Ricci-Flat Anti-Self-Dual Asymptotically Locally Euclidean 4-Manifolds

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Abstract of the Dissertation

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A classification result for Ricci-flat anti-self-dual asymptotically locally Euclidean 4-manifolds is obtained: they are either hyperkähler (one of the gravitational instantons classified by Kronheimer), or a cyclic quotient of a Gibbons-Hawking space. In the latter case, the action of the deck group is described in terms of the corresponding monopole set in $\mathbb{R}^3$, and it is shown that every such quotient is Kähler.

Contents

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Chapter 1

Overview

Given a sequence of compact 4-dimensional Einstein manifolds with volumes bounded below and diameters and second Betti numbers bounded above, a subsequence will converge in the Gromov-Hausdorff sense to an Einstein orbifold [1, 39]. To understand the singular points, we can "zoom in" on the points of maximum curvature by rescaling, and take a limit. The result will be a Ricci-flat, asymptotically locally Euclidean (ALE) 4-manifold: a "bubble" [1, 9]. Understanding these bubbles is thus an important step in understanding the moduli space of Einstein manifolds in dimension 4.

The simply-connected bubbles were classified by Kronheimer [36, 37], but even when examining a family of metrics on a fixed simply-connected manifold, it is difficult to rule out non-simply-connected bubbles a priori. Moreover, if one aims to show compactness by arguing that not enough curvature accumulates to form a bubble, then a complete list is needed, since quotients will have less total curvature than their covers.

To this end, we prove the following classification result:

Theorem A. *Every Ricci-flat anti-self-dual ALE 4-manifold which is not simply-connected and not flat is a finite quotient of a Gibbons-Hawking space by a cyclic group of isometries, and is actually Kähler.*

Remark 1.1. The Gibbons-Hawking spaces are parameterized by the finite subsets of $\mathbb{R}^3$, and the quotients of any given Gibbons-Hawking metric can be described explicitly in terms of the Euclidean isometries which preserve the corresponding set (see Chapter 7).

Remark 1.2. The fact that every Ricci-flat *Kähler* ALE 4-manifold is either hyperkähler or a cyclic quotient of a Gibbons-Hawking space was stated without proof by Nakajima [40], and a proof of that same fact has recently been given by Ioana Şuvaina [51]. Because our theorem does not require an a priori assumption that the manifold is Kähler, it applies to bubbles resulting from any sequence of Einstein 4-manifolds with W_+ bounded, rather than just from sequences of Kähler-Einstein manifolds.

The following "gap theorem" follows quickly as a corollary:

Corollary B. *If X is a Ricci-flat anti-self-dual ALE 4-manifold such that*

$$\int_X |\mathrm{Rm}|^2 \, d\mu < 6\pi^2,$$

or such that $b_2(X) > 0$ and

$$\int_X |\mathrm{Rm}|^2 \, d\mu < 8\pi^2 \left(b_2(X) + 1 - \frac{1}{b_2(X) + 1} \right),$$

then X is isometric to flat $\mathbb{R}^4$.

An orbifold compactness theorem similar to the one for Einstein manifolds has been proved for Kähler manifolds with constant scalar curvature by Tian and Viaclovsky [55, 56] (see [15] for the general extremal Kähler case). The following corollary is a partial step in the direction of understanding the scalar-flat Kähler bubbles that result:

Corollary C. *If X is a scalar-flat Kähler ALE 4-manifold and $b_2(X) = 0$, then X is actually Ricci-flat, and is a quotient of a Gibbons-Hawking space $\widetilde{X}$ by a cyclic group of order $\chi(\widetilde{X})$.*

Remark 1.3. See the comments before Lemma 18 in [14] for another argument that X must be Ricci-flat.

Outline

For the sake of being self-contained, in Chapter 2 we will review the definitions, constructions, and previous results which are needed to make sense of the statement of our main theorem. Then in Chapter 3, we recall some basic facts needed for the proof, and set up the overall strategy. In chapters 4 and 5, we prove that the three "exceptional" cases of Kronheimer's construction have no quotients, and then in Chapter 6, we prove it in the remaining infinite family of cases. The description of the quotients of the Gibbons-Hawking spaces is given in Chapter 7, and we close with a proof of the two corollaries in Chapter 8.

Chapter 2

Introduction

2.1 Ricci-Flat Anti-Self-Dual 4-Manifolds

2.1.1 Quaternions and (anti-)self-dual 2-forms

First, let us once and for all fix an identification between $\mathbb{R}^4$, $\mathbb{C}^2$, and the quaternions $\mathbb{H}$. Namely,

$$(x, y, z, w) \in \mathbb{R}^4 \mapsto (x + yi, z + wi) \in \mathbb{C}^2,$$
$$(a, b) \in \mathbb{C}^2 \mapsto a + bj \in \mathbb{H}.$$

These identifications are compatible with the standard metrics and orientations on the three spaces.

In all dimensions $n > 2$, the special orthogonal group $\mathrm{SO}(n)$ has fundamental group $\mathbb{Z}_2$, and its universal cover is called $\mathrm{Spin}(n)$. For $n = 4$, the spin group can be concretely represented as follows:

Let $\mathrm{Sp}(1)$ be the group of unit quaternions, and consider the action of

$\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on $\mathbb{H}$ given by:

$$(u_1, u_2) \cdot q = u_1\, q\, u_2^{-1}, \tag{2.1}$$

where juxtaposition represents multiplication as quaternions. One can check that this action is orthogonal and preserves the orientation, so via the above identification between the quaternions and $\mathbb{R}^4$, it gives a homomorphism

$$p\colon\ \mathrm{Sp}(1)^+ \times \mathrm{Sp}(1)^- \to \mathrm{SO}(4), \tag{2.2}$$

where the superscripts are simply labels to distinguish the two copies of $\mathrm{Sp}(1)$. The kernel of this action is exactly $\mathbb{Z}_2 = \pm(1,1)$, so it identifies $\mathrm{Spin}(4)$ with $\mathrm{Sp}(1)^+ \times \mathrm{Sp}(1)^-$.

With our chosen identification, the action of $\mathrm{Sp}(1)^-$ on $\mathbb{H}$ is mapped to the standard action of $\mathrm{SU}(2)$ on $\mathbb{C}^2$, via an isomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ which can be written explicitly as:

$$a + jb \mapsto \begin{bmatrix} \bar{a} & b \\ -\bar{b} & a \end{bmatrix}$$

To simplify notation, we will replace $\mathrm{Sp}(1)$ with $\mathrm{SU}(2)$ via this isomorphism, wherever the former group appears.

Since the universal cover of $\mathrm{SO}(4)$ splits as a Lie group, $\mathfrak{so}(4)$ splits as a Lie algebra: $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. In any dimension, the action of $\mathrm{SO}(n)$ on 2-forms is equivalent to the adjoint action on $\mathfrak{so}(n)$, so on an oriented Riemannian 4-manifold, we have a splitting of the bundle of 2-forms into two sub-bundles, called the self-dual and anti-self-dual 2-forms: $\bigwedge^2 = \bigwedge^+ \oplus \bigwedge^-$.

Reversing the orientation of M interchanges the factors.

Remark 2.1. Notice that the covering map from Spin(4) to SO(4) restricts to an injective homomorphism on each copy of SU(2), so we will consider $\mathrm{SU}(2)^{\pm}$ as subgroups of SO(4) as well (they are no longer complementary, but have intersection $\{\pm 1\}$). The representations corresponding to the self-dual and anti-self-dual 2-forms yield surjective homomorphisms $\mathrm{SO}(4) \to \mathrm{SO}(3)^{\pm}$ with respective kernels $\mathrm{SU}(2)^{\mp}$.

2.1.2 Anti-self-dual metrics

The curvature of a Riemannian metric can be thought of as an operator $\mathcal{R}$ on 2-forms, so the decomposition of 2-forms into self-dual and anti-self-dual parts splits the curvature operator into four components (see Equation 2.3). The trace-free parts of the two diagonal components are called the self-dual (W_+) and anti-self-dual (W_-) Weyl curvatures, respectively. Both are conformally invariant, and their sum is the usual Weyl curvature operator.

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \frac{s}{12} & \mathring{r} \\ \hline \mathring{r} & W_- + \frac{s}{12} \end{array} \right) \tag{2.3}$$

Definition 1. If (M^4, g) is an oriented Riemannian manifold with $W_+ = 0$, then we say that the manifold is *anti-self-dual (ASD)*. Similarly, if $W_- = 0$ we say that the manifold is *self-dual* (SD).

Warning. In the physics literature, a metric is said to have *self-dual curvature* when the curvature operator itself is self-dual, in the sense that the Hodge star sends $\mathcal{R}$ to itself: $\star\mathcal{R} = \mathcal{R}$. By Equation 2.3, this is equivalent to both W_-

and $\mathring{r}$ vanishing, so in our terminology this would be called self-dual Einstein.

Any anti-self-dual manifold is self-dual for the opposite orientation, but we will prefer to consider the ASD orientation because of the following fact:

Proposition 2.1 (Gauduchon [21]). *A Kähler manifold of real dimension 4 is anti-self-dual for the orientation defined by the complex structure if and only if it is scalar-flat.*

2.1.3 Local complex structures

Suppose that a Riemannian 4-manifold (M, g) is both anti-self-dual and Ricci-flat. Then from Equation 2.3, we see that the bundle of self-dual 2-forms is flat. This means that parallel transport around any contractible loop must act by an element in the kernel of the defining representation $\mathrm{SO}(4) \to \mathrm{SO}(3)^+$ of $\bigwedge^+$, which is exactly $\mathrm{SU}(2)^- < \mathrm{SO}(4)$ (see Remark 2.1). But from Equation 2.1, the action of this subgroup commutes with left-multiplication by quaternions, so there is a left $\mathbb{H}$-module structure on $\mathbb{R}^4$ which is preserved by the action.

From the bundle point of view, this means that in any simply-connected open set on M, we can find three almost-complex structures I, J, and K (coming from left-multiplication by the imaginary quaternions i, j, and k) which satisfy the conditions of the following definition:

Definition 2 (Calabi [12]). A *hyperkähler* manifold is a Riemannian manifold (M, g), equipped with three almost-complex structures I, J, and K which are parallel under the Levi-Civita connection of g, and which satisfy the quaternionic relation $IJK = -\mathrm{Id}$.

Thus, we see that:

Proposition 2.2. *Every simply-connected Ricci-flat ASD 4-manifold is hyperkähler.*

In particular, *every* Ricci-flat ASD 4-manifold is *locally* hyperkähler.

2.1.4 Global complex structures

A Ricci-flat anti-self-dual 4-manifold (M, g) must have three independent parallel complex structures locally near any point, but if M is not simply-connected, then none of them necessarily extends globally. If exactly one of the three extends, then the metric is Kähler with respect to that complex structure, so (M, g) is Ricci-flat Kähler. If two of them extend, then so does the third (by the relation $IJ = K$), and the metric is globally hyperkähler. Thus, we have the following relationships in dimension 4:

$$\{\text{hyperkähler}\} \subset \{\text{Ricci-flat Kähler}\} \subset \{\text{Ricci-flat anti-self-dual}\}.$$

All three classes are nonempty, and the inclusions are strict:

- *hyperkähler:* A K3 surface is a complex manifold (of real dimension 4), which is simply-connected and has vanishing first Chern class. By Yau's solution [58] to the Calabi conjecture [11], there exists a unique Ricci-flat metric in each Kähler class, which must be hyperkähler because the manifold is simply-connected.

