X h_{\omega_0}\omega_0^n + (\beta-\beta')J_{\omega_{\varphi_\epsilon}}(\psi_{\epsilon,\beta}) + (1-\beta)J_{\chi_\epsilon}(\psi_{\epsilon,\beta})$$

Since $||\varphi_\epsilon||_{L^\infty(X)}$ and $||\psi_{\epsilon,\beta}||_{L^\infty(X)}$ are uniformly bounded from above by C, the first term

$$\begin{aligned} &|\int_X \{-\beta\varphi_\epsilon - (1-\beta)\log(|S|_h^2+\epsilon)\}e^{-\beta\varphi_\epsilon + h_{\omega_0}} \frac{\omega_0^n}{(|S|_h^2+\epsilon)^{1-\beta}}| \\ &\leq C\int_X |S|_h^{2\beta-2}|\log|S|_h^{2\beta-2}|\omega_0^n \leq C \end{aligned}$$

and the other terms

$$\begin{aligned} J_{\omega_0}(\psi_{\epsilon,\beta} &\leq I(\psi_{\epsilon,\beta}) = \int_X \psi_{\epsilon,\beta}(\omega_0^n - \omega_{\psi_{\epsilon,\beta}}^n) \\ J_{\omega_{\varphi_\epsilon}}(\psi_{\epsilon,\beta}) &= J_{\omega_0}(\psi_{\epsilon,\beta}) + \int_X \varphi_\epsilon(\omega_{\psi_{\epsilon,\beta}}^n - \omega_0^n) \\ J_{\chi_\epsilon}(\psi_{\epsilon,\beta}) &= J_{\omega_0}(\psi_{\epsilon,\beta}) + \int_X \log(|S|_h^2+\epsilon)(\omega_{\psi_{\epsilon,\beta}}^n - \omega_0^n) \end{aligned}$$

are all bounded from above (independent of ϵ). Thus the values of $\widetilde{M}_{\epsilon,\beta'}$ at the initial points $\psi_{\epsilon,\beta}$ are uniformly bounded (independent of ϵ) from above.

The *modified log-Mabuchi-functional* is coercive (see Lemma 3.7):

$$\widetilde{\mathcal{M}}_{\epsilon,\beta'}(\phi) \geq (\beta-\beta')J_{\omega_0}(\phi) - C$$

Therefore along the interval $t \in [0,\gamma]$ where $\gamma \leq \beta'$ on which $\star_{\epsilon,t}^\beta$ can be solved,

$$J_{\omega_0}(\phi_{\epsilon,t}^\beta) \leq C$$

for C independent of $\epsilon \in (0,1]$ and $t \in [0,\gamma]$. Along the continuity path $\star^{\beta}_{\epsilon,t}$ for $t \in [\delta,\gamma]$, for some fixed $\delta > 0$, we have Ric $\omega_{\phi^{\beta}_{\epsilon,t}} \geq \delta\omega_{\phi^{\beta}_{\epsilon,t}}$, therefore the Poincaré and Sobolev constants of the family of Kähler metrics $\omega_{\phi^{\beta}_{\epsilon,t}}$ are uniformly bounded. By the third part of Propositon 2.14,

$$\text{Osc } \phi^{\beta}_{\epsilon,t} \leq C$$

Since they satisfy the Monge-Ampère equation:

$$\omega^n_{\phi^{\beta}_{\epsilon,t}} = e^{-t\phi^{\beta}_{\epsilon,t}-(\beta-t)\varphi_\epsilon+h_{\omega_0}} \frac{\omega_0^n}{(|S|_h^2+\epsilon)^{1-\beta}} \tag{3.7}$$

we can easily deduce that

$$||\phi^{\beta}_{\epsilon,t}||_{L^\infty(X)} \leq C$$

Lemma 3.3 and 3.7 give us the higher order estimates for $\phi^{\beta}_{\epsilon,t}$ (which may depend on ϵ), therefore the equation $\star^{\beta}_{\epsilon,t}$ can be solved up to $t = \beta'$ with uniform L^∞ bound on $\phi^{\beta}_{\epsilon,t}$ (independent of ϵ). □

Since β' is any parameter smaller than β, and the continuity path $\star^{\beta}_{\epsilon,t}$ does not depend on β', we immediately get the following corollary:

Corollary 3.6. *For all $\epsilon \in (0,1]$, the continuity path $\star^{\beta}_{\epsilon,t}$ can be solved for $t \in [0,\beta)$.*

3.1.3 Smooth approximation of weak conical KE

In this subsection, we will proceed to achieve the solution of $\star^{\beta}_{\epsilon,t}$ up to $t = \beta$, which gives smooth approximation of weak conical KE ω_{φ_β} with Ricci curvature bounded below by β. We first need a simple convergence property about Aubin's I functional (Eq. 2.14) that will be used several times later.

Lemma 3.7. *If $\phi^{\beta}_{\epsilon_j,t_j}$ converges to ϕ in the C^α sense globally on X, then*

$$\lim_{j\to\infty} I(\phi^{\beta}_{\epsilon_j,t_j}) = I(\phi).$$

Proof. We have the formula

$$I(\phi^{\beta}_{\epsilon_j,t_j}) = \int_X \phi^{\beta}_{\epsilon_j,t_j}\omega_0^n - \int_X \phi^{\beta}_{\epsilon_j,t_j} e^{-t_j\phi^{\beta}_{\epsilon_j,t_j}-(\beta-t_j)\varphi_{\epsilon_j}+h_{\omega_0}} \frac{\omega_0^n}{(|S|_h^2+\epsilon_j)^{1-\beta}}$$

and the second integrand is bounded by some L^1 function on X. The convergence claimed in the lemma follows from the Dominated Convergence Theorem.

□

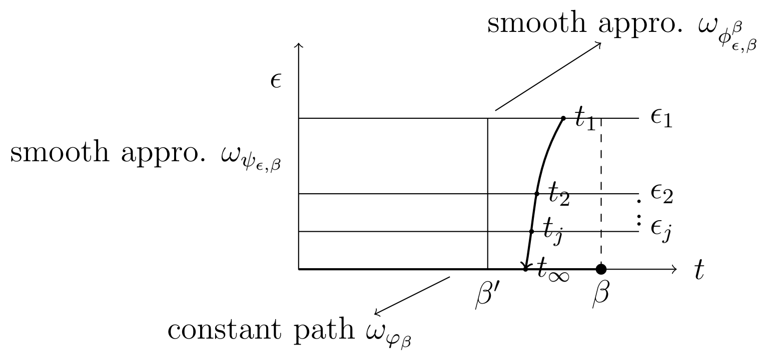

Figure 3.1: Two-parameter Continuity Path $\star^{\beta}_{\epsilon,t}$

Since Corollary 3.6 already assures that $\star^{\beta}_{\epsilon,t}$ can be solved on the open interval $[0,\beta)$ with uniform L^∞ bound on $\phi^{\beta}_{\epsilon,\beta'}$ for any fixed $\beta' < \beta$, what remains to be shown is a bound on $J_{\omega_0}(\phi^{\beta}_{\epsilon,t})$, uniform in ϵ and $t \in (\beta', \beta)$. Since Aubin's I functional is equivalent to J_{ω_0} by the inequality 2.16, we have

$$I(\phi^{\beta}_{\epsilon,\beta'}) \leq C \tag{3.8}$$

for C independent of ϵ. By a contradiction argument based on Berndtsson's Generalized Bando-Mabuchi Theorem [Bern, Theorem 6.6], the required uniform bound is achievable by the following proposition:

Proposition 3.8. *For any fixed $\beta' < \beta$, there exists $\epsilon_0 = \epsilon_0(\beta') > 0$, such that for all $\epsilon \in (0, \epsilon_0]$ we have*

$$\sup_{t\in[\beta',\beta)} \{I(\phi^{\beta}_{\epsilon,t}) - I(\phi^{\beta}_{\epsilon,\beta'})\} \leq 1.$$

Proof. Argue by contradiction. Suppose the claimed estimate does not hold. Then there exists a sequence $\epsilon_j \searrow 0$, with

$$\sup_{t\in[\beta',\beta)} \{I(\phi^{\beta}_{\epsilon_j,t}) - I(\phi^{\beta}_{\epsilon_j,\beta'})\} > 1$$

Let t_j be the first number $t \in (\beta', \beta)$ such that

$$I(\phi^{\beta}_{\epsilon_j,t_j}) - I(\phi^{\beta}_{\epsilon_j,\beta'}) = 1$$

By the inequality 3.8,

$$I(\phi^{\beta}_{\epsilon_j,t_j}) \le C+1$$

Then we get the uniform bound on $\phi^{\beta}_{\epsilon_j,t_j}$ and argue similarly as in the proof of Proposition 3.5: we have all the higher order bounds

$$||\phi^{\beta}_{\epsilon_j,t_j}||_{C^k(K)} \le C_{k,K}$$

on any $K \subset\subset X\backslash D$ and $k \in \mathbb{N}$. By taking a subsequence, we can assume that $\phi^{\beta}_{\epsilon_j,t_j}$ converges to the ϕ in C^{α} sense globally on X and in the C^{∞} sense away from D.

