Limits of Real-Normalized Differentials on Stable Curves

A Dissertation Presented

by

Chaya Rifka Norton

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

August 2014

Stony Brook University

The Graduate School

Chaya Rifka Norton

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Samuel Grushevsky – Dissertation Advisor
Professor, Department of Mathematics

Robert Lazarsfeld – Chairperson of Defense
Professor, Department of Mathematics

Leon Takhtajan
Professor, Department of Mathematics

Scott Wolpert
Professor, Department of Mathematics, University of Maryland

This dissertation is accepted by the Graduate School.

Charles Taber
Dean of the Graduate School

Abstract of the Dissertation

Limits of Real-Normalized Differentials on Stable Curves

by

Chaya Rifka Norton

Doctor of Philosophy

in

Mathematics

Stony Brook University

2014

A recent approach proposed by Grushevsky and Krichever uses real-normalized differentials, meromorphic differentials with all periods real, to study the geometry of the moduli space of curves. We describe the behavior of real-normalized differentials under degeneration of the Riemann surface, and this analysis allows us to study the limits of zeros of these differentials near the boundary of the Deligne-Mumford compactification of the moduli space of curves. Our explicit description of the behavior of real-normalized differentials near nodal curves provides a tool for understanding common zeros of such differentials which has applications for the study of plane curves. This analysis should have further applications to degenerations of holomorphic differentials which is of interest in Teichmüller dynamics.

To my grandparents,
for making this possible.

Contents

Acknowledgements

I owe a great debt of gratitude to my advisor, Professor Samuel Grushevsky, without whom this thesis would have been impossible. Thank you, Sam, for advising and encouraging me throughout my graduate studies and for introducing me to real-normalized differentials. It is difficult to express the many ways in which you have helped me to develop professionally.

I would like to thank Professor Igor Krichever for his guidance and insight, and for sharing his expertise.

Thank you to my defense committee; Professor Robert Lazarsfeld, Professor Leon Takhtajan, and Professor Scott Wolpert for taking the time to evaluate my thesis.

There have been times during graduate school when I doubted my abilities. Thank you, Professor Dusa McDuff, for being a powerful female role model for me and for all the time you spent encouraging and advising me.

Although I am primarily thanking those who have contributed to my academic career, Professor Lisa Berger deserves thanks both as a dear friend and mentor. Thank you, Lisa, for the countless times you've helped me by sharing your experience and expertise, and for 'showing me the ropes'.

I would like to thank Professor Claude LeBrun, who as Director of the graduate program for the past six years, has been extremely helpful/supportive. In addition, 'thank you' to all of my fellow graduate students for your help and friendship. Specifically, I would like to mention Mark Hughes, my office-mate of six years, who has helped me to format my thesis correctly.

Chapter 1

Introduction

Our approach toward the study of the moduli space of Riemann surfaces uses local coordinates and a foliation introduced originally in the context of Whitham theory. Here we use a uniquely chosen meromorphic differential with real periods, called a real-normalized differential, on each curve in the moduli space. This thesis provides a complete description of the behavior of real-normalized differentials of the second kind under degeneration of the Riemann surface.

In the thesis we overcome one main difficulty in understanding how real-normalized differentials behave under degenerations. The difficulty is due to the fact that depending on location of the prescribed singular points, the differential may vanish identically on some components of a nodal curve, and thus one initially loses all information on limits of zeros as well as absolute and relative periods. This same issue arises when studying how holomorphic differentials degenerate, as is of interest in Teichmüller dynamics.

From the point of Algebraic Geometry this difficulty can be thought of as an issue understanding limit linear series, namely understanding the behavior of sections of a line bundle under degeneration of the Riemann surface. To understand all possible limits when studying sections on a family of smooth curves which degenerates to a nodal curve, it may be necessary to tensor the line bundle with some irreducible components of the nodal fiber. This is well-understood for curves of compact type, but further issues arise when one would like to consider a general nodal curve with two irreducible components, and limit linear has not been fully understood in general.

Our construction describes explicitly the real-normalized differential on all smooth or less singular curves in the neighborhood of a nodal curve. Specifically we construct the real-normalized differentials on curves in the neighborhood of a singular curve by prescribing singularities of a bounded order at the nodes which satisfy a matching condition, which will thus provide a well-defined meromorphic differential on smooth curves where the neighborhood of the node has been altered by the plumbing deformation. We further define holomorophic differentials whose absolute periods are understood, such that there exists a symplectic basis on smooth curves where one can identify one cycle with a non-real period. The real-normalized differential is then given by the meromorphic differential which satisfies the matching condition at the nodes and a linear combination of these holomorphic differentials scaled by factors which are sector real-analytic in plumbing parameters.

One can describe the location of zeros as a corollary of our construction, and these are parameterized by the choice of the singular parts of the matching differential at the nodes. Alternatively one can view the limits of zeros as parametrized by the plumbing parameters, and in the case where there is a zero of some real-normalized differential of order m at a node, the limits of zeros is additionally parameterized by the behavior of m zero on curves in the neighborhood of the nodal curve.

Thus our degeneration analysis which provides information for locating zeros is a tool which can be further used to study common zeros of real-normalized differential. The study of the zeros of real-normalized differentials play a crucial role, while the common zeros of a pair of real-normalized differentials are even more important (with specific applications to tautological classes) as suggested in [14] and future work toward the geometry of plane curves [16].

In the thesis we restrict ourselves to studying real-normalized differentials of the second kind which have no residues at the singular points of the differential. There are few new complication if one would like to describe degenerations of real-normalized differentials with non-zero residues, and our analysis can be applied almost directly, but we leave this description for possible future work.

The structure of this document is as follows: In the second chapter we provide a quick and incomplete introduction to the moduli space of Riemann surfaces where we focus on the ideas important to the thesis. In addition we discuss briefly the situation in Teichmüller dynamics given by studying holomorphic differentials on a Riemann surface. This is included in order to highlight the many similarities to our study of real-normalized differential, and the hope that our tools can be further used for degenerations of holomorphic differentials.

In the second chapter we begin to discuss real-normalized differentials and the moduli space of curves with m-jets at labeled marked points. Here we include a description of local coordinates on moduli space originally proven in the more general context of Whitham theory, and we define a foliation on the moduli space.

We then proceed to outline previous work in chapter 4, specifically focusing on real-normalized differentials with one double pole. In this chapter we intended to motivate further use of real-normalized differentials by recording some of its application towards studying the geometry of moduli space. Specifically we include a new and innovative proof of Diaz's theorem given in [14]. The last section includes an initial result describing degenerations of real-normalized differentials proven in [14].

The final chapter is dedicated to presenting our new results. The chapter is structure such that we present successively more complicated nodal curves beginning with curves of compact type. As we will see, the constructions used to describe the real-normalized differentials on curves in a small neighborhood of curves of compact type and irreducible nodal curves is sufficient for understanding real-normalized differentials near a general nodal curve. In the later sections, we provide the details needed to verify that no new complications arise.

Chapter 2

Overview of $\mathcal{M}_g$

In this chapter we present a brief overview of the moduli space of Riemann surfaces where we attempt to give a quick outline of the various facts which are relevant to our study of the moduli space. Many of the important theorems will be stated without proof as the proofs are very involved and take us too far afield. For all the details we refer to for example [13] and [2], which were used by the author throughout the writing of this chapter.

2.1 Properties of Algebraic Curves

We begin by summarizing some basic properties of algebraic curves which are the main objects in this thesis. This section is comprised of various relevant definitions and statements which will be used later in this thesis. We include very little discussion outside of the definitions and theorems.

Definition 2.1.1. A *Riemann surface* is a one dimensional smooth, compact complex manifold. Equivalently an *algebraic curve* is a complete reduced projective algebraic variety of dimension one over $\mathbb{C}$.

The language of Riemann surfaces from complex analysis and of algebraic curves from algebraic geometry will be used in this thesis interchangeably.

Denote the space of homology classes of closed cycles on C as $H_1(C,\mathbb{Z})$, and for smooth curves the genus of C is half the dimension of $H_1(C,\mathbb{Z})$. There is a natural intersection pairing on the space of cycles on C, and under this pairing $H_1(C,\mathbb{Z})$ is a symplectic space. A basis $\{A_i, B_i\}_{i=1}^g$ of $H_1(C,\mathbb{Z})$ is called a *symplectic basis* if $A_i \cdot B_j = \delta_{i,j}$, $A_i \cdot A_j = B_i \cdot B_j = 0$ if $i \neq j$. For a given choice, we call these the A- and B-cycles correspondingly, noting that the A-cycles span a Lagrangian subspace disjoint from the Lagrangian subspace spanned by the B-cycles.

Recall that as a topological manifold, a Riemann surface of genus g is a sphere with g handles, and for any $s_0 \in C$ the fundamental group $\pi_1(C, s_0)$ has a presentation

$$\pi_1(C, s_0) = \langle \hat{A}_1, \ldots, \hat{A}_g, \hat{B}_1, \ldots, \hat{B}_g \rangle / \prod \hat{A}_i \hat{B}_i \hat{A}_i^{-1} \hat{B}_i^{-1} = 1.$$

We call a set of generators $\hat{A}_i, \hat{B}_i$ of $\pi_1(C, s_0)$ a *standard generating set* if they generate $\pi_1(C, s_0)$ with the only relation being the one above. Thinking of C as a sphere with g handles, we can think of $\hat{A}_i$ as a path going to the handle number a_i, and then around it lengthwise, and back while $\hat{B}_i$ also goes to i'th handle, but then circles around it in the other direction. In particular, we note that the paths representing the classes of a standard generating set can be chosen not to intersect outside of s_0.

If we cut C along such paths, we get a contractible region, which turns out to be a polygon with $4g$ sides, that can then be identified pairwise with opposite orientation to glue it back into a surface.

Definition 2.1.2. We call a *marked surface* a Riemann surface C together with a standard generating set for $\pi_1(C)$.

The result of cutting up a Riemann surface C along a standard generating set of paths is a contractible polygon, which can be mapped to any other such polygon. Therefore any marking on C can be mapped to any other marking on C by a homeomorphism. Obviously the set of homeomorphisms of C acts on the set of markings on C, and we call two markings *equivalent* if one of them can be mapped to another by a homeomorphism isotopic to the identity map of C to itself.

We recall further that the natural map $\pi_1(C) \to H_1(C, \mathbb{Z})$ is the abelianization of the group $\pi_1(C)$, and under it the image of a standard generating set of $\pi_1(C)$ must be a symplectic basis of $H_1(C, \mathbb{Z})$.

We now return to the complex analytic structure on a Riemann surface, and in a neighborhood U of any point in C, we denote a local holomorphic coordinate as $z : U \to \mathbb{C}$. A holomorphic 1-form is locally of the form $f(z)dz$ where $f(z)$ is a holomorphic function, and the 1-form transforms under coordinate change $z = z(z')$ as $f(z)dz = f(z')dz' \cdot (\frac{dz}{dz'})$.

Definition 2.1.3. On a smooth curve C the line bundle of holomorphic 1-forms is called the *canonical bundle* and denoted K_C.

Proposition 2.1.4. *The dimension of the space of sections of K_C is equal to the genus of the (smooth) curve C.*

To see this, recall that if $H^{p,q}(C)$ denotes the space of (p,q)-forms, $H^{1,0}(C)$ is the space of holomorphic one forms, and by Hodge decomposition $H^1(C,\mathbb{C}) = H^{0,1}(C,\mathbb{C}) \oplus H^{1,0}(C,\mathbb{C})$, while $H^{0,1}(C) = \overline{H^{1,0}}(C)$ is the complex conjugate. Thus $\dim_{\mathbb{C}} H^{1,0}(C,\mathbb{C}) = \frac{1}{2}\dim_{\mathbb{C}} H^1(C,\mathbb{C}) = g$.

This proposition can be used to prove the fundamental theorem of Riemann-Roch on the dimensions of sections of line bundles (note that we use the additive notation for line bundles throughout this thesis, so plus denotes the tensor product, and minus denotes the dual line bundle).

Theorem 2.1.5 (Riemann-Roch theorem)**.** *For any line bundle L on a smooth curve C,*

$$\dim H^0(C,L) = \dim H^0(C, K_C - L) + deg(L) - g + 1.$$

Definition 2.1.6. A basis $\omega_1, \ldots, \omega_g$ of $H^0(C, K_C)$ is called dual to a symplectic basis $A_1, \ldots, A_g, B_1, \ldots, B_g$ of $H_1(C,\mathbb{Z})$ if $\int_{A_i} \omega_j = \delta_{i,j}$.

Definition 2.1.7. Given a symplectic basis of $H_1(C,\mathbb{Z})$ and a dual basis of $H^0(C, K_C)$, the *period matrix* of C is the matrix whose i,j'th entry is $\int_{B_i} \omega_j$. The period matrix is denoted $\tau(C)$.

The following theorem is the main result in this section, and it is essential for the constructions in this thesis.

Theorem 2.1.8 (Riemann's bilinear relations)**.** *The period matrix $\tau(C)$ of a smooth curve C is symmetric, and its imaginary part is positive definite.*

2.2 Moduli Spaces

In this thesis we study the geometry of the moduli space of genus g smooth curves, and thus recall various facts about this moduli space. We do not discuss the details of the definition of a stack. We outline the basics of constructing $\mathcal{M}_g$ as the quotient of the simply connected Teichmüller space which is a complex manifold, where all of our local constructions take place — so that we do not have to deal with the stack structure. Thus we view $\mathcal{M}_{g,n}$ as an orbifold, which is a complex manifold with orbifold loci which are subvarieties where $\mathcal{M}_{g,n}$ is locally isomorphic to $\mathbb{C}^m$ quotient by a finite group G. From our point of view, in small analytic neighborhoods of orbifold points it will be sufficient to work on the manifold cover of moduli space, in other words on Teichmüller space.

Definition 2.2.1. The moduli space $\mathcal{M}_g$ is the space of equivalence classes up to biholomorphism of smooth projective connected algebraic curves of genus g. The moduli space $\mathcal{M}_{g,n}$ is the moduli space of smooth curves with n ordered, distinct, marked points up to biholomorphisms which map the marked points to marked points, preserving the numbering.

Let $\mathcal{C}_g$, which is equal to $\mathcal{M}_{g,1}$, be the moduli space of pairs (C, p) where $C \in \mathcal{M}_g$ and $p \in C$, which we call the *universal family* over $\mathcal{M}_g$. As a stack, the fiber of $\mathcal{C}_g$ over a curve C with automorphisms is the quotient of C by its group of automorphisms. Let $\mathcal{C}_{g,n}$ denote the moduli space of pairs $(C, p_1, \ldots, p_n; p)$ where $C, p_1, \ldots, p_n \in \mathcal{M}_{g,n}$ and p is any additional point in C which may be one of the marked points points $p_1, \ldots, p_n$. There is a natural forgetful map $\mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$ forgetting the point p, and again the fiber is $C/Aut(C)$.

Every Riemann surface $C \in \mathcal{M}_g$ has the same underlying topological structure, and therefore choosing a marking on a fixed base topological surface S and a homeomorphism $f : S \to C$ defines a marking on the Riemann surface C. By the definition of equivalence for markings, the pair (C, f) and (C', f') define the same marked Riemann surface if there exists a biholomorphism $h : C \to C'$ which is homotopic to $f' \circ f$ sending the marking on C to the marking on C'.

Definition 2.2.2. Given a fixed oriented genus g topological surface S, the *Teichmüller space*, $\mathcal{T}_g(S)$, is the space of all marked Riemann surfaces (C, f) of genus g where two marked surfaces are equivalent if they are the same Riemann surface and the markings are equivalent.

There is a forgetful map $\Pi : \mathcal{T}_g \to \mathcal{M}_g$ defined by forgetting the marking $\Pi(C, f) = C$.

Remark 2.2.3. In fact any homeomorphism $h : S' \to S$ of Riemann surfaces induces a map $\mathcal{T}(S) \to \mathcal{T}(S')$ by sending (C, f) to $(C, f \circ h)$, which is clearly bijective. It will turn out that the Teichmüller spaces are actually complex manifolds and that this bijective map is in fact a biholomorphism, and thus we'll denote $\mathcal{T}_g$ the Teichmüller space of any Riemann surface of genus g.

We introduce a topology on the Teichmüller space using the *Teichmüller metric*.

Definition 2.2.4. The *Teichmüller metric* is given as follows,
$d((C, f), (C', [f'])) := \frac{1}{2} \inf\{\ln K(h) | h : C \to C' \text{ for all } h \text{ homotopic to } f' \circ f^{-1}\}$
where $K(h)$ is the supremum over all points in C of $\frac{|\partial_z h| + |\partial_{\bar{z}} h|}{|\partial_z h| - |\partial_{\bar{z}} h|} \geq 1$.

Remark 2.2.5. Two genus g surfaces are diffeomorphic which implies there exists a map $h : C \to C'$ such that $K(h)$ is finite. Such a map is called a *qausiconformal* mapping from C to C'.

Note $K(h) = 0$ if h is holomorphic, and therefore equivalent marked surfaces have distance zero in the Teichmüller metric, and this metric is well defined on $\mathcal{T}_g$.

One can construct $\mathcal{M}_g$ as a quotient of $\mathcal{T}_g$ by the finitely generated modular group denoted Γ_g, which acts properly discontinuously on $\mathcal{T}_g$. As we will show $\mathcal{T}_g$ is simply connected and therefore the Teichmüller space is then the universal orbifold covering space of $\mathcal{M}_g$.

Definition 2.2.6. The Teichmüller *modular group*, also called the *mapping class group*, and denoted Γ_g, is the group of isotopy (in fact equivalently homotopy) classes of orientation preserving homeomorphism from S to its self.

The modular group acts on $\mathcal{T}_g$ as follows: Let $\gamma \in \Gamma_g$, $\gamma \cdot (C, f) = (C, f \circ \gamma^{-1})$. In addition the Teichmüller metric is preserved under the action of the modular group.

All markings on S can be obtained by acting on the surface by a homeomorphism, thus the set of points $\{(C, f \circ \gamma^{-1}) | (C, f) \in \mathcal{T}_g, \gamma \in \Gamma_g\}$ are the set of all markings on the surface C. Therefore the fibers of the forgetful map Π from Teichmüller space to moduli space are the orbits of the action by the modular group. This implies that as a set,

$$\mathcal{M}_g = \mathcal{T}_g / \Gamma_g.$$

The action of the modular group on $\mathcal{T}_g$ is not free; The stabilizer of any point $(C, f) \in \mathcal{T}_g$ is isomorphic to the group of automorphisms on C. We will now outline some steps used to define a complex structure on $\mathcal{T}_g$, and therefore we construct that moduli space is an orbifold whose complex structure is given by the map $\Pi : \mathcal{T}_g \to \mathcal{T}_g$.

We now state Teichmüller's theorem which then implies that $\mathcal{T}_g$ is homeomorphic to an open unit ball in $\mathbb{C}^{3g-3}$, and thus has real dimension $6g-6$. This follows by defining a global homeomorphism from the space of quadratic differentials onto $\mathcal{T}_g$.

A *quadratic differential*, denoted q, on a smooth curve C is a section of the square of the canonical bundle, $q \in H^0(C, 2K_C)$. Locally q is of the form $f(z)(dz)^2$ where $f(z)$ is holomorphic. This is to say that under a local change of coordinates $z = z(z')$, we must have $f(z)(dz)^2 = f(z(z'))(\frac{dz}{dz'})^2(dz')^2$. It follows directly from the Riemann-Roch theorem (since the line bundle $K_C - 2K_C = -K_C$ is negative and does not have any holomorphic sections) that the dimension of the space of quadratic differentials is $\dim H^0(C, 2K_C) = 3g - 3$.

We define an L^1 norm on the space of quadratic differentials by taking $||q|| := \int_C i/2|f|dz \wedge d\overline{z}$, and q is said to be *integrable* if $||q|| := \int_C i/2|f|dz \wedge d\overline{z}$ is finite. Let $Q(C)$ denote the space of all *integrable* quadratic differentials on C which is a Banach space with the L^1-norm given by $||q||$, and let $Q_1(C)$ denote the space of quadratic differentials with norm less than one.

A map $h : (C, f) \to (C', f')$ is called a *Teichmüller mapping* if $\frac{\partial_{\overline{z}} h}{\partial_z h} = k\frac{\overline{q}}{|q|}$ for some $k < 1$ and $q \in Q(S)$, and Teichmüller showed that such a map it is the unique map in its homotopy class which minimizes the Teichmüller distance.

Define a map π from $Q(C)_1$ to $\mathcal{T}_g$ as follows, $\pi(q) = (C', f')$ where f' is the Teichmüller mapping associated to the quadratic differential q with $k = ||q||$ and one can show that C' exists such that $C' := f'(C)$. The map π is injective by construction as the Teichmüller map is the unique extremal mapping between (C, f) and (C', f'). One can further show π is a continuous, surjection between spaces of equal dimension which implies π is a global homeomorphism.

Theorem 2.2.7. *For any smooth curve of genus g, there exists a global (bijective) homeomorphism $\pi : Q_1(C) \to \mathcal{T}_g$, and therefore the Teichmüller space is contractible of real dimension $6g - 6$.*

In addition the space of quadratic differentials is isomorphic to the cotangent space a point. This follows from a result in deformation theory which states that the space of infinitesimal deformations of C are given by the first cohomology group of C with coefficients in the tangent sheaf. By KodairaSerre duality there is an isomorphism between $H^1(C, T_C) = H^1(C, -K_C)$ and $H^0(C, 2K_C)$, and the theorem follows.

Theorem 2.2.8. *[2] The cotangent space of $\mathcal{T}_g$ at C is isomorphic to $Q(C)$.*

The Bers embedding endows the Teichmüller space the structure of a complex manifold by defining a map which realizes $\mathcal{T}_g$ as a bounded domain in $\mathbb{C}^{3g-3}$. For the details involved in defining this embedding which maps $\mathcal{T}_g$ to the space of holomorphic quadratic differentials on the lower half plane modulo the Fuchsian group of C.The mapping class group is a group of biholomorphism from $\mathcal{T}_g$ to itself with this complex structure.

In order to understand the moduli space $\mathcal{M}_g$ further, we turn our attention to the mapping class group. As we will see this group is generated by a finite number of homeomorphisms called *Dehn twists* which we now define.

Let c be a smooth, homotopically non-trivial, simple closed path on the smooth oriented surface S. A collar containing the path c is an annulus $c\times[0,1]$ where the curve c itself is embedded as $c\times\{1/2\}$. The twist map of the annulus to itself is defined as follows, $T:(r,\theta)\rightarrow(r,\theta+2\pi r)$, which fixes the boundary of the annulus. The image of the curve $\eta:=\{(r,0)\}$ which intersects the waist curve c at one point in the annulus is sent via T to a curve which wraps around waist of the annulus before exiting the annulus at the same point.

Definition 2.2.9. For a simple closed path c on S the *Dehn twist* around c, denoted δ_c, is the homeomorphism of S to itself obtained by applying the twist map in a collar around c and the identity outside of the collar.

The Dehn twist depends only on the isotopy class of the cycle c. As an example of a Dehn twist, let $\hat{A}_i$ be an element in a standard generating set for $\pi_1(C)$, then the Dehn twist around the cycle $\hat{A}_i$ results in a marking of S which leaves all elements of standard generating set fixed except $\hat{B}_i$ is now sent to $\hat{B}_i\hat{A}_i$.

We then have the following,

Theorem 2.2.10 (Dehn, Lickorish)**.** *The mapping class group Γ_g is generated by the isotopy classes of Dehn twists δ_c for a finite set of non-separating cycles c in $\pi_1(S)$.*

In summary we have shown that $\mathcal{M}_g$ is a $K(\pi,1)$, and the cohomology of $\mathcal{M}_g$ with coefficients in $\mathbb{Q}$ is the group cohomology of the mapping class group. The Teichmüller space is the universal covering space of the moduli space.

In order to further understand the geometry of the moduli space of genus g curves, we now turn out attention to constructing cohomology classes, specifically tautological classes, on $\mathcal{M}_g$ by using the characteristic classes of natural bundles on moduli space. For the following relevant line bundles which we now define, the automorphisms of a curve $C \in \mathcal{M}_g$ acts non-trivial on the fibers of the line bundle, and thus to correctly define these line bundles, one must work on the moduli stack. On the other hand if $g \geq 4$, the set of curves with non-trivial automorphism has codimension greater than or equal to 2, and the characteristic classes which we define in $H^2(\mathcal{M}_g, \mathbb{Q})$ don't see these curves, thus we continue to avoid discussing the correct definition of a line bundle on a stack.

