Compactness and Non-compactness for the Yamabe Problem on Manifolds With Boundary

A Dissertation Presented

by

Marcelo Mendes Disconzi

to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2012

Stony Brook University

The Graduate School

Marcelo Mendes Disconzi

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Marcus Khuri – Dissertation Advisor
Professor, Department of Mathematics

David Ebin – Chairperson of Defense
Professor, Department of Mathematics

Mchael Anderson
Professor, Department of Mathematics

Michael Douglas
Professor, Simons Center for Geometry and Physics

This dissertation is accepted by the Graduate School.

Charles Taber
Interim Dean of the Graduate School

Abstract of the Dissertation

Compactness and Non-compactness for the Yamabe Problem on Manifolds With Boundary

by

Marcelo Mendes Disconzi

Doctor of Philosophy

in

Mathematics

Stony Brook University

2012

We study the problem of conformal deformation of Riemannian structure to constant scalar curvature with zero mean curvature on the boundary. We prove compactness for the full set of solutions when the boundary is umbilic and the dimension $n \leq 24$. The Weyl Vanishing Theorem is also established under these hypotheses, and we provide counter-examples to compactness when $n \geq 25$. Lastly, our methods point towards a vanishing theorem for the umbilicity tensor, which is anticipated to be fundamental for a study of the nonumbilic case.

I dedicate this work to Alex, my companion and true friend.

Contents

Acknowledgements

I owe my deepest gratitude to

My advisor, Marcus Khuri, for his immense support. His teachings will forever be a cornerstone of my academic career.

David Ebin, Michael Anderson, Blaine Lawson and Dennis Sullivan. Each of them has helped me to see a different perspective of this beautiful field which is Mathematics.

To all my friends and colleagues who have, on one way or another, helped me to reach this point. Although they remain unnamed in these notes, I am sincerely thankful to all of them.

Finally, my acknowledgements would not be complete without mentioning my parents, Beatriz and Roboamo, for their love, care and encouragement throughout my life — including, and perhaps mainly, in its most difficult moments.

Chapter 1

Introduction

The Yamabe problem consists of finding a constant scalar curvature metric $\tilde{g}$ which is pointwise conformal to a given metric g on an n-dimensional ($n \geq 3$) compact Riemannian manifold M without boundary. This is equivalent to producing a positive solution to the following semilinear elliptic equation

$$L_g u + K u^{\frac{n+2}{n-2}} = 0, \text{ on } M, \tag{1.1}$$

where K is a constant, $L_g = \Delta_g - c(n)R_g$ is the conformal Laplacian for g with scalar curvature R_g, and $c(n) = \frac{n-2}{4(n-1)}$. If $u > 0$ is a solution of (1.1) then the new metric $\tilde{g} = u^{\frac{4}{n-2}} g$ has scalar curvature $c(n)^{-1}K$. This problem was solved in the affirmative through the combined works of Yamabe [1], Trudinger [2], Aubin [3] and Schoen [4] (see also [5] for a complete overview). From an analytic perspective the Yamabe problem has proven to be a rich source of interesting ideas. The complete solution of the problem was the first instance of a satisfactory existence theory for equations involving a critical exponent, where the standard techniques of the calculus of variations fail to apply.

A quantity that plays an important role in this context is the so-called Yamabe invariant, defined as

$$Y(M) = \inf_{\tilde{g}\in[g]} \frac{\int_M R_{\tilde{g}} dV_{\tilde{g}}}{\left(\int_M dV_{\tilde{g}}\right)^{\frac{n-2}{2}}}.$$

$Y(M)$ being positive (resp. negative, zero) implies that we can conformally deform the metric to one of constant positive (resp. negative, zero) scalar curvature, and this also corresponds to finding solutions to (1.1) where the constant K is positive (resp. negative, zero) [6].

When $Y(M) > 0$ — which can be shown to be equivalent to having the first eigenvalue of the conformal Laplacian positive — solutions to (1.1) are not unique, and it is known that the set of solutions can be quite large [7, 8]. Therefore it becomes natural to ask what can be said about the full set of solutions to (1.1) when $Y(M) > 0$. While this set is noncompact in the C^2 topology when the underlying manifold is S^n with the round metric (see [8]), when M is not conformally equivalent to the round sphere compactness was established in various cases, namely by Schoen [9] in the locally conformally flat case, Schoen and Zhang [10] in three dimensions, Druet [11] for $n \leq 5$, Marques [12] for $n \leq 7$, Li and Zhang [13, 14] for $n \leq 11$. However in a surprising turn of events, counterexamples to compactness were found by Brendle [15] when $n \geq 52$ and subsequently by Brendle and Marques [16] for $25 \leq n \leq 51$. Finally, Khuri, Marques and Schoen [17] proved that compactness does hold in all remaining cases, that is, for $n \leq 24$. See [18] for a survey of various compactness and non-compactness results for the Yamabe equation.

An obvious extension of such problems is to consider manifolds with boundary. In this case one would like to conformally deform a given metric to one which has not only constant scalar curvature but constant mean curvature as well. This problem is equivalent to showing the existence of a positive solution to the boundary value problem

$$\begin{cases} L_g u + K u^{\frac{n+2}{n-2}} = 0, & \text{in } M, \\ B_g u = \partial_{\nu_g} u + \frac{n-2}{2}\kappa_g u = \frac{n-2}{2} c u^{\frac{n}{n-2}}, & \text{on } \partial M, \end{cases} \tag{1.2}$$

where ν_g is the unit outer normal and κ_g is the mean curvature. If such a solution exists then the metric $\tilde{g} = u^{\frac{4}{n-2}} g$ has scalar curvature $c(n)^{-1}K$ and the boundary has mean curvature c. This Yamabe problem on manifolds with boundary was initially investigated by Escobar [19, 20], who solved the problem affirmatively in several cases. With contributions from several authors (see [21–29]), most of the cases have now been solved.

Notice that if $K \neq 0$ and $c \neq 0$ then both the equation and the boundary condition are nonlinear. In order to simplify the problem, it is customary to assume then that one of them is linear, that is, that either K or c is zero. Geometrically, this corresponds to deforming the manifold to one with either constant nonzero scalar curvature and zero mean curvature on the boundary ($K \neq 0$, $c = 0$) or zero scalar curvature and constant nonzero mean curvature on the boundary ($K = 0$, $c \neq 0$). In this paper we will focus on the first of these two cases.

In analogy to the case of manifolds without boundary, where the round sphere provides the canonical example of noncompactness, when the manifold has boundary and is not conformally equivalent to the round hemisphere, the question of compactness of solutions arises. Compactness was proven by Han and Li [24] when the scalar curvature is negative ($K < 0$) and the mean curvature is zero ($c = 0$), and also when the scalar curvature is positive ($K > 0$) with no restriction on the mean curvature but with the extra hypotheses that the manifold is locally conformally flat and the boundary is umbilic; by Felli and Ahmedou [30] when the scalar curvature is zero ($K = 0$), the mean curvature positive ($c > 0$), the manifold is locally conformally flat and the boundary umbilic (see also[31]); and by Almaraz [32] when the scalar curvature is zero ($K = 0$), $n \geq 7$, and a generic condition on the trace-free part of the second fundamental form holds.

It is natural to consider subcritical approximations to equation (1.2), where a priori estimates are readily available. Thus we define

$$\Phi_p = \Big\{ u > 0 \;\Big|\; L_g u + K u^p = 0 \text{ in } M,\ B_g u = 0 \text{ on } \partial M \Big\},$$

for $p \in [1, \frac{n+2}{n-2}]$. Furthermore, as the case $K < 0$ has already been treated in [24], we will assume from now on that $K > 0$. Then our main result may be stated as follows.

Theorem 1.1. *(Compactness) Let (M^n, g) be a smooth compact Riemannian manifold of dimension $3 \leq n \leq 24$ with umbilic boundary, and which is not conformally equivalent to the standard hemisphere (S^n_+, g_0). Then for any $\varepsilon > 0$ there exists a constant $C > 0$ depending only on g and ε such that*

$$C^{-1} \leq u \leq C \text{ and } \| u \|_{C^{2,\alpha}(M)} \leq C$$

for all $u \in \cup_{1+\varepsilon \le p \le \frac{n+2}{n-2}} \Phi_p$*, where* $0 < \alpha < 1$*.*

This theorem is established by a fine analysis of blow-up behavior at boundary points; such a fine analysis was carried out for interior blow-up points in [17]. The entire problem is reduced to showing the positivity of a certain quadratic form on a finite dimensional vector space, which may be analyzed in a similar manner as is done in the appendix of [17]. Of course this theorem also relies on the Positive Mass Theorem of General Relativity, in its usual form. That is, although we are concerned with manifolds having boundary, we are still able to use the standard Positive Mass Theorem by employing a doubling procedure.

Another key feature of our approach is to employ a version of conformal normal coordinates adapted to the boundary, which elucidates the dependence of various geometric quantities on the conformally invariant umbilicity tensor and Weyl tensor (see chapter 3). This coordinate system can be thought of as a good compromise between traditional conformal normal coordinates [5] and the so-called conformal Fermi coordinates [22]. This is because although the latter has been shown to be a powerful tool to study the Yamabe problem on manifolds with boundary, a critical part of the compactness result in [17] is the proof of the positivity of the quadratic form mentioned earlier. This proof makes substantial use of the the radial symmetry coming from normal coordinates and we would like to preserve as much as possible of that original argument.

In general, it is expected that wherever blow-up occurs, these conformally invariant quantities will vanish to high order because, up to a conformal change, the geometry of the manifold resembles that of a sphere near the blow-up. As we are assuming that the boundary is umbilic here, we focus on the Weyl tensor. In this regard we prove

Theorem 1.2. *(Weyl vanishing) Let g be a smooth Riemannian metric defined in the unit half n-ball B_1^+, $6 \leq n \leq 24$. Suppose that there is a sequence of positive solutions $\{u_i\}$ of*

$$\begin{cases} L_g u_i + K u_i^{p_i} = 0, & \text{in } B_1^+, \\ B_g u_i = 0, & \text{on } \overline{B_1^+} \cap \mathbb{R}^{n-1}, \end{cases}$$

$p_i \in (1, \frac{n+2}{n-2}]$, such that for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that $\sup_{B_1^+ \setminus B_\varepsilon^+} u_i \leq C(\varepsilon)$ and $\lim_{i\to\infty}(\sup_{B_1^+} u_i) = \infty$. Assume also that $\overline{B_1^+} \cap \mathbb{R}^{n-1}$ is umbilic. Then the Weyl tensor W_g satisfies

$$|W_g|(x) \leq C|x|^\ell$$

for some integer $\ell > \frac{n-6}{2}$.

Remark 1.3. It may appear that since the boundary is umbilic, the proofs of theorems 1.1 and 1.2 should follow directly from [17] by applying a reflection argument. However, the techniques employed in [17] require a higher degree of regularity than what is typically available from a simple reflection of the metric.

In analogy to the case without boundary, one wonders if theorem 1.1 is false when $n \geq 25$. We have also been able to answer this question.

Theorem 1.4. *Assume that $n \geq 25$. Then there exists a smooth Riemannian metric g on the hemisphere S^n_+ and a sequence of positive functions $u_i \in C^\infty(S^n_+)$, such that:*

(a) g is not conformally flat (so in particular (S^n_+, g) is not conformally equivalent to (S^n_+, g_0), where g_0 is the round metric),

(b) ∂S^n_+ is umbilic in the metric g,

(c) for each i, u_i is a positive solution of the boundary value problem:

$$\begin{cases} L_g u_i + K u_i^{\frac{n+2}{n-2}} = 0, & \text{in } S^n_+, \\ B_g u_i = 0, & \text{on } \partial S^n_+, \end{cases}$$

where K is a positive constant,

(d) $\sup_{S^n_+} u_i \to \infty$ as $i \to \infty$.

Together, theorems 1.1 and 1.4 give a complete answer to the question of compactness of solutions to the Yamabe problem on manifolds with umbilic boundary in the positive scalar curvature setting (Almaraz has proven an analogue to theorem 1.4 for scalar-flat manifolds [33]).

The proof of theorem 1.4 relies heavily on [15, 16]. In fact, with [15, 16] at hand, the idea to proof theorem 1.4 is not complicated. Brendle and Marques' construction is a perturbation of the round sphere (S^n, g_0). Although their solutions are constructed on S^n rather than S^n_+, they "almost" satisfy the boundary condition. We can therefore slightly modify Brendle and Marques' solutions in order to produce a blow-up sequence for the hemisphere.

One obvious consequence of theorem 1.1 is to give an alternative proof of the solution to the Yamabe problem, allowing us to compute the total Leray-

Schauder degree of all solutions to (1.2) (with $c = 0$), and to obtain more refined existence theorems. This is discussed at the end of the paper (see chapter 14).

As mentioned earlier, certain conformally invariant quantities are expected to vanish to high order at a blow-up point. In particular such behavior is expected for the umbilicity tensor when the boundary is not umbilic. In this regard, we expect the following.

Conjecture 1.5. *Let g be a smooth Riemannian metric defined in the unit half n-ball B_1^+, $4 \le n \le 24$. Suppose that there is a sequence of positive solutions $\{u_i\}$ of*

$$\begin{cases} L_g u_i + K u_i^{p_i} = 0, & \text{in } B_1^+, \\ B_g u_i = 0, & \text{on } \overline{B_1^+} \cap \mathbb{R}^{n-1}, \end{cases}$$

$p_i \in (1, \frac{n+2}{n-2}]$, such that for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that $\sup_{B_1^+ \setminus B_\varepsilon^+} u_i \le C(\varepsilon)$ and $\lim_{i\to\infty}(\sup_{B_1^+} u_i) = \infty$. Then the umbilicity tensor T_g satisfies

$$|T_g|(x) \le C|x|^m, \ x \in \overline{B_1^+} \cap \mathbb{R}^{n-1},$$

for some integer $m > \frac{n-4}{2}$. Moreover, if $n \ge 6$ we also have

$$|W_g|(x) \le C|x|^\ell, \ x \in B_1^+,$$

for some integer $\ell > \frac{n-6}{2}$.

Proving this conjecture would be a key step towards a compactness theorem for manifolds with non-umbilic boundary. In fact, one of the main ingredients of our proofs is to estimate several relevant quantities in terms of the umbilicity tensor and its derivatives at the origin. The vanishing of these terms should allow one, at least in principle, to adapt the ideas presented here to the non-umbilic case.

Chapter 2

Setting, notation, and basic definitions

Let M^n be a n-dimensional Riemannian manifold with smooth boundary, and let $\{g_i\}_{i=1}^{\infty}$ be a sequence of metrics on M converging in $C^k(M)$ to a metric g, where k is large and depends only on n. Let $\{u_i\}$ be a sequence of positive solutions of the boundary value problem

$$\begin{cases} L_{g_i}u_i + Kf_i^{-\delta_i}u_i^{p_i} = 0, & \text{in } M, \\ B_{g_i}u_i = \partial_{\nu_{g_i}}u_i + \frac{n-2}{2}\kappa_{g_i}u_i = 0, & \text{on } \partial M, \end{cases} \tag{2.1}$$

where $L_{g_i} = \Delta_{g_i} - c(n)R_{g_i}$, $c(n) = \frac{n-2}{4(n-1)}$, R_{g_i} is the scalar curvature of the metric g_i, $K = n(n-2)$, ν_{g_i} is the outer unit normal, κ_{g_i} is the mean curvature of the boundary, $\{f_i\}$ is a sequence of smooth positive functions converging in $C^2(M)$ to a smooth positive function f, $1 < p_i \le \frac{n+2}{n-2}$, $\delta_i = \frac{n+2}{n-2} - p_i$. L_g is referred to as the conformal Laplacian, and the boundary value problem (2.1) is conformally invariant (see proposition A.2).

In conformal normal coordinates (see proposition 3.1 and [5, 6, 34]) centered at a point p, we write $g(x) = \exp(h(x))$, where h is a smooth function taking values in the space of symmetric $n \times n$ matrices. From standard properties of conformal normal coordinates it then follows that $x^j h_{ij}(x) = 0$, and $\operatorname{tr} h_{ij}(x) = O(r^N)$, where $r = \operatorname{dist}_g(x,p)$ and N is arbitrarily large. We also have $\det g_{ij} = 1 + O(r^N)$.

In most of the text we will identify the center p of normal coordinates with the origin. We will write $u_i(x)$ instead of $u_i(\exp_p(x))$ and $|x|$ instead of $\operatorname{dist}_g(x,p)$. Since N in the above expressions is as large as we want, we will often ignore the $O(r^N)$ contribution in the volume element and write $d\operatorname{vol}_g(x) = dx$.

The proofs of theorems 1.1 and 1.2 depend crucially on finding a good approximation to the scalar curvature in terms of polynomials. To this end we define, in conformal normal coordinates

$$H_{ij}(x) = \sum_{2 \le |\alpha| \le n-4} h_{ij,\alpha} x^\alpha \tag{2.2}$$

where $h_{ij,\alpha}$ are the coefficients of the Taylor polynomials centered at the origin. Notice that we will sometimes use $_{,\alpha}$ to denote Taylor coefficients at the origin — which are multiples of derivatives evaluated at the origin rather than the derivatives themselves.

We then have $h_{ij} = H_{ij} + O(|x|^{n-3})$, $H_{ij} = H_{ji}$, $x^j H_{ij}(x) = 0$, and

$\operatorname{tr} H_{ij}(x) = 0$. Put also

$$H_{ij}^{(k)} = \sum_{|\alpha|=k} h_{ij,\alpha} x^\alpha, \tag{2.3}$$

$$|H^{(k)}|^2 = \sum_{ij} \sum_{|\alpha|=k} |h_{ij,\alpha}|^2, \tag{2.4}$$

and for $\varepsilon > 0$, set $x = \varepsilon y$ and define

$$\tilde{H}_{ij}^{(k)}(y) = H_{ij}^{(k)}(\varepsilon y). \tag{2.5}$$

We will make extensive use of the following standard rescaling argument. Let $\{\varepsilon_i\}_{i=1}^\infty$ be a given sequence of positive numbers converging to zero. Define M_i by $M_i^{\frac{p_i-1}{2}} = \varepsilon_i^{-1}$ and in normal coordinates put $y = M_i^{\frac{p_i-1}{2}} x = \varepsilon_i^{-1} x$ and

$$v_i(y) = M_i^{-1} u_i(x) = M_i^{-1} u_i(M_i^{-\frac{p_i-1}{2}} y) = \varepsilon_i^{\frac{2}{p_i-1}} u_i(\varepsilon_i y)$$

for $y \le \sigma M_i^{\frac{p_i-1}{2}} = \varepsilon_i^{-1}\sigma$, where $|x| \le \sigma$ belongs to the domain of definition of the normal coordinates. Then v_i satisfies

$$\begin{cases} L_{\tilde{g}_i} v_i + K \tilde{f}_i^{-\delta_i} v_i^{p_i} = 0, & \text{for } |y| \le \sigma M_i^{\frac{p_i-1}{2}}, \\ B_{\tilde{g}_i} v_i = \partial_{\nu_{\tilde{g}_i}} v_i + \frac{n-2}{2} \kappa_{\tilde{g}_i} v_i = 0, & \text{on } \partial M, \end{cases} \tag{2.6}$$

where $\tilde{f}_i(y) = f_i(M_i^{-\frac{p_i-1}{2}} y) = f_i(\varepsilon_i y)$, $(\tilde{g}_i)_{kl}(y) = (g_i)_{kl}(M_i^{-\frac{p_i-1}{2}} y) = (g_i)_{kl}(\varepsilon_i y)$ (see [12, 17]).

We recall some standard definitions (see [17, 24]). Consider a sequence $\{u_i\}$ of solutions of (2.1). A point $\bar{x} \in M$ is called a *blow-up* point for $\{u_i\}$ if

$u_i(x_i) \to \infty$ for some $x_i \to \bar{x}$.

Definition 2.1. A point $\bar{x} \in M$ is called an isolated blow-up point for $\{u_i\}$ if there exists a sequence $\{x_i\} \subset M$, $x_i \to \bar{x}$, where each x_i is a local maximum for u_i and

1) $u_i(x_i) \to \infty$ as $i \to \infty$,

2) $u_i(x) \leq C \operatorname{dist}_{g_i}(x, x_i)^{-\frac{2}{p_i - 1}}$ for $x \in B_\sigma(x_i)$ and some constants $\sigma, C > 0$.

Notice that the definition of isolated blow-up points is the same as for the boundaryless case ([24]).

Remark 2.2. If we change the metric by a uniformly bounded conformal factor $\phi > 0$ such that $\phi(x_i) = 1$ and $\nabla \phi(x_i) = 0$, then isolated blow-up points are preserved.

Definition 2.3. ([24]) Let $\{u_i\}$ and $\{x_i\}$ be as in definition 2.1. $x_i \to \bar{x}$ is an isolated simple blow-up point if for some $\rho \in (0, \sigma)$ and $C > 1$, where σ comes from the definition of isolated blow up point, the function

$$\hat{u}_i(r) = r^{\frac{2}{p_i - 1}} \bar{u}_i(r) = \frac{r^{\frac{2}{p_i - 1}}}{\operatorname{vol}_{g_i}(M \cap \partial B_r(x_i))} \int_{M \cap \partial B_r(x_i)} u(z) dS(z)$$

satisfies, for large i, $\hat{u}_i' < 0$ for r such that $C M_i^{-\frac{p_i - 1}{2}} \leq r \leq \rho$.

Observe that if $\bar{x}$ is an interior point then this definition agrees with the standard one (compare with [12]).

Throughout the paper we let $U : \mathbb{R}^n \to \mathbb{R}$ be the function $U(y) = (1 + |y|^2)^{\frac{2-n}{2}}$. U is known as the "standard bubble". From [17] we have the following.

Definition 2.4. Let $\tilde{z}_\varepsilon$ be the solution of

$$\Delta\tilde{z}_\varepsilon + n(n+2)U^{\frac{4}{n-2}}\tilde{z}_\varepsilon = c(n)\sum_{k=4}^{n-4}\partial_i\partial_j\tilde{H}_{ij}^{(k)}U \tag{2.7}$$

constructed in [17]. It is implicitly assumed that $\tilde{z}_\varepsilon \equiv 0$ if $n = 3, 4, 5$.

We recall estimate (4.4) of [17]

$$|\partial^\beta\tilde{z}_\varepsilon(y)| \le C\sum_{|\alpha|=4}^{n-4}\sum_{\ell k}\varepsilon^{|\alpha|}|h_{\ell k,\alpha}|(1+|y|)^{|\alpha|+2-n-|\beta|}, \tag{2.8}$$

which implies

$$|\partial^\beta\tilde{z}_\varepsilon(y)| \le C(1+|y|)^{2-n-|\beta|}, \quad \text{for } |y| \le \sigma\varepsilon^{-1}. \tag{2.9}$$

The role of $\tilde{z}_\varepsilon$ is to provide a sharp correction term for the usual approximation of the (rescaled) solutions u by U around a blow-up point. $\tilde{z}_\varepsilon$ was introduced in the context of manifolds without boundary, and one of the main challenges in our paper is to establish that the same $\tilde{z}_\varepsilon$ can be used in our setting. In other words, we need to show that $\tilde{z}_\varepsilon$ satisfies a natural boundary condition. In order to accomplish this, we use one of the key results of the paper, theorem 3.4, to show that the umbilicity of the boundary implies severe constraints on the behavior of the polynomials $\tilde{H}_{ij}^{(k)}$ on the boundary. Then we use the explicit construction of $\tilde{z}_\varepsilon$ in terms of $\tilde{H}_{ij}^{(k)}$ to show that it satisfies the desired boundary condition.

Notation and terminology used throughout the text:

(i) $d = [\frac{n-2}{2}]$.

(ii) If $x_i \to \bar{x}$ is an isolated blow-up point, we denote $M_i = u_i(x_i)$ and $\varepsilon_i^{-1} = M_i^{\frac{p_i-1}{2}}$.

(iii) x' denotes the first $n-1$ coordinate functions.

(iv) We use N to denote an integer that is arbitrarily large, coming typically from properties of conformal normal coordinates, such as $\det(g) = 1 + O(r^N)$.

(v) Let Ω be an open connected set that intersects ∂M. We then set $\partial'\Omega = \overline{\Omega} \cap \partial M$ and $\partial^+\Omega = \partial\Omega \backslash \partial'\Omega$.

(vi) In a coordinate system near the boundary, define $B_\sigma^G(0) = \{x \in B_\sigma(0) \mid x^n > G(x)\}$ for some real valued function G, then denote $\partial' B_\sigma^G(0) = \{(x', G(x))\}$, $\partial^+ B_\sigma^G(0) = \partial B_\sigma^G(0) \backslash \partial' G_\sigma^G(0)$ (see chapter 3 and corollary 3.8).

(vii) We will always assume that the blow-up points $\bar{x}$ lie on the boundary ∂M, since theorems 1.1 and 1.2 would otherwise follow from [17] (see chapter 12).