If $t_j \to t_{\infty}$, ϕ is then a solution to the current equation:

$$\text{Ric } \omega_{\phi} = t_{\infty}\omega_{\phi} + (\beta - t_{\infty})\omega_{\varphi_{\beta}} + 2\pi(1-\beta)[D]$$

On the other hand, the *weak conical KE metric* $\omega_{\varphi_{\beta}}$ is also a solution:

$$\text{Ric } \omega_{\varphi_{\beta}} = t_{\infty}\omega_{\varphi_{\beta}} + (\beta - t_{\infty})\omega_{\varphi_{\beta}} + 2\pi(1-\beta)[D]$$

Let $\Omega_1 = t_{\infty}\omega_{\phi}$, $\Omega_2 = t_{\infty}\omega_{\varphi_{\beta}}$ and $\theta = (\beta - t_{\infty})\omega_{\varphi_{\beta}} + 2\pi(1-\beta)[D]$. Then Ω_1, Ω_2 give two bounded solutions to the equation:

$$\text{Ric } \Omega = \Omega + \theta$$

in the fixed cohomology class $2\pi t_{\infty} c_1(X)$. Berndtsson's Generalized Bando-Mabuchi Theorem implies that there exists $f \in Aut(X)$ which is generated by some holomorphic vector field $\mathbf{V}$ on X such that

$$f^*\Omega_1 = \Omega_2$$

and

$$f^*\theta = \theta$$

Since $\theta = (\beta - t_{\infty})\omega_{\varphi_{\beta}} + 2\pi(1-\beta)[D]$ is the Siu's decomposition of a current, and $(\beta - t_{\infty})\omega_{\varphi_{\beta}}$ has zero *Lelong's number*,

$$f^*[D] = [D]$$

Therefore $\mathbf{V}$ is tangential to D. However, there are no holomorphic vector fields tangential to D according to Berman [Berm1, Theorem 1.5] (by proving the *properness* of the *log-Mabuchi-functional*) or Song-Wang [SW, Theorem 2.1] (by pure algebraic geometry). Hence $f = id$, which implies that $\Omega_1 = \Omega_2$, and in particular $\phi = \varphi_{\beta}$. And similarly $\phi^{\beta}_{\epsilon_j,\beta'}$ converges to φ_{β} in the C^{α} sense globally on X.

However, by Lemma 3.7

$$0 = I(\varphi_\beta) - I(\varphi_\beta) = \lim_{j\to\infty} \left(I(\phi^\beta_{\epsilon_j,t_j}) - I(\phi^\beta_{\epsilon_j,\beta'}) \right) = 1$$

which is a contradiction. □

With the uniform upper bound on the I functional of Proposition 3.8, the same strategy as in Proposition 3.5 would show the following:

Proposition 3.9. *There exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0]$, the continuity path $\star^\beta_{\epsilon,t}$ is solvable up to $t = \beta$ with a uniform bound $||\phi^\beta_{\epsilon,\beta}||_{L^\infty(X)}$ for $\epsilon \in (0, \epsilon_0]$.*

3.1.4 Deforming The Cone Angle from β

The endpoint $\omega_{\phi^\beta_{\epsilon,\beta}}$ of the continuity path $\star^\beta_{\epsilon,t}$ in previous sections will be the starting point of the new two-parameter continuity path $\star_{\epsilon,t}$ defined in Sect. 3.1.1, with $\epsilon \in (0, \epsilon_0]$:

$$\star_{\epsilon,t} : \begin{cases} \text{Ric } \omega_{u_{\epsilon,t}} &= t\omega_{u_{\epsilon,t}} + (1-t)\chi_\epsilon \\ u_{\epsilon,\beta} &= \phi^\beta_{\epsilon,\beta} \end{cases}$$

Since χ_ϵ is a strictly positive $(1,1)$ form, the linearized operator at $t = \beta$, which equals to $\Delta_{\omega_{u_{\epsilon,\beta}}} + \beta$, is invertible for some standard suitable Banach spaces. The standard *implicit function theorem* enables us to perturb t a little bit in both directions on $\star_{\epsilon,t}$ for $\epsilon \in (0, \epsilon_0]$.

Proposition 3.10. *In both directions, there is uniform upper bound on the I functional under small perturbation:*

- *There exists $\delta_1 > 0$ such that for all $\beta' \in (\beta - \delta_1, \beta)$, there exists $\epsilon_1 \in (0, \epsilon_0]$ such that for all $\epsilon \in (0, \epsilon_1)$*

$$\sup_{t\in(\beta',\beta]} \{I(u_{\epsilon,t}) - I(u_{\epsilon,\beta})\} \le 1$$

- *There exists $\delta_2 > 0$ such that for all $\beta'' \in (\beta, \beta + \delta_2)$, there exists $\epsilon_2 \in (0, \epsilon_0]$ such that for all $\epsilon \in (0, \epsilon_2)$*

$$\sup_{t\in[\beta,\beta'')} \{I(u_{\epsilon,t}) - I(u_{\epsilon,\beta})\} \le 1$$

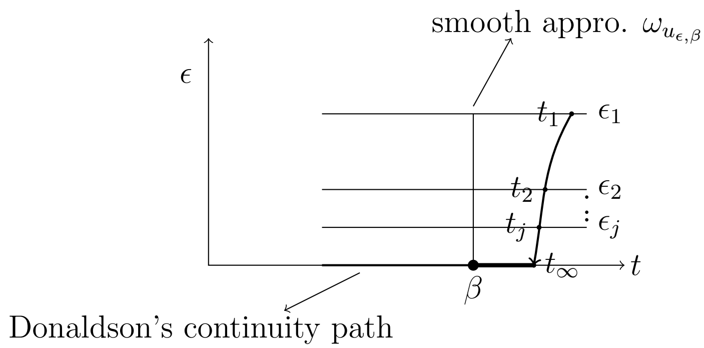

Figure 3.2: Two-parameter Continuity Path $\star_{\epsilon,t}$

Consequently, take $\delta = \min(\delta_1, \delta_2)$ *and* $\underline{\epsilon} = \min(\epsilon_1, \epsilon_2)$*, then* $u_{\epsilon,t}$ *will have uniform* L^∞ *bound for* $\epsilon \in (0, \underline{\epsilon}]$ *and* $t \in (\beta - \delta, \beta + \delta)$*.*

Proof. Since the two directions are similar, we only prove the right direction. The proof uses the same idea as in Proposition 3.8. We argue by contradiction based on normalization of the I functional. Assume the conclusion is not true, then we can find a sequence $\epsilon_j \searrow 0$ and $\beta_j'' = \beta + \delta_j$ with $\delta_j \searrow 0$, such that

$$\sup_{t \in [\beta, \beta_j'']} \{I(u_{\epsilon_j,t}) - I(u_{\epsilon_j,\beta})\} > 1$$

Let t_j be the first number bigger than β such that

$$I(u_{\epsilon_j,t_j}) - I(u_{\epsilon_j,\beta}) = 1$$

Since $I(u_{\epsilon_j,\beta})$ is uniformly bounded by Proposition 3.9,

$$I(u_{\epsilon_j,\beta}) \leq C,$$

which implies that

$$I(u_{\epsilon_j,t_j}) \leq C + 1$$

The standard argument gives the L^∞ bound since the Ricci curvature of this family of metrics is uniformly bounded below by some positive constant and the volume is a fixed topological constant.

The continuity path $\star_{\epsilon,t}$ corresponds to the Monge-Ampère equation

$$\omega_{u_{\epsilon,t}}^n = e^{-tu_{\epsilon,t}+h_{\omega_0}} \frac{\omega_0^n}{(|S|_h^2+\epsilon)^{1-t}} \tag{3.9}$$

Even though this equation is different from Eq. 3.7, the uniform L^∞ bound of u_{ϵ_j,t_j} will imply the R.H.S. is uniform in L^p for some $p > 1$, which yields the global C^α bound for some $\alpha \in (0,1)$ by [Ko08].

Now by using the analogous result of Lemma 3.3 and 3.7 for Eq. 3.9, we get all the higher order bounds of u_{ϵ_j,t_j} away from D. Then by taking the limit (subsequentially) u_{ϵ_j,t_j} converges to some u in the C^α sense globally on X and C^∞ sense away from D, and $u_{\epsilon_j,\beta}$ converges to some v in the same sense. We know that u and v are both solutions to the equation:

$$\text{Ric}\ \omega = \beta\omega + 2\pi(1-\beta)[D]$$

By the same argument as in Proposition 3.8,

$$u = v = \varphi_\beta.$$

Similar to Lemma 3.7, we have the convergence of the functional I by the Dominated Convergence Theorem:

$$0 = I(u) - I(v) = \lim_{j\to\infty} I(u_{\epsilon_j,t_j}) - I(u_{\epsilon_j,\beta}) = 1$$

Contradiction. □

As a consequence, we can finish the proof of Theorem 2.1.

Proof of Theorem 2.1. Proposition 3.10 gives a family of solutions with $\epsilon \in (0, \underline{\epsilon}]$ and $\beta' \in (\beta-\delta, \beta+\delta)$:

$$\omega_{u_{\epsilon,\beta'}}^n = e^{-\beta' u_{\epsilon,\beta'}+h_{\omega_0}} \frac{\omega_0^n}{(|S|_h^2+\epsilon)^{1-\beta'}} \tag{3.10}$$

with $||u_{\epsilon,\beta'}||_{L^\infty} \leq C$ independent of ϵ and β'. Then by taking a limit, $\varphi_{\beta'} = \lim_{\epsilon\to 0} u_{\epsilon,\beta'}$ is bounded on X and smooth away from D. Moreover it satisfies:

$$\omega_{\varphi_{\beta'}}^n = e^{-\beta'\varphi_{\beta'}+h_{\omega_0}}|S|_h^{2\beta-2}\omega_0^n.$$

Therefore $\omega_{\varphi_{\beta'}}$ is a *weak conical KE metric* of angle $2\pi\beta'$ along D. □

In the next subsection, we are going to show that the smooth deformed Kähler metrics $\omega_{\phi_{\epsilon,\beta}^\beta} = \omega_{\phi_\epsilon}$ which approximate *weak conical KE metrics* actually are uniformly quasi-isometric to the smooth models.