Consider the universal family $\pi : \mathcal{C}_g \to \mathcal{M}_g$ on which we have the *relative dualizing sheaf* $\omega_\pi := K_{\mathcal{C}_g} \otimes \pi^* K^{\vee}_{\mathcal{M}_g}$, which is a line bundle on the universal family. On any flat family of smooth curves ($\pi : X \to S$), ω_π is the line bundle whose fibers are the cotangent bundle on each curve $\pi^{-1}(s), s \in S$.

The *Hodge bundle* $\mathbb{E}$ is defined to be the direct image of the relative dualizing sheaf, i.e. $\mathbb{E} := \pi_*(\omega_\pi)$, which is to say that its fiber over some curve $C \in \mathcal{M}_g$ is $H^0(C, K_C)$. We denote $L := \wedge^g \mathbb{E}$ the *Hodge line bundle*, the determinant line bundle of the Hodge bundle. Let $\psi := c_1(\omega_\pi) \in H^2(\mathcal{C}_g, \mathbb{Q})$ be the first Chern class of ω_π.

Definition 2.2.11. The *kappa classes* are defined as the pushforwards of the $i+1$'st power of the chern class of the relative cotangent sheaf, i.e. $\kappa_i := \pi_*(\psi^{i+1}) \in H^{2i}(\mathcal{M}_g, \mathbb{Q})$.

The *lambda classes* are the characteristic classes of the Hodge bundle, $\lambda_i =$

$c_i(\mathbb{E}) \in H^{2i}(\mathcal{M}_g, \mathbb{Q})$.

When $n > 0$ there are other natural line bundles to consider, $\mathcal{L}_i$ whose fiber at an n-pointed curve $(C; p_1, \ldots, p_n)$ is the cotangent space to C at p_i. The marked points provide a section from $\mathcal{M}_{g,n}$ to $\mathcal{C}_{g,n}$, defined by $\sigma_i(C; p_1, \ldots, p_n) := ((C; p_1, \ldots, p_n), p_i)$, and we can pull back the relative dualizing sheaf by this section.

Definition 2.2.12. The *psi classes* are the chern classes $c_1(\mathcal{L}_i) \in H^2(\mathcal{M}_{g,n}, \mathbb{Q})$, which are denoted as ψ_i.

Definition 2.2.13. The *tautological ring* is the subring of $H^*(\mathcal{M}_{g,n}, \mathbb{Q})$ generated by all the kappa classes and all the psi classes.

Remark 2.2.14. One can define the tautological ring, usually denoted R^*, as the subring of the rational Chow ring generated by all the natural classes above. The usual notation for the tautological subring in homology is RH^*. All classes in the tautological ring are even, and the grading (as in Chow) is such that $RH^* \subset H^{2*}$, and the ψ classes which are in H^2 are in RH^1.

Mummford showed that $\lambda_1, \ldots, \lambda_g$ are polynomial in the kappa classes, and thus lie in the tautological ring. In addition he showed that κ_i for $i > g-1$ are polynomial in $\kappa_1, \ldots, \kappa_{g-1}$, and therefore the tautological ring in generated by the psi classes and the first $g-1$ kappa classes [27].

The following conjecture due to Faber roughly states that the tautological ring R^* looks like the cohomology ring of a compact $(g-2+n)$-dimensional real manifold. More precisely it says that R^* is a Gorenstein (sometimes called Poincaré duality) graded ring with a socle in dimension $g-2+n$:

Conjecture 2.2.15. *[10] The tautological ring of $\mathcal{M}_{g,n}$ has the following properties:*

1. $R^i(\mathcal{M}_{g,n}) = 0$ *for* $i > g-2+n$
2. $R^{g-2+n}(\mathcal{M}_{g,n}) \cong \mathbb{Q}$
3. *The map* $R^s(\mathcal{M}_{g,n}) \times R^{g-2+n-s}(\mathcal{M}_{g,n}) \to R^{g-2+n}(\mathcal{M}_{g,n}) = \mathbb{Q}$ *is a perfect pairing.*

The first two parts of the conjecture are known to hold.

Theorem 2.2.16 (Looijenga [25])**.** $R^i(\mathcal{M}_{g,n}) = 0$ *for* $i > g-2+n$*, i.e. part 1 of the conjecture holds.*

This and the statement that λ_1 is ample on $\mathcal{M}_g$ immediately implies a theorem of Diaz which bounds the dimension of a compact subvariety of $\mathcal{M}_g$. Indeed, the ampleness of λ_1 implies that for any d-dimensional complete subvariety $X \subset \mathcal{M}_g$, the intersection number $\lambda_1^d \cdot X$ is positive, but we know that $\lambda_1^{g-1} = 0$. Therefore for any compact subvariety $X \subset \mathcal{M}_g$, the dimension of X is at most $g-2$.

In addition Faber and Looijenga have proven the second statement in the conjecture.

Theorem 2.2.17 (Faber [9], Looijenga [25])**.** $R^{g-2+n}(\mathcal{M}_{g,n}) \cong \mathbb{Q}$.

Remark 2.2.18. Faber made a similar conjecture about the tautological ring of $\overline{\mathcal{M}}_{g,n}$ which we have not defined, but whose restriction to $\mathcal{M}_{g,n}$ gives the tautological ring defined above. In recent work of Petersen and Tommasi [29], it was shown that the perfect pairing statement (3) does *not* hold for

$R^{3+n}(\overline{\mathcal{M}}_{2,n})$, for some $20 \leq n \leq 22$, and further Petersen [28] showed that the perfect pairing also fails on the moduli space of curves of genus 2 of compact type for $n \geq 8$.

These curious vanishing properties of the tautological ring have led to the following further ambitious conjectures. Harer's bound on the homotopical dimension of $\mathcal{M}_g$ further supports these conjectures [18].

Conjecture 2.2.19 (Hain - Looijenga [17])**.** *The moduli space $\mathcal{M}_g$ has an affine cover with exactly $g-1$ open sets.*

There is a weaker conjecture which would similarly explain vanishing properties.

Conjecture 2.2.20 (Roth - Vakil [30])**.** *There exists an affine stratification (in the sense they define it) of $\mathcal{M}_{g,n}$ by $g - \delta_{n,0}$ locally closed strata S_i where $\overline{S_j} = \cup_{i\leq} S_i$.*

Remark 2.2.21. An affine stratification and cover for $\mathcal{M}_g$ when $g = 3, 4, 5$ are constructed in [11][12], while there are no conjectural candidates for either an affine cover or stratification of $\mathcal{M}_g$ for $g > 5$.

In the thesis we work with a foliation on $\mathcal{M}_g$ which is tangentially complex and was introduced in [14]. A *foliation* of a manifold is a decomposition into equidimensional leaves which are locally immersed submanifolds. The existence of such a foliation is certainly weaker than the existence of a conjectural stratification. Some evidence [14] [15] has been presented suggesting that this foliation explains various vanishing properties, specifically it provides a proof of Diaz's Theorem.

2.3 Compactifying the Moduli Spaces $\mathcal{M}_{g,n}$

The moduli space $\mathcal{M}_g$ is not compact, and it is natural to look for a compactification which parameterizes natural geometric objects and whose boundary is divisorial with simple normal crossings. In this section we introduce the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ which is a projective variety and parameterizes stable nodal curves. We begin the section with the definition and basic properties of nodal curves.

Definition 2.3.1. A complete algebraic, possibly singular, complex projective curve C is called a *nodal curve* if any point $p \in C$ is either a smooth point of C or is locally complex analytically isomorphic to a neighborhood of the origin given by the equation $\{xy = 0\} \subset \mathbb{C}^2$.

The normalization of a nodal curve is a smooth Riemann surface with possibly more than one irreducible components, denoted $\tilde{C} := \cup_i C_i$ obtained by disconnecting the nodes. The normalization map $n : \tilde{C} \to C$ is an isomorphism away from the nodes of C, and for p a node and U a sufficiently small neighborhood of p, $n^{-1}(U)$ is two disjoint coordinate neighborhoods of q_1 and q_2 where $n^{-1}(p) = \{q_1, q_2\}$. An *irreducible nodal curve* is a nodal curve with one irreducible component or equivalently $\tilde{C}$ is connected.

If C is a nodal curve with δ nodes and m irreducible components denoted $C_1, \ldots, C_m$, then the arithmetic genus of C is $g(C) = \sum_i g(C_i) + \delta - m + 1$ where $g(C_i)$ is the geometric genus of the normalization of each component C_i.

Definition 2.3.2. The *dual graph* of a nodal curve C is a graph whose vertices v_i correspond to irreducible components and such that for each node in C

whose preimage under the normalization map lies on irreducible components C_i and C_j, there in an edge in the dual graph joining vertices v_i and v_j (note: i may equal j). The dual graph of a marked nodal curve with m_i marked points on C_i has m_i legs (by which we mean half-edges starting from it) on the vertex v_i.

We now introduce the simplest type of nodal curve whose genus is exactly the sum of the genera of its irreducible components.

Definition 2.3.3. A nodal curve is a curve of *compact type* if its dual graph is a tree, i.e. its dual graph does not have any non-trivial closed cycles.

We now define the curves which will be parameterized in $\overline{\mathcal{M}}_g$.

Definition 2.3.4. A *stable* nodal curve $(C, p_1, \ldots, p_n)$ is a nodal curve whose group of automorphisms which fix the marked points is finite.

In fact all smooth curves of genus greater than one are stable and the size of the group of automorphisms is bounded by $84(g-1)$. Requiring a nodal curve to be stable amounts to requiring that any irreducible component C_i whose normalization is genus 0 has at least 3 marked points and (preimages of) nodes, in total, and excluding the case of smooth elliptic curves.

The *dualizing sheaf* on a nodal curve C, which plays the role of the canonical line bundle on a smooth curve (and satisfies the properties outlined in Serre's duality theorem), is the space of meromorphic differentials on the normalization $\tilde{C}$ which are holomorphic away from the node and may have simple poles with residues of opposite sign at the preimage of each node under the normalization map. Clearly if C is smooth, the dualizing sheaf is the sheaf of holomorphic differentials and is exactly the canonical bundle.

Curves with at worst nodal singularities are locally complete intersections, and by the residue condition the dualizing sheaf is invertible. In addition the dualizing sheaf is ample on each irreducible component of a stable curve precisely because of the stability condition, and alternatively one can define a curve as stable if the dualizing sheaf is ample on this curve.

The dualizing sheaf for nodal curves can be defined in families, and in fact the dualizing sheaf on a nodal curve is the unique extension in families of canonical bundle for smooth curves. One can show that if X^* is the set of smooth fibers of π, then to extend the relative canonical bundle over $X^* \to S^*$ one must allow simple poles at the nodes of the singular curve.

We now proceed to define the Deligne-Mumford compactification of $\mathcal{M}_g$.

Definition 2.3.5. The Deligne-Mumford compactification of $\mathcal{M}_{g,n}$, denoted $\overline{\mathcal{M}}_{g,n}$, is the moduli space of stable nodal curves up to biholomorphism.

It is a non-trivial statement to show that this is a compactification, and this follows from the stable reduction theorem which asserts that any flat family of smooth curves over a non complete base can be completed up to base change by allowing fibers which are stable nodal curves [5]. It is indeed surprising that one does not need to allow worse singularities.

Theorem 2.3.6 (Deligne-Mumford)**.** *The Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ is a projective variety, and the boundary $\partial\overline{\mathcal{M}}_{g,n}$ is a divisor in $\overline{\mathcal{M}}_{g,n}$ with simple normal crossing singularities.*

In order to understand a neighborhood of nodal curves, we remark that in fact for stable nodal curves there exist a universal deformation space (given by the Kuranishi family) which satisfies the property that given any other deformation and a sufficiently small neighborhood of the nodal curve, there exists a unique morphism of deformations to the Kuranishi family.

We do not discuss all the details here and refer to [2] for the details, and instead focus on constructing a specific deformation of nodal curves given by altering a neighborhood of each node which will provide local coordinates near the boundary of $\overline{\mathcal{M}}_g$. Given a nodal curve C with δ nodes, one can define a family $\mathcal{C} \to \Delta$ where $\Delta \subset \mathbb{C}^\delta$ which smooths each node individually with a complex parameter.

This is accomplished by a direct generalization of the one-parameter family of smooth curves degenerating to a nodal curve with one node called the *plumbing deformation*. Specifically we will describe a deformation of C with one node at p, over the disk $\Delta := \{s \in \mathbb{C} : |s| < \epsilon\}$ for ϵ sufficiently small, such that each fiber $\phi^{-1}(s)$ is a smooth curve for $s \neq 0$, while $C_0 = C$.

Let U be a neighborhood of the node p in C which is given by $V := \{z \in \mathbb{C} : |z| < \epsilon\}$ union $W := \{w \in \mathbb{C} : |w| < \epsilon\}$ with the origins identified. This description of a neighborhood of the node can be thought of as choosing coordinate neighborhoods of the points q_1 and q_2 in the preimage of p under the normalization map $n : \tilde{C} \to C$, with the node given by identifying $q_1 \sim q_2$.

Let $U_s := \{z \in \mathbb{C} : |z| < \sqrt{|s|}\} \cup \{w :\in \mathbb{C} : |w| < \sqrt{|s|}\}$ for $s \in \Delta$ which is an open neighborhood of the node and denote the curve with boundary ${C_s}^0 := C - U_s$. In addition let $\gamma_z := \{z \in \mathbb{C} : |z| = \sqrt{|s|}\}$ and similarly $\gamma_w := \{w \in \mathbb{C} : |w| = \sqrt{|s|}\}$ which are simple closed paths in V and W.

Definition 2.3.7. For any $s \in \Delta$, identify points $z' \in \gamma_z$ and $w' \in \gamma_w$ in the boundary of C_s^0 by the relation $z' \sim w'$ if $z'w' = s$, and denote the resulting smooth curve by C_s.

The family $\phi : \mathcal{C} \to \Delta$ with fibers $\phi^{-1}(s) = C_s$ is a one parameter deformation of C called *plumbing deformation* of C with complex parameter s. The path $\gamma_s := \gamma_z \sim \gamma_w$ on C_s is called the *seam*, and its homology classes is called the *vanishing cycles*.

If C has marked points, one can choose a neighborhood of the node which does not contain any marked points, and proceed in exactly the same way to define a deformation of $(C, p_1, \ldots, p_n)$ which smooths the node.

Let $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$, have δ nodes, and let $n : \tilde{C} \to C$ be the normalization of C whose irreducible components are $\cup_j C_j$ and the preimage of each node q_i is given by q_i^1 and q_i^2.

Theorem 2.3.8. *[3],[2][20] Any choice of local coordinates in a neighborhood of the smooth marked curves* $(\cup_j C_j, p_1, \ldots, p_n, q_1^1, q_1^2, \ldots, q_\delta^1, q_\delta^2)$ *on the moduli spaces to which they belong, and complex plumbing parameters* $(s_1, \ldots, s_\delta)$ *in the neighborhood of each node* q_i *defines a local coordinate system in a neighborhood of* $(C, p_1, \ldots, p_n) \in \partial\overline{\mathcal{M}}_{g,n}$.

Thus if $C \in \overline{\mathcal{M}}_{g,n}$ is a nodal curve, then the union of any set of local coordinates on the moduli space (or product of moduli spaces) where the normalization $\widetilde{C}$ lives, together with a complex plumbing coordinate for every node of C, gives local coordinates near C.

In the thesis we will be working in a small neighborhood of nodal curves, and one can consider our local computations as taking place on the Teichmüller space (in many instances the Torelli space will be enough). The very technical issue of constructing a bordification of the Teichmüller space to which the action of the mapping class group extends is considered in [1][2][20]. In the thesis, all we need is local plumbing coordinates and coordinates on the various moduli spaces which are local coordinates near the boundary given by plumbing coordinates, and all the details for this situation are given in [33].

2.4 Holomorphic Differentials

Recently there has been a lot of progress in Teichmüller dynamics, starting with the groundbreaking work of Kontsevich and Zorich [22]. In that context, many methods were developed for dealing with the moduli space of Riemann surfaces together with a holomorphic differentials, and the constructions used in the thesis are somewhat parallel, by generalizing to the case of meromorphic differentials, and then restricting to the real-normalized case (which excludes all holomorphic differentials). To motivate the real-normalized case and to explain the geometry of our situation, in this section we briefly summarize some of the constructions and results with holomorphic differentials.

Let $\mathcal{H}_g$ be the total space of the Hodge bundle $\mathbb{E} \to \mathcal{M}_g$, with the zero section removed, that is to say, a point $X \in \mathcal{H}_g$ is a Riemann surface $C \in \mathcal{M}_g$ together with a holomorphic differential ω on C. The fiber of the map $\mathbb{E} \to \mathcal{M}_g$ over C is $H^0(C, K_C)$, and thus $\mathcal{H}_g$ has complex dimension $4g - 3$.

Given a point in $\mathcal{H}_g$, we can either view it complex analytically as a pair (C, ω), with $\omega \in H^0(C, K_C)$, or as a structure of a flat surface on C — the notion we now recall.

Given a holomorphic one-form on a compact Riemann surface, near a point that is not a zero of ω, the abelian integral $\int \omega$ naturally gives a local coordinate on the Riemann surface, defined up to translation. This provides C with a flat metric away from the zeros of ω. If q is a zero of order n, then locally near q the differential $\omega = (n+1)z^n dz$ is the pullback $f^*(dz)$ where $f(z) = z^{n+1}$ is the ramified covering of the disc of degree $n+1$. Therefore $\int \omega$ defines a metric which has a conical singularity of angle $2\pi(n+1)$ near a zero of order n.

Therefore the pair (C, ω) defines a flat structures which realizes C as a polygon in the complex plane such that a pair of sides of the polygon are identified by translations (where we cut along geodesics in the flat metric which contain zeros of ω).

Definition 2.4.1. A collection of polygons oriented to the left in the real plane with edges appearing in pairs which are parallel and of equal length with opposite orientation is called a *flat surface* if gluing pairs of edges results in a connected surface.

Actually this definition of a flat surface given by polygons which are embedded in the real plane comes with a chosen direction to the "north". In general one would like to consider two flat surfaces as the same if one is simply given by rotating the other flat surface, and we can quotient by all rotations in the plane.

We have just constructed a map from $\mathcal{H}_g$ into the space of flat surfaces by describing how a curve equipt with a holomorphic differential prescribes a flat structure where the zeros of the differential map to the vertices. In fact we can also describe a map from the space of flat structures into $\mathcal{H}_g$, and this provides a bijection, and thus there is a dictionary between the language of complex analysis and geometry of flat structures.

Given a polygon in the plane where sides are identified by translation, the global differential dz on $\mathbb{C}$ defines a holomorphic one form on the glued surface whose abelian integral clearly gives rise to the flat structure one started with. Local coordinates in the plane near identified points are given by translation, i.e. $z = z' +$ constant, which implies $dz = dz'$, so the global differential indeed descends to define a holomorphic differential on the glued surface. The pair of flat surface and the image of dz under the gluing is therefore a point in $\mathcal{H}_g$.

Notice that near any point in a flat surface which is not a vertex of the polygon, a local coordinate chart has angle 2π (use the chosen direction to define angles), while the vertices of the polygon have total angle an integer multiple of 2π. Near a vertex with angle $2\pi(n+1)$, w provides a local coordinate chart of the glued surface where $z = w^{n+1}$ for z global coordinate in the plane, and therefore the global differential dz in the plane descends to $(n+1)w^n dw$ on the glued surface, and thus has a zero of order n.

We have thus outlined the steps in the following correspondence.

Proposition 2.4.2. *We can equivalently study the a pair (C, ω) where ω is a holomorphic differential or the corresponding flat surface of polygons in the real plane such that the global differential dz descends to ω on the glued surface.*

A holomorphic differential ω on a smooth, genus g surface has $2g-2$ zeros counted with multiplicity, and we stratify $\mathcal{H}_g$ by the multiplicities of the zeros of ω.

Definition 2.4.3. The stratum $\mathcal{H}_g(d_1, \ldots, d_m) \subset \mathcal{H}_g$, is defined to be the locus of $(C, \omega) \in \mathcal{H}_g$ where ω has m zeros with multiplicities prescribed by $d_1, \ldots, d_m$.

In general a holomorphic one-form has simple zeros, which implies that $\mathcal{H}_g(1, \ldots, 1)$ is open in $\mathcal{H}_g$, and thus has dimension $2g - 1 + (2g - 2)$. More generally, the dimension of the stratum $\mathcal{H}_g(d_1, \ldots, d_m)$ is $4g-3-(2g-2-m) = 2g - 1 + m$ as we would expect having fixed the location of $2g - 2 - m$ zeros.

Denote each edge of the polygon as a vector v_j with vertices q_j and q_{j+1}. Intuitively to describe a flat structure, one needs to prescribe the length of the vectors v_j, i.e. $\int_{q_j}^{q_{j+1}} \omega$, and distance between identified sides of the polygon. The combinatorics of the flat structure prescribes the number of singular points and their corresponding cone angles at each of the vertex in the resulting glued surface. In fact given one flat surface in $\mathcal{H}_g(1, \ldots, 1)$ one can get almost all flat surfaces in $\mathcal{H}_g(1, \ldots, 1)$ by varying these parameters, namely the length of v_j and distance between identified vectors.

Indeed we will now show that these parameters define a local coordinate system in $\mathcal{H}_g(d_1, \ldots, d_m)$.

Let $\gamma_1, \ldots, \gamma_{2g}$ be a basis of $H_1(C, \mathbb{Z})$ which can be realized in the polygon by choosing paths inside the polygon joining two identified points. Choose a set of path, $\gamma_{2g+1}, \ldots, \gamma_{2g+(m-1)}$, between the zeros q_j and q_{j+1}, which can be chosen to be some number of edges in the polygon. The collection of cycles $\gamma_1, \ldots, \gamma_{2g+(m-1)}$ is a basis of the first homology of C relative to the zeros of ω, i.e. of $H_1(C; \{q_1, \ldots, q_m\}, \mathbb{Z})$.

Definition 2.4.4. We call the periods $\int_{\gamma_i} \omega$ for $i \in \{1, \ldots, 2g\}$ the *absolute periods* of ω, and the periods $\int_{\gamma_j} \omega$ for $j \in \{2g + 1, \ldots, 2g + 1 - m\}$ are called the *relative periods* of ω.

Theorem 2.4.5 ([22])**.** *The absolute and relative periods of ω over a basis of the relative homology group $H_1(C; \{q_1, \ldots, q_m\}, \mathbb{Z})$ gives local coordinates on the stratum $\mathcal{H}_g(d_1, \ldots, d_m)$.*

There is a natural action of $SL(2, \mathbb{R})$ on $\mathcal{H}_g$ given by transforming the flat structure in the plane by the matrix action on the real plane. The new surface still glues to define a flat surface (parallel lines are sent to parallel lines), and the number and angle of vetrices is unchanged by this action. In order to verify that this action is well-defined one can check that it does not depend on the polygon representation of (C, ω), namely the choice of geodesic of the flat metric $\int \omega$ used to unwrap C.

Definition 2.4.6. For any $A \in SL(2, \mathbb{R})$ and $(C, \omega) \in \mathcal{H}_g$, $A \cdot (C, \omega)$ is the flat surface given by acting on the polygon representation of (C, ω) in the plane. If $(C, \omega) \in \mathcal{H}_g(d_1, \ldots, d_m)$, then $A \cdot (C, \omega) \in \mathcal{H}_g(d_1, \ldots, d_m)$.

We note that one can consider an action by $GL^+(2, \mathbb{R})$ on $\mathcal{H}_g$, but the SL-action preserves the area of the surface in the flat metric, and thus is well-defined on $\mathcal{H}_g(d_1, \ldots, d_m)_1$ of flat surfaces with unit area.

The action by $SL(2, \mathbb{R})$ or $GL^+(2, \mathbb{R})$ on $\mathcal{H}_g$ naturally leads to interesting questions regarding orbit closures and invariant subvarieties under this action or under the action by Borel subgroups. Of course the simplest case of a subvariety preserved by the action are the strata $H_g(d_1, \ldots, d_m)$.

The general philosophy (which is further outlined in [35]) is that one can understand some numerical characteristics of $(C, \omega) \in \mathcal{H}_g(d_1, \ldots, d_m)$ by studying the orbit closure of (C, ω) under the action of $GL^+(2, \mathbb{R})$. In this direction we state a theorem of Kontsevich and a very important breakthrough development by Eskin-Mirzkhani-Mohammadi.