- *strictly Ricci-flat Kähler:* An Enriques surface is a holomorphic $\mathbb{Z}_2$ quotient of a K3 surface. The first Chern class of an Enriques surface is nonzero but it has trivial image in real cohomology, so Yau's theorem again produces a Ricci-flat metric in each Kähler class. However, the canonical bundle (and thus the first Chern class) of a hyperkähler surface must be trivial, because if ω_J and ω_K are the Kähler forms corresponding to the complex structures J and K, then $\omega_J + i\omega_K$ is a non-vanishing holomorphic 2-form. Thus, an Enriques surface admits metrics which are Ricci-flat Kähler but not hyperkähler.

- *strictly Ricci-flat anti-self-dual:* Some Enriques surfaces admit a free antiholomorphic $\mathbb{Z}_2$ action. The Ricci-flat Kähler metric corresponding to any Kähler class which is preserved by this involution descends to the quotient, and the quotient metric must at least be Ricci-flat ASD, since this is a purely local notion. Since the metric upstairs is not hyperkähler, there is only one complex structure preserved by parallel transport. On the quotient, any loop which corresponds to a nontrivial element of the deck group will take this complex structure to its conjugate, so there is no globally parallel complex structure. Thus, such a metric is Ricci-flat ASD, but not Kähler.

In fact, Hitchin proves [29, Theorem 1] that these are the only *compact* 4-dimensional examples of Ricci-flat ASD metrics (excluding flat metrics).

2.2 Gravitational Instantons

Definition 3. A *gravitational instanton* is a complete, non-compact, hyperkähler 4-manifold (M, g) such that $\int_M |\mathrm{Rm}|^2 \, d\mu_g < \infty$.

These space were first studied by physicists, and the name "gravitational instanton" comes from looking at them as gravitational (i.e., metric) analogues of the Yang-Mills instantons which occur in gauge theory. Namely, a Yang-Mills instanton is a connection on a principal bundle with self-dual or anti-self-dual curvature (corresponding to the hyperkähler condition) which has finite energy (corresponding to the L^2 curvature condition).

Warning. The definition of gravitational instanton varies somewhat in the literature. In particular, some authors allow compact manifolds or relax the hyperkähler condition, but essentially all agree that they must be Einstein 4-manifolds and that the curvature of a non-compact gravitational instanton must decay at infinity.

Gravitational instantons are categorized by the asymptotic form of the metric at infinity. The simplest case is when the metric approaches a quotient of flat $\mathbb{R}^4$:

2.2.1 Asymptotically locally Euclidean metrics

Definition 4. We say that an oriented Riemannian manifold (M^4, g) is *asymptotically locally Euclidean* (ALE) of order $\tau > 0$ if

(i) there is a compact set $C \subset M$ and an orientation-preserving diffeomorphism $(\mathbb{R}^4 \setminus B)/\Gamma \to M \setminus C$, where B is some closed ball centered at the origin of $\mathbb{R}^4$, and Γ is a finite subgroup of $SO(4)$ which acts freely on the 3-sphere; and

(ii) g approximates the Euclidean metric in the sense that the pullback of the metric on $M \setminus C$ to the cover $\mathbb{R}^4 \setminus B$ satisfies

$$\partial^\alpha(g_{ij} - \delta_{ij}) = O(r^{-\tau-|\alpha|}) \quad \text{for all multi-indices } \alpha,$$

where $|\alpha|$ denotes the length of the multi-index, and r is the function given by the distance to 0 in the Euclidean metric.

Remark 2.2. The choice of asymptotic coordinates on $M \setminus C$ is not unique, but even without choosing coordinates we can identify the space form S^3/Γ at the "boundary" of M up to an orientation-preserving isometry. Thus, the group Γ is well-defined up to conjugacy in SO(4). It is called the *fundamental group at infinity* of M, and will be denoted by $\pi_1^\infty(M)$.

Some authors allow an ALE manifold to have several ends, requiring that each connected component of the complement of some compact set satisfies the above hypotheses. Since the ALE metrics we will be considering are Ricci-flat, our definition does not exclude any nontrivial spaces:

Proposition 2.3. *A complete Ricci-flat 4-manifold (M, g) which has more than one end must be flat.*

Proof. Since (M, g) is complete and has more than one end, it must contain a geodesic line (see for example [43, Lemma 41]). Roughly, one takes two sequences of points $\{p_i\}$, $\{q_i\}$ diverging to infinity in two different ends, forms a geodesic segment between p_i and q_i, and takes the limit of the segments, resulting in a line.

Then, since the manifold is Ricci-flat, the Cheeger-Gromoll splitting theorem [13] implies that M splits isometrically as a product $N \times \mathbb{R}$, where $\mathbb{R}$

is equipped with the flat metric, and N is a Ricci-flat 3-manifold. But on a 3-manifold, the Ricci tensor determines the full curvature [10, 1.119], so N and therefore M are flat. $\square$

The definition above may seem unwieldy, in that it requires us to make a choice of coordinates at infinity, but there is a beautiful characterization of ALE spaces in the Ricci-flat case by Bando, Kasue, and Nakajima (a special case of [9, Theorem 1.5]):

Theorem 2.1 (BKN)**.** *A Ricci-flat 4-manifold (M, g) is ALE if and only if it has Euclidean volume growth:*

$$\operatorname{vol} B(p,r) \geq Cr^4 \quad \textit{for some } p \in M \textit{ and } C > 0,$$

and finite energy:

$$\int_M |\mathrm{Rm}|^2 \, d\mu < \infty.$$

Moreover, one can always choose coordinates at infinity making the metric ALE of order $\tau = 4$.

See also [54, Theorem 1.3] for a related theorem in which the Ricci-flat hypothesis is weakened.

2.2.2 History of ALE gravitational instantons

The first nontrivial ALE gravitational instanton was constructed in the late 1970s by Eguchi and Hanson [19], while studying quantum gravity in the Euclidean signature. The underlying manifold is the cotangent bundle of S^2, and the metric can be written explicitly. Shortly after, Gibbons and Hawking [22] discovered an infinite family, parameterized by a choice of a finite set in $\mathbb{R}^3$ (see Chapter 7 for further details), of which the Eguchi-Hanson metric is just the simplest case. An alternative construction of these metrics was given by Hitchin [30], who also noted a connection between gravitational instantons and the finite subgroups of SU(2). Finally, Kronheimer found a construction [36] of the Gibbons-Hawking metrics and several additional families as hyperkähler quotients [32] of a flat vector space, and proved that his construction yields all ALE gravitational instantons [37].

2.3 Kronheimer's Construction

Let us recall some basic facts that will be needed in order to describe Kronheimer's classification of ALE gravitational instantons.

2.3.1 Finite subgroups of SU(2)

The adjoint action of SU(2) on its (3-real-dimensional) Lie algebra gives a homomorphism $\phi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$, which is a 2-fold covering map, and represents the universal cover Spin(3) of SO(3) concretely as SU(2). The nontrivial element of the kernel is -Id, which is the unique element of order two in SU(2).

Now, consider a finite subgroup Γ of SU(2). If Γ contains the kernel of ϕ, then Γ maps 2-to-1 onto a finite subgroup of SO(3). Otherwise, Γ must be of odd order, and map isomorphically onto a subgroup of SO(3). Thus, finding all finite subgroups of SU(2) reduces to enumerating the finite subgroups of SO(3), a very classical problem. Every such subgroup is conjugate to the symmetry group of a polygon (a cyclic group), a degenerate polyhedron with two faces (a dihedral group), or one of the Platonic solids (the tetrahedral, octahedral, and icosahedral groups). Pulling back to $\mathrm{SU}(2)$ gives the following classification, where the term "binary" refers to the pullback of a group under the 2-to-1 map ϕ:

Proposition 2.4. *Each finite subgroup of* $\mathrm{SU}(2)$ *is conjugate to one of the following:*

- *The cyclic group* $\mathbb{Z}_n$ *for* $n \geq 1$
- *The binary dihedral group* $\mathrm{D}^*_{4\mathrm{n}}$ *for* $n \geq 2$
- *The binary tetrahedral group* T^* *of order 24*
- *The binary octahedral group* O^* *of order 48*
- *The binary icosahedral group* I^* *of order 120*

Remark 2.3. To minimize ambiguity, subscripts in the names of groups will always denote the order of the group: e.g., the binary dihedral group of order 8 is D^*_8, not D^*_2.

2.3.2 Resolution of du Val singularities

Definition 5. Given a complex surface Y with an isolated singular point y, a *resolution* of Y is a proper, surjective, holomorphic map $f\colon X \to Y$ such that

- X is nonsingular, and
- the restriction of f to $X \setminus f^{-1}(y)$ is a biholomorphism.

The resolution is called *minimal* if the exceptional divisor $f^{-1}(y)$ contains no (-1)-curves.

The quotients $\mathbb{C}^2/\Gamma$ for Γ a finite subgroup of $\mathrm{SU}(2)$ are complex surfaces which are singular at the origin. This type of singular point is known as a *du Val singularity* (also simple surface singularity, Kleinian singularity, or rational double point), and each admits a unique minimal resolution $\pi\colon X \to \mathbb{C}^2/\Gamma$.

There is an amazing relationship between the representation theory of finite subgroups of $\mathrm{SU}(2)$, the exceptional divisors of the minimal resolutions, and the simply-laced Dynkin diagrams known as the McKay correspondence [38]:

Let $\{R_i\}_{i=0}^n$ be an enumeration of the irreducible complex representations of Γ, and let Q be the 2-dimensional representation defined by the inclusion $\Gamma \subset \mathrm{SU}(2)$. Define the *McKay quiver* as the directed graph with vertices $\{R_i\}$, and such that the number of edges from R_i to R_j is equal to the multiplicity of R_j in the decomposition of $Q \otimes R_i$ into irreducibles. Because $Q \cong Q^*$ as Γ-spaces, the edges will always occur in oppositely-directed pairs. If we remove the vertex corresponding to the trivial representation, and change every pair of opposite edges into a single undirected edge, we produce the *McKay graph.*

Proposition 2.5 (McKay [38])**.** *The map taking a group to its McKay graph gives a one-to-one correspondence between the (conjugacy classes of) finite subgroups of* $\mathrm{SU}(2)$ *and the simply-laced Dynkin diagrams (those with no multiple edges: types A_n, D_n, E_6, E_7, and E_8).*

Moreover, the Dynkin diagram corresponding to Γ also determines the minimal resolution of $\mathbb{C}^2/\Gamma$, in the sense that the preimage of 0 is a union of (-2)-curves, one for each vertex of the diagram, and two curves intersect (with multiplicity +1) exactly when there is an edge joining the corresponding vertices.

2.3.3 Classification of ALE gravitational instantons

We are now ready to state Kronheimer's classification theorem:

Theorem 2.2 (Kronheimer [36, 37])**.** *Let Γ be a finite subgroup of* $\mathrm{SU}(2)$*, and X be the smooth manifold underlying the minimal resolution of $\mathbb{C}^2/\Gamma$. Choose three cohomology classes $\alpha_1, \alpha_2, \alpha_3 \in H^2(X;\mathbb{R})$ such that for any Σ in $H_2(X;\mathbb{Z})$ with $\Sigma \cdot \Sigma = -2$, at least one α_i satisfies $\alpha_i(\Sigma) \neq 0$. Then there exists an ALE hyperkähler structure on X such that $[\omega_i] = \alpha_i$.*

Moreover, if (Y, g) is any ALE hyperkähler 4-manifold, then Y is diffeomorphic to $\mathbb{C}^2/\Gamma$ for some Γ, the Kähler classes of g satisfy the nondegeneracy condition above, and they determine the metric uniquely up to isometry.

