On the Partition Function for $\mathbb{CP}^1$ Instantons on a Flat Torus

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Abstract of the Dissertation

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The partition function for the free theory of $\mathbb{CP}^1$-valued fields on a flat two-dimensional torus is studied. The partition function localizes to an infinite series of finite-dimensional integrals of the form

$$\int_{N_d} (\det{}'\Delta_w)^{-1/2}\, d\mu_d(w)$$

where N_d is the space of all holomorphic maps w of topological degree d. Δ_w is the Laplace operator on $w^*(T\mathbb{P}^1)$ with respect to the pullback of teh Fubini-Study metric, and $d\mu_d$ is the induced measure.

Through the process of ζ-regularization of the determinant and variation of the conformal anomaly of the metric, the determinant is explicitly calculated in terms

of the zeroes $a_1, a_2, \ldots, a_d$, poles $b_1, b_2, \ldots, b_d$, and scaling factor c of w. For each degree, a simple family of bundle metrics are introduced, parametrized by the universal cover of the Jacobian variety, to act as the base points for the conformal variation.

The measure $d\mu_d(w)$ is computed exactly with respect to the same coordinates. This requires a careful examination of its dependence upon the complex structure of the torus.

Finally, we discuss the convergence properties of each of the above integrals.

To my wife.

Contents

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Chapter 1

Introduction

1.1 Physical Motivation

Let $M = \mathbb{R}^{1,3}$ be Minkowski spacetime with metric $-dt^2 + dx^2 + dy^2 + dz^2$. A field ϕ on spacetime taking values in some other space X is simply a section of a bundle over M with fibre X. For example, if X is $\mathbb{R}$ (or $\mathbb{C}$), then ϕ is called a real (or complex) scalar field. The most common example is the case where ϕ is a section of some tensor or spinor bundle over M.

A classical field theory is given by a functional S, called the action, on the space of fields under consideration:

$$S[\phi] = \int_M \mathcal{L}[\phi(t, \vec{x})]dtd^3\vec{x},$$

where $\mathcal{L}$, the Lagrangian density, is a scalar quantity that is constructed out of ϕ and its derivatives. The dynamics of the field theory are encoded into the action functional by "the principal of least action." This states that the physical fields are those for which the action is stationary (usually minimum); i.e. the only fields allowable in the theory are those that satisfy the cooresponding Euler-Lagrange equations. Thus, the dynamics of the physical fields are completely determined.

In quantum field theory, fields are no longer explicitly determined by the Euler-Lagrange equations; rather, they are self-adjoint operator-valued distributions, acting on the space of probability amplitudes, which can be thought of as a complex projective Hilbert space. The dynamics of the theory is contained in the correlation functions: the expectation values of these distributions evaluated at different points of spacetime. The classical action still determines the dynamics of the quantum theory in the following way. If $\tilde{F}$ is a quantum operator corresponding to the classical functional F, then its expected value is

$$\left\langle \tilde{F} \right\rangle := \frac{\int F[\phi] e^{iS[\phi]} \mathcal{D}\phi}{\int e^{iS[\phi]} \mathcal{D}\phi}$$

where the "integral" is performed over the space of classical fields ϕ.

In the classical realm, observables are built out of combinations of the fields and their derivatives, and we demand that this relationship carry over to the quantum world in some consistent way. Therefore, the major quantities of interest are the "n-point" correlation functions $\langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle$, where $x_1, \ldots, x_n$ are distinct time-like separated points of spacetime, with $x_1 > x_2 > \ldots > x_n$ in the time-ordering. These correlation functions are encoded in a generating functional called the partition function:

$$\begin{aligned} Z[J] &= \int e^{i\left(S[\phi] + \int_M J\phi\right)} \mathcal{D}\phi; \\ \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle &= (-i)^n \frac{1}{Z[0]} \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \cdots \frac{\delta}{\delta J(x_n)} Z[J]\bigg|_{J=0}. \end{aligned}$$

Some texts call $Z[0]$ the partition function and simply refer to $Z[J]$ as the generating functional, in order to better match terminology in statistical mechanics.