(viii) We will switch back and forth between problems (2.1) and (2.6), referring to them as "x-coordinates" and "y-coordinates".

(ix) If $x_0 \in \partial M$, then $B_\sigma(x_0)$ is a ball of radius σ and center x_0, i.e.,

$$B_\sigma(x_0) = \{x \in M \mid \operatorname{dist}(x, x_0) \leq \sigma\}.$$

Notice that $B_\sigma(x_0)$ will usually look more like a half-ball rather than like a full ball, but we will not denote it by $B_\sigma^+(x_0)$, reserving the latter for balls which explicitly satisfy the condition $x^n > 0$.

(x) For $T \geq 0$ we define $\mathbb{R}^n_{-T} = \{x \in \mathbb{R}^n \mid x^n > -T\}$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x^n > 0\}$.

Chapter 3

Estimates near the boundary and boundary conformal normal coordinates

In this chapter we will derive estimates for the second fundamental form, mean curvature, etc., in terms of the umbilicity tensor. Then we will use these estimates to modify the standard conformal normal coordinates construction in order to obtain conformal normal coordinates at the boundary with zero mean curvature.

We first recall a result of Escobar.

Proposition 3.1. *(Conformal normal coordinates at the boundary [19]) Assume ∂M is umbilic and let $x_0 \in \partial M$. For any $N > 0$ there exists a metric $\tilde{g}$*

conformal to g such that, in normal coordinates for $\tilde{g}$ centered at x_0

$$\det(\tilde{g}) = 1 + O(r^N),$$

where $r = |x|$. If $N \geq 5$ then $R_{\tilde{g}} = O(r^2)$, and $\Delta R_{\tilde{g}}(0) = -\frac{1}{6}|W_{\tilde{g}}|^2(0)$, and $\kappa_{\tilde{g}} = O(r^2)$. Here $W_{\tilde{g}}$ is the Weyl tensor.

Now take conformal normal coordinates at $x_0 \in \partial M$. We choose the coordinates so that ∂_i, $i \leq i \leq n-1$, are tangent to ∂M and ∂_n is normal (pointing inward) to ∂M at $x_0 = 0$. Near x_0 the boundary ∂M may be expressed as a graph $x^n = F(x')$, where $x' = (x^1, \ldots, x^{n-1})$ and since normal coordinates are defined up to a rotation we can assume that $F(0) = \nabla F(0) = 0$ and the tangent plane at 0 is the "horizontal" hyperplane $\{x^n = 0\}$. Then a basis for the tangent space $T_x\partial M$ is given by the vectors $X_i = \partial_i + F_{,i}\partial_n$, $1 \leq i \leq n-1$. The normal is given as a covector by $(\nu_g)_n = -1, (\nu_g)_i = F_{,i}$, $1 \leq i \leq n-1$, or as a vector by

$$(\nu_g)^i = g^{ij}(\nu_g)_j = -g^{in} + \sum_{i=1}^{n-1} g^{ij} F_{,j}. \tag{3.1}$$

If $g = e^h$ then we may write

$$(\nu_g)^i = -\delta^{in} + h_{in} + F_{,i} + O(|h|^2 + |h||\nabla F|).$$

Define the second fundamental form by

$$\kappa_{ij} = \kappa(X_i, X_j) = g(\nabla_{X_i}\nu_g, X_j), \tag{3.2}$$

then

$$\kappa_{ij} = g(\nabla_i \nu_g, \partial_j) + F_{,i} g(\nabla_n \nu_g, \partial_j) + F_{,j} g(\nabla_i \nu_g, \partial_n) + F_{,i} F_{,j} g(\nabla_n \nu_g, \partial_n) \tag{3.3}$$
$$= \Gamma^n_{ij} + F_{;ij} - F_{,i} \Gamma^l_{nj} (\nu_g)_l - F_{,j} \Gamma^l_{in} (\nu_g)_l - F_{,j} F_{,i} \Gamma^l_{nn} (\nu_g)_l$$
$$= \frac{1}{2}(-\partial_n h_{ij} + \partial_i h_{nj} + \partial_j h_{ni}) + F_{,ij} + O(|h||\nabla h| + |\nabla F||\nabla h|).$$

The mean curvature is given by

$$\kappa = g^{-1}(X_i, X_j)\kappa_{ij} = \Delta F + \sum_{i=1}^{n-1} \partial_i h_{ni} + \frac{1}{2}\partial_n h_{nn} \tag{3.4}$$
$$+ O(|h||\nabla h| + |\nabla F||\nabla h| + |h||\nabla^2 F| + |\nabla F|^2 |\nabla^2 F| + |x|^N),$$

where we have used $\sum_{i=1}^n h_{ii} = O(|x|^N)$. Finally the umbilicity tensor is given by

$$T(X_i, X_j) = T_{ij} = \kappa(X_i, X_j) - \frac{1}{n-1}\kappa g(X_i, X_j). \tag{3.5}$$

Notice that these quantities differ from the usual ones by a multiple of $|\nu_g|$ (since ν_g is not necessarily a unit vector). As we show below (see proposition 3.5 and corollary 3.10), this will immediately yield estimates for the standard (i.e., defined with respect to a unit vector) mean curvature, second fundamental form and umbilicity tensor, and it will suffice for our purposes. In fact, we will express all desired quantities in terms of T_{ij}, and the umbilicity of the boundary implies that T_{ij} defined with respect to (3.1) vanishes as well. We remark also that our definition of the mean curvature in this chapter differs from the standard one by a multiple of $(n-1)^{-1}$. However, in all other chapters of the paper we adopt the standard convention, unless otherwise specified.

The next theorem will be our main tool to produce estimates. Although its proof is long, the idea behind it is quite simple: from properties of conformal normal coordinates we can derive several identities involving geometric quantities and the functions h_{ij}. We restrict the obtained expressions to their Taylor polynomials, and successively solve these equations for one quantity in terms of the others, until we express all quantities in terms of the umbilicity tensor and an error.

Remark 3.2. It should be noted that in (3.3) and (3.4), as well as in the proof below, the expression $|h||\nabla h|$ appearing in the error only includes terms of the form $|h||\partial_i h_{nj}|$, $|h||\partial_n h_{ij}|$, $|h||\partial_n h_{nn}|$ or $|h_{ni}||\nabla h|$.

Remark 3.3. Since we will eventually restrict all expressions to their Taylor polynomials in theorem 3.4, and N is arbitrarily large, we will ignore the $O(|x|^N)$ contributions.

Theorem 3.4. *Take conformal normal coordinates at $x_0 \in \partial M$ as described above and choose a large integer N. Then there exists a constant C, depending only on N such that for any $\varepsilon > 0$ sufficiently small:*

$$\sum_{|\alpha|=2}^{N} |\kappa_{,\alpha}|\varepsilon^{|\alpha|} \leq C \sum_{|\alpha|=2}^{N} \sum_{i,j=1}^{n-1} |T_{ij,\alpha}|\varepsilon^{|\alpha|}$$

$$\sum_{|\alpha|=2}^{N} |\Delta F_{,\alpha}|\varepsilon^{|\alpha|} \leq C \sum_{|\alpha|=0}^{N-2} \sum_{i,j=1}^{n-1} |T_{ij,\alpha}|\varepsilon^{|\alpha|+2}$$

$$\sum_{|\alpha|=0}^{N} \sum_{i,j=1}^{n-1} |\kappa_{ij,\alpha}|\varepsilon^{|\alpha|} \leq C \sum_{|\alpha|=0}^{N} \sum_{i,j=1}^{n-1} |T_{ij,\alpha}|\varepsilon^{|\alpha|}$$

$$\sum_{|\alpha|=2}^{N} |F_{,\alpha}|\varepsilon^{|\alpha|} \leq C \sum_{|\alpha|=0}^{N-2} \sum_{i,j=1}^{n-1} |T_{ij,\alpha}|\varepsilon^{|\alpha|+2}$$

$$\sum_{|\alpha|=2}^{N} \sum_{j=1}^{n-1} |h_{nj,\alpha}|\varepsilon^{|\alpha|} \leq C \sum_{|\alpha|=1}^{N-1} \sum_{i,j=1}^{n-1} |T_{ij,\alpha}|\varepsilon^{|\alpha|+1}$$

$$\sum_{|\alpha|=1}^{N} \sum_{i,j=1}^{n-1} |\partial_n h_{ij,\alpha}|\varepsilon^{|\alpha|} \leq C \sum_{|\alpha|=1}^{N} \sum_{i,j=1}^{n-1} |T_{ij,\alpha}|\varepsilon^{|\alpha|}$$

$$\sum_{|\alpha|=1}^{N} |\partial_n h_{nn,\alpha}|\varepsilon^{|\alpha|} \leq C \sum_{|\alpha|=1}^{N} \sum_{i,j=1}^{n-1} |T_{ij,\alpha}|\varepsilon^{|\alpha|}$$

where α denotes partial derivatives in the variables $x^1, \ldots, x^{n-1}$ evaluated at the origin, and F is the local representation of the boundary as a graph as explained at the beginning of this chapter. Moreover $\kappa(0) = |\nabla\kappa|(0) = F(0) = |\nabla F|(0) = \Delta F(0) = 0$ and $|\nabla \Delta F|(0) \leq C\sum_{ij} |\nabla T_{ij}|(0)$.

Proof. We first record several useful calculations. When repeated indices i or j appear this signifies summation from 1 to $n-1$. Using familiar properties of conformal normal coordinates and (3.3) we have

$$\begin{aligned} x^i\kappa_{ij} &= \frac{1}{2}[-\partial_n(x^i h_{ij}) + x^i\partial_i h_{nj} - \delta^i_j h_{ni} + \partial_j(x^i h_{ni})] \\ &\quad + x^i F_{,ij} + O(|x||h||\nabla h| + |x||\nabla F||\nabla h|) \\ &= \frac{1}{2}[\partial_n(x^n h_{nj}) + x^i\partial_i h_{nj} - h_{nj} - \partial_j(x^n h_{nn})] \\ &\quad + x^i\partial_i F_{,j} + O(|x||h||\nabla h| + |x||\nabla F||\nabla h|) \\ &= \frac{1}{2}x^i\partial_i h_{nj} + x^i\partial_i F_{,j} + O(|x^n||\nabla h| + |x||h||\nabla h| + |x||\nabla F||\nabla h|). \end{aligned} \tag{3.6}$$

Furthermore

$$
\begin{aligned}
x^i x^j \kappa_{ij} &= x^i x^j F_{,ij} - \frac{1}{2} x^i \delta_i^j h_{nj} + \frac{1}{2} x^i \partial_i (x^j h_{nj}) \\
&+ O(|x||x^n||\nabla h| + |x|^2|h||\nabla h| + |x|^2|\nabla F||\nabla h|) \\
&= x^i x^j F_{,ij} + \frac{1}{2} x^n h_{nn} - \frac{1}{2} x^n x^i \partial_i h_{nn} \\
&+ O(|x||x^n||\nabla h| + |x|^2|h||\nabla h| + |x|^2|\nabla F||\nabla h|) \\
&= x^i x^j F_{,ij} + O(|x^n||h| + |x||x^n||\nabla h| + |x|^2|h||\nabla h| + |x|^2|\nabla F||\nabla h|),
\end{aligned} \tag{3.7}
$$

and

$$
\begin{aligned}
x^i x^j \kappa g(X_i, X_j) &= x^i x^j \kappa (g_{ij} + F_{,i} g_{jn} + F_{,j} g_{in} + F_{,i} F_{,j} g_{nn}) \\
&= |x'|^2 \kappa + O(|x|^2|h|||\kappa| + |x|^2||\nabla F|^2|\kappa|).
\end{aligned} \tag{3.8}
$$

Recalling the definition of the umbilicity tensor together with (3.7) and (3.8) yields

$$
\begin{aligned}
x^i x^j F_{,ij} = \frac{1}{n-1}|x'|^2 \kappa + x^i x^j T_{ij} + O(|x^n||h| + |x||x^n||\nabla h| \\
+ |x|^2|h||\nabla h| + |x|^2|\nabla F||\nabla h| + |x|^2|h||\kappa| + |x|^2|\nabla F|^2|\kappa|).
\end{aligned} \tag{3.9}
$$

Moreover since

$$
x^i \kappa g(X_i, X_j) = x^j \kappa + O(|x||h||\kappa| + |x||\nabla F|^2|\kappa|),
$$

we find that (using (3.6))

$$\begin{aligned}&\frac{1}{2}x^i\partial_i h_{nj}+x^i\partial_i F_{,j}\\&=\frac{1}{n-1}x^j\kappa+x^iT_{ij}\\&+O(|x^n||\nabla h|+|x||h||\nabla h|+|x||\nabla F||\nabla h|+|x||h||\kappa|+|x||\nabla F|^2|\kappa|).\end{aligned} \tag{3.10}$$

Eliminating κ from (3.9) and (3.10) produces

$$x^i\partial_i F_{,j}+\frac{1}{2}x^i\partial_i h_{nj}-x^iT_{ij}=|x'|^{-2}x^jx^ix^l(F_{,il}-T_{il})+\Omega_j, \tag{3.11}$$

where throughout this proof Ω_j denotes error which satisfies

$$\begin{aligned}\Omega_j&=O\Big(|x^n||\nabla h|+|x||h||\nabla h|+|x||\nabla F||\nabla h|+|x||h||\kappa|\\&+|x||\nabla F|^2|\kappa|+|x'|^{-1}|x^n||h|\Big)\\&=O\Big(|x^n||\nabla h|+|x||h||\nabla h|+|x||\nabla F||\nabla h|+|x||h||\nabla^2F|\\&+|x||\nabla F|^2|\nabla^2F|+|x'|^{-1}|x^n||h|\Big).\end{aligned} \tag{3.12}$$

Upon restricting attention to Taylor polynomials (3.11) simplifies to

$$\begin{aligned}&(k-1)|x'|^2F^{(k)}_{,j}-k(k-1)x^jF^{(k)}\\&=-\frac{1}{2}(k-1)|x'|^2h^{(k-1)}_{nj}+|x'|^2x^iT^{(k-2)}_{ij}-x^jx^ix^lT^{(k-2)}_{il}+|x'|^2\Omega^{(k-1)}_j\end{aligned} \tag{3.13}$$

where $h^{(k-1)}_{nj}$ denotes the $(k-1)$-degree Taylor polynomial in the variables $x^1,\dots,x^{n-1}$ and similarly for $F^{(k)}$, $T^{(k-2)}_{ij}$. We note that $h^{(k-1)}_{nj}$ is not the full Taylor polynomial in all the variables $x^1,\dots,x^n$ but rather just the portion involving the first $n-1$ coordinates, and the remainder involving x^n is relegated to the error term. Now apply ∂_j to (3.13) and sum over j to find an equation for $F^{(k)}$,

$$|x'|^2\Delta F^{(k)} + k(3-n-k)F^{(k)} = -\frac{1}{2}|x'|^2\partial_j h_{nj}^{(k-1)} \tag{3.14}$$
$$+(k-1)^{-1}\partial_j(|x'|^2 x^i T_{ij}^{(k-2)} - x^j x^i x^l T_{il}^{(k-2)}) + \partial_j(|x'|^2\Omega_j^{(k-1)})$$

where we used $x^i h_{ij} = -x^n h_{nj}$ to absorb this term in the error. Differentiate (3.10) with respect to x^j and sum over j to find

$$\frac{1}{2}\partial_j h_{nj} + \frac{1}{2}x^i\partial_i\partial_j h_{nj} + \Delta F + x^i\partial_i\Delta F = x^i\partial_j T_{ij} + \kappa + \frac{1}{n-1}x^j\partial_j\kappa + \partial_j\Omega_j$$

where we used that the error term in (3.10) has the form Ω_j. Then

$$\frac{1}{2}(k-1)\partial_j h_{nj}^{(k-1)} + (k-1)\Delta F^{(k)} = x^i\partial_j T_{ij}^{(k-2)} + \frac{n+k-3}{n-1}\kappa^{(k-2)} + \partial_j\Omega_j^{(k-1)}. \tag{3.15}$$

On the other hand (3.9) gives

$$k(k-1)F^{(k)} - x^i x^j T_{ij}^{(k-2)} = \frac{1}{n-1}|x'|^2\kappa^{(k-2)} + x^i\Omega_i^{(k-1)}. \tag{3.16}$$

Therefore using (3.14) and (3.16) in (3.15) produces

$$\partial_j h_{nj}^{(k-1)} = -\frac{2}{k-1}x^i\partial_j T_{ij}^{(k-2)} + \frac{x^j}{|x'|^2}\Omega_j^{(k-1)} + \partial_j\Omega_j^{(k-1)}. \tag{3.17}$$

Let B_1^{n-1} denote the unit ball with respect to $x^1,\ldots,x^{n-1}$, and let $\phi \in$

$C^\infty(S_1^{n-2})$. Extend ϕ radially so that it is defined on $B_1^{n-1}\backslash\{0\}$ and $\partial_r\phi = 0$ on S_1^{n-2}, where $r = |x'|$. Notice that even though ϕ is not defined at the origin, we can still integrate by parts against functions which vanish at zero, and so in particular against homogeneous polynomials.

Let ϕ be as above. From (3.10) we have

$$(k-1)\partial_i F^{(k)} = \frac{1}{n-1}x^i\kappa^{(k-2)} - \frac{k-1}{2}h_{ni}^{(k-1)} + x^jT_{ij}^{(k-2)} + \Omega_i^{(k-1)}. \tag{3.18}$$

Multiply (3.18) by $\partial_i\phi$, sum over i and integrate by parts to get

$$\begin{aligned}
&-(k-1)\int_{B_1^{n-1}}\phi\Delta F^{(k)} + (k-1)\int_{S_1^{n-2}}\phi\nu^i\partial_i F^{(k)}\\
&= -\frac{1}{n-1}\int_{B_1^{n-1}}\phi(n+k-3)\kappa^{(k-2)} + \frac{1}{n-1}\int_{S_1^{n-2}}\phi\nu^i x^i\kappa^{(k-2)}\\
&+\frac{k-1}{2}\int_{B_1^{n-1}}\phi\partial_i h_{ni}^{(k-1)} - \frac{k-1}{2}\int_{S_1^{n-2}}\phi\nu^i h_{ni}^{(k-1)}\\
&-\int_{B_1^{n-1}}\phi x^j\partial_i T_{ij}^{(k-1)} + \int_{S_1^{n-2}}\phi\nu^i x^j T_{ij}^{(k-2)} - \int_{B_1^{n-1}}\phi\partial_i\Omega_i^{(k-1)}\\
&+\int_{S_1^{n-2}}\phi\nu^i\Omega_i^{(k-1)}.
\end{aligned}$$

Integrating in polar coordinates produces

$$\begin{aligned}
&-(k-1)\int_0^1 r^{n+k-4}\int_{S_1^{n-2}}\phi\Delta F^{(k)} + (k-1)\int_{S_1^{n-2}}\phi\nu^i\partial_i F^{(k)} =\\
&-\frac{1}{n-1}\int_0^1 r^{k+n-4}\int_{S_1^{n-2}}\phi(n+k-3)\kappa^{(k-2)} + \frac{1}{n-1}\int_{S_1^{n-2}}\phi\nu^i x^i\kappa^{(k-2)}\\
&+\frac{k-1}{2}\int_0^1 r^{n+k-4}\int_{S_1^{n-2}}\phi\partial_i h_{ni}^{(k-1)} - \frac{k-1}{2}\int_{S_1^{n-2}}\phi\nu^i h_{ni}^{(k-1)}\\
&-\int_0^1 r^{n+k-4}\int_{S_1^{n-2}}\phi x^j\partial_i T_{ij}^{(k-1)} + \int_{S_1^{n-2}}\phi\nu^i x^j T_{ij}^{(k-2)}
\end{aligned}$$

$$
-\int_0^1 r^{n+k-4}\int_{S_1^{n-2}} \phi\partial_i\Omega_i^{(k-1)} + \int_{S_1^{n-2}} \phi\nu^i\Omega_i^{(k-1)},
$$

which implies (notice that the mean curvature terms cancel out)

$$
\begin{aligned}
\Delta F^{(k)} - \frac{k(n+k-2)}{|x'|^2}F^{(k)} = & -\frac{1}{2}\partial_i h_{ni}^{(k-1)} + \frac{1}{2}\frac{n+k-3}{|x'|^2}x^i h_{ni}^{(k-1)} \\
& + \frac{1}{k-1}x^j\partial_i T_{ij}^{(k-2)} - \frac{n+k-3}{k-1}\frac{x^i x^j}{|x'|^2}T_{ij}^{(k-2)} \\
& + \partial_i\Omega_i^{(k-1)} + \frac{x^i}{|x'|^2}\Omega_i^{(k-1)},
\end{aligned} \tag{3.19}
$$

where we have used that ϕ is an arbitrary smooth function on S_1^{n-2} and homogeneous polynomials are determined by their values on the sphere. Using (3.14) and (3.17) in (3.19) we find that

$$
x^i h_{ni}^{(k-1)} = -\frac{2}{(k-1)(n+k-3)}|x'|^2 x^j\partial_i T_{ij}^{(k-2)} + x^j\Omega_j^{(k-1)} + |x'|^2\partial_j\Omega_j^{(k-1)}. \tag{3.20}
$$

Similarly, multiplying (3.13) by $\partial_j\phi$ and integrating by parts yields

$$
\begin{aligned}
|x'|^2\Delta F^{(k)} + \frac{k(k-n-1)}{k-1}F^{(k)} = & \frac{1}{2}(n+k-3)x^j h_{nj}^{(k-1)} \\
& - \frac{1}{2}|x'|^2\partial_j h_{nj}^{(k-1)} + \frac{1}{k-1}|x'|^2 x^i\partial_j T_{ij}^{(k-2)} \\
& - \frac{n+k-3}{k-1}x^i x^j T_{ij}^{(k-2)} + x^j\Omega_j^{(k-1)} \\
& + |x'|^2\partial_j\Omega_j^{(k-1)}.
\end{aligned} \tag{3.21}
$$

Solving for $\Delta F^{(k)} + \frac{1}{2}\partial_j h_{nj}^{(k-1)}$ in (3.14) and using it along with (3.20) in (3.21) we obtain

$$F^{(k)} = \frac{n+k-3}{k(2n+3k-nk-k^2-3)} x^i x^j T_{ij}^{(k-2)} + x^j \Omega_j^{(k-1)} + |x'|^2 \partial_j \Omega_j^{(k-1)}. \tag{3.22}$$

Notice that the denominator of the first term on the right hand side is never zero since $k \geq 2$.

From (3.9) we have

$$k|x'|^{-2} F^{(k)} = \frac{1}{(n-1)(k-1)} \kappa^{(k-2)} + \frac{1}{k-1} |x'|^{-2} x^i x^j T_{ij}^{(k-2)} + |x'|^{-2} x^j \Omega_j^{(k-1)}. \tag{3.23}$$

Using (3.22) in (3.23) yields

$$\kappa^{(k-2)} = c(n,k) \frac{x^i x^j}{|x'|^2} T_{ij}^{(k-2)} + \frac{x^j}{|x'|^2} \Omega_j^{(k-1)} + \partial_j \Omega_j^{(k-1)}, \tag{3.24}$$

where $c(n,k)$ is a numerical factor depending on n and k only. Let $\mathcal{R}$ be the set of homogeneous polynomials that can be estimated in terms of the umbilicity tensor and an error (of the same degree) in Ω_j. Then (3.17), (3.20), (3.22) and (3.24) give that $\partial_i h_{ni}^{(k-1)}$, $x^i h_{ni}^{(k-1)}$, $F^{(k)}$, $\kappa^{(k-2)} \in \mathcal{R}$. From (3.19) it then follows that $\Delta F^{(k)} \in \mathcal{R}$ as well. From (3.13) and $F^{(k)} \in \mathcal{R}$ we get $h_{nj}^{(k-1)} \in \mathcal{R}$, and from (3.4) and $\Delta F^{(k)}$, $\partial_i h_{ni}^{(k-1)} \in \mathcal{R}$ it follows that $(\partial_n h_{nn})^{(k-2)} \in \mathcal{R}$. Using (3.5) along with $\kappa^{(k-2)} \in \mathcal{R}$ we get $\kappa_{ij}^{(k-2)} \in \mathcal{R}$ and from this, (3.3), $h_{nj}^{(k-1)} \in \mathcal{R}$ and $F^{(k)} \in \mathcal{R}$ we find that $(\partial_n h_{ij})^{(k-2)} \in \mathcal{R}$.

The inequalities of theorem 3.4 now follow with the help of remark 3.2.

By our construction of F and properties of conformal normal coordinates we have $\kappa(0) = |\nabla\kappa|(0) = F(0) = |\nabla F|(0)$. Hence in order to finish the theorem we only have to show that $\Delta F(0) = 0$ and $|\nabla \Delta F|(0) \leq C\sum_{ij} \nabla |T_{ij}|(0)$.

