Applications of the Seiberg-Witten equations to the Differential Geometry of non-compact Kähler manifolds

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Abstract of the Dissertation

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Soon after the introduction of the Seiberg-Witten equations, and their magnificent application to the differential topology of 4-manifolds, LeBrun [LeB95a] used these equations to study differential geometry and prove a rigidity theorem for compact complex hyperbolic manifolds. Biquard [Biq97] extended these results to non-compact, finite volume complex hyperbolic manifolds, and Rollin [Rol04] extended these techniques to $\mathbb{CH}^2$. Finally, Di Cerbo[DC12, DC11] applied Biquard's techniques to $\Sigma \times \Sigma_g$.

The main tool that allows one to use the Seiberg-Witten equations to

study differential geometry is an integral scalar curvature estimate. The principle difficulty in extending these methods to the non-compact case, which was overcome by Biquard, Rollin and Di Cerbo is the proof of the existence of a solution to the equations. Finally, in LeBrun used conformal rescaling of the Seiberg-Witten equations to prove an integral estimate that involves both the scalar and Weyl curvature.

In this thesis we extend these techniques to quasiprojective 4-manifolds which admit negatively curved, finite volume Kähler-Einstein metrics. Following Biquard's method we produce an irreducible solution to the Seiberg-Witten equations on the non-compact manifold as a limit of solutions on the compactification, and then use the Weitzenböck formula to obtain a scalar curvature estimate that is necessary for geometric applications.

Dedicated to my grandfather, Yuri Lyubich.

Contents

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Chapter 1

Introduction

The first appearance of gauge theory in physics was the theory of Electricity and Magnetism developed by Maxwell in [Max65], though, without the modern language of gauge theory or even modern vector calculus notation. The second appearance of gauge theory is Einstein's theory of general relativity in [Ein16]. It was Weyl in [Wey29] who in his attempt to unify Electricity and Magnetism with Einstein's Theory of Gravitation, enunciated the principle of *gauge invariance* or *Eichinvarianz* in German. The ascendance of gauge theory as the foundation of modern physics dates to the paper of Yang and Mills [YM54], which introduced non-abelian gauge theory. An account of the history of gauge theory can be found in [JO01].

On the mathematical side, gauge theory is the study of connections, initiated by Cartan in [Car23]. The development of the modern theory of connections on fiber bundles was initiated by Ehresmann in [Ehr51]. Since then the study of connections and their curvatures has led to a rich and beautiful theory. No attempt will be made to summarize this theory or even its basic definitions, for many such expositions exist in the literature, *e. g.* [Tay11b, DK90].

Instead, the first chapter consists of a concise exposition of the notions necessary to write down the Seiberg-Witten equations (1.3.1) and prove the basic theorems in Seiberg-Witten theory. This summary is aimed at a student of differential geometry, who may be interested in applications of Seiberg-Witten to differential geometry but who does not wish to delve into a detailed exposition of Seiberg-Witten theory. Section 1.1 contains a summary of the main results of the thesis, for the convenience of the reader.

1.1 Summary of results

The main results of this thesis are:

- Theorem 2.3.1, which shows existence of irreducible solutions to the Seiberg-Witten equations on quasi-projective manifolds with metrics C^2-close to the Kähler-Einstein metric.
- Theorem 2.5.3, which is a scalar curvature estimate which can be derived from the Weitzenböck formula.
- Theorem 2.5.6, which asserts the non-existence of Einstein metrics on k-fold blow-ups of quasi-projective manifolds, which are asymptotic to the Kähler-Einstein metric, with $k > \frac{2}{3}\left(c_1(K_{\bar{X}}) + c_1(L_D)\right)^2$.
- Proposition 3.2.1, which is a mixed Weyl and scalar curvature estimate, based on a conformal rescaling argument.

- Theorem 3.3.1, which improves the bound on k by a factor of 2 to $k > \frac{1}{3}\left(c_1(K_{\bar{X}}) + c_1(L_D)\right)^2$.

1.2 Preliminaries

1.2.1 Spin and Spin$^{\mathbb{C}}$ structures

A classic and comprehensive reference for Sections 1.2.1 – 1.2.4 is [LM89]. A much shorter and less comprehensive introduction, however, can be found in the expositions of Seiberg-Witten theory in [Mor96], and a very concise exposition can be found in [Tay11c, Chapter 10]. In the following, all manifolds will be assumed smooth, and a closed manifold will mean a compact manifold without boundary. Let M be a closed, orientable Riemannian manifold of dimension n and let E be an orientable rank r vector bundle over M. The choice of a bundle metric g on E reduces the structure group of E from $GL(r)$ to $O(r)$. Now let us choose an orientation on E. This has the effect of further reducing the structure group of E to $SO(r)$. We have, thus, reduced the structure group of E from a group with $\pi_0 = \mathbb{Z}_2$ to one with $\pi_0 = \{0\}$. It is then natural to ask whether we can further change the structure group to a simply connected group. E is called *spinnable* if we can lift the structure group of E from $SO(r)$ to $Spin(r)$, the double cover of $SO(r)$, and a choice of such a lift is called a *spin structure on* E, or just *spin structure*, in the case $E = TM$, the tangent bundle of M.

To describe the above construction in greater detail, let us denote by $P_G(E)$ the principal G-bundle associated to a vector bundle E with structure group G.

Definition 1.2.1. A spin structure on an orientable rank r vector bundle E, is $Spin(r)$-principal bundle whose total space, $P_{\mathrm{Spin}(r)}$ is a double cover of the total space $P_{SO(r)}(E)$ of the principal $SO(r)$-bundle associated to E, which restricts to the cover $Spin(r) \to SO(r)$ on each fiber. This definition is neatly summarized in the following commutative diagram:

$$
\begin{tikzcd}
 & Spin(r) \arrow[rr, "\pi"] \arrow[dd] & & SO(r) \arrow[dd] \\
\mathbb{Z}_2 \arrow[ur] \arrow[dr] & & & \\
 & P_{Spin(r)}(E) \arrow[rr, "\tilde{\pi}"] \arrow[dr] & & P_{SO(r)}(E) \arrow[dl] \\
 & & M &
\end{tikzcd}
$$

Figure 1.2.1

where $\tilde{\pi}$ is a double cover which restricts to π on each fiber, so that for $p \in P_{Spin(r)}(E)$ and $g \in Spin(r)$, $\tilde{\pi}(pg) = \tilde{\pi}(p)\pi(g)$. A spin structure on a manifold is a spin structure on its tangent bundle.

Not every orientable vector bundle is spinnable. In fact, an orientable vector bundle is spinnable if and only if $w_2(E)$, its second Stiefel-Whitney class is 0. Indeed, essentially by definition we can regard a principal $SO(r)$ bundle over M as an element of the Čech cohomology group $H^1(M; \mathcal{E}(SO(r)))$, where $\mathcal{E}(G)$ is the sheaf of functions on M with values in G. The short exact sequence of coefficient groups

$$\{0\} \to \mathbb{Z}_2 \to Spin(r) \to SO(r) \to \{0\}$$

gives rise to a long exact sequence of Čech cohomology groups, though, the long exact sequence terminates at the $H^2(M;\mathbb{Z}_2)$ term since $H^2(M;\mathcal{E}(G))$ is not defined for non-abelian G,

$$\{0\} \to H^1(M;\mathbb{Z}_2) \to H^1(M;\mathcal{E}(Spin(r))) \to H^1(M;\mathcal{E}(SO(r))) \xrightarrow{\delta} H^2(M;\mathbb{Z}_2)$$

The boundary map δ can be identified with w_2 by computation on the universal $SO(r)$ bundle since the long exact sequence is natural. This shows that w_2 is the obstruction to a bundle being spinnable.

There is another important exact sequence relevant to the discussion of spin structures that arises from the Serre spectral sequence of a fibration. See [Hat] for an introduction to spectral sequences in general, and the Serre spectral sequence in particular. Consider the E_2 page of the Serre spectral sequence for the fibration $SO(r) \overset{\iota}{\hookrightarrow} P_{SO(r)}(E) \to M$ with $\mathbb{Z}_2$ coefficients. We have $E_2^{p,q} = H^p(M;H^q(SO(r);\mathbb{Z}_2))$ and

1	$H^1(SO(r),\mathbb{Z}_2)$		
0	$\mathbb{Z}_2$	$H^1(M,\mathbb{Z}_2)$	$H^2(M,\mathbb{Z}_2)$
	0	1	2

where we have substituted $H^0(SO(r);\mathbb{Z}_2) = \mathbb{Z}_2$.

In this portion of the spectral sequence there is only one non-trivial differential $d_2^{0,1} : H^1(M;\mathbb{Z}_2) \to H^2(M;\mathbb{Z}_2)$. The Serre spectral sequence then tells us that $H^1(P_{SO(r)}(E);\mathbb{Z}_2)$ has a filtration with $\ker d_2^{0,1}$ and $H^1(M;\mathbb{Z}_2)$ as the successive quotients. In other words, there exists $\{0\} \subset F \subset H^1(P_{SO(r)}(E);\mathbb{Z}_2)$ such that $F/\{0\} \cong H^1(M;\mathbb{Z}_2)$ and $H^1(P_{SO(r)}(E);\mathbb{Z}_2)/F \cong \ker d_2^{0,1}$. Note that while the filtration is natural, the isomorphisms above are not. Hence, we get two natural sequences, the first is exact, and the second is exact at the first and second terms:

$$H^1(M;\mathbb{Z}_2) \hookrightarrow H^1(P_{SO(r)}(E);\mathbb{Z}_2) \twoheadrightarrow \ker d_2^{0,1} \tag{1.2.1}$$

$$\ker d_2^{0,1} \hookrightarrow H^1(SO(r);\mathbb{Z}_2) \xrightarrow{d_2^{0,1}} H^2(M;\mathbb{Z}_2) \tag{1.2.2}$$

which we can combine to get the long exact sequence

$$H^1(M;\mathbb{Z}_2) \hookrightarrow H^1(P_{SO(r)}(E);\mathbb{Z}_2) \to \underbrace{H^1(SO(r);\mathbb{Z}_2)}_{\mathbb{Z}_2} \xrightarrow{d_2^{0,1}} H^2(M;\mathbb{Z}_2) \tag{1.2.3}$$

Now given a topological space X, the double covers of X, which we denote $\tilde{X}$, are in one-to-one correspondence with conjugacy classes of index 2 subgroups of $\pi_1(X, x_0)$. Since an index 2 subgroup is automatically normal, double covers of X are actually in one to one correspondence with homomorphisms $\pi_1(X, x_0) \to \mathbb{Z}_2$. Further, since $\mathbb{Z}_2$ is abelian any homomorphism to $\mathbb{Z}_2$ automatically contains the commutator subgroup of the domain in the kernel, and since $H_1(X;\mathbb{Z})$ is isomorphic to $\pi_1(X, x_0)$ modulo the commutator subgroup, we see that double covers of X are actually in one to one correspondence with homomorphisms from $H_1(X;\mathbb{Z}) \to \mathbb{Z}_2$. But $\mathrm{Hom}(H_1(X;\mathbb{Z}), \mathbb{Z}_2) \cong H^1(X;\mathbb{Z}_2)$ by the universal coefficients theorem since $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}_2)$ vanishes. Hence, Spin structures, which are double covers of $P_{SO(r)}(E)$ which restrict to the double cover $\mathrm{Spin}(r) \to SO(r)$ on the fibers, correspond to those elements of $H^1(P_{SO(r)}(E);\mathbb{Z}_2)$ which map to the non-zero element in $H^1(SO(r);\mathbb{Z}_2) \cong \mathbb{Z}_2$ in (1.2.3). By considering (1.2.3) with $M = BSO(r)$, the classifying space $SO(r)$, we can see that $d_2^{0,1}(1) = w_2(E)$. Hence, E is spinnable if and only if $w_2(E) = 0$, and in that case, spin structures are a torsor for $H^1(M;\mathbb{Z}_2)$ by (1.2.3) (in other words the difference between any two spin structures, thought of as elements of $H^1(P_{SO(r)}(E),\mathbb{Z}_2)$ is an element of $H^1(M;\mathbb{Z}_2)$).

Every compact, orientable 3-manifold is also spinnable, since a compact, orientable 3-manifold is actually parallelizable. However, $\mathbb{CP}^2$ is not spinnable. Indeed, the second Stiefel-Whitney class is the mod 2 reduction of the first chern class for a complex vector bundle, so $w_2(T\mathbb{CP}^2) = c_1(T\mathbb{CP}^2) \mod 2 = 1 \in \mathbb{Z}_2 \cong H^2(\mathbb{CP}^2;\mathbb{Z}_2)$. A deep theorem of Rohlin [Roh52], in fact, states that the signature of a smooth, compact, spinnable 4-manifold is divisible by 16.

Spin structures have a closely related variation, the $\mathrm{Spin}^{\mathbb{C}}$ structure. A $\mathrm{Spin}^{\mathbb{C}}$ structure allows one to define the Dirac operator (see below), and $\mathrm{Spin}^{\mathbb{C}}$ structures are more abundant: any spinnable vector bundle has a $\mathrm{Spin}^{\mathbb{C}}$ structure, as does any complex vector bundle. The tangent bundle of an orientable 4-manifold also always admits a $\mathrm{Spin}^{\mathbb{C}}$ structure.

To define a $\mathrm{Spin}^{\mathbb{C}}$ structure, we first define the group $\mathrm{Spin}^{\mathbb{C}}(n)$ as $\mathrm{Spin}(n)\times_{\mathbb{Z}_2} S^1$, in other words, $\mathrm{Spin}^{\mathbb{C}}(n)$ is the product of $\mathrm{Spin}(n)$ and S^1 modulo the simultaneous $\mathbb{Z}_2$ action. We have

$$S^1 \hookrightarrow \mathrm{Spin}^{\mathbb{C}}(n) \overset{\pi}{\twoheadrightarrow} SO(n) \tag{1.2.4}$$

Now following definition 1.2.1, we can write down

Definition 1.2.2. A $\mathrm{Spin}^{\mathbb{C}}$ structure on oriented rank r vector bundle E is a $\mathrm{Spin}^{\mathbb{C}}(r)$-principal bundle whose total space, $P_{\mathrm{Spin}^{\mathbb{C}}(r)}$ is an S^1 bundle over the total space $P_{SO(r)}$ of the principal $SO(r)$-bundle associated to E, which restricts to the fibration (1.2.4) on each fiber.

Just as for the definition of a Spin structure we have diagram 1.2.1, we can conveniently present the definition of a $\mathrm{Spin}^{\mathbb{C}}$ structure by a commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Spin}^{\mathbb{C}}(r) \arrow[rr, "\pi"] \arrow[dd] & & SO(r) \arrow[dd] \\
S^1 \arrow[ur] \arrow[dr] & & & \\
 & P_{\mathrm{Spin}^{\mathbb{C}}(r)}(E) \arrow[rr, "\tilde{\pi}"] \arrow[dr] & & P_{SO(r)}(E) \arrow[dl] \\
 & & M &
\end{tikzcd}$$

Figure 1.2.2

The short exact sequence of groups (1.2.4) gives rise to a long exact sequence in Čech cohomology

$$\{0\} \to H^1(M; \mathcal{E}(S^1)) \to H^1(M; \mathcal{E}(\mathrm{Spin}^{\mathbb{C}}(r))) \to \\ \to H^1(M; \mathcal{E}(SO(r))) \xrightarrow{\delta} H^2(M; \mathcal{E}(S^1)) \quad (1.2.5)$$

But the short exact sequence $\mathbb{Z} \hookrightarrow \mathbb{R} \twoheadrightarrow S^1$ gives rise to a long exact sequence in cohomology

$$\underbrace{H^1(M;\mathcal{E}(\mathbb{R}))}_{0} \to H^1(M;\mathcal{E}(S^1)) \xrightarrow{c_1} H^2(M;\mathbb{Z}) \to \underbrace{H^2(M;\mathcal{E}(\mathbb{R}))}_{0} \to$$
$$\to H^2(M;\mathcal{E}(S^1)) \xrightarrow{W_3} H^3(M;\mathbb{Z}) \to \underbrace{H^3(M;\mathcal{E}(\mathbb{R}))}_{0} \quad (1.2.6)$$

Since $H^k(M;\mathcal{E}(\mathbb{R})) = \{0\}$, (1.2.6) shows that $H^1(M;\mathcal{E}(S^1))$ is isomorphic to $H^2(M;\mathbb{Z})$ (the isomorphism is the first Chern class) and $H^2(M;\mathcal{E}(S^1))$ is isomorphic to $H^3(M;\mathbb{Z})$ (the isomorphism is the so-called "integer Stiefel-Whitney class" W_3). The W_3 class is the image of w_2, the ordinary second Stiefel-Whitney class, under the Bockstein homomorphism in the cohomology long exact sequence which arises from the short exact sequence $\mathbb{Z} \overset{\times 2}{\hookrightarrow} \mathbb{Z} \twoheadrightarrow \mathbb{Z}_2$:

$$\cdots \to H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{Z}_2) \xrightarrow{\beta} H^3(M;\mathbb{Z}) \to \cdots$$

$W_3 = \beta(w_2)$. Now (1.2.5) becomes

$$\{0\} \to H^2(M;\mathbb{Z}) \to H^1(M;\mathcal{E}(\mathrm{Spin}^{\mathbb{C}}(r))) \to H^1(M;\mathcal{E}(SO(r))) \xrightarrow{W_3} H^3(M;\mathbb{Z}) \quad (1.2.7)$$