These integrals do not make sense in the in the usual measure-theoretic sense, as there is no translation-invariant measure $\mathcal{D}\phi$ on the infinite-dimensional space of fields. In order to make sense of this, we perform a Wick rotation–we formally rotate our time coordinate so that it takes imaginary values, replacing t by it. This changes the metric on M to one of Euclidean signature, and therefore changes the action accordingly:

$$S[\phi] \mapsto iS_E[\phi] = i\int_M \mathcal{L}_E[\phi(t,\vec{x})]dtd^3\vec{x}.$$

Thus, the partition function also changes, $Z_E[J] = \int \exp\left\{-\int_M \left(\mathcal{L}_E[\phi] + J\phi\right) d^4x\right\} \mathcal{D}\phi$, to one more well understood mathematically. In this case, the kinetic term in the exponent can be combined with $\mathcal{D}\phi$ to determine a Gaussian measure on the space of fields. If one can evaluate this Euclidean partition function, it can be analytically continued back to give the result in Minkowski signature.

It is physically interesting and important to consider spacetimes with other topologies and curvatures, where computing the partition function is more difficult. In addition, the partition function is a valuable tool in string theory, where the objects of interest are not fields on spacetime, but rather maps from the string's worldsheet to spacetime. The worldsheet may be any smooth orientable surface with a Minkowski signature metric. Therefore, it is important to be able to compute the partition function in a wide variety of situations. In many of these cases, Wick rotation is harder to define. For instance, in field theory it is only understood if spacetime is a totally hyperbolic manifold [24, 12]. Nevertheless, for computational reasons, it is still more desirable to work in the Euclidean signature than the Minkowski signature. In fact, many texts only define the partition function with the Euclidean convention [21, 5]. This is the approach taken in this thesis.

1.2 Previous Work

In the late 1970's Fateev, Frolov, Schwarz, and Tyupkin [9] computed the leading terms of the partition function for maps from $\mathbb{R}^2$ with the usual Euclidean metric to the Riemann sphere with the round metric, subject to the following "free" action:

$$S_f[w] := \frac{1}{2f}\int_{\mathbb{R}^2} g^{\alpha\beta}\partial_\alpha w^i \partial_\beta w^j \rho_{ij} d\mu_g,$$

where g is the metric on $\mathbb{R}^2$, ρ, the metric on the Riemann sphere, and f is a coupling constant. In order to make the theory conform to certain physical axioms, they restricted their attention to fields whose limit at ∞ was well-defined. In order to handle this restriction, they examined the case of fields on a sphere of radius R, and then let R tend toward ∞. Physically, the value $w(\infty)$ represents the vacuum expectation value of that field. Mathematically, some restriction of this sort on the space of fields is necessary in order for the partition function to converge.

They computed the partition function in the small-f limit by formally applying Laplace's method of steepest descent, which localized the integral onto the spaces of holomorphic and anti-holomorphic maps, indexed by topological degree:

$$Z[J] \approx \sum_{d\in\mathbb{Z}} (2\pi f)^{-(|d|+1)}\, e^{-\frac{\pi}{f}|d|} \int_{N_d} e^{-\int Jw} \left(\det \Delta_w\right)^{-1/2} d\mu_d(w),$$

where d indexes the topological degree of the field w, N_d is the finite-dimensional subspace of holomorphic fields of degree d, $\det \Delta_w$ is the ζ-regularized determinant of a w-dependent elliptic operator on a particular line bundle over S^2, and μ_d is the measure induced by a natural metric on N_d. Details of this derivation can be found in Chapter 2.

For any fixed degree d, they parametrized $w \in N_d$ in terms of its zeroes $a_1, \cdots, a_d$

and poles $b_1, \cdots, b_d$ and a multiplicative factor c, so that

$$w(z) = c\frac{(z-a_1)\cdots(z-a_d)}{(z-b_1)\cdots(c-b_d)}.$$

Δ_w is the Laplace operator on $w^*(TS^2)$ with respect to the bundle metric $w^*\rho$ and base metric g. They were able to compute the dependence of $\det \Delta_w$ on any conformal variations in either metric. Furthermore, since any two metrics on a given bundle over the sphere are conformally equivalent, by computing $\det \Delta$ with respect to a relatively simple metric, they were able to extrapolate to an expression of $\det \Delta_w$ in terms of the coordinates.