Remark 2.4. The three Kähler forms ω_1, ω_2, and ω_3 referred to in the theorem are the Kähler forms corresponding to the three different complex structures I, J, and K making up the hyperkähler structure.

Since $\mathbb{C}^2$ deformation retracts onto the origin, the smooth manifold underlying the resolution of a quotient $\mathbb{C}^2/\Gamma$ deformation retracts onto the exceptional divisor. Thus, the explicit description of those divisors given in Subsection 2.3.2 tells us everything we need to know about the topology of the ALE gravitational instantons. (Alternatively, we can think of the Dynkin diagram corresponding to Γ as a plumbing diagram for the underlying smooth manifold). In particular,

Proposition 2.6. *Let X be an ALE gravitational instanton. Then*

1. *X is simply-connected,*

2. *$b_1(X) = b_3(X) = b_4(X) = 0$, and*

3. *the intersection form of X is negative-definite.*

Proof. Both (1) and (2) follow immediately from the fact that X is homotopy equivalent to the exceptional divisor, which is a tree of 2-spheres. Each irreducible component of the exceptional divisor is a sphere of self-intersection -2, and they span $H^2(X)$, giving (3). □

This allows us to easily compute the Euler characteristic χ and the signature τ for each of these spaces:

Table 2.1: Invariants of gravitational instantons

Dynkin diagram	Γ	$\lvert\Gamma\rvert$	χ	τ
A_{k-1}	$\mathbb{Z}_k$	k	k	$-k+1$
D_{k+2}	D^*_{4k}	$4k$	$k+3$	$-k-2$
E_6	T^*	24	7	-6
E_7	O^*	48	8	-7
E_8	I^*	120	9	-8

Chapter 3

Setup and Strategy

Let X be a Ricci-flat anti-self-dual ALE 4-manifold which is not simply-connected and not flat. By a theorem of Anderson [2, Corollary 1.5], the fundamental group is finite; let $p\colon \widetilde{X} \to X$ be the universal covering map. Since this is a finite cover, the L^2 norm of the curvature of $\widetilde{X}$ is still finite, and the volume growth remains Euclidean, so by Theorem 2.1, $\widetilde{X}$ is also ALE. Since it is simply connected, by Proposition 2.2 it is hyperkähler, and thus is covered by Kronheimer's classification.

Remark 3.1. Because p is a finite covering map, the pullback p^* acting on real cohomology is injective [24, Proposition 3G.1]. We saw in Proposition 2.6 that the cohomology of $\widetilde{X}$ is trivial except in degrees 0 and 2, and that its intersection form is negative definite. Thus $\chi(X) = 1 + b_2(X)$ and $\tau(X) = -b_2(X) = 1 - \chi(X)$.

Kronheimer's classification implies that $\widetilde{X}$ is a Gibbons-Hawking space if and only if the fundamental group at infinity is cyclic. Thus, in order to prove the first part of Theorem A, we need to exclude all other possibilities for $\pi_1^\infty(\widetilde{X})$. We can reduce much of the problem to a matter of group theory via the following:

Proposition 3.1. *There are maps $\pi_1^\infty(\widetilde{X}) \to \pi_1^\infty(X)$ and $\pi_1^\infty(X) \to \pi_1(X)$ such that the sequence*

$$1 \to \pi_1^\infty(\widetilde{X}) \to \pi_1^\infty(X) \to \pi_1(X) \to 1 \tag{3.1}$$

is exact.

Proof. By the definition of ALE, we can find a compact exhaustion of X (a sequence of compact subsets $\{K_i\}$ with $K_i \subset \mathrm{int}(K_{i+1})$ and $\bigcup K_i = X$) such that for each i, the complement $X \setminus K_i$ is diffeomorphic to $\mathbb{R} \times (S^3/\pi_1^\infty(X))$.

A finite covering map is proper, so $\widetilde{K_i} = p^{-1}(K_i)$ is a compact exhaustion of $\widetilde{X}$. We know that both X and $\widetilde{X}$ have only one end by Proposition 2.3, so $\widetilde{X} \setminus \widetilde{K_i}$ and $X \setminus K_i$ are connected for sufficiently large i. Choose some k making both complements connected, and let $K = K_k$ and $\widetilde{K} = \widetilde{K}_k$

Now, since $\widetilde{X} \setminus \widetilde{K}$ is connected and covers a space diffeomorphic to $\mathbb{R} \times (S^3/\pi_1^\infty(X))$, it must also be of the form $\mathbb{R} \times (S^3/\Gamma)$, and this Γ must be the fundamental group at infinity of $\widetilde{X}$. The map of fundamental groups induced by the restriction of p identifies $\pi_1^\infty(\widetilde{X})$ with a normal subgroup of $\pi_1(X \setminus K) \cong \pi_1^\infty(X)$. Moreover, the quotient of $\pi_1(X \setminus K)$ by $\pi_1^\infty(\widetilde{X})$ is identified with the deck group of p, which is just $\pi_1(X)$. $\square$

Remark 3.2. A choice of coordinates at infinity represents $\pi_1^\infty(X)$ and $\pi_1^\infty(\widetilde{X})$ as finite subgroups of $\mathrm{SO}(4)$. In the proof, our chosen coordinates on X and $\widetilde{X}$ were related to each other by a covering map, so the inclusion $\pi_1^\infty(\widetilde{X}) \to$

$\pi_1^\infty(X)$ is compatible with the inclusions of these groups into $\mathrm{SO}(4)$. Moreover, by Kronheimer's theorem and the discussion in Section 2.1.1, $\pi_1^\infty(\widetilde{X})$ is contained in the subgroup $\mathrm{SU}(2)^- < \mathrm{SO}(4)$. This is a meaningful statement even though $\pi_1^\infty(\widetilde{X})$ is only defined up to conjugacy because $\mathrm{SU}(2)^-$ is a normal subgroup.

3.1 Space-Form Groups

If $\Gamma < \mathrm{SO}(4)$ is the fundamental group at infinity of an ALE manifold, then the action of Γ on $\mathbb{R}^4$ is free outside of the origin (otherwise the model space $(\mathbb{R}^4 \setminus B)/\Gamma$ would be singular), so by restricting the action to vectors of unit length, we can form a spherical space form S^3/Γ. Knowing the conjugacy class of Γ in $\mathrm{SO}(4)$ determines the space form up to oriented isometry, and vice versa.

The classification of 3-dimensional spherical space forms was first found by Hopf [34] and Seifert-Threlfall [52, 53] in the early 20th century. In modern terms, the classification can be described as follows: (see [50, 57])

Proposition 3.2. *Let Γ be a subgroup of $\mathrm{SO}(4)$ which acts freely on the 3-sphere, and let $p\colon\ \mathrm{SU}(2)\times\mathrm{SU}(2) \to \mathrm{SO}(4)$ be the universal covering of $\mathrm{SO}(4)$. Up to conjugation in $\mathrm{O}(4)$, one of the following holds:*

1. *Γ is cyclic;*
2. *$\Gamma = p(H_1 \times H_2)$, where H_2 is a non-cyclic binary polyhedral group, H_1 is cyclic of relatively prime order, and $\Gamma \cong H_1 \times H_2$;*

3. *Γ is a diagonal subgroup of index 3 in $p(H_1 \times H_2) \cong H_1 \times H_2$, where H_1 is cyclic of order $3^k \cdot n$ ($k > 1$ and n odd), and $H_2 = T^*$; or*

4. *The pullback of Γ under p is a diagonal subgroup of index 2 in $H_1 \times H_2$, where H_1 is cyclic of order $2^k \cdot m$ ($k > 2$) and $H_2 = D^*_{4b}$ (b odd and relatively prime to m).*

Remark 3.3. A *diagonal subgroup* of a direct product $H_1 \times H_2$ is a subgroup of the form $\{(h_1, h_2) \in H_1 \times H_2 \mid \phi_1(h_1) = \phi_2(h_2)\}$, where $\phi_1 \colon H_1 \to Q$ and $\phi_2 \colon H_2 \to Q$ are homomorphisms onto the same group. In particular, the projection of a diagonal subgroup onto either factor is surjective.

The following proposition will be used to identify the possibilities for $\pi_1^\infty(X)$ given $\pi_1^\infty(\widetilde{X})$.

Proposition 3.3. *Let $G < \mathrm{SO}(4)$ be a finite subgroup acting freely on the 3-sphere, and define $G^- = G \cap \mathrm{SU}(2)^-$. If G^- is not a cyclic group and not isomorphic to D^*_8, then $G = p(\mathbb{Z}_m \times G^-) \cong \mathbb{Z}_m \times G^-$ for some m relatively prime to $|G^-|$.*

Proof. Since G cannot by cyclic, we must be in case (2), (3), or (4) of the previous proposition. Thus, $G = p(H)$, where

- $H_1 < \mathrm{SU}(2)^+$ and $H_2 < \mathrm{SU}(2)^-$
- $\phi_1 \colon H_1 \to Q$, $\phi_2 \colon H_2 \to Q$ are surjective homomorphisms (in case (2), Q is the trivial group)
- $H < H_1 \times H_2$ is defined as $\{(h_1, h_2) \mid \phi_1(h_1) = \phi_2(h_2)\}$

Now, G^- can be described as $p((1 \times H_2) \cap H)$, but $(1, h_2) \in H_1 \times H_2$ is an element of H exactly when $\phi_2(h) = \phi_1(1) = 1$, so G^- is equal to $p(\operatorname{Ker} \phi_2)$ and isomorphic to $\operatorname{Ker} \phi_2$. According to [42, p. 111], the kernel of ϕ_2 is cyclic in case (4) and is isomorphic to D_8^* in case (3), so we must be in case (2). This means that ϕ_1 and ϕ_2 are the trivial maps and $H = H_1 \times H_2$. If we think of G^- as a subgroup of $\mathrm{SU}(2)^-$, then $H_2 = G^-$, and H_1 is a cyclic group of relatively prime order. $\square$

Chapter 4

Binary Icosahedral and Octahedral Cases

In this chapter, we begin proving the negative part of the main theorem. We consider the case where $\pi_1^\infty(\widetilde{X})$ is either binary icosahedral or binary octahedral, which are the simplest cases because of the following:

Claim 1. *O^* and I^* are maximal finite subgroups of* $\mathrm{SU}(2)$.

Proof. Let $G = O^*$ or $G = I^*$, and suppose that G is a proper subgroup of some finite $H < \mathrm{SU}(2)$. Then H is one of the groups from Proposition 2.4. Because G is nonabelian, H cannot be cyclic. If H is a binary dihedral group, then map G and H down to $\mathrm{SO}(3)$ by the universal covering map. If $G = O^*$, then its image is the symmetry group of an octahedron, and is isomorphic to the symmetric group S_4. If $G = I^*$, then its image is the symmetry group of an icosahedron, isomorphic to the alternating group A_5. But the image of H is a dihedral group, which has only cyclic and dihedral subgroups, so this case is impossible. Finally, by looking at the orders of the groups, we see that H cannot be any of the three exceptional groups T^*, O^*, or I^*, so we have a contradiction. $\square$

Suppose first that the fundamental group at infinity of $\widetilde{X}$ is the binary icosahedral group I^*. From Table 2.1, we know that $\chi(\widetilde{X}) = 9$, so by the multiplicativity of the Euler characteristic, the degree of the covering map $p\colon \widetilde{X} \to X$ is either 3 or 9. In the latter case, since every group with 9 elements has a normal subgroup with 3 elements, there is a covering space of X with fundamental group of order 3, and so in order to rule out this case, it suffices to assume that $\pi_1(X) = \mathbb{Z}_3$. This means that the exact sequence (3.1) takes the form

$$1 \to I^* \to G \to \mathbb{Z}_3 \to 1$$

for some spherical space-form group G.