It follows that a vector bundle has a $\mathrm{Spin}^{\mathbb{C}}$ structure if and only if its W_3 class vanishes. There is also a long exact sequence analogous to (1.2.3) that arises from the Serre spectral sequence for the fibration $SO(r) \hookrightarrow P_{SO(r)}(E) \twoheadrightarrow M$ with $\mathbb{Z}$ coefficients. In this case, the E_2 page of the spectral sequence is

2	$H^2(SO(r);\mathbb{Z}) \cong \mathbb{Z}$			
1	$H^1(SO(r);\mathbb{Z}) \cong \{0\}$	0	0	0
0	$\mathbb{Z}$	$H^1(M;\mathbb{Z})$	$H^2(M;\mathbb{Z})$	$H^3(M;\mathbb{Z})$
	0	1	2	3

There are no non-trivial differentials on the portion of the E_2 page shown, so this fragment survives to the E_3 page, where the only non-trivial differential in the portion of the spectral sequence that is shown is

$$d_3^{0,2} : H^2(SO(r);\mathbb{Z}) \to H^3(M;\mathbb{Z}).$$

Just as in the discussion concerning Spin structures, there is a filtration of $H^2(P_{SO(r)}(E);\mathbb{Z})$ with successive quotients isomorphic to $\ker d_3^{0,2}$ and $H^2(M;\mathbb{Z})$. In other words, there exists $\{0\} \subset F \subset H^2(P_{SO(r)}(E);\mathbb{Z})$ such that $F/\{0\} \cong H^2(M;\mathbb{Z})$ and $H^2(P_{SO(r)}(E);\mathbb{Z})/F \cong \ker d_3^{0,2}$. This gives us the long exact sequence:

$$H^2(M;\mathbb{Z}) \hookrightarrow H^2(P_{SO(r)}(E);\mathbb{Z}) \to H^2(SO(r);\mathbb{Z}) \xrightarrow{d_3^{0,2}} H^3(M;\mathbb{Z}) \tag{1.2.8}$$

By computation on the universal $SO(r)$ bundle $SO(r) \hookrightarrow ESO(r) \twoheadrightarrow BSO(r)$, we see that $d_3^{0,2}(1) = W_3(E)$. Now, $BS^1 \cong \mathbb{CP}^\infty$, the classifying space for complex line bundles, or equivalently for orientable rank 2 real vector bundles, is also the Eilenberg-MacLane space $K(\mathbb{Z},2)$, hence homotopy classes of maps $X \to \mathbb{CP}^\infty$, $[X,\mathbb{CP}^\infty] \cong H^2(X;\mathbb{Z})$. Or in other words, S^1 bundles on X are in one-to-one correspondence with elements of $H^2(X;\mathbb{Z})$. Spin$^\mathbb{C}$

structures are S^1 bundles over $P_{SO(r)}(E)$ which restrict to

$$S^1 \hookrightarrow \mathrm{Spin}^{\mathbb{C}}(r) \to SO(r),$$

as in 1.2.2. By (1.2.8) a $\mathrm{Spin}^{\mathbb{C}}$ structure exists if and only if $W_3(E)$ is 0, and in that case the set of $\mathrm{Spin}^{\mathbb{C}}$ structures has a free transitive action of $H^2(M;\mathbb{Z})$. The $\mathrm{Spin}^{\mathbb{C}}$ story is very much analogous to the one for Spin structures.

Let us consider a few examples of Spin and $\mathrm{Spin}^{\mathbb{C}}$ structures.

Example 1.2.3. Let $M = S^1$. There are two Spin structures on $E = TS^1$, the tangent bundle of S^1. $SO(1) = \{1\}$ and $\mathrm{Spin}(1) = \mathbb{Z}_2$. One Spin structure has $P_{\mathrm{Spin}(1)}(TS^1) = \mathbb{Z}_2 \times S^1 \to P_{SO(1)}(TS^1) = \{1\} \times S^1$ with the map being the only map $\mathbb{Z}_2 \to \{1\}$. The other Spin structure is given by the commutative diagram 1.2.3 where $\tilde{\pi}$ and the projection from $P_{\mathrm{Spin}(1)}(TS^1)$ to S^1 are given

$$\begin{tikzcd}
 & Spin(1) \cong \mathbb{Z}_2 \arrow[rr, "\pi"] \arrow[dd] & & SO(1) \cong \{1\} \arrow[dd] \\
\mathbb{Z}_2 \arrow[ur] \arrow[dr] & & & \\
 & P_{Spin(1)}(TS^1) \cong S^1 \arrow[rr, "\tilde{\pi}"] \arrow[dr] & & P_{SO(r)}(TS^1) \cong S^1 \arrow[dl] \\
 & & S^1 &
\end{tikzcd}$$

Figure 1.2.3

by $z \mapsto z^2$ (where $S^1 = \{\, z \in \mathbb{C} \mid |z| = 1 \,\}$).

Example 1.2.4. Let $M = \mathbb{CP}^2$. Then $c_1(TM) = 3x$, where x is the generator of $H^2(\mathbb{CP}^2; \mathbb{Z})$ which is Poincaré dual to $[\mathbb{CP}^1]$ linearly embedded in $\mathbb{CP}^2$. Hence, $w_2(TM) = 1 \in \mathbb{Z}_2 \cong H^2(M, \mathbb{Z}_2)$, since w_2 is the mod-2 reduction of c_1, and so $\mathbb{CP}^2$ is not spinnable.

Example 1.2.5. Let M^{2n} be any $2n$-dimensional almost complex manifold, with almost complex structure J. Equip M with a metric g, compatible with J. We can then view TM as a bundle with structure group $U(n)$. Now, in the diagram the dotted arrow defines a unique homomorphism since $SU(n)$ is

$$\begin{tikzcd}
& Spin(2n) \arrow[d] \\
SU(n) \arrow[ur, dotted] \arrow[r, hook] & SO(2n)
\end{tikzcd}$$

Figure 1.2.4

simply connected. Hence, we also have a homomorphism

$$U(n) \cong SU(n) \times_{\mathbb{Z}_2} S^1 \to Spin(2n) \times_{\mathbb{Z}_2} S^1 \cong \mathrm{Spin}^{\mathbb{C}}(2n).$$

This is a somewhat highfalutin way to see that a homotopy class of almost complex structures on a manifold canonically induces a $\mathrm{Spin}^{\mathbb{C}}$ structure.

Example 1.2.6. Let M be a spinnable manifold. Then to every Spin-structure $\mathfrak{s}$ there is an associated $\mathrm{Spin}^{\mathbb{C}}$-structure.

We end this introduction with the following

Proposition 1.2.7. *Every 4-manifold admits a* $\mathrm{Spin}^{\mathbb{C}}$ *structure.*

See [Mor96, Lemma 3.1.2] for a proof.

1.2.2 Clifford bundles and the Dirac operator

Dirac introduced, what we now call, the Dirac operator in [Dir28], to explain the splitting of the energy levels of the electron in the hydrogen atom, by introducing a relativistic equation to replace the Schrödinger equation. The latter cannot possibly be Lorentz invariant, since the Schrödinger equation is first order in the time variable but second order in the spatial variables. This led Dirac to search for a first order differential operator which squares to the Laplacian.

Such an operator must have matrix coefficients and act on vector-valued functions. For example, let $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ $\sigma_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, be the Pauli matrices, which satisfy $\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}I$, where I is the identity matrix. Then the operator $D = \sum_{i=1}^{3} \sigma_i\partial_i$, acting on functions $\psi : \mathbb{R}^3 \to \mathbb{C}^2$ satisfies $D^2 = \Delta$. The operator that Dirac wrote down in [Dir28] is $\sum_{\mu=1}^{4} \gamma_\mu\partial_\mu$, where γ_μ are 4 by 4 matrices, satisfying, $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}I$. This latter relation is, of course, the Clifford relation, and is the starting point for the following definitions and constructions leading to the generalization of the Dirac operator to a differential operator acting on Clifford bundles.

1.2.3 Weitzenböck Formulas

Since the square of the Dirac operator, $\mathcal{D}^*\mathcal{D}$ which we can call the *Dirac laplacian* has the same principal symbol as the *connection laplacian* $\nabla^*\nabla$ the difference of the two is a differential operator of order at most one. Remarkably, this difference is in fact a zeroth order operator, expressible in terms of the curvature. Formulas that express the difference between two second order differential operators that both have the same principal symbol as the laplacian are usually called *Weitzenböck Formulas*. A concise exposition of this can be found in [Tay11b, Section 10.4], [Roe98, Chapter 3], [Bou81a] or [LM89, Chapter 2 §8]. Less classical but nevertheless helpful expositions can also be found in [Pet, Lab07].

Indeed, let M be a Riemannian manifold, $E \to M$ be a Hermitian vector bundle, with a metric connection, and also a Clifford module over TM, as in the previous section. Let $\mathcal{D}$ be the Dirac operator on E, $\phi \in \Gamma(E)$, and $f \in C^\infty(M)$. We can directly compute

$$\mathcal{D}^* \mathcal{D}(f\phi) = f \mathcal{D}^* \mathcal{D} \phi - 2\nabla_{\operatorname{grad} f}\phi - (\Delta f)\phi.$$

But we can also compute ∇^*. Let $\alpha \in \Omega^1(M)$, then

$$\nabla^* f(\alpha \otimes \phi) = f\nabla^*(\alpha \otimes \phi) - \langle df, \alpha\rangle \phi$$

and if a vector field $V \in \Gamma(TM)$ is defined by $\langle V, W\rangle = \alpha(W)$, then

$$\nabla^*(\alpha \otimes \phi) = -\nabla_V \phi - (\operatorname{div} V)\phi.$$

Putting these together we see that

$$(\mathcal{D}^* \mathcal{D} - \nabla^*\nabla)(f\phi) = f(\mathcal{D}^* \mathcal{D} - \nabla^*\nabla)\phi.$$

This is a very low-brow way to see that in fact, the difference between the Dirac laplacian and the connection laplacian is a zeroth order operator. To find an expression for this difference in terms of the curvature, one must perform a more detailed computation, which can also be found in [Tay11b] or [Roe98]. The result is the following formula due to Lichnerowicz [Lic63]

$$\mathcal{D}^* \mathcal{D} = \nabla^* \nabla + F^E + \frac{1}{4}\,\mathrm{scal} \tag{1.2.9}$$

where F^E is an endomorphism of E, which is the *Clifford contraction of the twisting curvature of* E.

We can compute F^E for concrete Clifford modules. The resulting formulas are called Weitzenböck or Weitzenböck-Lichnerowicz formulas. For the Spin-Dirac operator, where $\Phi \in \Gamma(S)$ a spinor, we obtain

$$\mathcal{D}^* \mathcal{D}\, \Phi = \nabla^* \nabla \Phi + \frac{s}{4}\Phi. \tag{1.2.10}$$

While for the Spin$^{\mathbb{C}}$-Dirac operator we obtain

$$\mathcal{D}_A^* \mathcal{D}_A\, \Phi = \nabla_A^* \nabla_A \Phi + \frac{1}{2} F_A^+ \cdot \Phi + \frac{s}{4}\Phi \tag{1.2.11}$$

where, of course, $F_A^+ \cdot \Phi$ denotes Clifford multiplication, and A is a connection on the determinant line bundle of the Spin$^{\mathbb{C}}$ structure. On a closed manifold, we can the integrate by parts to obtain

$$\int_X |\mathcal{D}_A \Phi|^2 \; d\mathrm{vol}_g = \int_X |\nabla_A \Phi|^2 + \frac{1}{2}\int_X \left\langle F_A^+ \cdot \Phi, \Phi \right\rangle \, d\mathrm{vol}_g + \frac{1}{4}\int_X s\, |\Phi|^2 \; d\mathrm{vol}_g. \tag{1.2.12}$$

There are also classic formulas for the Hodge Laplacian. On 1-forms we have the classic Bochner formula [Boc46] for $\alpha \in \Omega^1$

$$(dd^* + d^*d)\alpha = \nabla^*\nabla\alpha + \mathrm{Ric}(\alpha, \cdot). \tag{1.2.13}$$

For two forms $\eta \in \Omega^2$, we have [Bou81a, Bou81b]

$$(dd^* + d^*d)\eta = \nabla^*\nabla\eta - 2W_+(\eta, \cdot) + \frac{s}{3}\eta. \tag{1.2.14}$$

1.2.4 Differential Geometry in dimension 4

Geometry in dimension 4 enjoys several interesting properties, which we briefly describe below. The exceptional geometry of dimension 4 can be ascribed to the fact that the Lie Algebra of $SO(4)$ is the direct sum of two copies of the Lie Algebra of $SO(3)$, $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. We can see this isomorphism by explicitly describing $SU(2) \times SU(2)$ as the double cover of $SO(4)$, and $SU(2)$ as the double cover of $SO(3)$. In other words, $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$ and $\mathrm{Spin}(3) \cong SU(2)$.

To describe these covering maps, we note that if i, j and k are the unit quaternions, then a quaternion $q = a + bi + cj + dk$ can be written as $z + wj$, where $z = a + bi$ and $w = c + di$ are complex numbers. In fact, $q \mapsto \left(\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right)$ is an isomorphism of the algebra of quaternions $\mathbb{H}$ with the subalgebra of 2 by 2 complex matrices of the form $\left(\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right)$. Under this isomorphism, $\|q\|^2 = \det\left(\begin{smallmatrix} z & w \\ -\bar{w} & \bar{z} \end{smallmatrix}\right)$. Then, the unit quaternions are identified with $SU(2)$. The unit quaternions, and hence $SU(2)$ acts on $\mathbb{H}$ by conjugation: for $q \in \mathbb{H}$, $v \in \mathbb{H}$, $A_q(v) = q \cdot v \cdot q^{-1}$. Now, the reals $\mathbb{R} \subset \mathbb{H}$ are left invariant under this action, and the action preserves the quaternionic norm. So the action by conjugation preserves the orthogonal complement to the reals, which is the 3-dimensional space of purely imaginary quaternions, which we temporarily denote by $\mathbb{R}^3_{i,j,k} = \{ai + bj + ck \mid a, b, c \in \mathbb{R}\}$. Then for $q \in SU(2)$, thought of as a unit quaternion, $q \mapsto A_q$ defines a 3-dimensional representation $SU(2) \to SO(\mathbb{R}^3_{i,j,k})$. To see that this is the spin representation, we just need to check that the kernel is $q = \pm 1$.

Indeed, if $q \in SU(2)$ and $A_q(v) = v$ for $v \in \mathbb{R}^3_{i,j,k}$, then q commutes with all the imaginary unit quaternions, and thus, commutes with all quaternions. But the center of the algebra of quaternions is $\mathbb{R} \subset \mathbb{H}$, so $q \in \mathbb{R}$. Further, since $\|q\| = 1$, $q = \pm 1$ as required. So the action of the unit quaternions on the purely imaginary quaternions by conjugation, gives rise to the spin representation of $SU(2)$, and exhibits $SU(2)$ as the double cover (and hence, the universal cover) of $SO(3)$.

Similarly, if q_1 and q_2 are two unit quaternions, $q_1, q_2 \in \mathbb{H}$, $\|q_1\| = \|q_2\| = 1$, and $v \in \mathbb{H}$, then $(q_1, q_2) \mapsto A_{q_1,q_2}$ where $A_{q_1,q_2}(v) = q_1 \cdot v \cdot q_2^{-1}$ defines an orthogonal representation of $SU(2) \times SU(2)$. The kernel is $\pm(1,1)$, since if $q_1 \cdot v \cdot q_2^{-1} = v$ for any $v \in \mathbb{H}$, then in particular, this holds for $v = 1$, so $q_1 = q_2$. Then as above, $q_1 = \pm 1$, since q_1 commutes with all quaternions, so is in the center of $\mathbb{H}$ and has norm 1. Hence, $SU(2) \times SU(2)$ is the double cover of $SO(4)$.

1.3 Seiberg-Witten equations

In [Wit94], drawing on previous joint work with Nathan Seiberg [SW94], Edward Witten introduced the remarkable Seiberg-Witten equations. These equations were immediately used to prove heretofore inaccessible results in 4-manifold topology, notably the Thom conjecture [KM94]. In this section we can now write down the Seiberg-Witten equations and study their solutions. Plenty of references are now available for this material: the classic is [Mor96], but [Moo01] may be a bit more digestible for the beginner, and [Sal99] is very long and comprehensive; finally, [Nic00] (at least the electronic version) contains topics that are not otherwise available in book form, in particular, a discussion of gluing formulas in general, and the blow-up formula in particular. We use the latter in Section 2.5.4 to show non-existence of Einstein metrics on blow-ups.

1.3.1 The equations and basic properties

We can now write down the Seiberg-Witten equations on a 4-manifold X equipped with a $\mathrm{Spin}^{\mathbb{C}}$ structure.

$$\mathcal{D}_A \Phi = 0 \tag{1.3.1}$$

$$F_A^+ = (\Phi \otimes \Phi^*)_0 \tag{1.3.2}$$

In the second equation we are implicitly using the isomorphism between complexified self-dual two forms and traceless endomorphisms of the positive spinor bundle, $\Omega^2_+ \otimes \mathbb{C} \cong \mathrm{End}(S^+)$. $(\Phi \otimes \Phi^*)_0 = \Phi \otimes \Phi^* - \frac{1}{2}|\Phi| \,\mathrm{Id}$ is the traceless part of the rank one endomorphism $\Phi \otimes \Phi^*$. Note, that under this isomorphism, $(\Phi \otimes \Phi^*)_0$ corresponds to a purely imaginary self-dual two form.