3.2 Uniform C^2 bound

This section is reproduced from [Yao2]. Now we are at a situation where there is a family of smooth Kähler metrics ω_{ϕ_ϵ}, satisfying the Ricci curvature equation:

$$\text{Ric } \omega_{\phi_\epsilon} = \beta\omega_{\phi_\epsilon} + (1-\beta)\chi_\epsilon$$

or equivalently, the Monge-Ampère equation:

$$\omega_{\phi_\epsilon}^n = e^{-\beta\phi_\epsilon + h_{\omega_0}} \frac{\omega_0^n}{(|S|_h^2 + \epsilon)^{1-\beta}} \tag{3.11}$$

such that $||\phi_\epsilon||_{C^0}$ is uniformly bounded. The next theorem gives the uniform C^2 bound for this family.

Theorem 3.11. *There exists $C > 0$ such that near any point $p \in D$, the family of metrics ω_{ϕ_ϵ} are uniformly quasi-isometric to the standard local model metrics, i.e.*

$$C^{-1}\omega_{(\beta,\epsilon)} \leq \omega_{\phi_\epsilon} \leq C\omega_{(\beta,\epsilon)}$$

where $\omega_{(\beta,\epsilon)} = \sqrt{-1}(\beta^2|z|^{2\beta-2}\mathrm{d}z\wedge\mathrm{d}\bar{z} + \sum_{i=2}^n \mathrm{d}z_i\wedge\mathrm{d}\bar{z}_i)$ *for D locally defined by $\{z = 0\}$.*

By [CDS1, Theorem 2.2], this family has a *rough C^2 bound*

$$C^{-1}\omega_0 \leq \omega_{\phi_\epsilon} \leq \frac{C}{(|S|_h^2 + \epsilon)^{1-\beta}}\omega_0 \tag{3.12}$$

Comparing to this *rough C^2 bound*, the *uniform C^2 bound* in the stated theorem could be seen as an improvement along the tangential direction of the divisor. To prove this estimate, we need a *rough geometry* from the *rough C^2 bound* first. It is shown in [CDS1, Proposition 2.4] that the family of metric spaces $(X, \omega_{\phi_\epsilon})$ has uniform diameter upper bound.

The next lemma shows that the distance function of ω_{ϕ_ϵ} is Hölder continuous with respect to the smooth background Kähler metric ω_0.

Lemma 3.12. *Near any point $p \in D$,*
1. For any $R > 0$, $B_{\omega_{\phi_\epsilon}}(p, R) \subset B_{\omega_0}(p, \sqrt{C}R)$;
2. For any $R > 0$, let $\delta(R) = \tilde{C}R^{\frac{1}{\beta}}$, then $\forall\epsilon < \delta(R)^2$, $B_{\omega_0}(p, \delta(R)) \subset B_{\omega_{\phi_\epsilon}}(p, R)$, here $\tilde{C} = \{(\frac{2}{\beta} + 2)\sqrt{C}\}^{-\frac{1}{\beta}}$ is just some uniform unimportant constant.

Proof. The first part is just a straightforward consequence of inequality $\omega_0 \leq C\omega_{\phi_\epsilon}$, since any curve initiating from p measuring under the metric ω_{ϕ_ϵ} with length less than R will have length less than $\sqrt{C}R$ when measured using ω_0.

The second part of the containing relationship is also not so difficult. Since ω_0 is a fixed background smooth Kähler metric, there is no loss of generality assuming ω_0 is Euclidean metric near p. Let $B_{Euc}(p,\delta) = \{(z, z_2, \cdots, z_n)||z| \le \delta, |z_i| \le \delta\}$ be a Euclidean ball centered at p, pick any point q in this ball, then q is joined with p by three line segments $\overline{qq'}, \overline{q'p'}, \overline{p'p}$, where q' is the unique point on the boundary of the Euclidean ball whose$z_2, \cdots, z_n$ coordinates are the same as q and whose z coordinate is just the radial projection of the z coordinate of q to the boundary, i.e. $\frac{z}{|z|}\delta$, and p' is the projection of q' to the complex line $\{z_2 = 0, \cdots, z_n = 0\}$. Measured under the metric ω_{ϕ_ϵ},

$$|\overline{qq'}|_{\phi_\epsilon} \le \int_{r=0}^{r=\delta} \sqrt{C}(r^2+\epsilon)^{\frac{\beta-1}{2}} \mathrm{d}r \le \sqrt{C}\frac{1}{\beta}\delta^\beta$$

and similarly

$$|\overline{pp'}|_{\phi_\epsilon} \le \sqrt{C}\frac{1}{\beta}\delta^\beta$$

and

$$|\overline{q'p'}|_{\phi_\epsilon} \le \int_{z'\in\overline{q'p'}} \sqrt{C}(\delta^2+\epsilon)^{\frac{\beta-1}{2}}|\mathrm{d}z'| \le \sqrt{C}(\delta^2+\epsilon)^{\frac{\beta-1}{2}}\delta \le \sqrt{C}(\delta^2+\epsilon)^{\frac{\beta}{2}}$$

Let $\epsilon = \delta^2$ and choose δ such that $(\frac{2}{\beta}+2)\sqrt{C}\delta^\beta = R$, then

$$d_{\omega_{\phi_\epsilon}}(q,p) \le |\overline{qq'}|_{\phi_\epsilon} + |\overline{pp'}|_{\phi_\epsilon} + |\overline{q'p'}|_{\phi_\epsilon} \le R$$

□

The above *rough geometry* gives us a way to cover D by coordinate balls where *local Moser's Iteration* will be applied. Firstly, around each point $p_\alpha \in D$, there exists coordinate ball $\mathcal{U}_\alpha(p_\alpha) = \tau_\alpha(B_\alpha(0, r_\alpha))$ where

$$B_\alpha(0, r_\alpha) = \{(z^\alpha = z_1^\alpha, z_2^\alpha, \cdots, z_n^\alpha)||z^\alpha|, |z_2^\alpha|, \cdots, |z_n^\alpha| < r_\alpha\} \subset \mathbb{C}^n$$

such that $\mathcal{U}_\alpha \cap D$ is defined by $\{z^\alpha = 0\}$. The first part of Lemma 3.12 says that $\mathcal{U}_\alpha$ contains ω_{ϕ_ϵ}-metric ball $U_\alpha = B_{\phi_\epsilon}(p_\alpha, \frac{r_\alpha}{\sqrt{C}})$, and the second part of Lemma 3.12 says that this metric ball contains Euclidean ball $B_{Euc}(p_\alpha, \delta(\frac{r_\alpha}{\sqrt{C}}))$ and this further contains ω_{ϕ_ϵ}-metric ball $V_\alpha = B_{\phi_\epsilon}(p_\alpha, \frac{1}{\sqrt{C}}\delta(\frac{r_\alpha}{\sqrt{C}}))$, V_α contains Euclidean ball $\mathcal{V}_\alpha(p_\alpha) = B_{Euc}(p_\alpha, s_\alpha)$, for $s_\alpha = \delta(\frac{1}{\sqrt{C}}\delta(\frac{r_\alpha}{\sqrt{C}}))$. Because D is compact, it is convered by finitely many $\mathcal{V}_\alpha(p_\alpha)$'s, and then choose the weight for the Hermitian metric e^{-h_α} such that they satisfy

$$|h_\alpha| \le C_\alpha, |\frac{\partial h_\alpha}{\partial z_i^\alpha}| \le C_\alpha, |\frac{\partial^2 h_\alpha}{\partial z_i^\alpha \partial \bar{z}_i^\alpha}| \le C_\alpha.$$

On each chart $\mathcal{U}_\alpha$, we choose the smooth Kähler metrics:

$$\omega^\alpha_{(\beta,\epsilon)} = \sqrt{-1}\{\beta^2(|z^\alpha|^2+\epsilon)^{\beta-1}\mathrm{d}z^\alpha\wedge \mathrm{d}\bar{z}^\alpha + \sum_{j=2}^{n}\mathrm{d}z^\alpha_j\wedge \mathrm{d}\bar{z}^\alpha_j\}$$

as the smooth model metrics (cf. the beginning of Sect. 2) . The index α is going to be suppressed for simplicity of notation.

3.2.1 local Differential Inequality

A simply calculation shows that for those local model metrics,

$$R_{1\bar{1}1\bar{1}} = \epsilon\beta^2(1-\beta)(|z|^2+\epsilon)^{\beta-3};$$
$$R_{i\bar{j}k\bar{l}} = 0.$$

if one of $i,j,k,l\neq 1$. This means that $\omega_{(\beta,\epsilon)}$ has nonnegative holomorphic bisectional curvature, i.e. $R(\xi,\bar{\xi},\eta,\bar{\eta}) = |\xi^1|^2|\eta^1|^2R_{1\bar{1}1\bar{1}}\geq 0$ for any pair of $(1,0)$-vector fields $\xi = \xi^1\frac{\partial}{\partial z}+\sum_i\xi^i\frac{\partial}{\partial z^i}$, and $\eta = \eta^1\frac{\partial}{\partial z}+\sum_i\eta^i\frac{\partial}{\partial z^i}$. To get the comparison between ω_{ϕ_ϵ} and the local model metrics, we first derive a local differential inequality and then use *Moser's Iteration* of local type to proceed. Define $\sigma_{\epsilon,\beta} = \mathrm{tr}_{\omega_{(\beta,\epsilon)}}\omega_{\phi_\epsilon}$, then the following inequality holds.