For $w \in N_d$, a tangent vector $v \in T_w N_d$ can be associated to a holomorphic vector field $v(z) \in \Gamma(w^*(TS^2))$. A metric can then be defined on $T_w N_d$ by

$$\langle v_1, v_2 \rangle = \int_{S^2} w^*\rho\,(v_1(z), v_2(z))\, d\mu_g(z).$$

This defines a metric on N_d, which induces a measure μ_d. By changing from the coordinate basis of $T_w N_d$ to the vector fields defined by the monomials $1, z, \ldots, z^d$, some clever linear algebra allowed them to compute $d\mu_d(w)$ in terms of the local coordinates.

After taking the limit as $R \to \infty$, they were led to the following expression for the measure in the partition function formula:

$$\begin{aligned}(\det \Delta_w)^{-1/2}\, d\mu_d(w) \quad &= \quad \pi^{2d+1} e^{-d(\Gamma'(1)+2)} \left(|d|!\right)^2 \left(1+|c|^2\right)^{-2} \\ & \qquad \prod_{1\le j<k\le d} |a_j - a_k|^2 \, |b_j - b_k|^2 \prod_{1\le l,m\le d} |a_l - b_m|^{-2} \\ & \qquad |dc|^2 \prod_{1\le i\le d} |da_i|^2\, |db_i|^2\end{aligned}$$

Rather than compute the partition function explicitly, they drew parallels between this system, a neutrally charged classical Coulomb system, and a sine-Gordon system. This allowed them to solve for the two-point correlation function explicitly.

The work of Fateev et al raises a question which has long been unanswered: Can the partition function be calculated in the case of $\mathbb{CP}^1$-valued fields on other Riemann surfaces? There has been much work done on circle-valued fields; for example see [13, 2, 4, 3]. However, there have been very few mentions of $\mathbb{CP}^1$-valued fields. Sutcliffe discussed $\mathbb{CP}^1$-valued instantons (the higher degree holomorphic maps) on the torus as a tool for approximating certain periodic sine-Gordon solitons, but he did not attempt to compute the partition function for these fields [23]. This thesis aims to be a first step toward filling this gap. Also, literature on the classical Coulomb system or the sine-Gordon model on a torus is scarce. It is hoped that the analogy in the genus 0 case still holds, and that the results in this thesis may shed some light into these other models.

Chapter 2

Background

In this chapter, we describe the background for the main results of this thesis and establish notation, following closely the description given by Fateev, Frolov, Schwarz, and Tyupkin [9]. The material in this chapter describes the field theory on any compact Riemann surface.

2.1 The Action Functional

Let Σ be a compact Riemann surface with Riemannian metric g and associated measure $d\mu_g$. Let $\mathcal{W}$ be the space of smooth maps $w : \Sigma \to \mathbb{P}^1$, later called fields. We define the free action functional S on $\mathcal{W}$ by

$$S[w] := \frac{1}{2f} \int_\Sigma g^{\alpha\beta} \partial_\alpha w^i \partial_\beta w^j \rho_{ij} d\mu_g, \tag{2.1.1}$$

where ρ is the Fubini-Study metric on $\mathbb{P}^1$ and $\dfrac{1}{2f}$ is the coupling constant. In terms of a local holomorphic coordinate z on Σ and the usual coordinate on $\mathbb{P}^1$, this can be expressed as

$$S[w] = \frac{1}{f} \int_\Sigma \frac{|\partial_z w|^2 + |\partial_{\bar{z}} w|^2}{\left(1 + |w|^2\right)^2} |dz|^2 ,$$

where $|dz|^2$ is shorthand for the form $\frac{i}{2}dz \wedge d\bar{z}$. The topological charge, or degree, of a map w is given by the integral

$$\begin{aligned} \deg w : &= \frac{1}{Vol(\mathbb{P}^1)} \int_\Sigma w^* \left(d\mu_\rho\right) \\ &= \frac{1}{\pi} \int \frac{|\partial_z w|^2 - |\partial_{\bar{z}} w|^2}{\left(1+|w|^2\right)^2} |dz|^2 . \end{aligned}$$

Therefore, we see that

$$\begin{aligned} S[w] &= \frac{\pi \deg w}{f} + \frac{2}{f} \int_\Sigma \frac{|\partial_{\bar{z}} w|^2}{\left(1+|w|^2\right)^2} |dz|^2 && (2.1.2) \\ &= -\frac{\pi \deg w}{f} + \frac{2}{f} \int_\Sigma \frac{|\partial_z w|^2}{\left(1+|w|^2\right)^2} |dz|^2 . && (2.1.3) \end{aligned}$$

This shows that when restricted to $\mathcal{W}_d$, the maps of fixed degree d, $S[w]$ achieves a minimum value of $\frac{\pi}{f}|d|$ on the finite-dimensional submanifold N_d, consisting of holomorphic functions when $d \geq 0$ and antiholomorphic functions when $d \leq 0$. In particular, N_0 is the set of constant maps.