It is also necessary to consider the perturbed Seiberg-Witten equations

$$\mathcal{D}_A \Phi = 0 \tag{1.3.3}$$

$$F_A^+ - (\Phi \otimes \Phi^*)_0 = \varpi^+ \tag{1.3.4}$$

where ϖ^+ is a self-dual two form.

We can define the Seiberg-Witten map as

$$SW : \mathcal{A} \times \Gamma(S^+) \to \Omega^2_+ \times \Gamma(S^-) \tag{1.3.5}$$

$$SW(A, \Phi) = \left(F_A^+ - (\Phi \otimes \Phi^*)_0, \mathcal{D}_A \Phi\right) \tag{1.3.6}$$

It is important to note that these equations have an infinite dimensional symmetry group, namely, the group of automorphisms of the principal $\mathrm{Spin}^{\mathbb{C}}$-bundle $P^{\mathbb{C}}_{\mathrm{Spin}}(TM)$ that cover the identity of the frame bundle $P_{SO}(TM)$ under $\tilde{\pi}$, in the commutative diagram 1.2.1. An automorphism of a $\mathrm{Spin}^{\mathbb{C}}$-principal bundle is given by a smooth map $u : M \to \mathrm{Spin}^{\mathbb{C}}$. If an automorphism given by u covers the identity on $P_{SO}(TM)$, then $\pi \circ u : M \to \mathrm{Spin}^{\mathbb{C}}(4) \to SO(4)$ is the identity, and so $u : M \to U(1)$. Now given a change of gauge $u : M \to U(1)$, (A, Φ) transforms to

$$u \cdot (A, \Phi) = (A - u^{-1} du, u\Phi). \tag{1.3.7}$$

The power of these equations comes from the gained control on the F_A^+ term in the Weitzenböck formula (1.2.11). Substituting $F_A^+ \cdot \Phi = (\Phi \otimes \Phi^*)_0 \Phi =$

$\frac{1}{2} |\Phi|^2 \Phi$ into (1.2.11) we obtain

$$D_A^* D_A \Phi = \nabla_A^* \nabla \Phi + \frac{s}{4} \Phi + \frac{1}{4} |\Phi|^2 \Phi. \tag{1.3.8}$$

Note that

$$\langle \nabla_A^* \nabla_A \Phi, \Phi \rangle = \frac{1}{2} \Delta |\Phi|^2 + |\nabla_A \Phi|^2 . \tag{1.3.9}$$

So taking the inner product with Φ in (1.3.8), we obtain for a solution of the Seiberg-Witten equations (1.3.1)

$$0 = 2\Delta |\Phi| + 4 |\nabla_A \Phi|^2 + s |\Phi|^2 + |\Phi|^4 \tag{1.3.10}$$

Let X^4 be a 4-dimensional manifold with, possibly non-empty, boundary $Y^3 = \partial X^4$, and a Riemannian metric g. Let $(A, \Phi) \in \mathcal{A} \times \Gamma(S^+)$. Following Kronheimer and Mrowka, [KM07, Definition 4.5.4, page 96], we define the topological energy $\mathcal{E}^{\text{top}}(A, \Phi)$ and the analytical energy $\mathcal{E}^{\text{an}}(A, \Phi)$

$$\begin{aligned} \mathcal{E}^{\text{an}}(A, \Phi) = \frac{1}{4} \int_X |F_A|^2 \, d\text{vol}_g + \int_X |\nabla_A \Phi|^2 \, d\text{vol}_g + \\ \frac{1}{4} \int_X \left(|\Phi|^2 + \frac{s}{2} \right)^2 d\text{vol}_g - \int_X \frac{s^2}{16} \, d\text{vol}_g \end{aligned} \tag{1.3.11}$$

$$\begin{aligned} \mathcal{E}^{\text{top}}(A, \Phi) = \frac{1}{4} \int_X F_A \wedge F_A - \int_Y \langle \Phi, \mathcal{D}_B \, \Phi \rangle \, d\text{vol}_{g|_Y} + \\ \frac{1}{2} \int_Y H |\Phi|^2 \, d\text{vol}_{g|_Y} . \end{aligned} \tag{1.3.12}$$

Note that if X is a compact 4-manifold without boundary, then

$$\mathcal{E}^{\text{top}}(A, \Phi) = \int_X F_A \wedge F_A = -\pi^2 c_1^2(S^+)[X]$$

is a topological invariant, by Chern-Weil theory.

With these definitions integration by parts yields

$$\mathcal{E}^{\mathrm{an}}(A,\Phi) = \mathcal{E}^{\mathrm{top}}(A,\Phi) + \|SW(A,\Phi)\|^2 . \tag{1.3.13}$$

In particular, for any $(A,\Phi) \in \mathcal{A} \times \Gamma(S^+)$, we have $\mathcal{E}^{\mathrm{an}}(A,\Phi) \geq \mathcal{E}^{\mathrm{top}}(A,\Phi)$, with equality if and only if (A,Φ) is a solution of the Seiberg-Witten equations (1.3.1).

It is interesting to note that it follows from (1.3.13) that solutions of the Seiberg-Witten equations are absolute minima of the analytic energy functional, $\mathcal{E}^{\mathrm{an}}(A,\Phi)$. The Euler-Lagrange equations for this functional are second order, and can be found in [JPW96] or [Jos11, Section 9.2]. But the Seiberg-Witten equations are first order.

This is typical in gauge theory. For example, a similar phenomenon occurs in Yang-Mills theory. Take E to be a $SU(2)$ bundle over a 4-manifold X, and let A be a connection on E. The Euler-Lagrange equation of the Yang-Mills energy

$$\mathcal{YM}(A) = \int_X |F_A|^2 \, d\mathrm{vol} = \int_X F_A \wedge *F_A \tag{1.3.14}$$

is $d * F_A = 0$. This is a second order equation in A. But the anti-self dual equation $F_A^+ = 0$ is a first order equation for the absolute minima of $\mathcal{YM}(A)$. This follows by an elementary calculation:

$$\begin{aligned}\int_X |F_A|^2 \, d\mathrm{vol} = \int_X \left|F_A^+\right|^2 \, d\mathrm{vol} + \int_X \left|F_A^-\right|^2 \, d\mathrm{vol} = \\ 2\int_X \left|F_A^+\right|^2 \, d\mathrm{vol} - \int_X F_A \wedge F_A = 2\int_X \left|F_A^+\right|^2 \, d\mathrm{vol} - 8\pi^2 c_2(E).\end{aligned}$$

See [DK90] for an account of the anti-self dual equation, and the four manifold invariants that arise from it.

To proceed we prove a gauge-fixing lemma. This choice of gauge is known as *Coulomb gauge*.

Lemma 1.3.1. *Let A_0 be a C^∞ connection on some Hermitian line bundle on X. Then for any L^2_k connection A, there is a L^2_{k+1} change of gauge $u : X \to S^1$, such that $u \cdot A = A_0 + \alpha$, where $\alpha \in L^2_l(T^*X)$ and $d^*\alpha = 0$.*

Proof. By (1.3.7), $u \cdot A = A - u^{-1}du$, we want to solve $d^*(A - A_0 - u^{-1}du) = 0$ for u. Or in other words, $d^*(u^{-1}du) = d^*(A - A_0) \in L^2_{k-1}$. Now by Hodge Theory, there is a solution $f \in L^2_{k+1}$ to $d^*df = d^*(A - A_0)$ (recall the difference between two connections on a Hermitian line bundle is a purely imaginary one-form, so f is a purely imaginary function). Let $u = e^f$. Then

$$d^*(u^{-1}du) = d^*(e^{-f}de^f) = d^*df = d^*(A - A_0)$$

as desired. So the existence of Coulomb gauge is just a consequence of the solvability of the Poisson equation with right-hand side orthogonal to the locally constant functions. □

Lemma 1.3.2. *Let A_0, A, α and u be as in the previous lemma. There exists a constant C such that we can find a further change of gauge $v : X \to S^1$ such that $v \cdot A = v \cdot (A_0 + \alpha) = A_0 + \alpha - v^{-1}dv = A_0 + \tilde{\alpha}$, with $d^*\tilde{\alpha} = 0$ and if we let γ denote the harmonic projection of $\tilde{\alpha}$, then $\|\gamma\|_{L^2_k} < C$.*

Proof. We can write $\alpha = \beta + \gamma$, where $d^*\beta = d^*\gamma = d\gamma = 0$ and β is orthogonal to the harmonic 1-forms. Now suppose $\gamma_0 \in \mathcal{H}^1(X)$ is a harmonic 1-form with periods in $2\pi\mathbb{Z}$. In other words, for any loop $l : S^1 \to X$, $\int_l \gamma_0 \in 2\pi\mathbb{Z}$. Then there exists a function $v_0 : X \to S^1$ such that $dv_0 = \gamma_0$. Indeed, after choosing a basepoint in $\tilde{X}$, the universal cover of X, integration of the pull-back of γ_0 to $\tilde{X}$ yields a map to $i\mathbb{R}$, $\tilde{v}_0 : \tilde{X} \to i\mathbb{R}$. Note that $d\tilde{v}_0 = \pi^*\gamma_0$, where $\pi : \tilde{X} \to X$. Since the periods of γ_0 are integer multiples of 2π, this map descends to a map $v_0 : X \to \mathbb{R}/2\pi\mathbb{Z} = S^1$, $v_0 = \exp(\tilde{v}_0)$. By construction, $v_0^{-1}dv_0 = \gamma_0$.

Since $\mathcal{H}^1(X)/\left\{\gamma_0 \in \mathcal{H}^1(X) \,|\, \text{periods of } \gamma \text{ are in } 2\pi\mathbb{Z}\right\}$ is a compact torus, we can write any $\gamma \in \mathcal{H}^1(X)$ as $\gamma = \gamma_0 + \gamma_1$ where γ_0 has periods in $2\pi\mathbb{Z}$ and $\|\gamma_1\|_{L^2_k(X)} < C$. So we can use v_0 constructed above to change gauge to remove the γ_0 term, and we are left with $\gamma = \gamma_1$, which has the requisite $L^2_k(X)$ bound. $\square$

Now we give an argument for the compactness of the space of solutions to the Seiberg-Witten equations based on an energy bound. This argument is different from the usual argument, which is based on a C^0-bound for the spinor, which is derived from the Weitzenböck formula, see [Mor96]. However, this argument is not at all original. For example, a version of this argument for a manifold with boundary can be found in [KM07]. Further, energy bounds and compactness, or the lack of compactness, in the case of bubbling, is one of the most important technical aspects of gauge theory.

The argument below establishes compactness for the space of solutions modulo gauge to the Seiberg-Witten equations on a closed 4-manifold. Since the argument is well known, we provide it here since it is a simpler version of the argument needed to prove Theorem 2.3.1. The main difference is that in the proof of Theorem 2.3.1, we must consider a sequence of metrics g_j, whereas in the current setting, the metric g is fixed.

Proposition 1.3.3. *Let $\mathfrak{s}$ be a* $\mathrm{Spin}^{\mathbb{C}}$ *structure, and A_0 a fixed smooth connection on* $\det(\mathfrak{s})$*. Let $(A_j, \Phi_j) \in \mathcal{A} \times \Gamma(S^+)$, and $SW(A, \Phi) = (\varpi_j^+, 0)$, for some perturbations $\varpi_j^+ \in \Omega_+^2$, with $\left\|\varpi_j^+\right\|_{L_1^2} < C$. And assume that $A_j = A_0 + \alpha_j$ which $\alpha_j \in L_2^2(\Omega^1(X))$ and $\Phi \in L_2^2(S^+)$. Then, there is a sequence of changes of gauge $u_j : X \to S^1$, $u_j \in L_3^2$, such that $u_j \cdot (A_j, \Phi_j)$ has a convergent subsequence.*

Note that the group of L_3^2 changes of gauge is an infinite dimensional Lie Group thanks to the Sobolev multiplication $L_3^2 \times L_3^2 \to L_3^2$, and it is thanks to the Sobolev multiplication $L_3^2 \times L_2^2 \to L_2^2$ that this Lie Group acts on our configurations, see A.3.

Before we proceed with the proof of the theorem, we need several Lemmas.

Lemma 1.3.4. *Let $\beta \in \Gamma(T^*X)$ be a 1-form on a closed Riemannian manifold (X, g). Suppose $\beta \perp \mathcal{H}^1(X, g)$. Then*

$$\|\beta\|_{L_1^2} \leq C \left(\|d\beta\|_{L^2} + \|d^*\beta\|_{L^2}\right). \tag{1.3.15}$$

Proof. We need to control the L^2-norm of the 1-form and its derivative. To achieve the latter, we note that the Bochner formula for the Hodge laplacian on 1-forms says

$$(dd^* + d^*d)\beta = \nabla^*\nabla\beta + \mathrm{Ric}_g(\beta, \cdot)$$

hence,

$$\begin{aligned}
\|\nabla\beta\|_{L^2}^2 &= \|d\beta\|_{L^2}^2 + \|d^*\beta\|_{L^2}^2 - \int_X \mathrm{Ric}_g(\beta,\beta)\, d\mathrm{vol}_g \\
\|\nabla\beta\|_{L^2}^2 &\leq \|d\beta\|_{L^2}^2 + \|d^*\beta\|_{L^2}^2 + C\,\|\beta\|_{L^2}^2 \qquad (1.3.16)
\end{aligned}$$

It remains to establish control of the L^2-norm of the 1-form itself. Arguing by contradiction, suppose that the right hand side of (1.3.15) does not control $\|\beta\|_{L^2}$. Then, there exists a sequence β_j of 1-forms, orthogonal to $\mathcal{H}^1$, with $\|\beta_j\|_{L^2} = 1$ and with $(\|d\beta_j\|_{L^2} + \|d^*\beta_j\|_{L^2}) \to 0$. But then (1.3.16) tells us that $\|\beta_j\|_{L^2_1} < C$ and so there is a subsequence β_{j_k} which weakly converges to β in L^2_1, and converges strongly in L^2. In particular, $\|\beta\|_{L^2} = 1$. But, weak convergence in L^2_1 implies that $d\beta = d^*\beta = 0$ so $\beta \in \mathcal{H}^1$. On the other hand, β is also orthogonal to $\mathcal{H}^1$, so $\beta = 0$, which contradicts $\|\beta\|_{L^2} = 1$. $\square$

Proof of Proposition 1.3.3. By equation (1.3.13) we have $\mathcal{E}^{\mathrm{an}}(A_j,\Phi_j) < C$ uniformly in j, and this immediately gives

$$\begin{aligned}
\left\|F_{A_j}\right\|_{L^2} &< C \\
\|\Phi_j\|_{L^4} &< C \qquad (1.3.17) \\
\left\|\nabla_{A_j}\Phi_j\right\|_{L^2} &< C \text{ all uniformly in } j.
\end{aligned}$$

By the gauge fixing Lemmas 1.3.1 and 1.3.2 we can assume $d^*\alpha_j = 0$, and if we write $\alpha_j = \beta_j + \gamma_j$ where β_j is orthogonal to $\mathcal{H}^1(X,g)$ and $\gamma_j \in \mathcal{H}^1(X,g)$, we can also assume that $\|\gamma_j\|_{L^2_1} < C$.

Then $F^+_{A_j} = F^+_{A_0} + d^+\beta_j$ and since A_0 is a fixed connection $\left\|F^+_{A_0}\right\|_{L^2} < C$

uniformly in j. Hence, (1.3.17) gives us $\|d^+\beta_j\|_{L^2} < C$. But

$$0 = \int_X d(\beta \wedge d\beta) = \left\|d^+\beta_j\right\|_{L^2} - \left\|d^-\beta_j\right\|_{L^2}.$$

Hence, we have $\|d\beta_j\|_{L^2} = \sqrt{2}\,\|d^+\beta_j\|_{L^2} < C$. Lemma 1.3.4 then gives us $\|\beta_j\|_{L^2_1} < C$. We can pass to a subsequence which converges weakly in L^2_1, $\beta_j \rightharpoonup \beta \in L^2_1$. Since $\|\gamma_j\|_{L^2_1} < C$, we can pass to a subsequence which converges weakly in L^2_1, so $\gamma_j \rightharpoonup \gamma$. And hence $\alpha_j = \beta_j + \gamma_j \rightharpoonup \beta + \beta = \alpha$ converges weakly in L^2_1. Let $A = A_0 + \alpha$, so A_j converge weakly in L^2_1 to A.