Lemma 3.13 (local differential inequality)**.** *The following uniform bounds hold:*

1. $\Delta_{\omega_{\phi_\epsilon}}\log\sigma_{\epsilon,\beta}\geq -C(|z|^2+\epsilon)^{-\beta}-C$;
2. $\Delta_{\omega_{\phi_\epsilon}}\log\sigma_{\epsilon,\beta'}\geq -C(|z|^2+\epsilon)^{-\beta'}-C$ *for* $\beta'<\beta$ *sufficiently close.*

Proof. Let us prove the item *1* of this lemma first. On $\mathcal{U}_\alpha$, write $\omega_{(\beta,\epsilon)} = \omega_0+\sqrt{-1}\partial\bar{\partial}\psi_\epsilon$ with suitable normalization on the potential ψ_ϵ such that $||\psi_\epsilon||_{C^0}\leq C$. Rewrite the Eq. 3.11 locally,

$$(\omega_{(\beta,\epsilon)}+\sqrt{-1}\partial\bar{\partial}\tilde{\phi}_\epsilon)^n = e^{F_{\epsilon,\beta}}\omega^n_{(\beta,\epsilon)}$$

for $F_{\epsilon,\beta} = -\beta\phi_\epsilon + h_{\omega_0}+(1-\beta)h+(1-\beta)\log\frac{|z|^2+\epsilon}{|z|^2+\epsilon e^h}+\log\frac{\omega_0^n}{\omega^n_{Euc}}$.

Following Yau's calculation on the C^2 estimate [Yau, Eq. 2.9], let $g_{i\bar{j}}$ denote the background metric $\omega_{(\beta,\epsilon)}$ and $g'_{i\bar{j}}$ denote ω_{ϕ_ϵ}, then under the local normal coordinate for which $g_{i\bar{j}} = \delta_{i\bar{j}}, g'_{i\bar{j}} = (1+\phi_{i\bar{j}})\delta_{i\bar{j}}$,

$$
\begin{aligned}
\Delta_{\omega_{\phi_\epsilon}}\sigma_{\epsilon,\beta} &= \Delta_{\omega_{(\beta,\epsilon)}}F_{\epsilon,\beta} + \sum_{i,j,k,l,n} g'^{k\bar{j}}g'^{i\bar{n}}\phi_{k\bar{n}l}\phi_{i\bar{j}\bar{l}} + \sum_{k,l} R_{k\bar{k}l\bar{l}}\frac{(\phi_{k\bar{k}}-\phi_{l\bar{l}})^2}{(1+\phi_{k\bar{k}})(1+\phi_{l\bar{l}})} \\
&\geq \Delta_{\omega_{(\beta,\epsilon)}}F_{\epsilon,\beta} + \sum_{i,j,k}\frac{1}{1+\phi_{i\bar{i}}}\frac{1}{1+\phi_{j\bar{j}}}|\phi_{i\bar{j}k}|^2
\end{aligned}
$$

Because h_{ω_0}, h and $\log\frac{\omega_0^n}{\omega_{Euc}^n}$ are smooth functions, and because of the inequality

$$
\Delta_{\omega_{(\beta,\epsilon)}}\phi_\epsilon = \mathrm{tr}_{\omega_{(\beta,\epsilon)}}(\omega_{\phi_\epsilon}-\omega_0) \leq \sigma_{\epsilon,\beta}
$$

the main troublesome term is the lower bound on $\Delta_{\omega_{(\beta,\epsilon)}}\log\frac{(|z|^2+\epsilon)}{(|z|^2+\epsilon e^h)}$.

$$
\begin{aligned}
&\sqrt{-1}\partial\bar{\partial}\log(|z|^2+\epsilon) = \sqrt{-1}\frac{\epsilon \mathrm{d}z\wedge\mathrm{d}\bar{z}}{(|z|^2+\epsilon)^2} \\
&\sqrt{-1}\partial\bar{\partial}\log(|z|^2+\epsilon e^h) \\
&\quad = \sqrt{-1}\{\frac{\epsilon e^h + \epsilon e^h|z|^2\frac{\partial^2 h}{\partial z\partial\bar{z}} + \epsilon^2 e^{2h}\frac{\partial^2 h}{\partial z\partial\bar{z}} + \epsilon e^h|z|^2|\frac{\partial h}{\partial z}|^2 - \epsilon e^h(\bar{z}\frac{\partial h}{\partial\bar{z}} + z\frac{\partial h}{\partial z})}{(|z|^2+\epsilon e^h)^2}\mathrm{d}z\wedge\mathrm{d}\bar{z} \\
&\quad + \sum_{i=2}^n\frac{\epsilon e^h|z|^2\frac{\partial^2 h}{\partial\bar{z}_i\partial z_i} + \epsilon^2 e^{2h}\frac{\partial^2 h}{\partial\bar{z}_i\partial z_i} + \epsilon e^h|z|^2\frac{\partial h}{\partial\bar{z}_i}\frac{\partial h}{\partial z_i}}{(|z|^2+\epsilon e^h)^2}\mathrm{d}z_i\wedge\mathrm{d}\bar{z}_i\} \\
&\quad + \text{mixed terms} \\
&\quad \leq C\sqrt{-1}\{\frac{\epsilon}{(|z|^2+\epsilon)^2}\mathrm{d}z\wedge\mathrm{d}\bar{z} + \sum_{i=2}^n\mathrm{d}z_i\wedge\mathrm{d}\bar{z}_i\} + \text{mixed terms} \\
&\quad \leq C\sqrt{-1}\{\frac{1}{|z|^2+\epsilon}\mathrm{d}z\wedge\mathrm{d}\bar{z} + \sum_{i=2}^n\mathrm{d}z_i\wedge\mathrm{d}\bar{z}_i\} + \text{mixed terms}
\end{aligned}
$$

Because $\omega_{(\beta,\epsilon)}$ is a diagonal metric, the mixed terms do not contribute when taking the trace, thus we get the inequality

$$
\begin{aligned}
\Delta_{\omega_{(\beta,\epsilon)}}\log\frac{|z|^2+\epsilon}{|z|^2+\epsilon e^h} &\geq -C\{(|z|^2+\epsilon)^{1-\beta}(|z|^2+\epsilon)^{-1} + (|z|^2+\epsilon)^{1-\beta}\} \\
&\geq -C(|z|^2+\epsilon)^{-\beta} - C
\end{aligned}
$$

and moreover,

$$
\begin{aligned}
|\nabla\sigma_{\epsilon,\beta}|^2_{\omega_{\phi_\epsilon}} &= \sum_{i,j,k,l,p,q} g'^{i\bar{j}} g^{k\bar{l}}\phi_{k\bar{l}i} g^{p\bar{q}}\phi_{p\bar{q}\bar{j}} = \sum_i \frac{1}{1+\phi_{i\bar{i}}} |\sum_k \phi_{k\bar{k}i}|^2 \\
&= \sum_i \frac{1}{1+\phi_{i\bar{i}}} |\sum_k \frac{\phi_{k\bar{k}i}}{(1+\phi_{k\bar{k}})^{1/2}}(1+\phi_{k\bar{k}})^{1/2}|^2 \\
&\le \sum_i \frac{1}{1+\phi_{i\bar{i}}} \sum_k \frac{|\phi_{k\bar{k}i}|^2}{1+\phi_{k\bar{k}}} \sum_l (1+\phi_{l\bar{l}}) \\
&= \sigma_{\epsilon,\beta} \sum_{i,k} \frac{1}{1+\phi_{i\bar{i}}} \frac{1}{1+\phi_{k\bar{k}}} |\phi_{k\bar{k}i}|^2 \\
&\le \sigma_{\epsilon,\beta} \sum_{i,j,k} \frac{1}{1+\phi_{i\bar{i}}} \frac{1}{1+\phi_{j\bar{j}}} |\phi_{i\bar{j}k}|^2
\end{aligned}
$$

The above inequalities together with the lower bound on $\sigma_{\epsilon,\beta} = \mathrm{tr}_{\omega_{(\beta,\epsilon)}}\omega_{\phi_\epsilon} \ge (n-1)C$ (we assume $n \ge 2$) gives us the following inequality

$$
\begin{aligned}
\Delta_{\omega_{\phi_\epsilon}} \log\sigma_{\epsilon,\beta} &= \frac{\Delta_{\omega_{\phi_\epsilon}}\sigma_{\epsilon,\beta}}{\sigma_{\epsilon,\beta}} - \frac{|\nabla\sigma_{\epsilon,\beta}|^2}{\sigma^2_{\epsilon,\beta}} \\
&\ge \frac{-C\sigma_{\epsilon,\beta} - C}{\sigma_{\epsilon,\beta}} + \frac{1}{\sigma_{\epsilon,\beta}} \sum_{i,j,k} \frac{1}{1+\phi_{i\bar{i}}} \frac{1}{1+\phi_{j\bar{j}}} |\phi_{i\bar{j}k}|^2 - \frac{|\nabla\sigma_{\epsilon,\beta}|^2}{\sigma^2_{\epsilon,\beta}} - C\frac{(|z|^2+\epsilon)^{-\beta}}{\sigma_{\epsilon,\beta}} \\
&\ge -C - \frac{C}{\sigma_{\epsilon,\beta}} - \frac{(|z|^2+\epsilon)^{-\beta}}{\sigma_{\epsilon,\beta}} \\
&\ge -C - C(|z|^2+\epsilon)^{-\beta}
\end{aligned}
$$

which finishes the proof of item *1* in the Lemma.

Let us continue to derive the differential inequality in item *2* of the Lemma. We will first drive the upper bound of $\sigma_{\epsilon,\beta'}$ for $\beta' < \beta$ to give a better control on the directions tangential to D and then in turn conclude the *correct* upper bound on the direction perpendicular.