By Remark 3.2, this I^* lies in the subgroup $G^- = G \cap \mathrm{SU}(2)^-$. Since I^* is maximal in $\mathrm{SU}(2)$, we must actually have $G^- = I^*$. Thus, by Proposition 3.3, $G = \mathbb{Z}_3 \times I^*$. This is not a space-form group, because 3 is not relatively prime to $|I^*| = 120$, so the binary icosahedral case is impossible.

Now suppose that $\pi_1^\infty(\widetilde{X})$ is binary octahedral. In this case, $\chi(\widetilde{X}) = 8$, so the degree of the covering must divide 8. However, there are only a few groups with order dividing 8, and one can check that they all have normal subgroups of order 2. Arguing exactly as before, this implies that $G = \mathbb{Z}_2 \times O^*$, which is not a space-form group, so this case is also eliminated.

Chapter 5

Binary Tetrahedral Case

We now suppose that $\pi_1^\infty(\widetilde{X})$ is the binary tetrahedral group T^*. Since $\chi(\widetilde{X}) = 7$ (Table 2.1), the only possibility for $\pi_1(X)$ is $\mathbb{Z}_7$, and the exact sequence becomes

$$1 \to T^* \to G \to \mathbb{Z}_7 \to 1.$$

The binary tetrahedral group is not maximal in SU(2), so we will need a different argument to show that G is a direct product. Recall the following result from group theory:

Theorem 5.1 (Schur-Zassenhaus [46, 9.1.2])**.** *If G is a finite group, and N is a normal subgroup whose order is relatively prime to the other order of the quotient G/N, then G is a semidirect product of N and G/N.*

This means that $G = \mathbb{Z}_7 \ltimes_\phi T^*$, for some homomorphism $\phi \colon \mathbb{Z}_7 \to \mathrm{Aut}(T^*)$. But the automorphism group of T^* is the symmetric group S_4 [48, p. 176], which has order 24. This is relatively prime to $|\mathbb{Z}_7|$, so ϕ must be the trivial map, and $G = \mathbb{Z}_7 \times T^*$.

Unfortunately, this actually *is* a space-form group, so we will need to rule out this case in another way. This will require a brief detour into 3- and 4-dimensional topology.

5.1 Some Low-Dimensional Topology

5.1.1 Seifert-fibered 3-manifolds

This section follows very closely the paper of Neumann and Raymond [41]:

Definition 6. A *Seifert-fibered space* is a closed, oriented 3-manifold with a fixed-point-free action of S^1 (nontrivial isotropy is allowed).

Warning. The most common definition of "Seifert-fibered" in the literature requires only a foliation by circles with a certain local structure near each fiber. The definition above is more restrictive. In particular, the S^1 action gives a consistent orientation on the circle fibers, which is not possible for a general foliation.

The quotient of a Seifert fibered space M by its S^1 action is homeomorphic to a closed, oriented surface Σ. Let $p\colon M \to \Sigma$ be the quotient map, and choose a collection of orbits $\{O_1, \ldots, O_n\}$, containing at least all orbits of nontrivial isotropy. Let x_i be the image in Σ of the orbit O_i, and choose a set of disjoint open discs $\{D_i\} \subset \Sigma$ with $x_i \in D_i$. Letting $\Sigma_0 = \Sigma \backslash (\bigcup D_i)$ and $M_0 = p^{-1}(\Sigma_0)$, the restriction $p\colon M_0 \to \Sigma_0$ is a principal S^1 bundle. Since Σ_0 is a compact surface with boundary, we have $H^2(\Sigma_0; \mathbb{Z}) = 0$, so the Euler class of the bundle is trivial and we can find a section $s\colon \Sigma_0 \to M_0$. The intersection of s with the boundary of one of the solid tori T_i is a curve s_i on ∂T_i. Modulo some orientation conventions, we can define the Seifert invariants (α_i, β_i) of the orbit O_i by noting that s_i is homologous to some multiple β_i of the central fiber O_i, and letting α_i be the order of the isotropy group of the orbit.

Definition 7. The *unnormalized Seifert invariant* of M is the collection of numbers

$$(g; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)), \tag{5.1}$$

where g is the genus of the quotient surface, and (α_i, β_i) are the Seifert invariants of the chosen orbits.

A Seifert-fibered space is determined up to an orientation- and fiber-preserving homeomorphism by this invariant, but we made some choices in defining it. Different choices lead to invariants which are related in a simple way:

Theorem 5.2 ([41, Theorem 1.1])**.** *Any two unnormalized Seifert invariants for the same Seifert-fibered space can be joined by a sequence of the following operations:*

1. *permuting the indices*
2. *adding or deleting a pair* $(1, 0)$
3. *replacing* $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ *by* $(\alpha_1, \beta_1 + m\alpha_1), (\alpha_2, \beta_2 - m\alpha_2)$

5.1.2 The Rokhlin invariant

Rokhlin's theorem is a fundamental result in the study of smooth 4-manifolds:

Theorem 5.3 (Rokhlin's Theorem [47]). *If X is a smooth, closed, spin 4-manifold, then $\tau(X) \equiv 0 \pmod{16}$.*

This statement about 4-manifolds also leads to an invariant of 3-manifolds. Let M be a closed, oriented 3-manifold. Then M is automatically spin (see for example [35, p. 46]), and its spin structures are in one-to-one correspondence with the elements of $H^1(M;\mathbb{Z}_2)$. In the special case that M is a $\mathbb{Z}_2$-homology sphere (i.e., its $\mathbb{Z}_2$ homology is trivial except in degrees 0 and 3), it has a unique spin structure.

We can always find a smooth, oriented, spin 4-manifold X with $\partial X = M$, such that the orientation and spin structure on X restrict to the chosen ones on M. If we chose a different such 4-manifold X', then we could glue X and an orientation-reversed copy of X' along their common boundary M to form a *closed* spin 4-manifold Y. Rokhlin's theorem applies to Y and says that $\tau(Y) \equiv 0 \pmod{16}$. Since M may have nontrivial homology over $\mathbb{Z}$, we can't simply use the Mayer-Vietoris sequence to split the intersection form of Y between X and X', but a theorem of Novikov says that the signatures are related anyway:

Theorem 5.4 (Novikov Additivity [35, Theorem 5.3]). *If X_1 and X_2 are oriented 4-manifolds with $\partial X_1 \cong \overline{\partial X_2}$, and X is the result of gluing X_1 and X_2 along their common boundary, then:*

$$\tau(X) = \tau(X_1) + \tau(X_2).$$

In our case, since we've reversed the orientation of X', this means that $\tau(Y) = \tau(X) - \tau(X')$, and so Rokhlin's theorem tells us that $\tau(X) - \tau(X') \equiv 0$

(mod 16). This proves that the rational residue

$$\mu(M) = \frac{\tau(X)}{8} \pmod 2 \tag{5.2}$$

is independent of the choice of X, so is an invariant on $\mathbb{Z}_2$-homology 3-spheres, the *Rokhlin invariant* [49, Chapter 2].

5.2 Exclusion of the $\mathbb{Z}_7 \times T^*$ Case

Now let us return the situation we were considering at the beginning of the chapter. We have a 7-fold covering map $p\colon \widetilde{X} \to X$, where $\pi_1^\infty(\widetilde{X}) = T^*$ and $\pi_1^\infty(X) = \mathbb{Z}_7 \times T^*$. Since X is ALE we can represent it (as a smooth manifold, not metrically) as the interior of a compact 4-manifold with boundary $Y = S^3/\pi_1^\infty(\widetilde{X})$.

Now $p^* w_2(X) = w_2(\widetilde{X}) = 0$, and p^* is injective on $H^2(X, \mathbb{Z}_2)$ because the degree of the covering is odd, so X is spin. Thus, we must have

$$\mu(Y) \equiv \frac{\tau(X)}{8} \pmod 2.$$

But since $\chi(\widetilde{X})$ is equal to the degree of the covering map, we have $\chi(X) = 1$, and so by Remark 3.1 the signature of X must be 0. Thus, if $\widetilde{X}$ has such a quotient, then $\mu(Y)$ must vanish. We will compute the Rokhlin invariant of Y explicitly and see that this is not the case.

5.2.1 Computation of $\mu(Y)$

It turns out that every space-form group in dimension 3 commutes with some S^1 action on the 3-sphere (up to conjugation, they all preserve the Hopf fibration), so every 3-dimensional spherical space form is Seifert-fibered. In our case, an unnormalized Seifert invariant for Y is $(0; (2,1), (3,1), (3,1))$ [42, p. 112].

If M has Seifert invariant $(0; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$, then

$$|H_1(M;\mathbb{Z})| = |\alpha_1 \cdots \alpha_n \cdot e(M)|, \tag{5.3}$$

whenever this value is nonzero [49, p. 31]. The quantity $e(M) = -\sum_{i=1}^{n} \beta_i/\alpha_i$ is the so-called *Euler number* of the Seifert fibration. (When the fibration is a true S^1 bundle, this is the usual Euler number).

In our case, since $|H_1(Y;\mathbb{Z})| = 21$ is odd, Y is a $\mathbb{Z}_2$-homology sphere and has a well-defined Rokhlin invariant. We can compute it explicitly using the following theorem of Neumann and Raymond:

Theorem 5.5 ([41, Theorem 6.2])**.** *Given a Seifert-fibered $\mathbb{Z}_2$-homology 3-sphere N with unnormalized Seifert invariant $(0; (\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3))$ such that exactly one α_i is even and each $\alpha_i - \beta_i$ is odd, the Rokhlin invariant of N is given by*

$$\mu(N) \equiv \frac{\operatorname{sign} e(N) + \sum_{i=1}^{3} c(\alpha_i - \beta_i, \alpha_i)}{8} \pmod 2.$$

Remark 5.1. The notation $c(a, b)$ represents the integer-valued function defined implicitly by:

1. $c(a, \pm 1) = 0$ for odd a,

2. $c(a \pm 2b, b) = c(a, b)$,

3. $c(a, b + a) = c(a, b) + \operatorname{sign} b(b + a)$, and

4. $c(a, b) = -c(-a, b) = -c(a, -b)$.

We can put the Seifert invariant $(0; (2, 1), (3, 1), (3, 1))$ into the form required by the theorem using two applications of operation (3) from Theorem 5.2:

$$\begin{aligned}
(0; (2,1), (3,1), (3,1)) &\approx (0; (2, 1 - 1 \cdot 2), (3, 1 + 1 \cdot 3), (3, 1)) \\
&\approx (0; (2, 1 - 1 \cdot 2 - 1 \cdot 2), (3, 1 + 1 \cdot 3), (3, 1 + 1 \cdot 3)) \\
&\approx (0; (2, -3), (3, 4), (3, 4))
\end{aligned}$$

Thus, by Theorem 5.5 we have

$$\mu(Y) \equiv \frac{\operatorname{sign} e(Y) + c(5, 2) + c(-1, 3) + c(-1, 3)}{8} \pmod 2.$$

We can easily compute $e(M) := -\sum_{i=1}^{n} \beta_i/\alpha_i = -7/6$, and either from the implicit definition or the table on p. 185 of [41], we see that $c(5, 2) = 1$ and $c(-1, 3) = -2$. Putting it all together, we have $\mu(Y) \equiv -1/2 \pmod 2$. This is nonzero, so by our previous discussion the $\pi_1^\infty(\widetilde{X}) = T^*$ case has no Riemannian quotient.