2.2 The Partition Function

The main goal of this thesis is to give meaning to and calculate the normalized partition function

$$Z[\Phi] := \frac{\int_{\mathcal{W}} \Phi[w] e^{-S[w]} \mathcal{D}w}{\int_{\mathcal{W}} e^{-S[w]} \mathcal{D}w},$$

where Φ is a function on the space of fields. If we take $\Phi[w] = -\int_\Sigma Jw d\mu_g$, then $Z[\Phi]$ is equivalent to the quantity $Z_E[J]/Z_E[0]$ described in Chapter 1. As stated in Chapter 1, $\mathcal{D}w$ does not make sense as a translation-invariant measure on the space of fields. However, the integral is of a form familiar to physicists, and most likely can be calculated via a lattice renormalization procedure in which the coupling constant f becomes dependent upon the lattice cutoff. ($f \to 0$ as the lattice cutoff is removed.) We will not pursue this course, but rather simply define $Z[\Phi]$ in the small f limit by formally applying Laplace's method of steepest descent.

Let us briefly recall how this works in finite dimensions [11]. Suppose $S : M^m \to \mathbb{R}$ is a smooth function that attains its minimum value γ on an n-dimensional submanifold N. Suppose further that N is a nondegenerate stationary manifold; that at every point $x \in N$, 0 is an eigenvalue of $S''(x)$ with multiplicity n. Then

$$\begin{aligned} \int_M g(x)\exp(-S(x)/f)d^m x &= (2\pi f)^{(m-n)/2}\exp(-\gamma/f)\int_N g(x)\left(\det S''(x)\right)^{-1/2} d^n x \\ &\quad + O\left(f^{(m-n)/2+1}\right). \end{aligned}$$

Here we are taking the regularized determinant, the product of the non-zero eigenvalues, of the Hessian $S''(x)$. It will be useful to scale the Hessian by some constant, k, to be determined later:

$$\int_M g(x)\exp(-S(x)/f)d^m x \approx (2\pi f k)^{(m-n)/2}\exp(-\gamma/f)\int_N g(x)\left(\det(kS''(x))\right)^{-1/2} d^n x$$

If we formally apply this to the partition function, we arrive at

$$\begin{aligned} Z[\Phi] &\approx \frac{\sum_d (2\pi f k)^{(\dim\mathcal{W}-\dim N_d)/2}\, e^{-\frac{\pi}{f}|d|}\int_{N_d}\Phi[w]\left(\det kS''(w)\right)^{-1/2} d\mu_d(w)}{\sum_d (2\pi f k)^{(\dim\mathcal{W}-\dim N_d)/2}\, e^{-\frac{\pi}{f}|d|}\int_{N_d}\left(\det kS''(w)\right)^{-1/2} d\mu_d(w)} \\ &= \frac{\sum_d (2\pi f k)^{-\dim N_d/2}\, e^{-\frac{\pi}{f}|d|}\int_{N_d}\Phi[w]\left(\det kS''(w)\right)^{-1/2} d\mu_d(w)}{\sum_d (2\pi f k)^{-\dim N_d/2}\, e^{-\frac{\pi}{f}|d|}\int_{N_d}\left(\det kS''(w)\right)^{-1/2} d\mu_d(w)}. \end{aligned} \tag{2.2.1}$$