Using the definition of κ_{ij}, and recalling that $\nu_n = -1$ and $\nu_j = F_{,j}$, $1 \leq j \leq n-1$, we obtain

$$\begin{aligned}\kappa_{ij} = F_{,ij} - \Gamma^k_{ij}F_{,k} + \Gamma^n_{ij} - F_{,j}\Gamma^k_{in}F_{,k} + F_{,j}\Gamma^n_{in} \\ - F_{,i}\Gamma^k_{nj}F_{,k} + F_{,i}\Gamma^n_{nj} - F_{,i}F_{,j}\Gamma^k_{nn}F_{,k} + F_{,i}F_{,j}\Gamma^n_{nn}\end{aligned} \tag{3.25}$$

where we have used $F_{,n} = 0$ and $\sum_{k=1}^{n-1}\Gamma^k_{ij}F_{,k} = \Gamma^k_{ij}F_{,k}$ since F does not depend on x^n. Evaluating (3.25) at 0 and using $\Gamma^k_{ij}(0) = 0 = |\nabla F|(0)$, we have $\kappa_{ij}(0) = F_{ij}(0)$. Taking a trace produces $\Delta F(0) = \kappa(0) = 0$. Finally notice that (3.12) gives

$$\Omega_i = O(|x^n||x| + |x|^4 + |x|^3 + |x'|^{-1}|x^n||x|^2),$$

so we can compute directly from (3.22) to find $|\nabla \Delta F(0)| \leq C\sum_{ij}|\nabla T_{ij}(0)|$, finishing the proof. □

Now with the help of theorem 3.4 we improve the properties of conformal normal coordinates at the boundary by showing that we can also require zero mean curvature. We call these coordinates *boundary conformal normal coordinates* to avoid confusion with the usual conformal normal coordinates at a point on the boundary.

Proposition 3.5. *(Boundary conformal normal coordinates) Let (M, g_0) be a Riemannian manifold with umbilic boundary and $x_0 \in \partial M$. Fix an integer $N \geq 5$. Then there exists a metric $\tilde{g}$ conformal to g_0 such that, in $\tilde{g}$-normal* *coordinates centered at x_0: (i)* $\det \tilde{g} = 1 + O(r^N)$, *(ii)* $R_{\tilde{g}} = O(r^2)$, *(iii)* $\Delta_{\tilde{g}} R_{\tilde{g}}(0) = -\frac{1}{6}|W_{\tilde{g}}|^2(0)$ *and (iv)* $\kappa_{\tilde{g}} = 0$ *near* x_0, *where* $r = |x|$.

Proof. Using conformal normal coordinates at x_0 we obtain a metric g which satisfies properties (i)-(iii) in a ball $B_\sigma(0)$. Our task is to show that we can perform a further conformal change in the metric in order to obtain property (iv) while maintaining (i)-(iii).

We write all quantities as explained above (see equation (3.1) and what follows); in particular we denote by κ_g the mean curvature defined as in (3.4), and by $\widehat{\kappa}_g$ the mean curvature defined in the usual way, i.e., with respect to a unit vector.

If $\tilde{g} = e^{2f} g$ then $\widehat{\kappa}_{\tilde{g}} = e^{-f}(\widehat{\kappa}_g + \frac{\partial f}{\partial \nu_g})$. We will choose f appropriately.

Because the boundary is umbilic, T_{ij} vanishes identically and therefore theorem 3.4 gives $\kappa_{g,\alpha}(0) = 0$ for $|\alpha| = 0, \dots, N$, where α denotes derivatives with respect to $x^1, \dots, x^{n-1}$. In other words, we obtain that $\kappa_g = O(|x'|^N)$, from which it follows that $\widehat{\kappa}_g = O(|x'|^N)$ as well. Now we choose an extension of $\widehat{\kappa}_g$ to $\widetilde{\kappa}_g$ in a neighborhood of x_0, with $\widetilde{\kappa}_g$ satisfying $\widetilde{\kappa}_g = O(|x|^N)$ and $\frac{\partial \widetilde{\kappa}_g}{\partial \nu_g} = 0$. Such an extension is possible because $\widehat{\kappa}_g = O(|x'|^N)$.

Now pick a smooth function $\widetilde{f}$ such that $\frac{\partial \widetilde{f}}{\partial \nu_g} = -1$ near x_0 and put $f = \widetilde{f} \widetilde{\kappa}_g$. With this choice of f we then have $\widehat{\kappa}_{\tilde{g}} = 0$ in a neighborhood of x_0.

By construction we have $f = O(|x|^N)$, and so we obtain the desired result as the remaining properties all follow from $\det(\tilde{g}) = 1 + O(r^N)$ (after choosing a smooth extension of f to the whole of M). □

Remark 3.6. We stress a point made in the introduction. The so-called conformal Fermi coordinates [22] have been used with great success in the study of the Yamabe problem for manifolds with boundary (see references mentioned in the introduction). This expresses the fact that cylindrical coordinates generally work better than spherical ones for Neumann-type of problems. However, a critical part of the compactness result of Khuri, Marques and Schoen [17] for boundaryless manifolds is the proof of the positivity of a quadratic form on Taylor polynomials of the scalar curvature which naturally arises in the problem. Their proof makes substantial use of the the radial symmetry coming from normal coordinates and we would like to preserve as much as possible of that original argument. Boundary conformal normal coordinates preserve the radial symmetry while displaying features similar to the good properties of Fermi coordinates, as it is shown below.

Boundary conformal normal coordinates have the following useful property.

Corollary 3.7. *In boundary conformal normal coordinates centered at* $x_0 \in \partial M$ *the boundary is given by* $x^n = 0$*. Moreover,* $g_{in}(x', 0) = O(|x'|^N)$, $1 \leq i \leq n-1$.

Proof. Since the boundary is umbilic and $\kappa_g \equiv 0$, it is also totally geodesic (i.e. $\kappa_{ij} \equiv 0$) for the metric g, and therefore the boundary is given by $x^n = 0$ in normal coordinates. The second statement follows from theorem 3.4 as $g = e^h$. □

Now we want to extend the previous results for the case of interior points. Assume that $x_0 \in \mathring{M}$ is an interior point which is sufficiently close to ∂M, and take conformal normal coordinates at x_0. Denote by $\tilde{x}_0 \in \partial M$ the closet point to x_0. We can still write the boundary as a graph $x^n = F(x')$, and since normal coordinates are defined up to a rotation we can assume that $F(0) = -|\tilde{x}_0|$ where $|\tilde{x}_0| = \operatorname{dist}(x_0, \tilde{x}_0)$ (so that $\tilde{x}_0 = (0, \dots, 0, -|\tilde{x}_0|)$) and the tangent plane is horizontal there, so $|\nabla F(0)| = 0$. Moreover, by the Gauss lemma we also have $\frac{\partial}{\partial \nu_g}\Big|_{\tilde{x}_0} = g^{nn}\, \partial_n|_{\tilde{x}_0}$.

If we "translate the boundary", i.e., define

$$G(x') = F(x') + |\tilde{x}_0|$$

we have $G(0) = |\nabla G(0)| = 0$ and $\partial^\alpha G = \partial^\alpha F$. Set $B_\sigma^G = \{x \in B_\sigma(0) | x^n > G(x')\}$. Notice then that a basis for the tangent space at a point on $\partial' B_\sigma^G = \{(x', G(x'))\}$ is $X_i = \partial_i + G_{,i}\partial_n = \partial_i + F_{,i}\partial_n$, $1 \leq i \leq n-1$ (since we are simply translating the boundary). We can then consider all geometric quantities induced on the boundary $\partial' B_\sigma^G$. In this situation, theorem 3.4 holds with G replacing F and all quantities being defined with respect to the boundary $\partial' B_\sigma^G$, except for the conclusions that depend on $\kappa = O(r^2)$, since the boundary $\partial' B_\sigma^G$ need not be umbilic. We state this as a corollary.

Corollary 3.8. *Let $x_0 \in \mathring{M}$, take conformal normal coordinates at x_0 and assume that x_0 is sufficiently close to ∂M as to have $\partial M \cap B_\sigma(0) \neq \emptyset$, where $B_\sigma(0)$ is the domain of definition of the conformal normal coordinates. Let $F = F(x')$ be the local representation of the boundary as explained above. Define $G(x') = F(x') + |\tilde{x}_0|$, $B_\sigma^G = \{x \in B_\sigma(0) | x^n > G(x')\}$, and $\partial' B_\sigma^G = \{(x', G(x'))\}$. Then there exists a constant C, depending only on N, such that* *for any $\varepsilon > 0$ sufficiently small:*

$$\sum_{|\alpha|=2}^{N} |\widetilde{\kappa}_{,\alpha}|\varepsilon^{|\alpha|} \le C \sum_{|\alpha|=2}^{N} \sum_{i,j=1}^{n-1} |\widetilde{T}_{ij,\alpha}|\varepsilon^{|\alpha|}$$
$$\sum_{|\alpha|=2}^{N} |\Delta G_{,\alpha}|\varepsilon^{|\alpha|} \le C \sum_{|\alpha|=0}^{N-2} \sum_{i,j=1}^{n-1} |\widetilde{T}_{ij,\alpha}|\varepsilon^{|\alpha|+2}$$
$$\sum_{|\alpha|=0}^{N} \sum_{i,j=1}^{n-1} |\widetilde{\kappa}_{ij,\alpha}|\varepsilon^{|\alpha|} \le C \sum_{|\alpha|=0}^{N} \sum_{i,j=1}^{n-1} |\widetilde{T}_{ij,\alpha}|\varepsilon^{|\alpha|}$$
$$\sum_{|\alpha|=2}^{N} |G_{,\alpha}|\varepsilon^{|\alpha|} \le C \sum_{|\alpha|=0}^{N-2} \sum_{i,j=1}^{n-1} |\widetilde{T}_{ij,\alpha}|\varepsilon^{|\alpha|+2}$$
$$\sum_{|\alpha|=2}^{N} \sum_{j=1}^{n-1} |h_{nj,\alpha}|\varepsilon^{|\alpha|} \le C \sum_{|\alpha|=1}^{N-1} \sum_{i,j=1}^{n-1} |\widetilde{T}_{ij,\alpha}|\varepsilon^{|\alpha|+1}$$
$$\sum_{|\alpha|=1}^{N} \sum_{i,j=1}^{n-1} |\partial_n h_{ij,\alpha}|\varepsilon^{|\alpha|} \le C \sum_{|\alpha|=1}^{N} \sum_{i,j=1}^{n-1} |\widetilde{T}_{ij,\alpha}|\varepsilon^{|\alpha|}$$
$$\sum_{|\alpha|=1}^{N} |\partial_n h_{nn,\alpha}|\varepsilon^{|\alpha|} \le C \sum_{|\alpha|=1}^{N} \sum_{i,j=1}^{n-1} |\widetilde{T}_{ij,\alpha}|\varepsilon^{|\alpha|}$$

where $\widetilde{\kappa}$, $\widetilde{\kappa}_{ij}$ and $\widetilde{T}_{ij}$ are respectively the mean curvature, second fundamental form and umbilicity tensor of $\partial' B_\sigma^G$, all defined with respect to the outer normal

$$(\nu_g)^i = g^{ij}(\nu_g)_j = -g^{in} + \sum_{i=1}^{n-1} g^{ij} G_{,j}$$

(which is not necessarily a unit normal) and α denotes partial derivatives in the variables $x^1, \ldots, x^{n-1}$ evaluated at the origin. Moreover $G(0) = |\nabla G|(0) = 0$ and $|\nabla \Delta G|(0) \le C \sum_{ij} |\nabla \widetilde{T}_{ij}|(0)$.

As before, estimates on quantities defined with respect to ν_g, with ν_g not necessarily a unit vector, will suffice for our purposes.

Because $\partial^\alpha G = \partial^\alpha F$, estimates for G from corollary 3.8 translate into estimates for F.

Corollary 3.9. *Let $x_0 \in \overset{\circ}{M}$ and $\tilde{x}_0 \in \partial M$ be the closest point to x_0. Take conformal normal coordinates at x_0, choose a large integer N and let F be the local representation of the boundary as a graph. Then there exists a constant C, depending only on N such that for any $\varepsilon > 0$ sufficiently small:*

$$\sum_{|\alpha|=2}^{N} |F_{,\alpha}|\varepsilon^{|\alpha|} \leq C \sum_{|\alpha|=0}^{N-2} \sum_{i,j=1}^{n-1} |\widetilde{T}_{ij,\alpha}|\varepsilon^{|\alpha|+2}$$
$$\sum_{|\alpha|=2}^{N} |\Delta F_{,\alpha}|\varepsilon^{|\alpha|} \leq C \sum_{|\alpha|=0}^{N-2} \sum_{i,j=1}^{n-1} |\widetilde{T}_{ij,\alpha}|\varepsilon^{|\alpha|+2}$$

where $\widetilde{T}_{ij}$ is the umbilicity tensor of the boundary $\partial' B_\sigma^G$ as in corollary 3.8, and α denotes partial derivatives in the variables $x^1, \ldots, x^{n-1}$ evaluated at the origin. Moreover $|\nabla F|(0) = 0$, $|\nabla \Delta F|(0) \leq C \sum_{ij} |\nabla \widetilde{T}_{ij}(0)|$ and $F(0) = -|\tilde{x}_0|$.

The following corollary will finish the treatment of interior points in this chapter.

Corollary 3.10. *(Boundary conformal normal coordinates for an interior point) Let (M, g_0) be a Riemannian manifold with umbilic boundary and $x_0 \in \overset{\circ}{M}$. Fix an integer $N \geq 5$. If x_0 is sufficiently close to ∂M, then there exists a metric $\tilde{g}$ conformal to g_0 such that, in $\tilde{g}$-normal coordinates centered at x_0: (i) $\det \tilde{g} = 1 + O(r^N)$, (ii) $R_{\tilde{g}} = O(r^2)$, (iii) $\Delta_{\tilde{g}} R_{\tilde{g}}(0) = -\frac{1}{6}|W_{\tilde{g}}|^2(0)$ and (iv) $\kappa_{\tilde{g}} = 0$ near $\tilde{x}_0$, where $r = |x|$ and $\tilde{x}_0 \in \partial M$ is such that $\mathrm{dist}_{g_0}(x_0, \tilde{x}_0) = \mathrm{dist}_{g_0}(x_0, \partial M)$.*

Proof. Let $\tilde{x}_0 \in \partial M$ be the closet point to x_0. Denote by $\{\tilde{x}^i\}$ conformal normal coordinates centered at $\tilde{x}_0$, $\{x^i\}$ conformal normal coordinates centered at x_0, $\widetilde{\kappa}$ the mean curvature of ∂M in $\{\tilde{x}^i\}$-coordinates, κ the mean curvature of ∂M in $\{x^i\}$-coordinates. When $x_0 \to \tilde{x}_0$ we have $x^i \to \tilde{x}^i$, and $\partial_\alpha \kappa(x_0) \to \partial_{\tilde{\alpha}} \widetilde{\kappa}(\tilde{x}_0)$ where α denotes partial derivatives with respect to $x^1, \ldots, x^{n-1}$ and $\tilde{\alpha}$ denotes partial derivatives with respect to to $\tilde{x}^1, \ldots, \tilde{x}^{n-1}$.

By theorem 3.4 we have that $\partial_{\tilde{\alpha}} \kappa(\tilde{x}_0) = 0$ for $|\tilde{\alpha}| \leq N$ since the boundary is umbilic. Because $\partial_\alpha \kappa(x_0) \to \partial_{\tilde{\alpha}} \kappa(\tilde{x_0})$ as $x_0 \to \tilde{x}_0$, if x_0 is sufficiently close to $\tilde{x}_0$ we can choose an extension of κ to $B_\sigma(x_0)$ (with σ small) which is $O(|x - x_0|^N)$. The rest of the argument now is similar to the proof of proposition 3.5. ☐

We finish this chapter with several remarks.

Remark 3.11. One of the key ingredients of our proof is to show that the blow-up sequence x_i lies on the boundary (possibly after passing to a subsequence, see chapter 7). Before showing that, however, we have to deal with both the case of a blow-up sequence belonging to the boundary and the case of a blow-up sequence belonging to the interior of the manifold. It will therefore be implicitly understood that when $x_i \in \mathring{M}$, all quantities κ, κ_{ij} and T_{ij} are for the boundary $\partial' B^G_\sigma$, as described above, i.e., we will drop $\tilde{}$ from the interior quantities for the sake of notation. F, however, will always be the representation of ∂M as a graph unless stated otherwise.

Remark 3.12. Suppose that $x_0 \in \partial M$ or that it is sufficiently close to the boundary, and in boundary conformal normal coordinates centered at x_0 consider $x = (0, x^n)$. If we translate the boundary by $|x^n|$ instead of $|\tilde{x}_0|$,

$$G(x') = F(x') + |x^n|,$$

we can, for each $|x^n|$, consider geometric quantities induced on the boundary $\partial' B_\sigma^G$ as before. In another words, we have a foliation of a small neighborhood of the boundary by copies of ∂M. In particular, we can then think of T_{ij} as defined in a neighborhood of ∂M, allowing us to take derivatives with respect to x^n, Taylor expand T_{ij} in the x^n direction, etc.

Remark 3.13. Since boundary conformal normal coordinates are a special case of conformal normal coordinates, the results of this chapter stated for conformal normal coordinates, in particular theorem 3.4, are still valid if we choose boundary conformal normal coordinates instead.

Chapter 4

Higher order estimates

Our next goal is to extend the results of theorem 3.4 to higher order derivatives of h in the normal direction. Throughout this chapter we will work with boundary conformal normal coordinates centered at a point on the boundary; all definitions are as in chapter 3. We will use Greek letters to denote indices running up to n, Latin letters to denote indices running up to $n-1$, and x' to denote the first $n-1$ coordinates. Notice that in light of corollary 3.7 we have that the boundary is given by $x^n = 0$, and as in chapter 3, by a "translation" we can consider quantities defined on the neighborhood of the boundary, so that

$$\frac{\partial}{\partial \nu_g} = -g^{n\tau}\partial_\tau, \tag{4.1}$$

$$g_{ni}|_{\partial M} = g_{ni}(x', 0) = O(|x'|^N), \tag{4.2}$$

$$\left.\frac{\partial}{\partial \nu_g}\right|_{\partial M} = -g^{nn}(x', 0)\partial_n + \sum_{\ell=1}^{n-1} O(|x'|^N)\partial_\ell. \tag{4.3}$$

Theorem 4.1. *In boundary conformal normal coordinates at a point on the* *boundary,*

$$h_{nn}|_{\partial M} = h_{nn}(x', 0) = O(|x'|^N). \tag{4.4}$$

Proof. We will compute $\nabla_i \nu^n$ in two different ways. First,

$$\nabla_i \nu^n = -\partial_i g^{nn} - g^{nn}\Gamma^n_{in} - g^{nl}\Gamma^n_{il}. \tag{4.5}$$

Notice that (3.3) becomes in our coordinates $\Gamma^n_{ij}(x', 0) = \kappa_{ij}(x', 0) = 0$, and hence (4.5) gives

$$\begin{aligned} \nabla_i \nu^n|_{\partial M} &= -\partial_i g^{nn} - \frac{1}{2}(g^{nn})^2 \partial_i g_{nn} - \frac{1}{2} g^{nn} g^{nl}(\partial_i g_{nl} + \partial_n g_{li} - \partial_l g_{in}) \\ &= -\partial_i g^{nn} - \frac{1}{2}(g^{nn})^2 \partial_i g_{nn} + O(|x'|^{2N-1}), \end{aligned} \tag{4.6}$$

where we used theorem 3.4. In order to simplify notation, here and in the rest of the chapter we use the following convention. When an equality is restricted to the boundary we write $\cdot|_{\partial M}$ or $(\cdot)(x', 0)$ on one side of the equation, and it is implicitly understood that the remaining quantities on the other side are restricted as well.

Now differentiate $g^{nn} g_{nn} + g^{nl} g_{nl} = 1$ with respect to i to obtain $g^{nn}\partial_i g_{nn} =$

$-g_{nn}\partial_i g^{nn} - \partial_i(g^{nl}g_{nl})$ and so (4.6) becomes

$$\begin{aligned}\nabla_i \nu^n|_{\partial M} &= -\partial_i g^{nn} + \frac{1}{2}g^{nn}g_{nn}\partial_i g^{nn} - \partial_i(g^{nl}g_{nl}) + O(|x'|^{2N-1}) \\ &= -\partial_i g^{nn} + \frac{1}{2}(1 - g^{nl}g_{nl})\partial_i g^{nn} - \partial_i(g^{nl}g_{nl}) + O(|x'|^{2N-1}) \\ &= -\frac{1}{2}\partial_i g^{nn} + O(|x'|^{2N-1}),\end{aligned} \tag{4.7}$$

where we used theorem 3.4 again. Combining (4.6) and (4.7) gives

$$\partial_i g^{nn} + (g^{nn})^2\partial_i g_{nn} = O(|x'|^{2N-1}).$$

Using $g = e^h$ this becomes

$$-(1-(g^{nn})^2)\partial_i h_{nn} + \partial_i O_{nn}(-h) + (g^{nn})^2\partial_i O_{nn}(h) = O(|x'|^{2N-1}) \tag{4.8}$$

where

$$O_{\mu\sigma}(h) = \sum_{\ell=2}^{\infty}\frac{(h^\ell)_{\mu\sigma}}{\ell!}.$$

But

$$O_{nn}(-h) = \frac{1}{2}(h_{nn}h_{nn} + h_{nl}h_{nl}) + \sum_{\ell=2}^{\infty}\frac{(h^{2\ell})_{nn}}{(2\ell)!} - \sum_{\ell=1}^{\infty}\frac{(h^{2\ell+1})_{nn}}{(2\ell+1)!}$$

and

$$O_{nn}(h) = \frac{1}{2}(h_{nn}h_{nn} + h_{nl}h_{nl}) + \sum_{\ell=2}^{\infty}\frac{(h^{2\ell})_{nn}}{(2\ell)!} + \sum_{\ell=1}^{\infty}\frac{(h^{2\ell+1})_{nn}}{(2\ell+1)!},$$

and since by theorem 3.4 $h_{nl}(x',0)h_{nl}(x',0) = O(|x'|^{2N})$, (4.8) becomes

$$(-1+(g^{nn})^2+h_{nn}+(g^{nn})^2h_{nn})\partial_i h_{nn}+(1+(g^{nn})^2)\partial_i\sum_{\ell=2}^{\infty}\frac{(h^{2\ell})_{nn}}{(2\ell)!} \qquad (4.9)$$
$$+(1-(g^{nn})^2)\partial_i\sum_{\ell=1}^{\infty}\frac{(h^{2\ell+1})_{nn}}{(2\ell+1)!}=O(|x'|^{2N-1}).$$

We will now show inductively that $h_{nn} = O(|x'|^k)$ implies $h_{nn} = O(|x'|^{3k})$, $k \leq N$. Notice that we already know that $h_{nn} = O(|x'|^2)$. Also, as before, the terms h_{nl} appearing in $(h^{2\ell})_{nn}$, $\ell \geq 2$, and $(h^{2\ell+1})_{nn}$, $\ell \geq 1$, can be estimated by theorem 3.4 and hence they can be absorbed in the error; in other words we can replace $(h^\ell)_{nn}$ by $(h_{nn})^\ell$ up to and error $O(|x'|^{2N-1})$ (notice that due to the rules of multiplication of matrices, the terms h_{nl} appearing in $(h^{2\ell})_{nn}$, $\ell \geq 2$, $(h^{2\ell+1})_{nn}$, $\ell \geq 1$, or in the expansion of g^{nn} must be multiplied by another h_{nl} and hence such errors are of the same order of the right hand side of (4.9)).

Since $g^{nn} \geq C > 0$ near the origin, if $h_{nn} = O(|x'|^k)$ then

$$-1+(g^{nn})^2 = -(1+g^{nn})(1-g^{nn}) = -(1+g^{nn})O(|x'|^k) = O(|x'|^k)$$

and also

$$h_{nn}+(g^{nn})^2h_{nn} = O(|x'|^k),$$

so

$$-1+(g^{nn})^2+h_{nn}+(g^{nn})^2h_{nn} = O(|x'|^k). \qquad (4.10)$$

But

$$(1+(g^{nn})^2)\partial_i \sum_{\ell=2}^{\infty} \frac{(h^{2\ell})_{nn}}{(2\ell)!} = (1+(g^{nn})^2)\partial_i \sum_{\ell=2}^{\infty} \frac{(h_{nn})^{2\ell}}{(2\ell)!} + O(|x'|^{2N-1}) \tag{4.11}$$
$$= O(|x'|^{4k-1}) + O(|x'|^{2N-1})$$

and

$$(1-(g^{nn})^2)\partial_i \sum_{\ell=2}^{\infty} \frac{(h^{2\ell})_{nn}}{(2\ell)!} = (1+g^{nn})(1-g^{nn})\partial_i \sum_{\ell=2}^{\infty} \frac{(h_{nn})^{2\ell}}{(2\ell)!} + O(|x'|^{2N-1}) \tag{4.12}$$
$$= (1+g^{nn})O(|x'|^k)O(|x'|^{3k-1}) + O(|x'|^{2N-1})$$
$$= O(|x'|^{4k-1}) + O(|x'|^{2N-1}).$$

Therefore (4.9)-(4.12) give

$$\begin{aligned}\partial_i h_{nn} &= \frac{1}{(-1+(g^{nn})^2+h_{nn}+(g^{nn})^2 h_{nn})}(1+(g^{nn})^2)\partial_i \sum_{\ell=2}^{\infty} \frac{(h^{2\ell})_{nn}}{(2\ell)!} \\ &+ \frac{1}{(-1+(g^{nn})^2+h_{nn}+(g^{nn})^2 h_{nn})}(1-(g^{nn})^2)\partial_i \sum_{\ell=1}^{\infty} \frac{(h^{2\ell+1})_{nn}}{(2\ell+1)!} \\ &= O(|x'|^{3k-1}) + O(|x'|^{2N-1-k}),\end{aligned}$$

provided that $(-1+(g^{nn})^2+h_{nn}+(g^{nn})^2h_{nn})(x',0)$ is not zero and $k<N$. Since $h(0)=0$ we conclude that $h_{nn}(x')=O(|x'|^{3k})$. Repeating the argument we obtain the result.