Theorem 2.4.7 (Kontsevich[21])**.** *If the orbit closure of $GL^+(2,\mathbb{R}) \cdot (C,\omega) \subset \mathcal{H}_g(d_1,\dots,d_m)$ is a complex analytic subvariety then it is an affine variety in coordinates given by absolute and relative periods, i.e. $H_1(C;\{q_1,\dots,q_m\},\mathbb{Z})$.*

Theorem 2.4.8 (Eskin-Mirzkhani-Mohammadi[8])**.** *Every $SL(2,\mathbb{R})$ obit closure is an affine invariant* subvariety *in the stratum, which is a subvariety defined by real linear equations in local period coordinates.*

One can define a natural holomorphic foliation in any stratum $\mathcal{H}_g(d_1,\dots,d_m)$ called the *kernel*, *period*, or *isoperiodic* foliation [35][26]. The foliation is defined by fixing all absolute periods and varying the relative periods, and a leaf of this foliation is an affine subvariety in period coordinates.

Definition 2.4.9. The stratum $\mathcal{H}_g(d_1,\dots,d_m)$ is foliated by $m-1$ dimensional leaves defined by fixing all absolute periods and varying the relative periods.

Local holomorphic coordinates on the leaves are given by the relative periods, and the leaves are immersed submanifolds. Geometrically this foliation is given by moving the edges of the polygons in the plane relative to each other keeping the distance between the identified edges fixed. One can ask questions regarding the distribution of these leaves in $\mathcal{H}_g(d_1,\dots,d_m)$, namely is there a dense leaf and does there exist a leaf which is an embedded complex manifold.

Remark 2.4.10. The study of flat surfaces and Teichmüller dynamics is a rich area with connections to many areas of mathematics. What has been summarized in this section is only a small subset of the current work with flat surfaces. We have focused our summary almost exclusively on areas which will be generalized to the set-up in the thesis. Namely the local coordinates in each stratum $H_g(d_1, \ldots, d_m)$ as well as the period foliation can be defined for the case of meromorphic differentials with real periods, and this will be done carefully in the next chapter.

We would like to note that many questions which have been studied for holomorphic differentials regarding ergodicity question such as studying the Teichmüller flow in each leaf of the period foliation may be generalized to the set-up used in the thesis, and these questions may be worth further study. We would like to point out recent work in this direction [4] for general meromorphic differentials which may be further applied to real-normalized differentials, a subset in the space of all meromorphic differentials.

Meromorphic differentials naturally appear on the boundary of $\mathcal{H}_g$, and our techniques used to understand degenerations of real-normalized differentials may shed light on analogous question in Teichmüller dynamics regarding a correct compactification of $\mathcal{H}_g$.

Chapter 3

Real-Normalized Meromorphic Differentials

In this chapter we introduce the notion of a real-normalized differential and review previous work specifically of Grushevsky and Krichever suggesting that real-normalized differentials provide a useful approach toward understanding the geometry of $\mathcal{M}_g$. Using real-normalized differentials one can define local coordinates, a foliation, and globally well-defined continuous real-analytic functions on the moduli space of curves endowed with some extra data. This material relates directly to the thesis, and therefore we attempt to address all issues carefully.

Definition 3.0.11. A meromorphic differential ω on a smooth curve C is called *real-normalized* if the periods of ω over any cycle on C are real, i.e. $\int_\gamma \omega \in \mathbb{R}$ for all $\gamma \in H_1(C, \mathbb{Z})$.

3.1 Local Coordinates

We begin by introducing notations for the various bundles and the moduli spaces we will consider, as well as forgetful maps between these spaces. In addition we define local coordinates on each of these spaces– recall that by local coordinates we are constructing local coordinates on the cover near orbifold points.

Let $\underline{m} := (m_1, \ldots, m_n)$ be any set of positive integers and let $|\underline{m}| := \sum_{i=1}^{n}(m_i + 1)$. Recall $\sigma_i : \mathcal{M}_{g,n} \to \mathcal{C}_{g,n}$ is the section given by the i'th marked point, and we define the following bundle on $\mathcal{M}_{g,n}$.

Definition 3.1.1. The direct image of the relative dualizing sheaf ω_π twisted by $\sum(m_i + 1)\sigma_i$ is denoted $\Omega\mathcal{M}_{g,n}^{\leq \underline{m}}$.

The fibers of $\Omega\mathcal{M}_{g,n}^{\leq \underline{m}}$ over each curve with marked points $(C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n}$ is $H^0(C, K_C + \sum_i(m_i + 1)p_i)$, and thus a section of the bundle $\Omega\mathcal{M}_{g,n}^{\leq \underline{m}}$ over each point $(C, p_1, \ldots, p_n)$ is (C, ω) where ω is a *meromorphic* differential with at worst poles of order $m_i + 1$ at p_i.

We consider the open subset of this bundle where the order of singularities is fixed to be exactly $k_i + 1$:

Definition 3.1.2. Define $\Omega\mathcal{M}_{g,n}^{\underline{m}} := \{(C, p_1, \ldots, p_n) \in \mathcal{M}_{g,n}, \omega \in H^0(K_C + \sum_i(m_i+1)p_i) - \cup_{j=1}^{n} H^0(K_C + \sum_i(m_i+1)p_i - p_j)\}$, the moduli space of smooth curves with n marked points and a meromorphic differential with singularities of order exactly $m_i + 1$ at p_i which are holomorphic elsewhere.

By Riemann-Roch it easily follows that for any C we have $\dim H^0(C, K_C + \sum_{i=1}^{n}(m_i + 1)p_i) = g - 1 + |\underline{m}|$. Therefore the complex dimension of $\Omega\mathcal{M}_{g,n}^{\underline{m}}$, which is an open subset of $\Omega\mathcal{M}_{g,n}^{\leq m}$, is $4g-4+n+|\underline{m}|$, and for each point X in $\Omega\mathcal{M}_{g,n}^{m}$, the set of zeros of ω counted with multiplicity is of degree $2g-2+|\underline{m}|$ and denoted $\sum_{s=1}^{2g-2+|\underline{m}|} q_s$ where q_s are not necessarily distinct.

The abelian integral, $\int \omega$, which is defined locally on C, has critical points at the zeros of ω, and thus the critical values of $\int \omega$ are $\int^{q_s} \omega$. On $X = (C, p_1, \ldots, p_n, \omega) \in \Omega\mathcal{M}_{g,n}^{m}$, choose a symplectic basis for $H_1(C, \mathbb{Z})$, an ordering of the zeros, and paths between q_1 and q_s which do not intersect marked points $p_1, \ldots, p_n$.

The following definition is an exact replica of the absolute and relative periods defined for a holomorphic differential:

Definition 3.1.3. The *absolute periods* of a meromorphic differential ω (with no residues) on a smooth curve C are given by $\alpha_i(\omega) := \int_{A_i} \omega$ and $\beta_i(\omega) := \int_{B_i} \omega$ where $\{A_i, B_i\}_{i=1}^{g}$ is a symplectic basis of $H_1(C, \mathbb{Z})$ that does not pass through any of the marked points.

The *relative periods* of a meromorphic differential ω are defined to be $\phi_s(\omega) := \int_{q_1}^{q_s} \omega$ for some ordering of the zeros and a choice of path between q_1 and q_s which avoids the poles.

Remark 3.1.4. If ω has non-zero residues, the absolute periods are given by integrating ω over a basis of $H_1(C \setminus \{p_i\}, \mathbb{Z})$ which we denote as $\{A_i, B_i\}_{i=1}^{g}$ and $\gamma_1, \ldots, \gamma_n$ where $\{A_i, B_i\}$ are $2g$ nontrivial cycles on $C \setminus \{p_1, \ldots, p_n\}$ with $A_i \cdot A_j = B_i \cdot B_j = 0$ and $A_i \cdot B_j = \delta_{i,j}$ and each γ_i is a small loop around the marked point p_i.

The absolute and relative periods will describe local holomorphic coordinates on $\Omega\mathcal{M}_{g,n}^{m}$, and thus we must define how to vary the relative periods in the case where some of the zeros of Ψ_X occur with multiplicity, i.e. if $q_{s_1} = \ldots = q_{s_r}$. The zeros may cease to be multiple under a small perturbation, and thus in order to define local coordinates which vary holomorphically in a neighborhood of X, we consider the symmetric polynomials of the ϕ_s's corresponding to each of the multiple zeros.

Let s_ℓ denote the symmetric polynomial of degree ℓ in r variables for $\ell \in \{1, \ldots, r\}$, then $\sigma_\ell(\omega) := s_\ell(\phi_2, \ldots, \phi_r)$ deforms holomorphically in the neighborhood of the point with a zero of multiplicity r. For each point $X \in \Omega\mathcal{M}_{g,n}^{\underline{m}}$, and for each zero with corresponding multiplicity r_i, the collection of symmetric polynomials in r_i variables of the critical values for all zeros denote collectively as $\sigma_1(\omega), \ldots, \sigma_{2g-3+|\underline{m}|}(\omega)$. These deforms holomorphically in the neighborhood of $X \in \Omega\mathcal{M}_{g,n}^{\underline{m}}$.

A basis of cycles of $H_1(C \setminus \{p_i\}, \mathbb{Z})$ and paths between ordered zeros which do not intersect marked points can be chosen to vary continuously on curves X_t in a small neighborhood of X.

Let $\rho_i(\omega) := res_{p_i}\omega = 2\pi\sqrt{-1}\int_{\gamma_i}\omega$, the residue of the meromorphic differential near the singular points of the differential ω.

Theorem 3.1.5. *[14] In a small neighborhood of any point $X \in \Omega\mathcal{M}_{g,n}^{\underline{m}}$ the absolute periods, $\alpha(t) = (\alpha_1(\omega_t), \ldots, \alpha_g(\omega_t)), \beta(t) = (\beta_1(\omega_t), \ldots, \beta_g(\omega_t))$, the relative periods, $\sigma(t) = (\sigma_1(\omega_t), \ldots, \sigma_{2g-3+|\underline{m}|}(\omega_t))$ and residues $\rho(t) = (\rho_1(\omega_t), \ldots, \rho_n(\omega_t))$ have linearly independent gradients, and thus they define a holomorphic local coordinate system in a neighborhood of X in $\mathcal{M}_{g,n}(\underline{m})$.*

Remark 3.1.6. Recall from the holomorphic case, the neighborhood of a point in $\mathcal{H}_g(d_1, \ldots, d_m)$ local coordinates are given by the relative and absolute periods of the holomorphic differential with fixed order of zeros. In our case of real-normalized meromorphic differentials, where the multiplicity of zeros is no longer fixed, we need to introduce symmetric functions in order to define local functions which deform holomorphically. In addition, because the differential is meromorphic, we will need to include the globally defined functions given by the residues.

The following proof of 3.1.5, which we outline, originally appears in [24] with only small modifications to prove the statement above. The ideas in this proof will be used for a number of statements below with few changes, and thus we record it here and later indicate how is will be used to prove other statements. The following proof uses a choice of π_1 on curves in a small neighborhood of C_0, and thus we are working on the the cover of moduli space near orbifold points. If one traces the steps of the following proof, we actually define a differential on the curve C_0 though it was initially only defined on the cut curve.

Proof. The residues are clearly independent, and we show that the assumption that relative periods and absolute periods are linearly dependent leads to a contradiction.

Assume for contradiction that these functions are linearly dependent at X_0 which implies there exists a one dimensional deformation of X_0, i.e. a one-dimensional family $X_t = (C_t, \omega(t))$, such that the derivative with respect to t of the absolute and relative periods of $\omega(t)$ are zero at $t = 0$. Therefore we assume for contradiction that there is a one-dimensional family X_t with $\frac{\partial}{\partial t}|_{t=0}\alpha(t) = \frac{\partial}{\partial t}|_{t=0}\beta(t) = \frac{\partial}{\partial t}|_{t=0}\sigma(t) = 0$.

Locally choose a continuously varying symplectic basis of $H_1(C_t, \mathbb{Z})$, and let $\omega_j(t), j \in \{1, \ldots, g\}$ be a basis of $H^0(C_t, K_{C_t})$ dual to the cycles A cycles. Let Γ_t be the family of cut curves, which is the family of C_t cut along the chosen paths which define a basis of $H_1(C_t, \mathbb{Z})$. Consider the abelian integral $f_t(p) := \int_{q_1(t)}^{p} \omega_t$ which is well-define on the the cut curve, and in order to fix a point on C_t, use the level sets of f_t. In other words if $x = f_0(p)$ for $p \in C_0$, let $p_t = f_t^{-1}(x)$ which is a marked point in C_t. Now let $F_{t,j}(p)$ be the abelian integral $\int_{q_1(t)}^{p} \omega_j(t)$ which depends on the path chosen in C_t but is well-defined on the cut curve Γ_t.

We define the following differential which is initially defined on the cut curve Γ_0 and may have simple poles:

$$\frac{\partial}{\partial t} F_{t,j}(f_t^{-1}(x))|_{t=0}$$

In fact, as we will show, $\frac{\partial}{\partial t} F_{t,j}(f_t^{-1}(x))|_{t=0}$ is a holomorphic differential on C_0 with zero periods around the cycles A_i. This in turn implies that the differential $\frac{\partial}{\partial t} F_{t,j}(f_t^{-1}(x))|_{t=0}$ is identically zero, and thus has zero periods around the cycles B_i. Therefore $\frac{\partial}{\partial t} \int_{B_j} \omega_j(t) = 0$ for all i and j, and thus the torelli map $\tau : \mathcal{M}_g \to \mathcal{A}_g$, $\tau(C)$=period matrix of C, is not an embedding near C_0. By the infitesimal Torelli theorem, C_0 must be hyperelliptic and the family C_t is transverse to the hyperelliptic locus (the kernel of $d\tau$ near C_0 is one dimensional).

Clearly the differential $\frac{\partial}{\partial t} F_{t,j}(f_t^{-1}(x))|_{t=0}$ is holomorphic on C_0 as long as $f_t^{-1}(x)$ is non-singular. The singularity of $f_t^{-1}(x)$ occur when $p = f_0^{-1}(x)$ is a zero of ω. By assumption $\frac{\partial}{\partial t} \int_{q_1(t)}^{p} \omega|_{t=0} = 0$ because the critical values do not change with respect to t. One can show that the coefficient of the singular part of $\frac{\partial}{\partial t}F_{t,j}(f_t^{-1}(x))|_{t=0}$ is zero because it involves the term $\frac{\partial}{\partial t}\int_{q_1(t)}^{p}\omega|_{t=0} = 0$ which is zero.

By the same argument the jumps of $F_{t,j}(f_t^{-1}(x))|_{t=0}$ along the edges of the cut surface Γ_t are fixed with respect to t because $\frac{\partial}{\partial t}|_{t=0}\alpha(t) = \frac{\partial}{\partial t}|_{t=0}\beta(t) = 0$. Therefore $\frac{\partial}{\partial t}F_{t,j}(f_t^{-1}(x))|_{t=0}$ has no jumps along the edges of Γ_0, and it defines a holomorphic differential on C_0.

To finish the argument one must show that if C_0 is hyperelliptic then $\frac{\partial}{\partial t}|_{t=0}$ is tangent to the hyperelliptic locus. Thus providing a contradiction with the infinitesimal Torelli theorem. This is done carefully in [24] by showing that if C_0 is hyperelliptic, then up to $O(t^2)$ the curves C_t are also hyperelliptic. □

We will now define the bundle over $\mathcal{M}_{g,n}$ of curves with jets on which we will work directly in the thesis.

Definition 3.1.7. An m-jet of a local coordinate for $m \geq 1$ at a marked point $p \in C$ is an equivalence class of local coordinates at p such that $z \sim z'$ if $z' = z + O(z^{m+1})$.

In particular, a 1-jet of a local coordinate at p is equivalent to the choice of a non-zero tangent vector at p.

In general, if $z' = z + az^{m+1}$, then $\frac{dz'}{z'^{m+1}} = \frac{1+a(m+1)z^m}{z^{m+1}(1+az^m)^{m+1}}dz = \frac{1}{z^{m+1}}dz + O(1)dz$ because $\frac{1}{(1+az^m)^{m+1}}$ is regular at p where $z(p) = 0$. Therefore if $z \sim z'$ as an m-jet then $\frac{dz'}{z'^{m+1}} = \frac{dz}{z^{m+1}} + O(1)dz$. This implies the following simple lemma.

Lemma 3.1.8. *If $m > 1$, prescribing an m-jet of local coordinates near p is* *equivalent to fixing the singular part of a meromorphic differential with a pole of order $m+1$ at p.*

In order to fix the singular part of a differential with a simple pole, no jet is needed because the residue is globally well-defined and does not depend on the choice of coordinate system near p. In all the following constructions we will only consider differentials with purely imaginary residues, and define a zero jet to be no extra data at the marked point. To describe a differential with residue, we will specify the choices of $a \in i\mathbb{R}$ with the condition that the sum of residues is zero.

In order to fix the singular part of a meromorphic differential with a double pole at p one is required to choose a 1-jet.

Definition 3.1.9. Let $\mathcal{M}_{g,n}^{m}$ be the moduli space of smooth genus g curves C with n marked points $p_1, \ldots, p_n$ and a choice of m_i-jet of local coordinate denoted z_i near each marked point p_i.

There is naturally a forgetful map from $\mathcal{M}_{g,n}^{m}$ to $\mathcal{M}_{g,n}$ by forgetting the jets which exhibits $\mathcal{M}_{g,n}^{m}$ as $\prod(\mathbb{C}^{m_i} \setminus \mathbb{C}^{m_i-1})$ bundle over $\mathcal{M}_{g,n}$. Thus $\mathcal{M}_{g,n}^{m}$ naturally has a complex structure as the total space of a bundle over $\mathcal{M}_{g,n}$. For example $\mathcal{M}_{g,1}^{(1)}$ is a $\mathbb{C}^*$ bundle over $\mathcal{M}_{g,1}$ or alternatively it is the total space of the bundle whose fiber over (C,p) is the tangent space to C at the marked point p (i.e. the dual of $\mathbb{L}_1$) with the zero section removed.

The following result is a straightforward application of Riemann's bilinear relations, and it will be used to define a real-analytic section from $\mathcal{M}_{g,n}^{m}$ to $\Omega\mathcal{M}_{g,n}^{m}$.

Theorem 3.1.10. *Given any point $X \in \mathcal{M}_{g,n}^{m}$ and any complex polynomials $R_i = \sum_{\alpha=0}^{m_i} r_{i,\alpha} z_i^{-\alpha-1}$, for each $i \in \{1, \ldots, n\}$ with $r_{i,0} \in i\mathbb{R}$ and $\sum_i r_{i,0} = 0$, there exists a unique real normalized differential on C which is holomorphic away from the marked points and has singular part at the marked points p_i of the form $R_i dz_i + O(1)dz_i$ in the the local m_i-jet z_i.*

Proof. Assume for contradiction ω and ω' were both real-normalized differentials with singularities at the marked points prescribed by R_i. The difference $\omega - \omega'$ is holomorphic and real-normalized, and thus because the period matrix is non-degenerate, $\omega - \omega'$ is identically zero which proves uniqueness.

To prove existence, we construct a real-normalized differential with prescribed singularities explicitly starting with ϕ, any differential with prescribed singularities. If $\{\omega_1, \ldots, \omega_g\}$ is a normalized basis of holomorphic differentials on C with respect to a symplectic basis A_i, B_i of $H_1(C, \mathbb{Z})$, then $\tilde{\phi} := \phi - \sum_{i=1}^{g} (\int_{A_i} \phi) \omega_i$ has zero periods around the cycles A_i.

Let $\boldsymbol{c} := (c_1, \ldots, c_g)$ where $c_i = \operatorname{Im} \int_{B_i} \tilde{\phi}, i \in \{1, \ldots, g\}$ and $\operatorname{Im} \tau$ be the imaginary part of the period matrix. By Riemann's bilinear relations, specifically the positive definiteness of the imaginary part of the period matrix, there exists a vector $\boldsymbol{b} = (b_1, \ldots, b_g)$ such that $\operatorname{Im} \tau \boldsymbol{b} = \boldsymbol{c}$.

Clearly $\tilde{\phi} - \sum_{i=1}^{g} b_i \omega_i$ is a real-normalized differential with the prescribed singularities. □

It follows that if all $m_i > 0$ then for each point $X \in \mathcal{M}_{g,n}^{m}$, there exists a unique real-normalized differential with singularities $\frac{dz_i}{z_i^{m_i+1}}$.

Definition 3.1.11. Assume all $m_i > 0$ and $X \in \mathcal{M}_{g,n}^{m}$, let Ψ_X be the unique real-normalized differential on X with singularities $\frac{dz_i}{z_i^{m_i+1}}$ near marked points

p_i. This defines a real-analytic section from $\mathcal{M}_{g,n}^m$ to $\Omega\mathcal{M}_{g,n}^{\leq m}$ which by abuse of notation will be denoted Ψ as well.

Remark 3.1.12. In previous work [14] [15] two key examples were considered. A *dipole* differential is a differential with two simple poles. It follows from the previous theorem that given a point $(C, p_-, p_+) \in \mathcal{M}_{g,2}$, there exists a unique real-normalized differential with simple poles of residue $\pm i$ at marked points $p_\pm$. In the case of a dipole differential, we can work directly on the moduli space, and the situation is completely algebraic.

Another example of importance is $\mathcal{M}_{g,1}^{(1)}$. For each point $X \in \mathcal{M}_{g,1}^{(1)}$ there is a unique real-normalized differential Ψ_1 with one double pole whose singular part in the local 1-jet is $\frac{dz}{z^2}$. In addition, there is a unique real-normalized differential Ψ_2 whose singular part is given by $i\frac{dz}{z^2}$.

Using the section Ψ from $\mathcal{M}_{g,n}^m$ to $\Omega\mathcal{M}_{g,n}^m$ and exact replica of the proof of theorem 3.1.5 where the connection on the universal cut curve is given by the real-normalized differentials, as opposed to a general meromorphic differential, one can prove that relative and absolute periods are local coordinates on $\mathcal{M}_{g,n}^m$. The proof of theorem 3.1.5 concludes by showing a that if such function are not local coordinates on the moduli of curves with a prescribed real-normalized differentials (the locus under Ψ of $\mathcal{M}_{g,n}^m$ in $\Omega\mathcal{M}_{g,n}^m$), there is a family of curves along which the Torelli theorem is contradicted, therefore this provides local coordinates on the moduli of curves with jets as well.

Theorem 3.1.13. *For any $X \in \mathcal{M}_{g,n}^m$, the absolute and relative periods of the real-normalized differentials Ψ_{X_t} give a local* real-analytic *coordinate system in a neighborhood of $X \in \mathcal{M}_{g,n}^m$.*

Proof. The proof follows exactly along the lines of the proof 3.1.5. The only change required is to replace the connection defining points on nearby curves to be given by Ψ, the real-normalized differential, instead of a general meromorphic differential ω with fixed singularities. $\square$

The coordinates are no longer holomorphic in a neighborhood of $X \in \mathcal{M}_{g,n}^m$ because the absolute periods are all real, and Ψ is a real-analytic section.

3.2 Foliation

In this section we introduce a foliation on the moduli space $\mathcal{M}_{g,n}^m$ by fixing all absolute periods of Ψ. There is certainly a well-defined foliation on the Torelli space (or alternatively the Teichmüller space) where there is a chosen symplectic basis of $H_1(C, \mathbb{Z})$ on each curve. As we will verify this descends to define a foliation on $\Omega\mathcal{M}_{g,n}^m$ which in turn restricts to a well-defined foliation of $\mathcal{M}_{g,n}^m$.

A *foliation* of a manifold M is a set of equi-dimensional leaves such that through any point x in M there exist a unique leaf containing x.

Let $\Omega\mathcal{T}_{g,n}^m$ be the moduli space of points in $\Omega\mathcal{M}_{g,n}^m$ with additionally a chosen basis of $H_1(C \setminus \{p_1, \ldots, p_n\}, \mathbb{Z})$. One can foliate the moduli space $\Omega\mathcal{T}_{g,n}^m$ by embedded submanifolds defining each leaf to be the locus where absolute periods and residues are fixed.

Let $\underline{r} := (r_1, \ldots, r_n), \underline{a} := (a_1, \ldots, a_g), \underline{b} := (b_1, \ldots, b_g)$ be any set of fixed complex numbers.

Definition 3.2.1. A leaf of the foliation on the space $\Omega\mathcal{T}_{g,n}^m$ is given by the locus, $\mathcal{L}_{\underline{r},\underline{a},\underline{b}}$ on which the residues and absolute periods around the chosen basis of $H_1(C \setminus \{p_1, \ldots, p_n\}, \mathbb{Z})$ are fixed to be $\underline{r}, \underline{a}, \underline{b}$ respectively.

We now verify that this descends to define a foliation on $\Omega\mathcal{M}^{m}_{g,n}$ although we no longer have a chosen basis of $H_1(C \setminus \{p_1, \ldots, p_n\}, \mathbb{Z})$.