Here, S'' is the operator-valued function on N_d determined by

$$S[w+v] = S[w] + \frac{1}{2f}\langle v, S''(w)v\rangle + o\left(\|v\|^2\right), \tag{2.2.2}$$

for any infinitesimal variation v of w. To calculate $S''(w)$ for (anti-)holomorphic w, we first naturally identify the space of infinitesimal variations of w with the space, $\Gamma\left(w^*\left(T\mathbb{P}^1\right)\right)$, of sections of the pullback via w of the holomorphic tangent bundle, $T\mathbb{P}^1$, as follows: Let v be an infinitesimal variation of w, that is, $v \in T_w\mathcal{W}_d$. Therefore, there is a smooth curve $t \mapsto w_t$ in $\mathcal{W}_d$ satisfying $w_0 = w$ and $\left.\frac{d}{dt}\right|_{t=0} w_t = v$. For each fixed $z \in \Sigma$, $w_t(z)$ is a smooth curve in $\mathbb{P}^1$, so that $\left.\frac{d}{dt}\right|_{t=0} w_t(z) \in T_{w(z)}\mathbb{P}^1 \cong \left(w^*\left(T\mathbb{P}^1\right)\right)_z$. Thus, we can define

$$v(z) := \left.\frac{d}{dt}\right|_{t=0} w_t(z),$$

so that $v(z)$ is a smooth section of $w^*\left(T\mathbb{P}^1\right)$.

Through this identification, we can express the inner product in (2.2.2) as

$$\begin{aligned}\langle v_1, v_2\rangle &:= \int_\Sigma \left(v_1(z), v_2(z)\right)_w d\mu_g \\ &= \frac{1}{2}\int_\Sigma \frac{v_1\bar{v}_2 + \bar{v}_1 v_2}{\left(1+|w|^2\right)^2} d\mu_g,\end{aligned}$$

where $(,)_w$ is the Riemannian metric on $w^*\left(T\mathbb{P}^1\right)$ associated to the pullback of the Fubini-Study metric on $\mathbb{P}^1$.

Let w be any field. The first variation of S at w is, in essence, the derivative of S at w. Let $v \in T_w\mathcal{W}$, and let $w(u) : (-\epsilon, \epsilon) \to \mathcal{W}$ be a smooth curve satisfying $w(0) = w$ and $\left.\frac{d}{du}\right|_{u=0} w(u) = v$. $S[w(u)]$ is a smooth real-valued function of u, and first variation of S at w in the direction of v is defined by the formula $\partial_u S[w]|_{u=0}$. Assume for now that $\deg w \geq 0$ so that S is given by (2.1.2). Since $\deg w$ is a local invariant, the first variation can be expressed as

$$\partial_u S[w] \;=\; 2\int_\Sigma \frac{\partial_u\partial_{\bar{z}}w\partial_z\bar{w} + \partial_u\partial_z\bar{w}\partial_{\bar{z}}w}{\left(1+|w|^2\right)^2} - 2\frac{\partial_{\bar{z}}w\partial_z\bar{w}\left(\partial_u w\bar{w} + w\partial_u\bar{w}\right)}{\left(1+|w|^2\right)^3}|dz|^2$$

$$
\begin{aligned}
= \quad & -2\int_{\Sigma} \partial_u w \partial_{\bar{z}} \left(\frac{\partial_z \bar{w}}{\left(1+|w|^2\right)^2} \right) + \partial_u \bar{w} \partial_z \left(\frac{\partial_{\bar{z}} w}{\left(1+|w|^2\right)^2} \right) \\
& +2\frac{\partial_{\bar{z}} w \partial_z \bar{w} \left(\partial_u w \bar{w} + w \partial_u \bar{w}\right)}{\left(1+|w|^2\right)^3} \, |dz|^2 \,,
\end{aligned}
$$

where we have integrated by parts to obtain the second formula. From here we clearly see that $\partial_u S[w]|_{u=0}$ vanishes, independently of the value of $\partial_u w|_{u=0}$, whenever w is holomorphic, as we expected.

$S''[w]$ is defined by the second variation of S at w, which is analogous to the Hessian matrix in finite dimensions. Let v_1 and v_2 be any two tangent vectors based at w; we can find a two parameter function $w(u_1, u_2) : (-\epsilon, \epsilon)^2 \to \mathcal{W}$ where $w(0,0) = w$, $\partial_{u_1} w|_{(u_1,u_2)=(0,0)} = v_1$, and $\partial_{u_2} w|_{(u_1,u_2)=(0,0)} = v_2$. We can compute the second variation $\partial_{u_1}\partial_{u_2} S[w]$ in much the same way as first variation. Evaluating it in the case where w is a holomorphic field of degree d, we find the formula

$$
\partial_{u_1}\partial_{u_2} S[w]|_{(u_1,u_2)=(0,0)} = -4\left\langle \partial_{u_2} w, \frac{1}{\det |g|}\left(1+|w|^2\right)^2 \partial_z \left(\frac{\partial_{\bar{z}}\partial_{u_1} w}{\left(1+|w|^2\right)^2} \right) \right\rangle .
$$

Therefore, $S''(w) = 4\Delta_w$, where

$$
\Delta_w \quad := \quad -\frac{1}{\det |g|}\left(1+|w|^2\right)^2 \partial_z \left(\frac{\partial_{\bar{z}}}{\left(1+|w|^2\right)^2} \right) \tag{2.2.3}
$$

is the $\bar{\partial}$-Laplace operator acting on sections of $w^*(T\mathbb{P}^1)$.