Now we have to show that the result is still true if $(-1+(g^{nn})^2+h_{nn}+(g^{nn})^2h_{nn})(x',0)$ vanishes or is $O(|x'|^N)$, and it is enough to consider this latter case. So suppose that $(-1+(g^{nn})^2+h_{nn}+(g^{nn})^2h_{nn})(x',0) = O(|x'|^N)$. Multiply it by g_{nn} and use $1 = g^{nn}g_{nn}+g^{nl}g_{nl} = g^{nn}g_{nn}+O(|x'|^{2N})$ to get

$$-g_{nn}+g^{nn}+(g_{nn}+g^{nn})h_{nn} = O(|x'|^N),$$

which implies $-O_{nn}(h)+O_{nn}(h)+(O_{nn}(h)+O_{nn}(-h))h_{nn} = O(|x'|^N)$ and therefore

$$-2\sum_{\ell=1}^{\infty}\frac{(h^{2\ell+1})_{nn}}{(2\ell+1)!}+2h_{nn}\sum_{\ell=1}^{\infty}\frac{(h^{2\ell})_{nn}}{(2\ell)!} = O(|x'|^N).$$

As before we can ignore contributions from h_{nl} and replace $(h^\ell)_{nn}$ by $(h_{nn})^\ell$, which gives

$$-\frac{1}{3}(h_{nn})^3-2\sum_{\ell=2}^{\infty}\frac{(h_{nn})^{2\ell+1}}{(2\ell+1)!}+\Big((h_{nn})^2+2\sum_{\ell=2}^{\infty}\frac{(h_{nn})^{2\ell}}{(2\ell)!}\Big)h_{nn} = O(|x'|^N).$$

This gives $(h_{nn})^3 = O((h_{nn})^5)+O(|x'|^N)$. Since $h = O(|x|^2)$ we obtain $(h_{nn})^3 = O(|x'|^{10})$. But then

$$\begin{aligned}(h_{nn})^3 &= O((h_{nn})^5)+O(|x'|^N) = O((h_{nn})^3(h_{nn})^2)+O(|x'|^N)\\ &= O(|x'|^{10}(h_{nn})^2)+O(|x'|^N) = O(|x'|^{14}).\end{aligned}$$

Repeating the argument produces $(h_{nn})^3 = O(|x'|^N)$, which gives the result since N is as large as we want. □

Theorem 4.2. *In boundary conformal normal coordinates centered at a point* *on the boundary we have*

$$\partial_n^2 h_{nj,\alpha'}(0) = 0, \quad |\alpha'| \leq N \tag{4.13}$$

$$\partial_n^3 h_{nn,\alpha'}(0) = 0, \quad |\alpha'| \leq N \tag{4.14}$$

where α' *denotes derivatives with respect to* $x^1, \ldots, x^{n-1}$*. In other words* $\partial_n^2 h_{nj}|_{\partial M} = O(|x'|^N)$ *and* $\partial_n^3 h_{nn}|_{\partial M} = O(|x'|^N)$*.*

Proof. Denote by $h_{nl}^{(m)}$ the m^{th} Taylor polynomial of h_{nl}. Let $\phi \in C^\infty(S_+^{n-1})$ and extend it radially similarly to what was done in theorem 3.4 (notice however that here we have the full, i.e., including x^n, Taylor polynomial). Integration by parts yields

$$\int_{B_+} \phi^2 \partial_n h_{nl}^{(m)} = -2 \int_{B_+} \phi \partial_n \phi h_{nl}^{(m)} + \int_{S_+^{n-1}} \phi^2 x^n h_{nl}^{(m)} - \int_{B_1^{n-1}} \phi^2 h_{nl}^{(m)}$$

where B_+ is the half unit ball and B^{n-1} the unit ball in x' coordinates. Since $h_{nl}(x', 0) = O(|x'|^N)$ by theorem 3.4, we obtain that the integral over B^{n-1} vanishes. Integrating in polar coordinates as in theorem 3.4 shows that

$$\int_{S_+^{n-1}} \phi^2 \partial_n h_{nl}^{(m)} = -2 \frac{m+n-1}{m+n} \int_{S_+^{n-1}} \phi \partial_n \phi h_{nl}^{(m)} + (m+n-1) \int_{S_+^{n-1}} \phi^2 x^n h_{nl}^{(m)}.$$

And since this is true for any $\phi \in C^\infty(S_+^{n-1})$, we conclude

$$\phi \partial_n h_{nl}^{(m)} = -2 \frac{m+n-1}{m+n} \partial_n \phi h_{nl}^{(m)} + \phi(m+n-1) x^n h_{nl}^{(m)} \quad \text{on } S_+^{n-1}.$$

Using theorem 3.4 again, or, alternatively, choosing a non-zero test function such that $\partial_n \phi = 0$ on $S^{n-2} = \partial B^{n-1}$, it follows that $\partial_n h_{nl}^{(m)}(x', 0) = 0$, from which we conclude

$$\partial_n h_{nl}(x', 0) = O(|x'|^N). \tag{4.15}$$

Now with (4.15) in hand, we repeat the integration by parts argument with $\partial_n^2 h_{nl}$ in place of $\partial_n h_{nl}$ and conclude (4.13).

To obtain (4.14), argue similarly to the above, integrate $\partial_n^2 h_{nn}$ by parts and use theorem 3.4 to conclude $\partial_n^2 h_{nn}(x', 0) = O(|x'|^N)$; then repeat the argument, expressing $\partial_n^3 h_{nn}$ in terms of $\partial_n^2 h_{nn}$. □

Remark 4.3. Since theorem 3.4 gives $\partial_n h_{nn}(x', 0) = O(|x'|^N)$, using an argument similar to that of theorem 4.2 we can relate $\partial_n h_{nn}(x', 0)$ and $h_{nn}(x', 0)$, obtaining in this way an alternative proof of theorem 4.1.

Chapter 5

Boundary condition for the correction term

In this chapter we use the results of chapters 3 and 4 to show that the correction term $\tilde{z}_\varepsilon$ (see definition 2.4) satisfies the correct boundary condition. The idea is to use the results of chapters 3 and 4 to show that certain homogeneous polynomials that appear in the (explicit) construction of $\tilde{z}_\varepsilon$ satisfy the boundary condition and so will $\tilde{z}_\varepsilon$ itself. Throughout this chapter we work with boundary conformal normal coordinates centered at a point on the boundary. This chapter relies heavily on the appendix of [17] and we will often refer to it.

Lemma (A.6) of [17] gives the decomposition

$$H_{ij}^{(k)} = W_{ij}^{(k)} + \sum_{q=1}^{[\frac{k-2}{2}]} (\widehat{H}_q^{(k)})_{ij} \tag{5.1}$$

where $W_{ij}^{(k)}$ satisfies $\partial_{ij} W_{ij}^{(k)} = 0$. In [17] it is also computed that

$$\begin{aligned} \partial_{ij}(\widehat{H}_q^{(k)})_{ij} = \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.2) \\ \frac{n-2}{n-1}(k-2q)(k-2q-1)(n+k-2q-1)(n+k-2q-2)|x|^{2q-2}p_{k-2q} \\ = C_{n,k,q}|x|^{2q-2}p_{k-2q}, \end{aligned}$$

where p_{k-2q} is a harmonic polynomial of degree $k-2q$.

Since $H_{ij}^{(k)}(x) = \sum_{|\alpha|=k} h_{ij,\alpha} x^\alpha$, we obtain

$$\partial_n \partial_{ij} H_{ij}^{(k)}(x) = \sum_{|\alpha|=k} h_{ij,\alpha} \partial_{nij} x^\alpha.$$

Now we consider $\partial_n \partial_{ij} H_{ij}^{(k)}\Big|_{\partial M}$ and identify the terms that do not necessarily vanish (recall that the boundary is given by $x^n = 0$).

Consider the case $i, j < n$. In this case if

$$\sum_{|\alpha|=k} h_{ij,\alpha}\, \partial_{ijn} x^\alpha|_{x^n=0} \neq 0,$$

then the non-zero terms have $\alpha_n = 1$, i.e., we can write the multi-index α (of the non-zero terms) as $\alpha = (\alpha', 1)$. Hence the coefficients of the non-vanishing terms are all of the form

$$h_{ij,\alpha} = \frac{1}{\alpha!}\partial^\alpha h_{ij}(0) = \frac{1}{\alpha!}\partial_n h_{ij,\alpha'}(0), \qquad |\alpha'| = k-1. \tag{5.3}$$

Similarly if $i = n$ and $j < n$ then the coefficients of the non-vanishing terms are all of the form

$$h_{ij,\alpha} = \frac{1}{\alpha!}\partial_n^2 h_{nj,\alpha'}(0), \qquad |\alpha'| = k-2, \tag{5.4}$$

and if $i = j = n$ the coefficients of the non-vanishing terms are all of the form

$$h_{ij,\alpha} = \frac{1}{\alpha!}\partial_n^3 h_{nn,\alpha'}(0), \qquad |\alpha'| = k-3, \tag{5.5}$$

where α' is a multi-index with $\alpha'_n = 0$. Since $k \leq n-4$, we have by theorems 3.4 and 4.2 that (5.3), (5.4) and (5.5) all vanish, and therefore

$$\partial_n \partial_{ij} H_{ij}^{(k)}(x', 0) = 0. \tag{5.6}$$

Combining (5.1) and $\partial_{ij} W_{ij}^{(k)} = 0$ with (5.2) and (5.6) gives

$$\sum_{q=1}^{[\frac{k-2}{2}]} C_{n,k,q} |x'|^{2q-2} \partial_n p_{k-2q}(x', 0) = 0,$$

and it then follows from usual decomposition theorems for homogeneous polynomials (see e.g. [35]) that each $\partial_n p_{k-2q}(x', 0)$ vanishes separately.

To compute $\partial_n \tilde{z}_\varepsilon|_{\mathbb{R}^{n-1}}$ it is enough to compute the derivative of $Z((\widehat{H}_q^{(k)})_{ij})$ — the solution to (2.7) with $(\widehat{H}_q^{(k)})_{ij}$ instead of $\sum_k H_{ij}^{(k)}$. Such a solution takes the form (see [17])

$$Z((\widehat{H}_q^{(k)})_{ij}) = c(n)\alpha_{k-2q}(1+|x|^2)^{-\frac{n}{2}} \sum_{j=1}^{q+1} \Gamma(k,q,j)|x|^{2j} p_{k-2q}$$

where α_{k-2q} and $\Gamma(k,q,j)$ are numerical coefficients. Computing we find

$$\partial_n Z((\widehat{H}_q^{(k)})_{ij})(x',0) = c(n)\alpha_{k-2q}(1+|x'|^2)^{-\frac{n}{2}} \sum_{j=1}^{q+1} \Gamma(k,q,j)|x'|^{2j}\partial_n p_{k-2q}(x',0).$$

But we showed above that $\partial_n p_{k-2q}(x',0) = 0$ and hence $\partial_n Z((\widehat{H}_q^{(k)})_{ij})(x',0)$. Therefore, we have proven

Proposition 5.1. *Take boundary conformal normal coordinates at a point on the boundary and let $\tilde{z}_\varepsilon$ be as in definition 2.4. Then it satisfies*

$$\begin{cases} \Delta \tilde{z}_\varepsilon + n(n+2)U^{\frac{4}{n-2}}\tilde{z}_\varepsilon = c(n)\sum_{k=4}^{n-4} \partial_i\partial_j \tilde{H}_{ij}^{(k)} U, & \text{in } \mathbb{R}^n_+, \\ \partial_n \tilde{z}_\varepsilon = 0, & \text{on } \mathbb{R}^{n-1}. \end{cases} \tag{5.7}$$

Chapter 6

Basic convergence results

Here we prove some basic convergence results. Most of the results are either known or modifications of similar results for manifolds without boundary.

Lemma 6.1. *Suppose $x_i \to \bar{x}$ is an isolated blow-up point. Take normal coordinates at x_i, and rescale coordinates to y-coordinates. Then in the limit $i \to \infty$ the boundary becomes a hyperplane.*

Proof. The metric $\tilde{g}_i$ is obtained from g by (i) a rescaling $g_{1_i} = M_i^{p_i-1} g_i = \varepsilon_i^{-2} g$ and then (ii) by the change of coordinates $y = \varepsilon^{-1} x$. If we write $g_{1_i} = M_i^{p_i-1} g_i$ in the standard form $g_{1_i} = \phi_i^{\frac{4}{n-2}} g_i$ we have $\phi_i^{\frac{2}{n-2}} = M_i^{\frac{p_i-1}{2}} = \varepsilon_i^{-1}$, so transformation law (A.4) gives $(\kappa_{1_i})_{kj} = \varepsilon_i^{-1}(\kappa_i)_{kj}$. The second fundamental form transforms as

$$(\tilde{\kappa}_i)_{kj}(y) = \frac{\partial x^p}{\partial y^k}\frac{\partial x^q}{\partial y^j}(\kappa_{1_i})_{pq}(x) = \varepsilon_i^2 \delta^{kp}\delta^{qj}(\kappa_{1_i})_{pq}(x)$$
$$= \varepsilon_i^2(\kappa_{1_i})_{kj}(x) = \varepsilon_i^2\varepsilon_i^{-1}(h_i)_{kj}(x) = \varepsilon_i(\kappa_i)_{kj}(x)$$

when we change coordinates from x to y via $x = \varepsilon_i y$. Now the sequence

$(\kappa_1)_{kj}(x)$ is bounded because in x-coordinates the metrics converge in C^k (k large), and therefore the second fundamental form goes to zero in y-coordinates. But $\tilde{g}_{ij}(y) \to \delta_{ij}$ since

$$\tilde{g}_{ij}(y) = g_{ij}(\varepsilon_i y) = \delta_{ij} + \varepsilon_i^2 O(|y|^2)$$

and therefore in the limit the boundary is a hyperplane (see also [24]). $\square$

Lemma 6.2. *Let $x_i \to \bar{x} \in \partial M$ be an isolated blow-up point. There exists a constant $C > 0$ such that for all i and all $|y| \le \sigma M_i^{\frac{p_i - 1}{2}}$ we have $|v_i(y)| \le C$ (where σ comes from the definition of isolated blow-up point).*

Proof. The proof is similar to the first claim in proposition 4.3 of [12] and it uses the maximum principle, the Harnack inequality and the definition of isolated blow-up point. In fact, from the definition of v_i and isolated blow-up points we have that

$$\begin{cases} v_i(0) = 1, \ \nabla v_i(0) = 0 \\ 0 < v_i(y) \le C|y|^{-\frac{2}{p_i - 1}} \quad \text{for } |y| \le \sigma M_i^{\frac{p_i-1}{2}} = l_i. \end{cases} \tag{6.1}$$

From these properties it follows that $v_i(y) \le C$ for $1 \le |y| \le l_i$. Since $L_{\tilde{g}_i} v_i = -K v_i^{p_i} \le 0$, using the maximum principle (corollary A.4) we have that there exists a constant $C > 0$ such that for every i,

$$\min_{|y| \le r} v_i(y) \ge C^{-1} \min_{|y| = r} v_i(y)$$

with $0 < r \leq 1$. Using the Harnack inequality (lemma A.6) we get

$$\max_{|y|=r} v_i(y) \leq C \min_{|y|=r} v_i(y),$$

so that

$$\max_{|y|=r} v_i(y) \leq C \min_{|y|=r} v_i(y) \leq C \min_{|y|\leq r} v_i(y) \leq C v_i(0) \leq C$$

for $0 < r \leq 1$, and the claim follows. □

The next proposition is the analogue of proposition 4.3 of [12] and of proposition 1.4 of [24].

Proposition 6.3. *Let $x_i \to \bar{x} \in \partial M$ be an isolated blow-up point and assume that $R_i \to \infty$ and $\epsilon_i \to 0$ are given. Then $p_i \to \frac{n+2}{n-2}$ and, after passing to a subsequence*

$$\| v_i - U \|_{C^2(B_{R_i}(0))} \leq \epsilon_i$$

and

$$\frac{R_i}{\log M_i} \to 0.$$

Proof. Let $R > 0$ and $\epsilon > 0$ be given. From lemma 6.2 we have $|v_i(y)| \leq C$.

Therefore, by standard elliptic estimates there exists a subsequence of v_i

converging in C^2_{loc} to a limit v which satisfies

$$\begin{cases} \Delta v + K v^p = 0 & \text{in } \mathbb{R}^n_{-T}, \\ \frac{\partial v}{\partial y^n} = 0 & \text{on } \partial \mathbb{R}^n_{-T} \text{ if } T < \infty, \\ v(0) = 1 & \text{and } y = 0 \text{ is a local maximum of } v, \end{cases}$$

where T is the limit of a subsequence of

$$T_i = M_i^{\frac{p_i - 1}{2}} \operatorname{dist}_{g_i}(x_i, \partial M) = M_i^{\frac{p_i - 1}{2}} |\tilde{x}_i| = |\tilde{y}_i|.$$

If $T = \infty$ then the proposition follows from the well known result of Caffarelli, Gidas, and Spruck ([36]). If $T < \infty$ then the boundary converges to a hyperplane when $i \to \infty$ by lemma 6.1, and the result follows from proposition A.1. □

The following lemma is analogous to lemma 2.1 of [24]. As in the the proof of [24] — where they assume conformal flatness — the idea is to show that if the $M_i^{\frac{p_i - 1}{2}} \operatorname{dist}_{g_i}(x_i, \partial M)$ does not stay bounded, then after rescaling the solutions we obtain an *interior* blow-up point, in which case the machinery of [17] can be applied (of course, in [24] they could not use [17] since such results had not yet been known, but they could still apply whatever was known about blow-up points in conformally flat manifolds without boundary; the idea here is similar). The proof does not require change to y-coordinates but we will keep track of the expression in y-coordinates for future use.

Lemma 6.4. *Let $x_i \to \bar{x} \in \partial M$ be an isolated simple blow-up point, with*

$x_i \in \mathring{M}$. *Then*

$$M_i^{\frac{p_i-1}{2}} \operatorname{dist}_{g_i}(x_i, \partial M)$$

stays bounded.

Proof. Let $\tilde{x}_i$ be such that $\operatorname{dist}_{g_i}(x_i, \partial M) = \operatorname{dist}_{g_i}(x_i, \tilde{x}_i)$. The proof is by contradiction. Consider a subsequence such that

$$M_i^{\frac{p_i-1}{2}} \operatorname{dist}_{g_i}(x_i, \partial M) = M_i^{\frac{p_i-1}{2}} |\tilde{x}_i| \to \infty$$

i.e., $|\tilde{y}_i| \to \infty$. Put $T_i = M_i^{\frac{p_i-1}{2}} |\tilde{x}_i| = |\tilde{y}_i|$ and take normal coordinates at x_i. For $|z| \leq |\tilde{x}_i|^{-1}\sigma$ (where σ comes from the definition of isolated blow-up point) define

$$\xi_i(z) = N_i^{-1} u_i(N_i^{-\frac{p_i-1}{2}} z)$$

where $N_i^{-1} = |\tilde{x}_i|^{\frac{2}{p_i-1}}$.

Notice that ξ_i has the same form as v_i with N_i in place of M_i, so if $(\tilde{g}_i)(z)_{kl} = (g_i)_{kl}(N_i^{-\frac{p_i-1}{2}} z)$ we see that ξ_i satisfies

$$\begin{cases} L_{\tilde{g}_i}\xi_i + K\tilde{f}_i^{-\delta_i}\xi_i^{p_i} = 0 & \text{for } |z| \leq |\tilde{x}_i|^{-1}\sigma, \\ B_{\tilde{g}_i}\xi_i = \partial_{\nu_{\tilde{g}_i}}\xi_i + \frac{n-2}{2}\kappa_{\tilde{g}_i}\xi_i = 0 & \text{on } \partial M, \end{cases}$$

where $\tilde{f}_i(z) = f_i(N_i^{-\frac{p_i-1}{2}} z)$. Since x_i is an isolated simple blow-up point for u_i we have $u_i(x) \leq C|x|^{-\frac{2}{p_i-1}}$ and then $\xi_i(z) \leq C|z|^{-\frac{2}{p_i-1}}$. This, together with the fact that $\xi(0) = |\tilde{x}_i|^{\frac{2}{p_i-1}} u_i(0) = |\tilde{x}_i|^{\frac{2}{p_i-1}} M_i = T_i^{\frac{2}{p_i-1}} \to \infty$ as $i \to \infty$

implies that $\{0\}$ is an *interior* isolated blow-up point for ξ_i, hence we can use corollary 2.6 of [17] (with N_i instead of M_i and ξ_i instead of u_i) and conclude that $\xi_i(0)\xi_i \to w$ in $C^2_{loc}(\mathbb{R}^n_{-1} - \{0\})$, where $w > 0$ is the Euclidean Green's function for the Laplacian centered at 0 (Euclidean because $\tilde{g}_i$ converges to the Euclidean metric) and $\mathbb{R}^n_{-1} = \{z^n > -1\}$. It also follows that $B_{\tilde{g}_i}\xi_i = \partial_{\nu_{\tilde{g}_i}}\xi_i + \frac{n-2}{2}\kappa_{\tilde{g}_i}\xi_i = 0$ becomes in the limit $\frac{\partial w}{\partial z^n} = 0$ on $\partial\mathbb{R}^n_{-1}$. We have (see for instance [24])

$$w(z) = a|z|^{2-n} + A + O(|z|),\ A > 0.$$

Define

$$B(r, z, \xi, \nabla\xi) = \frac{n-2}{2}\xi\frac{\partial\xi}{\partial\nu} - \frac{r}{2}|\nabla\xi|^2 + r\big(\frac{\partial\xi}{\partial\nu}\big)^2.$$

Because 0 is an interior blow-up point, we can use theorem 7.1 of [17] to get

$$\liminf_{r\to 0}\int_{|z|=r} B(r, z, w, \nabla w) \geq 0$$

and a direct computation gives

$$\liminf_{r\to 0}\int_{|z|=r} B(r, z, w, \nabla w) = -\frac{n-2}{2}A|S^{n-1}|,$$

contradicting $A > 0$. $\square$

Suppose $x_i \to \bar{x} \in \partial M$ is an isolated simple blow-up point. In the notation of lemma 6.4, write $T_i = M_i^{\frac{p_i-1}{2}}\,\mathrm{dist}_{g_i}(x_i, \partial M) = M_i^{\frac{p_i-1}{2}}|\tilde{x}_i|$. In y-coordinates this becomes $T_i = M_i^{\frac{p_i-1}{2}}|\tilde{x}_i| = |\tilde{y}_i|$. By lemma 6.4 we cannot have $T_i \to \infty$, and passing to a subsequence we have $T_{i_j} \to T < \infty$. Corresponding to the subsequence $\{T_{i_j}\}$ there is a subsequence $\{v_{i_j}\}$. Applying proposition 6.3 to the $\{v_{i_j}\}$ yields $T = 0$. Hence, we can hereafter assume that

$$|\tilde{y}_i| \to 0. \tag{6.2}$$

Proposition 6.5. *Let $x_i \to \bar{x} \in \partial M$ be an isolated simple blow-up point for the sequence $\{u_i\}$ of positive solutions to (2.1). Then there exist constants $C > 0$, $\sigma > 0$ independent of i such that*

$$M_i u_i(x) \geq C^{-1} G_i(x, x_i), \quad M_i^{\frac{p_i - 1}{2}} \leq |x| \leq \sigma,$$
$$M_i u_i(x) \leq C|x|^{2-n}, \quad |x| \leq \sigma,$$

where $G_i(x, x_i)$ is the Green's function for L_{g_i} centered at x_i with boundary condition $B_{g_i} G_i(x, x_i) = 0$ on $\partial' B_\sigma(x_i)$. Moreover, after passing to a subsequence $M_i u_i(x) \to G(x, \bar{x})$ in $C^2_{loc}(B_\sigma(\bar{x}) \backslash \{\bar{x}\})$, where $G(x, \bar{x})$ is the Green's function for L_g centered at $\bar{x}$ with boundary condition $B_g G(x, \bar{x}) = 0$ on $\partial' B_\sigma(\bar{x})$.