Chapter 6

Binary Dihedral Case

Suppose now that $\pi_1^\infty(\widetilde{X})$ is the binary dihedral group D_{4b}^*. Just as before, we have an exact sequence

$$1 \to 1 \times D_{4b}^* \to G \to H \to 1,$$

where G is a space-form group.

Consider the group $G^- = G \cap \mathrm{SU}(2)^-$. By assumption, G^- contains D_{4b}^*. Notice that $D_{4b}^* \triangleleft G$ and $\mathrm{SU}(2)^- \triangleleft \mathrm{SO}(4)$, so $D_{4b}^* \triangleleft G^-$. To identify G^- we need to understand the lattice of normal subgroups in $\mathrm{SU}(2)$:

Proposition 6.1. *The non-cyclic normal subgroups of the binary polyhedral groups are:*

- $D_{4b}^* \triangleleft D_{8b}^*$ *for all* b*,*
- $D_8^* \triangleleft T^*$ *and* $D_8^* \triangleleft O^*$*, and*
- $T^* \triangleleft O^*$.

Proof. Let B^* be a non-cyclic binary polyhedral group, and B be its projection in SO(3) under the covering map $\mathrm{SU}(2) \to \mathrm{SO}(3)$. If N is a normal subgroup of B^*, then either N contains $\{\pm 1\}$ and projects onto a normal subgroup of B, or it has odd order and must be cyclic. Thus, it suffices to find the non-cyclic normal subgroups of the polyhedral groups. The images of D^*_{4n}, T^*, O^*, and I^* are isomorphic to D_{2n}, A_4, S_4, and A_5 respectively, and the normal subgroups of dihedral, alternating, and symmetric groups are all well known. $\square$

Remark 6.1. If $b = 2$, we can rule out the extra possibilities as follows: We see from Table 2.1 that $\chi(\widetilde{X}) = 5$, so D^*_8 must be a subgroup of index 5 in G. Since $D^*_8 < G^- < G$, this means that the index of D^*_8 in G^- is either 1 or 5. This, together with the proposition, imply that $G^- = D^*_8$.

If $G^- = D^*_{4b}$, then Proposition 3.3 implies that $G = \mathbb{Z}_m \times D^*_{4b}$ for some m relatively prime to $4b$. The other possibility is $G^- = D^*_{8b}$, in which case $G = \mathbb{Z}_m \times D^*_{8b}$. But in this case, $\pi_1(X) = G/D^*_{4b}$ is isomorphic to $\mathbb{Z}_m \times \mathbb{Z}_2$. The $\mathbb{Z}_m$ is a normal subgroup of index 2, so by replacing X with the covering space corresponding to this subgroup, we may assume in this case that $G = D^*_{8b}$ and $\pi_1(X) = \mathbb{Z}_2$.

This is the furthest we can go by looking only at the boundary of X: both $\mathbb{Z}_m \times D^*_{4b}$ and D^*_{8b} actually are the fundamental group of some quotient of $\partial\widetilde{X}$. We will again need to look at the signature of X itself, this time by using the η-invariant. This requires another detour.

6.1 The η-Invariant

6.1.1 The signature theorem

The signature of a compact 4-manifold without boundary can be expressed in terms of characteristic classes using Hirzebruch's signature theorem:

Theorem 6.1 ([26, p. 86]). *Let M be a closed, oriented 4-manifold. Then*

$$\tau(M) = \int_M \frac{1}{3} p_1(TM).$$

We can in fact think of this as a special case of the Atiyah-Singer index theorem [7, 6, 8] (though it was discovered earlier): Denote the space of smooth, complex p-forms by Ω^p, and define an involution t on the space $\Omega^* = \oplus_{i=1}^4 \Omega^i$ by

$$t\alpha = -i^{p(p-1)} * \alpha \quad \text{ for } \alpha \in \Omega^p,$$

where $*$ is the Hodge star. Let $\Omega_\pm$ be the ± 1 eigenspaces of t. (The intersections with Ω^2 are exactly the complexified self-dual and anti-self-dual 2-forms).

Let δ be the formal adjoint of d (explicitly, $\delta = - * d*$), and consider the differential operator $D = d + \delta$. This is a self-adjoint elliptic operator with the same kernel as the Hodge Laplacian (in fact, $D^2 = \Delta$). It splits into two pieces, $D_+ \colon \Omega_+ \to \Omega_-$ and $D_- \colon \Omega_- \to \Omega_+$, which are adjoints of each other. The component D_+ is often called the *signature operator*.

The kernel of $D_\pm$ is the intersection of $\Omega_\pm$ with the space of harmonic forms; call this space $\mathcal{H}_\pm$. Thus,

$$\begin{aligned}\mathrm{Ind}(D_+) &= \dim \mathrm{Ker}\, D_+ - \dim \mathrm{Coker}\, D_+ \\ &= \dim \mathrm{Ker}\, D_+ - \dim \mathrm{Ker}\, D_- \\ &= \dim \mathcal{H}_+ - \dim \mathcal{H}_-\end{aligned}$$

Finally, note that if α is a harmonic form of degree $p \neq 2$, then $\alpha + t\alpha$ and $\alpha - t\alpha$ are nonzero elements of $\mathcal{H}_+$ and $\mathcal{H}_-$, respectively. Thus, the difference $\dim\mathcal{H}_+ - \dim\mathcal{H}_-$ cancels except in degree 2, so $\mathrm{Ind}(D_+)$ is the dimension of the space of harmonic self-dual 2-forms minus the dimension of the space of harmonic anti-self-dual 2-forms, which is the signature of M.

The Atiyah-Singer theorem expresses the signature of the elliptic operator D_+ in terms of characteristic numbers, which in this case takes the form of Equation 6.1.

Curvature integral

The integral in the signature theorem is just a notation for pairing a cohomology class with the fundamental homology class of M, but it is suggestive. If M is given a Riemannian metric, then Chern-Weil theory allows us to re-express this equation in terms of an actual integral involving the curvature of the metric:

$$\tau(M) = \frac{1}{12\pi^2} \int_M |W_+|^2 - |W_-|^2 \, d\mu_g \tag{6.1}$$

In particular, this makes it clear that τ is multiplicative under finite coverings: just pull back the metric and integrate.

The signature theorem with boundary

Formula 6.1 does not in general hold for a 4-manifold with boundary, but by gluing together two 4-manifolds along a common boundary, we see that the defect

$$\mathrm{def}(M,g) = \tau(M) - \frac{1}{12\pi^2}\int_M |W_+|^2 - |W_-|^2 \, d\mu_g$$

depends only on the geometry of ∂M (at least if the glued metric is sufficiently smooth across the boundary).

Atiyah, Patodi, and Singer proved an extension of the index theorem to manifolds with boundary, which when specialized to the case of the signature operator identifies $\mathrm{def}(M,g)$ with a spectral invariant of the boundary. Let us review their theorem.

The η-invariant of the signature operator

Let A be a first-order self-adjoint elliptic operator on a closed manifold N. Then its eigenvalues $\{\lambda\}$ are real numbers, and for complex numbers s with sufficiently large real part, the series

$$\eta_A(s) := \sum_{\lambda \neq 0} \frac{\mathrm{sign}\lambda}{|\lambda|^s}$$

converges. In fact, it extends to a meromorphic function on the entire complex plane, and one can prove that $\eta_A(0)$ is finite. This value, called the η-invariant of A, is a way to make sense of "the number of positive eigenvalues minus the number of negative eigenvalues" when both quantities are infinite.

Now, consider the case where M is a compact, oriented, Riemannian 4-

manifold with boundary $\partial M = N$. Suppose that the metric on M is isomorphic to a product $[0,1] \times N$ near the boundary. Then for some bundle isomorphism σ, the restriction of the signature operator to $[0,1] \times N$ can be written as

$$D_+ = \sigma \left(\frac{\partial}{\partial u} + A \right),$$

where $\partial/\partial u$ is the inward-pointing normal, and A is an elliptic operator on N. The restriction of Ω_+ to the boundary yields *all* differential forms on N, and the operator A preserves the parity of forms on N, so it splits into even and odd pieces: $A = A^{ev} + A^{odd}$. Let $\eta(N)$ be the η-invariant corresponding to A^{ev}. Since we've assumed that the metric on M is a product near the boundary, this value depends only on the metric on N.

The signature of M is then given by

Theorem 6.2 (Atiyah-Patodi-Singer [5]).

$$\tau(M) = \frac{1}{12\pi^2} \int_M |W_+|^2 - |W_-|^2 \, d\mu_g - \eta(N).$$

The signature theorem for ALE spaces

So far, our discussion of the η-invariant has assumed that we were working on a compact manifold equipped with a product metric near the boundary. We will now sketch an argument that the same formula applies to ALE manifolds. (See [31] for more details).

The Euclidean metric on $\mathbb{R}^4$ can be written in polar coordinates as

$$g_{\mathrm{Ecl}} = dr^2 + r^2 g_{S^3}.$$

Away from the origin, consider the conformally equivalent metric $r^{-2}g_{Ecl} = r^{-2}dr^2 + g_{S^3}$. Via the substitution $\rho = \log r$, this becomes the cylindrical metric $d\rho^2 + g_{S^3}$.

Now, suppose that (M, g) is ALE with coordinates at infinity of order $\tau > 0$, and let u be a smooth function on M which is equal to r^{-1} outside of a compact set. Then by the above, the conformally related metric $u^2 g$ is asymptotically cylindrical. For all r sufficiently large, let K_r be a compact subset of M which in the coordinates at infinity corresponds to the closed ball of radius r centered at the origin. The signature theorem on K_r with respect to the metric $u^2 g$ takes the form

$$\tau(K_r) = \frac{1}{12\pi^2}\int_{K_r} |W_+|^2 - |W_-|^2\, d\mu + \frac{1}{24\pi^2}\int_{\partial K_r} \mathrm{Tr}(\mathcal{R} \wedge \Pi) - \eta(\partial K_r),$$

where $\mathcal{R}$ is the curvature 2-form, and Π is the second fundamental form of ∂K_r. The new integral term appears because the metric on K_r is not a product near the boundary [18]. We wish to take a limit of this equation as $r \to \infty$.

All of the spaces K_r are diffeomorphic to each other and to M, so the left-hand side of the equation becomes $\tau(M)$. By construction, the union of all K_r is M, so the integrals over K_r converge to the integral over M. Moreover, the integrand is conformally invariant, so we can replace $u^2 g$ with the original metric. One can check that the ALE condition implies that $\Pi \to 0$, and the curvature stays bounded, so the boundary integral goes to zero. Finally, since we've rescaled the metric to be asymptotically cylindrical, the metric on ∂K_r converges smoothly to the round metric on the space form S^3/Γ at infinity, and one can check that this implies convergence of the η-invariants. (For an arbitrary elliptic operator, the η-invariant of a smooth family of metrics can have integer jumps, but this does not occur with the signature operator). Thus the signature theorem for ALE spaces is:

Proposition 6.2. *If (M, g) is a 4-dimensional ALE manifold, then*

$$\tau(M) = \frac{1}{12\pi^2} \int_M |W_+|^2 - |W_-|^2 \, d\mu_g - \eta(S^3/\pi_1^\infty). \tag{6.2}$$

6.1.2 The G-signature theorem with boundary

As one might expect, the description of $\eta(N)$ in terms of the spectrum of a differential operator is not very conducive to calculation. However, there is a related generalization of the G-signature theorem of Atiyah-Singer [8, (6.12)] to manifolds with boundary due to Donnelly [17] which will allow us to calculate the η-invariants of the space forms we are interested in.