Lemma 3.2.2. *The moduli space $\Omega\mathcal{M}^{m}_{g,n}$ is foliated by locally immersed leaves $\mathcal{L}_{\underline{r},\underline{a},\underline{b}}$ defined by fixing residues and fixing all absolute periods to be locally constant.*

Proof. Two symplectic bases of $H_1(C, \mathbb{Z})$ differ by an element in the symplectic group $Sp(2g, \mathbb{Z})$, and if all absolute periods are locally constant in one basis then they are locally constant in another basis as well. The action of $Sp(2g, \mathbb{Z})$ permutes the leaves of the foliation on $\Omega\mathcal{T}^{m}_{g,n}$, and thus we have a well-defined leaf on $\Omega\mathcal{M}^{m}_{g,n}$ defined by fixing absolute periods locally.

Local holomorphic coordinates on $\mathcal{L}$ are given by relative periods, and the absolute periods and residues define holomorphic coordinates in the direction transverse to the leaves. □

The notation above is deceiving because a symplectic basis on a fixed curve is only determined up to an action of $Sp(2g, \mathbb{Z})$, and thus one can not actually determine the complex values $\underline{a}, \underline{b}$ on a fixed leaf. The exception is the leaf of exact differentials, $L_{\underline{r},\underline{0},\underline{0}}$, where all periods are zero as the action of $Sp(2g, \mathbb{Z})$ fixes the periods $\underline{0}, \underline{0}$.

Each leaf has codimension $2g + n - 1$ in $\Omega\mathcal{M}^{m}_{g,n}$ as it is defined by fixing $2g + n - 1$ independent holomorphic functions.

Now it is clear that there is in fact a well-defined foliation of the moduli space $\mathcal{M}^{m}_{g,n}$ by leaves with all absolute periods real (the residues are purely imaginary and fixed). We can consider the subset of points in $\Omega\mathcal{M}^{m}_{g,n}$ where ω is real-normalized. Any leaf $\mathcal{L}$ defined by fixing all absolute periods to be real and residues purely imaginary which intersects this locus of real-normalized differentials in $\Omega\mathcal{M}^{m}_{g,n}$ is in fact by definition contained in this locus. Recall that on $\mathcal{M}^{m}_{g,n}$ the residues are automatically fixed as we defined it, the section Ψ maps $\mathcal{M}^{m}_{g,n}$ into this space of real-normalized differentials, and the leaves $\mathcal{L}$ provided a foliation on the space $\mathcal{M}^{m}_{g,n}$.

Definition 3.2.3. The moduli space $\mathcal{M}^{m}_{g,n}$ is foliated by locally complex leaves $\mathcal{L}$ defined fixing the all absolute periods $\mathcal{L}_{\underline{a},\underline{b}}$.

Each leaves of the foliation $\mathcal{L}_{\underline{a},\underline{b}} \subset \mathcal{M}^{m}_{g,n}$ are locally smooth embedded complex submanifold with local coordinates give by relative periods of Ψ. On the other hand, the foliation is real-analytic in the direction transverse to the leaves, given by absolute periods which are all real. Thus $\mathcal{L}$ is an foliation of $\mathcal{M}^{m}_{g,n}$ with complex leaves which is real-analytic in the transverse direction.

Remark 3.2.4. In fact we can now view the theorem providing local coordinates as a generalization of the Lyashko-Looijenga coordinates on the Hurwitz space. When $n > 1$ or $n = 1$ but $m > 1$ there is a leaf $\mathcal{L}_{\underline{0}}$ where all absolute periods are zero. This is the leaf of exact differentials $\Psi = df$ for a meromorphic function f unbranched as a map to $\mathbb{P}^1$ except over ∞ where fixing the singularities of Ψ is equivalent to fixing the branching behavior of points over ∞.

Therefore a point $X \in \mathcal{L}_{\underline{0}}$ is a curve with an exact differential df such that the map $f : C \to \mathbb{P}^1$ has prescribed branching over one point in $\mathbb{P}^1$, and this realizes $\mathcal{L}_{\underline{0}}$ as a $\mathbb{C}^*$ bundle over the Hurwitz space. The Lyashko-Looijenga mapping, defined as the critical values of f corresponds exactly to the relative periods of df.

The benefit of working with real-normalized differentials and viewing the Hurwitz space as one leaf is the ability to locally perturb the value of an absolute period [15].

Remark 3.2.5. Recall an equivalently defined foliation of $\mathcal{H}_g(m_1, \ldots, m_n)$ by fixing the absolute periods of the holomorphic abelian differential.

It may be surprising that one can now easily define global continuous functions on $\mathcal{M}_{g,n}^{m}$ which are harmonic when restricted to a leaf of our foliation $\mathcal{L}$. Choose the base point of integration q_0 to be a zero of Ψ such that $\int_{q_0}^{q_s} \Psi \geq 0$ for all s, or alternatively a base point can be chosen so that $\sum_i \mathrm{Im} \int^{q_i} \Psi = 0$.

Lemma 3.2.6. *The imaginary part of the relative periods,* $f_s(X) := \mathrm{Im} \int^{q_s} \Psi_X$ *is a well-defined continuous function from* $\mathcal{M}_{g,n}^{m} \to \mathbb{R}$ *for* $s \in \{1, \ldots, 2g-2+|m|\}$*. Clearly* $f_i|_{\mathcal{L}}$ *is harmonic on any leave* $\mathcal{L}$ *as it is the imaginary part of a holomorphic function.*

The proof is extremely simple, but we highlight this fact as it provides a powerful new tool (not available in the holomorphic case of $\mathcal{H}_g(d_1, \ldots, d_m)$) for understanding the geometry of $\mathcal{M}_{g,n}$. These functions are the main ingredient in a simple and straightforward new proof of Diaz's theorem bounding the dimension of compact subvarities of $\mathcal{M}_{g,n}$.

Proof. Let γ_1 and γ_2 denote two paths between the base point and q_s. The integral of Ψ over any closed cycle is real, $\int_{\gamma_1} \Psi = \int_{\gamma_2} \Psi + c$ for $c = \int_{\gamma_1 \cup \gamma_2^-} \Psi \in \mathbb{R}$, and thus $\mathrm{Im} \int^{q_s} \Psi$ is well-define independent of the choice of path between end points. $\square$

Certainly having an foliation on the moduli space $\mathcal{M}^m_{g,n}$ is a lot weaker than having an affine stratification with $g-1$ strata on $\mathcal{M}_{g,n}$ as conjectured by Roth and Vakil. The current work using this affine foliation suggests that having an affine foliation is in fact useful toward understanding the geometry of $\mathcal{M}_{g,n}$.

In fact one can directly prove various known vanishing properties using this foliation. For example Diaz's theorem is a basic application of this setup. In future work we suggest that new vanishing results may be proven using the real-normalized perspective on $\mathcal{M}_{g,n}$.

Chapter 4

Meromorphic Differentials with One Double Pole: $\mathcal{M}_{g,1}^{(1)}$

In the thesis we focus on the case where the unique real-normalized differential is of the second kind, and from now on we assume all residues are zero unless it is explicitly stated otherwise. In this chapter we recall here previous work of Grushevsky and Krichever on $\mathcal{M}_{g,1}^{(1)}$ which motivated our interest in real-normalized differentials.

Throughout this chapter $X = (C, p, z)$ is a point in $\mathcal{M}_{g,1}^{(1)}$, and recall notation for two real-analytic sections from $\mathcal{M}_{g,1}^{(1)}$ to $\Omega\mathcal{M}_{g,1}^{(1)}$ which were defined earlier.

Definition 4.0.7. For $X \in \mathcal{M}_{g,1}^{(1)}$, let $\Psi_1(X)$ be the unique real-normalized differential on C with one double pole at p locally of the from dz/z^2, and $\Psi_2(X)$ denotes the unique real-normalized differential locally near p of the form idz/z^2.

Remark 4.0.8. Alternatively the second differential, $\Psi_2(X)$, can be viewed as an imaginary-normalized differential with one double pole locally of the form $\frac{dz}{z^2}$. An *imaginary-normalized* differential is a differential with prescribed singularities and all absolute periods purely imaginary.

Any real-normalized differential Ψ with one double pole at p is a $\mathbb{R}$-linear combination of Ψ_1 and Ψ_2. If Ψ is locally of the form $(a+ib)\frac{dz}{z^2}$ in the jet z, then $\Psi = a\Psi_1 + b\Psi_2$. It follows therefore if q is a zero of both Ψ_1 and Ψ_1, it is also a zero of any real-normalized differential on C with a double pole at p, and one can define a locus of common zeros of Ψ_1 and Ψ_2 which is no longer dependent on the 1-jet.

For any divisor $E = \sum_{i=1}^{n} d_i q_i$ on the curve C, $\Psi \in H^0(C, K_C + 2p - E)$ if Ψ has zeros of order at least d_i at q_i.

Definition 4.0.9. Let $\hat{\mathcal{D}}_n \subset \mathcal{M}_{g,1+n}^{(1,0\ldots,0)}$ be the locus of points $(C, p, q_1, \ldots, q_n, z)$, such that the n marked points $(q_1, \ldots, q_n)$ are zeros of both $\Psi_1(C,p,z)$ and $\Psi_2(C,p,z)$ for z a 1-jet, i.e. $\Psi_j(C,p,z) \in H^0(C, K_c - \sum q_i)$ for $j \in \{1,2\}$.

We consider the two forgetful maps, $\pi_1 : \mathcal{M}_{g,1+n}^{(1,0,\ldots,0)} \to \mathcal{M}_{g,1}^{(1)}$ and $\pi_2 : \mathcal{M}_{g,1+n}^{(1,0,\ldots,0)} \to \mathcal{M}_{g,1+n}$. The first map forgets the n marked points, and the second map forgets the 1-jet at the first marked point.

The image of $\hat{\mathcal{D}}_n$ under π_1 is denoted $\hat{\mathcal{D}}$ which is the locus of curves $X \in \mathcal{M}_{g,1}^{(1)}$ such that $\Psi_1(X)$ and $\Psi_2(X)$ have at least n distinct common zeros. The image of $\hat{\mathcal{D}}_n$ under π_2 is denoted $\mathcal{D}_n$ and is the locus of curves $(C, p, q_1, \ldots, q_n) \in \mathcal{M}_{g,1+n}$ such that any real-normalized differential with a double pole at the first marked point, has a zeros at each one of the other marked points $(q_1, \ldots, q_n)$. Equivalently this is the locus such that for any jet,

z, $\Psi_1(C, p, z)$ and $\Psi_2(C, p, z)$ with double poles at the first marked point both have zeros at each one of the other marked points.

Remark 4.0.10. In was shown in [15] that the cohomology class of the locus $\mathcal{D}_n \in \mathcal{M}_{g,1+n}$ is tautological. A zero of Ψ_i at q_i is a section of $\mathcal{L}_i$ (viewed as a bundle over $\mathcal{M}_{g,1+n}^{(1,0,\ldots,0)}$), in other words this is ψ_i, and a common zero is ψ_i^2 which is the section squared. Therefore it is clear that the locus of common zeros $\hat{\mathcal{D}}_n$ in $\mathcal{M}_{g,1+n}^{(1,0,\ldots,0)}$ is $\prod_{i=1}^{n} \psi_i^2$ which is tautological. In addition this locus is independent of the chosen jet at the first marked point, and one can further verify that $\mathcal{D}_n$ is tautological.

4.1 Map on the Tangent Space at a point X in $\mathcal{M}_{g,1}^{(1)}$

In order to achieve our goal of recording some of the previous work on $\mathcal{M}_{g,1}^{(1)}$, this section will contain a very brief and incomplete outline of some results obtained in [15]. The following map on the tangent space $T_X(\mathcal{M}_{g,1}^{(1)})$ originally appeared in [**?**] where is was shown to be an isomorphism of the tangent space at points where $\Psi_1(X)$ and $\Psi_2(X)$ have no common zeros on C and a space of differentials on a cut surface with one simple pole at the marked point. In [15] it was further shown that the loci of common zeros, $\hat{\mathcal{D}}_n$, can be viewed as degeneracy loci of this map on $T_X(\mathcal{M}_{g,1}^{(1)})$.

For $X = (C, p, z) \in \mathcal{M}_{g,1}^{(1)}$ and the curve C, choose a standard generating set for $\pi_1(C)$ and denote the cut curve C^* obtain by removing this set of path which is a polygon with $2g$ sides. Such a generating set for $\pi_1(C)$ can be chosen consistently in neighborhood of $C \in \mathcal{M}_g$, and locally we can consider the universal cut curve, which is given by cutting each curve in a small neighborhood of C along a chosen generating set of π_1 which varies smoothly.

Definition 4.1.1. Let $\hat{\mathcal{T}}(X)$ be the space of differentials with are holomorphic inside the $2g$-gon $C^* \setminus \{p\}$, continuous on C^* with at worst a simple pole at p and jumps along the identified sides of the polygon given by an $\mathbb{R}$-linear combination of the absolute periods of $\Psi_1(X)$ and $\Psi_1(X)$ along these cycles.

Let v be a tangent vector in $T_X(\mathcal{M}_{g,1}^{(1)})$ and $X_{tv} = (C_{tv}, \Psi_1(X_{tv}))$ denote a corresponding family of points in a neighborhood of X whose tangent direction is v. Choose a generating set for $\pi_1(C_{tv})$ consistently and denote the corresponding cut curves as C_{tv}^*. Following a very similar idea to the one used in the proof of 3.1.5, and using similar notation, we denote the abelian integral $F_{tv,1}(q) := \int^q \Psi_1(X_{tv})$ on C_{tv}^*, and use $F_1(q) = \int^q \Psi_1(X)$ to define a connection on the universal cut curve. Equivalently fixing the value of $F_{tv,1}(q)$ to be $F_1(q)$ defines a way to vary points on the cut curves C_{tv}^* in a small neighborhood of C^*.

In fact to correctly define $F_{tv,1}$, a 2-jet is needed to fix the constant of integration, and we will not address this issue and remark here that everything done here can be lifted to $\mathcal{M}_{g,1}^{(2)}$ where this map is actually well-defined.

Consider the partial derivative of $\int \Psi_2$ in the direction of v where $\int \Psi_1$ is fixed. For each v, let $F_{tv,2}(F_{tv,1}^{-1}(x)) := \int^{F_{tv,1}^{-1}(x)} \Psi_2(X_{tv})$ with $x := \int^q \Psi_1(X)$, and the partial derivative $\partial v F_{tv,2}(F_{tv,1}^{-1}(x))$ defines a differential at each point $q \in C^*$.

Definition 4.1.2. Let τ be the $\mathbb{R}$-linear homomorphism from $T_X(\mathcal{M}_{g,1}^{(1)})$ to

$\hat{\mathcal{T}}(X)$ defined as follows:

$$\tau(v) = \partial_v F_{tv,2}(F_{tv,1}^{-1}(x))\Psi_1(X).$$

As in 3.1.5, $\partial_v F_2(F_1^{-1}(x))$ may have simple poles at the zeros of $\Psi_1(X)$, and as shown there $\partial_v F_2(F_1^{-1}(x))\Psi_1(X)$ is actually holomorphic on C^* except possibly at p, the singular point of the differential Ψ. Additionally the jumps are a $\mathbb{R}$-linear combination of $\Psi_1(X)$ and $\Psi_2(X)$. If $v \in T_X(\mathcal{L})$, where $\mathcal{L}$ is the foliation by fixing the absolute periods of Ψ_1, then $\tau(v)$ is a multiple of the absolute periods of Ψ_2 only.

For $(C, p, q_1, \ldots, q_n) \in \mathcal{D}_n \setminus \mathcal{D}_{n+1}$ the real-normalized differential $\Psi_1(C, p, z)$ and $\Psi_2(C, p, z)$ have exactly n common simple zeros at $(q_1, \ldots, q_n)$ where z is any 1-jet at p.

Definition 4.1.3. For any $(C, p, z) \in \mathcal{M}_{g,1}^{(1)}$ such that there exists n marked points $(q_1, \ldots, q_n)$ and $(C, p, q_1, \ldots, q_n) \in \mathcal{D}_n \setminus \mathcal{D}_{n+1}$ define,

$$\hat{\mathcal{T}}_n := \{\Omega \in \hat{\mathcal{T}} | \Omega(q_1) = \ldots = \Omega(q_n) = 0\}.$$

The following theorem is a generalization of the statement: For points $X \in \mathcal{M}_{g,1}^{(1)}$ such that $\Psi_1(X)$ and $\Psi_2(X)$ have no common zeros, the map τ is an isomorphism between $T_X(\mathcal{M}_{g,1}^{(1)})$ and $\hat{\mathcal{T}}$.

Theorem 4.1.4. *[15] There is an open subset $\mathcal{D}_n^* \subset \mathcal{D}_n$, such that for any jet z and $X := (C, p, z)$, τ is an isomorphism between the tangent space $T_X(\mathcal{M}_{g,1}^{(1)})$ and $\hat{\mathcal{T}}_n$.*

Remark 4.1.5. This generalization is the main tool used to calculate the real dimensions of the loci $\mathcal{D}_n \subset \mathcal{M}_{g,1+n}$, and the theorem can be used to further prove that the relative periods of the non-common zeros of Ψ_1 or Ψ_2 define local coordinates on these loci.

4.2 Dual Periods

A meromorphic differential with one double pole has $2g$ zeros counted with multiplicity, and in this section we introduce a set of cycle, cycles dual to the zeros of Ψ, which spans $H_1(C, \mathbb{Z})$. For a general point in $\mathcal{M}_{g,1}^{(1)}$ the dual cycles canonically define a *basis* of $H_1(C, \mathbb{Z})$.

The abelian integral of Ψ on C, denoted $F(q) := \int^q \Psi$, depends on a choice of path between the base point and q, but $\operatorname{Im} F(q)$ is a well-defined harmonic function on $C \setminus \{p\}$ which by 3.2.6 does not depend on a choice of path.

The real part of the abelian integral, $\operatorname{Re} F(q)$, is locally well-defined and multi-valued on C changing when one goes around a cycle in $H_1(C, \mathbb{Z})$. On the other hand, near every point in $C \setminus \{p\}$ aside from the zeros of Ψ, the direction along which $\operatorname{Re} F(q)$ remains locally constant is well-defined.

In a neighborhood of a zero q_s of Ψ with multiplicity m, there are $m+1$ directions along which $\operatorname{Re} F(q)$ remains locally constant.

Definition 4.2.1. Define a foliation on the Riemann surface $C \setminus \{p\}$ by defining each leaf to be the integral lines of the directions along which $\operatorname{Re} \int \Psi$ remains locally constant and oriented such that $\operatorname{Im} \int \Psi$ is increasing in a positively oriented direction. The singularities of this foliation are the zeros of Ψ.

Definition 4.2.2. For any point $q \in C \setminus \{p\}$, an *imaginary ray beginning at* q is a ray of the foliation beginning at q and ending at p.

The next lemma will show that the closure in C of every leaf of the foliation on the Riemann surface $C \setminus \{p\}$ includes the singular point p of Ψ. In other words the integral directions of $\mathrm{Re} \int \Psi$ tend to the pole of Ψ.

Lemma 4.2.3. *For any point $q \in C \setminus \{p\}$, the closure in C of the leaf though q contains the marked point p.*

Proof. Let η be an imaginary ray beginning q along which $\mathrm{Im}\,\Psi$ is increasing. By definition η can not close up to define a closed cycle (the imaginary part is uniquely defined and is increasing along η). Assume η does not tend toward the pole, which implies there is a small neighborhood U in C, not containing any zeros or poles, such that η returns to U twice. Let $\tilde{\eta}$ be the segment of η from the first point where η enters U until it returns a second time.

The lines along which $\mathrm{Re}\,\Psi$ is fixed locally in U are orthogonal to the directions which fix $\mathrm{Im}\,\Psi$. Let γ be a closed cycle given by the imaginary ray $\tilde{\eta}$ and part of the orthogonal trajectory. The period $\mathrm{Im} \int_\gamma \Psi = \int_{\tilde{\eta}} \Psi$ which is non-zero and thus contradicts the assumption that Ψ is real-normalized. $\square$

Definition 4.2.4. Let $\Sigma \subset C$ be a graph on C given by the union of all imaginary rays which begin at some zero of Ψ.

For a general curve, the imaginary rays emanating from one zero are not contained in an imaginary ray emanating from another zero. In this case, if q is a simple zero of Ψ, then there are four imaginary rays beginning at q and ending at p. Two of them are oriented positively from q to p, or in other words $\mathrm{Im} \int \Psi$ increases from $\mathrm{Im} \int_*^q \Psi$ to $+\infty$ along two rays, while two are oriented negatively.

Similarly at a zero q of order m, there are $m+1$ imaginary rays beginning at the zero and ending at p with positive orientation. Denote these rays γ_q^i, $i \in \{1, \ldots, m+1\}$.

Remark 4.2.5. If a zeros of Ψ lies on the imaginary ray emanating from another zero, i.e. $q_i \in \gamma_{q_j}^l$ and $i \neq j$, there is at least one more imaginary ray emanating from q_j than the general case. To illustrate why there are extra imaginary rays, assume for simplicity that q_i and q_j are simple zeros and let $\overline{q_i q_j}$ denote a segment of $\gamma_{q_j}^1$ between the points q_i and q_j. There are at least three imaginary rays oriented positively emanating from q_j which include $\overline{q_i q_j} \cup \gamma_{q_i}^1$ and $\overline{q_i q_j} \cup \gamma_{q_i}^2$, as well as at least one other imaginary ray emanating from q_j which may or may not contain q_i.

There are a finite number of imaginary rays oriented positive on any Riemann surface.

Definition 4.2.6. A *dual period* for the zero q is the union of two imaginary rays γ_q^i and γ_q^j beginning at q and ending at p with positive orientation. The dual period is oriented by reversing the orientation along one of the rays, and an *oriented dual period* is thus $\gamma_q^i \cup -\gamma_q^j$ which will be denoted $\sigma_q^{i,j}$.

Remark 4.2.7. In general there are exactly $2g$ dual periods on C; each zero defines one period. In specific cases, when the zeros are in a special configuration relative to each other, there may be more than $2g$ dual cycles, and thus the set of dual periods is no longer a basis.

The differential Ψ has zero residue, therefore the absolute period $\int_{\sigma_q} \Psi$ is well-defined as we can consider cycles homologous to σ_q which do not contain the singular point p. Let r_q^i be the value of $\operatorname{Re}\int\Psi$ defined (up to an additive constant) along the imaginary ray γ_q^i. The absolute period over the cycle $\sigma_q^{i,j}$ is denoted as follows, $\pi_q^{i,j} := \int_{\sigma_q^{i,j}} \Psi = r_q^i - r_q^j$.

Theorem 4.2.8. *[23] For any $X \in \mathcal{M}_{g,1}^{(1)}$ and for all zeros q_s of Ψ, the homology classes of all dual periods $\sigma_{q_s}^{i,j}$ spans $H_1(C, \mathbb{Z})$.*

Proof. The proof of this theorem is an almost exact replica of the proof of 3.1.5. If dual periods do not span $H_1(C, \mathbb{Z})$, there exists a one parameter family (C_t, Ψ_t) such that the derivative of all absolute periods around the dual cycles and relative periods vanish at (C_0, Ψ_0).

Locally in a neighborhood of C_0, one can choose a fixed basis of cycles, and choose a dual basis of holomorphic one forms $\omega_1(t), \ldots, \omega_g(t)$. Then the abelian integrals $F_{i,t}(p) := \int^p \omega_i(t)$ is well-defined on C_t/Σ_t by defining it continuously along imaginary rays which by definition of Σ_t do not contain zeros of Ψ_t. Certainly it may have jumps along the edges of Σ_t, and these jumps are linear combinations of the absolute periods of $\omega_i(t)$ around the dual cycles.

Then one can show for all i, the differential $\frac{\partial}{\partial t} F_{i,t}(x)|_{t=0}$ for $C_t \ni x := \int^p \Psi_0$ is a holomorphic differential on C_0 with zero A-periods. The proof follows all the remaining steps in 3.1.5 without any change. $\square$

Remark 4.2.9. The construction above can be used to define dual periods in the more general case where the real-normalized differential has one pole of order $k > 1$ [23], and again the dual periods span $H_1(C, \mathbb{Z})$ in this case as well.

4.3 Diaz's Theorem

Following an innovative proof originally appearing in [14] which was modified very slightly in [23], we now use the set-up provided by real-normalized differentials to give a simple proof of Diaz's theorem. The key feature of this proof is the continuous functions defined in 3.2.6.