Proof. The proof is an adaptation of the ideas from [12] and [24] using lemma A.6 and proposition 6.3. □

Chapter 7

A further estimate on $\mathrm{dist}_{g_i}(x_i, \partial M)$.

Let $x_i \to \bar{x} \in \partial M$ be an isolated simple blow-up point. As in the boundaryless case, one of the main features of our proofs is the usual use of coordinates centered at the points x_i. If $x_i \in \mathring{M}$, lemma 6.4 then gives an estimate for the distance of x_i to the boundary. Unfortunately this estimate is not enough for our purposes. In fact the results of chapters 4 and 5 require the center of the coordinate system to be on the boundary. We therefore have to prove that we can pass to a subsequence such that $x_i \in \mathring{M}$.

Proposition 7.1. *Suppose $x_i \to \bar{x}$ is an isolated simple blow-up point. Then in boundary conformal normal coordinates at x_i, there exist constants $\sigma, C > 0$, independent of i, such that*

$$|v_i - U|(y) \le C\varepsilon_i$$

for every $|y| \le \sigma M_i^{\frac{p_i-1}{2}}$.

Proof. The idea of the proof is as follows. Using the fact that the boundary is totally geodesic in boundary conformal normal coordinates, we can reflect all quantities across the boundary and then mimic the proofs of [12]. In order to simplify notation the index i will be dropped from all quantities when no confusion arises, and the metric $\tilde{g}$ in y-coordinates will simply be denoted as g. Similarly $\tilde{f}$ will be denoted by f. We will use Greek letters to denote indices running up to n and Latin letters for indices running up to $n-1$, and write as usual $y = (y', y^n)$. Let $l = \sigma\varepsilon^{-1}$.

If $y_i \in \partial M$, take Fermi coordinates $(z^1, \ldots, z^n)$ at y_i. If $y_i \in \mathring{M}$ then take Fermi coordinates $(z^1, \ldots, z^n)$ at $\tilde{y}_i$, where $\tilde{y}_i \in \partial M$ is the closest point to y_i. Then in these coordinates $g_{nn} \equiv 1$ and $g_{ni} \equiv 0$. Shrinking the domain if necessary, we can assume that the domain of definition of the Fermi coordinates contains the domain of definition of the boundary conformal normal coordinates. Define the extensions

$$\overline{g}(z', z^n) = \begin{cases} g(z', z^n), & z^n \ge 0 \\ g(z', -z^n), & z^n < 0 \end{cases} \quad \text{and} \quad \overline{v}(z', z^n) = \begin{cases} v(z', z^n), & z^n \ge 0 \\ v(z', -z^n), & z^n < 0. \end{cases} \tag{7.1}$$

Recall that in boundary conformal normal coordinates the mean curvature vanishes and hence the boundary condition for v is just a Neumann condition. Moreover the umbilicity of ∂M gives that the second fundamental form vanishes as well. Therefore the above extensions are C^2, and are in fact smooth in the z' direction. Notice also that we are performing a change of coordinates to Fermi coordinates, but we are not making a conformal change of the metric, and hence the vanishing of κ and κ_{ij} are still true in Fermi coordinates. Mimicking a standard one-dimensional argument then shows that $\partial_n(\partial_n^2\overline{v})$ and $\partial_n(\partial_n^2\overline{g})$ exist in the weak sense, so in particular the extensions are $C^{2,\alpha}$. Of course, the extended metric also satisfies $\overline{g}_{nn} \equiv 1$ and $\overline{g}_{ni} \equiv 0$.

A simple calculation shows that

$$R_{\overline{g}}(z', z^n) = R_g(z', -z^n), \quad z^n < 0, \tag{7.2}$$

and

$$\Delta_{\overline{g}}\overline{v}(z', z^n) = \Delta_g v(z', -z^n), \quad z^n < 0. \tag{7.3}$$

Now extend the function f across the boundary by $\overline{f}(z', z^n) = f(z', -z^n)$ if $z^n < 0$. Notice that $\overline{f}$ and $R_{\overline{g}}$ are $C^{0,\alpha}$. Combining (7.2) and (7.3) produces

$$\begin{aligned} L_{\overline{g}}\overline{v}(z', z^n) &= L_g v(z', -z^n) = -K f^{-\delta}(z', -z^n) v^p(z', -z^n) \\ &= -K\overline{f}^{-\delta}(z', z^n)\overline{v}^p(z', z^n) \end{aligned}$$

for $z^n < 0$, i.e., the extended quantities also satisfy the equation. It follows that the extended equation holds in the original y-coordinates,

$$L_{\overline{g}}\overline{v}(y) + K\overline{f}^{-\delta}\overline{v}^p(y) = 0 \ \text{ in } \ \widetilde{B}_l(0), \tag{7.4}$$

where $\widetilde{B}_l(0)$ is a full ball in $\mathbb{R}^n$, i.e., $\widetilde{B}_l(0) = \{y \in \mathbb{R}^n \mid |y| < l\}$. From $\det g = 1 + O(r^N)$ in $B_l(0)$ we obtain $\det \overline{g} = 1 + O(r^N)$ in $\widetilde{B}_l(0)$ as well.

Now that the problem is defined in the full ball $\widetilde{B}_l(0)$, to prove the proposition, proceed with almost identical arguments as in the proofs of lemmas 5.1, 5.2 and 5.3 of [12]. There are, however, three differences that we now discuss.

First, unlike in [12] the coefficients, of the PDE are not smooth. However, they are sufficiently regular to apply elliptic estimates.

Second, we need the estimate $\overline{v} \leq CU$. In [12] this arises from the fact that the blow-up is isolated simple. In the current situation it is not necessarily true that 0 is an isolated simple blow-up point for $\overline{v}$ on $\widetilde{B}_l(0)$. Nevertheless, we will show that $\overline{v}(y) \leq CU(y)$ still holds for all $y \in \widetilde{B}_l(0)$. Notice that we do not need to make an extension of U since it is a priori defined on the whole of $\mathbb{R}^n$.

To see why this is the case, first notice that since y_i is an isolated simple blow-up point for v on $B_l(0)$, we have $v \leq CU$ there. For $p \in B_l(0)$, let $\overline{p} \in \widetilde{B}_l(0) \backslash \overline{B_l(0)}$, be the reflected point. If $y_i \in \partial M$ then $d_{\overline{g}}(y_i, \overline{p}) = d_{\overline{g}}(y_i, p)$, where $d_{\overline{g}}$ means $\mathrm{dist}_{\overline{g}}$. If $y_i \notin \partial M$, then in y coordinates the boundary is given by a graph $y^n = F(y')$, but $F(y') \to 0$ as $i \to 0$ (see (6.2) and lemma 6.1), which then implies $d_{\overline{g}}(y_i, \overline{p}) = d_{\overline{g}}(y_i, p) + o(1)$. Therefore

$$\overline{v}(\overline{p}) = v(p) \leq CU(p) = C(1 + d_{\overline{g}}(y_i, p)^2)^{\frac{2-n}{2}} \leq C_1(1 + d_{\overline{g}}(y_i, \overline{p})^2)^{\frac{2-n}{2}} = C_1 U(\overline{p}),$$

as desired.

Finally, the third difference with [12] is that there, the scalar curvature satisfies $R_g = O(r^2)$, which comes from the Taylor formula and properties of conformal normal coordinates. Here, since $R_{\overline{g}}$ is $C^{0,\alpha}$ only, we avoid the Taylor expansion. Without $R_{\overline{g}} = O(r^2)$ the proof in [12] yields a weaker estimate, but since we only need $|v-U|(y) \leq C\varepsilon$, the hypothesis $R_{\overline{g}} = O(r^2)$ is not necessary. In [12] the better estimate $|v-U|(y) \leq C\varepsilon^s$, with $s > 1$, is established. □

Remark 7.2. Observe that as in [12], the proof of proposition 7.1 produces the estimate $\delta_i \leq C\varepsilon_i$.

In the proof of the next proposition, we retain the notation for the reflected quantities that appears in the proof of proposition 7.1.

Proposition 7.3. *Under the same hypotheses of proposition 7.1, there exists a constant C_0, independent of i, such that*

$$\| v_i - U \|_{C^{2,\alpha}(\widetilde{B}_{\frac{l_i}{4}}(0))} \leq C_0 \varepsilon_i,$$

where $l_i = \sigma\varepsilon_i^{-1}$.

Proof. It is sufficient to establish the desired estimate for $w_i = \overline{v}_i - U$. We have

$$L_{\tilde{g}_i} w_i + b_i w_i = Q_i \text{ in } \widetilde{B}_{l_i}(0)$$

with

$$\begin{aligned} b_i(y) &= K\overline{f}^{-\delta_i} \frac{\overline{v}_i^{p_i} - U^{p_i}}{\overline{v}_i - U}(y), \\ Q_i(y) &= \Big(c(n)\varepsilon_i^2 R_{g_i}(\varepsilon_i y) U(y) + K(U^{\frac{n+2}{n-2}} - \overline{f}^{-\delta_i} U^{p_i}) \\ &\quad + \varepsilon_i^{N+1} O(|y|^N)|y|(1+|y|^2)^{-\frac{n}{2}} \Big). \end{aligned}$$

Use (7) to find $|b_i(y)| \leq c(1+|y|)^{-4}$. Then the representation formula gives, for any and $|y| \leq \frac{l_i}{4}$,

$$w_i(y) = \int_{\widetilde{B}_{l_i}(0)} G_i(y,z)(b_i w_i - Q_i)(z)dz - \int_{\partial \widetilde{B}_{l_i}(0)} \frac{\partial G_i}{\partial \nu_{\tilde{g}_i}}(y,z) w_i(z) dS(z), \quad (7.5)$$

where G_i is the Green's function for the conformal Laplacian with Dirichlet boundary condition. The proof is now similar to standard estimates for the Newtonian potential, and therefore we will only indicate the main steps (see for example [37]).

First notice that unlike the Newtonian potential case, there is a boundary integral in the representation formula (7.5). Nevertheless, this boundary integral is easily estimated using standard properties of the Green's function and $\overline{v}_i \leq CU$, since the singularities occur within the radius $\frac{l_i}{4}$.

For the interior integral, write $\gamma_i = b_i w_i - Q_i$. This quantity plays the role of the inhomogeneous term in potential theory. Therefore standard potential theoretic arguments yield

$$[D^2 w_i]_{\alpha, \widetilde{B}_{\frac{l_i}{4}}(0)} \leq \frac{C}{l_i^\alpha}\Big(\parallel \gamma_i \parallel_{C^0(\widetilde{B}_{\frac{l_i}{2}}(0))} + l_i^\alpha [\gamma_i]_{\alpha, \widetilde{B}_{\frac{l_i}{2}}(0)} \Big) \quad (7.6)$$

where $[\cdot]_{\alpha,\Omega}$ is the Hölder semi-norm on Ω. Next, observe that by interpolation

$$[\gamma_i]_{\alpha, \widetilde{B}_{\frac{l_i}{2}}(0)} \leq C\Big([b_i]_{\alpha, \widetilde{B}_{\frac{l_i}{2}}(0)} [w_i]_{\alpha, \widetilde{B}_{\frac{l_i}{2}}(0)} + [Q_i]_{\alpha, \widetilde{B}_{\frac{l_i}{2}}(0)} \Big). \quad (7.7)$$

In order to estimate $[w_i]_{\alpha, \widetilde{B}_{\frac{l_i}{2}}(0)}$ the representation formula (7.5) may again be employed along with standard properties of G_i and proposition 7.1. However, control of the boundary term relies on y staying away from the boundary, that is why we choose an estimate on $\widetilde{B}_{\frac{l_i}{2}}(0)$ (giving then a final estimate on $\widetilde{B}_{\frac{l_i}{4}}(0)$).

Moreover, using remark 7.2 and $U^{\frac{n+2}{n-2}} - \overline{f}^{-\delta_i} U^{p_i} = U^{\frac{n+2}{n-2}} O((|\log f| + |\log U|)\delta_i)$, it follows that

$$\begin{aligned}
[Q_i]_{\alpha, \widetilde{B}_{\frac{l_i}{2}}(0)} &\leq C\Big(\varepsilon_i^2 [R_{g_i}(\varepsilon_i y) U(y)]_{\alpha, \widetilde{B}_{\frac{l_i}{2}}(0)} + [U^{\frac{n+2}{n-2}} - \overline{f}^{-\delta_i} U^{p_i}]_{\alpha, \widetilde{B}_{\frac{l_i}{2}}(0)} \\
&\quad + \varepsilon_i [\varepsilon_i^N O(|y|^N)|y|(1+|y|^2)^{-\frac{n}{2}}]_{\alpha, \widetilde{B}_{\frac{l_i}{2}}(0)}\Big) \\
&\leq C\varepsilon_i.
\end{aligned} \tag{7.8}$$

Finally, the term $\| \gamma_i \|_{C^0(\widetilde{B}_{\frac{l_i}{2}}(0))}$ is estimated in a similar manner

$$\| \gamma_i \|_{C^0(\widetilde{B}_{\frac{l_i}{2}}(0))} \leq C\varepsilon_i.$$

Combining this with (7.6), (7.7) and (7.8) yields $[D^2 w_i]_{\alpha, \widetilde{B}_{\frac{l_i}{4}}(0)} \leq C\varepsilon_i$. The remaining lower order terms of the $C^{2,\alpha}$ norm may be estimated in an analogous way. □

The analogous of the following result is already known for scalar-flat manifolds [32].

Theorem 7.4. *Suppose $x_i \to \bar{x}$ is an isolated simple blow-up point. Then in boundary conformal normal coordinates at x_i, for all i sufficiently large and possibly after passing to a subsequence, we have $x_i \in \partial M$.*

Proof. The proof is by contradiction. Therefore assume that $x_i \in \partial M$ occurs only for finitely many i. Hence passing to a subsequence, still denoted x_i, we can assume that

$$x_i \in \overset{\circ}{M} \quad \text{for all } i. \tag{7.9}$$

Take boundary conformal normal coordinates at x_i (see corollary 3.10), rescale all quantities to y coordinates as explained at the beginning of the text, and denote by $\tilde{y}_i \in \partial M$ the closest point to y_i, where y_i is identified with the origin. The closure of the ball of radius $|\tilde{y}_i|$ will be denoted by $\overline{B_{|\tilde{y}_i|}(0)}$. Furthermore, for any domain Ω, denote by $[\cdot]_{1+\alpha,\Omega}$ the $C^{1,\alpha}$ Hölder semi-norm, and by $[\cdot]_{1,\Omega}$ the C^1 Hölder semi-norm.

Let $w_i = v_i - U$, then

$$\frac{|\partial_n(v_i - U)(0) - \partial_n(v_i - U)(\tilde{y}_i)|}{|\tilde{y}_i - 0|^\beta} \leq [w_i]_{1+\beta,\overline{B_{|\tilde{y}_i|}(0)}}. \tag{7.10}$$

As explained in chapter 3, the coordinates may be arranged such that $\left.\frac{\partial}{\partial \nu_{\tilde{y}_i}}\right|_{\tilde{y}_i} = g^{nn}\,\partial_n|_{\tilde{y}_i}$. Observe that the boundary condition for v_i implies that $\partial_n v_i(\tilde{y}_i) = 0$, since the mean curvature vanishes. Notice also that we have $\nabla v_i(0) = \nabla U(0) = 0$. On the other hand a direct calculation gives

$$\partial_n U(\tilde{y}_i) = (2-n)(1+|\tilde{y}_i|^2)^{-\frac{n}{2}}\tilde{y}_i^n. \tag{7.11}$$

Hence (7.10) becomes

$$|\tilde{y}_i^n| = |\tilde{y}_i| \leq \frac{1}{n-2}(1+|\tilde{y}_i|^2)^{\frac{n}{2}}|\tilde{y}_i|^\beta[w_i]_{1+\beta,\overline{B_{|\tilde{y}_i|}(0)}},$$

since $\tilde{y}_i = (0,\ldots,0,\tilde{y}_i^n)$. By (6.2), $\frac{1}{n-2}(1+|\tilde{y}_i|^2)^{\frac{n}{2}} \leq C_1$ for a constant C_1 independent of i, so

$$|\tilde{y}_i| \leq C_1|\tilde{y}_i|^\beta[w_i]_{1+\beta,\overline{B_{|\tilde{y}_i|}(0)}}. \tag{7.12}$$

By proposition 7.3, w_i converges to zero in $C^{2,\alpha}$, so there exists a small $r > 0$, independent of i, such that the Taylor formula for w_i holds in $B_r(0)$ for all i. By (6.2) we can assume that $\overline{B_{|\tilde{y}_i|}(0)} \subset B_r(0)$. Therefore for any $y \in \overline{B_{|\tilde{y}_i|}(0)}$,

$$\partial_k w_i(y) = \partial_k w_i(0) + \mathcal{R}_l(y) y^l = \mathcal{R}_l(y) y^l,$$

where we used $\nabla w_i(0) = 0$. The remainder term satisfies, for each $l = 1, \ldots, n$,

$$|\mathcal{R}_l(y)| \leq \sup_{z \in \overline{B_{|\tilde{y}_i|}(0)}} |\nabla^2 w(z)| \leq \| w_i \|_{C^{2,\alpha}(\overline{B_{|\tilde{y}_i|}(0)})} \leq C_0 \varepsilon_i,$$

where proposition 7.3 has been used. Hence $|\partial_k w_i(y)| \leq |\mathcal{R}_l(y) y^l| \leq n C_0 \varepsilon_i |y|$, and therefore

$$[w_i]_{1, \overline{B_{|\tilde{y}_i|}(0)}} \leq n C_0 \varepsilon_i |\tilde{y}_i|. \tag{7.13}$$

Let Ω be a convex domain. The following inequality is standard (see e.g. [37])

$$[u]_{1+\beta,\Omega} \leq \Lambda \rho^{\alpha-\beta} [u]_{1+\alpha,\Omega} + \Lambda \rho^{-\beta} [u]_{1,\Omega} \ , \tag{7.14}$$

where the constant Λ depends only on the dimension, $0 < \beta < \alpha < 1$, and $\rho > 0$ is any positive number. Also, using the mean value inequality, there is a constant A depending only on the dimension, such that $|\partial_i u(p) - \partial_i u(q)| \leq A|p-q|[u]_{2,\Omega} = A|p-q|^{\alpha}|p-q|^{1-\alpha}[u]_{2,\Omega}$. From this it follows that

$$[u]_{1+\alpha,\Omega} \leq A \operatorname{diam}(\Omega)^{1-\alpha} [u]_{2,\Omega}. \tag{7.15}$$

Because the constants C_0, C_1, A and Λ do not depend on i, we can, with the help of (6.2) and the definition of ε_i, choose i so large that

$$|\tilde{y}_i| < \frac{1}{2}, \tag{7.16}$$

$$\varepsilon_i < 1, \tag{7.17}$$

$$\max\left\{C_0 C_1 \Lambda A,\ nC_0 C_1 \Lambda\right\} \varepsilon_i^{\frac{\alpha^2}{p}} < \frac{1}{2}, \tag{7.18}$$

$$C_0 C_1 \varepsilon_i^{1-\alpha} < 1, \tag{7.19}$$

where $p > 1$ is a large number chosen such that

$$\alpha \frac{p+1}{p} < 1. \tag{7.20}$$

This is possible since $\alpha < 1$; notice that p does not depend on i.

Now fix an $i_0 = i_0(n, C_0, C_1, A, \Lambda)$ such that (7.16)-(7.19) hold. From (7.15) we have

$$[w_{i_0}]_{1+\alpha, \overline{B_{|\tilde{y}_{i_0}|}(0)}} \leq A [w_{i_0}]_{2, \overline{B_{|\tilde{y}_{i_0}|}(0)}}. \tag{7.21}$$

Moreover the constants C_0, C_1, A and Λ do not depend on the choice of β, as can be seen from the derivation of inequalities (7.12), (7.14), (7.15), and the proof of proposition 7.1. Therefore the inequalities (7.12)-(7.21) hold for any β such that $0 < \beta < \alpha$.

We are now in a position to prove the theorem. It will show by induction that

$$|\tilde{y}_{i_0}| \leq \varepsilon_{i_0}^{k\alpha} \tag{7.22}$$

for all $k = 0, 1, 2, 3, \ldots$. Since $\varepsilon_{i_0} < 1$ this would imply $|\tilde{y}_{i_0}| = 0$ so that $x_{i_0} \in \partial M$, contradicting (7.9).

For $k = 0$ (7.22) is true by (7.16). For $k = 1$, recall that $\partial_n v_{i_0}(\tilde{y}_{i_0}) = 0$, and observe that

$$|\partial_n U(\tilde{y}_{i_0})| = |\partial_n (v_i - U)(\tilde{y}_{i_0})| \leq [w]_{1, \overline{B_{|\tilde{y}_{i_0}|}(0)}}.$$

Then (7.11), proposition 7.1 and (7.19) give

$$|\tilde{y}_{i_0}| \leq C_1 [w_{i_0}]_{1, \overline{B_{|\tilde{y}_{i_0}|}(0)}} \leq C_0 C_1 \varepsilon_{i_0} = C_0 C_1 \varepsilon_{i_0}^{1-\alpha} \varepsilon_{i_0}^{\alpha} < \varepsilon_{i_0}^{\alpha}.$$

So assume that (7.22) holds for some $k \geq 1$. Combining (7.12) and (7.14) gives

$$\begin{aligned} |\tilde{y}_{i_0}| \leq C_1 |\tilde{y}_{i_0}|^{\beta} [w_{i_0}]_{1+\beta, \overline{B_{|\tilde{y}_{i_0}|}(0)}} \leq C_1 \Lambda |\tilde{y}_{i_0}|^{\beta} \Big(& \rho^{\alpha - \beta} [w_{i_0}]_{1+\alpha, \overline{B_{|\tilde{y}_{i_0}|}(0)}} \\ & + \rho^{-\beta} [w_{i_0}]_{1, \overline{B_{|\tilde{y}_{i_0}|}(0)}} \Big). \end{aligned} \tag{7.23}$$

Choose $\beta = \frac{\alpha}{pk}$ (which is less than α by the choice of p). If we also choose $\rho = \varepsilon_{i_0}^{k}$ then (7.23) becomes

$$|\tilde{y}_{i_0}| \leq C_1 \Lambda |\tilde{y}_{i_0}|^{\frac{\alpha}{pk}} \Big(\varepsilon_{i_0}^{k\alpha - \frac{\alpha}{p}} [w_{i_0}]_{1+\alpha, \overline{B_{|\tilde{y}_{i_0}|}(0)}} + \varepsilon_{i_0}^{-\frac{\alpha}{p}} [w_{i_0}]_{1, \overline{B_{|\tilde{y}_{i_0}|}(0)}} \Big).$$

By (7.13), (7.21), proposition 7.1, the induction hypothesis (7.22) and the fact that

$$[w_{i_0}]_{2,\overline{B_{|\tilde{y}_{i_0}|}(0)}} \leq \| w_{i_0} \|_{C^{2,\alpha}(\overline{B_{|\tilde{y}_{i_0}|}(0)})},$$

we obtain

$$\begin{aligned}
|\tilde{y}_{i_0}| &\leq \max\left\{C_0C_1\Lambda A,\ nC_0C_1\Lambda\right\}\varepsilon_{i_0}^{\frac{\alpha^2}{p}}\left(\varepsilon_{i_0}^{k\alpha+1-\frac{\alpha}{p}} + \varepsilon_{i_0}^{-\frac{\alpha}{p}+1+k\alpha}\right) \\
&= 2\max\left\{C_0C_1NA,\ nC_0C_1N\right\}\varepsilon_{i_0}^{\frac{\alpha^2}{p}}\varepsilon_{i_0}^{(k+1)\alpha}\varepsilon_{i_0}^{1-\alpha-\frac{\alpha}{p}} \\
&\leq \varepsilon_{i_0}^{(k+1)\alpha}\varepsilon_{i_0}^{1-\alpha-\frac{\alpha}{p}}.
\end{aligned}$$

where (7.18) has been employed. Finally, $\varepsilon_{i_0}^{1-\alpha-\frac{\alpha}{p}} < 1$ by (7.17) and (7.20). □

Chapter 8

Symmetry estimates

In this chapter we derive sharp estimates for the behavior of solutions u_i in the neighborhood of an isolated simple blow-up point. The proofs are an adaptation of the results of [17] and we will often refer the reader to it for details.

Throughout this chapter, let $x_i \to \bar{x} \in \partial M$ be an isolated simple blow-up point. By theorem 7.4 we can assume that $x_i \in \partial M$. We will be using boundary conformal normal coordinates at x_i (see proposition 3.5) and rescale all the quantities to y-coordinates as explained at the beginning of the text. Notice that because in boundary conformal normal coordinates we have $\kappa_g = 0$ in the neighborhood of the origin, the boundary condition becomes a Neumann condition. Moreover, since the boundary is umbilic we obtain that it is totally geodesic in the neighborhood of the origin.