Consider the same situation as before: (M, g) is a compact Riemannian 4-manifold with $\partial M = N$, such that g is a product metric near the boundary, and let A^{ev} be the even component of the tangential signature operator on N. But now suppose that we have a compact group G of orientation-preserving isometries on M such that the action is a product near the boundary. Any isometry $f \in G$ defines a map on the even-dimensional forms of N which commutes with A^{ev}, so in particular we have a linear map $f_\lambda^\#$ on each eigenspace. Define the equivariant η-function on $G \times \mathbb{C}$ by

$$\eta_f(N, s) := \sum_{\lambda \neq 0} \frac{\mathrm{sign}\lambda \cdot \mathrm{Tr} f_\lambda^\#}{|\lambda|^s}. \tag{6.3}$$

Just as before, one can prove that $\eta_f(N, 0)$ is always finite; it will be denoted by $\eta_f(N)$.

In a similar way, each $f \in G$ defines a map on the harmonic forms which commutes with the involution t from Section 6.1.1 and thus preserves the ± 1 eigenspaces. We can define a signature corresponding to f by

$$\tau(f, M) = \mathrm{Tr}(f|\mathcal{H}_+) - \mathrm{Tr}(f|\mathcal{H}_-).$$

As before, the difference always cancels except on forms of degree 2. In particular, if f is the identity then this gives the usual signature of M.

If we assume that the boundary of M is empty, Atiyah and Singer's G-signature theorem says [8, (6.12)]:

$$\tau(f, M) = \sum_{F \in \Omega} \Psi(F, f), \tag{6.4}$$

where the sum is taken over the connected components of the fixed-point set of f, and $\Psi(F, f)$ is the integral over F of a cohomology class built up from the action of f on the normal bundle. Donnelly's generalization to manifolds with boundary identifies the difference between the two sides of this equation with the equivariant η-invariant [17, I.4]:

$$\tau(f, M) = \sum_{F \in \Omega} \Psi(F, f) - \eta_f(N). \tag{6.5}$$

Moreover, in the special case where $\hat{N} \to N$ is a finite, regular covering with deck group G, he proves that [17, I.6]:

$$\eta(\hat{N}) - |G|\eta(N) = -\sum_{f\neq 1} \eta_f(N), \tag{6.6}$$

where the η-invariants on the left-hand side are the ordinary (non-equivariant) invariants defined in Section 6.2. This relates the η-invariants of S^3 and S^3/π_1^∞ and will allow us to compute the latter one.

Isolated fixed points

The formula for $\Psi(F, f)$ on an arbitrary component of the fixed point set is quite complicated, but we will need it only in the case where F is an isolated fixed point. The G-signature theorem in this special case was discovered by Atiyah and Bott [3, 4].

Suppose that $f\colon M \to M$ is an orientation-preserving isometry of a Riemannian 4-manifold, with an isolated fixed point at some $p \in M$. By choosing a basis, we can represent the action of f on T_pM as an element of SO(4). Any element of SO(4) can be conjugated into the maximal torus, which consists of block matrices of the form

$$\begin{bmatrix} R(\theta_1) & 0 \\ 0 & R(\theta_2) \end{bmatrix},$$

where $R(\theta)$ is the 2x2 clockwise rotation matrix with angle θ. If $f \in \mathrm{U}(2)$, then the angles θ_i are exactly the arguments of the two complex eigenvalues.

With respect to these two angles, Ψ is given by [4, (6.25)]:

$$\Psi(p, f) = -\cot(\theta_1/2)\cot(\theta_2/2). \tag{6.7}$$

(This is finite because if either angle was 0, then the fixed point would not be isolated).

6.1.3 Dedekind sums

From equations 6.7 and 6.6, the η-invariant of a space form S^3/G can be expressed in terms of $\eta(S^3)$ and a sum of products of cotangents. Such sums occur in a surprising array of places in mathematics (see the book by Hirzebruch and Zagier for many examples [28]). We will now describe a connection to number theory.

Let $((.)) \colon \mathbb{R} \to \mathbb{R}$ be the "sawtooth" function given by:

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

where $\lfloor x \rfloor$ means the floor of x (the greatest integer $\leq x$). Then for integers a, b, and c which are pairwise relatively prime, we define the *Dedekind sum* $D(a, b; c)$ by

$$D(a, b; c) = \sum_{i=1}^{c-1} \left(\left(\frac{ai}{c}\right)\right) \left(\left(\frac{bi}{c}\right)\right). \tag{6.8}$$

These sums are named after Richard Dedekind because he first studied the special case $s(p, q) := D(1, p; q)$, in relation to transformation properties of modular forms.

Our interest in Dedekind sums is that they can also be expressed as a cotangent sum [45]:

$$D(p,q;r) = \frac{1}{4r}\sum_{k=1}^{r-1} \cot\left(\frac{\pi pk}{r}\right)\cot\left(\frac{\pi qk}{r}\right). \tag{6.9}$$

6.1.4 Properties of Dedekind sums

To calculate the η-invariant of a space form, we will need to compute some Dedekind sums. Let us review some of the properties that will allow us to complete the calculation:

1. If $a \equiv a' \pmod{c}$ and $b \equiv b' \pmod{c}$, then $D(a,b;c) = D(a',b',c)$.

2. If z is relatively prime to c, then $D(a,b;c) = D(az,bz;c)$.

3. $D(-a,b;c) = D(a,-b;c) = -D(a,b;c)$

4. For any a and b, we have $D(a,b;1) = 0$. (In particular, $s(a,1) = 0$.)

5. Rademacher Reciprocity [44]:

$$D(a,b;c) + D(b,c;a) + D(c,a;b) = \frac{1}{12}\frac{a^2+b^2+c^2}{abc} - \frac{1}{4}$$

6. Dedekind Reciprocity [16, vol. 1, pp. 159-173]:

$$s(b,c) + s(c,b) = \frac{1}{12}\left(\frac{b}{c} + \frac{1}{bc} + \frac{c}{b}\right) - \frac{1}{4}$$

Proof of 1-4. Since $((.))$ has period 1, each sawtooth factor in every term is unaltered by changing a or b by a multiple of c, giving property 1. This implies property 2, because multiplying by a relatively prime integer is an invertible operation, so multiplying a and b by z just changes the order of the terms in the sum without changing the result. The sawtooth function is odd and vanishes at every integer, giving properties 3 and 4, respectively. $\square$

6.2 The η-Invariant of $S^3/(\mathbb{Z}_u \times D^*_{4v})$

Consider a 3-dimensional spherical space form $Y = S^3/G$ with fundamental group $G = \mathbb{Z}_u \times D^*_{4v}$.

In the signature theorem with boundary,

$$\tau(M) = \frac{1}{12\pi^2} \int_M |W_+|^2 - |W_-|^2 \, d\mu_g - \eta(\partial M),$$

reversing the orientation negates both the signature of M and the curvature integral, so it must also negate η. Since S^3 admits an orientation-reversing isometry, this implies that $\eta(S^3) = 0$, and Equation 6.6 becomes

$$\eta(Y) = -\frac{1}{|G|} \sum_{f \neq 1} \cot\left(\frac{\theta_1(f)}{2}\right) \cot\left(\frac{\theta_2(f)}{2}\right).$$

For notational convenience, define $\mathrm{r}(\theta) = e^{2\pi i \theta}$, and for any element A of SO(4), denote by $\eta(A)$ the quantity $-\cot(\theta_1/2)\cot(\theta_2/2)$ described above. We will only need to work with matrices that are unitary, and either diagonal or anti-diagonal as complex matrices. The second case turns out to be trivial:

Claim 2. *For any $A \in U(2)$ of the form*

$$\begin{bmatrix} 0 & \mathrm{r}(a) \\ \mathrm{r}(b) & 0 \end{bmatrix},$$

we have $\eta(A) = 1$.

Proof. This matrix has eigenvalues $\mathrm{r}(\frac{a+b}{2})$ and $\mathrm{r}(\frac{a+b}{2} + \frac{1}{2})$. Thus

$$\eta(A) = -\cot\left(\frac{\pi(a+b)}{2}\right)\cot\left(\frac{\pi(a+b)}{2} + \frac{\pi}{2}\right).$$

But the angle-addition formula for cotangent gives $\cot(x + \frac{\pi}{2}) = -1/\cot(x)$, and the claim follows. $\square$

Consider first the case where v is even. According to Wolf [57], up to equivalence and automorphisms of the group, the only representation of this group into SO(4) which acts freely on the 3-sphere has image in $\mathrm{U}(2) < \mathrm{SO}(4)$ generated by the matrices

$$A = \begin{bmatrix} \mathrm{r}\left(\frac{2v+u}{2uv}\right) & 0 \\ 0 & \mathrm{r}\left(\frac{2v-u}{2uv}\right) \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

This representation comes from the presentation

$$\langle\, A, B \mid A^{2uv} = 1, B^2 = A^{uv}, BAB^{-1} = A^k \,\rangle,$$

where k is chosen so that $k \equiv -1 \pmod{2v}$ and $k \equiv 1 \pmod{u}$. (This choice is unique by the Chinese remainder theorem).

We wish to compute the quantity $\eta(Y) = \frac{1}{4uv} \sum_{f \neq 1} \eta(f)$, identifying G with the image of the representation above. Notice that every element of G can be uniquely written as $A^p B^q$ for some $p \in \mathbb{Z}_{2uv}$ and $q \in \mathbb{Z}_2$. The elements with $q = 0$ are diagonal, and those with $q = 1$ are anti-diagonal. Because of Claim 2, each anti-diagonal element contributes 1 to the above sum, and there are $2uv$ of them, so $\eta(Y) = \frac{1}{2} + \frac{1}{4uv} \sum_{j=1}^{2uv-1} \eta(A^j)$. But since each A^j is diagonal, this latter sum is equal to

$$-\frac{1}{4uv} \sum_{j=1}^{2uv-1} \cot\left(\frac{\pi j(2v+u)}{2uv}\right) \cot\left(\frac{\pi j(2v-u)}{2uv}\right),$$

We can write this in terms of Dedekind sums using Equation 6.9. Thus the η invariant of Y is just given by

$$\eta(Y) = \frac{1}{2} - 2D(2v+u, 2v-u; 2uv).$$

We will simplify this in a moment, but let us first examine the case where v is odd. There is again just one orthogonal representation up to equivalence and automorphisms, given in a complex basis by the generators

$$A = \begin{bmatrix} \mathrm{r}\left(\frac{1}{v}\right) & 0 \\ 0 & \mathrm{r}\left(-\frac{1}{v}\right) \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 1 \\ \mathrm{r}\left(\frac{1}{2u}\right) & 0 \end{bmatrix}.$$

Every element of the group can be written uniquely as $A^s B^t$ with $s \in \mathbb{Z}_v$ and $t \in \mathbb{Z}_{4u}$. It is easy to see that B^2 commutes with A, and that the set of elements with t even is a cyclic subgroup generated by

$$AB^2 = \begin{bmatrix} \mathrm{r}\left(\frac{v+2u}{2uv}\right) & 0 \\ 0 & \mathrm{r}\left(\frac{v-2u}{2uv}\right) \end{bmatrix}.$$

The elements with t odd are all anti-diagonal, and by the same reasoning as in the even case, we obtain the formula

$$\eta(Y) = \frac{1}{2} - 2D(v+2u, v-2u; 2uv) = \frac{1}{2} + 2D(2u+v, 2u-v; 2uv).$$

Thus, if we can compute $D(2x+y, 2x-y; 2xy)$ for $2x$ and y relatively prime, then we will have a formula for both cases. Rademacher reciprocity (property 5 from Section 6.1.4) expresses the sum $D(a,b;c)+D(b,c;a)+D(c,a;b)$ in closed form, which reduces our problem to that of computing $D(2x-y, 2xy; 2x+y)$ and $D(2xy, 2x+y; 2x-y)$. We will sketch the calculation of the first; the second is similar.