In the case of $\deg w < 0$ so that the critical points are anti-holomorphic functions, the same computations carry through with the only change being the switching of ∂_z and $\partial_{\bar{z}}$. Therefore, for anti-holomorphic w, we define:

$$
\Delta_w := -\frac{1}{\det |g|}\left(1+|w|^2\right)^2 \partial_{\bar{z}} \left(\frac{\partial_z}{\left(1+|w|^2\right)^2} \right) .
$$

Ergo, if we set $k = \frac{1}{4}$ in (2.2.1), we see

$$Z[\Phi] \approx \frac{\sum_d (2^{-1}\pi f)^{-\dim N_d/2} e^{-\frac{\pi}{f}|d|} \int_{N_d} \Phi[w] (\det \Delta_w)^{-1/2} \, d\mu_d(w)}{\sum_d (2^{-1}\pi f)^{-\dim N_d/2} e^{-\frac{\pi}{f}|d|} \int_{N_d} (\det \Delta_w)^{-1/2} \, d\mu_d(w)}. \tag{2.2.4}$$

In the rest of this thesis, we will explore this quantity in the case when Σ is a flat complex torus, with $\det|g| = 1$. In this case, there are no (anti-)holomorphic maps of degree (-)1, and therefore the integrals over $\mathcal{W}_{\pm 1}$ are suppressed in the small-f limit. Thus, the sums in (2.2.4) range over all $d \neq \pm 1$. In Chapters 3 and 4, we will compute the terms corresponding to $d \geq 2$. In Chapter 3, we will concentrate on the calculation of $\det \Delta_w$, while Chapter 4 will cover the computation of the metric $d\mu_d$. In Chapter 5, we will complete the computation by examining $d \leq -2$ and $d = 0$ separately. We will then discuss the convergence of the partition function.

Chapter 3

Determinant of Δ_w

In this chapter, we compute the dependence of $\det \Delta_w$ on w, where Δ_w given by (2.2.3) is an operator on the space of sections of the holomorphic line bundle $w^*(T\mathbb{P}^1)$. We will restrict our attention to the case when $d \geq 2$; i.e. w is holomorphic. This computation has three main parts. First, the entire problem is reframed in terms of the theory of multipliers. Second, the dependence on the conformal anomaly of the bundle metric is computed in order to reduce the problem to a computationally easier metric. Third, the spectrum of this reduced metric will be computed exactly.

Let Σ be the complex torus $\mathbb{C}/\Lambda$, where Λ is the lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$ with $\Im(\tau) > 0$. As a matter of convention in this chapter, variables without decoration such as z and a_j will denote points of $\mathbb{C}$, and the same variables with tildes, $\tilde{z}$ and $\tilde{a}_j$, will denote the corresponding point on the torus. Furthermore, we will denote the real part and imaginary part of any complex number A by A_x and A_y, respectively. The only exception to this rule is the complex variable z, which we will decompose in the traditional way as $x + iy$.

3.1 Review of the Geometry of Line Bundles and Multipliers

In this section, we describe holomorphic line bundles on Σ in terms of their multipliers. This is a reformulation of material covered in [10], in particular Chapter 2, Section 6, but this section sets the notational conventions for the remainder of the thesis.