Theorem 4.3.1 (Diaz)**.** *[6] Given a compact subvariety X in $\mathcal{M}_g$, the dimension of X is at most $g-2$.*

This proof for Diaz's theorem follows by pulling $X \subset \mathcal{M}_g$ a compact subvariety of dimension $g-1$ back to a moduli space of curves with real-normalized differentials $Z \subset \mathcal{M}_{g,1}^{(1)}$, showing that Z must have dimension zero intersection with the leaves of $\mathcal{L}$ which implies $\dim Z < g+1$. The fibers of the forgetful map $\pi : \mathcal{M}_{g,1}^{(1)} \to \mathcal{M}_g$ are not compact, and therefore Z is not a compact subvariety of $\mathcal{M}_{g,1}^{(1)}$, and the main technical part of this proof of Diaz's Theorem is defining an appropriate way to compactify.

There is an action of $\mathbb{R}^+$ on the space $\mathcal{M}_{g,1}^{(1)}$, acting on the jet by multiplication of a real number. Notice that altering the jet with multiplication by an element $r \in \mathbb{R}^+$, changes the real-normalized differential by a real number.

Definition 4.3.2. Let $\mathcal{P}_{g,1}^{(1)}$ be the factor space $\mathcal{M}_{g,1}^{(1)}/\mathbb{R}^+$. Let $[\mathcal{L}] \subset \mathcal{P}_{g,1}^{(1)}$ be a foliation of $\mathcal{P}_{g,1}^{(1)}$ defined by fixing the ratio of any two absolute periods to be locally constant.

The fiber of the projection map $p : \mathcal{P}_{g,1}^{(1)} \to \mathcal{M}_{g,1}$ is $\mathbb{C}^*/\mathbb{R}^+ = S^1$, and thus $\hat{Z} := p(Z)$ in $\mathcal{P}_{g,1}^{(1)}$ is compact. In addition the foliation $[\mathcal{L}]$ is the image of the foliation $\mathcal{L}$ on $\mathcal{M}_{g,1}^{(1)}$ under the factor map $\mathcal{M}_{g,1}^{(1)} \to \mathcal{P}_{g,1}^{(1)}$.

Lemma 4.3.3. *The preimage of any leaf $[\mathcal{L}]$ in $\mathcal{P}_{g,1}^{(1)}$ is a collection of disjoint leaves $\mathcal{L} \subset \mathcal{M}_{g,1}^{(1)}$, any two related by multiplication by a real number.*

Proof. There is no leaf of exact differentials $\mathcal{L}_0$ for real-normalized differentials with one double pole,and thus the action of $\mathbb{R}^+$ on $\mathcal{M}_{g,1}^{(1)}$ permutes the leaves of $\mathcal{L}$ by sending one leaf to another. Therefore each leaf $[\mathcal{L}]$ is locally isomorphic to any one of the leaves in the preimage under the action by multiplication with real numbers. Notice if $\Psi' = r\Psi$ then the absolute periods of Ψ' and Ψ are proportional to each other up the real number r, and thus the foliation on $\mathcal{P}_{g,1}^{(1)}$ is correctly defined as the locus where the ratio of absolute periods are locally constant. $\square$

The imaginary parts of the relative periods are globally well-defined on $\mathcal{M}_{g,1}^{(1)}$, see 3.2.6, and using the dual periods one can construct homogeneous functions on $\mathcal{M}_{g,1}^{(1)}$, i.e. a function which is fixed by the action of $\mathbb{R}^+$, which thus descends to a function on $\mathcal{P}_{g,1}^{(1)}$.

Recall notation introduction in 3.2.6, $f_s(X) := \operatorname{Im}\int_*^{q_s} \Psi_X$ where the base point $*$ is chosen such that $\sum_s f_s = 0$ as well as notation $\pi_q^{i,j}(X) := \int_{[\sigma_q^{i,j}]} \Psi = r_q^i - r_q^j$ which is the absolute period of Ψ_X over the dual cycle $[\sigma_q^{i,j}]$.

Definition 4.3.4. For each zero q_s, let $\hat{\phi}_s(X) := \frac{f_s(X)}{\pi_s(X)}$ where $\pi_s(X) := \min_{i,j}\{\pi_{q_s}^{i,j}(X) \neq 0\}$.

For any two points $X = (C, \Psi)$ and $X' = (C, r\Psi)$ in the same orbit under the $\mathbb{R}^+$ action on $\mathcal{M}_{g,1}^{(1)}$, $f_s(X') = rf_s(X)$ and $\pi_{q_s}(X') = r\pi_{q_s}(X)$, thus $\hat{\phi}_s(X) = \hat{\phi}_s(X')$. Therefore $\hat{\phi}_s(X)$ is a homogeneous function which descends to define a function on $\mathcal{P}_{g,1}^{(1)}$ which we denote $\hat{\phi}_s$ as well.

The absolute periods of the dual cycles $\pi^{i,j}_{q_s}$ have finite jump discontinuities, and π_{q_s} are by definition less than or equal to π_{q_s} at all nearby points. Therefore π_{q_s} is a lower semi-continuous function where the discontinuities occur when the number of imaginary rays change, i.e. when one varies the location of a zero which lies on an imaginary ray emanating from another zero. On an open set denoted $\mathcal{K} \subset \mathcal{M}^{(1)}_{g,1}$, where the zeros are in general position and do not lie on an imaginary rays emanating from other zeros, $\pi^{i,j}_{q_s}$ are continuous.

Let's order the critical values $g_1 := \max_s\{\hat{\phi}_s\}$ and $g_k := \max_s\{\hat{\phi}_s - \{g_1, \ldots, g_{k-1}\}\}$, so that we have

$$g_1 \geq g_2 \ldots \geq g_{2g}.$$

The functions g_s are positive, continuous functions on $\mathcal{K}$ and g_1 is subharmonic on $\mathcal{K}$ when restricted to any leaf $\mathcal{L}$. They are upper semi-continuous on $\mathcal{P}^{(1)}_{g,1}$, and therefore must achieve a maximum on any compact subset of $\mathcal{P}^{(1)}_{g,1}$.

We now show that g_1 restricted to the leaves $\mathcal{L}$ satisfies the maximum principle.

Lemma 4.3.5. *If g_1 achieves a local maximum on a subvariety of $\mathcal{L}$, then g_1 in constant on this subvariety.*

Proof. The function q_1 is subharmonic at points in $\mathcal{L} \cap \mathcal{K}$, and therefore if g_1 achieves a maximum here, it is constant. Locally in a neighborhood of a discontinuities of g_1, the real part of the relative periods are varying, and f_1 does not have local maximum (subharmonic) must vary locally in the transverse direction. This implies that the maximum occurs at a point in $\mathcal{K} \cap \mathcal{L}$ which is a contradiction. □

Notice if g_1 is constant on any subvariety contained in a leaf of the foliation, then g_2 is subharmonic on $\mathcal{K} \cap \mathcal{L}$, and thus the lemma applies to show g_2 also satisfies the maximum principle.

Diaz's Theorem. Let $\hat{Z}$ be the image of Z in $\mathcal{P}_{g,1}^{(1)}$, which is compact, and therefore there exist a nonempty set of points, denoted $\hat{Z}_1$, on which g_1 achieves a maximum. By construction at any point in the pre-image of $\hat{Z}_1$ in $\mathcal{M}_{g,1}^{(1)}$ the function g_1 also achieves a maximum. We denoted the pre-image of $\hat{Z}_1$ in $\mathcal{M}_{g,1}^{(1)}$ as Z_1.

For the leaf $\mathcal{L}$ which passes through a point $(C, \Psi) \in Z_1$, because g_1 achieves its supremum at points in Z_1, the lemma implies that g_1 is therefore constant and any point in a connected component of $Z \cap \mathcal{L}$ containing (C, Ψ) is contained in Z_1, and the leaves of $\mathcal{L}$ foliate Z_1 which implies $\dim_{\mathbb{C}} Z \cap \mathcal{L} = \dim_{\mathbb{C}} Z_1 \cap \mathcal{L}$.

We can apply the same argument to show that g_2 achieves a maximum on a compact subset $\hat{Z}_2 \subset \hat{Z}_1$, and thus there is a nonempty subset $Z_2 \subset Z_1 \subset \mathcal{M}_{g,1}^{(1)}$ whose connected components are foliated by leaves of $\mathcal{L}$ which in turn implies $\dim_{\mathbb{C}} Z \cap \mathcal{L} = \dim_{\mathbb{C}} Z_2 \cap \mathcal{L}$.

By continuing the argument, there is a nonempty subset of points Z_{2g-1} on which all the function $g_1, \ldots, g_{2g-1}, g_{2g}$ (by the normalization condition g_{2g} is constant as well) achieves a maximum, and are therefore all constant on $Z_{2g-1} \cap \mathcal{L}$ for any leaf $\mathcal{L}$. This in turn implies that all the $\mathrm{Re} \int^{q_s} \Psi$ are constant as they are the real parts of a holomorphic function with constant imaginary part.

Therefore local coordinate ϕ_s are all constant on $Z_{2g_1} \cap \mathcal{L}$ which in turn implies that $\dim_{\mathbb{C}} Z_{2g-1} \cap \mathcal{L} = 0$. By assumption $\dim Z = g+1$ which implies $\dim_{\mathbb{C}} Z_{2g-1} \cap \mathcal{L} \geq 1$, and we have a contradiction. □

4.4 Degenerations: Starting Point

In this section we begin to discuss how real-normalized differentials degenerate on families of curves which develop nodal singularities. The results in this section appeared in [14] for the case of one marked point and the result there immediately generalize to describe how real-normalized differentials with multiple singular points degenerate as long as one assumes all residues are zero. The following proposition is the starting point for the complete analysis which is carried out in the thesis (where we will in fact reprove it).

The moduli space of smooth curves with marked points admits the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$, and since in the Deligne-Mumford compactification the marked points are not allowed to coincide with the nodes, the bundle of 1-jets (i.e. the relative tangent bundle to the curve) extends to a bundle $\overline{\mathcal{M}}_{g,n}^{1,\ldots,1} \to \overline{\mathcal{M}}_{g,n}$.

Definition 4.4.1. We denote $\overline{\mathcal{M}}_{g,n}^{1,\ldots,1}$ and call the Deligne-Mumford compactification of the moduli space of curves with points and jets the total space of the Cartesian product over $\overline{\mathcal{M}}_{g,n}$ of the relative cotangent line bundles at each of the marked points, with zero sections removed. We then have the forgetful map $\overline{\mathcal{M}}_{g,n}^{1,\ldots,1} \to \overline{\mathcal{M}}_{g,n}$ with fibers $(\mathbb{C}^*)^n$.

The following result states that the section given by real-normalized differentials Ψ over $\mathcal{M}_{g,n}^{\underline{m}}$ extends to a continuous section of the sheaf $\omega_C(\sum 2p_i)$

over $\overline{\mathcal{M}}_{g,n}^m$, where ω_C is the relative dualizing sheaf. Moreover, it was shown, very surprisingly, that the section Ψ does not develop poles at the nodes of stable curves, so that Ψ can in fact be extended to a section of $\Omega_C(\sum 2p_i)$, where Ω_C denotes the sheaf of relative Kähler differentials over $\overline{\mathcal{M}}_{g,n}^m$ (we note that this sheaf is a pullback from $\overline{\mathcal{M}}_{g,n}$, but the section depends on the jet).

Proposition 4.4.2. *The real-normalized differential Ψ extends to a* contin-uous *section of the twisted relative dualizing sheaf $\omega_C + 2\sum p$ over $\overline{\mathcal{M}}_{g,1}^{(1,\ldots,1)}$. For a nodal curve $C = \cup C_j$, where C_j are the connected components of the normalization of C, the continuous limit $\Psi_C|_{C_j}$ is the unique real-normalized differential on C_j with the prescribed double poles at those marked points that lie in C_j.*

The argument in [14] for $n = 1$ applies verbatim for the case $n > 1$, as only the local structure near the nodes is relevant, and we now recap the proof present there.

Proof. Choose a base point for integration, and let $f(p) := \operatorname{Im}\int_*^p \Psi$ which is a well-defined harmonic function on $C \setminus \{p\}$ which blows up at the singular point of Ψ.

On any family $(C_t, p_t, z_t) \in \mathcal{M}_{g,1}^{(1)}$ of smooth curves degenerating to (C_0, p_0, z_t) where C_0 is a nodal curve, choose a small neighborhood $p_t \in D_\epsilon$ of size ϵz_t which does not contain any nodes (which is possible by the definition of the Deligne-Mumford compactification).

By the maximum principle for harmonic functions $f_t(p) := \int_*^p \Psi_t$ on $C_t \setminus D_\epsilon$ achieve it maximum and minimum on the boundary of D_ϵ. On the other hand, the value of Ψ_t in ∂D_ϵ is prescribed and equal to $\frac{dz_t}{z_t^2}$ and is thus bounded independent of t which implies $\lim_{t\to 0} f_t(p) := \int^p \Psi_0$ is bounded on $C_0 \setminus D_\epsilon$.

Let Φ be the real-normalized differential on $\widetilde{C}_0$ with one double pole at p_0 which is *holomorphic* at the preimage of any node in C_0. The difference $\int^q(\Phi - \Psi_0)$ is bounded on $C_0 \setminus D_\epsilon$ by the above argument, and in addition, it is bounded in D_ϵ because it is holomorphic inside the disk. Therefore $\int^q(\Phi - \Psi_0)$ is a bounded harmonic function on C_0 which implies $\Phi = \Psi_0$. □

Notice that on an irreducible component C_j which does not contain any marked point, the differential is real-normalized and holomorphic, thus it is identically zero.

Definition 4.4.3. For a stable curve $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}$ we call an irreducible component C_j of C a *null component* if it contains no marked points; otherwise we call such a component *non-null*.

The proposition above does not provide sufficient information about the behavior of real-normalized differentials near nodal curves in the boundary. For many applications we will want to locate the limits of the zeros of a (or a pair of) real-normalized differentials in a family of smooth curves degenerating to a nodal curves. The results in this thesis provides further information on the behavior of the real-normalized differentials near the boundary of $\overline{\mathcal{M}}_{g,n}^{1,\ldots,1}$. In particular, we will explicitly describe Ψ near nodal curves and can thus investigate the limits of the zeros of the real-normalized differential — this is a non-trivial question when there is a component in the normalization of C with no marked points.

Remark 4.4.4. This proposition stating that new poles do not form at the nodes is a priori very surprising, as limits of abelian differentials (holomorphic 1-forms) on stable curves can develop poles at the nodes (this is what is allowed for the relative dualizing sheaf). The fact that real normalization prevents the development of the residues is crucial for this entire study. We note that the limiting real-normalized differential on a stable nodal curve is not in fact real-normalized in the sense that its integrals over cycles passing through the nodes may not be real — the simplest example of this would be a "banana" curve that has two irreducible components, one without any marked points. Then in the limit the real-normalized differential becomes identically zero on that component and its integral on the cycle intersecting the nodes is equal to its integral from one node to the other on the non-null component — which does not have to be real.

Chapter 5

Degenerations of Real-Normalized Differentials of the Second Kind

In this chapter we will provide much more information about the degenerations of real-normalized differentials with no residues. Given an arbitrary stable curve, we construct explicitly the real-normalized differentials everywhere in its neighborhood, from which it is easy to see that the limit is continuous and does not develop poles at the node. In fact we reprove here that real-normalized differentials of the second kind do not develop poles at the nodes in a family of smooth curves degenerating to a nodal curve.

We will construct the real-normalized differentials near a nodal curve explicitly by taking the difference of a meromorphic differential with prescribe singularities and a linear combination of some holomorphic differentials which have a non-real period over one cycle in some symplectic basis of $H_1(C,\mathbb{Z})$.

These differentials are obtained from inverting and manipulating the real and imaginary parts of the period matrix. By the work of Taniguchi [31], Yamada [34], the period matrix is analytic in plumbing coordinates and their logarithms, therefore our construction will be real analytic in s, $\ln s$ and their inverses. Such functions were called sector real-analytic by Wolf and Wolpert [32], and thus we see a priori that the real-normalized differential extends to a sector real-analytic section over $\overline{\mathcal{M}}_{g,n}^{m}$ in terms of plumbing coordinates.

Throughout this chapter we work with $X_0 \in \overline{\mathcal{M}}_{g,n}^{1,\dots,1}$ in order to simplify the notation. One can check directly that these results immediately extend to the more general situation describing how real-normalized differentials of the second kind degenerate in the neighborhood of a nodal curve $X_0 \in \overline{\mathcal{M}}_{g,n}^{m}$ where each $m_i > 0$.

The results in this chapter will constitute a paper with S. Grushevsky and I. Krichever.

5.1 General setup for analyzing the degeneration of real-normalized differentials

In the following sections, we will analyze in more detail the behavior of real-normalized differentials near various points of the boundary of $\overline{\mathcal{M}}_{g,n}$, starting with the generic points of boundary divisors, proceeding to some standard more complicated examples, and then in general. In this section we introduce the general notation and the setup for such an analysis.

We start with a nodal curve

$$X_0 := (C_0, p_1, \ldots, p_n, z_1, \ldots, z_n) \in \partial\overline{\mathcal{M}}_{g,n}^{1,\ldots,1},$$

where $(C_0, p_1, \ldots, p_n) \in \partial\overline{\mathcal{M}}_{g,n}$, and let $(\widetilde{C}_0, p_1, \ldots, p_n)$ by abuse of notation be its possibly disconnected normalization. Let $r_1, \ldots, r_k$ be all the nodes of C_0, and let then $q_1^i, q_2^i \in \widetilde{C}_0$ be the preimages of r_i. Let $\underline{s} := (s_1, \ldots, s_k)$ be the k-tuple of complex numbers that give plumbing parameters. Further, let $\underline{v}$ be any system of local coordinates on the moduli space for $(\widetilde{C}_0, p_1, \ldots, p_n)$, so that $(\underline{s}, \underline{v})$ gives local coordinates on $\overline{\mathcal{M}}_{g,n}$ near $(C_0, p_1, \ldots, p_n)$. Since $\overline{\mathcal{M}}_{g,n}^{1,\ldots,1}$ is the total space of a locally trivial $(\mathbb{C}^*)^n$ fibration, it follows that $\underline{t} := (\underline{s}, \underline{v}, \underline{z})$, where $\underline{z} := (z_1, \ldots, z_n)$, gives local complex coordinates on $\overline{\mathcal{M}}_{g,n}^{1,\ldots,1}$ near X_0, and we denote by $X_{\underline{t}}$ the point with corresponding coordinates.

For further use we will denote by $C_{\underline{v}} := C_{\underline{0},\underline{v}}$ the curve in a neighborhood of C_0 whose nodes correspond 1-to-1 to the nodes of C_0 (which is recorded by the fact that the plumbing coordinates are zero), and let $\widetilde{C}_{\underline{v}}$ be its normalization. In addition, the plumbing deformation of $\widetilde{C}_{\underline{v}}$ at every node except r_i is denoted $C_{s_i=0,\underline{s}} := C_{1,\ldots,s_{i-1},0,s_i,\ldots,s_k,\underline{v}}$ (or more generally if a number of s_j are zero).

Note that in general the cohomology group $H_1(\widetilde{C}_0)$ may not be equal to $H_1(C_{\underline{s},\underline{v}})$, as thinking of the family as $\underline{s} \to 0$ as a degeneration, some of the cycles on the smooth Riemann surface get pinched to nodes (are vanishing cycles). Indeed, the cohomology is the same if and only if C_0 is a nodal curve of compact type — which will be the simplest situation for our analysis. Otherwise we will need to ensure the reality of periods of the meromorphic differential that we construct over the vanishing cycles on $C_{\underline{s},\underline{v}}$ by a separate argument.

In order to simplify the language later and in light of 4.4.2, we make the following definition.

Definition 5.1.1. A differential on a nodal curve is called real-normalized if it is real-normalized on the normalization of the nodal curve.

Therefore a differential describes the real-normalized differential in a neighborhood of C_0 implies that on any nodal curve in a neighborhood of C_0, this differential is a real-normalized differential on its normalization.

5.2 Gluing Differentials

We will construct the real-normalized differentials in a neighborhood of a nodal curve by explicitly patching them together in the plumbing construction, that is to say by gluing a differential on $C_{\underline{s}}$ from the differential(s) on the connected component(s) of $\widetilde{C}_0$.

Definition 5.2.1. We say that the values of a meromorphic differential Φ *match locally near* $q_1, q_2 \in C$ (where C is possibly disconnected) *with respect to plumbing with parameter* s *in chosen local coordinates* z_i *around* q_i if $\Phi(z_1) = \Phi(z_2)$ for any $|z_1| = \sqrt{|s|}$ and $z_2 = s/z_1$ — that is to say, if the values of $\widetilde{\Phi}$ match on the glued points of the seams $\gamma_{1,s}$ and $\gamma_{2,s}$.

Remark 5.2.2. This notion is a generalization of matching residues: If Φ has a simple pole at q_1 with residue c, so that locally near q_1 we have $\Phi(z_1) =$

$c\,dz_1/z_1$, then we must have

$$\Phi(z_2) = \Phi(s\,z_2^{-1}) = c\,d(s\,z_2^{-1})/(s\,z_2^{-1}) = c\,z_2 d(z_2^{-1}) = -c\,dz_2/z_2$$

so the matching condition is that Φ has a simple pole at q_2 with residue $-c$. More generally, if Φ has a pole of order $m > 1$ at q_1, then the matching condition is for Φ to have a zero of order $m-2$ at q_2, of the form prescribed by Φ at q_1.

Remark 5.2.3. We note that for any fixed s and a choice of local coordinates near q_1, a meromorphic differential Φ given locally in a neighborhood of q_1 in the chosen local coordinates defines uniquely the matching local behavior of a meromorphic differential in a neighborhood of q_2. Indeed, let z_1 be a local coordinate on near q_1 such that $\Phi = z_1^m dz_1$, and z_2 any local coordinates near q_2 such that a neighborhood of the node is described by the locus $z_1 z_2 = 0$. The values of Φ on $\gamma_{1,s}$ defines the matching values of Φ, with respect to parameter s, to be $-s\frac{s^m}{z_2^m}\frac{dz_2}{z_2^2}$ on $\gamma_{2,s}$, and the meromorphic differential on a disk with prescribed pole order at zero is determined uniquely by its boundary values. Thus if $q_1 \in C_1$ and $q_2 \in C_2$, with $C_1 \neq C_2$, then a differential Ψ on C_1 holomorphic at q_1 in a chosen local coordinate defines a unique matching singular part of a meromorphic differential on C_2 at q_2 with finite laurent expansion.

The reason for the definition of matching is of course the following:

Lemma 5.2.4. *In the situation as above, if $U_{i,s}$ are small neighborhoods of q_i as in the plumbing construction, the differential $\Phi|_{C\setminus(U_{1,s}\sqcup U_{2,s})}$ extends across* *the seam γ_s to define a meromorphic differential Φ on C_s* holomorphic on the seam *if and only if the values of Φ near q_1, q_2 in local coordinates z_1, z_2 match for plumbing parameter s.*

Proof. The fact that the resulting differential on C_s is meromorphic away from the seam is automatic. By definition of complex structure on C_s, it is meromorphic on C_s if and only if it is continuous on the seams — which is exactly when $\Phi(z_1) = \Phi(z_2)$ for any points $z_1 \in \gamma_{1,s}$ and $z_2 \in \gamma_{2,s}$ glued to each other. By definition in coordinate z_1 it has no poles on $\gamma_{1,s}$ and is thus holomorphic in the neighborhood of the seam. □

Since away from the seams Ψ and $\widetilde{\Psi}$ are the same differential, in particular we get the following important

Corollary 5.2.5. *If the meromorphic differential $\widetilde{\Psi}$ is real-normalized on (every component of) $\widetilde{C}_{\underline{v}}$ and satisfies the matching condition at each node $q_1^i \sim q_2^i$ with respect to the plumbing parameter s_i, then the glued differential Ψ on $C_{\underline{s},\underline{v}}$ has a real period on any cycle that does not intersect any seams.*

In what follows we will describe differentials which satisfy the matching condition with respect to a plumbing parameter on curves in a full neighborhood of a nodal curve with respect to local coordinates on moduli space given by the plumbing deformation, and thus must verify that choosing a local coordinate neighborhood z_1 of q_1 such that a real-normalized differential Ψ is locally of the form $z_1^m dz_1$ can be done consistently on all curves in a neighborhood, i.e. allowing the coordinates $\underline{u}$ and $\underline{z}$ to vary. The differential Ψ defines a real-analytic section on the moduli space, so certainly if q_1 is not a zero of Ψ, the local coordinate z_1 such that $\Psi = dz_1$ varies real-analytically in a family.