Now we establish convergence for the spinor. Equation (1.3.17) gives us an L^4 bound on the spinor and an L^2 bound on the covariant derivative of the spinor, but with a variable connection. We use the following extra argument to deal with this minor problem. $\nabla_{A_0}\Phi_j = \nabla_{A_j}\Phi_j - \alpha_j \otimes \Phi_j$, so

$$\begin{aligned}\|\nabla_{A_0}\Phi_j\|_{L^2} \leq \left\|\nabla_{A_j}\Phi_j\right\|_{L^2} + \|\alpha_j \otimes \Phi_j\|_{L^2} \leq \\ C + \|\alpha_j\|_{L^4}\|\Phi_j\|_{L^4} \leq C + C'\|\alpha_j\|_{L^2_1}\|\Phi_j\|_{L^4} \leq C''\end{aligned}$$

where the first inequality is the triangle inequality, the second inequality is (1.3.17) and the fact that the L^2-norm of the product is controlled by the product of the L^4 norms of the factors by Cauchy-Schwartz, and the third inequality is the Sobolev embedding $L^2_1 \hookrightarrow L^4$ in dimension 4 (see Theorem A.2). Hence, we have a subsequence Φ_j weakly convergent in L^2_1, $\Phi_j \rightharpoonup \Phi$.

After possibly passing to a further subsequence, we then have: F_{A_j}, $\nabla_{A_j}\Phi_j$ and $(\Phi_j \otimes \bar{\Phi}_j^*)_0$ converge weakly in L^2, while (A_j, Φ_j) and the perturbations $\varpi_j^+ \rightharpoonup \varpi^+$ converge weakly in L^2_1 and hence, strongly in L^2. Now, $\alpha \mapsto$

$F^+_{A_0+\alpha}$ is a continuous map from L^2_1 to L^2, and weak convergence is preserved under continuous maps, hence the weak limit of $F^+_{A_j}$ is F^+_A. To proceed, we need to show that the weak limit of $\nabla_{A_j}\Phi_j$ is $\nabla_A\Phi$ and the weak limit of $(\Phi_j \otimes \bar{\Phi}^*_j)_0$ is $(\Phi \otimes \bar{\Phi}^*)_0$. Now for the former, note that $A_j = A_0 + \alpha_j$ so $\nabla_{A_j}\Phi_j = \nabla_{A_0}\Phi_j + \alpha_j \otimes \Phi_j$. Since Φ_j converges weakly in L^2_1 to Φ, $\nabla_{A_0}\Phi_j$ converges weakly in L^2 to $\nabla_{A_0}\Phi$. Further, since α_j and Φ_j converge to α and Φ respectively strongly in L^2, $\alpha_j \otimes \Phi_j$ converges strongly in L^1 to $\alpha \otimes \Phi$. Similarly, $(\Phi_j \otimes \bar{\Phi}^*_j)_0$ converges strongly to $(\Phi \otimes \bar{\Phi}^*)_0$ in L^1 since Φ_j converges strongly to Φ in L^2. Hence, we can conclude that (A, Φ) solves the Seiberg-Witten equations, $SW(A, \Phi) = (\varpi^+, 0)$. □

Proposition 1.3.3 says that the set of solutions to the Seiberg-Witten equations modulo change of gauge is compact. This feature makes Seiberg-Witten theory considerably simpler than Donaldson's theory or the theory of pseudo-holomorphic curves, where solutions can "degenerate" or "bubble," and this phenomenon must be understood.

There is actually an L^∞ bound for Φ, if (A, Φ) solves the unperturbed Seiberg-Witten equations.

Proposition 1.3.5. *Suppose $SW(A, \Phi) = 0$. Then $\|\Phi\|_{L^\infty} \leq \max(-s)$, where s is the scalar curvature.*

Proof. The proof is based on the Weitzenböck formula (1.3.10). Indeed, let p be a point where $|\Phi|$ attains its maximum, then $\Delta\,|\Phi|_p \leq 0$, so by (1.3.10) $s\,|\Phi|^2 + |\Phi|^4 \leq 0$ and hence, if $|\Phi|_p \neq 0$, $|\Phi|^2 \leq -s \leq \max(-s_-)$, where $f_- = \min(f, 0)$, for any function f. □

1.3.2 The Seiberg-Witten equations on Kähler manifolds

On a compact, Kähler manifold, we can directly find the solutions to (1.3.1); in fact, this computation was done in Section 4 of Witten's original paper [Wit94], and [Mor96, Chapter 7] has an excellent discussion, which we follow here.

$$S^+ \cong \Omega^0(X;\mathbb{C}) \oplus \Omega^{0,2}(X;\mathbb{C})$$
$$S^- \cong \Omega^{0,1}(X;\mathbb{C})$$
$$\mathcal{D} = \sqrt{2}\left(\bar{\partial} + \bar{\partial}^*\right)$$

where the Dirac operator is for the $\mathrm{Spin}^{\mathbb{C}}$-structure associated to the complex structure and the Chern connection on its determinant line bundle, K_X^{-1}. $\Phi = (\alpha,\beta) \in \Omega^0(X;\mathbb{C}) \oplus \Omega^{0,2}(X;\mathbb{C})$.

$$\bar{\partial}_{A_0}\alpha + \bar{\partial}^*\beta = 0 \tag{1.3.18}$$
$$(F_A)^{1,1} = \frac{i}{4}(|\alpha|^2 - |\beta|^2)\omega \tag{1.3.19}$$
$$F_A^{0,2} = \frac{\bar{\alpha}\beta}{2} \tag{1.3.20}$$

Theorem 1.3.6. *The Seiberg-Witten invariant of a Kähler manifold is non-zero for the canonical* $\mathrm{Spin}^{\mathbb{C}}$ *structure with determinant line bundle the anticanonical bundle. Therefore, for that* $\mathrm{Spin}^{\mathbb{C}}$ *structure, the Seiberg-Witten equations have a solution on a Kähler manifold with respect to any metric and a generic perturbation.*

Chapter 2

Quasiprojective manifolds

Let $\bar{X}$ be a smooth, projective manifold, $D \subset \bar{X}$ a smooth divisor, and L_D its associated line bundle. Let $K_{\bar{X}}$ be the canonical bundle and let $X = \bar{X} \setminus D$. Assume that $K_{\bar{X}} \otimes L_D$ is ample. In [Kob84] it is shown that X admits a unique, up to scaling, complete Kähler-Einstein metric g_{KE} with negative Ricci curvature and finite volume.

2.1 Cheng-Yau Hölder Spaces

The proof in [Kob84] (also see [TY87] proceeds by using the continuity method in the Cheng-Yau Hölder spaces, introduced in [CY80] by Cheng and Yau to prove the existence of complete Kähler-Einstein metrics on bounded, strictly pseudo-convex domains in $\mathbb{C}^n$.

To define the Cheng-Yau Hölder spaces we it is first necessary to introduce "quasi-coordinates." The model is again the Poincaré punctured disk $0 < |z| < 1$

with the form $\frac{dz\wedge d\bar{z}}{|z|^2\ln^2|z|}$. For any $0<\eta<1$ let $z=\phi_\eta(w)=\exp\left(\frac{\eta+1}{\eta-1}\frac{w+1}{w-1}\right)$ and

$$\frac{dz\wedge d\bar{z}}{|z|^2\ln^2|z|}=\frac{dw\wedge d\bar{w}}{\left(1-|w|^2\right)^2}.$$

The punctured disk is covered by (infinitely many) disks in the w plane $B_{1/2}(0)=\{w\mid |w|<1/2\}$, and the point is that the Laplacian is uniformly elliptic on each of the disks $B_{1/2}(0)$ in the w-plane.

In the general case, there is a collection of quasi-coordinates, $V_\eta\subset\mathbb{C}$, $\Phi_\eta: V_\eta\times B_1(0)\to X$, and the Cheng-Yau Hölder spaces $C^{k,\lambda}(X)$ are defined as the usual Hölder spaces but with respect to these quasi-coordinates, see [Kob84] Lemma 2.

The construction of the Kähler-Einstein metric proceeds by the continuity method starting with the Carlson-Griffiths metric

$$\omega_{\mathrm{CG}}=\omega_0-\partial\bar{\partial}\log(-\log\|\sigma\|^2)$$

2.2 Approximating metrics

We construct a sequence of metrics g_t on $\bar{X}$ such that $g_t\to g_{\mathrm{KE}}$ on compact subsets of X as $t\to\infty$ and g_t have uniformly bounded curvature, and finite volume. This is Proposition 2.2.2. Similarly, in Proposition 2.2.3, for any metric g on X, which is close to g_{KE} at infinity in C^2, we construct a sequence $\tilde{g}_t$ of metrics on $\bar{X}$ such that $\tilde{g}_t\to g$ and also have uniformly bounded curvature and finite volume.

The model for the Kähler-Einstein metrics constructed on X in [Kob84] is the Poincare metric $g_{\mathbb{H}} = \frac{dz\,d\bar{z}}{|z|^2(\log|z|)^2}$ on the punctured open disk $D = \{z\,|\,|z| < 1, z \neq 0\}$. As motivation for the construction in Proposition 2.2.2, we construct a sequence of metrics g_t on the unpunctured disk, which converge to $g_{\mathbb{H}}$ on compact subsets of the punctured open disk, have uniformly bounded scalar curvature, and the volume of a compact subset of the unpunctured open disk is bounded independent of t. We do this by smoothly transitioning the metric $g_{\mathbb{H}}$ to the flat metric on an annulus of appropriate radius for $1/t^2 < |z| < 1/t$. Then we cap off the flat annulus with an appropriate "standard" cap.

Let $g_{E,R} = R^2 \frac{1}{|z|^2} dz\,d\bar{z}$ be the flat metric on the punctured plane $\mathbb{R}^2 - \{(0,0)\}$. Let S^1_R be a circle of radius R. $F : (\mathbb{R}^2 - \{(0,0)\}, g_{E,R}) \to (S^1_R \times \mathbb{R}, d\theta^2 + du^2)$, $F(z) = (R \arg z, -R \ln|z|)$ is an isometry. Let $\chi(x)$ be a smooth function which satisfies $\chi(x) = 1$ for $0 \leq x \leq 1$, $\chi(x) = 0$ for $2 \leq x$ and is monotonically decreasing, with $\chi'(x)$ and $\chi''(x)$ bounded. Finally, let

$$\begin{aligned}\widetilde{g}_t = (1 - \chi(-\ln|z|/\ln t))\, g_{E,1/\ln t} + \chi(-\ln|z|/\ln t)\, g_{\mathbb{H}} =& \\ \frac{1 - \chi(-\ln|z|/\ln t)}{|z|^2(\ln t)^2} dz\,d\bar{z} + \frac{\chi(-\ln|z|/\ln t)}{|z|^2(\ln|z|)^2} dz\,d\bar{z} =& \\ \frac{1}{r^2}\frac{(1-\chi)\ln^2 r + \chi \ln^2 t}{\ln^2 t \ln^2 r} dz\,d\bar{z} = f^2 dz\,d\bar{z}&\end{aligned} \tag{2.2.1}$$

where in the last line we set $r = |z|$, $\chi = \chi(-\ln|z|/\ln t)$, and the coefficient in front of $dz\,d\bar{z}$ is denoted by f^2 for convenience. $\widetilde{g}_t$ is equal to g_H for $|z| > 1/t$ and $\widetilde{g}_t$ is equal to $g_{E,1/\ln t}$ for $|z| < 1/t^2$, and there is a smooth transition region, $R_t = \{z\,|\,1/t^2 < |z| < 1/t\}$. Note that the width, i.e. the distance between boundary components, of R_t with respect to $g_{\mathbb{H}}$ is $\ln 2$, independent of t, and the width of R_t with respect to $g_{E,1/\ln t}$ is 1, also independent of t. The volume form of the metric $f^2 dz\,d\bar{z}$ is $\frac{i}{2} f^2 dz \wedge d\bar{z}$, so the volume (or one might say the area) of R_t with respect to $g_{\mathbb{H}}$ is $\pi/\ln t$ and with respect to $g_{E,1/\ln t}$ is $2\pi/\ln t$.

The scalar curvature of a metric $f^2 dz\, d\bar{z}$ which is twice the Gaussian curvature, is $(8/f^2)\,\partial^2_{z\bar{z}}\,(\ln f)$. So we compute the scalar curvature of $\widetilde{g}_t$ to obtain

$$
\begin{aligned}
s = &-\frac{2}{f^2}\left(\frac{1}{r}\partial_r\left(r\partial_r \ln f\right) + \frac{1}{r^2}\partial^2_{\theta\theta}\ln f\right) = \\
&- \frac{\ln^2 r \ln^2 t}{\left((1-\chi)\ln^2 r + \chi \ln^2 t\right)^2}\left(4\rho\chi' - \rho^2\chi'' + 2(1-\chi) + \chi''\right) \\
&- \frac{\ln^2 r \ln^2 t}{\left((1-\chi)\ln^2 r + \chi \ln^2 t\right)^3}\ln^2 r\left(\rho\chi' + 2(1-\chi) - \chi'/\rho\right)^2 \\
&- \frac{2\ln^2 t}{(1-\chi)\ln^2 r + \chi\ln^2 t}
\end{aligned}
\tag{2.2.2}
$$

where we denote $\ln r/\ln t$ by ρ for brevity, and where χ, χ' and χ'' are all evaluated at $-\rho$. Note that for $z \in R_t$, $-2 < \rho < -1$. Now let $0 < a \leq b$, then $h(\chi) = \frac{ab}{\left((1-\chi)a+\chi b\right)^2}$ is a monotonically decreasing function of χ, which decreases from b/a to a/b as χ varies from 0 to 1. Hence, by examining (2.2.2) we see that s is bounded on R_t.

Now to complete the construction, we construct a metric $\widetilde{\widetilde{g}}_t$ on the closed unit disk. Let $\widetilde{\widetilde{g}}_t = g_{E,1/\ln t}$ for $1/2 < |z| < 1$. Let $\varphi : \{z \,|\, 1/4 < z < 1/2\} \to \{z \,|\, 1/t^2 < |z| < 1/2\}$ be a diffeomorphism by scaling radially, and let $\widetilde{\widetilde{g}}_t = \varphi^* \widetilde{g}_t$. Note that for $|z| = 1/4$, $\widetilde{\widetilde{g}}_t(z) = g_{\mathbb{H}}(1/2)$. Finally, let $\widetilde{\widetilde{g}}$ be an arbitrary metric on $\{z \,|\, |z| \leq 1/4\}$ that matches $g_{\mathbb{H}}(1/2)$ smoothly on the boundary.

The metrics $\widetilde{g}_t$ and $\widetilde{\widetilde{g}}_t$ glue smoothly on the boundary, since both are equal to $g_{E,1/\ln t}$ in an open collar around the boundary, yielding a metric g_t on the unpunctured unit disk. By the above computation, $\mathrm{scal}(g_t)$ is uniformly bounded, and since $g_t(z) = g_{\mathbb{H}}(z)$ for $|z| > 1/t$, $g_t \to g_{\mathbb{H}}$ on compact subsets of D as $t \to \infty$. We also have $\operatorname{vol}(R_t)_{\widetilde{g}_t} \leq \operatorname{vol}(R_t)_{g_{E,1/\ln t}} + \operatorname{vol}(R_t)_{g_{\mathbb{H}}} = 3\pi/\ln t$. Hence, for K a compact subset of the open, unpunctured disk, $\operatorname{vol}(K)_{g_t}$ is uniformly bounded.

Now suppose g is any Riemannian metric on D, which approaches $g_{\mathbb{H}}$ in C^2 as $z \to 0$. Let

$$h_t = \chi\left(-\ln|z|/\ln t\right) g + \big(1 - \chi\left(-\ln|z|/\ln t\right)\big) g_{\mathbb{H}} \tag{2.2.3}$$

This transitions g to $g_{\mathbb{H}}$ over the region $R_t = \{z \mid 1/t^2 < |z| < 1/t\}$ as before. $\operatorname{scal} h_t$ is uniformly bounded on R_t, and $\operatorname{vol}(R_t)_{h_t}$ is uniformly bounded as well.

This motivates a similar construction in the quasiprojective case. The construction is possible, thanks to the asymptotics of g_{KE} found in [Sch98]. For the convenience of the reader, we quote the asymptotics from [Sch98, Sch02]:

Theorem 2.2.1 (Schumacher 1998)**.** *Let σ be a suitably chosen section of L_D, defining D, and $\|\cdot\|$ a suitably chosen Hermitian metric on L_D. In suitable local coordinates (σ, z), ω_{KE}, the Kähler form associated to g_{KE} has the following asymptotics:*

$$\begin{aligned}\omega_{KE} = &\left(\frac{2h}{\|\sigma\|^2 (\log\|\sigma\|)^2}\left(1 + \frac{\mu}{(\log\|\sigma\|)^{\alpha}}\right)\right) d\sigma \wedge d\bar{\sigma} + \\ &O\left(\frac{1}{\|\sigma\|\,(\log\|\sigma\|)^{1+\alpha}}\right) d\sigma \wedge d\bar{z} + O\left(\frac{1}{\|\sigma\|\,(\log\|\sigma\|)^{1+\alpha}}\right) d\bar{\sigma} \wedge dz + \\ &f dz \wedge d\bar{z}\end{aligned} \tag{2.2.4}$$

where $f dz \wedge d\bar{z}$ converges to the Kähler-Einstein metric on D and $\mu \in C^{k,\lambda}(X)$.