By reducing the first two arguments modulo the third,

$$D(2x-y, 2xy; 2x+y) = D(-2y, 2xy; 2x+y).$$

Since $2x$ and y are relatively prime, the sum $2x+y$ is relatively prime to $2y$, so we can divide the first two arguments by it:

$$D(-2y, 2xy; 2x + y) = D(-1, x; 2x + y).$$

Then, we can pull out the negative sign from the first argument, and it becomes equivalent to a 2-argument Dedekind sum:

$$D(-1, x; 2x + y) = -D(1, x; 2x + y) = s(x, 2x + y).$$

Dedekind reciprocity expresses this 2-argument sum in terms of $s(2x + y, x)$, which by reducing the first argument modulo y is equal to $s(y, x)$. Putting it all together, we get the following:

$$D(2x + y, 2x - y; 2xy) = \frac{1}{12xy} + \frac{y}{6x} - \frac{1}{4} - 2s(y, x).$$

Applying this in the odd case, we get:

$$\eta(Y) = \frac{1}{2} + 2D(2u + v, 2u - v; 2uv) = \frac{1}{6uv} + \frac{v}{3u} - 4s(v, u),$$

and in the even case:

$$\eta(Y) = \frac{1}{2} - 2D(2v + u, 2v - u; 2vu) = 1 - \frac{1}{6vu} - \frac{u}{3v} + 4s(u, v).$$

But by applying Dedekind reciprocity to the Dedekind sum in this last expression, we get exactly the same formula as in the odd case. Thus, we have:

Lemma 1. *If Y is a 3-dimensional spherical space form with fundamental group $\mathbb{Z}_u \times D^*_{4v}$, then*

$$\eta(Y) = \frac{1}{6uv} + \frac{v}{3u} - 4s(v, u). \tag{6.10}$$

6.3 Exclusion of the Remaining Cases

Let us now return to the situation at the beginning of the chapter. We have an ALE gravitational instanton $\widetilde{X}$ with $\pi_1^\infty(\widetilde{X}) = D^*_{4b}$, and a Riemannian quotient with $G = \pi_1^\infty(X)$ given by the exact sequence

$$1 \to 1 \times D^*_{4b} \to G \to H \to 1.$$

By looking at the topology of the boundary, we excluded all cases except

- $G = \mathbb{Z}_m \times D^*_{4b}$ and $H = \mathbb{Z}_m$ (with m relatively prime to $4b$), or
- $G = D^*_{8b}$ and $H = \mathbb{Z}_2$.

Now, since the metric on X is anti-self-dual, the signature formula (6.2) reduces to

$$\tau(X) = -\frac{1}{12\pi^2} \int_X |W_-|^2 \, d\mu - \eta(S^3/G).$$

We can compute the first two terms of this equation directly. Let d be the degree of the covering map. We know that $\chi(\widetilde{X}) = b + 3$ from Table 2.1, so we must have $\chi(X) = (b+3)/d$. By Remark 3.1, this means that $\tau(X) = 1 - (b+3)/d$. The Gauss-Bonnet formula for Ricci-flat anti-self-dual ALE spaces is

$$\chi(M) = \frac{1}{8\pi^2}\int |W_-|^2\, d\mu + \frac{1}{|\pi_1^\infty|}, \tag{6.11}$$

which implies that

$$\int_{\widetilde{X}} |W_-|^2\, d\mu = 8\pi^2\left(b + 3 - \frac{1}{4b}\right).$$

We know that $\int_X |W_-|^2\, d\mu = \frac{1}{d}\int_{\widetilde{X}} |W_-|^2\, d\mu$, so putting everything together, we get that the eta invariant of the boundary of X must be given by

$$\eta(\partial X) = \frac{b}{3d} + \frac{1}{d} + \frac{1}{6bd} - 1. \tag{6.12}$$

Now, the case $G = D^*_{8b}$ corresponds to Equation 6.10 with $u = 1$ and $v = 2b$. The Dedekind sum in the formula vanishes by property 4, so

$$\eta(\partial X) = \frac{1}{12b} + \frac{2b}{3}.$$

The degree of the covering is 2, so setting this equal to Equation 6.12 and substituting $d = 2$, we have

$$\frac{1}{12b} + \frac{2b}{3} = \frac{b}{6} + \frac{1}{12b} - \frac{1}{2}.$$

The only solution of this equation is negative ($b = -1$), so this case is impossible.

In the case $G = \mathbb{Z}_m \times D^*_{4b}$, the space $\widetilde{X}$ has Euler characteristic $b+3$ (Table 2.1), and the group of covering transformations is $\mathbb{Z}_m$, so $b+3$ must be divisible by m. This implies that $b \equiv -3 \pmod{m}$, so that the Dedekind sum in Equation 6.10 is equal to $s(-3,m) = -s(3,m)$. The Dedekind reciprocity formula allows us to write this last sum in terms of either $s(0,3)$, $s(1,3)$, or $s(2,3)$, depending on the residue class of m modulo 3, but these three sums are easy to evaluate by hand. Thus, we determine that

$$\eta(\partial X) = \frac{1}{6mb} + \frac{b}{3m} + \begin{cases} \frac{(m-10)(m+1)}{9m} & m \equiv 0 \pmod 3 \\ \frac{(m-10)(m-1)}{9m} & m \equiv 1 \pmod 3 \\ \frac{(m-5)(m-2)}{9m} & m \equiv 2 \pmod 3 \end{cases} \cdot \tag{6.13}$$

If we set this equal to Equation 6.12 and set $d = m$, we have

$$\frac{1}{m} - 1 = \begin{cases} \frac{(m-10)(m+1)}{9m} & m \equiv 0 \pmod 3 \\ \frac{(m-10)(m-1)}{9m} & m \equiv 1 \pmod 3 \\ \frac{(m-5)(m-2)}{9m} & m \equiv 2 \pmod 3 \end{cases} \cdot$$

It is easily verified that the only positive integer solution of these equations is $m = 1$, so there are no non-trivial quotients in this case. Thus, the binary dihedral gravitational instantons have no quotients.

Chapter 7

Cyclic Case

We will work from the other direction in this case: starting with the universal covering and determining all isometric quotients.

7.1 The Gibbons-Hawking Ansatz

Let (M, g, π) be a Gibbons-Hawking space with k monopoles, i.e., given by the following ansatz:

Theorem 7.1 (Gibbons-Hawking [22]). *Let $\pi \colon M_0 \to \mathbb{R}^3 - \{p_1, \dots, p_k\}$ be the principal S^1-bundle whose first Chern class yields -1 when paired with the generators of second homology given by small 2-spheres about the "monopole" points $\{p_1, \dots, p_k\}$. Let V be the function on $\mathbb{R}^3$ given by*

$$V(x) = \frac{1}{2} \sum_{i=1}^{k} \frac{1}{|x - p_i|}.$$

If we equip the bundle M_0 with the connection 1-form ω defined by $d\omega =$

$\pi^*(*dV)$, then the metric

$$g = V(dx_1^2 + dx_2^2 + dx_3^2) + V^{-1}\omega^2$$

on M_0 can be smoothly completed, yielding a hyperkähler manifold M.

The difference $M \setminus M_0$ consists of k points $\{\tilde{p}_1, \ldots, \tilde{p}_k\}$, and the map π on M_0 extends to a map $\pi\colon M \to \mathbb{R}^3$ such that $\pi(\tilde{p}_i) = p_i$. To see the 2-sphere of parallel complex structures, first let ∂_θ denote the vertical vector field on M_0 defined by $\omega(\partial_\theta) = 1$. Then the horizontal lifts of the vector fields $V^{1/2}\partial_\theta$ and $V^{-1/2}\partial_{x_i}$ give an orthonormal trivialization of the tangent bundle of M_0. Thus an orthogonal almost-complex structure I on M_0 which is compatible with the orientation is uniquely determined by where it sends $V^{1/2}\partial_\theta$, which must be a unit vector in $\mathrm{Span}\{V^{-1/2}\partial_{x_i}\} \cong \mathbb{R}^3$. One needs only check that these almost-complex structure all extend to M, and that the result is integrable.

Notice also that the S^1 action coming from the principal bundle structure on M_0 extends to an action on M by leaving $M \setminus M_0$ fixed pointwise, so that ∂_θ extends by zero to M. One can check from the above description that this S^1 action is actually triholomorphic. It also follows from the construction that π is both the (ordinary) quotient map and the hyperkähler moment map for this S^1 action (see [32]).

We will determine the isometries of M by looking at holomorphic curves in M:

Proposition 7.1. *Every real surface $C \subset M$ which is a (-2)-curve with respect to some parallel complex structure is the preimage under π of a straight line segment between monopoles.*

Proof. Equip M with the parallel complex structure I that makes C into a (-2)-curve. Since ∂_θ is triholomorphic, the translations of C along that field are also (-2)-curves, and they are homologous to the original, so they must actually coincide. Thus, C is the union of fibers of π. For $p \in \mathbb{R}^3$ such that C contains $\pi^{-1}(p)$, we see that $I(\partial_\theta)_x$ is tangent to C for each $x \in \pi^{-1}(p)$. But by construction of M, $\pi_*(I(\partial_\theta)_x)$ is independent of $x \in \pi^{-1}(p)$. Let l be the line in $\mathbb{R}^3$ containing p and tangent to $\pi_*(I(\partial_\theta)_x)$. Then $\pi^{-1}(l)$ is a non-compact holomorphic curve with respect to I, and its intersection with C contains a circle, so it must be all of C. Thus $C \subset \pi^{-1}(l)$. If $\{p_{i_1}, \ldots, p_{i_m}\}$ are the monopoles lying on l, then $\pi^{-1}(l)$ is a chain of $m-1$ spheres intersecting transversely, together with two discs intersecting the end spheres transversely. It follows that C must be the preimage of a segment between monopoles. $\square$

Corollary 1. *If $k > 2$, then the identity component of the group of triholomorphic isometries of M is isomorphic to $U(1)$.*

Proof. Take 3 distinct monopoles p, q, r and assume that the line segments pq and qr intersect only at q (by reordering). The preimage of these two segments will be two spheres intersecting transversely.

Now, any isometry of M takes a parallel complex structure to another one, so it must permute the union of all real surfaces which are (-2)-curves for some parallel complex structure. Thus the identity component of the triholomorphic isometry group acts by homeomorphisms isotopic to the identity on the union of these curves. In particular, every element must fix the point $\pi^{-1}(q)$. Since isometries are determined by their value and differential at any single point, we have a faithful representation of the identity component of the triholomorphic isometry group into $GL(T_{\pi^{-1}q}M)$. Identify this tangent space with $\mathbb{C}^2$ in a way that is compatible with the parallel complex structure making the preimage of the segment between p and q into a (-2)-curve. Then the image of the representation must lie in $SU(2)$. Moreover, it must take the tangent spaces of the two intersecting spheres into themselves. Thus the image must actually lie in $\{\, \mathrm{diag}(a, a^{-1}) \mid a \in U(1) \,\} \cong U(1)$. The identity component also *contains* a $U(1)$, namely the subgroup generated by ∂_θ, so it follows that it must be isomorphic to $U(1)$. $\square$

7.2 Isometries of Gibbons-Hawking Metrics

Let f be an isometry of a Gibbons-Hawking space with $k > 1$ monopoles. By Table 2.1, the signature of M is nonzero, so f must preserve the orientation.