Let's rewrite the Eq. 3.11 as

$$
\omega^n_{\phi_\epsilon} = e^{F_{\epsilon,\beta'}} \omega^n_{(\beta',\epsilon)} \tag{3.13}
$$

where β' is a number a little bit smaller than β, and $F_{\epsilon,\beta'} = F_{\epsilon,\beta} + (\beta - \beta')\log(|z|^2+\epsilon)$.

By the inequality

$$\begin{aligned}\sqrt{-1}\partial\bar{\partial}F_{\epsilon,\beta'} &= \sqrt{-1}\partial\bar{\partial}F_{\epsilon,\beta} + \sqrt{-1}(\beta-\beta')\partial\bar{\partial}\log(|z|^2+\epsilon)\\ &\geq \sqrt{-1}\partial\bar{\partial}F_{\epsilon,\beta}\\ &\geq -\sqrt{-1}C\{\frac{1}{|z|^2+\epsilon}\mathrm{d}z\wedge\mathrm{d}\bar{z}+\sum_i \mathrm{d}z_i\wedge\mathrm{d}\bar{z}_i\} + \text{mixed terms}\end{aligned}$$

we get

$$\begin{aligned}\Delta_{\omega_{(\beta',\epsilon)}}F_{\epsilon,\beta'} &= \mathrm{tr}_{\omega_{(\beta',\epsilon)}}\sqrt{-1}\partial\bar{\partial}F_{\epsilon,\beta'}\\ &\geq -C - C(|z|^2+\epsilon)^{1-\beta'}(|z|^2+\epsilon)^{-1}\\ &= -C - C(|z|^2+\epsilon)^{-\beta'}\end{aligned}$$

□

Use two local auxiliary functions $C(\beta-\beta')^{-2}(|z|^2+\epsilon)^{\beta-\beta'}$ and $C\beta^{-2}(|z|^2+\epsilon)^{\beta}$, and use the *rough bound* $\omega_{\phi_\epsilon} \leq C\frac{\omega_0}{(|z|^2+\epsilon)^{1-\beta}}$, we finally arrive at a differential inequality that suits for the purpose of *local Moser's Iteration*.

Proposition 3.14. *Let* $f_\epsilon = \log\sigma_{\epsilon,\beta'} + C(\beta-\beta')^{-2}(|z|^2+\epsilon)^{\beta-\beta'} + C\beta^{-2}(|z|^2+\epsilon)^{\beta}$, *then*

$$\Delta_{\omega_{\phi_\epsilon}} f_\epsilon \geq 0$$

Proof. This is a simple consequence of combination of the following two inequalities

$$\begin{aligned}&\Delta_{\omega_{\phi_\epsilon}}\{\log\sigma_{\epsilon,\beta'} + C(\beta-\beta')^{-2}(|z|^2+\epsilon)^{\beta-\beta'}\}\\ &\quad\geq C(|z|^2+\epsilon)^{-\beta'} - C(|z|^2+\epsilon)^{-\beta'} - C\\ &\quad\geq -C\end{aligned}$$

and

$$\Delta_{\omega_{\phi_\epsilon}}\{C\beta^{-2}(|z|^2+\epsilon)^{\beta}\} \geq C$$

□

3.2.2 local Moser's Iteration

Since $\sigma_{\epsilon,\beta'} = \mathrm{tr}_{\omega_{(\beta',\epsilon)}}\omega_{\phi_\epsilon} \geq \mathrm{tr}_{\omega_{(\beta',\epsilon)}}C^{-1}\omega_0 \geq (n-1)C^{-1}$, we can add a constant to f_ϵ to make it positive. We will do *Moser's Iteration* for f_ϵ on the pair of ω_{ϕ_ϵ}-metric balls V_α and U_α to conclude the L^∞ bound. By the compactness of D, there is no loss of generality to assume V_α is radius 1 and U_α is radius 2. The L^∞ norm of f_ϵ on metric 1-ball will be bounded by the L^2 norm of f_ϵ on metric 2-ball. The calculation is quite standard. However, for the completeness, we include it here and suppress ϵ for simplicity of notation:

$$\begin{aligned}\frac{4p}{(p+1)^2}\int_X \eta^2|\nabla f^{\frac{p+1}{2}}|^2\omega_\phi^n &= p\int_X \eta^2 f^{p-1}|\nabla f|^2\omega_\phi^n \\ &= \int_X \{\nabla(\eta^2 f^p\nabla f) - 2\eta f^p\nabla\eta\cdot\nabla f - \eta^2 f^p\Delta_{\omega_\phi} f\}\omega_\phi^n \\ &\leq \int_X -2\eta f^p\nabla\eta\cdot\nabla f\omega_\phi^n \\ &= \int_X -\frac{4}{p+1}f^{\frac{p+1}{2}}\nabla\eta\cdot\eta\nabla f^{\frac{p+1}{2}} \quad \omega_\phi^n \\ &\leq \int_X \frac{2}{p+1}\{\delta\eta^2|\nabla f^{\frac{p+1}{2}}|^2 + \delta^{-1}f^{p+1}|\nabla\eta|^2\}\omega_\phi^n\end{aligned}$$

Taking $\delta = \frac{p}{p+1}$, we end up with the control

$$\int_X \eta^2|\nabla f^{\frac{p+1}{2}}|^2\omega_\phi^n \leq (\frac{p+1}{p})^2\int_X f^{p+1}|\nabla\eta|^2\omega_\phi^n \leq 4\int_X f^{p+1}|\nabla\eta|^2\omega_\phi^n$$

Since our manifold $(M,\omega_{\phi_\epsilon})$ has a uniform positive lower bound on the Ricci curvature, and constant volume, there is a uniform Sobolev constant, which implies:

$$\begin{aligned}\{\int_X |\eta f^{\frac{p+1}{2}}|^{\frac{2n}{n-1}}\omega_\phi^n\}^{\frac{n-1}{n}} &\leq C_S\int_X \{\eta^2 f^{p+1} + |f^{\frac{p+1}{2}}\nabla\eta + \eta\nabla f^{\frac{p+1}{2}}|^2\}\omega_\phi^n \\ &\leq C_S\int_X \{\eta^2 f^{p+1} + 2(\eta^2|\nabla f^{\frac{p+1}{2}}|^2 + f^{p+1}|\nabla\eta|^2)\}\omega_\phi^n \\ &\leq 10C_S\int_X (\eta^2 + |\nabla\eta|^2)f^{p+1}\omega_\phi^n\end{aligned}$$

Taking a suitable cut-off function on the real line η and composing it with the distance function on the manifold gives us a cut-off function which satisfies $\eta \equiv 0$ outside the metric ball $B_\phi(R)$ and $\eta \equiv 1$ inside the metric ball $B_\phi(S)$, and $|\nabla\eta| \leq \frac{C}{R-S}$. Plugging η to the above inequality, we have:

$$||f||_{L^{(p+1)\frac{n}{n-1}}(B_\phi(S))} \leq (10C_S)^{\frac{1}{p+1}}\{1 + \frac{C}{(R-S)^2}\}^{\frac{1}{p+1}}||f||_{L^{p+1}(B_\phi(R))}$$

The general *Moser's Iteration* technique uses γ_m to replace $p+1$ in the above inequality, and uses the pair of radius $r_m = 1 + 2^{-m}$ and $r_{m+1} = 1 + 2^{-(m+1)}$ to replace R and S at the m-th step. Eventually,

$$||f_\epsilon||_{L^\infty(B_{\phi_\epsilon}(1))} \leq C||f_\epsilon||_{L^2(B_{\phi_\epsilon}(2))} \tag{3.14}$$

Because $\sigma_{\epsilon,\beta'} \leq C(1 + (|z|^2+\epsilon)^{\beta-1}) \leq C(|z|^2+\epsilon)^{\beta-1}$, we see that the L^2 norm of f_ϵ on the ω_{ϕ_ϵ}-metric 2-ball is controlled by the following calculations:

$$\begin{aligned}
&\int_{B_{\phi_\epsilon}(p,2)} |\log \sigma_{\epsilon,\beta'}|^2 \omega_{\phi_\epsilon}^n \\
&\qquad \leq C \int_{|z|\leq 2\sqrt{C}} |\log(|z|^2+\epsilon)|^2 (|z|^2+\epsilon)^{\beta-1}\sqrt{-1}\mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&\qquad \leq C \\
&\int_{B_{\phi_\epsilon}(p,2)} |\log \sigma_{\epsilon,\beta'}|(|z|^2+\epsilon)^{\beta-\beta'} \omega_{\phi_\epsilon}^n \\
&\qquad \leq C \int_{|z|\leq 2\sqrt{C}} |\log(|z|^2+\epsilon)|(|z|^2+\epsilon)^{2\beta-\beta'-1}\sqrt{-1}\mathrm{d}z \wedge \mathrm{d}\bar{z} \leq C \\
&\int_{B_{\phi_\epsilon}(p,2)} |\log \sigma_{\epsilon,\beta'}|(|z|^2+\epsilon)^{\beta} \omega_{\phi_\epsilon}^n \\
&\qquad \leq C \int_{|z|\leq 2\sqrt{C}} |\log(|z|^2+\epsilon)|(|z|^2+\epsilon)^{2\beta-1}\sqrt{-1}\mathrm{d}z \wedge \mathrm{d}\bar{z} \\
&\qquad \leq C
\end{aligned}$$

The consequence is that on each of those balls $\mathrm{tr}_{\omega_{(\beta',\epsilon)}}\omega_{\phi_\epsilon} \leq C$. This yields us a finer control about the metric ω_{ϕ_ϵ} on the tangential directions. $\mathrm{tr}_{\omega_{(\beta',\epsilon)}}\omega_{\phi_\epsilon} \leq C$ implies that $\omega_{\phi_\epsilon} \leq C\omega_{(\beta',\epsilon)}$, thus we have uniform bound on the divisor direction, i.e. $\lambda_2, \cdots, \lambda_n \leq C$. Combined with the previous lower bound $\lambda_2, \cdots, \lambda_n \geq C^{-1}$ and the asymptotic behavior of the volume form $\lambda_1\lambda_2\cdots\lambda_n \sim \frac{1}{(|z|^2+\epsilon)^{1-\beta}}$, we conclude that $\lambda_1 \sim \frac{1}{(|z|^2+\epsilon)^{1-\beta}}$. Finally we get the desired uniform control about the metric ω_{ϕ_ϵ}, finishing the proof of Theorem 3.11.