Assume q_1 is a zero of Ψ of order m, on nearby curves there is no guarantee that q_1 continues to be a zero of order m. Therefore for local coordinate z_1 such that $\Psi = z_1^m dz_1$, we may not be able to fix local coordinates at q_1 on nearby curves so that Ψ is exactly $z^(m')dz$ (where m' is the order of the zero on nearby curves), but certainly a local coordinate z_1 can be chosen consistently such that $\Psi = \prod_{i=1}^{m}(z_1 - y_i)dz_1$ where $y_i := y_i((\underline{u}, \underline{z})$ records the location of the m zeros on nearby curves. The matching differential with respect to a plumbing parameter s on curves in a neighborhood C_0 is given by $s(y_i z_2 - s)\frac{dz_2}{z_2^m}$ on the seam $\gamma_{2,s}$ which therefore depends on the plumbing parameter as well as the behavior of the zeros at q_1.

In other words, if Ψ has a zero of order m at q_1, we can choose local coordinates near q_1 consistently such that the matching differential with respect to a plumbing parameter has finite laurent expansion near q_2 with a pole of order $m + 2$.

5.3 Stable curves of compact type

In this section we investigate the behavior of real-normalized differentials in a neighborhood of a stable curve of compact type. A priori we expect this situation to be easier than the case of a general stable curve, as the theory of limit linear series is much better behaved in this case (see [19],[2]). As limit linear series are not available (though seemingly doable, and perhaps worth developing) for real-analytic families, we proceed by explicitly gluing the real-normalized differential. For concreteness, we do the case of curves with two irreducible components in detail, and the general case then follows once the irreducible components are ordered properly and the general notation is set up. In this situation we already see that the limits of zeros of real-normalized differentials depend on the degenerating family, and not just on the limit stable nodal curve.

5.3.1 Stable curve of compact type with one node

Suppose $\widetilde{C}_0 = C_1 \sqcup C_2$, with C_1 and C_2 connected, of genera h and $g-h$ respectively (where we may have $h=0$ or $h=g$ if the corresponding component contains at least two p_i's). Let $q_i \in C_i$ be the two preimages of the node $r \in C_0$, and denote the plumbing parameter $s := s_1$. Our goal is to construct the real-normalized meromorphic differential on $X_{\underline{t}} = X_{s,\underline{v},\underline{z}}$ in a neighborhood of X_0. We denote $C_{\underline{v}} = C_1(\underline{v}) \cup C_2(\underline{v})$ the nodal stable curves in this neighborhood C_0, with marked points $p_i(\underline{v}) \in C_{\underline{v}}$.

We have two different cases to consider depending on whether one component is null (contains no marked points) — and the gluing construction is slightly different for them.

We first deal with the case when one component is null — that is, without loss of generality we assume $p_1, \ldots, p_n \in C_1$, so that C_2 is a null component. In this case in a neighborhood let $\Psi_1(\underline{t}) = \Psi_1(\underline{v}, \underline{z})$ be the real-normalized differential on $(C_1(\underline{v}), p_1(\underline{v}), \ldots, p_n(\underline{v}), \underline{z})$ with prescribed singularities — notice that it is holomorphic at q_1 as $q_1 \neq p_i(\underline{v})$. By remark 5.2.3 the local behavior of Ψ_1 near q_1 in a chosen local coordinate and for any plumbing parameter s, one can define uniquely the local behavior of a meromorphic differential defined locally near q_2 (and with a pole at q_2 of order equal to two plus the vanishing order of Ψ_1 at q_1) matching Ψ_1. We then let $\Psi_2(\underline{t}) = \Psi_2(s, \underline{v}, \underline{z})$ be the *real-normalized differential on* $C_2(\underline{v})$ with this local singularity at q_2 and holomorphic elsewhere.

Proposition 5.3.1. *The differential $\Psi_1(\underline{v}, \underline{z}) \sqcup \Psi_2(\underline{t})$ on $C_1(\underline{v}) \sqcup C_2(\underline{v})$ glues to define a real-normalized differential $\Psi_{\underline{t}}$ on $X_{\underline{t}}$ with prescribed singularities.*

Proof. Indeed, this is a special case of corollary 5.2.5: since the only node on $C_{\underline{v}}$ is separating, there are no vanishing cycles in homology, and thus the glued differential $\Psi_{\underline{t}}$ ends up being real-normalized. $\square$

For applications to cohomology of the moduli space of curves or to studying cusps of plane curves in [16], it is important to be able to determine the limits of zeroes of real-normalized differentials as a curve degenerates to a stable curve — here the difficulty is determining which points on a null component are limits of zeroes. From the results above, and from the more general constructions below in the neighborhood of an arbitrary stable curve, we will be able to track zeroes in any family suitably given in plumbing coordinates, but for now to show that the issue is essential we give an example to show that the limits of zeroes indeed depend on the *degenerating family* and not only on the *limit stable curve*. This is in accordance with the ideas of limit linear series, but we see this explicitly for our situation. Indeed, suppose in the situation above the point $q_1 \in C_1$ is simple zero of the real-normalized differential Ψ_1 on C_1, and we construct two families of smooth curves converging to this point, essentially one where the zero moves, and the other where it does not.

Example 5.3.2. The first family $X_{\underline{t}}$ degenerating to X_0 will be given simply by the plumbing deformation: $\underline{t} := (s, 0, 0)$, i.e. we plumb C_0 with parameter

s, and do nothing else, so that $q_1(\underline{t}) := q_1$. The differential Ψ_1 has $2h$ zeroes on C_1; we choose ε sufficiently small so that q_1 is the unique zero of Ψ_1 that lies in the plumbing neighborhood (note that $(C_1, p_1, \ldots, p_n)$ and Ψ_1 are fixed). Then the remaining $2h-1$ zeroes of Ψ_1 lie outside the plumbing neighborhood and thus are zeroes of the glued differential $\Psi(\underline{t})$. Since they do not depend on $\underline{t}$, they are limits of the zeroes of $\Psi_{\underline{t}}$ under degeneration. On C_2, the real-normalized differential Ψ_2 with behavior near q_2 matching with respect to s that of Ψ_1 near q_1 has a triple pole at q_2 for any $s \neq 0$, and thus has $2(g-h)+1$ zeroes on C_2, all of which are outside of the plumbing neighborhood, and tend to some points on C_2 in the limit, distinct from q_2.

Example 5.3.3. The second family $X_{\underline{t}}$ degenerating to X_0 we consider is where in addition to plumbing we also vary the point of attachment of C_2 along C_1, letting in fact $q_1(\underline{t}) := 2\sqrt{|s|}$ in the local coordinate centered at q_1 (so that $q_1 = q_1(0)$ is the origin). In this family locally Ψ_1 is of the form $(z - 2\sqrt{|s|})dz$, so that for $s \neq 0$ the differential Ψ_1 is regular at $q_1(\underline{t}) \neq q_1$, and all of its $2h$ zeroes on C_1, including q_1, lie outside the plumbing neighborhood, and are thus zeroes of $\Psi(\underline{t})$. On the other hand, for $s \neq 0$ the matching real-normalized differential Ψ_2 with respect to s on C_2 has a double pole at q_2, and thus has $2(g-h)$ zeroes, all outside the plumbing neighborhood. Thus of the zeroes of $\Psi(\underline{t})$ in the limit $2h$ of them lie on C_1 (including q_1), and $2(g-h)$ limits of zeroes lie on C_2.

We note that in the above examples, while the limits of the zeroes are different, by taking more general degenerating families one situation deforms to the other, as one of the limits of the zeroes on C_2 can approach q_2 (which is the same as q_1) as we vary the family. By tracing through the construction we can see precisely what happens in this case, and give the following description of the space of possible limits of zeroes of differentials (this is the first case of the general picture described in full generality by our main result, theorem 5.7.3).

Theorem 5.3.4. *The set of all possible limits of zeroes of real-normalized differentials on families of smooth curves $X_{\underline{t}} \subset \mathcal{M}_{g,n}^{1,\dots,1}$ converging as $t \to 0$ is the stable curve of compact type X_0 as above, where the real-normalized differential Ψ_1 on C_1 has a zero of order m at q_1, is parameterized by the real projective space $\mathbb{RP}^{2m+1}$, which is the set of parameters for the singular part of order at most $m+2$ of a real-normalized differential on C_2, modulo multiplication by a real number.*

Remark 5.3.5. In other words, for any family of stable curves degenerating to X_0, the limits of zeros are the zeros of Ψ_1 that are distinct from q_1, the point q_1 with multiplicity $m' \leq m$, and the zeros of some real-normalized differential on C_2 with pole of order $(m - m' + 2)$ at q_2, with no residue.

Proof. The theorem is basically a restatement of the construction of the real-normalized differential we gave above on stable curves of compact type with two components, as also seen in the examples. Indeed, in a sufficiently small neighborhood of X_0 the multiplicity of the zero of $\Psi_1(\underline{v}, \underline{z})$ at the node $q_1(\underline{v}, \underline{z})$ of $C_{0,\underline{v},\underline{z}}$ is at most equal to m — the multiplicity of zero is a semicontinous function. Thus for any non-zero s the differential $\Psi_2(\underline{t}) = \Psi_2(s, \underline{v}, \underline{z})$ on the null component of $C_{s,\underline{v},\underline{z}}$ before gluing will have the pole of order at most $m+2$ at q_2. The differential $\Psi_2(\underline{t})$ is then the unique real-normalized differential with this prescribed singular part, and the data of such a singular part is exactly a non-zero point in $\mathbb{C}^{m+1}$ (recall that there is no residue). Further noting that rescaling s by a non-zero real number leads to rescaling the singular part by this real number, thus rescaling $\Psi_2(\underline{t})$ by a real number — without changing the location of its zeroes — we see that the locations of zeroes are parameterized by $\mathbb{C}^{m+1} \setminus \{0\}/\mathbb{R}^* = \mathbb{RP}^{2m+1}$ as claimed. $\square$

Remark 5.3.6. The theorem above indicates that if one were to try to construct a compactification of $\mathcal{M}_{g,n}^{1,\dots,1}$ on which the real-normalized meromorphic differentials would extend without becoming identically zero, then along the open part of the boundary stratum δ_h of $\overline{\mathcal{M}}_g$ one should take a real-analytic blowup corresponding to the vanishing order of the real-normalized differential at the node. Defining this rigorously (and more generally) presents additional challenges that we will not deal with here — but which may be of interest for further use in dealing especially with holomorphic differentials.

The case of a stable curve with two irreducible components neither of which is null turns out to be very similar. Here we automatically have the non-zero real-normalized differentials on the two components C_1 and C_2 with prescribed singularities at those p_i that lie on that component. However, these differentials would be regular at q_1 and q_2, and thus do not glue to a differential on the plumbed curve $C_{s,\underline{v}}$. One may be tempted to glue by allowing simple poles at q_1 and q_2, but this is of course impossible as the sum of residues on both C_1 and on C_2 has to be zero. We thus break the symmetry of the two components, and while letting $\Psi_1(\underline{t}) = \Psi_1(\underline{v}, \underline{z})$ still be the real-normalized differential on $C_{\underline{v}}$ with prescribed singularities at the marked points p_i that lie on C_1, we now choose $\Psi_2(\underline{t}) = \Psi_2(s, \underline{v}, \underline{z})$ to be the real-normalized differential on $C_2(\underline{v})$ with prescribed singularities at those p_i that lie on C_2 *and with a pole at* q_2*, matching with respect to a plumbing parameter, the values of* Ψ_1 *at* q_1.

Then, as above, these differentials glue:

Proposition 5.3.7. *The differential* $\Psi_1(\underline{v}, \underline{z}) \sqcup \Psi_2(\underline{t})$ *on* $C_1(\underline{v}) \sqcup C_2(\underline{v})$ *glues to define a real-normalized differential on* $X_{\underline{t}}$ *with prescribed singularities.*

Remark 5.3.8. The construction above appears to be manifestly non-symmetric in that we have taken the real-normalized differential on $C_1(\underline{v})$, and then modified the differential on $C_2(\underline{v})$ to match it. Notice, however, that the resulting differential on $C_{s,\underline{v}}$ does not depend on this choice. Indeed, we have shown the resulting differential is real-normalized regardless of how we match and thus is unique. The values of the differentials dw and $d(s/w)$ agree on the circle $|w| = \sqrt{|s|}$, namely at points w_1 and w_2 where $w_1 \cdot w_2 = s$ then $dw_1 = d(s/w_2)$.

5.3.2 Ordering the irreducible components of C_0 for a general stable curve

To describe the limits of real-normalized differentials on stable curves of compact type, we apply the construction as above step by step starting from the non-null components. This is straightforward except that we need to introduce terminology for ordering the irreducible components of the stable curve, which will also be used for the general statement of our main theorem. This ordering is in a sense by the order of the vanishing of the real-normalized differential on the component (and is parallel to the understanding of the degenerations of flat structures in Teichmüller dynamics, see [7]), though we define it more directly.

Definition 5.3.9. We order the irreducible components of C_0 inductively. A component is called of *order 0* (we call it order zero, not one, as the real-normalized generically does not vanish) if it is non-null; that is an irreducible component is called of order 0 if and only if it contains at least one marked point p_i. Then we inductively call a component of *order* $i+1$ if and only if is attached via a node to *some* component of *order* i. Since C_0 is connected, all its irreducible components have finite order. Explicitly, the order of an irreducible component is the shortest distance from the corresponding vertex of the dual graph to a vertex corresponding to a non-null component (that is, to a vertex having a marked point, aka a "leg" on the dual graph). We denote by $C^i := \cup_j C^i_j$ the union of all irreducible components of C_0 of order i.

5.3.3 Real-normalized differentials near general stable curves of compact type

The only complication here is that the dual graph of such a curve is more complicated, which is really just a notational issue taken care of by the definition above.

Indeed, we start with the possibly disconnected $C^0(\underline{v})$, the union of non-null irreducible components $C^0_j(\underline{v})$. Whenever two components C^0_j and C^0_i are connected by a node $C^0_j \ni q_j \sim q_i \in C^0_i$, as above in the construction for proposition 5.3.7 we need to choose on which component we would allow an extra pole, matching the value at the other glued point. More precisely, we take the real-normalized differential $\Psi_1^0(\underline{v},\underline{z})$ on C_1^0 to have prescribed singularities at those p_ℓ that lie on it. Then inductively on $C_{i+1}^0(\underline{v})$ we define the real-normalized differential $\Psi_{i+1}^0(\underline{t}) = \Psi_{i+1}^0(\underline{s},\underline{v},\underline{z})$ that has prescribed singularities at all the point p_ℓ that lie on $C_{i+1}^0(\underline{v})$, and also at any node where $C_{i+1}^0(\underline{v})$ intersects some $C_j^0(\underline{v})$ with $j \leq i$ has a singularity matching with respect to a plumbing parameter the singularity of the already defined $\Psi_j^0(\underline{t})$ at the point glued to the node, with the given plumbing parameter at that node. In particular if $C_{i+1}^0(\underline{v})$ is disjoint from $\cup_{j=1}^i C_j^0(\underline{v})$, we simply get the real-normalized differential on it with singularities only at p_ℓ. Then similarly to proposition 5.3.7 the differentials $\Psi_i^0(\underline{t})$ glue to define a real-normalized meromorphic differential $\Psi^0(\underline{t})$ on $C^0(\underline{s},\underline{v})$ with prescribed singularities at all the p_ℓ that lie on it, and no further singularities. Also, as before, $\Psi^0(\underline{t})$ does not depend on the arbitrary choice of the numbering of the components of $C^0(\underline{v})$. We then use the gluing as for the case for the null components to deal with $C^1(\underline{v})$, by defining the real-normalized differential $\Psi^1(\underline{t})$ on $C^1(\underline{v})$ (which, recall, is a union of null components, and contains no points p_ℓ) , by requiring it to have poles at all nodes that connect $C^1(\underline{v})$ to $C^0(\underline{v})$, and matching the values of $\Psi^0(\underline{t})$ with respect to the corresponding plumbing parameters. Proceeding inductively in this way, we define the differentials $\Psi^2(\underline{t}),\dots$ and finally get

Proposition 5.3.10. *Let $X_0 = (C_0, p_1, \dots, p_n, z_1, \dots, z_n) \in \overline{\mathcal{M}}_{g,n}^{1,\dots,1}$, with C_0 a curve of compact type. Then the differential $\cup_j \Psi^j(\underline{t})$ glues to define the real-normalized differential for any curve in a neighborhood of $X_0 \in \overline{\mathcal{M}}_{g,n}^{1,\dots,1}$ in plumbing coordinates.*

Remark 5.3.11. By tracing through the construction above, we can see what the singularities of the real-normalized differentials on each component of $C_{\underline{v}}$ taken for gluing depend on, and thus what data is needed to determine the location of the limits of the zeroes. The full statement is as complicated as the general one given in our main theorem 5.7.3 as for example a component of order 1 can intersect multiple components of order zero, and thus we would need prescribed information of singularities at many points, etc.

5.4 Irreducible stable curves

Next we consider the situation of an irreducible stable curve, when there is still no issue with limit linear series in algebraic geometry. We will again construct the differential explicitly in plumbing coordinates, though in this case we will need to glue differentials with simple poles, which automatically match. This is a minor modification of the construction, while the main new issue to be dealt with is that the differential thus glued may no longer be real-normalized, as its period over a cycle intersecting the seam may not be real. We will thus need to subtract from it a suitable *holomorphic* differential, for which this is the only non-real period, which we call an *almost real-normalized* differential. Determining the limits of zeroes of the difference of these two differentials will thus involve studying which of them goes to zero faster as the plumbing parameters go to zero. We thus define the main player of this section.

Definition 5.4.1. Given any stable nodal curve $C_0 \in \partial\overline{\mathcal{M}}_g$ and a non-separating node r_i on it, let $C_{s_i=0,\underline{s},\underline{v}}$ where $s_j > 0$ for all $j \neq i$ as before be the plumbing deformation of $C_{\underline{v}}$ at the other nodes, and let $q_1, q_2 \in \widetilde{C}_{s_i=0,\underline{s},\underline{v}}$

be the two preimages of r_i on its normalization. Now q_1 and q_2 lie on the same irreducible component of $\widetilde{C}_{s_i=0,\underline{s},\underline{v}}$, and we let $\widetilde{\omega}$ be the unique real-normalized meromorphic differential on $\widetilde{C}_{s_i=0,\underline{s},\underline{v}}$ whose only singularities are simple poles at q_1, q_2 of residue $\pm\sqrt{-1}$.

Then the *almost real-normalized* holomorphic differential $\omega_{r_i}(\underline{s},\underline{v})$ on $C_{\underline{s},\underline{v}}$ is defined to be the result of gluing $\widetilde{\omega}$ at q_1, q_2 with plumbing parameter s_i. We note that it is a holomorphic differential defined up to sign given by the choice of signs of the residues of $\widetilde{\omega}$, and when no confusion is possible we will denote it $\omega_i(\underline{s},\underline{v})$.

Remark 5.4.2. The motivation for calling this the almost real-normalized differential is the easy observation that the integral of ω_i over any loop in $H_1(C_{\underline{s},\underline{v}})$ that does not intersect the seam γ_{s_i} is real, being the integral of the real-normalized $\widetilde{\omega}$ over it. Moreover, the integral $\int_{\gamma_{s_i}} \omega_r = \pm 2\pi$, and thus the integral of ω_i may not be real only over cycles with a non-zero intersection number with γ_{s_i}. While it would be just as natural to define the notion of an almost real-normalized holomorphic differential to say that given a choice of a basis of $H_1(C_{\underline{s},\underline{v}})$, its only non-real period is over the first of the basis elements, the above construction has the great advantage of *canonically* associating an almost real-normalized differential to a node of a stable curve, avoiding the issue of choosing a basis of homology in a neighborhood of a nodal curve.

In what follows we will have to consider what happens to the almost real-normalized differential when some nodes form, and understand how $\omega_i(\underline{s},\underline{v})$ behaves on nodal curves $C_{\underline{s},\underline{v}}$ where $s_i > 0$, but various other plumbing parameters degenerate to zero. Notice that if C_0 is an irreducible nodal curve then

$\widetilde{C}_{\underline{s},\underline{v}}$ is connected for any set of plumbing parameters equal to zero.

Proposition 5.4.3. *For any* $s_j = 0$ *such that* $\widetilde{C}_{s_i=0,s_j=0,\underline{s},\underline{v}}$ *is connected, the almost real-normalized differential* $\omega_i(\underline{s},\underline{v})$ *on* $C_{s_j=0,\underline{s},\underline{v}}$ *where* $s_i > 0$ *does not have poles at any of the nodes.*

Proof. Let B be any cycle which intersects the seam γ_{s_j} and does not enter a neighborhood of the node r_i. By construction of the almost real-normalized differential, the imaginary part of the absolute period $\operatorname{Im}\int_B \omega_i(\underline{s},\underline{v})$ is zero for any curve on which $s_j > 0$.

Again by construction $\operatorname{Im}\int_{\gamma_{s_j}} \omega_i(\underline{s},\underline{v})$ is also zero for all $s_j > 0$, therefore if $\omega_i(\underline{s},\underline{v})$ develops a simple pole at the node r_j the residue must be purely imaginary. On the other hand, if $\omega_i(\underline{s},\underline{v})$ develops a singularity at the node with purely imaginary residue, $\sqrt{-1}r$, this would imply $\operatorname{Im}\int_B \omega_i(\underline{s},\underline{v})$ blows up like $r\ln(|s_j|)$ where on computes $\operatorname{Im}\int \sqrt{-1}r\frac{dz}{z}$ in an ε-neighborhood of the seam γ_{s_j}. By assumption $\operatorname{Im}\int_B \omega_i(\underline{s},\underline{v})$ vanishes throughout and thus $r = 0$. □

This will be important throughout the paper whenever we use a linear combination of almost real-normalized differentials; In specific it will allow us to claim that new poles do not form at the nodes.

For concreteness, and to explain the geometry with simpler notation (where don't have to deal with less singular curves in the neighborhood of this curve– thus no need for 5.4.3), we first discuss the case of irreducible curves with one node (so we drop the index $i = 1$) in detail, showing how the almost real-normalized holomorphic differential is used.

5.4.1 An irreducible stable curve with one node

Let C_0 be an irreducible stable curve with one node $q_1 \sim q_2$. We then let $\widetilde{\Psi}'(\underline{t}) = \Psi'(s, \underline{v}, \underline{z})$ be the meromorphic differential on the normalization $\widetilde{C}_{\underline{v}}$ that has singularities prescribed by $\underline{z}$ at the marked points p_i, *and also has simple poles at* q_1, q_2 *with residues* $\sqrt{-1}\,|s|$ *respectively.* Since simple poles with opposite residues match, $\widetilde{\Psi}'(\underline{t})$ glues to define a meromorphic differential $\Psi'(\underline{t})$ on $C_{s,\underline{v}}$ whose only singularities are at the points p_i.

Let B be any cycle on $C_{s,\underline{v}}$ such that its intersection number with the seam γ is equal to one. We introduce the following notion:

Definition 5.4.4. We call the ratio $\beta(\underline{t}) := \frac{\operatorname{Im}\int_B \Psi'(\underline{t})}{\operatorname{Im}\int_B \omega(s,\underline{v})}$ the (real) *normalizing factor* associated to the node.

Remark 5.4.5. We note that the integral of a differential over a cycle only depends on the homology class of such a cycle. The symplectic pairing on the first homology of any smooth curve is non-degenerate, and thus identifies H_1 with its dual space. As noted above, the periods of ω may not be real only over the loops intersecting the seam, and are completely determined by the intersection number of class in H_1 with the class in H_1 of the seam. Thus the normalizing factor above is well-defined. More generally, to define the normalizing factor we will need to take a class in $H_1(\widetilde{C}_{s_i=0,\underline{s},\underline{v}}, \mathbb{Z})$ such that its intersection number with the seam γ_{s_i} is equal to one

To construct the real-normalized differential on $X_{\underline{t}}$, we combine these two differentials:

Proposition 5.4.6. *The differential given by* $\Psi(\underline{t}) := \Psi'(\underline{t}) - \beta(\underline{t})\omega(s, \underline{v})$ *is real-normalized on* $X_{\underline{t}}$.

Proof. Indeed, the periods of $\Psi'(\underline{t})$ over all cycles which do not intersect the seam are real by construction, as are the periods of $\omega(s, \underline{v})$ over them. Similarly the periods of both summands over γ_s are real, while finally the coefficient $\beta(\underline{t})$ is precisely to ensure that the linear combination above has a purely real period over B. □

Remark 5.4.7. Note that we have prescribed the residues of $\widetilde{\Psi}'$ in an ad hoc manner — this choice will in fact make it easier to analyze the behavior of the linear combination in general, as some other node is pinched. However, the resulting Ψ is independent of this choice, as it can also be thought of as a differential glued from a differential with matching residues, where the residues are chosen precisely to ensure real normalization — and thus no choice is involved. For determining the behavior of the real-normalized differential as the curve degenerates it turns out to be much simpler to separate the two summands as we did, and then to determine the behavior of the normalizing factor explicitly by a direct computation.