We first consider the case $k > 2$. Since the pushforward f_* must send ∂_θ to another triholomorphic Killing field of unit length, by Corollary 1 we must have $f_*\partial_\theta = \pm\partial_\theta$. Notice that since ∂_θ is exactly tangent to the fibers of π, f must take fibers of π to fibers, and so we get an induced map $\hat{f} \colon \mathbb{R}^3 \to \mathbb{R}^3$ on the quotient space. The orientation on the circle fibers is preserved (resp. reversed) when f_* preserves (resp. reverses) ∂_θ.

If instead $k = 2$ (which is the Eguchi-Hanson metric [20]), then the triholomorphic isometry group is $SO(3)$ (see [23]), so this same argument doesn't work. From Proposition 7.1, it follows that f must preserve the 2-sphere S given by the preimage of the segment between the two monopoles and act freely on it, so f must have order 2. Let $A \in GL(3; \mathbb{R})$ be the corresponding action on the space of triholomorphic Killing fields $\mathfrak{so}(3) \cong \mathbb{R}^3$. Then A must have a nonzero eigenvector, but $A^2 = \mathrm{Id}$ implies that the only possible eigenvalues are ± 1. This means that G must send *some* triholomorphic Killing field to itself or its negative. However, consider the explicit form

$$g = \left[1 - \left(\frac{a}{4}\right)^4\right] dr^2 + r^2 \left[1 - \left(\frac{a}{4}\right)^4\right] (d\psi + \cos\theta d\phi)^2 + \frac{r^2}{4}(d\theta^2 + \sin^2\theta d\phi^2)$$

for the metric, where θ, ϕ, and ψ are Euler angles on $SO(3)$. The triholomorphic $SO(3)$ in this picture is exactly the standard $SO(3)$ acting on the round 2-sphere $r = a$, which corresponds to S in the previous picture. It then follows that the triholomorphic isometry group acts transitively on the unit sphere of the triholomorphic Killing fields, so by conjugating f by some triholomorphic isometry, we may assume that f sends ∂_θ to itself or its negative. Thus, we are in exactly the same situation as before: $f_*\partial_\theta = \pm\partial_\theta$, and there is an induced map $\hat{f}\colon \mathbb{R}^3 \to \mathbb{R}^3$.

Since π is also the hyperkähler moment map for the S^1 action, there are parallel complex structures I, J, and K with Kähler forms ω_I, ω_J, and ω_K such that the 1-forms defined by $dx = \omega_I(\partial_\theta, \cdot)$, $dy = \omega_J(\partial_\theta, \cdot)$, $dy = \omega_K(\partial_\theta, \cdot)$ are actually the differentials of the component functions of $\pi\colon M \to \mathbb{R}^3$.

Now, f acts on the space of parallel self-dual 2-forms by pullback, so in the basis for this space given by ω_I, ω_J, and ω_K, we can think of this action as being an element $\rho(f)$ of $SO(3)$. Let $d\vec{x}$ be the $\mathbb{R}^3$-valued 1-form on M with components (dx, dy, dz), and $\vec{\omega}$ the $\mathbb{R}^3$-valued 2-form with components

$(\omega_I, \omega_J, \omega_K)$. We compute:

$$\begin{aligned}
(f^* d\vec{x})(v) &= f^*(\vec{\omega}(\partial_\theta, \cdot))(v) \\
&= \vec{\omega}(\partial_\theta, f_* v) \\
&= \pm(f^*\vec{\omega})(\partial_\theta, v) \\
&= \pm(\rho(f)\cdot\vec{\omega})(\partial_\theta, v) \\
&= \pm\rho(f)\cdot d\vec{x}(v),
\end{aligned}$$

where the sign in the same as in $f_*\partial_\theta = \pm\partial_\theta$.

Thus, if we go back to the induced map $\hat{f}\colon \mathbb{R}^3 \to \mathbb{R}^3$, we see that it has constant differential, given by $\pm\rho(f)$, so $\hat{f}$ is an affine transformation. The monopoles are the only points in $\mathbb{R}^3$ whose fibers are single points, so $\hat{f}$ must permute them. If we let o be the center of mass of the monopoles for the Euclidean metric on $\mathbb{R}^3$, this means that $f(o) = o$. By modifying π by a translation, we may assume that o is the origin, making f an element of $\mathrm{O}(3)$.

If $\hat{f}$ fixes any monopole p, then since the fiber over p contains only one point, it must be fixed by f. In particular, if o is a monopole point, then every isometry of M has a fixed point. If we assume that this is not the case, then we can completely identify the isometry group of M via some general arguments of Honda and Viaclovsky:

Theorem 7.2 ([33]). *Let $\hat{f}\colon \mathbb{R}^3 \to \mathbb{R}^3$ be an isometry which preserves the monopole points. Then $\hat{f}$ lifts to an orientation-preserving isometry of M which preserves (resp. reverses) the orientation of the circle fibers when $\hat{f}$ is orientation-preserving (resp. orientation-reversing). The lift is unique up to* *the S^1 action generated by ∂_θ.*

Thus, we have an exact sequence

$$1 \to S^1 \to \mathrm{Iso}(M) \to G \to 1$$

where G is the subgroup of O(3) which takes monopoles to monopoles. We claim that this sequence admits a splitting homomorphism $\phi\colon G \to \mathrm{Iso}(M)$. First, fix an origin for the circle fiber lying over the origin, so that it is identified with the circle group $\{\lambda \in \mathbb{C}\colon |\lambda| = 1\}$. Then a lifting of $f \in G$ is uniquely determined by its action on this fiber. Let $\phi(f)$ be the unique lifting which acts trivially on the fiber if f is orientation-preserving, and acts by complex conjugation if f is orientation-reversing. This identifies $\mathrm{Iso}(M)$ with the semidirect product $S^1 \ltimes G$, where $f \in G$ acts on $\lambda \in S^1$ by $f\lambda f^{-1} = \lambda^{\det f}$.

7.2.1 Free quotients

We are now ready to determine the quotients of M. Let H be a subgroup of $\mathrm{Iso}(M)$ which acts freely. As we saw before, this implies that H descends to a subgroup $\hat{H} < \mathrm{O}(3)$ which acts freely on the monopole points. If $\hat{h} \in \hat{H}$ reverses the orientation of $\mathbb{R}^3$, then h reverses the orientation of the circle fiber over o. But every orientation-reversing isometry of a circle has fixed points, so this is impossible. Since every element of $\hat{H}$ is orientation-preserving, the group of liftings in $\mathrm{Iso}(M)$ is actually a *direct* product $S^1 \times \hat{H}$: the projection onto S^1 is given by the action on the fiber over o.

Now, if the projection of H onto $\hat{H}$ were not injective, then H would contain a nontrivial element of the form $(\lambda, 1) \in S^1 \times \hat{H}$. But every such element fixes the points of M lying over the monopoles, so this is impossible. This means that H is the graph of a homomorphism $c\colon \hat{H} \to S^1$ (in particular, H is isomorphic to $\hat{H}$). In a similar way, every element of the form $(1, \lambda)$ fixes the circle lying over the origin, so c is also injective and thus $\hat{H}$ is a cyclic group.

Every cyclic subgroup of SO(3) is a group of rotations about some fixed axis, so the action of H will preserve some constant vector field w on $\mathbb{R}^3$. Since it permutes the monopoles, it will also preserve the function V, so $V^{-1/2}w$ will be invariant. By our previous argument, we see that H preserves ∂_θ and the orientation, so the action of H must actually preserve the complex structure given by $I(\partial_\theta) = V^{-1/2}w$, and therefore the quotient M/H is always Kähler. This concludes the proof of Theorem A.

This also gives us an explicit description of the quotients. Given a fixed cyclic group $\hat{H} < \mathrm{SO}(3)$ which acts freely on the monopoles, the liftings H which act freely on M are determined by the injective homomorphism $c\colon \hat{H} \to S^1$. The circle contains only one finite group of each order n, the n-th roots of unity, so c is determined by the choice of a primitive n-th root of unity. Some of these choices yield isometric quotients, however. In particular, if two groups have the same projection in SO(3) and the corresponding maps c_1 and c_2 are complex conjugates of one another, then the groups are conjugate in $\mathrm{Iso}(M)$, and the quotients are isometric. In rare cases, it is even possible that the choice of $\hat{H}$ is not uniquely determined by the order of H. Consider the case of an even number of equally-spaced collinear monopoles. This space admits an S^1 worth of different $\mathbb{Z}_2$ actions which act freely on the monopoles, but all of them are conjugate in the full isometry group.

Chapter 8

Consequences

We now use the classification theorem to prove the two corollaries mentioned earlier:

Corollary B. *If X is a Ricci-flat anti-self-dual ALE 4-manifold such that*

$$\int_X |\mathrm{Rm}|^2 \, d\mu < 6\pi^2,$$

or such that $b_2(X) > 0$ and

$$\int_X |\mathrm{Rm}|^2 \, d\mu < 8\pi^2 \left(b_2(X) + 1 - \frac{1}{b_2(X)+1} \right),$$

then X is isometric to flat $\mathbb{R}^4$.

Proof. If X is flat, then it must be isometric to $\mathbb{R}^4$: The universal cover of X must be $\mathbb{R}^4$, and the results of Anderson [2] imply that $\pi_1(X)$ is finite. But every finite group of isometries of $\mathbb{R}^4$ has a fixed point (just take the center of mass of any orbit), so we must have $\pi_1(X) = 1$.

If X is not flat, then the only non-vanishing component of the curvature tensor of X is W_-, so in this case the Gauss-Bonnet formula (6.11) gives

$$\int_X |\mathrm{Rm}|^2 \, d\mu = 8\pi^2 \left(\chi(X) - \frac{1}{|\pi_1^\infty(X)|} \right),$$

which by Remark 3.1 is equal to

$$8\pi^2 \left(b_2(X) + 1 - \frac{1}{|\pi_1^\infty(X)|} \right).$$

Thus to find the minimum curvature, it is enough to find the minimum of $|\pi_1^\infty|$ for each fixed b_2. From Table 2.1, it is clear that the Gibbons-Hawking spaces dominate the other gravitational instantons in this respect, so we need only compare the Gibbons-Hawking spaces and their quotients. From the same table, we see that a d-fold quotient of a Gibbons-Hawking space with k monopoles has $|\pi_1^\infty| = dk$ and $b_2 = (k/d) - 1$, so $|\pi_1^\infty| = d^2(b_2 + 1)$. Thus, for each b_2, the minimum $|\pi_1^\infty|$ occurs when $d = 1$, the Gibbons-Hawking space itself. Note, however, that the 1-monopole Gibbons-Hawking space is just flat $\mathbb{R}^4$, so the non-flat metric with smallest $|\pi_1^\infty|$ for the $b_2 = 0$ case occurs when $k = d = 2$, a 2-fold quotient of the Eguchi-Hanson metric. Plugging these minimizing parameters back into the curvature formula above yields the minimum curvatures listed in the theorem. $\square$

Corollary C. *If X is a scalar-flat Kähler ALE 4-manifold and $b_2(X) = 0$, then X is actually Ricci-flat, and is a quotient of a Gibbons-Hawking space $\widetilde{X}$ by a cyclic group of order $\chi(\widetilde{X})$.*

Proof. Since X is scalar-flat, the Ricci form ρ of X must be harmonic [10, 2.33], and since X is ALE, it must moreover be square-integrable. However, on an ALE 4-manifold, the space of L^2 harmonic forms is isomorphic to the image of $H^2(X, \partial X; \mathbb{R})$ in $H^2(X; \mathbb{R})$ [25, Theorem 1A]. Therefore, if $b_2(X) = 0$, then ρ must be the zero form, and so X is Ricci-flat. The last assertion then follows immediately from the main theorem and Remark 3.1. $\square$

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