These asymptotics are restated as follows in [Sch02]. Let $z^1 = \sigma$, and

$z^2 = z$, a coordinate on D. There exists α, $0 < \alpha \leq 1$, such that for all $k \in \mathbb{N}$ and all λ, $0 < \lambda < 1$,

$$g_{\mathrm{KE}} = g_{i\bar{j}} dz^i dz^{\bar{j}} \tag{2.2.5}$$

where

$$\begin{aligned} g_{1\bar{1}} &= \frac{1}{2\,|\sigma| \ln^2 |\sigma|} \left(1 + \frac{g^0_{1\bar{1}}}{\ln^\alpha \sigma}\right) & g_{1\bar{2}} &= \frac{g^0_{1\bar{2}}}{|\sigma| \ln^{1+\alpha} |\sigma|} \\ g_{2\bar{1}} &= \frac{g^0_{2\bar{1}}}{|\sigma| \ln^{1+\alpha} |\sigma|} & g_{2\bar{2}} &= g^\infty_{2\bar{2}} \left(1 + \frac{g^0_{2\bar{2}}}{\ln^\alpha |\sigma|}\right) \end{aligned}$$

$g^0_{i\bar{j}} \in C^{k,\lambda}(X)$, $g^\infty_{2\bar{2}} \in C^\infty(\bar{X})$ and $g_D = g^\infty_{2\bar{2}} dz d\bar{z}$ is the constant -1 curvature metric on D.

Proposition 2.2.2. *There exists a sequence of metrics ${}_jg$ on $\bar{X}$, and a sequence of open sets $U_j \subset X$, $\bar{U_j} \subset U_{j+1}$, such that $\bigcup_j U_j = X$, and ${}_jg|U_j = g_{KE}|U_j$. Further, the curvature of ${}_jg$ is bounded uniformly in j as is the volume.*

The unfortunate left subscript on ${}_jg$ is necessitated by the fact in the sequel we will need the right subscripts to refer to the components of ${}_jg_{\alpha\bar{\beta}}$ and its inverse ${}_jg^{\bar{\beta}\alpha}$

Proof. There is some $\epsilon_0 > 0$ small enough, so that the tubular neighborhood of $N_{\epsilon_0}(D) = \left\{ p \in \bar{X} \;\middle|\; \|\sigma(p)\| < \epsilon_0 \right\}$ is diffeomorphic to an open disk bundle over D. Let $U_j = \bar{X} \setminus \overline{N_{\epsilon_j}(D)}$, where $\epsilon_j > 0$ is a decreasing sequence of positive numbers, $\epsilon_j \to 0$.

Let $\chi(t)$ be a smooth function such that $\chi(t) = 0$ for $t \leq 0$, $\chi(t) = 1$ for $t \geq 1$, $0 \leq \chi'(t) \leq 2$ and $|\chi''(t)| < 4$. Let $\chi_j(t) = \chi\left(\frac{t-\epsilon_{j+1}}{\epsilon_j - \epsilon_{j+1}}\right)$.

We define

$$g_{\text{prod}} = \frac{1}{2\,|\sigma|\ln^2|\sigma|}d\sigma d\bar{\sigma} + p^*(g_D) = \frac{1}{2\,|\sigma|\ln^2|\sigma|}d\sigma d\bar{\sigma} + g^{\infty}_{2\bar{2}}dzd\bar{z}$$

where $p : N_{\epsilon_0} \to D$ and

$$\begin{aligned}
{}_j\tilde{g} = \chi_j(|\sigma|)g_{\text{KE}} + (1-\chi_j(|\sigma|))\,g_{\text{prod}} = \\
\frac{1}{2\,|\sigma|\ln^2|\sigma|}\left[1+\chi_j\frac{g^0_{1\bar{1}}}{\ln^{\alpha}|\sigma|}\right]dz^1 d\bar{z}^1 + \chi_j\frac{g^0_{1\bar{2}}}{|\sigma|\ln^{1+\alpha}|\sigma|}dz^1 d\bar{z}^2 + \chi_j\frac{g^0_{2\bar{1}}}{|\sigma|\ln^{1+\alpha}|\sigma|}dz^2 d\bar{z}^1 + \\
g^{\infty}_{2\bar{2}}\left[1+\chi_j\frac{g^0_{2\bar{2}}}{\ln^{\alpha}|\sigma|}\right]dz^2 d\bar{z}^2
\end{aligned} \tag{2.2.6}$$

We need to show that the curvature and volume of ${}_j\tilde{g}$ are bounded. To that end we compute, keeping in mind the fact that for $f \in C^{k,\lambda}(X)$, there is some $\tilde{f} \in C^{k-1,\lambda}(X)$ such that $\frac{\partial f}{\partial \sigma} = \frac{\tilde{f}}{\sigma \ln|\sigma|}$. Also, we adopt the common convention that upper case latin indices range over the set $\{1,2,\bar{1},\bar{2}\}$, whereas lowercase Greek indices range over the set $\{1,2\}$. An index following a comma denotes a partial derivative. We use the usual formulas

$$2\Gamma^{A}{}_{BC} = g^{AD}\left(g_{DB,C} + g_{DC,B} - g_{BC,D}\right) \tag{2.2.7}$$

$$\begin{aligned}
2R_{ABCD} = g_{AD,BC} + g_{BC,AD} - g_{AC,BD} - g_{BD,AC} + \\
2g_{EF}\left(\Gamma^{E}{}_{BC}\Gamma^{F}{}_{AD} - \Gamma^{E}{}_{BD}\Gamma^{F}{}_{AC}\right).
\end{aligned} \tag{2.2.8}$$

The results of the computation are relegated to the Appendix. A more elegant proof, which avoids these messy computations is presented in Section 2.4.

Of course, ${}_j\tilde{g}$ is not the sequence of metrics we seek, since we need a sequence of metrics on $\bar{X}$. To finish the construction, the metric ${}_j\tilde{\tilde{g}}$ is constructed from

${}_j\tilde{g}$ by transitioning the "Poincaré" factor $1/2\,|\sigma|\ln^2|\sigma|$ to a cylindrical metric as in the discussion at the beginning of the Section. To complete the construction of the metric compactification, we take $N_{\epsilon_0}(D)\setminus N_{\epsilon_{j+1}}$, with the cylindrical end, and glue it back on to U_{j+1} with its cylindrical end. Then, we can smoothly cap off $N_{\epsilon_0}(D)$ with an arbitrary metric on the total space of the disk bundle $N_{\epsilon_0}(D)$ which is fixed once and for all throughout the construction. □

Proposition 2.2.3. *Let g be a metric on X which is C^2 close to g_{KE} at infinity. In other words, for any $\epsilon > 0$, there is a compact set $K \subset X$, such that $\|g - g_{KE}\|_{C^2(X\setminus K, g_{KE})} < \epsilon$. Then there exists a sequence of metrics g_j on $\bar{X}$, and a sequence of open sets $U_j \subset X$, $\bar{U}_j \subset U_{j+1}$, such that $\bigcup_j U_j = X$, and $g_j|U_j = g|U_j$.*

Proof. Take χ_j and U_j as above. And let $\tilde{g}_j = \chi_j(|\sigma|)g + (1-\chi_j(|\sigma|))g_{\mathrm{KE}}$. We get a sequence of metrics such that $\tilde{g}_j|U_j = g|U_j$ and $\tilde{g}_j|(X\setminus U_{j+1}) = g_{\mathrm{KE}}|(X\setminus U_{j+1})$. Now we can approximate $\tilde{g}_j$ by a metric g_j on $\bar{X}$ as in Proposition 2.2.2. □

2.3 Existence theory for the Seiberg-Witten equations

We follow Biquard's argument [Biq97] and produce an irreducible solution to the Seiberg-Witten equations (1.3.1) on (X, g) by considering a sequence of solutions to the perturbed equations on $(\bar{X}, g_j)$, where the g_j are the approximating metrics obtained in Proposition 2.2.3. This argument was also used by Rollin [Rol02] and Di Cerbo[DC11, DC12].

Theorem 2.3.1. *As throughout this Chapter, let $\bar{X}$ be a smooth, projective manifold, $b^2_+(\bar{X}) \geq 2$, $D \subset \bar{X}$ a smooth divisor, and L_D its associated line bundle. Let $K_{\bar{X}}$ be the canonical bundle and let $X = \bar{X} \setminus D$. Assume that $K_{\bar{X}} \otimes L_D$ is ample. Let g_{KE} be the unique complete Kähler-Einstein metric of Ricci curvature -1. Let g be any metric that is C^2 close to g_{KE} at infinity as in Proposition 2.2.3. Then there exists an irreducible (*i. e. $\Phi \not\equiv 0$*) solution $(A, \Phi) \in \mathcal{A} \times S^+$ on (X, g) in the* Spin$^{\mathbb{C}}$ *structure $\mathfrak{s}$ which is the restriction to X of the* Spin$^{\mathbb{C}}$ *structure on $\bar{X}$ with determinant line bundle $K_{\bar{X}}$. Further, if A_0 is a fixed C^∞ connection on $\bar{X}$, then $A = A_0 + \alpha$, where $\alpha \in \Gamma(T^*X)$, $\alpha \in L^2_1(X, g)$ and $d^*\alpha = 0$. Further,* $\sup \|\Phi\| < C$*..*

The condition $b^2_+(\bar{X}) \geq 2$ can be relaxed to $b^2_+(\bar{X}) \geq 1$, however, we then need to keep track of chambers in working with the Seiberg-Witten invariant. Before we proceed with the proof of the theorem, we need several Lemmas.

Lemma 2.3.2. *Let $\beta \in \Gamma(T^*M)$ be a 1-form on a closed Riemannian manifold (M, g). Suppose $\beta \perp \mathcal{H}^1(M, g)$. Then*

$$\|\beta\|_{L^2_1(M,g)} \leq C \left(\|d\beta\|_{L^2(M,g)} + \|d^*\beta\|_{L^2(M,g)} \right). \tag{2.3.1}$$

Proof. We need to control the L^2-norm of the 1-form and its derivative. To achieve the latter, we note that the Bochner formula for the Hodge Laplacian on 1-forms says

$$(dd^* + d^*d)\beta = \nabla^*\nabla\beta + \mathrm{Ric}_g(\beta, \cdot)$$

hence,

$$\begin{aligned}\|\nabla\beta\|^2_{L^2(M,g)} &= \|d\beta\|^2_{L^2(M,g)} + \|d^*\beta\|^2_{L^2(M,g)} - \int_M \mathrm{Ric}_g(\beta,\beta)\, d\mathrm{vol}_g \\ \|\nabla\beta\|^2_{L^2(M,g)} &\leq \|d\beta\|^2_{L^2(M,g)} + \|d^*\beta\|^2_{L^2(M,g)} + C\,\|\beta\|^2_{L^2(M,g)} \end{aligned} \tag{2.3.2}$$

It remains to establish control of the L^2-norm of the 1-form itself. Arguing by contradiction, suppose that the right hand side of (2.3.1) does not control $\|\beta\|_{L^2(M,g)}$. Then, there exists a sequence β_j of 1-forms, orthogonal to $\mathcal{H}^1(M,g)$, with $\|\beta_j\|_{L^2(M,g)} = 1$ and with $(\|d\beta_j\|_{L^2(M,g)} + \|d^*\beta_j\|_{L^2(M,g)}) \to 0$. But then (2.3.2) tells us that $\|\beta_j\|_{L^2_1(M,g)} < C$ and so there is a subsequence β_{j_k} which weakly converges to β in $L^2_1(M,g)$, and converges strongly in $L^2(M,g)$. In particular, $\|\beta\|_{L^2(M,g)} = 1$. But, weak convergence in $L^2_1(M,g)$ implies that $d\beta = d^*\beta = 0$ so $\beta \in \mathcal{H}^1(M,g)$. On the other hand, β is also orthogonal to $\mathcal{H}^1(M,g)$, so $\beta = 0$, which contradicts $\|\beta\|_{L^2(M,g)} = 1$. □

Proof of Theorem 2.3.1. Let $U_j \subset X$ be the open sets and g_j metrics on $\bar{X}$ constructed in Proposition 2.2.3. Since $\bar{X}$ admits a Kähler metric, even though g_j are most certainly not Kähler, by Theorem 1.3.6, the perturbed Seiberg-Witten equations (1.3.3) admit a solution with an arbitrary metric and perturbation.

Since L_D is trivial on $X = \bar{X} \setminus D$, we can take B_j to be a smooth connection on L_D, which is flat on U_j, where the U_j are the open sets as in Proposition 2.2.2. Let $\varpi_j^+ = F^+_{B_j}$ be the perturbation.

So we have a sequence $(A_j, \Phi_j) \in \mathcal{A} \times S^+$ such that $SW_{\bar{X},g_j}(A_j,\Phi_j) = (\varpi_j^+, 0)$, where the Seiberg-Witten map is defined in (1.3.5). Note that

$\left\|\varpi_j^+\right\|_{L_1^2(\bar{X},g_j)} < C$ uniformly in j. By equation (1.3.13) we then get $\mathcal{E}^{\mathrm{an}}_{\bar{X},g_j}(A_j,\Phi_j) < C$ uniformly in j, and this immediately gives

$$\begin{aligned}\left\|F_{A_j}\right\|_{L^2(\bar{X},g_j)} &< C\\ \left\|\Phi_j\right\|_{L^4(\bar{X},g_j)} &< C\\ \left\|\nabla_{A_j}\Phi_j\right\|_{L^2(\bar{X},g_j)} &< C \text{ all uniformly in } j.\end{aligned}$$

Now let us write $A_j = A_0 + \alpha_j$, by the gauge fixing Lemma 1.3.1 we can assume $d^{*_j}\alpha_j = 0$, where d^{*_j} is the adjoint of the exterior derivative with respect to g_j, and by the Hodge decomposition we can further write $\alpha_j = \beta_j + \gamma_j$ where β_j is orthogonal to $\mathcal{H}^1(\bar{X},g_j)$ and $\gamma_j \in \mathcal{H}^1(\bar{X},g_j)$. Further, we can assume that $\|\gamma_j\| < C$.

Then $F^+_{A_j} = F^+_{A_0} + d^+\beta_j$ and since A_0 is a fixed connection $\left\|F^+_{A_0}\right\|_{L^2(\bar{X},g_j)} < C$ uniformly in j. Hence, (1.3.17) gives us $\|d^+\beta_j\|_{L^2(\bar{X},g_j)} < C$. But

$$0 = \int_{\bar{X}} d(\beta\wedge d\beta) = \left\|d^+\beta_j\right\|_{L^2(\bar{X},g_j)} - \left\|d^-\beta_j\right\|_{L^2(\bar{X},g_j)}. \tag{2.3.3}$$

Hence, we have $\|d\beta_j\|_{L^2(\bar{X},g_j)} = \sqrt{2}\,\|d^+\beta_j\|_{L^2(\bar{X},g_j)} < C$. Lemma 2.3.2 then gives us $\|\beta_j\|_{L_1^2(\bar{X},g_j)} < C$. By a diagonal argument, we can pass to a subsequence which converges weakly in $L_1^2(X,g)$, $\beta_j \rightharpoonup \beta \in L_1^2(X,g)$. Indeed, firstly, recall that by the construction in Proposition 2.2.3, $g_j|U_j = g|U_j$. So $\|\beta_j\|_{L_1^2(U_k,g)} = \|\beta_j\|_{L_1^2(U_k,g_k)} \le \|\beta_j\|_{L_1^2(\bar{X},g_k)} < C$. Define $\beta_{1,j}$ as a subsequence of β_j which converges weakly in $L_1^2(U_1,g)$. Then assuming that for $k \ge 1$, $\beta_{k,j}$ converges weakly in $L_1^2(U_k,g)$, define $\beta_{k+1,j}$ as a subsequence of $\beta_{k,j}$ which converges weakly in $L_1^2(U_{k+1},g)$. Then a subsequence of $\beta_{j,j} \rightharpoonup \beta \in L_1^2(X,g)$, converges weakly as claimed.

Now we establish convergence for the spinor. (1.3.17) gives us an L^4 bound on the spinor and an L^2 bound on the covariant derivative of the spinor, but with a variable connection, and a variable metric so we need the following extra argument. $\nabla_{A_0}\Phi_j = \nabla_{A_j}\Phi_j - \alpha_j \otimes \Phi_j$, so

$$\begin{aligned} &\|\nabla_{A_0}\Phi_j\|_{L^2(\bar{X},g_j)} \leq \left\|\nabla_{A_j}\Phi_j\right\|_{L^2(\bar{X},g_j)} + \|\alpha_j \otimes \Phi_j\|_{L^2(\bar{X},g_j)} \leq \\ &C + \|\alpha_j\|_{L^4(\bar{X},g_j)} \|\Phi_j\|_{L^4(\bar{X},g_j)} \leq C + C' \|\alpha_j\|_{L^2_1(\bar{X},g_j)} \|\Phi_j\|_{L^4(\bar{X},g_j)} \leq C'' \end{aligned} \quad (2.3.4)$$

where the first inequality is the triangle inequality, the second inequality is (1.3.17) and the fact that the L^2-norm of the product is controlled by the product of the L^4 norms of the factors by Cauchy-Schwartz, and the third inequality is the Sobolev embedding $L^2_1 \hookrightarrow L^4$ in dimension 4 (see Theorem A.2). A diagonalization argument, just as for the 1-forms above, now gives us a subsequence Φ_j weakly convergent in $L^2_1(X, g)$. $\square$

2.4 Alternate Approach to Existence of Solutions of the Seiberg-Witten equtions

In this section we present an alternative proof of Theorem 2.3.1. Instead of approximating g_{KE}, the Kähler-Einstein metric on $X = \bar{X} \setminus D$, by a smooth sequence of metrics on $\bar{X}$, we approximate g_{KE} by a sequence of orbifold Kähler-Einstein metrics with orbifold singularity of cone angle $2\pi\beta$ along D, with $\beta = 1/p$, $p \in \mathbb{N}$, $\beta \to 0$.