3.2.3 Finishing proof of *Donaldson's Openness Theorem*

Theorem 3.11 implies that the limit *weak conical KE metrics* $\omega_{\varphi'_\beta}$ is quasi-isometric to the standard local conical Kähler metrics around D. Thanks to the beautiful local regularity result of [CW, Theorem 1.6], or more generally the global regularity result of [GP, Theorem A], these weak conical KE metrics actually possesses the maximal regularity, i.e. $\varphi'_\beta \in C^{2,\alpha,\beta'}$ in Donaldson's sense. Our perturbation results therefore gives a new proof to *Donaldson's Openness Theorem*, which was originally proved by a delicate implicit function theorem about varying Banach spaces.

3.3 Deformation along SNC pluri-anticanonical Divisors

In this subsection, we use the same idea to deform the cone angles of *weak conical Kähler-Eistein metrics* along a simple normal crossing (SNC) pluri-anti-canonical divisors on a smooth Fano manifold.

Let X be a smooth Fano manifold with a smooth background Kähler metric ω_0, and let $D_1, D_2, \cdots, D_k$ be a collection of smooth hypersurfaces in X with simple normal crossing (this means that at any intersecting point of this collection of divisors, the divisors involved in are intersecting transversally such that they could be represented by coordinate hyperplanes in some local holomorphic coordinates) at any intersection points. Let $\mathbf{b} = (\beta_1, \cdots, \beta_k)$ be a k-tuple of numbers in $(0, 1)$. Let S_i be the defining section of D_i, and choose a smooth Hermitian metric h_i on the holomorphic line bundle L_{D_i} with smooth curvature form α_i. Let $2\pi[D_i] = \alpha_i + \sqrt{-1}\partial\bar{\partial}\log|S_i|^2_{h_i}$ be the *current of integration* along D_i. Generalizing Definition (2.6), we can define *weak conical KE metrics* with angles $2\pi\beta_i$ along D_i (see [GP] for a more general definition for a KLT pair).

Definition 3.15. Let $\omega_{\varphi_\mathbf{b}}$ be a smooth Kähler metric on $X \backslash \cup_i D_i$ with the potential $\varphi_\mathbf{b}$ bounded on X. If it satisfies:

$$\omega^n_{\varphi_\mathbf{b}} = e^{-\mu\varphi_\mathbf{b} + h_{\omega_0}} \frac{\omega_0^n}{\Pi_{i=1}^k |S_i|^{2-2\beta_i}_{h_i}} \tag{3.15}$$

or equivalently the equation:

$$\text{Ric } \omega_{\varphi_\mathbf{b}} = \mu\omega_{\varphi_\mathbf{b}} + 2\pi\sum_{i=1}^k (1-\beta_i)[D_i] \tag{3.16}$$

it is called a *weak conical KE metric* on X with angle $2\pi\beta_i$ along D_i.

Notice that in this situation the necessary *cohomological condition* holds:

$$2\pi c_1(X) = \mu[\omega_{\varphi_{\mathbf{b}}}] + 2\pi \sum_{i=1}^{k}(1-\beta_i)[D_i] \tag{3.17}$$

Proof of Theorem 2.2 If $D_i \in |-\lambda_i K_X|$ with $\lambda_i > 0$, then α_i can be chosen to be $\lambda_i\omega_0$ by suitable choice of h_i. Let

$$\chi_\epsilon^i = \lambda_i\omega_0 + \sqrt{-1}\partial\bar{\partial}\log(|S_i|_{h_i}^2 + \epsilon)$$

(with $\epsilon \in (0,1]$) be the smoothing of the current of integration

$$2\pi[D_i] = \lambda_i\omega_0 + \sqrt{-1}\partial\bar{\partial}\log|S_i|_{h_i}^2,$$

and let $\mu = r(\mathbf{b}) := 1 - \sum_{i=1}^k \lambda_i(1-\beta_i)$. The first step is also to approximate $\omega_{\varphi_{\mathbf{b}}}$ by smooth Kähler metrics $\omega_{\varphi_{\epsilon,\mathbf{b}}}$ in the C^γ sense. For any $\mu' < \mu$, we introduce the analogous *modified log-Mabuchi-functional*

$$\widetilde{\mathcal{M}}_{\epsilon,\mu'} = \mathcal{M}_{\omega_0} + \sum_{i=1}^{k}(1-\beta_i)J_{\chi_\epsilon^i} + (\mu-\mu')J_{\omega_{\varphi_{\epsilon,\mathbf{b}}}}$$

where $J_{\chi_\epsilon^i}$ is the J_α functional defined in Definition 2.8 with $\alpha = \chi_\epsilon^i$.

This functional can be used to solve the continuity path $\star_{\epsilon,t}^{\mathbf{b}}$:

$$\text{Ric}\ \omega_{\phi_{\epsilon,t}^{\mathbf{b}}} = t\omega_{\phi_{\epsilon,t}^{\mathbf{b}}} + (\mu-t)\omega_{\varphi_{\epsilon,\mathbf{b}}} + \sum_{i=1}^{k}(1-\beta_i)\chi_\epsilon^i \tag{3.18}$$

up to $t = \mu$ with uniform L^∞ bound on $\phi_{\epsilon,\mu}^{\mathbf{b}}$. One remark is that along the continuity path $\star_{\epsilon,t}^{\mathbf{b}}$, in the case of negative or zero Ricci curvature, the L^∞ bound is obtained by the maximum principle or Kołodziej's estimate [Ko08]. Comparatively, in the case of positive Ricci curvature, we use *Moser's Iteration*, which depends on the uniform bound of the Sobolev constant and Poincaré constant (both hold on the continuity path). Another remark is that the condition *simple normal crossing* is required since Kołodziej's estimate requires uniform L^p bound of the R.H.S. of Eq. 3.15 some some $p > 1$. And in the contradiction argument here we need to use again Berndtsson's Generalized Bando-Mabuchi Theorem [Bern, Theorem 6.6] for weak Kähler-Einstein metrics.

We can then deform the angle parameters $\beta_i's$ as in section 4. The potentials $\phi_{\epsilon,\mu}^{\mathbf{b}}$ give the starting points at $t = \mu$ for the continuity paths:

$$\text{Ric } \omega_{u_{\epsilon,t}} = t\omega_{u_{\epsilon,t}} + \sum_{i=1}^{k}(1-\beta_{i,t})\chi_\epsilon^i \tag{3.19}$$

The *openness property* on those smooth continuity paths holds is the same reason as before, the linearized operator $\Delta_{\omega_{u_{\epsilon,\mu}}} + \mu$ is invertible because of Ric $\omega_{u_{\epsilon,\mu}}$. Since there are k parameters $\beta_1, \cdots, \beta_k$ to vary and only one time parameter, we need to use successive continuity paths to achieve this, i.e. firstly deform β_1 to β_1' nearby, and then start from this new weak conical KE metric to deform the second angle β_2 to β_2' nearby, and so on $\cdots$. After k steps, we can deform $(\beta_1, \beta_2, \cdots, \beta_k)$ to $(\beta_1', \beta_2', \cdots, \beta_k')$ nearby. This finishes the proof of Theorem 2.2.

4 Deformation of weak conical KE metric on $\mathbb{Q}$-Fano variety

The starting point in our proof of *Donaldson's Openness Theorem* in last section is an approximation of the weak conical KE metric by smooth Kähler metrics, where the approximation is achieved by finding Kähler metrics with prescribed volume forms (the content of Calabi conjecture solved by Yau). As already explained above, to approximate *weak conical KE metric* on a $\mathbb{Q}$-Fano variety X, we need to assume X sits inside a good family of smoothing, namely $\mathbb{Q}$-Gorenstein smoothing $\mathcal{X}$. And our weak conical KE metric is assumed to sit on a pair $(X, (1-\beta)D)$ where D is cut from a global Cartier divisor $\mathcal{D} \in |-\lambda K_{\mathcal{X}/\Delta}|$ such that $D = \mathcal{D}|_{X_0}$ and $(X_t, (1-\kappa)\mathcal{D}|_{X_t})$ is KLT for any $\kappa \in (0, 1]$. We assume $\lambda > 1$.

4.1 Starting point and GH continuity

It is shown in [SSY] that there is this kind of similar starting conical KE metrics on all of the smooth fiber X_t in this $\mathbb{Q}$-Gorenstein smoothing (cf. Definition 2.12), i.e. the existence in family of (weak) conical KE metrics for sufficiently small (but uniform) values of the cone angles will be established.

Theorem 4.1 ([BBEGZ, Oda]). *There exist $\underline{\beta} > 1-\lambda^{-1}$ such that for any $\beta \in (0, \underline{\beta}]$ and $t \in \Delta$, there exists a unique weak conical KE metric on $(X_t, (1-\beta)D_t)$, which is genuinely conical when $t \neq 0$.*

We just remark that those weak conical KE metrics with small angles are obtained via the combination of a uniform alpha-invariant estimate in family (observation of Yuji Odaka) and the variational approach developed by [BBEGZ] based on the properness of log-Ding/Mabuchi-functional.