Also, by (4.3), proposition 5.1 gives that for $|y| \leq \sigma\varepsilon_i^{-1}$ we have, with

$\tilde{z}_i = \tilde{z}_{\varepsilon_i}$,

$$\begin{cases} \Delta \tilde{z}_i + n(n+2)U^{\frac{4}{n-2}}\tilde{z}_i = c(n)\sum_{k=4}^{n-4}\partial_i\partial_j\tilde{H}_{ij}^{(k)}U, & \text{for } |y| \le \sigma\varepsilon_i^{-1}, \\ \frac{\partial \tilde{z}_i}{\partial \nu_{\tilde{g}_i}} = \sum_{l=1}^{n-1} g^{nl}\partial_l\tilde{z}_i = \varepsilon_i^N O(|y'|^N(1+|y'|)^{1-n}), & \text{on } \partial M \end{cases} \tag{8.1}$$

where (2.9) has also been used. Notice that since $\partial_n U(y',0) = 0$, in these coordinates U also satisfies the boundary condition

$$\frac{\partial U}{\partial \nu_{\tilde{g}_i}} = \sum_{l=1}^{n-1} g^{nl}\partial_l U = \varepsilon_i^N O(|y'|^N(1+|y'|)^{1-n}) \text{ on } \partial M. \tag{8.2}$$

Proposition 8.1. *Suppose $x_i \to \bar{x}$ is an isolated simple blow-up point. Then in boundary conformal normal coordinates at x_i, there exist constants $\sigma, C > 0$ such that*

$$|v_i - U - \tilde{z}_i| \le C \max_{2\le k\le d-1}\{\varepsilon_i^{2k}|H^{(k)}|^2(x_i), \varepsilon_i^{n-3}, \delta_i\}$$

for every $|y| \le \sigma M_i^{\frac{p_i-1}{2}}$.

Proof. Put $\Lambda_i = \max_{|y|<l_i}|v_i - U - \tilde{z}_i| = |v_i - U - \tilde{z}_i|(y_i)$. Then as in the boundaryless case we get a stronger inequality if there exists a constant c such that $|y_i| \ge cl_i$ for every i. In fact, using that $\bar{x}$ is an isolated simple blow-up point we get the inequality $v \le CU \le C|y|^{2-n}$, and using estimate (2.9) we get $\Lambda_i = |v_i - U - \tilde{z}_i|(y_i) \le C|y_i|^{2-n} \le \varepsilon_i^{n-2}$. Hence we can assume $|y_i| \le \frac{l_i}{2}$.

If the proposition is false we have

$$\frac{1}{\Lambda_i}\max_{2\le k\le d-1}\{\varepsilon_i^{2k}|H^{(k)}|^2(x_i)\} \to 0, \quad \frac{1}{\Lambda_i}\varepsilon_i^{n-3} \to 0, \quad \frac{1}{\Lambda_i}\delta_i \to 0. \tag{8.3}$$

Define

$$w_i(y) = \frac{1}{\Lambda_i}(v_i - U - \tilde{z}_i)(y).$$

Then $|w_i(y)| \le 1$, and

$$\begin{cases} L_{\tilde{g}_i} w_i + b_i w_i = Q_i & \text{in } B_{l_i}(0) \\ w_i = O(\Lambda^{-1}\varepsilon_i^{n-2}) & \text{on } \partial^+ B_{l_i}(0) \\ \frac{\partial w_i}{\partial \nu_{\tilde{g}_i}} = \Lambda^{-1}\varepsilon_i^N O(|y'|^N(1+|y'|)^{1-n}) & \text{on } \partial' B_{l_i}(0), \end{cases} \tag{8.4}$$

where (8.1), (8.2) and the boundary condition for v_i have been used; Q_i and b_i are as in the boundaryless case

$$\begin{aligned} b_i(y) &= K\tilde{f}^{-\delta_i}\frac{v_i^{p_i} - (U+\tilde{z}_i)^{p_i}}{v_i - U - \tilde{z}_i}(y), \\ Q_i(y) &= \frac{1}{\Lambda_i}\Big\{c(n)\varepsilon_i^2\big(R_{g_i} - \sum_{\ell=2}^{n-6}(\partial_j\partial_k H_{jk})^{(\ell)}\big)(\varepsilon_i y)U(y) + (\Delta - L_{\tilde{g}_i})(\tilde{z}_i) \\ &\quad + O(|\tilde{z}_i|^2 U^{\frac{6-n}{n-2}}) + K((U+\tilde{z}_i)^{\frac{n+2}{n-2}} - \tilde{f}^{-\delta_i}(U+\tilde{z})^{p_i}) \\ &\quad + M_i^{-(1+N)\frac{p_i-1}{2}} O(|y|^N)|y|(1+|y|^2)^{-\frac{n}{2}}\Big\}, \end{aligned}$$

and they satisfy the estimates (see [17])

$$|b_i(y)| \le c(1+|y|)^{-4}$$

and

$$|Q_i(y)| \le C\frac{1}{\Lambda_i}\Big\{\max_{2\le k\le d-1}\{\varepsilon_i^{2k}|H^{(k)}|^2(x_i)\}(1+|y|)^{2d-2-n}$$

$$+ \varepsilon_i^{n-3}(1+|y|)^{-3} + M_i^{-(1+N)\frac{p_i-1}{2}} O(|y|^N)|y|(1+|y|^2)^{-\frac{n}{2}}$$
$$+ \delta_i(|\log(U+\tilde{z}_i)| + |\log \tilde{f}_i|)(1+|y|)^{-n-2}\Big\}.$$

Let G_i be the Green's function for the conformal Laplacian with boundary condition $G_i = 0$ on $\partial^+ B_{l_i}(0)$ and $B_{\tilde{g}_i} G_i = \frac{\partial G_i}{\partial \nu_{\tilde{g}_i}} = 0$ on $\partial' B_{l_i}(0)$. The representation formula then gives

$$w_i(y) = \int_{B_{l_i}(0)} G_i(y,\eta)(b_i w_i - Q_i)(\eta)d\eta - \int_{\partial^+ B_{l_i}(0)} w_i(\eta)\frac{\partial G_i(y,\eta)}{\partial \nu_{\tilde{g}_i}} dS(\eta)$$
$$+ \int_{\partial' B_{l_i}(0)} G_i(y,\eta)\frac{\partial w_i(\eta)}{\partial \nu_{\tilde{g}_i}} dS(\eta)$$

for $|y| \leq \frac{l_i}{2}$. The first two integrals are estimated as in the boundaryless case (see [17]). For the third one we use (8.4) to find

$$\Big|\int_{\partial' B_{l_i}(0)} G_i(y,\eta)\frac{\partial w_i(\eta)}{\partial \nu_{\tilde{g}_i}} d\eta'\Big| \leq C\varepsilon_i^{n-2}. \tag{8.5}$$

Hence,

$$|w_i(y)| \leq C\Big((1+|y|)^{-2} + \frac{1}{\Lambda_i}\max_{2\leq k\leq d-1}\{\varepsilon_i^{2k}|H^{(k)}|^2(x_i), \varepsilon_i^{n-3}, \delta_i\}\Big). \tag{8.6}$$

It then follows from (8.3), (8.6), and standard elliptic estimates that w_i is bounded in C^2_{loc} and has a subsequence, still denoted w_i, converging to a limit

w_∞, which satisfies

$$\begin{cases} \Delta w_\infty + n(n+2)U^{\frac{4}{n-2}} w_\infty = 0 & \text{on } \mathbb{R}^n_+ \\ \frac{\partial w_\infty}{\partial y^n} = 0 & \text{on } \mathbb{R}^{n-1} \\ \lim_{|y|\to\infty} w_\infty(y) = 0. & \end{cases}$$

Note that for the boundary condition we used that $\Lambda^{-1}\varepsilon_i^N |y'|^N (1+|y'|)^{1-n} \leq C\varepsilon_i^{n-3}$ for $|y'| \leq \sigma\varepsilon_i^{-1}$. Lemma A.5 then gives

$$w = c_0\Big(\frac{n-2}{2}U(y) + y \cdot \nabla U\Big) + \sum_{j=1}^{n-1} c_j \partial_j U.$$

However $w_i(0) = |\nabla w_i|(0) = 0$ implies $w_\infty(0) = |\nabla w_\infty|(0) = 0$, from which we conclude that $w_\infty \equiv 0$. It then follows that $|y_i| \to \infty$. This combined with (8.3) contradicts (8.6), as $w_i(y_i) = 1$. □

The proofs of the next two results are similar to those in [17], making the necessary adaptations to the boundary case with ideas described in proposition 8.1.

Proposition 8.2. *Under the hypotheses of proposition 8.1,*

$$\delta_i \leq C \max_{2\leq k\leq d-1} \{\varepsilon_i^{2k} |H^{(k)}|^2(x_i), \varepsilon_i^{n-3}\}$$

for every $|y| \leq \sigma M_i^{\frac{p_i-1}{2}}$.

Proof. If the proposition is false we have

$$\frac{1}{\delta_i} \max_{2 \le k \le d-1} \{\varepsilon_i^{2k} |H^{(k)}|^2(x_i)\} \to 0, \quad \frac{1}{\delta_i} \varepsilon_i^{n-3} \to 0. \tag{8.7}$$

Hence from proposition 8.1,

$$|v_i - U - \tilde{z}_i|(y) \le C\delta_i.$$

Define

$$w_i = \frac{1}{\delta_i}(v_i - U - \tilde{z}_i),$$

and argue as in proposition 8.1, with δ_i replacing Λ_i, to obtain $w_i \to w_\infty$ in C^2_{loc}, where $\partial_n w_\infty = 0$ on $\mathbb{R}^{n-1}$. Define $\Psi(y) = \frac{n-2}{2}U(y) + y^j \partial_j U(y)$. Now we argue as in [17], except possibly for the extra boundary terms

$$\int_{\partial' B_{\frac{l_i}{2}}(0)} \Psi \frac{\partial w_i}{\partial \nu_{\tilde{g}_i}} \quad \text{and} \quad \int_{\partial' B_{\frac{l_i}{2}}(0)} w_i \frac{\partial \Psi}{\partial \nu_{\tilde{g}_i}}.$$

But as before, $\frac{\partial w_i}{\partial \nu_{\tilde{g}_i}} = \varepsilon_i^N O(|y'|^N (1+|y'|)^{1-n})$, and a direct computation gives $\frac{\partial \Psi}{\partial \nu_{\tilde{g}_i}} = \varepsilon_i^N O(|y'|^N (1+|y'|)^{1-n})$, which is enough to handle the boundary integrals as in proposition 8.1. □

Proposition 8.3. *Under the hypotheses of proposition 8.1,*

$$|\nabla^m (v_i - U - \tilde{z}_i)|(y) \le C \sum_{k=2}^{d-1} \varepsilon_i^{2k} |H^{(k)}|^2(x_i)(1+|y|)^{2k+2-n-m} + \varepsilon_i^{n-3}(1+|y|)^{-1-m}$$

for every $|y| \le \sigma\varepsilon^{-1}$, $m = 0, 1, 2$.

Proof. Arguing similarly to [17] with the necessary modifications as in propositions 8.1 and 8.2, we obtain the result with $m = 0$. To obtain the result for the derivatives, we invoke standard elliptic theory, which gives the estimate provided that we can bound the $C^{1,\alpha}$ norm of $\partial_{\nu_{\tilde{g}_i}}(v_i - U - \tilde{z}_i)$ on the boundary. Since $\partial_{\nu_{\tilde{g}_i}} v_i = 0$ and $\partial_n \tilde{z}_i|_{y^n=0} = 0 = \partial_n U|_{y^n=0}$, it is enough to show that

$$\| \sum_{l=1}^{n-1} g^{nl} \partial_l (\tilde{z}_i + U) \|_{C^{1,\alpha}(\partial' B_{l_i}(0))} \leq C\varepsilon_i^{n-3}. \tag{8.8}$$

From (2.9), (5.1), properties of boundary conformal normal coordinates (in particular corollary 3.7) and the explicit form of U we have

$$\Big| \sum_{l=1}^{n-1} g^{nl} \partial_l (\tilde{z}_i + U)(y', 0) \Big| \leq C\varepsilon_i^N |y'|^N (1 + |y'|)^{1-n},$$

which is bounded by $C\varepsilon_i^{n-3}$ for $|y'| \leq \sigma\varepsilon_i^{-1}$.

Differentiating $\sum_{l=1}^{n-1} g^{nl} \partial_l (\tilde{z}_i + U)$ with respect to y^k, $k \leq n-1$, using again (2.9), (5.1), and properties of boundary conformal normal coordinates yields

$$\Big| \partial_k \big(\sum_{l=1}^{n-1} g^{nl} \partial_l (\tilde{z}_i + U) \big)(y', 0) \Big| \leq C\varepsilon_i^{n-3} \quad \text{for} \quad |y'| \leq \sigma\varepsilon_i^{-1}.$$

Differentiating again and repeating the argument gives (8.8). Now the pointwise estimate follows by standard arguments. □

Chapter 9

Weyl vanishing

In this chapter we will work mostly in x-coordinates and take boundary conformal normal coordinates at x_i. In these coordinates, estimate (2.8) and the estimate of proposition 8.3 become, for $|x| \leq \sigma$,

$$|\nabla^m z_i(x)| \leq \varepsilon_i^{\frac{n-2}{2}} \sum_{|\alpha|=4}^{n-4} \sum_{jl} |h_{jl,\alpha}|(\varepsilon_i + |x|)^{|\alpha|+2-n-m} \tag{9.1}$$

$$\begin{aligned}|\nabla^m (u_i - u_{\varepsilon_i} - z_i)(x)| \leq C\varepsilon_i^{\frac{n-2}{2}} \sum_{k=2}^{d-1} |H^{(k)}|^2(x_i)(\varepsilon_i + |x|)^{2k+2-n-m} \\ + \varepsilon_i^{\frac{n-2}{2}}(\varepsilon_i + |x|)^{-m-1},\end{aligned} \tag{9.2}$$

where both z_i and the sum with $|H^{(k)}|^2(x_i)$ appear only when $n \geq 6$, and

$$z_{\varepsilon_i} = z_i(x) = \varepsilon^{\frac{2-n}{2}} \tilde{z}_i(\varepsilon_i^{-1} x)$$
$$u_{\varepsilon_i}(x) = \varepsilon_i^{\frac{n-2}{2}} (\varepsilon_i^2 + |x|^2)^{\frac{2-n}{2}}.$$

Throughout this chapter it will be assumed that (M^n, g) is a Riemannian manifold of dimension $3 \leq n \leq 24$ with umbilic boundary. The index i will be dropped from all quantities in several estimates below. Note also that by theorem 7.4 we can assume that $x_i \in \partial M$, therefore the boundary is given by $\partial M = \{x^n = 0\}$. We will use the notation $B_\rho^+ = \{x \in B_\rho(x_i) \bigm| x^n \geq 0\}$, where $\rho \leq \sigma$ — of course, B_ρ^+ is the same as $B_\rho(0)$, but the first notation will be emphasized since it better suits the Pohozaev identity. Furthermore, the unit normal will be denoted by $\nu = \nu_g = \nu_{g_i}$ when no confusion arises, and ν_δ will denote the Euclidean normal.

We can now state one of the main estimates of the paper.

Proposition 9.1. *Suppose* $6 \leq n \leq 24$ *and that* $x_i \to \overline{x} \in \partial M$ *is an isolated simple blow-up point. Then*

$$\sum_{|\alpha|=2}^{d} \sum_{i,j=1}^{n} |h_{ij,\alpha}|^2 \varepsilon^{2|\alpha|} |\log \varepsilon|^{\theta_{|\alpha|}} \leq C\varepsilon^{n-2},$$

where $\theta_k = 1$ *if* $k = \frac{n-2}{2}$ *and* $\theta_k = 0$ *otherwise.*

Before giving a proof of proposition 9.1, some consequences are derived, in particular the Weyl vanishing theorem.

Theorem 9.2. *(Weyl vanishing) Let* $x_i \to \bar{x}$ *be an isolated simple blow-up point and* $6 \leq n \leq 24$*, then*

$$|\nabla_{g_i}^l W_g|^2(x_i) \leq C\varepsilon_i^{n-6-2l} |\log \varepsilon_i|^{-\theta_{l+2}},$$

for every $0 \leq l \leq \left[\frac{n-6}{2}\right]$*, where* $\theta_k = 1$ *if* $k = \frac{n-2}{2}$ *and* $\theta_k = 0$ *otherwise. In particular* $|\nabla_g^l W_g|^2(\bar{x}) = 0$ *for* $0 \leq l \leq \left[\frac{n-6}{2}\right]$.

Proof. Proposition 9.1 gives the same estimate as in the boundaryless case, the argument then is similar (see [17]). □

Corollary 9.3. *Under the same hypotheses of the Weyl vanishing theorem,*

$$|\nabla^m(v_i - U - \tilde{z}_i)(y)| \leq C\varepsilon_i^{n-3}(1+|y|)^{-m-1}$$

or, in x-coordinates

$$|\nabla^m(u_i - u_{\varepsilon_i} - z_{\varepsilon_i})(x)| \leq C\varepsilon_i^{\frac{n-2}{2}}(\varepsilon+|x|)^{-m-1}.$$

Proof. This is straightforward from proposition 8.3 and theorem 9.2. □

We now proceed with the proof of proposition 9.1. The proof will involve an application of the Pohozaev identity (A.6) in a half ball B_ρ^+.

Write $\phi = \frac{n-2}{2}u + x^k\partial_k u$ and $\phi_\varepsilon = \frac{n-2}{2}u_\varepsilon + x^k\partial_k u_\varepsilon$. In the proofs below extensive use will be made of the inequalities $|\nabla^m u| \leq C\varepsilon^{\frac{n-2}{2}}|x|^{2-n-m}$ and $|\nabla^m \phi| \leq C\varepsilon^{\frac{n-2}{2}}|x|^{2-n-m}$, which follow from proposition 6.5.

Proof of proposition 9.1: First it will be shown that there exists a constant C such that

$$\begin{aligned} &C\Big(\sum_{|\alpha|=2}^{d}\sum_{ij=1}^{n}|h_{ij,\alpha}|^2\varepsilon^{2|\alpha|+1} + \sum_{|\alpha'|=0}^{N}\sum_{i,j=1}^{n-1}|T_{ij,\alpha'}|\varepsilon^{|\alpha'|+1} + \varepsilon^{n-2}\Big) \qquad (9.3)\\ &\geq -\int_{\partial' B_\rho^+}\phi_\varepsilon H_{in}\partial_i z_\varepsilon dx + \int_{B_\rho^+} c(n)\phi_\varepsilon u_\varepsilon \partial_{ij}h_{ij}dx \\ &+\int_{B_\rho^+} c(n)(\phi_\varepsilon z_\varepsilon + u_\varepsilon(\frac{n-2}{2}z_\varepsilon + x^k\partial_k z_\varepsilon))\partial_{ij}h_{ij}dx \end{aligned}$$

$$+\int_{B_\rho^+} c(n)\phi_\varepsilon u_\varepsilon(-\partial_j(H_{ij}\partial_l H_{il}) + \frac{1}{2}\partial_j H_{ij}\partial_l H_{il} - \frac{1}{4}\partial_l H_{ij}\partial_l H_{ij})dx.$$

where by α' we mean derivatives along x' only.

Start with the Pohozaev identity (A.6). On its left hand-side the integrals over the hemisphere $S_+^{n-1}(\rho)$ are of order ε^{n-2} and the boundary terms with $x^k\nu_\delta^k$ vanish on $\partial' B_\rho^+$, so the remaining term on the left hand side of (A.6) is

$$\int_{\partial' B_\rho^+}(\frac{n-2}{2}u + x^k\partial_k u)\frac{\partial u}{\partial \nu_\delta}dx'.$$

Since $\partial_\nu u = 0$ and g^{nn} is bounded away from zero near the origin

$$\frac{\partial u}{\partial \nu_\delta} = -\partial_n u = \frac{1}{g^{nn}}\sum_{l=1}^{n-1} g^{nl}\partial_l u,$$

therefore

$$\Big|\int_{\partial B_\rho^+}(\frac{n-2}{2}u + x^k\partial_k u)\frac{\partial u}{\partial \nu_\delta}dx'\Big| \leq$$
$$C\Big|\int_{\partial B_\rho^+}(\frac{n-2}{2}u + x^k\partial_k u)\sum_{l=1}^{n-1} g^{nl}\partial_l(u-u_\varepsilon-z_\varepsilon)dx'\Big|$$
$$+C\Big|\int_{\partial B_\rho^+}(\frac{n-2}{2}u + x^k\partial_k u)\sum_{l=1}^{n-1} g^{nl}\partial_l(u_\varepsilon+z_\varepsilon)dx'\Big|.$$

Using (4.2), (9.1), (9.2), and theorem 3.4, we find that the above integrals are bounded by $C\varepsilon^{n-2}$ and terms involving the umbilicity tensor. Now (9.3) follows from (see [17])

$$|R_g - \partial_{ij}h_{ij} + \partial_j(H_{ij}\partial_l H_{il}) - \frac{1}{2}\partial_j H_{ij}\partial_l H_{il} + \frac{1}{4}\partial_l H_{ij}\partial_l H_{ij}| \leq$$
$$C\sum_{|\alpha|=2}^{d}\sum_{ij=1}^{n}|h_{ij,\alpha}|^2|x|^{2|\alpha|} + C|x|^{n-3}.$$

The next step is to show that, as in the standard case of a full ball, the first interior term on the right hand side of (9.3) may be absorbed into the error. To see this observe that theorem 3.4 implies

$$\begin{aligned}\int_{B_\rho^+}\phi_\varepsilon u_\varepsilon \partial_{ij}h_{ij} &= -\int_{\partial' B_\rho^+}\phi_\varepsilon u_\varepsilon(\sum_{j=1}^{n-1}\partial_j H_{nj} + \partial_n H_{nn}) + O(\varepsilon^{n-2})\\ &= O\Big(\sum_{|\alpha'|=0}^{N}\sum_{i,j=1}^{n-1}|T_{ij,\alpha'}|\varepsilon^{|\alpha'|+1} + \varepsilon^{n-2}\Big).\end{aligned}$$

Therefore after an integration by parts (9.3) becomes

$$\begin{aligned}&C\Big(\sum_{|\alpha|=2}^{d}\sum_{ij=1}^{n}|h_{ij,\alpha}|^2\varepsilon^{2|\alpha|+1} + \sum_{|\alpha'|=0}^{N}\sum_{i,j=1}^{n-1}|T_{ij,\alpha'}|\varepsilon^{|\alpha'|+1} + \varepsilon^{n-2}\Big) \qquad (9.4)\\ &\geq \int_{\partial' B_\rho^+}(c(n)\phi_\varepsilon u_\varepsilon H_{in}\partial_l H_{il} - \phi_\varepsilon H_{in}\partial_i z_\varepsilon)\\ &-2\int_{B_\rho^+}c(n)u_\varepsilon z_\varepsilon(1 + \frac{1}{2}x^k\partial_k)\partial_{ij}h_{ij}\\ &+\int_{B_\rho^+}c(n)\phi_\varepsilon u_\varepsilon(\frac{1}{2}\partial_j H_{ij}\partial_l H_{il} - \frac{1}{4}\partial_l H_{ij}\partial_l H_{ij}).\end{aligned}$$

Furthermore the boundary integral on the right hand side of (9.4) may be absorbed into the error term with the help of theorem 3.4,

$$\int_{\partial' B_\rho^+} (c(n)\phi_\varepsilon u_\varepsilon H_{in}\partial_l H_{il} - \phi_\varepsilon H_{in}\partial_i z_\varepsilon) = O\Big(\sum_{|\alpha'|=0}^{N}\sum_{i,j=1}^{n-1}|T_{ij,\alpha'}|\varepsilon^{|\alpha'|+1} + \varepsilon^{n-2}\Big).$$

The remaining interior integrals are the same as those that appear in the original Weyl vanishing proof [17], except that the domain of integration is a half ball instead of the full ball. At this point we may follow the original proof to obtain the desired conclusion, as long as the following two facts hold: (i) the necessary integration by parts may be performed with the extra boundary integrals (along $\partial' B_\rho^+$) being absorbed into the error, (ii) an orthogonality condition among harmonic polynomials holds on the half ball.