The definition of an orbifold with singularity along a surface, a review of differential topology and Hodge theory on orbifolds, and most importantly, the definition of the Seiberg-Witten invariant in this context, can be found in [LeB13a]. For the convenience of the reader we quote some of the necessary results and definitions here. Given $\bar{X}$, a smooth, compact 4-manifold, and $D \subset \bar{X}$, a smoothly embedded surface, and $\beta = 1/p$, an orbifold $(\bar{X}, D, \beta)$ is defined as follows. Near any point $q \in D$, there is a coordinate neighborhood U_α with coordinates (w_α, z_α), where $U \cap D = \{ (w_\alpha, z_\alpha) \mid z_\alpha = 0 \}$, and we have $\tilde{U}_\alpha \to U_\alpha$, $(w, \zeta) \mapsto (w, \zeta^p / \|\zeta\|^{p-1})$. The action on ζ is the standard isometric $\mathbb{Z}_p$ action on $\mathbb{R}^2 \cong \mathbb{C}$. $\tilde{U}_\alpha$ is an orbifold chart. The transition functions must lift as diffeomorphisms of $\mathbb{R}^4$. In [LeB13a] this is referred to as the "origami" model. The model more frequently used in holomorphic geometry, is $(w, \zeta) \mapsto (w, \zeta^p)$. These two models are related by an appropriate homeomorphism of $\bar{X}$ to istelf, which is homotopic to the identity, and differs from the identity only in a tubular neighborhood of D. Note that ordinarily, an orbifold is defined more generally, where the finite group may vary from orbifold chart to orbifold chart, there is no restriction on dimension, and one uses real coordinates rather than complex ones. However, we state the definition for the application in the sequel. A smooth object, such as a metric, or a differential form is smooth on the orbifold $(\bar{X}, D, \beta)$, if it is smooth in the orbifold charts $\tilde{U}_\alpha$ and $\mathbb{Z}_p$-invariant.

A crucial further concept is one of V-bundle, which is a total space E, with a projection $p : E \to \bar{X}$, however, unlike the smooth case, in the orbifold case the projection is not required to be locally trivial. Rather, we require that for some fixed vector space, V, there is a system of orbifold charts, such that there is an isomorphism $p^{-1}(U_\alpha) \to (U_\alpha \times V)/\mathbb{Z}_p$, where the action of $\mathbb{Z}_p$

on U_α is the given orbifold action, and the action on V is some linear action. The transition functions on the overlaps are required to act linearly, as in the smooth case. Note, however, that the fiber over a singular point may be a quotient of V. The definition of sections of V-bundles, as well as tensor and exterior products carry over from the smooth case. A connection on a V-bundle E is a map $\nabla : \Gamma(E, U) \to \Gamma(E \otimes T^*\bar{X}, U)$ for any open set $U \subset \bar{X}$. In our setting, the finite groups figuring in the definition of orbifold, is $\mathbb{Z}_p$ over every singular point. Hence, since $L^{\otimes p}$ is then locally trivial, we can define, as in [LeB13a], $c_1^{\mathrm{orb}}(L) = \frac{1}{p} c_1(L^{\otimes p}) \in H^2(\bar{X}, \mathbb{Q})$. The deRham theory and the Hodge Theorem carry over to this context, as does the Chern-Weil theorem that for a connection ∇, $[\frac{i}{2\pi} F_\nabla] = c_1^{\mathrm{orb}}(L)$, for a line V-bunlde L, a connection ∇ on L, and F_∇ its curvature. The reader not familiar with these in the orbifold setting is referred to Section 2 of [LeB13a].

The Seiberg-Witten invariant, in this context is defined assuming $\bar{X}$ is almost complex and D is a pseudo-holomorphic curve. The definition of the Seiberg-Witten invariant in this setting, can be found in Sections 2.4, 2.5, 2.6 and 3 of [LeB13a]. We summarize this construction in the discussion below. First, we quote the following proposition,

Proposition 2.4.1 ([LeB13a], pg. 17)**.** *Let $\bar{X}$, D, β be as above: $\bar{X}$ a smooth, almost complex 4-manifold, with almost complex structure J_0, $D \subset \bar{X}$ a pseudo-holomorphic curve (with respect to J_0), and $\beta = 1/p$, $p \in \mathbb{N}$, $p \geq 2$. Then, there exists an almost complex structure J on $\bar{X}$, which is homotopic to J_0, coincides with J_0 along D and outside a tubular neighborhood of D, and is integrable in a tubular neighborhood of D.*

This allows us to associate to an almost-complex structure J_0 on $\bar{X}$, which by the above without loss of generality we can assume integrable in a tubular neighborhood of D, an orbifold almost complex structure J on $(\bar{X}, D, \beta)$. On $X = \bar{X} \setminus D$, J is simply J_0, and on orbifold charts (in the holomorphic geometry model) around points of D, we take J to be the usual complex structure, with respect to the orbifold charts. J can also be understood in the usual orbifold charts (the "origami" model), as is explained in [LeB13a].

Making the almost complex structure integrable in a tubular neighborhood of D, and working in the holomorphic geometry model, allows one to prove

$$c_1^{\mathrm{orb}}(\bar{X}, J_0, D, \beta) = c_1(\bar{X}, J_0) + (\beta - 1)[D], \qquad \beta = 1/p, p \in \mathbb{N}$$

Now, as in Example 1.2.5, outside a tubulur neighborhood of D, the almost complex structure J determines a Spin$^{\mathbb{C}}$ structure $\mathfrak{s}$ with determinant line bundle $L = \det \mathfrak{s} = \Lambda^{0,2}$. Using the holomorphic geometry model, this can be extended to the orbifold setting, though now, the bundles $\Lambda^{p,q}$ are V-bundles. In particular, the Spin$^{\mathbb{C}}$-V-bundles of spinors, just as in the smooth case, are $S^+ = \Lambda^{0,0} \oplus \Lambda^{0,2}$ and $S^- = \Lambda^{0,1}$. If the orbifold metric is Kähler and we take the Chern connection on $L = \det S^+ = \det S^- = K^{-1}$, then the associated Spin$^{\mathbb{C}}$-Dirac operator is $\mathcal{D} = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$. The Dirac operator for more general metrics will differ from the latter by a 0^{th}-order term, and hence, have the same principal symbol, and hence the same index. The latter can be computed in the holomorphic geometry model and is $\frac{1}{4}(\chi + \sigma)$, where χ is the Euler characteristic of $\bar{X}$ and σ is the signature, same as in the smooth case. Note that in our application, $\bar{X}$ is a complex manifold, and D is a divisor.

With spinors and the Dirac operator carried over to the orbifold setting we can write down the Seiberg-Witten equations. Note, that the special choice of orbifold-(almost)-complex structure on $(\bar{X}, D, \beta)$, and the resulting choice of $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}$ on the orbifold, is key to computing the index of the Dirac operator $\mathcal{D}$ on the orbifold to be the same as on $\bar{X}$, $\frac{1}{4}(\chi + \sigma)$. We then have the following orbifold version of the Seiberg-Witten invariants, developed in [LeB13a].

Theorem 2.4.2 ([LeB13a], Proposition 3.2)**.** *Let $(\bar{X}, J_0)$ be almost complex, $D \subset \bar{X}$ a pseudo-holomorphic curve, $p \in \mathbb{N}$, $p \geq 2$, $\beta = 1/p$. Equip the orbifold $(\bar{X}, D, \beta)$ with the orbifold almost complex structure J and corresponding* $\mathrm{Spin}^{\mathbb{C}}$ *structure as in the discussion above. Then for any orbifold metric g, and generic perturbation ϖ^+ the space of solutions to the perturbed Seiberg-Witten equations, modulo gauge equivalence, is finite. Further, if $b_+(\bar{X}) \geq 2$, then the two spaces of solutions modulo gauge for different choices of metric and perturbation are cobordant. This allows us to define the mod-2 Seiberg-Witten invariant of $(\bar{X}, D, \beta)$ as the mod-2 count of points in the space of solutions module gauge, with respect to any metric, and generic perturbation (the perturbations simply need to be a regular value of the SW map.*

Note that if the Seiberg-Witten invariant is non-zero, the above implies the existence of solutions for any metric and perturbation. Indeed, if there were no solutions for some perturbation, this perturbation would be a regular point, and we could use it to compute the invariant, and conclude that it is 0. Of course, we are *not* saying that one can use any perturbation to compute the invariant. Merely, that a solution must exist for any perturbation.

The invariant would not be terribly useful, if the non-vanishing results did not also carry over to the orbifold setting, but fortunately, they do, and we have the orbifold version of Taubes famous non-vanishing result of the Seiberg-Witten invariant

Theorem 2.4.3 (Taubes). *In the setting of Theorem 2.4.2, let us assume that $\bar{X}$ is actually a symplectic orbifold with symplectic form ω, J_0 is compatible with the symplectic structure, and ω restricts as an area form to D. Then, if $b_+(\bar{X}) \geq 2$, the Seiberg-Witten invariant of $\bar{X}$, with respect to the* $\mathrm{Spin}^{\mathbb{C}}$ *structure associated to J as above, is non-zero.*

Theorems 2.4.2 and 2.4.3 imply the following existence theorem, which we need to start the proof of Theorem 2.3.1.

Theorem 2.4.4. *Let $\bar{X}$ be a smooth, compact, Kähler 4-manifold, $b_+ \geq 2$, $D \subset \bar{X}$ a smooth divisor, $p \in \mathbb{N}$, $p \geq 2$, $\beta = 1/p$. Then for any orbifold metric g on $\bar{X}$, and with the* $\mathrm{Spin}^{\mathbb{C}}$ *structure $\mathfrak{s}$ as above, there exists a solution to the Seiberg-Witten equations with arbitrary perturbation.*

In addition to the above theory of the Seiberg-Witten equations on almost Kähler manifolds with orbifold singularities along a surface, the other crucial ingrdient in the proof of existence is an approximation of the Kähler-Einstein metric g_{KE} on $X = \bar{X} \setminus D$ by orbifold, Kähler-Einstein metrics on $\bar{X}$ with orbifold singularity along D. This approximation is furnished by the following theorem due to Henri Guenancia [Gue].

Theorem 2.4.5 (Henri Guenancia). *Let $\bar{X}$ be a smooth, projective complex manifold, and D a smooth divisor such that $K_{\bar{X}} \otimes L_D > 0$. Then there is a* *sequence of Kähler-Einstein, orbifold metrics g_j on $\bar{X}$ with conical singularity along D, with cone angle β_j going to 0, and $g_j \to g_{KE}$ on compact subsets of $X = \bar{X} \setminus D$.*

Note, that this theorem provides a sequence of Kähler-Einstein conical metrics; whereas, for our purposes it would be sufficient to have a sequence of conical metrics, which are merely Kähler, and have Ricci curvature uniformly bounded below. These can be constructed directly, but instead we quote Guenancia's much more powerful result above.

With the approximating conical metrics in hand, we can proceed with the alternate proof of existence. Firstly, we note that the Poincaré type lemma for 1-forms, Lemma 2.3.2 and the Gauge fixing lemma, Lemma 1.3.2 apply in the orbifold setting. Indeed, the Poincaré type lemma is proved by the Bochner formula and integration by parts, both of which carry over to the orbifold setting. The Gauge fixing lemma is proved by Hodge theory, which again carries over to orbifolds.

Now we can proceed to the proof of existence of solutions to the Seiberg-Witten equations (1.3.1) on X, with respect to any metric g which is C^2-close to g_{KE} at infinity, Theorem 2.3.1.

Proof of existence. Let $\tilde{g}_j$ be the conical KE metrics furnished by Theorem 2.4.5 (decorated with a tilde, since we reserve the designation g_j for metrics that we will construct momentarily). Now let $U_j \subset X$ be a sequence of open sets, with $\bigcup U_j = X$, $\overline{U_j} \subset U_{j+1}$. For definiteness, we can let $U_j = \{\, \|s\| > 1/j \,\}$, where s is a defining section for D. Then let χ_j be a smooth function on $\bar{X}$, such that $\chi_j(q) = 1$ for $q \in U_j$ and $\chi_j(q) = 0$ for $q \in U_{j+2}$. Let $g_j = \chi_j g + (1 - \chi_j)\tilde{g}_j$.

These g_j are orbifold metrics on $\bar{X}$, which converge to g on compact subsets of X. Theorem 2.4.4 then furnishes us with an irreducible solution (A_j, Φ_j) of the Seiberg-Witten equations with respect to the conical metric g_j. Note, that here A_j is a connection on the V-bundle $\det(\mathfrak{s})$, where $\mathfrak{s}$ is the $\mathrm{Spin}^{\mathbb{C}}$ structure described above, and $\Phi_j \in \Gamma(S^+)$ is a spinor, but also on the orbifold.

Now, just as in the case of a smooth compactification, the solutions have uniform apriori bounds (1.3.17), namely L^2 bounds on the curvature of A, L^4 bounds on the spinor Φ, and an L^2 bound on $\left\|\nabla_{A_j}\Phi_j\right\|_{L^2(\bar{X},g_j)}$. Indeed, these bounds are a consequence of (1.3.13), which is obtained by integrating by parts $\|SW(A_j,\Phi_j)\|^2_{L^2(\bar{X},g_j)}$. Since Stoke's Theorem holds for orbifolds, so does the relation (1.3.13)

$$\mathcal{E}^{\mathrm{an}}_{\bar{X},g_j}(A_j,\Phi_j) = \frac{1}{4}\int_{\bar{X}} \left|F_{A_j}\right|^2 d\mathrm{vol}_g + \int_{\bar{X}} \left|\nabla_{A_j}\Phi_j\right|^2 d\mathrm{vol}_{g_j} + \frac{1}{4}\int_{\bar{X}} \left(|\Phi_j|^2 + \frac{s_j}{2}\right)^2 d\mathrm{vol}_{g_j}$$
$$- \int_{\bar{X}} \frac{s_j^2}{16}\, d\mathrm{vol}_{g_j} = \mathcal{E}^{\mathrm{top}}_{\bar{X},g_j}(A_j,\Phi_j) + \|SW(A_j,\Phi_j)\|^2_{L^2(\bar{X},g_j)} =$$
$$\frac{1}{4}\int_{\bar{X}} F_{A_j}\wedge F_{A_j} + \|SW(A_j,\Phi_j)\|^2_{L^2(\bar{X},g_j)}\,.$$

If $SW(A_j,\Phi_j) = \varpi_j^+$, where ϖ_j^+ is a sequence of perturbations bounded in L^2, $\left\|\varpi_j^+\right\|_{L^2(\bar{X},g_j)} < C$, then we obtain the a priori bounds claimed above. These a priori bounds, in turn, imply compactness, just as in the smooth case.

Indeed, by the Gauge-Fixing Lemma, if we fix a smooth orbifold connection A_0, then $A_j = A_0 + \alpha_j$, and we can assume $d^*\alpha_j = 0$, *i. e.* we are in Coulomb gauge. Further, just as in the smooth case, a further change of gauge allows us to assume that not only $d^*\alpha_j = 0$, but $\alpha_j = \beta_j + \gamma_j$, where β_j is orthogonal to $\mathcal{H}^1(\bar{X},g_j)$ and $\gamma_j \in \mathcal{H}^1(\bar{X},g_j)$ and $\|\gamma_j\|_{L^2_1} < C$. Now, $F^+_{A_j} = F^+_{A_0} + d^+\beta_j$, and hence, the uniform L^2 bounds on the curvature F_{A_j} gives us uniform bounds on $\|d^+\beta_j\|$. Since Stokes Theorem holds for orbifolds, equation (2.3.3) also holds on the orbifold, and we obtain a bound on $\|d\beta_j\|_{L^2(\bar{X},g_j)}$. Finally, the Poincaré Lemma, then gives us a uniform bound on $\|\beta_j\|_{L^2_1(\bar{X},g_j)}$. The same diagonal argument as in the smooth case allows us to extract a weakly convergent in $L^2_1(X,g)$ subsequence, which by abuse of notation, we continue to call $\beta_j \rightharpoonup \beta$. Now, (2.3.4) holds on the orbifold as well, which along with the discussion which follows this equation in the smooth case gives convergence on a subsequence for the spinor as well. □

Solutions of the Seiberg-Witten equations are actually smooth, as can be shown by standard elliptic bootstrapping.

We will also need the following classical C^0 bound for the spinor.