Next, we want to show that the conical KE metrics on the smooth fiber $(X_t,(1-\beta)D_t,\omega_{t,\beta})$ actually converges to $(X_0,(1-\beta)D_0,\omega_{0,\beta})$ in the GH sense (cf. Definition 2.15). In our proof of *Donaldson's Openness Theorem*, the uniform L^∞ bound (actually C^α) on the potentials of approximated Kähler metrics gives all higher order estimates outside the divisor. So, the approximated Kähler metrics satisfying the smoothing of the Monge-Ampère equation really converges to conical KE metrics in a nice sense. In the current situation of a varying family of manifolds, we could not simply apply the Evan-Krylov theory since our family of Monge-Ampère equation involves the term of Ricci potentials (of background Fubini-Study metrics) which is in general not uniformly bounded. The GH continuity under L^∞ bound of the Kähler potentials could be viewed as an replacement for the higher order bound.

We may assume that the $\mathbb{Q}$-Gorenstein smoothing $\pi:\mathcal{X}\to\Delta$ of a $\mathbb{Q}$-Fano variety is λ-plurianticanonically embedded in $\mathbb{P}^{N(\lambda)}\times\mathbb{C}$.

Theorem 4.2 ([SSY]). *Let $\pi:(\mathcal{X},\mathcal{D})\to\Delta$ be a $\mathbb{Q}$-Gorenstein smoothing as above. Let $\beta\in(1-\lambda^{-1},1]$, assume that for any $t\in\Delta$, there is a weak conical KE metric $\omega_{t,\beta}$ on $(X_t,(1-\beta)D_t)$, which is genuinely conical for $t\neq 0$ and such that $\omega_{t,\beta}=\omega_{t,FS}+\sqrt{-1}\partial\bar\partial\phi_{t,\beta}$ with $|\phi_{t,\beta}|_{L^\infty}$ uniformly bounded. Then the conical KE metrics on the smooth fibers converge to the weak KE metric on the central fiber in the GH topology. Moreover, we have that $|\nabla_{\omega_{t,\beta}}\phi_{t,\beta}|$ is uniformly bounded and all higher derivatives of $\phi_{t,\beta}$ are uniformly bounded away from the singular set and the divisor.*

The key point of the proof to this theorem is a comparison between the algebraic embedding $X_t\subset\mathbb{P}^{N(\lambda)}$ and the L^2 embedding of X_t, denoted as Y_t. The first embedding could be viewed as an orthonormal embedding under some Hermitian inner product H on $H^0(X_t,K_{X_t}^{-\lambda})$, while the L^2-embedding (usually called *Bergman embedding* in the literature) is the orthonormal embedding under the Hermitian inner product H_t defined by

$$H_t(\sigma_1,\sigma_2)=\int_{X_t}h_t(\sigma_1,\sigma_2)\omega_{t,\beta}^n$$

where h_t is the singular Hermitian metric on $K_{X_t}^{-\lambda}$ with curvature $\lambda\omega_{t,\beta}$. The L^∞ bound on $\phi_{t,\beta}$ implies the Hermitian metrics H_t and H are uniformly equivalent, which in turn implies the two embedding images inside $\mathbb{P}^{N(\lambda)}$ are uniformly bounded from each other, i.e. $Y_t=A_t.X_t$ for a bounded family of matrices A_t in $GL(N(\lambda),\mathbb{C})$. The theorem of Chen-Donaldson-Sun [CDS2, Theorem 1] says the GH limit of $\omega_{t,\beta}$

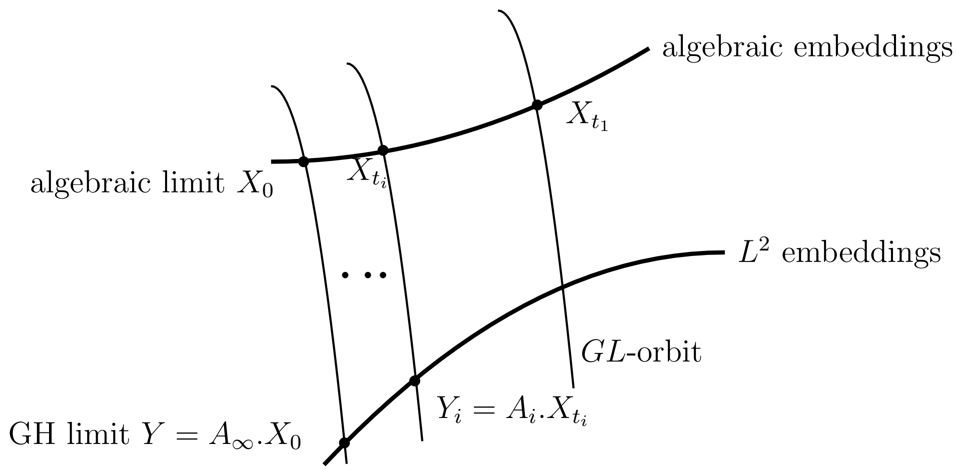

Figure 4.1: GH convergence under L^∞ bound

could be realized as the limit Y of the family Y_t inside the Chow variety, and the conical KE metric converges to the weak conical KE metric also in the sense of current, meanwhile D_i converges to Δ as algebraic cycles . Therefore, the GH limit is actually isomorphic to X_0, the algebraic limit of X_t. Figure 3.1.3 is an illustration for this sketch of proof.

Remark 4.3. It follows from the proof below and the theorem in [DS] that the same conclusion is true if the cone angles of the conical KE metrics vary and stay bounded below by $1-\lambda^{-1}+\delta$ for some $\delta > 0$ since they all have a uniform diameter upper bound.

By this theorem, to show the GH continuity, it suffices to establish the uniform L^∞ estimate of the conical KE potentials. This is related to the log-Ding/Mabuchi-functional that was introduced in the first section.

Suppose we have a family $(\mathcal{X}, \mathcal{D})$ over Δ, which is embedded into $\mathbb{P}^N \times \mathbb{C}$ by $|-\lambda K_{\mathcal{X}/\Delta}|$. We denote by $\omega_{t,FS}$ the restriction of $\lambda^{-1}\omega_{FS}$ on X_t and by h_t the restriction of $h_{FS}^{1/\lambda}$ on $K_{\mathcal{X}/\Delta}|_{X_t}$. We define $F_{t,\beta}$ (respectively $M_{t,\beta}$) to be the infimum of the log-Ding-functional (respectively log-Mabuchi-functional) on X_t with base metric $\omega_t = \omega_{t,FS}$. When X_t admits a weak conical KE metric on the pair $(X_t, (1-\beta)D_t)$ for $\beta \in (0,1)$, this functional is finite according to [BBEGZ, Theorem 4.8] and $F_{t,\beta} = \mathcal{F}_{\omega_t,(1-\beta)D_t}(\phi_{t,\beta})$ is achieved at the unique conical KE metric $\omega_{t,\beta} = \omega_t + \sqrt{-1}\partial\bar{\partial}\phi_{t,\beta}$.

Theorem 4.4 ([SSY]). *Suppose for a fixed $\beta \in (1-\lambda^{-1}, 1)$, or $\beta = 1$ if $Aut(X_0)$ is discrete, there are weak conical KE metrics $\omega_{t,\beta}$ on $(X_t, (1-\beta)D_t)$ for all $t \in \Delta$, which are genuinely conical for $t \neq 0$. Then we have that $\limsup_{t\to 0} F_{t,\beta} > -\infty$ and $\limsup_{t\to 0} M_{t,\beta} > -\infty$. Moreover, $(X_t, (1-\beta)D_t, \omega_{t,\beta})$ converges to $(X_0, (1-\beta_0)D_0, \omega_{0,\beta})$ in GH sense as $t \to 0$.*

The next lemma shows that the I functional is continuous under a kind of convergence of the Kähler potentials which frequently appears in our situation.

Lemma 4.5 (Strong Convergence of I functional). *Suppose we have $t_j \to t_0 \in \Delta$ and suppose we have a sequence of potentials ϕ_j on X_{t_j} with $\omega_{t_j} + \sqrt{-1}\partial\bar{\partial}\phi_j \geq 0$ and $|\phi_{t_j}|_{L^\infty}$ is uniformly bounded. Furthermore assume ϕ_j is C^2 on $X_{t_j} \setminus D_{t_j}$ and ϕ_j converges smoothly away from $\mathcal{X}^{sing} \cup \mathcal{D}$. Then we have*

$$\lim_{j\to\infty} I_{\omega_{t_j}}(\phi_j) = I_{\omega_{t_0}}(\phi_0).$$

Proof. We assume $t_0 = 0$. The other case is simpler. We write

$$I_{\omega_{t_j}}(\phi_j) = \int_{U_{t_j}} \phi_j(\omega_{t_j}^n - (\omega_{t_j} + i\partial\bar{\partial}\phi_j)^n) + \int_{X_{t_j}\setminus U_{t_j}} \phi_j(\omega_{t_j}^n - (\omega_{t_j} + i\partial\bar{\partial}\phi_j)^n).$$

where U_{t_j} is cut from an open subset $\mathcal{X}\backslash(\mathcal{X}^{sing} \cup \mathcal{D})$. It is clear that the first term converges to zero. For the second term, we notice that we can choose U_0 in X_0 so that $\int_{X_0\setminus U_0} \omega_0^n$ and $\int_{X_0\setminus U_0}(\omega_0 + i\partial\bar{\partial}\phi_0)^n$ arbitrarily small since the complement is arbitrarily close to the volume of X_0. □

We are ready now to perform a continuity method argument to show that cone angle of a weak conical KE metric on $(X_0, (1-\beta)D_0)$ could be perturbed.