An inspection of the original proof shows that (i) is valid, since any integrand along $\partial' B_\rho^+$ will contain quantities that either appear in theorem 3.4 (and thus may be estimated by the umbilicity tensor) or involve $\partial_n z_\varepsilon$ — which vanishes by proposition 5.1. Furthermore consider the decomposition (5.1), then in the notation of [17]

$$(\widehat{H}_q)_{ij}^{(k)} = \mathrm{Proj}(\partial_i\partial_j p_{k-2q}|x|^{2q+2}).$$

In chapter 5 it was shown that $\partial_n p_{k-2q}|_{\partial' B_\rho^+}$ vanishes, therefore it follows that $(\widehat{H}_q)_{in}$, $\partial_n(\widehat{H}_q)_{nn}$, and $\partial_n(\widehat{H}_q)_{ij}$ can be estimated in terms of the umbilicity tensor. This implies that the corresponding elements of W_{ij} can also be estimated in terms of the umbilicity tensor (since the corresponding elements of H_{ij} have this property by consequence of theorem 3.4). Hence $((\widehat{H}_q)_{ij}, W_{ij})$ can be absorbed into the error, where the inner product is taken over the half sphere. Similarly

$$\int_{S_+^{n-1}(\rho)} (l-k)p_l p_k = \int_{\partial' B_\rho^+} (p_k \partial_n p_l - p_l \partial_n p_k) = 0,$$

so that $p_l \perp p_k$, $l \neq k$, that is, (ii) is valid. This finishes the proof of proposition 9.1.

Chapter 10

Sign restriction

Define

$$P'(r,w) = \int_{\partial B_r^+(x_i)} \big(\frac{n-2}{2} w \frac{\partial w}{\partial \nu_\delta} + x^k \partial_k w \frac{\partial w}{\partial \nu_\delta} - \frac{1}{2} x^k \nu_\delta^k |\nabla w|^2\big) ds.$$

Proposition 10.1. *(Sign restriction) Let $x_i \to \bar{x}$ be an isolated simple blow-up point and assume that $3 \le n \le 24$. If $M_i u_i(x) \to w$ away from the origin then*

$$\liminf_{r\to 0} P'(r,w) \ge 0.$$

Proof. Define

$$P(r,u_i) = \int_{\partial B_r^+(x_i)} \big(\frac{n-2}{2} u_i \frac{\partial u_i}{\partial \nu_\delta} + x^k \partial_k u_i \frac{\partial u_i}{\partial \nu_\delta} - \frac{1}{2} x^k \nu_\delta^k |\nabla u_i|^2 + \frac{1}{p_i+1} K(x) x^k \nu_\delta^k u_i^{p_i+1}\big) ds.$$

If r is sufficiently small, the Pohozaev identity (proposition A.7) gives

$$P(r,u_i) \geq -\int_{B_r^+(x_i)} (\frac{n-2}{2}u_i + x^k\partial_k u_i)\big((g_i^{lj}-\delta^{lj})\partial_{lj}u_i + \partial_l g_i^{lj}\partial_j u_i)dx$$
$$+\int_{B_r^+(x_i)} (\frac{n-2}{2}u_i + x^k\partial_k u_i)R_{g_i}u_i dx.$$

Notice that

$$\int_{B_r^+(x_i)} x^k\partial_k u_i R_{g_i}u_i dx = -\int_{B_r^+(x_i)} (x^k\partial_k R_{g_i} + nR_{g_i})u_i^2 dx$$
$$-\int_{B_r^+(x_i)} x^k\partial_k u_i R_{g_i}u_i dx + \int_{\partial B_r^+(x_i)} x^k\nu_\delta^k R_{g_i}u_i^2.$$

Since $x^k\nu_\delta^k = 0$ on $\partial' B_r^+(x_i)$ and $\nu_\delta^k = x^k/r$ on $\partial^+ B_r^+(x_i)$ we obtain

$$\int_{B_r^+(x_i)} x^k\partial_k u_i R_{g_i}u_i dx = -\frac{1}{2}\int_{B_r^+(x_i)} (x^k\partial_k R_{g_i} + nR_{g_i})u_i^2 dx$$
$$+\frac{r}{2}\int_{\partial^+ B_r^+(x_i)} R_{g_i}u_i^2 ds,$$

so

$$c(n)\int_{B_r^+(x_i)} (\frac{n-2}{2}u_i + x^k\partial_k u_i)R_{g_i}u_i dx \tag{10.1}$$
$$= -c(n)\int_{B_r^+(x_i)} (\frac{1}{2}x^k\partial_k R_{g_i} + R_{g_i})u_i^2 dx + c(n)\frac{r}{2}\int_{\partial^+ B_r^+(x_i)} R_{g_i}u_i^2 ds,$$

and then

$$P(r,u_i) \geq -\int_{B_r^+(x_i)} (\frac{n-2}{2}u_i + x^k\partial_k u_i)\big((g_i^{lj}-\delta^{lj})\partial_{lj}u_i + \partial_l g_i^{lj}\partial_j u_i\big)dx$$
$$- c(n)\int_{B_r^+(x_i)} (\frac{1}{2}x^k\partial_k R_{g_i} + R_{g_i})u_i^2 dx + c(n)\frac{r}{2}\int_{\partial^+ B_r^+(x_i)} R_{g_i}u_i^2 ds$$

$$= \quad A_i(r) + c(n)\frac{r}{2}\int_{\partial^+ B_r^+(x_i)} R_{g_i} u_i^2 ds$$

where $A_i(r)$ is defined by the above equality. Now observe that

$$\int_{\partial B_r^+(x_i)} K(x) M_i^2 x^k \nu_\delta^k u_i^{p_i+1} \to 0.$$

In fact, the integral over $\partial' B_r^+(x_i)$ vanishes as $x^k \nu_\delta^k = 0$ there. On $\partial^+ B_r^+(x_i)$ we have $x^k \nu_\delta^k = r$, hence, using the equation satisfied by u_i produces

$$\int_{\partial^+ B_r^+(x_i)} K(x) M_i^2 u_i^{p_i+1} = -\int_{\partial^+ B_r^+(x_i)} M_i u L_{g_i} M_i u_i \to -\int_{\partial^+ B_r^+(x_i)} w L_g w = 0.$$

Therefore $M_i^2 P(r, u_i) \to P'(r, w)$, so

$$\begin{aligned} P'(r,w) &= \lim_{i\to\infty} M_i^2 P(r,u_i) \geq \lim_{i\to\infty} M_i^2 A_i(r) + \lim_{i\to\infty} c(n)\frac{r}{2}\int_{\partial^+ B_r^+(x_i)} R_{g_i}(M_i u_i)^2 ds \\ &= \lim_{i\to\infty} M_i^2 A_i(r) + c(n)\frac{r}{2}\int_{\partial^+ B_r^+(x_i)} R_{g_i} w^2 ds. \end{aligned}$$

We now proceed to analyze $M_i^2 A_i(r)$, noticing that since theorem 9.2 and corollary 9.3 give the same estimates as in the boundaryless case, the same analysis can be carried out, except for an extra boundary term that appears in $\hat{A}_i(r)$ when integration by parts is performed, where

$$\begin{aligned} \hat{A}_i(r) = &-c(n)\int_{B_r^+(x_i)} (\frac{1}{2}x^k \partial_k R_{g_i} + R_{g_i})(u_{\varepsilon_i} + z_{\varepsilon_i})^2 dx \\ &-\int_{B_r^+(x_i)} (\frac{n-2}{2}(u_{\varepsilon_i} + z_{\varepsilon_i}) + x^k \partial_k (u_{\varepsilon_i} + z_{\varepsilon_i}))(\Delta_{g_i} - \Delta_\delta)(u_{\varepsilon_i} + z_{\varepsilon_i}) dx. \end{aligned} \tag{10.2}$$

Corollary 9.3 implies that $\varepsilon_i^{2-n}|A_i(r) - \hat{A}_i(r)| \leq Cr$, so $\lim_{i\to\infty} \varepsilon_i^{2-n}(A_i(r) -$

$\hat{A}_i(r)) \geq -Cr$. Notice that since $M_i = \varepsilon_i^{-\frac{2}{p_i-1}}$ and $-\frac{4}{p_i-1} \to 2-n$ we can replace M_i^2 by ε_i^{2-n} and obtain

$$P'(r,w) \geq -Cr + c(n)\frac{r}{2}\int_{\partial^+ B_r^+(x_i)} R_{g_i} w^2 ds + \lim_{i\to\infty} \varepsilon_i^{2-n}\hat{A}_i(r). \tag{10.3}$$

Using the symmetries of u_{ε_i}

$$\begin{aligned} &\int_{B_r^+(x_i)} (\frac{n-2}{2}(u_{\varepsilon_i}+z_{\varepsilon_i}) + x^k\partial_k(u_{\varepsilon_i}+z_{\varepsilon_i}))(\Delta_{g_i}-\Delta_\delta)(u_{\varepsilon_i}+z_{\varepsilon_i})dx \\ &= \int_{B_r^+(x_i)} (\frac{n-2}{2}z_{\varepsilon_i} + x^k\partial_k z_{\varepsilon_i})(\Delta_{g_i}-\Delta_\delta)z_{\varepsilon_i}dx \\ &+ \int_{\partial B_r^+(x_i)} (\frac{n-2}{2}u_{\varepsilon_i} + x^k\partial_k u_{\varepsilon_i})(\frac{\partial z_{\varepsilon_i}}{\partial \nu_{g_i}} - \frac{\partial z_{\varepsilon_i}}{\partial \nu_\delta})ds \\ &- \int_{\partial B_r^+(x_i)} z_{\varepsilon_i}(\frac{\partial L u_{\varepsilon_i}}{\partial \nu_{g_i}} - \frac{\partial L u_{\varepsilon_i}}{\partial \nu_\delta})ds, \end{aligned} \tag{10.4}$$

where $L = \frac{n-2}{2} + x^k\partial_k$. The integrals over $\partial^+ B_r^+(x_i)$ vanish by properties of normal coordinates, so consider the integrals over $\partial' B_r^+(x_i)$. Observe that $\frac{\partial z_{\varepsilon_i}}{\partial \nu_\delta} = 0 = \partial_n z_{\varepsilon_i}$ by proposition 5.1. Then using (4.2), the definition of u_{ε_i}, and (9.1), we obtain

$$\begin{aligned} &\Big|\int_{\partial' B_r^+(x_i)} (\frac{n-2}{2}u_{\varepsilon_i} + x^k\partial_k u_{\varepsilon_i})\frac{\partial z_{\varepsilon_i}}{\partial \nu_{g_i}}ds\Big| = \\ &\Big|\int_{\partial' B_r^+(x_i)} (\frac{n-2}{2}u_{\varepsilon_i} + x^k\partial_k u_{\varepsilon_i})\sum_{l=1}^{n-1} g_i^{nl}\partial_l z_{\varepsilon_i} ds\Big| \\ &\leq \int_{\partial' B_r^+(x_i)} (\varepsilon_i + |x'|)^{2-n}(\varepsilon_i+|x'|)^{6-n-1}|x'|^N dx' \\ &\leq C\varepsilon_i^{n-2} r. \end{aligned} \tag{10.5}$$

For the other boundary integral notice that

$$\frac{\partial Lu_{\varepsilon_i}}{\partial \nu_{g_i}} - \frac{\partial Lu_{\varepsilon_i}}{\partial \nu_\delta} = (-g_i^{n\sigma}\partial_\sigma + \partial_n)Lu_{\varepsilon_i} = (-g_i^{nn}+1)Lu_{\varepsilon_i} - \sum_{l=1}^{n-1} g_i^{nl}\partial_l Lu_{\varepsilon_i}.$$

Since $|\nabla Lu_{\varepsilon_i}| \leq \varepsilon_i^{\frac{n-2}{2}}(\varepsilon_i + |x|)^{-n}$, using (9.1), theorem 4.1, theorem 3.4, and (4.2), it follows that

$$\Big|\int_{\partial' B_r^+(x_i)} z_{\varepsilon_i}(\frac{\partial Lu_{\varepsilon_i}}{\partial \nu_{g_i}} - \frac{\partial Lu_{\varepsilon_i}}{\partial \nu_\delta})ds\Big| \leq C\varepsilon_i^{n-2}r. \tag{10.6}$$

Combining (10.2), (10.3), (10.4), (10.5), and (10.6) yields

$$\begin{aligned} P'(r,w) \geq & -Cr\int_{B_r^+(x_i)}(\frac{n-2}{2}z_{\varepsilon_i} + x^k\partial_k z_{\varepsilon_i})(\Delta_{g_i} - \Delta_\delta)z_{\varepsilon_i}dx \\ & + c(n)\frac{r}{2}\int_{\partial^+ B_r^+(x_i)} R_{g_i}w^2 ds \\ & - c(n)\lim_{i\to\infty}\varepsilon_i^{2-n}\int_{B_r^+(x_i)}(\frac{1}{2}x^k\partial_k R_{g_i} + R_{g_i})(u_{\varepsilon_i} + z_{\varepsilon_i})^2 dx. \end{aligned} \tag{10.7}$$

We can now proceed as in the boundaryless case. The first integral on the right hand side of (10.7) as well as $\frac{r}{2}\int_{\partial^+ B_r^+(x_i)} R_{g_i}w^2 ds$ are estimated using theorem 9.2. Theorem 9.2 and corollary 9.3 may be used to estimate $\int_{B_r^+(x_i)}(\frac{1}{2}x^k\partial_k R_{g_i} + R_{g_i})z_{\varepsilon_i}^2 dx$. Finally the estimate of proposition 9.1 is used to handle $\int_{B_r^+(x_i)}(\frac{1}{2}x^k\partial_k R_{g_i} + R_{g_i})(u_{\varepsilon_i}^2 + 2u_{\varepsilon_i}z_{\varepsilon_i})dx$. □

Chapter 11

Blow-up set

In this chapter we show that the set of blow-up points is finite and consists only of isolated simple blow-up points. The proofs are very similar to the boundaryless case ([17]) and the locally conformally flat case with boundary ([24]), and therefore we will go through them rather quickly, indicating the necessary modifications.

The following proposition is proven in [24] (proposition 1.1, see also [12, 17]).

Proposition 11.1. *Given $\delta > 0$ sufficiently small and $R > 0$ sufficiently large, there exists a constant $C = C(\delta, R) > 0$ such that if u is a positive solution of (2.1) with $\max u > C$, then there exists $\{x_1, \dots, x_N\} \subset M$, $N = N(u) > 1$, where $\frac{n+2}{n-2} - p < \delta$ and each x_i is a local maximum of u such that:*

1) $\{B_{r_i}(x_i)\}_{i=1}^N$ is a disjoint collection if $r_i = Ru(x_i)^{-\frac{p-1}{2}}$,

2) in normal coordinates centered at x_i

$$\| u_i(x_i)^{-1} u(u_i(x_i)^{-\frac{p-1}{2}} y) - U(y) \|_{C^2(B_R(0))} < \delta$$

where $y = u(x_i)^{\frac{p-1}{2}} x$,

3) $u(x) \le C d_g(x, \{x_1, \ldots, x_n\})^{-\frac{2}{p-1}}$ *for all* $x \in M$ *and*

$$d_g(x_i, x_j)^{\frac{2}{p-1}} u(x_j) \ge C^{-1}$$

for $x_i \neq x_j$.

Lemma 11.2. *Let* $x_i \to \bar{x}$ *be an isolated blow-up point for the sequence* $\{u_i\}$ *of positive solutions of (2.1). Then* $\bar{x}$ *is an isolated simple blow-up point.*

Proof. We argue as in [17] to obtain a subsequence w_i such that

$$w_i(0)w_i(y) \to h(y) = a|y|^{2-n} + b(y) \text{ in } C^2_{loc}(\mathbb{R}^n_+ \backslash \{0\}),$$

where $b(y)$ is harmonic in $\mathbb{R}^n_+$ and satisfies $\partial_n b = 0$ on $\mathbb{R}^{n-1}$. Therefore, extending b across $\mathbb{R}^{n-1}$ and using Liouville's theorem shows that $a = b > 0$. Arguing as in [17] this leads to a contradiction with proposition 10.1. □

Proposition 11.3. *Let* δ, R, u, $C(\delta, R)$, *and* $\{x_1, \ldots, x_N\}$ *be as in proposition 11.1. If* δ *is sufficiently small and* R *sufficiently large, then there exists a constant* $\bar{C}(\delta, R) > 0$ *such that if* $\max_M u \ge C$ *then* $d_g(x_j, x_l) \ge \bar{C}$ *for all* $1 \le j \neq l \le N$.

Proof. Again we argue as in [17], making the necessary modifications along the lines of [24] as in lemma 11.2. □

The following is an immediate consequence.

Corollary 11.4. *Let* $\{u_i\}$ *be a sequence of solutions of (2.1) with* $\max_M u_i \to$

∞. *Then* $p_i \to \frac{n+2}{n-2}$ *and the set of blow-up points is finite and consists only of isolated simple blow-up points.*

Chapter 12

Compactness

Now that we have the Weyl vanishing theorem and sign restriction, the remaining arguments for the proof of theorems 1.1 and 1.2 are similar to those of the boundaryless case. In fact, the results of this chapter will be an adaptation of [17, 19, 28], and therefore as in chapter 11, we will go through the proofs very briefly.

Proof of theorem 1.1: From the results of chapter 11 $p_i \to \frac{n+2}{n-2}$, and there exists a finite number $N > 0$ of isolated simple blow-up points $x_i^{(1)} \to \bar{x}^{(1)}, \ldots, x_i^{(N)} \to \bar{x}^{(N)}$. If none of the $\bar{x}_\ell$ belong to the boundary then the compactness result follows from [17], so assume that at least one of them belongs to ∂M. It may also be assumed without loss of generality that $\bar{x}_\ell \in \partial M$, $\ell = 1, \ldots, N-k$ and $\bar{x}_\ell \notin \partial M$, $\ell = N-k+1, \ldots, N$, for some $k \leq N-1$. Furthermore let

$$u_i(x_i^{(1)}) = \min\{u_i(x_i^{(1)}), \ldots, u_i(x_i^{(N-k)})\}$$

for all i.

Set $w_i = u_i(x_i^{(1)})u_i$. A standard estimate gives that away from the blow-up points $w_i \to \sum_{j=1}^N a_j G_{\bar{x}^{(j)}}$, where $a_j \geq 0$, $a_1 > 0$ and $G_{\bar{x}^{(j)}}$ is the Green's function for the conformal Laplacian with singularity at $\bar{x}^{(j)}$. Now argue as in [6] (see [19, 28] as well) to obtain the asymptotic expansion

$$G(x, \bar{x}^{(1)}) = |x|^{2-n}\Big(1 + \sum_{k=d+1}^{n-2} \psi_k\Big) + A + O(|x| \log |x|), \tag{12.1}$$

where $G = G_{\bar{x}^{(1)}}$, ψ_k are homogeneous polynomials of degree k and A is a constant. The sum between parenthesis starts at $k = d+1$ because $h_{ij,\alpha}(\bar{x}) = 0$ at a blow-up point $\bar{x} \in \partial M$, by the Weyl vanishing theorem. We remark that when the boundary is not umbilic an extra singular term appears in this expansion (see e.g. [38]). Also notice that standard properties of conformal normal coordinates, theorem 3.4, and the umbilicity of the boundary, imply that $\int_{S_+^{n-1}} \partial_{ij} H_{ij} = 0$, from which it follows that

$$\int_{S_+^{n-1}} \psi_k = 0 = \int_{S_+^{n-1}} x_i \psi_k. \tag{12.2}$$

Now put $\widehat{g} = G^{\frac{4}{n-2}} g$. Then $(M \backslash \{\bar{x}^{(1)}\}, \widehat{g})$ is scalar flat and its boundary is totally geodesic. If we introduce the asymptotic coordinates $y = |x|^{-2}x$, then the expansion (12.1) and the Weyl vanishing theorem give $\widehat{g}_{ij} = \delta_{ij} + O(|y|^{-d-1})$. Therefore the doubling of $(M \backslash \{\bar{x}^{(1)}\}, \widehat{g})$ is asymptotically flat and has a well defined ADM mass ([39], compare also with [28]).

The rest of the argument now is standard. The positive mass theorem (see remark below) along with (12.2) and the Weyl vanishing give that $A > 0$ (as in [17], using the hypothesis that the manifold is not conformally equivalent to the round hemisphere we can rule out the $A = 0$ case). This contradicts the sign restriction of theorem 10.1, finishing the proof.

Remark 12.1.

1) Strictly speaking, we did not show how to prove a positive mass theorem (PMT) for manifolds with boundary, as the mass of such manifolds was never defined. What is referred to as the PMT for manifolds with boundary is actually the statement that the constant term in the asymptotic expansion of the Green's function is non-negative, which in turn is implied by the positivity of the mass of the doubled manifold (see [12]).

2) The PMT is known to hold up to dimension 7 [6, 40, 41] and in arbitrary dimensions if the manifold is spin [5, 42]. Therefore, our result for $n \geq 8$ in the case of non-spin manifolds is true provided that the PMT holds under such hypotheses.

Proof of theorem 1.2: This follows from lemma 11.2 and theorem 9.2.

Chapter 13

Blow-up of solutions for $n \geq 25$

In this chapter we prove theorem 1.4. We assume $n \geq 25$ throughout. As we mention in the introduction, the proof relies heavily on the constructions of Brendle [15] and Brendle and Marques [16], and we refer the reader to them on several occasions.

We start collecting facts from [15, 16] that will be of direct use in our proof. Their main results is

Theorem 13.1. *(Brendle and Marques, [15, 16]) Assume that $n \geq 25$. Then there exists a metric g on S^n (of class C^∞) and a sequence of positive functions $u_i \in C^\infty(S^n)$ with the following properties:*

(a) g is a small perturbation of the round metric g_0 which is not conformally flat, and $g = g_0$ near and beyond the equator,

(b) for each i, u_i is a solution of the Yamabe equation

$$L_g u_i + K u_i^{\frac{n+2}{n-2}} = 0,$$

where $K = n(n-2)$ *is a positive constant,*

(c) $E_g(u_i) < Y(S^n)$ *for all* $i \in \mathbb{N}$*, and* $E_g(u_i) \to Y(S^n)$ *as* $i \to \infty$*, where* $E_g(u_i)$ *is the Yamabe energy of* u_i *and* $Y(S^n)$ *is the Yamabe invariant of the round sphere,*

(d) $\sup_{S^n} u_i \to \infty$ *as* $i \to \infty$*.*

The scalar curvature of the metric g satisfies

$$R_g \geq c > 0. \tag{13.1}$$

for some constant c, since g is a small perturbation of the round metric. In particular this guarantees the coercivity of L_g, which allows us to use the the C^0-blow-up theory developed by Druet, Hebey and Robert [43]. From their results and estimate (c) of theorem 13.1 it then follows (theorem 5.2 of [43], see also discussion at the end of section 5.1) that u_i has only one blow-up point, and it is apparent from [15, 16] that this is the south pole (from the point of view of stereographic projection). Moreover, up to a subsequence the following estimate holds (again theorem 5.2 of [43])

$$Q^{-1} u_{\varepsilon_i, x_i}(x) \leq u_i(x) \leq Q u_{\varepsilon_i, x_i}(x), \tag{13.2}$$

for some constant $Q > 1$ independent of i and for all $x \in S^n$; here $\varepsilon_i = (\sup_{S^n} u_i)^{-\frac{2}{n-2}} = u_i(x_i)^{-\frac{2}{n-2}}$, and $u_{\varepsilon_i, x_i} = \varepsilon_i^{\frac{n-2}{2}} (\varepsilon_i^2 + |x - x_i|^2)^{\frac{2-n}{2}}$, $|x - x_i| = \mathrm{dist}_g(x, x_i)$.

Consider now the south hemisphere S^n_-, which we identify with the unit ball in $\mathbb{R}^n$ via stereographic projection. Since $g = g_0$ on a neighborhood ∂S^n_-, we have that ∂S^n_- is totally geodesic, and in particular $B_g = \partial_{\nu_g}$. Combining (13.2) with the Harnack inequality implies that away from the south pole, $\varepsilon_i^{\frac{2-n}{2}} u_i$ converges in C^2 to a positive Green's function for the conformal Laplacian (possibly after passing to a subsequence). We claim that for large i

$$\frac{\partial u_i}{\partial \nu_g} \leq 0. \tag{13.3}$$

To see this, denote by δ the Euclidean metric so that $g_0 = 4U^{\frac{4}{n-2}}\delta$. Let G_{g_0} and G_δ be the corresponding Green's functions with singularity at zero. Their relation is given by $G_{g_0} = 4^{-\frac{n-2}{2}} U^{-1} G_\delta$. Using (A.3) and the fact that the mean curvature of ∂S^n_- vanishes , we have

$$\frac{\partial G_{g_0}}{\partial \nu_{g_0}} = B_{g_0} G_{g_0} = 4^{-\frac{n-2}{2}} U^{-\frac{n}{n-2}} B_\delta G_\delta < 0$$

on ∂S^n_-, where the inequality follows by direct calculation. Therefore $\frac{\partial G_g}{\partial \nu_g} < 0$ by theorem 13.1(a), so that (13.3) holds.