Proposition 2.4.6. *Let (A, Φ) be a solution of the Seiberg-Witten equations on (X, g). Then $\|\Phi\|_{L^\infty} \leq \sup_X -s_-$, where $s_- = \min(s, 0)$, where s is the scalar curvature of g.*

Proof. We have equation (1.3.10) □

It remains to show that the solution obtained is irreducible, which we do in the following proposition:

Proposition 2.4.7. *The solution obtained above is irreducible,* i. e. *$\Phi \not\equiv 0$.*

Proof. First, we need the following uniform lower bound on the L^4 norm of the spinor on $\bar{X}$.

$$0 < 4\pi^2 c_1^2(K_{\bar{X}} \otimes L_D) = \int_{\bar{X}} F_{A_j} \wedge F_{A_j} = \int_{\bar{X}} \left|F_{A_j}^+\right|^2 d\mathrm{vol}_{g_j} - \int_{\bar{X}} \left|F_{A_j}^-\right|^2 d\mathrm{vol}_{g_j} = \\ \mathrm{const}\, \|\Phi_j\|^4_{L^4(\bar{X}, g_j)} - \left\|F_{A_j}^-\right\|^2_{L^2(\bar{X}, g_j)}. \quad (2.4.1)$$

Hence,

$$0 < \delta + \frac{1}{C} \left\|F_{A_j}^-\right\|^2_{L^2(\bar{X}, g_j)} < \|\Phi_j\|^4_{L^4(\bar{X}, g_j)}.$$

Let $N_j(D) = \{\, p \in X \mid \|\sigma(p)\| < \epsilon_0 \,\}$ for σ a defining section for D, and let $U_j = X \setminus N_j(D)$. So that $X = N_j \bigcup U_j$. Then we have

$$\|\Phi_j\|^4_{L^4(U_j, g_j)} = \|\Phi_j\|^4_{L^4(\bar{X}, g_j)} - \|\Phi_j\|^4_{L^4(N_j(D), g_j)} \geq \delta - \max(s_-)\,\mathrm{vol}(N_j(D), g_j) > \delta/2,$$

for j sufficiently large, where we used Proposition 2.4.6 for the uniform L^∞ bound on the spinor. Thus, we have a uniform lower bound on the L^4 norm of the spinors on U_j, and the proposition follows. □

2.5 Geometric Consequences

2.5.1 L^2-cohomology

To obtain geometric consequences for the existence of solutions to the Seiberg-Witten equations, we need to quote some facts about L^2-cohomology of complete manifolds. Our exposition follows [Car02, And88].

Let (X, g) be a complete, oriented Riemannian manifold. Let $\Omega^k_{L^2(g)}$ be the Hilbert completion of the space of smooth k-forms on X such that $\int_X \alpha \wedge *^g \alpha < \infty$ with respect to the inner product $\langle \alpha, \beta \rangle = \int_X \alpha \wedge *^g \beta$. The maximal domain of the exterior derivative d_k on k-forms is defined as

$$D_k = \left\{ \alpha \in \Omega^k_{L^2(g)} \;\middle|\; d\alpha \in \Omega^{k+1}_{L^2(g)} \right\}$$

so we have

$$d_k : D_k \to \operatorname{Im} d_k \subset \Omega^{k+1}_{L^2(g)}$$

and the reduced L^2-cohomology groups are defined as usual as the quotient of the closed forms by the *closure* of the coboundaries:

$$H^k_{L^2(g)} = \frac{\left\{ \alpha \in \Omega^k_{L^2(g)} \;\middle|\; d_k\alpha = 0 \right\}}{\overline{\operatorname{Im} d_{k-1}}}.$$

Note that under the assumption that X is complete, $\overline{\operatorname{Im} d_{k-1}}$ is the equal to the closure of the image of d applied to compactly supported smooth forms.

Further, the harmonic forms are defined as usual as well

$$\mathcal{H}^k_g = \left\{ \alpha \in \Omega^k_{L^2(g)} \;\middle|\; d\alpha = 0, d^*\alpha = 0 \right\} = \left\{ \alpha \in \Omega^k_{L^2(g)} \;\middle|\; (dd^* + d^*d)\alpha = 0 \right\}$$

The Hodge decomposition carries over to this setting:

$$\Omega^k_{L^2(g)} = \mathcal{H}^k_g \oplus \overline{\operatorname{Im} d_{k-1}} \oplus \overline{\operatorname{Im} d^{*_g}_{k+1}}$$

and so

$$H^k_{L^2(g)} = \mathcal{H}^k_g.$$

The Hodge star realizes the Poincare duality isomorphism $*^g : \mathcal{H}^k_g \to \mathcal{H}^{n-k}_g$; and moreover, if the dimension of X is a multiple of 4, then the Hodge star gives an isomorphism $*^g : \Omega^{\dim X/2}_{L^2(g)} \to \Omega^{\dim X/2}_{L^2(g)}$ which squares to 1. The self-dual and anti-self-dual forms are then the $+1$ and the -1 eigenspaces of $*^g$ on $\Omega^{\dim X/2}_{L^2(g)}$. If $\mathcal{H}^{\dim X/2}_g$ is finite-dimensional, then the difference of the dimensions of the self-dual and anti-self-dual subspaces is the L^2-signature.

Now let L be a line bundle on X, and let (X, g) be a finite volume 4-manifold. Let A be a connection L, such that $F_A \in \Omega^2_{L^2(g)}$. Chern-Weil theory extends to this situation, and we can define $c_1(L) = \frac{i}{2\pi}[F_A] \in H^2_{L^2(g)}$. If B is a different connection, such that $\alpha = B - A \in D^1 \subset \Omega^1_{L^2(g)}$, then $F_B = F_A + d\alpha$ and so $[F_B] = [F_A] \in H^2_{L^2(g)}$. Hence, the L^2-Chern class is independent of the connection, as long as the connections only differ by L^2 1-forms in the closure of the domain of d. Further,

Lemma 2.5.1. *$c_1^2(L) = \int_X F_A \wedge F_A$ is also independent of the connection.*

Proof.

$$\int_X F_B \wedge F_B = \int_X (F_A + d\alpha) \wedge (F_A + d\alpha) = \int_X F_A \wedge F_A + \int_X 2d\alpha \wedge F_A + d\alpha \wedge d\alpha =$$
$$\int_X F_A \wedge F_A + \int_X d\left(\alpha \wedge (2F_A + d\alpha)\right) = \int_X F_A \wedge F_A.$$

$\square$

Proposition 2.5.2. *Let L be a line bundle on (X, g), a finite-volume 4-manifold, and A a connection on L with $F_A \in \Omega^2_{L^2(g)}$. Then*

$$\int_X \left|F_A^+\right|^2 \, \mathrm{dvol}_g \geq 4\pi^2 (c_1^+(L))^2$$

Proof. Since $F_A \in \Omega^2_{L^2(g)}$, there is a $\phi \in \mathcal{H}^2_g$ (so $d\phi = d^*\phi = 0$) such that $[F_A] = [\phi] \in H^2_{L^2(g)}$, *i. e.* there is a $\alpha \in \Omega^1_{L^2_1(g)}$ such that $F_A = \phi + d\alpha$. Since ϕ is harmonic, ϕ minimizes the L^2-norm in its cohomology class, and hence,

$$\int_X |F_A|^2 \, \mathrm{dvol}_g \geq \int_X |\phi|^2 \, \mathrm{dvol}_g$$

Now we have

$$\begin{aligned} 4\pi c_1^2(L) = \int_X F_A \wedge F_A &= \int_X \left|F_A^+\right|^2 \, \mathrm{dvol}_g - \int_X \left|F_A^-\right|^2 \, \mathrm{dvol}_g \\ \int_X |F_A|^2 \, \mathrm{dvol}_g &= \int_X \left|F_A^+\right|^2 \, \mathrm{dvol}_g + \int_X \left|F_A^-\right|^2 \, \mathrm{dvol}_g \end{aligned}$$

so

$$\int_X \left|F_A^+\right|^2 \, \mathrm{dvol}_g = \frac{1}{2} \int_X |F_A|^2 \, \mathrm{dvol}_g + 2\pi c_1^2(L) \geq \frac{1}{2} \int_X |\phi|^2 \, \mathrm{dvol}_g + 2\pi c_1^2(L)$$

But just as above for F_A, we have

$$\int_X \left|\phi^+\right|^2 \, d\mathrm{vol}_g = \frac{1}{2}\int_X |\phi|^2 \, d\mathrm{vol}_g + 2\pi c_1^2(L)$$

so

$$\int_X \left|F_A^+\right|^2 \, d\mathrm{vol}_g \geq \int_X \left|\phi^+\right|^2 \, d\mathrm{vol}_g = 4\pi^2 \left(c_1^+(L)\right)^2.$$

$\square$

2.5.2 Scalar curvature estimate

Theorem 2.5.3. *As above, let $\bar{X}$ be a smooth, projective manifold, $b_+^2(\bar{X}) \geq 2$, $D \subset \bar{X}$ a smooth divisor, and L_D its associated line bundle. Let $K_{\bar{X}}$ be the canonical bundle and let $X = \bar{X} \setminus D$. Assume that $K_{\bar{X}} \otimes L_D$ is ample, so that we have the unique Kähler-Einstein metric g_{KE}. Then, any metric g on X which is C^2-close to g_{KE} at infinity satisfies the following integral scalar curvature estimate with equality if and only if g is Kähler with respect to some complex structure*

$$\int_X s^2 \, d\mathrm{vol} \geq 32\pi^2 \left(c_1^+(K_{\bar{X}}^{-1} \otimes L_D^{-1})\right)^2. \tag{2.5.1}$$

Proof. The proof follows [LeB95a, Biq97, Rol02, DC11, DC12].Theorem 2.3.1 furnishes us with an irreducible solution $(A, \Phi) \in \mathcal{A} \times \Gamma(S^+)$ to the Seiberg-Witten equations (1.3.1) . The Weitzenböck formula (1.3.8) and (1.3.9) gives us the following pointwise equality for (A, Φ)

$$\frac{1}{2}\Delta |\Phi|^2 + |\nabla_A \Phi|^2 + \frac{s}{4}|\Phi|^2 + \frac{1}{4}|\Phi|^4 = 0.$$

Let U_j be as in the previous section, complements of shrinking tubular neighborhoods of D, then dropping the positive $|\nabla_A \Phi|^2$ term and integrating we obtain

$$\int_{U_j} (-s) |\Phi|^2 \, d\mathrm{vol}_g \geq \int_{U_j} |\Phi|^4 \, d\mathrm{vol}_g + 2 \int_{U_j} \Delta |\Phi|^2 \, d\mathrm{vol}_g =$$
$$\int_{U_j} |\Phi|^4 \, d\mathrm{vol}_g + 4 \int_{\partial U_j} \langle \nabla_{A,n} \Phi, \Phi \rangle \, d\mathrm{vol}_{g|U_j}.$$

Now, since the energy bound gives a bound on $|\nabla_A \Phi|$, the volume of (X, g) is finite, and the mean curvature of ∂U_j is bounded, we have

$$\lim_{j \to \infty} \int_{\partial U_j} \langle \nabla_{A,n} \Phi, \Phi \rangle \, d\mathrm{vol}_{g|U_j} = 0 \tag{2.5.2}$$

so together with the Cauchy-Schwartz inequality we obtain

$$\int_x s^2 \, d\mathrm{vol}_g \int_X |\Phi|^4 \geq \left(\int_X |\Phi|^4 \, d\mathrm{vol}_g \right)^2$$

or using the Seiberg-Witten equation for F_A^+,

$$\int_X s^2 \, d\mathrm{vol}_g \geq \int_X |\Phi|^4 \, d\mathrm{vol}_g = 8 \int_X \left| F_A^+ \right|^2 \, d\mathrm{vol}_g.$$

We now obtain the desired estimate by referring to Theorem 2.5.2. □

2.5.3 Gauss-Bonnet and Hirzebruch

On a closed 4-manifold (X, g), the Chern-Gauss-Bonnet theorem says

$$\chi(X) = \frac{1}{8\pi^2} \int_X \left(\|W_+\|^2 + \|W_-\|^2 + \frac{s^2}{24} - \frac{1}{2} \|\mathrm{Ric}_0\|^2 \right) \, d\mathrm{vol}_g \tag{2.5.3}$$

where $W_\pm$ are the self-dual and anti-self-dual projections of the Weyl tensor and Ric_0 is the traceless Ricci curvature tensor.

The Hirzebruch signature theorem says

$$\tau(X) = \frac{1}{12\pi^2} \int_X \left(\|W_+\|^2 - \|W_-\|^2 \right) \, d\mathrm{vol}_g \tag{2.5.4}$$

These two equations yield

$$2\chi(X) \pm 3\tau = \frac{1}{4\pi^2} \int_X \left(2 \|W_\pm\|^2 + \frac{s^2}{24} - \frac{1}{2} \|\mathrm{Ric}_0\|^2 \right) d\mathrm{vol}_g. \tag{2.5.5}$$

Note that these two formulæ immediately imply the Hitchin-Thorpe inequality for Einstein manifolds, $\chi \geq \frac{3}{2} |\tau|$.

In our case, (X, g) is a finite-volume, complete Riemannian manifold, with bounded curvature, so we can still consider the integrals in (2.5.3), (2.5.4)

$$\begin{aligned}
\chi(X, g) &= \frac{1}{8\pi^2} \int_X \left(\|W_+\|^2 + \|W_-\|^2 + \frac{s^2}{24} - \frac{1}{2} \|\mathrm{Ric}_0\|^2 \right) \, d\mathrm{vol}_g \\
\tau(X, g) &= \frac{1}{12\pi^2} \int_X \left(\|W_+\|^2 - \|W_-\|^2 \right) \, d\mathrm{vol}_g
\end{aligned}$$

Theorem 2.5.4. *Let X be as above, $X = \bar{X} \setminus D$, and let g be a metric which* *is C^2 asymptotic to g_{KE}. Then*

$$
\begin{aligned}
\chi(X,g) &= \chi(X) \\
\tau(X,g) &= \tau(\bar{X}) - \frac{1}{3} D \cdot D
\end{aligned}
$$

2.5.4 Einstein metrics on blow-ups

An obstruction to the existence of Einstein metrics on blow-ups coming from Seiberg-Witten theory was first observed in [LeB96], based on the scalar curvature estimate. This result was successively improved in [LeB98], where a Weyl curvature estimate was first introduced, and then in [LeB01].

Let $\bar{X}$, $D \subset \bar{X}$, $X = \bar{X} \setminus D$ be as above, and take $\{ p_1, \ldots, p_k \} \subset X$ k points outside the divisor D. Let $\widetilde{\bar{X}}$ be the blow-up of $\bar{X}$ at the k points $\{ p_1, \ldots, p_k \}$, and let $\widetilde{X} = \widetilde{\bar{X}} \setminus D$. By [FM97, Proposition 2.5], (see [Nic00, Theorem 4.6.7] for an exposition) if $E_1, \ldots, E_k$ are the exceptional divisors, then there are 2^k $\mathrm{Spin}^{\mathbb{C}}$-structures, $\mathfrak{s}_{i_1,\ldots,i_k}$, $i_j = \pm 1$, such that $\det(\mathfrak{s}) = K_{\bar{X}}^{-1} \otimes \bigotimes_{l=1}^{k} L_{E_l}^{i_l}$, such that the Seiberg-Witten invariant of $\mathfrak{s}_{i_1,\ldots,i_k}$ is non-zero. We can use this to prove the following

Proposition 2.5.5. *With notation as above, we have the following scalar curvature estimate*

$$
\frac{1}{32\pi^2} \int_{\widetilde{X}} s^2 \, d\mathrm{vol}_g \geq \left(c_1^+ (K_{\bar{X}} \otimes L_D) \right)^2
$$

Proof. Follows similarly to the proof of Theorem 2.5.3. □

Theorem 2.5.6. *With notation as above, if $k > \frac{2}{3} \left(c_1(K_{\bar{X}}) + c_1(L_D) \right)^2$, then*

$\widetilde{X}$ does not admit an Einstein metric, which is C^2-close to the Kähler-Einstein metric on $\widetilde{X}$ at infinity.

Proof. By Theorem 2.5.4 we have

$$\begin{aligned}
\chi(\widetilde{X}, g) &= \chi(\bar{X}) - \chi(D) + k \\
\tau(\widetilde{X}, g) &= \tau(\bar{X}) - \frac{1}{3} c_1^2(L_D)[\bar{X}] - k
\end{aligned}$$

and by adjunction $\chi(D) = -c_1(D) \cup c_1(K_{\bar{X}} \otimes L_D)[\bar{X}]$. Now let g be an Einstein metric, *i. e.* $\mathrm{Ric}_0(g) = 0$ in (2.5.3). Then

$$\begin{aligned}
c_1^2(K_{\bar{X}} \otimes L_D)[\bar{X}] - k &= \left(c_1^2(K_{\bar{X}}) + 2c_1(K_{\bar{X}})c_1(L_D) + c_1^2(L_D)\right)[\bar{X}] - k \\
&= 2\chi(\bar{X}) + 3\tau(\bar{X}) - 2\chi(D) - c_1^2(L_D)[\bar{X}] + 2k - 3k \\
&= 2\chi(\widetilde{X}, g) + 3\tau(\widetilde{X}, g) \\
&= \frac{1}{4\pi^2} \int_{\widetilde{X}} \left(2\left\|W_+\right\|^2 + \frac{s^2}{24}\right) d\mathrm{vol}_g \\
&\geq \frac{1}{96\pi^2} \int_{\widetilde{X}} s^2 \, d\mathrm{vol}_g \\
&\geq \frac{1}{3} c_1^2(K_{\bar{X}} \otimes L_D)[\bar{X}]
\end{aligned} \tag{2.5.6}$$

Hence $k \leq \frac{2}{3} c_1^2(K_{\bar{X}} \otimes L_D)$. Or in other words, if $k > \frac{1}{3} c_1^2(K_{\bar{X}} \otimes L_D)$, there is no Einstein metric on $\widetilde{X} = X \sharp k \overline{\mathbb{CP}^2}$ which is C^2 asymptotic to the Kähler-Einstein metric at infinity. □

Chapter 3

Weyl Curvature Estimates

3.1 Compact Case

In [LeB01] (also see [LeB09b] for an exposition) LeBrun proved the following Weyl curvature estimate. Let (X, g) be a smooth, compact Riemannian manifold and $\mathfrak{s}$ a Spin$^{\mathbb{C}}$ structure. Suppose that $\mathfrak{s}$ admits a solution of the Seiberg-Witten equations (1.3.1), and let s be the scalar curvature of g and W_+ the self-dual part of the Weyl curvature tensor of g. Then

$$\int_X \left(s - \sqrt{6}\,|W_+|\right)^2 dx \geq 72\pi^2 c_+^2(\mathfrak{s}) \tag{3.1.1}$$

this is a different estimate than the scalar curvature estimate (2.5.1) one usually has when working with the Seiberg-Witten equations. The key to obtaining (3.1.1) is considering what happens to the Seiberg-Witten equations when the metric is conformally rescaled. To that end, consider $\hat{g} = f^{-2}g$, where f is some non-vanishing function on X. If $(A, \widehat{\Phi})$ is a solution of (1.3.1) with respect to g, then we obtain a solution (A, Φ) to

$$\begin{aligned} D_A\Phi &= 0 && (3.1.2) \\ F_A^+ &= f(\Phi \otimes \Phi^*)_0 && (3.1.3) \end{aligned}$$

(note the appearance of the function f as compared to (1.3.1)).