We also note that we cannot choose any other pole orders for $\widetilde{\Psi}'$ at q_1, q_2, as then the differential would be regular at one of these two points, and its value there would depend on the singularity at the other point, so that we could not prescribe it a priori.

In order to verify that no new poles appear at the node as it develops, reproving proposition 4.4.2 for the case of an irreducible curve with one node, and for future use, we compute $\beta(\underline{t})$ explicitly in the plumbing parameter s.

Lemma 5.4.8. *The limit of the normalizing factor $\beta(\underline{t})$ is non-zero exactly when $\Psi'(\underline{t})$ is not real-normalized. In addition the normalizing factor vanishes* *when* $s \to 0$ *and therefore* $\Psi(0, \underline{v}, \underline{z}) = \Psi'(0, \underline{v}, \underline{s})$.

More precisely, we will show that when $\operatorname{Im} \int_B \Psi'(\underline{t})$ is not zero throughout we have the following,

$$\beta(\underline{t}) = \frac{|s| \ln(\sqrt{|s|}) - 2\,|s| \ln(\epsilon) + O(1)}{\ln(|s|) - 2 \ln(\epsilon) + O(1)}$$

as $s \to 0$.

Proof. We first note that if $\operatorname{Im} \int_B \Psi'(\underline{t}) = 0$, then the normalizing factor β is zero, and so we only have to deal with the case when $\operatorname{Im} \int_B \Psi'(\underline{t})$ is non-zero.

Let B be a cycle which intersects the ϵ-neighborhood of the node as the real segments $[\sqrt{|s|}, \epsilon] \subset \mathbb{C}$ on both U_1 and U_2. We denote the part of B outside of U_1 and U_2 by B^o and compute

$$\operatorname{Im} \int_B \Psi'(\underline{t}) = 2 \int_\epsilon^{\sqrt{|s|}} |s| \, \frac{dz}{z} + \int_{B^o} \Psi_s(\underline{t}) = |s| \ln(\sqrt{|s|}) - 2\,|s| \ln(\epsilon) + \int_{B^o} \Psi'(\underline{t}).$$

We similarly compute

$$\operatorname{Im} \int_B \omega(s, \underline{v}) = 2 \int_\epsilon^{\sqrt{|s|}} \frac{dz}{z} + \int_{B^o} \omega(s, \underline{v}) = \ln(|s|) - 2 \ln(\epsilon) + \int_{B^o} \omega(s, \underline{v}).$$

and dividing one by the other yields the desired formula for the normalizing factor β. $\square$

5.4.2 A general irreducible curve

The construction above can be easily generalized to the case of an irreducible curve with multiple nodes, for which we use many almost real-normalized differentials and normalizing factors, indexed by the nodes. Indeed, let $X_0 \in \overline{\mathcal{M}}_{g,n}^{1,\dots,1}$ with C_0 being an irreducible curve with k nodes $r_1, \dots, r_k$, obtained by gluing its normalization $\widetilde{C}_0$ at k pairs of points q_1^i, q_2^i. In order to reprove 4.4.2 one must verify that the various almost real-normalized differentials do not develop poles when any number of plumbing parameters vanish and this follows directly from 5.4.3.

Similarly to the case of one node, let $\widetilde{\Psi}'(\underline{t})$ be the real-normalized differential on $\widetilde{C}_{\underline{v}}$ which has prescribed singularities at $\{p_1, \dots, p_n\}$, and simple poles with residues $\pm\sqrt{-1}\,|s_i|$ at q_1^i, q_2^i for all i (as per the remark above, this is an ad hoc choice that does not influence the resulting Ψ, but is more convenient for computations). For any $i \in [1, \dots, k]$ let $B_i \in H_1(C_{\underline{s},\underline{v}})$ be a cycle with intersection number one with γ_{s_i}, and not intersecting any other seam γ_{s_j}. Let $\omega_i(\underline{s}, \underline{v})$ be the almost real-normalized differential associated to the node r_i, and let $\beta_i(\underline{t}) := \frac{\operatorname{Im}\int_{B_i}(\Psi'(\underline{t}))}{\operatorname{Im}\int_{B_i}(\omega_i(\underline{s},\underline{v}))}$ be the corresponding normalizing factor — which does not depend on the choice of B_i.

Similarly to the above, we have

Proposition 5.4.9. *For any smooth curve in a neighborhood of X_0, the formula*

$$\Psi(\underline{t}) := \Psi'(\underline{t}) - \sum_i \beta_i(\underline{t})\omega_i(\underline{s}, \underline{v}) \tag{5.1}$$

defines the real-normalized differential on $C_{\underline{s},\underline{v}}$ with prescribed singularities.

We will now verify that the formula above also works when any number of the s_i is zero, i.e. that it can be used to define the limits of real-normalized differentials on all stable curves in the neighborhood of X_0 as well. This amounts to showing that for each normalizing factor which is not zero in the whole neighborhood (i.e. when $\mathrm{Im}\int_{B_i}\Psi'(\underline{t})$ is not zero throughout-which would amount to the matching differential being already real-normalized over this new cycle.) vanishes *exactly* when $s_i = 0$. This is basically clear by our construction, and is given by almost an exact replica of the proof for one node with few new details here.

Lemma 5.4.10. *As $s_i \to 0$, the normalizing factor (if it does not vanish identically) has the following asymptotics:*

$$\beta_j(\underline{t}) = \frac{|s_i|\ln(|s_i|) - 2\,|s_i|\ln(\epsilon) + O(1)}{\ln(|s_i|) + O(1)}.$$

If $\mathrm{Im}\int_{B_i}\Psi'(\underline{t})$ is non-zero, then the limit $\lim_{s_j\to 0}\beta_i(\underline{t}) = 0$ if and only if $j = i$.

Proof. The formula for $\beta_i(\underline{t})$ as stated in the lemma follows exactly from our computation done in the one node case where we compute with B_i intersecting an ϵ-neighborhood of the node r_i in positive real-rays. Let B_i^o denote the part of B_i outside of the neighborhood of the node r_i, which by definition does not enter a neighborhood of any other node r_j. Therefore $\mathrm{Im}\int_{B_i^o}\Psi'(\underline{s},\underline{v})$ and $\mathrm{Im}\int_{B_i^o}\omega_i(\underline{s},\underline{v})$ have finite limits as $s_j \to 0$. In specific the denominator blows up precisely when $s_i = 0$ and is finite otherwise, and the asymptotics of $\beta_i(\underline{t})$ follows. □

Corollary 5.4.11. *The formula* (5.1) *defines the real-normalized differentials on all smooth curves in a neighborhood of X_0, and the limits of real-normalized differentials on all nodal curves (i.e. the real-normalized differentials on the normalizations) in the neighborhood of X_0 in $\overline{\mathcal{M}}_{g,n}^{1,\ldots,1}$.*

The above description allows us to determine the limits of zeros of real-normalized differentials under degenerations to irreducible stable curves. We note, that unlike the compact type case, here there are no null components, and the limits of the zeroes are unique, depending only on the limiting nodal curve, and independent of the degenerating family.

Proposition 5.4.12. *For any family X_t degenerating to an irreducible stable curve $X_0 \in \overline{\mathcal{M}}_{g,n}^{1,\ldots,1}$, the zeros of the real-normalized differentials on X_t in the limit tend to the zeroes of the real-normalized differential Ψ_0' on the normalization $\widetilde{C}_0$, together with the nodes, with a pair of zeros tending to each node.*

Proof. Because the normalizing factor vanishes as $\underline{s} \to 0$ (or if it is zero throughout), the zeros of $\Psi(\underline{t})$ tend the zeros of $\Psi'(\underline{t})$. Recall $\Psi'(\underline{t})$ has $2(g-k-1)+2n+2k$ zeros on $\widetilde{C}_{\underline{v}}$. The expansion of $\widetilde{\Psi}'(\underline{t})$ near each node is $i\,|s_i|\,\frac{dz}{z}$, and thus as $s_i \to 0$, two zeros approach the node (this is one more reason why our choice of the residues of Ψ' is convenient). □

5.5 Further cases: the banana and triangle curves

We now consider the more complicated cases, where the theory of limit linear series is very complicated (banana) or unavailable (triangle). As we will see, the situation is rather straightforward for real-normalized differentials — in fact the techniques already introduced above suffice to construct the real-normalized differentials.

To set notation, the *banana curve* is the nodal curve has two nodes and two irreducible components, so that $\widetilde{C}_0 = C_1 \cup C_2$, with nodes given by $q_1^2 \sim q_2^1$ and $q_1^1 \sim q_2^2$, where $q_1^1, q_1^2 \in C_1$ and $q_2^1, q_2^2 \in C_2$. Similarly, the *triangle curve* is the stable curve with three nodes and three irreducible components, so that $\widetilde{C}_0 = C_1 \cup C_2 \cup C_3$, with nodes given by $q_1^1 \sim q_2^2$, $q_2^1 \sim q_3^2$, and $q_3^1 \sim q_1^2$ where $q_i^1 \in C_i$. We note that in a sense a banana curve is a "2-gon" curve, to emphasize which we chose the different notation for the nodes than in general; note also that the "k-gon" curve will be completely analogous to what we do in this section. For both the banana and the triangle curve, we note that plumbing all but one nodes gives an irreducible stable curve, while normalizing all but one nodes gives a stable curve of compact type — thus it is natural to expect here the behavior that somehow generalizes both cases considered above and can specialize to either one under a suitable degeneration.

In contrast to 5.4.3 (where the degenerate curve was connected), if $\widetilde{C}_{\underline{s},\underline{v}}$ is disconnected for some $s_j = 0$ and $s_i = 0$ the almost real-normalized differential does develop poles at the node which by the sum of residues is zero, it must have residue $\pm\sqrt{-1}$. Therefore $\omega_i(\underline{s}, \underline{v})$ is alternatively given by gluing the unique real-normalized differentials on $\widetilde{C}_{s_i=0,s_j=0,\underline{s},\underline{v}}$ with simple poles of residue $\pm\sqrt{-1}$ at each node.

Proposition 5.5.1. *If $\widetilde{C}_{s_i=0,s_j=0,\underline{s},\underline{v}}$ is disconnected where r_i and r_j are each non-separating nodes, the almost real-normalized differential $\omega_i(\underline{s}, \underline{v})$ on on the curve $C_{s_j=0,\underline{s},\underline{v}}$ has simple poles of residue $\pm\sqrt{-1}$ at the node r_j .*

Proof. This follows from simply noting the sum of the residues on any irreducible component must be non-zero. Assume for contradiction that $\omega_i(\underline{s}, \underline{v})$ does not have simple poles when $s_j = 0$ and $s_i > 0$. Therefore $\omega_i(\underline{s}, \underline{v})$ is holomorphic on a family $C_{s_i>0,s_j=0,\underline{s},\underline{v}}$ where $s_1 \to 0$. By construction $\omega_i(\underline{s},\underline{v})$ has simple poles at r_i when $s_i = 0$, and thus $\omega_i(\underline{s},\underline{v})$ on $C_{s_1=0,s_2=0,\underline{s},\underline{v}}$ has two irreducible components with one simple at the preimages of r_i which lie on separate components. This is impossible because the sum of residues must equal zero on any irreducible component which implies we must have poles at r_j when $s_j = 0$. Therefore $\omega_i(\underline{s},\underline{v})$ can equivalently be constructed by gluing the real-normalized differential on $C_{s_i=0,s_j=0,\underline{s},\underline{v}}$ with simple poles at the node r_i and r_j. $\square$

Notice that in the case of a banana (resp. triangle) we have initially two (resp. three) different constructions of the almost real-normalized differentials, corresponding to each of the nodes. By the proposition on the banana curve $\omega_1(\underline{s},\underline{v}) = \omega_2(\underline{s},\underline{v})$ (also equal to $\omega_3(\underline{s},\underline{v})$ for the triangle), and this differential $\omega_1(\underline{s},\underline{v})$ is equal to the result of gluing real-normalized differentials on $C_{\underline{v}}$ (here all $s_i = 0$) with simple poles of residue $\pm\sqrt{-1}$ at each of the two (resp. three) nodes. We will thus denote it as $\omega(\underline{s},\underline{v})$ where we drop the index.

5.5.1 The banana curve

We have two possibilities depending on whether one of the components is null. Similarly to the compact type case, even if neither component is null, we will choose one component to determine by matching the condition on the other.

Indeed, let $\Psi_1(\underline{t})$ be the real-normalized differential on $C_{1,\underline{v}}$ with the required singularities as determined by local coordinates $\underline{z}$ corresponding to those p_i that lie on C_1. If C_2 is a null component, we then let $\Psi_2(\underline{t})$ be the real-normalized differential whose only singularities are at q_2^1 and q_2^2, determined by the matching condition from q_1^1 and q_1^2, for the plumbing parameters s_1 and s_2, respectively. If C_2 is a non-null component, we let $\Psi_2(\underline{t})$ be the real-normalized differential on $C_{2,\underline{v}}$ whose singularities at q_2^1 and q_2^2 are determined by matching, and which in addition has prescribed singularities at all the points p_i that lie on C_2.

Similarly to the case of stable curves of compact type, in either case the differentials $\Psi_1(\underline{t})$ and $\Psi_2(\underline{t})$ glue to define a meromorphic differential $\Psi'(\underline{t})$ on $C_{\underline{s},\underline{v}}$ with prescribed singularities at all the points p_i, and no other poles. However, similarly to the case of irreducible stable curves, this differential $\Psi'(\underline{t})$ may not be real-normalized, as its period may not be real over a cycle that intersects both seams.

Let B be a non-trivial cycle on $C_{\underline{s},\underline{v}}$ with intersection pairing $B \cdot \gamma_{s_1} = 1$ (and since the two seams are homologous, also $B \cdot \gamma_{s_2} = 1$), and define the normalizing factor $\beta(\underline{t}) := \frac{\operatorname{Im} \int_B \Psi'(\underline{t})}{\operatorname{Im} \int_B \omega(\underline{s},\underline{v})}$. As before, we have

Proposition 5.5.2. *The meromorphic differential*

$$\Psi(\underline{t}) := \Psi'(\underline{t}) - \beta(\underline{t})\omega(\underline{s}, \underline{v}) \tag{5.2}$$

is the real-normalized differential with prescribed singularities on any smooth $X_{\underline{t}}$ *in a neighborhood of the banana curve.*

Here we encounter the new feature of the situation: the locus of banana curves is of complex codimension two in $\overline{\mathcal{M}}_{g,n}$, and a neighborhood of C_0 contains nodal stable curves that are not bananas but irreducible curves with one node, corresponding to the case when exactly one of the two plumbing parameters s_1 and s_2 is zero. While the above proposition describes Ψ on any smooth curve in the neighborhood of C_0, proposition 4.4.2 states that Ψ on C_0 is also the continuous limit of real-normalized differentials on irreducible one-nodal curves in the neighborhood. To reprove this and obtain further information about the geometry of the situation and in particularly about the zeroes, we study the behavior of the normalizing factor as one or both plumbing coordinates go to zero.

Lemma 5.5.3. *We have* $\lim_{s_i \to 0} \beta(s_1, s_2, \underline{u}, \underline{z}) = 0$ *for* $i = 1, 2$.

Remark 5.5.4. By this lemma, we can interpret the construction for the banana as a combination of the cases of curves of compact type and of irreducible nodal curves. Indeed, if we consider a family X_t for which s_i is identically zero, then in this family the limits of real-normalized differentials are real-normalized differentials on the partial normalization of the curves at s_i, and the degeneration when $s_j \to 0$ then corresponds to the normalization degeneration to a stable curve of compact type. In this family indeed the normalizing factor is identically zero, and the real-normalized differential on the normalization is simply Ψ', which we note is constructed the same way as in the compact type case.

On the other hand, if s_i is non-zero and fixed, while $s_j \to 0$, we have a family of smooth curves degenerating to an irreducible stable curve with one node, and the construction is identical to the one we had for that case.

Proof. The computation of the normalizing factor is very similar to the case of irreducible curves, the only modification being that B now intersects both seams and by 5.5.1 the almost real-normalized differential develops poles at both node. Here we need to be careful to integrate in both of the plumbing neighborhoods. Let B be any cycle on $C_{\underline{s},\underline{v}}$ which intersects each seam γ_{s_i} in one point and which contains the four real segments $\left[\sqrt{|s_i|}, \epsilon\right] \subset \mathbb{C}$ on each neighborhood of each q_i^j; as before, let B^o be the part of B outside the plumbing neighborhoods. We then compute, noticing that Ψ' is regular on C_1,

$$\int_B \Psi'(\underline{t}) = \operatorname{Im} \sum_{i=1}^{2} \frac{s_i^{m_i+1}}{\sqrt{|s_i|}} - \operatorname{Im} \sum_{i=1}^{2} \frac{s_i^{m_i+1}}{\epsilon^{m_i+1}} + \operatorname{Im} \int_{B^o} \Psi'(\underline{t}).$$

We similarly calculate (though noticing that now ω has simple poles on both C_1 and C_2)

$$\beta(\omega(\underline{s}, \underline{v})) = 2 \sum_{i=1}^{2} (\ln(|s_i|)) - 4 \ln(\epsilon) + \int_{B^o} \omega(\underline{s}, \underline{v}).$$

Therefore as the numerator is exactly zero when $\Psi'(\underline{t})$ is real-normalized and the denominator blows up for either node forming, the limit of $\beta(\underline{t})$ as either s_i approaches zero is zero. □

Corollary 5.5.5. *Formula* (5.2) *defines the real-normalized differential on any smooth curve in the neighborhood of a banana, and also gives the limit of real-normalized differentials (i.e. the real-normalized differential on the normalization) on any irreducible curve with one node in the neighborhood of the banana.*

We now determine the limits of the zeros of real-normalized differentials on families degenerating to the banana curve. The case when C_2 is non-null is clear: then the limits of the zeros are the zeros of $\Psi_1(\underline{t})$, $\Psi_2(\underline{t})$, and the nodes, to each of which two zeros tend in the limit.

The main new and interesting feature of the case of the banana is that if C_2 is a null component, then $\Psi_2(\underline{t})$ and the normalizing factor $\beta(\underline{t})$ both vanish as $s_i \to 0$, and the limit of the zeros of their linear combination on C_2 depends on which term vanishes faster as $s_i \to 0$.

Proposition 5.5.6. *The set of all possible limits of zeros of real-normalized differentials on families of smooth curves $X_{\underline{t}} \subset \mathcal{M}_{g,n}^{1,\dots,1}$ converging as $t \to 0$ is the banana curve X_0 as above, where the real-normalized differential Ψ_1 on C_1 has zeros at q_1^1 and q_1^2 of orders m_1 and m_2 respectively, is parameterized by the real projective space $\mathbb{RP}^{2m_1+2m_2+4}$, which is the set of parameters for the singular parts at q_2^1 and q_2^2 of orders at most m_1+2 and m_2+2, respectively,* not both simultaneously zero, and allowed to have purely imaginary residues*, of a real-normalized differential on C_2, modulo multiplication by a real number.*

Remark 5.5.7. In other words, for any family of stable curves degenerating to X_0, the limits of zeros are the zeros of Ψ_1 that are distinct from q_1^1 and q_1^2, the points q_1^1 and q_1^2 with multiplicities $m_1' \leq m_1$ and $m_2' \leq m_2$ respectively, and the zeros of some real-normalized differential on C_2 with poles of order $(m_i - m_i' + 2)$ at q_2^i, which *is allowed to have purely residues equal in value but opposite in sign* at q_2^1 and q_2^2. Still more explicitly, we see that the limits of the zeros are the limits of the zeros of the linear combination $\Psi'(\underline{t}) + \beta(\underline{t})\omega(\underline{s}, \underline{v})$, where both summands go to zero as s_i goes to zero, but one of the following could happen: if $\operatorname{Im}\int_{q_1^1}^{q_1^2} \Psi_1 \neq 0$, then $\beta(\underline{t}) \neq 0$, and goes to zero slower than $\Psi'(\underline{t})$, so that the limit of the zeros are the zeros of ω, or it could be that $\operatorname{Im}\int_{q_1^1}^{q_1^2} \Psi_1 = 0$, so that $\beta(\underline{t})$ is identically zero, so that the limit of the zeros are the zeros of $\Psi'(\underline{t})$ on C_2, or it could be that s_1 and s_2 are chosen to go to zero at such speeds (roughly as $s^1 = e^{1/s_2}$) that $\Psi'(\underline{t}) \sim$ lowest order and $\beta(\underline{t}) \sim$ lowest order go to zero at the same rate, and the limit of the zeros are the zeros of this non-trivial linear combination. We note that this set of zeros is precisely the fiber of the map $\mathcal{M}_{g_2,2}^{\leq m_1+1, \leq m_2+1} \times \mathbb{R} \to \mathcal{M}_{g_2,2}$ (where we multiplied by $\mathbb{R}$ to allow purely imaginary residues) over (C_2, q_2^1, q_2^2), with the zero section removed, modulo multiplication by a real number.

5.5.2 "Triangle" curves

We now deal with the triangle curve that has three irreducible components. This is in fact completely analogous to the case of the banana, and this subsection just serves to confirm that we encounter no further difficulties in dealing these stable curves where the theory of limit linear series in unavailable.

To start, we order the irreducible components of C_0 as described in section 5.3.2. There are now three possible situations: all of order zero (no null components), two components of order zero and one component of order one (one null component), or one component of order zero and two null components (automatically both of order one). Regardless of the case, we choose $\Psi_i(\underline{t})$ to be the real-normalized differential on $C_{i,\underline{v}}$ with prescribed singularities which satisfies the matching condition at the nodes with respect to the plumbing parameters $\underline{s}$ and the ordering on the components. If two irreducible curves of the same order share a node, there is a choice which we prescribe to match with the other, and the result is independent of this choice.

Let $\Psi'(\underline{t})$ be the result of gluing $\Psi_1(\underline{t})$, $\Psi_2(\underline{t}), \Psi_3(\underline{t})$ on $C_{\underline{s},\underline{v}}$ with respect to plumbing parameters at each node, which as before will have real periods on all cycles contained in one irreducible component of C_0, but not necessarily over the cycle which is represented in the dual graph, intersecting each of the seams. As discussed in the previous subsection on the banana, the three almost real-normalized differentials associated to the three nodes are equal, and we denote them $\omega(\underline{s},\underline{v}) = \omega_1(\underline{s},\underline{v})$. Choosing the cycle B to have intersection one with each of the three seams, we have the usual normalizing factor: $\beta(\underline{t}) := \frac{\operatorname{Im}\int_B \Psi'(\underline{t})}{\operatorname{Im}\int_B \omega(\underline{s},\underline{v})}$, so that again the differential $\Psi(\underline{t}) := \Psi'(\underline{t}) - \beta(\underline{t})\omega(\underline{s},\underline{v})$ is real-normalized on any smooth curve in the neighborhood of the triangle curves. The asymptotics of $\beta(\underline{t})$ as some s_i go to zero are computed exactly the same way as for the banana curve (and 5.5.1), and we see that $\beta(\underline{t})$ vanishes whenever any $s_i = 0$. More precisely, we have

$$\beta(\underline{t}) = \frac{\operatorname{Im}\frac{s_1^{m_1+1}}{\sqrt{|s_1|}^{m_1+1}} + \operatorname{Im}\frac{s_2^{m_2+1}}{\sqrt{|s_2|}^{m_2+1}} + \operatorname{Im}\frac{s_3^{m_3+1}}{\sqrt{|s_3|}^{m_3+1}} + \operatorname{Im}\int_{B^o}\Psi'(\underline{t})}{\ln|s_1| + \ln|s_2| + \ln|s_3| + \operatorname{Im}\int_{B^o}\omega(\underline{s},\underline{v})}.$$

and it thus follows that the formula for $\Psi(\underline{t})$ describes the limits of real-normalized differentials on all stable curves in the neighborhood of the triangle curve — which now includes the banana curves for the case when precisely one s_i vanishes.