4.2 Deformation of weak conical KE metrics

Let $\mathcal{D} \in |-\lambda K_{\mathcal{X}/\Delta}|$ be as in the setting of Theorem 2.13, we define the following functions, which measure the "maximal" cone angle:

$\beta_t := \sup\{\beta \in (1-\lambda^{-1}, 1] \,|\, \exists$ conical KE metric on $(X_t, (1-\beta)D_t)\}$, for $t \neq 0$;
$\beta_0 := \sup\{\beta \in (1-\lambda^{-1}, 1] \,|\, \exists$ weak conical KE metric on $(X_0, (1-\kappa)D_0)$ for $\forall \kappa \leq \beta\}$.

Notice that by Theorem 4.1, we may assume $\beta_t \geq \underline{\beta} > 1 - \lambda^{-1}$ for all $t \in \Delta$. For $t \neq 0$, it follows from [CDS1, CDS2, CDS3, Berm2] that the existence of KE metrics on X_t with cone angle $2\pi\beta$ ($\beta \in (0, 1]$) along D_t corresponds to the K-polystability of $(X_t, (1-\beta)D_t)$ and the latter condition satisfies an obvious interpolation property for β. In particular, for $t \neq 0$, there indeed exists a KE metric on X_t with cone angle $2\pi\beta$ along D_t for all $\beta \in (0, \beta_t)$. Since on the central fiber it is one of our goals to establish the existence result, at this stage we do not have the interpolation property yet. This is the reason that in the above definition we distinguish between the case $t \neq 0$ and $t = 0$.

For a KLT pair (W, Δ), we use $Aut(W, \Delta)$ to denote the group of holomorphic automorphisms of W which preserve the divisor Δ.

Proposition 4.6 ([SSY]).

- *(Lower-Semicontinuity): β_t is a lower semi-continuous function of $t \in \Delta$;*
- *(Uniqueness of Limit): For any sequence $t_i \to 0$ and $\beta_i \to \hat{\beta} < 1$, the conical KE pair $(X_{t_i}, (1-\beta_i)D_{t_i}, \omega_{t_i,\beta_i})$ converges to $(X_0, (1-\hat{\beta})D_0, \omega_{0,\hat{\beta}})$ in GH sense, and moreover the potential of ω_{t_i,β_i} relative to the L^2 embeddings are uniformly bounded.*

The proof of Theorem 2.13 is more involved than the proof for *Donaldson's Openness Theorem* since it uses also the notion of K-polystability. The existence of weak conical KE metrics on the pair $(X_0, (1-\kappa)D_0)$ for β_* and the small angle κ_0 implies the K-polystability for those pairs and thus the K-polystability of the pair for any κ in between κ_0 and β_* by the linearity of K-polystability on the parameters.

Define $A = \{\beta \leq \beta_* | \exists$ a weak conical KE metric on $(X_0, (1-\kappa)D_0)$, $\forall \kappa \leq \beta\}$.

As in the work of [CDS1, CDS2, CDS3], we will also use the method of deforming the cone angles. It suffices to show A is both open and closed in $[\underline{\beta}, \beta_*]$, where $\underline{\beta}$ is the number in Theorem 4.1.

Proof of Theorem 2.13. For any $\hat{\beta} \in A$, there is a weak conical KE pair $(X_0, (1-\kappa)D_0, \omega_{0,\kappa})$ for any $\kappa \leq \hat{\beta}$. We use the lower semicontinuity of β_t (Proposition 4.6) to prove that nearby smooth pairs $(X_t, (1-\hat{\beta})D_t)$ all admit weak conical KE metrics, i.e. $\beta_t > \hat{\beta}$ (this is the analogy of Proposition 3.9 in Sect. 3.1.2). Suppose it is not the case. Then we have a subsequence $t_i \to 0$, $\beta_{t_i} \leq \hat{\beta}$ and $\lim_{i\to\infty} \beta_{t_i} = \hat{\beta}$. By the weak compactness result of [CDS2], for each fixed i, the conical KE pair $(X_{t_i}, (1-\kappa)D_{t_i}, \omega_{t_i,\beta})$ converges by subsequence to a limit $(W_i, (1-\beta_{t_i})\Delta_i, \omega_i)$ as $\kappa \to \beta_{t_i}$, with $Aut_0(W_i, \Delta_i)$ containing a non-trivial one parameter subgroup. By Prop. 4.6, this sequence of limits must converge to $(X_0, (1-\hat{\beta})D_0, \omega_{0,\hat{\beta}})$, contradicting the fact that $Aut_0(X_0, D_0) = \{1\}$.

The second item in Prop. 4.6 implies that $(X_t, (1-\hat{\beta})D_t, \omega_{t,\hat{\beta}})$ converges to $(X_0, (1-\beta)D_0, \omega_{0,\hat{\beta}})$ in the GH sense and the potential $\phi_{t,\hat{\beta}}$ of $\omega_{t,\hat{\beta}}$ relative to the

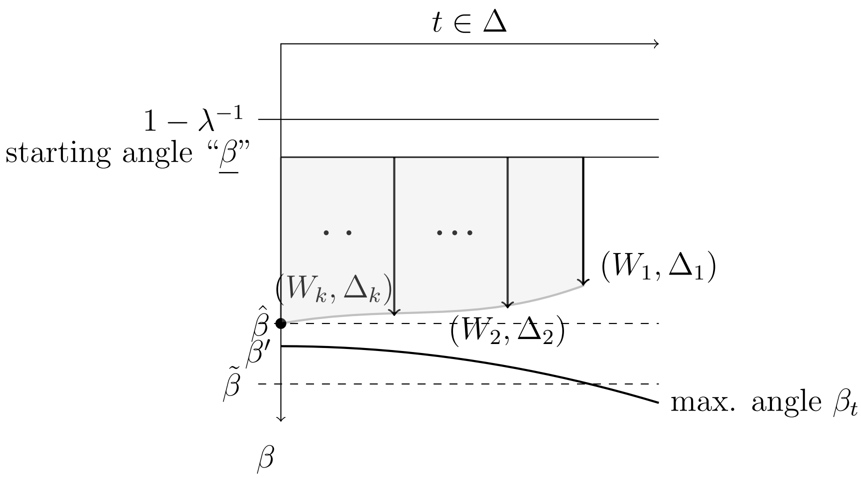

Figure 4.2: conical KE's in a $\mathbb{Q}$-Gorenstein family

induced Fubini-Study metric ω_t with respect to the L^2 holomorphic embedding is uniformly bounded in L^∞. This implies that $I_{\omega_t}(\phi_{t,\hat{\beta}})$ is uniformly bounded. This fact together with that $\beta_t > \hat{\beta}$ imply that there is a $\tilde{\beta} > \hat{\beta}$ such that $I_{\omega_t}(\phi_{t,\beta'})$ is uniformly bounded for $\forall \beta' \in [\hat{\beta}, \tilde{\beta}]$ and for all t sufficiently small. Otherwise we could find a subsequence $t_i \to 0, \hat{\beta} < \kappa_i < \beta_{t_i}$ that converges to $\hat{\beta}$ and a sequence of weak conical KE pairs $(X_{t_i}, (1-\kappa_i)D_{t_i}, \omega_{t_i,\kappa_i})$ with $I_{\omega_{t_i}}(\phi_{t_i,\kappa_i}) - I_{\omega_{t_i}}(\phi_{t_i,\hat{\beta}}) = C$ for a fixed large constant C, by the *strong convergence of I functional* (Lem. 4.5) this would lead to a contradiction with the *Uniqueness of weak conical KE metrics* [BBEGZ, Theorem 5.1] and the arguments that we have used frequently before. From the uniform bounds of $I_{\omega_t}(\phi_{t,\beta'})$ it follows from Theorem 4.2 that $(X_t, (1-\beta')D_t, \omega_{t,\beta'})$ converges by subsequence to some weak conical KE pair $(X_0, (1-\beta')D_0, \omega_{0,\beta'})$ as $t \to 0$. This proves the openness of A.

Then we proceed to show the *closedness* part. Take any sequence $\{\beta_j\}_{j=1,2,\cdots} \subset A$ which strictly increases to a number $\beta_\infty \leq \beta_*$. By Proposition 4.6, for any j, $\beta_t \geq \beta_j$ for t small enough and the weak conical KE pairs $(X_0, (1-\beta_j)D_0, \omega_{0,\beta_j})$ is the GH limit of genuinely conical KE metrics on the smooth fibers. This enables us to take sequential GH limit of $(X_0, (1-\beta_j)D_0, \omega_{0,\beta_j})$ from which we get a weak conical KE pair $(Y, (1-\beta_\infty)\Delta, \omega)$ with $Aut(Y, \Delta)$ reductive. As in [CDS3], if (Y, Δ) is not isomorphic to (X_0, D_0) then there exists a nontrivial test configuration of (X_0, D_0) with central fiber (Y, Δ) and Donaldson-Futaki invariant vanishing. This shows that

$(X_0, (1-\beta_\infty)D_0)$ is not K-polystable. Contradiction.

$\square$

Finally, we remark that we only need to prove the *openness* at parameter $\beta_* < 1$ to proceed the continuity method. Therefore, K-polystable $\mathbb{Q}$-Gorenstein smoothable $\mathbb{Q}$-Fano variety admits a weak KE metric (without assuming the triviality of the automorphism group of the variety). On the other hand, if indeed $Aut(X_0)$ is discrete, then all nearby smooth Fano manifolds X_t's admit smooth Kähler-Einstein metrics and this family converges to the weak Kähler-Einstein metric on X_0 in GH sense (cf. [SSY, Theorem 1.1]).

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