The Weitzenböck formula (1.2.11) for the Spin$^{\mathbb{C}}$ Dirac operator then yields a formula analogous to (1.3.8)

$$D_A^* D_A \Phi = \nabla_A^* \nabla \Phi + \frac{s}{4}\Phi + f\frac{1}{4} \|\Phi\|^2 \Phi \qquad (3.1.4)$$

We then obtain

$$0 = 2\Delta \|\Phi\| + 4 \|\nabla_A \Phi\|^2 + s \|\Phi\|^2 + f \|\Phi\|^4 , \qquad (3.1.5)$$

again note the appearance of f as compared to (1.3.10).

3.2 Non-compact Case

In this section we aim to prove in this setting LeBrun's Weyl curvature estimate, [LeB09b, Proposition 3.8]

Proposition 3.2.1. *Let (X, g) be a smooth, non-compact, finite volume 4-manifold with $\mathfrak{s}$ a* $\mathrm{Spin}^{\mathbb{C}}$ *structure on X and suppose that the Seiberg-Witten equations (1.3.1) admit a solution (Φ, A), with $\|\Phi\|$ bounded and $A = A_0 + a$, where $a \in L^2_1$ and $d^* a = 0$, for any metric which is C^2 asymptotic to g at infinity. Further, assume that outside a compact set $\left|s - \sqrt{6}\,|W_+|\right| \to C > 0$,* i.e. *there is a sequence of $U_j \subset X$, $\bigcup U_j = X$, $\overline{U_j} \subset U_{j+1}$, with* $\lim_{j\to\infty} \sup_{X\setminus U_j} \left|\left|s - \sqrt{6}\,|W_+|\right| - C\right| = 0$. *Then,*

$$\int_X \left(s - \sqrt{6}\,\|W_+\|\right)^2 d\mathrm{vol}_g \geq 72\pi^2 (c_1^+(\mathfrak{s}))^2. \tag{3.2.1}$$

Note, that we do not analyze the very interesting question of what are the consequences of equality in (3.2.1). Also, note that the condition that $\left|s - \sqrt{6}\,|W_+|\right| \to C > 0$ outside a compact set will be satisfied in our application to metrics that are asymptotic to a Kähler-Einstein metric. The proof is very similar to the compact case. The key is to consider the conformally rescaled Seiberg-Witten equations (3.1.2), and the Weitzenböck formlæ. Just as in the compact case, consider $\hat{g} = f^{-2} g$, where $f > 0$ is some positive function on X. Further, assume that f approaches in C^2 some positive constant at infinity. This condition guarantees that a solution to the Seiberg-Witten equations with respect to $\hat{g}$ exists, by the assumption in the Proposition. If $(\widehat{\Phi}, A)$ is a solution of the Seiberg-Witten equations (1.3.1) with respect to $\hat{g}$, then we obtain a solution (Φ, A) to the rescaled Seiberg-Witten equations (3.1.2) with respect to g.

Let us set $\psi = (\Phi \otimes \bar{\Phi})_0$. Then the Weitzenböck formula (1.2.12) yields

$$\int_X \left(\|\nabla \psi\|^2 + s \|\psi\|^2 + f \|\psi\|^3 \right) d\mathrm{vol}_g \leq 0$$

We also have the Weitzenböck formula for the Hodge Laplacian (1.2.14) which yields

$$\int_X \|\nabla \psi\|^2 \, d\mathrm{vol}_g \geq \int_X \left(-2\sqrt{\frac{2}{3}} \|W_+\| - \frac{s}{3} \right) \|\psi\|^2 \, d\mathrm{vol}_g.$$

These two inequalities added together yield

$$\int_X \left(\left(\frac{2}{3}s - 2\sqrt{\frac{2}{3}} \|W_+\| \right) \|\psi\|^2 + f \|\psi\|^3 \right) d\mathrm{vol}_g \leq 0.$$

To simplify notation, let $\phi = \frac{3}{2}\psi$, and then rewriting the above inequality we get

$$\int_X \left(\sqrt{6} \|W_+\| - s \right) f^{-2/3} (f^{2/3} \|\phi\|^2) \, d\mathrm{vol}_g \geq \int_X f \|\phi\|^3 \, d\mathrm{vol}_g$$

Now let us apply the Holder inequality with $p = 3/2$ and $q = 3$ to obtain

$$\left(\int_X \left| s - \sqrt{6} \|W_+\| \right|^3 f^{-2} \, d\mathrm{vol}_g \right)^{1/3} \left(\int_X f \|\phi\|^3 \, d\mathrm{vol}_g \right)^{2/3} \geq \int_X f \|\phi\|^3 \, d\mathrm{vol}_g$$

and so

$$\int_X \left| s - \sqrt{6} \|W_+\| \right|^3 f^{-2} \, d\mathrm{vol}_g \geq \int_X f \|\phi\|^3 \, d\mathrm{vol}_g. \tag{3.2.2}$$

The Hölder inequality with the same p and q also tells us

$$\left(\int_X f^4 \, d\mathrm{vol}_g\right)^{1/3} \left(\int_X f \left\|\phi\right\|^3 \, d\mathrm{vol}_g\right)^{2/3} \geq \int_X f^3 \left\|\phi\right\|^2 \, d\mathrm{vol}_g. \tag{3.2.3}$$

Combining (3.2.2) and (3.2.3)

$$\left(\int_X f^4 \, d\mathrm{vol}_g\right)^{1/3} \left(\int_X \left|s - \sqrt{6}\left\|W_+\right\|\right|^3 f^{-2} \, d\mathrm{vol}_g\right)^{2/3} \geq \int_X f^2 \left\|\phi\right\|^2 \, d\mathrm{vol}_g. \tag{3.2.4}$$

Now we use the rescaled Seiberg-Witten equation (3.1.2), $f\phi = 3\sqrt{2}F_A^+$ to get

$$\int_X f^2 \left\|\phi\right\|^2 \, d\mathrm{vol}_g = 18 \int_X \left|F_A^+\right|^2 \, d\mathrm{vol}_g \geq 18(2\pi c_1^+(\mathfrak{s}))^2 = 72\pi^2(c_1^+(\mathfrak{s}))^2,$$

where the inequality follows by Proposition 2.5.2.

$$\left(\int_X f^4 \, d\mathrm{vol}_g\right)^{1/3} \left(\int_X \left|s - \sqrt{6}\left\|W_+\right\|\right|^3 f^{-2} \, d\mathrm{vol}_g\right)^{2/3} \geq 72\pi^2(c_1^+(\mathfrak{s}))^2. \tag{3.2.5}$$

Now we choose a sequence $f_j \to \sqrt{\left|s - \sqrt{6}\left\|W_+\right\|\right|}$ uniformly from above. Since $f_j^2 \geq \left|s - \sqrt{6}\left\|W_+\right\|\right|$, we have

$$\begin{aligned}\int_X f_j^4 \, d\mathrm{vol}_g \geq \int_X \left|s - \sqrt{6}\left\|W_+\right\|\right|^2 \, d\mathrm{vol}_g = \\ = \int_X \left(f_j^{4/3}\right) \cdot \left(\left|s - \sqrt{6}\left\|W_+\right\|\right|^2 f_j^{-4/3}\right) \, d\mathrm{vol}_g \geq \\ \geq \left(\int_X f_j^4 \, d\mathrm{vol}_g\right)^{1/3} \left(\int_X \left|s - \sqrt{6}\left\|W_+\right\|\right|^3 f_j^{-2} \, d\mathrm{vol}_g\right)^{2/3}\end{aligned}$$

where the last inequality follows by once again applying Hölder inequality with

$p = 3/2$ and $q = 3$. The right hand side in the inequality above is the left hand side in (3.2.5), so we can combine the last inequality with (3.2.5) to obtain

$$\int_X f_j^4 \geq 72\pi^2 (c_1^+(\mathfrak{s}))^2. \tag{3.2.6}$$

Now, since $f_j^4 \to \left| s - \sqrt{6} \left\| W_+ \right\| \right|^2$ uniformly on M, we obtain the desired estimate:

$$\int_X \left(s - \sqrt{6} \left\| W_+ \right\| \right)^2 d\mathrm{vol}_g \geq 72\pi^2 (c_1^+(\mathfrak{s}))^2.$$

Corollary 3.2.2. *With (X, g), $\mathfrak{s}$ as in Proposition 3.2.1, we have*

$$\frac{1}{4\pi^2} \int_X \left(\frac{s^2}{24} + 2 \left| W_+ \right|^2 \right) d\mathrm{vol}_g \geq \frac{2}{3} \left(c_1^+(\mathfrak{s}) \right)^2. \tag{3.2.7}$$

Proof. Following [LeB01]

$$\begin{aligned}
\frac{9}{8} \int_X \left(s^2 + 48 \left| W_+ \right|^2 \right) d\mathrm{vol}_g &= \frac{9}{8} \left(\|s\|_{L^2}^2 + 48 \left\| W_+ \right\|_{L^2}^2 \right) \\
\text{(using the } L^2 \text{ norm in } \mathbb{R}^2) &= \left\| \left\langle 1, 1/\sqrt{8} \right\rangle \right\|^2 \left\| \left\langle \|s\|_{L^2}, \sqrt{48} \left\| W_+ \right\|_{L^2} \right\rangle \right\|^2 \\
\text{(by Cauchy-Schwarz in } \mathbb{R}^2) &\geq \left(\left\langle 1, 1/\sqrt{8} \right\rangle \cdot \left\langle \|s\|_{L^2}, \sqrt{48} \left\| W_+ \right\|_{L^2} \right\rangle \right)^2 \\
&= \left(\|s\|_{L^2} + \sqrt{6} \left\| W_+ \right\|_{L^2} \right)^2 \\
&\geq \left\| s - \sqrt{6} \left| W_+ \right| \right\|_{L^2}^2 \\
&= \int_X \left(s - \sqrt{6} \left| W_+ \right| \right)^2 d\mathrm{vol}_g \\
&\geq 72\pi^2 \left(c_1^+(\mathfrak{s}) \right)^2
\end{aligned} \tag{3.2.8}$$

and the inequality follows. $\square$

3.3 Non-existence of Einstein metrics on blow-ups

We can now easily attain an improvement by a factor of 2 in Theorem 2.5.6.

Theorem 3.3.1. *With notation as above, if $k > \frac{1}{3}\left(c_1(K_{\bar{X}}) + c_1(L_D)\right)^2$, then $\widetilde{X}$ does not admit an Einstein metric, which is C^2-close to the Kähler-Einstein metric on $\widetilde{X}$ at infinity.*

Proof. Indeed, in (2.5.6), rather than simply dropping the positive $2\left|W_+\right|^2$ term in the second to last line, we can use (3.2.7) to obtain

$$\begin{aligned}
c_1^2(K_{\bar{X}} \otimes L_D)[\bar{X}] - k &= \left(c_1^2(K_{\bar{X}}) + 2c_1(K_{\bar{X}})c_1(L_D) + c_1^2(L_D)\right)[\bar{X}] - k \\
&= 2\chi(\bar{X}) + 3\tau(\bar{X}) - 2\chi(D) - c_1^2(L_D)[\bar{X}] + 2k - 3k \\
&= 2\chi(\widetilde{X}, g) + 3\tau(\widetilde{X}, g) \\
&= \frac{1}{4\pi^2}\int_{\widetilde{X}} \left(2\left\|W_+\right\|^2 + \frac{s^2}{24}\right) d\mathrm{vol}_g \\
&\geq \frac{2}{3}c_1^2(K_{\bar{X}} \otimes L_D)[\bar{X}]
\end{aligned} \tag{3.3.1}$$

which yields $k \leq \frac{1}{3}\left(c_1(K_{\bar{X}}) + c_1(L_D)\right)^2$. □

Note that this improvement by a factor 2 applies to the non-existence theorems in [DC11, DC12, DC13].

Appendix A

Analysis

For a general introduction to Sobolev spaces see [Tay11a, Chapter 4], for the L^2 theory and [Tay11c, Chapter 13] for the L^p theory. Another excellent reference is [Eva10], or the Appendix to [DK90]. For the convenience of the reader we quote the basic definitions of Sobolev spaces and the two basic theorems regarding them.

Definition A.1. $L^p_k(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n) \mid \partial^m_{i_1,\ldots,i_m} f \in L^p(\mathbb{R}^n), 0 \leq m \leq k \right\}$ where the partial derivatives are understood in a weak sense. Naturally, the L^p_k-norm is defined as $\|f\|_{L^p_k(\mathbb{R}^n)} = \sum_{|\alpha|\leq k} \left\| \partial^{|\alpha|}_\alpha f \right\|_{L^p(\mathbb{R}^n)}$, where $\alpha = (i_1, \ldots, i_n)$, $|\alpha| = \sum i_l$, is a multi-index. It is easy to see that $C^\infty_0(\mathbb{R}^n)$ is dense in $L^p_k(\mathbb{R}^n)$, hence, an a posteriori equivalent definition of $L^p_k(\mathbb{R}^n)$ is the completion of $C^\infty_0(\mathbb{R}^n)$ with respect to the L^p_k-norm. Further, since multiplication by a compactly supported smooth function sends $L^p_k(\mathbb{R}^n)$ to itself, by using a partition of unity, we can define $L^p_k(M)$ on a manifold.

Note that in the case $p = 2$, there is an equivalent definition of L^2_k using the

Fourier transform, $\|f\|_{L^2_k(\mathbb{R}^n)} = \left\|(1+t^2)^{k/2}\hat{f}(t)\right\|_{L^2(\mathbb{R}^n)}$, where $\hat{f}$ is the Fourier transform of f.

Theorem A.2 (Sobolev Embedding)**.** *Suppose $k > l$ and $1 \leq p < q \leq \infty$ and $(k-l)p < n$. Then $L^p_k(\mathbb{R}^n) \hookrightarrow L^q_l(\mathbb{R}^n)$ for $\frac{1}{q} = \frac{1}{p} - \frac{k-l}{n}$ and the embedding is compact if $k > l$ and $k - n/p > l - n/q$.*

Suppose $\frac{k-r-\alpha}{n} = \frac{1}{p}$, then $L^p_k(\mathbb{R}^n) \hookrightarrow C^{r,\alpha}(\mathbb{R}^n)$. Where $C^{r,\alpha}(\mathbb{R}^n)$ are the usual Hölder spaces.

The compactness statement is known as the Rellich-Kondrachov Theorem. These statements also hold on a compact n-dimensional manifold, a compact n-dimensional manifold with Lipschitz boundary, and a complete Riemannian manifold with bounded sectional curvature and injectivity radius bounded away from zero. Note the special cases we need, $L^2_1(X^4) \hookrightarrow L^4(X^4)$ and $L^2_2(X^4) \hookrightarrow C^0(X^4)$ for a compact 4-manifold X^4.

Theorem A.3 (Sobolev Multiplication)**.** *Suppose $1/p - k/n > 0$ and $1/q - l/n > 0$ and $0 < m/n + (1/p - k/n) + (1/q - l/n) \leq 1/r \leq 1$. Then $L^p_k \times L^q_l \to L^r_m$. If $k/n - 1/p > 0$, $k \geq l$ and $k/n - 1/p \geq l/n - 1/q$ then $L^p_k \times L^q_l \to L^q_l$.*

Note the particular case we need in dimension 4, $L^2_3 \times L^2_2 \to L^2_2$ and $l \geq 3$, $L^2_l \times L^2_l \to L^2_l$.

Theorem A.4 (Gårding Inequality)**.** *Let E and F be two vector bundles over some manifold X and suppose $D : \Gamma(E) \to \Gamma(F)$ is an elliptic differential operator of order l. Then for $s \in \Gamma(E)$*

$$\|s\|_{L^2_{k+l}} \leq C\left(\|Ds\|_{L^2_k} + \|s\|_{L^2}\right)$$

Further, if s is orthogonal to the kernel of D, then

$$\|s\|_{L^2_{k+l}} \leq C \|Ds\|_{L^2_k}.$$

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