For the zeros we then have the statements analogous to and generalizing the previous situations. Indeed, if all of the components of the triangle are non-zero, then the limits of the zeros are simply the zeros of the real-normalized differentials on each of the components, together with two nodes approaching each of the nodes. If two components are non-null, then on them the limits of zeros are the zeros of the corresponding real-normalized differentials, together with two zeros approaching the node between them (we note for this and other cases like it that though when gluing two components of the same order, in particular two non-null components, the singular part of the pole arising there vanishes for $s = 0$, and thus does not matter for the limit on the non-null component), while on the null component we have in the limit the zeros of a real-normalized differential with poles of orders at most $m_1 + 2$ and $m_2 + 2$ at q_3^1 and q_3^2, where m_1 is the vanishing order of Ψ_1 at q_1^2, and m_2 is the vanishing order of Ψ_2 at q_2^1. Finally, if C_2 and C_3 are null components, then the limits of the zeros are the zeros of Ψ_1 on C_1, together with the zeros of the real-normalized differentials on C_2 and C_3 with poles at their nodes connecting to C_1 of orders and expansion matching the values on C_1, and where at the node between C_2 and C_3 we glue a regular value to a double pole with small singularity proportional to $|s|$ as before.

5.6 The dollar curve: more than one cycle in the dual graph

We now deal with the "dollar curve" — the nodal curve that has two irreducible components joined at three nodes. Again, it turns out that the techniques already developed suffice to obtain a complete description of real-normalized differentials in the neighborhood of such a curve, and this section of the text just serves to verify this. The main complication here is that the dual graph of the stable curve has more than one cycle (the first homology of the dual graph is two-dimensional); if we were associating holomorphic almost real-normalized differentials to homology bases on smooth curves, this would cause complications. However, as our construction of an almost real-normalized differential is associated to nodes, we do not encounter this problem.

To fix notation, we denote the dollar curve $C_0 = C_1 \sqcup C_2/(q_1^1 \sim q_2^1, q_1^2 \sim q_2^2, q_1^3 \sim q_2^3)$. As usual, we denote $\Psi_1(\underline{t})$ the real-normalized differential on $C_{1,\underline{v}}$ with required singularities at p_i, and let $\Psi_2(\underline{t})$ have prescribed singularities (or with no singularities if C_2 is a null component) and three new poles at q_2^i matching, with respect to plumbing parameters s_1, s_2, s_3 at each node, the values of $\Psi_1(\underline{t})$ at q_1^i; as always $\Psi_1(\underline{t})$ and $\Psi_2(\underline{t})$ glue to define a meromorphic differential $\Psi'(\underline{t})$ on $C_{\underline{s},\underline{v}}$ with singularities only at p_i as prescribed, which again may not be real-normalized.

We can then follow the same approach used for the banana and triangle curves above (where the dual graph had one non-trivial cycle in the dual graph and we used one almost real-normalized differential)— thus we choose two almost real-normalized differentials.

Initially there are three almost real-normalized $\omega_1(\underline{s}, \underline{v}), \omega_2(\underline{s}, \underline{v}), \omega_3(\underline{s}, \underline{v})$ associated to the three nodes of the dollar curve. In the triangle curve case these were all equal corresponding to the fact that there is one non-trivial cycle in the dual graph.

In the case of the dollar curve their linear span will be two-dimensional, which corresponds to the fact that there are two independent cycles in the dual graph of the curve (i.e. that the geometric genus of C_0 is equal to $g-2$). This is to say that there exists one linear relation among $\omega_i(\underline{s}, \underline{v})$ on $C_{\underline{t}}$. Therefore we continue to carefully state all the details of this case where one may initially be concerned that this will present a problem.

Let η_1 and η_2 be a basis for cycles in the dual graph of C_0, where η_1

intersects the nodes r_1 and r_3 and η_2 intersects the nodes r_2 and r_3. On any smooth curve $C_{\underline{s},\underline{v}}$, let B_i be any cycle which intersects the seam γ_{s_i} in one point. In addition let $B_{i,j}$ be a cycle which has intersects the seams γ_{s_i} and γ_{s_j} in one point and does not intersect the third seam. Note that $B_{1,2}$ and $B_{2,3}$ degenerate to η_1 and η_2 respectively on the dollar curve $C_{\underline{v}}$.

Define the following normalizing factors (where by construction of the almost real-normalized differentials, we only have to fix the pairing with one seams as denoted in the notation): $\beta_1(\underline{t}) := \frac{\operatorname{Im}\int_{B_{1,3}} \Psi'(\underline{t})}{\operatorname{Im}\int_{B_1} \omega_1(\underline{s},\underline{v})}$ and $\beta_2(\underline{t}) := \frac{\operatorname{Im}\int_{B_{2,3}} \Psi'(\underline{t})}{\operatorname{Im}\int_{B_2} \omega_2(\underline{s},\underline{v})}$.

The following proposition is in fact simple to verify, and for reference we record the details here.

Lemma 5.6.1. *The differential* $\Psi(\underline{t}) := \Psi'(\underline{t}) - \sum \beta_i(\underline{t})\omega_i(\underline{s},\underline{v})$ *is real-normalized on any smooth curve in the neighborhood of the dollar curve.*

Proof. Notice $\operatorname{Im}\int_{B_{j,3}} \omega_i(\underline{s},\underline{v}) = 0$ for $i \neq j$ by definition of $\omega_i(\underline{s},\underline{v})$ which is real-normalized over any cycle not intersecting the corresponding seam γ_{s_i} and in addition $\operatorname{Im}\int_{B_j} \omega_j(\underline{s},\underline{v}) = \operatorname{Im}\int_{B_{j,3}} \omega_j(\underline{s},\underline{v})$. Therefore, $\operatorname{Im}\int_{B_{j,3}} \Psi(\underline{t}) = \operatorname{Im}\int_{B_{j,3}} \Psi'(\underline{t}) - \frac{\operatorname{Im}\int_{B_{j,3}} \Psi'(\underline{t})}{\operatorname{Im}\int_{B_j} \omega_j(\underline{s},\underline{v})} \operatorname{Im}\int_{B_{j,3}} \omega_j(\underline{s},\underline{v}) = \operatorname{Im}\int_{B_{j,3}} \Psi'(\underline{t}) - \operatorname{Im}\int_{B_{j,3}} \Psi'(\underline{t}) = 0$ for $j \in \{1,2\}$. Any cycle $B_{1,2}$ which intersects the first two seams is homologous $B_{1,3} + B_{2,3}$, and thus clearly $\operatorname{Im}\int_{B_{1,2}} \Psi(\underline{t}) = 0$ by the above calculation. Thus we have verified that $\Psi(\underline{t})$ is real-normalized on smooth curves over any new cycle, and there are no issues with this choice of almost real-normalized differentials. $\square$

In order to ascertain that this formula also works for nodal curves near the dollar, we need to determine the behavior of the normalizing factors β_j as some nodes are formed, i.e. as some $s_i \to 0$, for arbitrary i and j. Here we will crucially use 5.4.3 and 5.5.1 to understand how $\omega_i(\underline{s}, \underline{v})$ behave.

Let $C_{\underline{s},\underline{v}}$ be a family of where all $s_i > 0$ and one node $s_j \to 0$ forms. It follows from 5.4.3, the differential $\omega_i(\underline{s}, \underline{v})$ does not develop poles at the node when $s_j \to 0$ for one node $j \neq i$ while all other plumbing parameters remain non-zero.

Lemma 5.6.2. *On any family $C_{\underline{s},\underline{v}}$ where $s_j \to 0$ for $j \in \{1, 2, 3\}$ and $s_i > 0$ for all other nodes, the differential $\Psi(\underline{t})$ is real-normalized.*

Proof. If $j \in \{1, 2\}$ by construction $\omega_j(\underline{s}, \underline{v})$ *has a poles* at the forming node $s_j \to 0$ and following a computation exactly as in the n-gon case, the denominator of $\beta_j(\underline{t})$ has a term $\ln(|s_j|)$ which implies the normalizing factor vanishes when $s_j = 0$ and the differential becomes $\Psi(\underline{t}) = \Psi'(\underline{t}) - \beta_i(\underline{t})\omega_i(\underline{s}, \underline{v})$ ($i \in \{1, 2\}$ and $\neq j$) which is clearly real-normalized and is exactly the formula given for the banana curve.

If $j = 3$, both normalizing factors survive as $s_3 \to 0$ because both almost real-normalized differentials ω_1 and ω_2 do not has singularities at the node $r_3 = 0$. In addition $\omega_1(s_3 = 0, \underline{s}, \underline{v})$ and $\omega_2(s_3 = 0, \underline{s}, \underline{v})$ are equal up to a real constant. In order to show that $\Psi'(\underline{t}) = \Psi'(\underline{t}) - \frac{\operatorname{Im} \int_{B_{2,3}} \Psi'(\underline{t})}{\operatorname{Im} \int_{B_2} \omega_2(\underline{s},\underline{v})} \operatorname{Im} \int_{B_{1,2}} \omega_2(\underline{s}, \underline{v})$ is real-normalized for $s_3 = 0$, we must integrate over the cycle $B_{1,2}$, but this is clearly real by noticing $B_{1,2}$ is homologous to the sum of paths $B_{1,3} + B_{2,3}$ (here $B_{i,3}$ are no longer close on $\widetilde{C}_{s_3=0,\underline{s},\underline{v}}$–when working on nodal curves, we are always verifying that the differential is real-normalized on the normalization as per our earlier definition regarding real-normalized differentials on nodal curves). ∎

For a family where two nodes r_j and $r_{j'}$ form (we are in the case of compact type), recall by 5.5.1 both normalizing factors vanish and $\Psi(\underline{t}) = \Psi'(\underline{t})$ which is automatically real-normalized on $C_{s_j=0,s_{j'}=0,\underline{s},\underline{v}}$.

We have thus finally proved the following,

Proposition 5.6.3. *The formula*

$$\Psi(\underline{t}) := \Psi'(\underline{t}) - \beta_1(\underline{t})\omega_1(\underline{s}, \underline{v}) - \beta_2(\underline{t})\omega_2(\underline{s}, \underline{v})$$

defines the real-normalized differential on any stable curve $X_{\underline{t}}$ (including nodal curves) in the neighborhood of the dollar curve.

5.7 General stable curves

The methods we have developed so far apply to deal with an arbitrary stable curve C_0. Recall that in section 5.3.2 we have defined an ordering of irreducible components of an arbitrary stable curves, by denoting $C^0 := \cup C_j^0$ the union of non-null components (which contain points p_ℓ), and then denoting $C^k = \cup C_j^k$ the union of all irreducible components distance k away from a non-null component in the dual graph. Similarly to how we used this to deal with a general stable curve of compact type, we will now explain how to use the techniques developed above to construct a real-normalized differential in a small plumbing neighborhood of any stable curve, by inductively constructing the differentials on C^k that glue, and then arguing as for the dollar case that we can then add a suitable holomorphic differential, glued from real-normalized differentials with simple poles, to ensure the reality of all periods.

We start by constructing a meromorphic differential on each component C_j^k with prescribed singularities. In other words we choose the differential on C_j^k to be the unique real-normalized differential which satisfies the matching condition with all curves of order $i-1$ to which its identified via a node and at any node where C_j^k intersects curves $C_1^k, \ldots, C_{j-1}^k$ of the same order with respect to the corresponding plumbing parameter. For curve of order zero, C_j^0, we additionally require the differential to have prescribed singularities at p_i. Each curve C_j^k is irreducible by definition and if C_0 has any nodes formed by identifying points two points in C_j^k we also require the differential to be the real-normalized differential on $\widetilde{C}_j^k$ with prescribed singularities as above and simple poles of residue $\pm\sqrt{-1}\,|s|$ where s is the corresponding plumbing parameter at this node. Thus we have just described how to choose differential $\Psi_j^k(\underline{t})$ on each irreducible components which have prescribed singularities and satisfy the matching condition at each node.

By construction, the differential $\Psi'(\underline{t}) := \cup_{j,k}\Psi_j^k(\underline{t})$ satisfies the gluing condition at every node and defines a meromorphic differential on $C_{\underline{s},\underline{v}}$ with poles exactly as prescribed at the marked points p_i and no new poles. To make it real-normalized, we proceed as by generalizing all previous constructions.

Let V be the space of real-normalized differentials on the normalization $\widetilde{C}_{0,\underline{v}}$ with purely imaginary matching residues at each of the nodes. Such a differential is completely described by the residues, and thus the dimension of the real vector space V is equal to the number of nodes — at each we can take a residue — minus the number of connected components of $\widetilde{C}_0$, on each of which the residues must sum to zero. Clearly these conditions are independent, and thus the dimension of V is equal to the number of edges of the dual graph of C_0 minus the number of vertices of the dual graph of C_0, which by Euler's formula is imply equal to the dimension of H_1 of the dual graph of C_0, i.e. the number of independent cycles in it (which for the case of the dollar curve was two, and which in general is the dimension of the space of vanishing cycles in homology).

Given a basis for cycles in the dual graph of C_0, we will proceed by making our choice of almost real-normalized differential and corresponding normalizing factors explicit. The corresponding degeneration arguments regarding the normalizing factors simply follow from 5.4.3 and 5.5.1 (where in addition once $\beta(\underline{t}) = 0$ for some number of nodes formed it is defined to be zero under any further degenerations of the curve).

Let $r_1, \ldots, r_\ell$ be all nodes which are irreducible nodes on the curves C_j^k and let $J := \{1, \ldots, \ell\}$. For each node r_j where $j \in J$, let $\omega_j(\underline{s}, \underline{v})$ be the corresponding almost real-normalized differential given by the gluing the real-normalized differential with simple poles of residue $\pm\sqrt{-1}$ on $C_{s_j=0,\underline{s},\underline{v}}$. The corresponding normalizng factor $\beta_j(\underline{t})$ is define by integrating over any cycle B_j is such that $B_j \cdot \gamma_{s_j} = 1$ and which has zero intersection with all other seams. We have verified how this degenerates in the irreducible nodal curve section, and it is exactly the same here.

Choose any basis of m simple cycles which span the space of cycles in the dual graph of C_0 normalized at each node in J and for simplicity we additionally normalize at every separating node. Let $I := \{1, \ldots, m\}$ and for each cycle $i \in I$, denote the nodes which lie on the cycle as $r_{i(1)}, \ldots, r_{i(n)}$. For different cycles i and i' the corresponding list of nodes may not be disjoint, but we claim that there is at least one node distinct to each cycle which we choose and denote as $r_i, i \in I$. (Note: As we saw in n-gon case the result won't depend on this choice.)

For each $i \in I$, let $\omega_i(\underline{s}, \underline{v})$ be the result of gluing the real-normalized differential with simple poles of residue $\pm i$ at preimages of the node r_i on $\widetilde{C}_{s_i=0,\underline{s},\underline{v}}$ where we assume all other nodes are smooth. In addition for each $i \in I$, let $B_{i(1),\ldots,i(n)}$ be any cycle on $C_{\underline{s},\underline{v}}$ which intersects each seam $\gamma_{i(1)}, \ldots, \gamma_{i(n)}$ in one point *and* has zero intersection with the other seams. This cycle is any cycle which degenerates to the chosen loop in the dual graph. In addition, let B_i be any non-trivial cycle on $C_{\underline{s},\underline{v}}$ which intersects the seam γ_i in one point (where γ_i is the seam at the node r_i for $i \in I$).

Now define $\beta_i(\underline{t}) := \frac{\operatorname{Im} \int_{B_{i(1),\ldots,i(n)}} \Psi'(\underline{t})}{\operatorname{Im} \int_{B_i} \omega_i(\underline{s},\underline{v})}$ our corresponding normalizing factors, and again one can check that by construction the real values in the normalizing factor do not depend any further on the choice of cycles $B_{i(1),\ldots,i(n)}$ for $\Psi'(\underline{t})$ and B_i for $\omega_i(\underline{s}, \underline{v})$.

Theorem 5.7.1. *The formula*

$$\Psi(\underline{t}) := \Psi'(\underline{t}) - \sum_{j \in J} \beta_j(\underline{t})\omega_j(\underline{s}, \underline{v}) - \sum_{i=1}^{m} \beta_i(\underline{t})\omega_i(\underline{s}, \underline{v})$$

defines the real-normalized differential on any stable curve $X_{\underline{t}}$ (including nodal curves) in the neighborhood of the a general nodal curve C_0.

Remark 5.7.2. Clearly the differential $\Psi(\underline{t}) := \Psi'(\underline{t}) - \sum_{j\in J} \beta_j(\underline{t})\omega_j(\underline{s}, \underline{v}) - \sum_{i=1}^{m} \beta_i(\underline{t})\omega_i(\underline{s}, \underline{v})$ is real-normalized for smooth curves. In fact one can now check this very explicitly by integrating over any cycle $B_{i(1),\ldots,i(n)}$ for $i \in \{1, \ldots, m\}$ and also over cycles B_j for $j \in J$. In addition one can verify using 5.4.3 and 5.5.1 that the normalizing factors $\beta_\ell(\underline{t})$ vanish when $s_\ell = 0$ for

$\ell \in J$ or $\ell \in J$, and we now claim that $\Psi(\underline{t})$ is real-normalized on any nodal curve in the neighborhood of C_0.

Theorem 5.7.3. *The set of all possible limits of zeros of real-normalized differentials on families of smooth curves $X_{\underline{t}} \subset \mathcal{M}_{g,n}^{1,\ldots,1}$ converging as $t \to 0$ to a fixed arbitrary stable curve X_0 is parameterized by a suitable product of real projective spaces, which is the set of parameters for the singular parts of meromorphic differentials at the nodes where irreducible components of C_0 of order $i+1$ meet components of order i (in the ordering given by definition 5.3.9, i.e. if one component is further in the dual graph from the marked points than the other). At each such node, the order of the pole of the meromorphic real-normalized differential on the component of order $i+1$ is at most two greater than the vanishing order of the (inductively constructed) real-normalized differential on the component of order i, including the possibility of the differential having a non-trivial purely imaginary residue there — so the set of parameters for the singular part is $\mathbb{R} \times \mathbb{C}^{m+1}$. For each irreducible component of order $i+1$, the real-normalized differential constructed on it must be non-identically zero, i.e. must have at least one non-zero singular part, and the sum of its residues must be equal to zero (so the set of parameters is the product of the above, subject to one linear condition on the residues lying in $\mathbb{R}$, and taking out the point zero of the product). Finally, the real-normalized differential on each irreducible component is taken up to multiplication by a non-zero real number (so we take the quotient of the above by $\mathbb{R}^*$).*

Remark 5.7.4. We note that for applications in [16] it will be necessary to consider for $X \in \mathcal{M}_{g,n}^{1,\ldots,1}$ a pair of real-normalized differentials Ψ_X and Ψ'_X, where for some fixed $(a_1, \ldots, a_n) \in \mathbb{C}^n$ the differential Ψ'_X has double poles of the form $\Psi'_X = (a_i z_i^{-2} + O(1))dz_i$. In the current paper we describe the behavior of Ψ_X near the boundary of the moduli space. The degeneration of Ψ'_X would then of course be described similarly, and in [16] we will study the behavior of common zeroes of Ψ_X and Ψ'_X under degeneration — which requires a detailed investigation in the case some irreducible component of the stable curve is null for both Ψ_X and Ψ'_X.

Remark 5.7.5. All our results apply to degenerations of real-normalized differentials on $\overline{\mathcal{M}}_{g,n}^m$ in full generality as long as residues are zero. If one attempts to perform similar analysis for differentials of the third kind, with non-zero residues, then the main problem is determining the residues of the limit of Ψ on a stable curve. While at any separating node the residue is determined from the sum of the residues on any irreducible component being zero, we have no analog of proposition 4.4.2 in this case, and there may arise arbitrary residues at non-separating nodes. Once the residues are determined, an analysis similar to the above can be attempted, but we leave the details for a future investigation.

Bibliography

[1] William Abikoff. *The real analytic theory of Teichmüller space*, volume 820 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[2] E. Arbarello, M. Cornalba, and P. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[3] L. Bers. Spaces of degenerating Riemann surfaces. In *Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973)*, pages 43–55. Ann. of Math. Studies, No. 79. Princeton Univ. Press, Princeton, N.J., 1974.

[4] Corentin Boissy. Connected components of the strata of the moduli sapce of meromorphic differentials. 2012. preprint arXiv:math/1211.4951.

[5] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[6] S. Diaz. A bound on the dimensions of complete subvarieties of $\mathcal{M}_g$. *Duke Math. J.*, 51(2):405–408, 1984.

[7] A. Eskin, M. Kontsevich, and A. Zorich. Sum of lyapunov exponents of the hodge bundle with respect to the teichmuller geodesic flow. 2011. preprint arXiv:1112.5872.

[8] A. Eskin, M. Mirzakhani, and A. Mohammadi. Isolation, equidistribution, and orbit closures for the $\mathrm{SL}(2,\mathbb{R})$ action on moduli space. 2014. preprint arXiv:math/1305.3015.

[9] C. Faber. A non-vanishing result for the tautological ring of $\mathcal{M}_g$. 1997. preprint arXiv:math/9711219.

[10] Carel Faber. A conjectural description of the tautological ring of the moduli space of curves. In *Moduli of curves and abelian varieties*, Aspects Math., E33, pages 109–129. Vieweg, Braunschweig, 1999.

[11] Claudio Fontanari and Eduard Looijenga. A perfect stratification of $\mathcal{M}_g$ for $g \leq 5$. *Geom. Dedicata*, 136:133–143, 2008.

[12] Claudio Fontanari and Stefano Pascolutti. An affine open covering of $\mathcal{M}_g$ for $g \leq 5$. 2011. preprint arXiv:math/1103.2646.

[13] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[14] S. Grushevsky and I. Krichever. The universal Whitham hierarchy and the geometry of the moduli space of pointed Riemann surfaces. In *Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces*, volume 14 of *Surv. Differ. Geom.*, pages 111–129. Int. Press, Somerville, MA, 2009.

[15] S. Grushevsky and I. Krichever. Foliations on the moduli space of curves, vanishing in cohomology, and Calogero-Moser curves. 2011. preprint arXiv:1108.4211, under revision.

[16] S. Grushevsky and I. Krichever. Real-normalized differentials and cusps of plane curves. 2014. in preparation.

[17] Richard Hain and Eduard Looijenga. Mapping class groups and moduli spaces of curves. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 97–142. Amer. Math. Soc., Providence, RI, 1997.

[18] John L. Harer. The cohomology of the moduli space of curves. In *Theory of moduli (Montecatini Terme, 1985)*, volume 1337 of *Lecture Notes in Math.*, pages 138–221. Springer, Berlin, 1988.

[19] J. Harris and I. Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, 1998.

[20] John H. Hubbard and Sarah Koch. An analytic construction of the deligne-mumford compactification of the moduli space of curves. 2013. preprint arXiv:math/1301.0062.

[21] M. Kontsevich. Lyapunov exponents and Hodge theory. In *The mathematical beauty of physics (Saclay, 1996)*, volume 24 of *Adv. Ser. Math. Phys.*, pages 318–332. World Sci. Publ., River Edge, NJ, 1997.

[22] M. Kontsevich and A. Zorich. Connected components of the moduli spaces of Abelian differentials with prescribed singularities. *Invent. Math.*, 153(3):631–678, 2003.

[23] I. Krichever. Real normalized differentials and Arbarello's conjecture. *Funktsional. Anal. i Prilozhen.*, 46(2):37–51, 2012.

[24] I. Krichever and D. Phong. On the integrable geometry of $N = 2+$ supersymmetric gauge theories and soliton equations. *J. Differential Geometry*, 45:445–485, 1997.

[25] Eduard Looijenga. On the tautological ring of $\mathcal{M}_g$. *Invent. Math.*, 121(2):411–419, 1995.

[26] C. McMullen. Moduli spaces of isoperiodic forms on Riemann surfaces. 2012. preprint.

[27] David Mumford. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 271–328. Birkhäuser Boston, Boston, MA, 1983.

[28] Dan Petersen. The tautological ring of the space of pointed genus two curves of compact type. 2014. preprint arXiv:math/1310.7369.

[29] Dan Petersen and Orsola Tommasi. The Gorenstein conjecture fails for the tautological ring of $\overline{\mathcal{M}}_{2,n}$. *Invent. Math.*, 196(1):139–161, 2014.

[30] M. Roth and R. Vakil. *The affine stratification number and the moduli space of curves*, volume 38 of *CRM Proc. Lecture Notes*, pages 213–227. Amer. Math. Soc., Providence, RI, 2004.

[31] M. Taniguchi. On the singularities of the periods of abelian differentials with normal behavior under pinching deformation. *J. Math. Kyoto Univ.*, 31(4):1063–1069, 1991.

[32] M. Wolf and S. Wolpert. Real analytic structures on the moduli space of curves. *Amer. J. Math.*, 114(5):1079–1102, 1992.

[33] S. Wolpert. Infinitesimal deformations of nodal stable curves. *Adv. Math.*, 244:413–440, 2013.

[34] A. Yamada. Precise variational formulas for abelian differentials. *Kodai Math. J.*, 3(1):114–143, 1980.

[35] Anton Zorich. Flat surfaces. In *Frontiers in number theory, physics, and geometry. I*, pages 437–583. Springer, Berlin, 2006.