We conclude that

$$\begin{cases} L_g u_i + K u_i^{\frac{n+2}{n-2}} = 0 \;, & \text{in } S^n_-, \\ B_g u_i \leq 0, & \text{on } \partial S^n_-. \end{cases}$$

That is, u_i is a sub-solution of the boundary value problem

$$\begin{cases} L_g v + K v^{\frac{n+2}{n-2}} = 0, & \text{in } S^n_-, \\ B_g v = 0, & \text{on } \partial S^n_-. \end{cases} \tag{13.4}$$

Actual solutions to (13.4) will be constructed by finding appropriate super-solutions. The super-solutions will satisfy the equation with a different constant K, and this will require a slight modification of the standard sub-super-solutions argument.

Theorem 13.2. *For all sufficiently large i there exists a solution v_i of (13.4) satisfying $u_i \leq v_i$. In particular $\sup_{S^n_-} v_i \to \infty$ as $i \to \infty$.*

Proof. Because of (13.1), we can choose $\delta > 0$ so small that

$$L_g \delta + K \delta^{\frac{n+2}{n-2}} = -c(n) R_g \delta + K \delta^{\frac{n+2}{n-2}} \leq 0.$$

Put $w_i = A_i \delta$, where $A_i > 1$ is a constant chosen so large that

$$u_i \leq A_i \delta, \tag{13.5}$$

and

$$u_i^{\frac{n+2}{n-2}} - A_i \delta^{\frac{n+2}{n-2}} \leq 0 \tag{13.6}$$

By the choice of δ

$$\begin{cases} L_g w_i + \widetilde{K}_i w_i^{\frac{n+2}{n-2}} \leq 0, & \text{in } S^n_-, \\ B_g w_i = 0, & \text{on } \partial S^n_-, \end{cases} \tag{13.7}$$

where $\widetilde{K}_i = A_i^{-\frac{4}{n-2}} K$. So w_i is a super-solution of the problem with constant $\widetilde{K}_i$. As pointed out before, (L_g, B_g) is invertible and therefore the operators

T and P_i given by

$$Tz = t \Leftrightarrow \begin{cases} L_g t = -K z^{\frac{n+2}{n-2}}, & \text{in } S^n_-, \\ B_g t = 0, & \text{on } \partial S^n_-, \end{cases} \tag{13.8}$$

and

$$P_i w = p_i \Leftrightarrow \begin{cases} L_g p_i = -\widetilde{K}_i w^{\frac{n+2}{n-2}}, & \text{in } S^n_-, \\ B_g p_i = 0, & \text{on } \partial S^n_-, \end{cases} \tag{13.9}$$

are well defined. By the maximum principle T and P_i are monotone in the sense that $z_0 \leq z_1 \Rightarrow Tz_0 \leq Tz_1$, and analogously for P_i.

Now we put $u_i^0 = u_i$, $w_i^0 = w_i$ and define inductively $u_i^{\ell+1} = Tu_i^\ell$ and $w_i^{\ell+1} = P_i w_i^\ell$. Since u_i is a sub-solution we obtain $u_i^0 \leq u_i^1$ and inductively $u_i^\ell \leq u_i^{\ell+1}$. Analogously $w_i^\ell \geq w_i^{\ell+1}$ since w_i is a super-solution.

We have $u_i^0 \leq w_i^0$ by (13.5), and claim that $u_i^\ell \leq w_i^\ell$ for every ℓ (the difference from the standard sub-super-solutions argument is that the equations involved in the definition of T and P_i are not exactly the same due to the different constants K and $\widetilde{K}_i$). The difference $u_i^{\ell+1} - w_i^{\ell+1}$ satisfies

$$\begin{cases} L_g(u_i^{\ell+1} - w_i^{\ell+1}) = -(K(u_i^\ell)^{\frac{n+2}{n-2}} - \widetilde{K}_i(w_i^\ell)^{\frac{n+2}{n-2}}), & \text{in } S^n_-, \\ B_g(u_i^{\ell+1} - w_i^{\ell+1}) = 0, & \text{on } \partial S^n_-. \end{cases} \tag{13.10}$$

In order to apply the maximum principle we need the right hand side of (13.10) to be non-negative. To show this, recall the definition of w_i and $\widetilde{K}_i$, use the monotonicity of the sequences u_i^ℓ and w_i^ℓ, as well as (13.6) to find

$$K(u_i^\ell)^{\frac{n+2}{n-2}} - \widetilde{K}_i(w_i^\ell)^{\frac{n+2}{n-2}} \geq K(u_i^0)^{\frac{n+2}{n-2}} - \widetilde{K}_i(w_i^0)^{\frac{n+2}{n-2}} = Ku_i^{\frac{n+2}{n-2}} - KA_i\delta^{\frac{n+2}{n-2}} \leq 0,$$

It follows that $u_i^\ell \leq w_i^\ell$.

Now a standard argument produces the desired solution u_i^∞ of (13.4) such that $u_i \leq u_i^\infty$. The proof also yields a w_i^∞ solving (13.4) with $\widetilde{K}_i$ in place of K, and such that and $w_i^\infty \leq w_i$, but this is not the solution we are looking for due to the different i-dependent constant $\widetilde{K}_i$. □

Chapter 14

Leray-Schauder degree of solutions

Here we discuss some consequences of theorem 1.1. Throughout this chapter we assume $3 \le n \le 24$. The results here are very similar to the cases of manifolds without boundary and locally conformally flat with boundary, so we refer the reader to [17] and [24] for details.

As we pointed out in the introduction, one obvious consequence of theorem 1.1 is to give an alternative proof of the solution to the Yamabe problem. This follows from the fact that standard variational methods can be used to give solutions to the subcritical problem

$$\begin{cases} L_g u + K u^p = 0, & \text{in } M, \\ B_g u = 0, & \text{on } \partial M, \end{cases} \tag{14.1}$$

with $1 < p < \frac{n+2}{n-2}$. More generally, the compactness theorem allows us to compute the total Leray-Schauder degree of all solutions to equation (14.1), and to obtain more refined existence theorems which we now discuss.

Without loss of generality we can assume that $R_g > 0$ and $\kappa_g = 0$. Then we can write (14.1) as

$$\begin{cases} L_g u + E(u)u^p = 0, & \text{in } M, \\ \frac{\partial u}{\partial \nu_g} = 0, & \text{on } \partial M, \end{cases} \tag{14.2}$$

where

$$E(u) = \int_M (|\nabla_g u|^2 + c(n)R_g u^2)dV_g$$

is the energy of u (there is no boundary term since $\kappa_g = 0$). Notice that the Neumann problem for the conformal Laplacian is invertible in that $R_g > 0$. Defining

$$\Omega_\Lambda = \{u \in C^{2,\alpha}(M) \mid \ \| u \|_{C^{2,\alpha}} (M) < \Lambda,\ u > \Lambda^{-1}\}$$

we obtain a map $F_p : \overline{\Omega}_\Lambda \to C^{2,\alpha}(M)$ given by $F_p(u) = u + L_g^{-1}(E(u)u^p)$.

From elliptic theory, we know that the map $u \mapsto L_g^{-1}(E(u)u^p)$ is a compact map from $\overline{\Omega}_\Lambda$ into $C^{2,\alpha}(M)$. Thus F_p is of the form I+compact, and we may define the Leray-Schauder degree (see [44]) of F_p in the region Ω_Λ with respect to $0 \in C^{2,\alpha}(M)$, denoted by $\deg(F_p, \Omega_\Lambda, 0)$, provided that $0 \notin F_p(\partial\Omega_\Lambda)$. The degree is an integer which counts with multiplicity the number of times that the value 0 is taken on by the map F_p. Notice that $F_p(u) = 0$ if and only if u is a solution of (14.2). Furthermore, the homotopy invariance of the degree tells us that $\deg(F_p, \Omega_\Lambda, 0)$ is constant for all $p \in [1, \frac{n+2}{n-2}]$ provided that $0 \notin F_p(\partial\Omega_\Lambda)$ for all $p \in [1, \frac{n+2}{n-2}]$. Moreover, in the linear case when $p = 1$, it is not difficult to calculate, by an argument similar to what is done in [9], that $\deg(F_1, \Omega_\Lambda, 0) = -1$ for all Λ sufficiently large. Therefore, theorem 1.1 allows us to calculate the degree for all $p \in [1, \frac{n+2}{n-2}]$. Since it follows from the a priori estimates we derived that 0 does not belong to $F_p(\partial\Omega_\Lambda)$, we obtain

Theorem 14.1. *Let (M^n, g) satisfy the assumptions of theorem 1.1. Then for all Λ sufficiently large and all $p \in [1, \frac{n+2}{n-2}]$, we have* $\deg(F_p, \Omega_\Lambda, 0) = -1$.

In the case that all solutions of the Yamabe problem are nondegenerate, our previous results assert that there will be a finite number of solutions of the variational problem. Moreover, the strong Morse inequalities will hold for the Yamabe problem since these inequalities hold for subcritical equations, and theorem 1.1 shows that all critical points converge as $p \to \frac{n+2}{n-2}$. It follows that

$$(-1)^\lambda \leq \sum_{\mu=0}^{\lambda} (-1)^{\lambda-\mu} C_\mu, \ \lambda = 0, 1, 2, \ldots$$

where C_μ denotes the number of solutions of Morse index μ. Since there is a finite number of solutions, we then obtain:

Theorem 14.2. *Let (M^n, g) satisfy the assumptions of theorem 1.1, and suppose that all critical points in $[g]$ are nondegenerate. Then there is a finite number of critical points $g_1, \ldots, g_k$, and we have*

$$1 = \sum_{j=1}^{k} (-1)^{I(g_j)}$$

where $I(g_j)$ denotes the Morse index of the variational problem with volume constraint.

Bibliography

[1] H. Yamabe. On a deformation of riemannian structures on compact manifolds. *Osaka Math. J.*, 12:21–37, 1960.

[2] N. Trudinger. Remarks concerning the conformal deformation of riemannian structures on compact manifolds. *Ann. Scuola Norm. Sup. Pisa Cl. Sci*, 22(3):165–274, 1968.

[3] T. Aubin. équations différentielles non linéaires et probléme de yamabe concernant la courbure scalaire. *J. Math. Pures Appl.*, 55:269–296, 1976.

[4] R. Schoen. Conformal deformaiton of a riemannian metric to constant scalar curvature. *J. Differential Geometry*, 20:479–495, 1984.

[5] J. M Lee and T. H. Parker. The yamabe problem. *Bulletin (New Series) of the American Mathematical Society*, 17(1), 1987.

[6] R. Schoen and S.-T Yau. *Lectures on Differential Geometry*. 1991.

[7] D. Pollack. Nonuniqueness and high energy solutions for a conformally invariant scalar curvature equation. *Comm. Anal. and Geom. 1*, pages 347–414, 1993.

[8] Obata. The conjectures on conformal transformations of riemannian manifolds. *J. Diff. Geom.*, 6:247–258, 1972.

[9] R. Schoen. On the number of constant scalar curvature metrics in a conformal class. In H.B. Lawson and K. Teneblat, editors, *Differential Geometry: A symposium in honor of Manfredo do Carmo*, pages 311–320. Wiley, 1991.

[10] R. Schoen and D. Zhang. Prescribed scalar curvature on the n-sphere. *Calc. Var.*, 4:1–15, 1996.

[11] O. Druet. Compactness for yamabe metrics in low dimensions. *Math. Res. Not.*, 23:1143–1191, 2004.

[12] F. C. Marques. A priori estimates for the yamabe problem in the non-locally conformally flat case. *Journal of Differential Geometry*, 71(2): 315–346, 2005.

[13] Y. Y. Li and L. Zhang. Compactness of solutions to the yamabe problem ii. *Calculus of Variations*, 24:185–237, 2005.

[14] Y. Y. Li and L. Zhang. Compactness of solutions to the yamabe problem iii. *Journal of Functional Analysis*, 245:438–474, 2007.

[15] S. Brendle. Blow-up phenomena for the yamabe equation. *J. Amer. Math. Soc.*, 21:951–979, 2008.

[16] S. Brendle and F. C. Marques. Blow-up phenomena for the yamabe equation ii. *J. Differential Geom.*, 81(2):225–250, 2009.

[17] M. A. Khuri, F. C. Marques, and R. M. Schoen. A compactness theorem for the yamabe problem. *Journal of Differential Geometry*, 81(1):143–196, 2009.

[18] S. Brendle and F. C. Marques. Recent progress on the yamabe problem. *arXiv:1010.4960v1 [math.DG]*, 2010.

[19] J. F. Escobar. The yamabe problem on manifolds with boundary. *Journal of Differential Geometry*, 35(1):21–84, 1992.

[20] J. F. Escobar. Conformal deformations of a riemannian metric to a scalar flat metric with constant mean curvature on the boundary. *Annals of Mathematics, Second Series*, 136(1):1–50, 1992.

[21] J. F. Escobar. Conformal metrics with prescribed mean curvature on the boundary. *Calc. var.*, 4:559–592, 1996.

[22] F. C. Marques. Existence results for the yamabe problem on manifolds with boundary. *Indiana University Mathematics Journal*, 54(6), 2005.

[23] F. C. Marques. Conformal deformations to scalar-flat metrics with constant mean curvature on the boundary. *Comm. Anal. and Geom.*, 15(2): 381–405, 2007.

[24] Z. Hao and Y. Li. The yamabe problem on manifolds with boundary: existence and compactness results. *Duke Mathematical Journal*, 99(3): 489–541, 1999.

[25] Z.-C. Han and Y.Y. Li. The existence of conformal metrics with constant scalar curvature and constant boundary mean curvature. *Comm. Anal. and Geom.*, 8(4):809–869, 2000.

[26] M. O. Ahmedou. A riemannian mapping type theorem in higher dimensions, part i: the conformally flat case with umbilic boundary. *Progr. Nonlinear Differential Equations Appl*, 54:1–18, 2003.

[27] S. Almaraz. An existence theorem of conformal scalar-flat metrics on manifolds with boundary. *arXiv:0907.0798v1 [math.DG]*, 2009.

[28] S. Brendle and S. Chen. An existence theorem for the yamabe problem on manifolds with boundary. *arXiv:0908.4327v2 [math.DG]*, 2009.

[29] S.S. Chen. Conformal deformation to scalar flat metrics with constant mean curvature on the boundary in higher dimensions. *arXiv: 0912.1302v2 [math.DG]*, 2009.

[30] V. Felli and M. O. Ahmedou. Compactness results in conformal deformations of riemannian metrics on manifolds with boundary. *Mathematische Zeitschrift*, 244:175–210, 2003.

[31] V. Felli and M. O. Ahmedou. On a geometric equation with critical nonlinearity on the boundary. *Pacific Journal of Mathematics*, (218): 75–99, 2005.

[32] S. Almaraz. A compactness theorem for scalar flat metrics on manifolds with boundary. *arXiv: 0906.0927v1 [math.DG]*, 2010.

[33] S. Almaraz. Blow-up phenomena for scalar-flat metrics on manifolds with boundary. *arXiv: 1011.4235 [math.DG]*, 2010.

[34] J. Cao. Existence of generalized isothermal coordinates for higher dimensional riemannian manifolds. *Transactions of the American Mathematical Society*, 324(2):901–920, 1991.

[35] S. Axler, P. bourdon, and W. Ramey. *Harmonic function theory*. Springer.

[36] L. Caffarelli, B. Gidas, and J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical sobolev growth. *Comm. Pure. Appl. Math.*, 42:271–297, 1989.

[37] D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order*. Springer.

[38] Y. Y Li. Fine analysis of blow up and applications. In *First International Congress of Chinese Mathematicians (Beijing, 1998)*. Amer. Math. Soc., 2001.

[39] R. Bartnik. The mass of an asymptotically flat manifold. *Comm. Pure. Appl. Math.*, 39:661–693, 1986.

[40] R. Schoen. Variational theory for the total scalar curvature functional for riemannian metrics and related topics. In *Topics in Calculus of Variations, Lecture Notes in Mathematics*, page 1365. Springer-Verlag, 1989.

[41] R. Schoen and S.-T. Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, page 1979.

[42] E. Witten. A new proof of the positive energy theorem. *Comm. Math. Phys.*, 80:381–402, 1981.

[43] O. Druet, E. Hebey, and F. Robert. *Blow-up theory for Elliptic PDEs in Riemannian Geometry*. Princeton University Press, 2004.

[44] L. Nirenberg. *Topics in Nonlinear Functional Analysis*. Courant Institute publication.

[45] Y. Y. Li and M. Zhu. Uniqueness theorems through the method of moving spheres. *Duke Math J.*, 80:383–417, 1995.

[46] C.-C. Chen and C.-S. Lin. Estimates of the scalar curvature equation via the method of moving planes ii. *J. Differential Geom.*, 49:115–178, 1998.

Appendix A

Auxiliary results

In this chapter we state several auxiliary results that are either well known or slight modifications of standard results. Therefore proofs, when provided, will be rather short.

The following proposition is analogous to a well known theorem of Caffarelli, Gidas, and Spruck ([36]):

Proposition A.1. *Let $T \geq 0$ and $\mathbb{R}^n_{-T} = \{y \in \mathbb{R}^n \mid y^n > -T\}$. Consider the problem*

$$\begin{cases} \Delta u + K u^p = 0, \ u > 0 & \text{in } \mathbb{R}^n_{-T}, \\ \frac{\partial u}{\partial x_n} = 0 & \text{on } \partial \mathbb{R}^n_{-T}, \\ u(0) = 1, \quad 0 \text{ is a local maximum of } u, \end{cases}$$

where $p \in (1, \frac{n+2}{n-2}]$. If $p < \frac{n+2}{n-2}$ then this problem has no solution. If $p = \frac{n+2}{n-2}$ then

$$u(x', x_n) = \left(\frac{1}{1 + |(x', x_n)|^2} \right)^{\frac{n-2}{2}} = U(x)$$

in which case $T = 0$ necessarily.

Proof. [45] (see also the proof of proposition 2.4 in [30], and [24] p. 498). □

Now we recall some transformation laws.

Proposition A.2. *Let (M, g) be a Riemannian manifold with boundary and $\phi > 0$ a smooth function. Let $\tilde{g} = \phi^{\frac{4}{n-2}} g$, then*

$$L_{\tilde{g}}(\phi^{-1} u) = \phi^{-\frac{n+2}{n-2}} L_g u \tag{A.1}$$

$$R_{\tilde{g}} = -c(n)^{-1} \phi^{-\frac{n+2}{n-2}} L_g \phi \tag{A.2}$$

$$B_{\tilde{g}}(\phi^{-1}u) = \phi^{-\frac{n}{n-2}} B_g u \tag{A.3}$$

$$\tilde{\kappa}_{ij} = \phi^{\frac{2}{n-2}} \kappa_{ij} + \frac{2}{n-2} \phi^{\frac{4-n}{n-2}} \frac{\partial \phi}{\partial \nu_g} g_{ij} \tag{A.4}$$

$$\tilde{\kappa} = \frac{2}{n-2} \phi^{-\frac{n}{n-2}} B_g \phi \tag{A.5}$$

where quantities with ˜ *refer to the metric* $\tilde{g}$*,* κ_{ij} *and* κ *are the second fundamental form and the mean curvature, respectively.*

Proof. Direct calculation (see [12, 17, 19] for example). □

Proposition A.3. *Up to a conformal change we can assume that in small balls the scalar curvature is positive and that the mean curvature of* ∂M *vanishes.*

Proof. The idea of the proof is to perform two conformal changes on the metric, one to produce a metric with zero mean curvature and a further one to achieve positive scalar curvature. Denote by $\phi_1 > 0$, the first eigenfunction of the conformal Laplacian with boundary condition $B_g \phi_1 = 0$, i.e.,

$$\begin{cases} L_g \phi_1 + \lambda_1 \phi_1 = 0, & \text{in } M, \\ B_g \phi_1 = 0, & \text{on } \partial M. \end{cases}$$

See [19] for the existence of ϕ_1; the fact that $\phi_1 > 0$ follows from a standard calculus of variation argument. By transformation law (A.5), the metric $g_1 = \phi_1^{\frac{4}{n-2}} g$ has zero mean curvature.

Now let $x_0 \in \partial M$ and consider a small ball $B_{2\delta}(x_0)$ near the boundary. Denote by $\psi_1 > 0$, the first eigenfunction of the Laplacian Δ_{g_1} with the boundary condition as below:

$$\begin{cases} \Delta_{g_1} \psi_1 + \mu_1 \psi_1 = 0, & \text{in } B_{2\delta}(x_0), \\ \psi_1 = 0, & \text{on } \partial^+ B_{2\delta}(x_0), \\ B_{g_1} \psi_1 = \frac{\partial \psi_1}{\partial \nu_{g_1}} = 0, & \text{on } \partial' B_{2\delta}(x_0). \end{cases}$$

The existence and positivity of ψ_1 again follows from a standard calculus of variations argument. Consider the metric $\tilde{g} = \psi_1^{\frac{4}{n-2}} g_1$ on $B_{2\delta}(x_0)$. Then from (A.2),

$$R_{\tilde{g}} = -c(n)^{-1} \psi^{\frac{n+2}{n-2}} L_{g_1} \psi_1 = -c(n)^{-1} \psi_1^{\frac{n+2}{n-2}} (\Delta_{g_1} \psi_1 - R_{g_1} \psi_1).$$

Since $\mu_1 \to \infty$ as $\delta \to 0$ we can choose $\delta > 0$ so small that

$$\Delta_{g_1} \psi_1 - R_{g_1} \psi_1 = -\mu_1 \psi_1 - R_{g_1} \psi_1 < 0,$$

and therefore $R_{\tilde{g}} > 0$ on $B_\delta(x_0)$. Notice that shrinking $B_{2\delta}(x_0)$ does not affect R_{g_1} as ϕ_1 is defined on the whole of M. Finally, the mean curvature for $\tilde{g}$ is $\tilde{\kappa} = \frac{2}{n-2}\psi^{-\frac{n}{n-2}} B_{g_1}\psi = 0$. □

The next result immediately follows.

Corollary A.4. *Up to a conformal change the maximum principle holds for the conformal Laplacian in small balls. More precisely, if $L_g u \geq 0$ in $B_\sigma(x_0)$, $u > 0$, then there exists a constant $C > 0$, independent of u, such that $\sup_{B_\sigma(x_0)} u \leq C \sup_{\partial B_\sigma(x_0)} u$, provided σ is small enough.*

Lemma A.5. *Let ψ be a solution of*

$$\begin{cases} \Delta\psi + n(n+2)U^{\frac{4}{n-2}}\psi = 0, & \text{in } \mathbb{R}^n_+, \\ \frac{\partial\psi}{\partial y^n} = 0, & \text{on } \partial\mathbb{R}^{n-1}, \\ \lim_{|y|\to\infty}\psi(y) = 0. \end{cases}$$

Then it takes the form

$$\psi(y) = c_0\Big(\frac{n-2}{2}U + y\cdot\nabla U\Big) + \sum_{j=1}^{n-1} c_j\partial_j U,$$

for some constants $c_0, \ldots, c_{n-1}$.

Proof. Since $\partial_n\psi = 0$ on $\mathbb{R}^{n-1}$, we can make a C^2 reflection across $\mathbb{R}^{n-1}$ and then the result follows from [46]. □

The following is a Harnack-type inequality.

Lemma A.6. *Let $x_i \to \bar{x}$ be an isolated blow-up point and assume that $\bar{r}$ is sufficiently small. Then for all r such that $0 < r < \bar{r}$ we have*

$$\sup_{B_r(x_i)\setminus B_{r/2}(x_i)} u_i \leq C \inf_{B_r(x_i)\setminus B_{r/2}(x_i)} u_i,$$

for some constant C independent of i and r.

Proof. It follows from a combination of lemma A.1 of [24], the Harnack inequality, and the definition of isolated blow-up points. □

Proposition A.7. *(Pohozaev identity) Let $u > 0$ be a solution of $L_g u + K f_i^{-\delta} u^p = 0$ on $B^+_\rho = \{x \in B_\rho(0) \mid x^n \geq 0\}$. Then*

$$\int_{\partial B^+_\rho} \Big(\big(\frac{n-2}{2}u + x^k\partial_k u\big)\frac{\partial u}{\partial\nu_0} - \frac{1}{2}x^k\nu^k_0|\nabla_0 u|^2 + \frac{1}{p+1}K(x)x^k\nu^k_0 u^{p+1}\Big)d\sigma \quad \text{(A.6)}$$

$$
\begin{aligned}
&= -\int_{B_\rho^+} \big(\frac{n-2}{2}u + x^k\partial_k u\big)\big((g^{ij}-\delta^{ij})\partial_{ij}u + \partial_j g^{ij}\partial_i u\big)dx \\
&+ \int_{B_\rho^+} c(n)\big(\frac{n-2}{2}u + x^k\partial_k u\big)Rudx + \frac{1}{p+1}\int_{B_\rho^+} x^k\partial_k K(x)u^{p+1}dx \\
&+ \Big(\frac{n}{p+1} - \frac{n-2}{2}\Big)\int_{B_\rho^+} K(x)u^{p+1}dx,
\end{aligned}
$$

where quantities with $_0$ *refer to the Euclidean metric and* $K(x) = Kf^{-\delta}(x)$.

Proof. Standard integration by parts argument. □