It is a well known fact [10] that the Picard group of equivalence classes of line bundles over Σ is parametrized by the group of divisors modulo linear equivalence. As Σ is one-dimensional, $Div(\Sigma)$ is simply the group of formal $\mathbb{Z}$-linear combinations of points of Σ. Given a divisor $D = \sum_{j=1}^{J} n_j \tilde{z}_j$, the corresponding line bundle, L_D, has a section 1_D, which we shall call a canonical section of the bundle, with a zero of order n_j at $\tilde{z}_j$. Zeroes of negative order are of course interpreted as poles of order $|n_j|$. 1_D defines a trivialization of L_D over the open set $U_0 := \Sigma \setminus \{\tilde{z}_j\}_{j=1}^{J}$. Furthermore, if U_j is a small open disc around $\tilde{z}_j$ for each j, then we can trivialize L_D over U_j in such a way that the transition map from U_0 to U_j is given by $(\tilde{z} - \tilde{z}_j)^{n_j}$. Furthermore, this construction is a group homomorphism: $L_{D+D'} \cong L_D \otimes L_{D'}$.

If $\pi : \mathbb{C} \to \Sigma$ is the quotient map, and L_D is a line bundle corresponding to a divisor $D = \sum_{j=1}^{J} n_j \tilde{z}_j$, then $\pi^{-1}(L_D)$ is trivializable. In fact, a holomorphic trivialization can be chosen uniquely (up to a scaling) so that for any continuous section, s, of L_D, $\pi^* s(z+1) = \pi^* s(z)$. More explicitly, let z_j be a preimage of $\tilde{z}_j$ under π. Let $n = \sum_{j=1}^{J} n_j$, and let $z_D = \sum_{j=1}^{J} n_j z_j$. Define the function $f_{n,z_D} : \mathbb{C} \to \mathbb{C}$ by

$$f_{n,z_D}(z) = \exp\left\{-\eta\left(nz^2 - 2z_D z\right) + i\pi n z\right\}.$$

Here, η is defined by the equation $\zeta(z+1) = \zeta(z) + 2\eta$, where $\zeta(z)$ is the Weierstrass ζ-function. Then if $\sigma(z)$ is Weierstrass's σ-function, we can define a holomorphic trivialization such that, for a given choice of canonical divisor of L_D:

$$s_D(z) := \pi^*(1_D)(z) = f_{n,z_D}(z) \prod_{j=1}^{J} \sigma\left(z - z_j\right)^{n_j}.$$

Here, we have abused notation. s_D is dependent not only on the divisor D, but also on the specific preimanges z_j of the points in D. Changing the choice of preimages will change s_D by an scaling factor. From here, a quick computation using the periodicity relations of the σ-function [1], we see

$$\begin{aligned} s_D(z+1) &= s_D(z); \\ s_D(z+\tau) &= (-1)^n \exp\left\{-2\pi i\left(nz - z_D\right)\right\} s_D(z). \end{aligned} \tag{3.1.1}$$

These quasi-periodicity relations must be satisfied for any section of L_D. It is important to note that the multiplier $(-1)^n \exp\left\{-2\pi i\left(nz - z_D\right)\right\}$ only depends on the representatives $\{z_j\}$ chosen for the points in the divisor D through the combination z_D. This is a consequence of the fact that the Jacobian variety of an elliptic curve is naturally isomorphic to the elliptic curve, itself. Due to this isomorphism, the line bundle L_D is determined up to equivalence precisely by its degree n and the point on the elliptic curve $\tilde{z}_D$.

Another useful fact is that $\log f_{n,z_D}$ is additive in n and z_D;

$$f_{n_1,z_{D1}} f_{n_2,z_{D2}} = f_{n_1+n_2,z_{D1}+z_{D2}}.$$

This implies a group structure on the set of functions $\{s_D\}$:

$$s_{D+D'}(z) = s_D(z) s_{D'}(z).$$

This arises from the fact that tensoring two canonical sections yields a third: If $D = \sum_j n_j \tilde{z}_j$

and $D' = \sum_k m_k \tilde{w}_k$ are two divisors, 1_D and $1_{D'}$ canonical sections of the corresponding line bundles, then $1_D \otimes 1_{D'}$ is a canonical section of $L_D \otimes L_{D'} \cong L_{D+D'}$.

Finally, if h is a Hermitian metric on the line bundle L_D, for any continuous sections f and g, $h(f,g)$ is a well-defined function on the torus. Pulling back to a holomorphic trivialization as above, we see that $h(z)\overline{f(z)}g(z)$ must be a doubly-periodic function, and so $h(z)$ must be represented by a nonzero function, transforming under the action of the lattice by:

$$\begin{aligned} h(z+1) &= h(z); \\ h(z+\tau) &= \exp\left\{-4\pi\left(ny - y_D\right)\right\} h(z). \end{aligned} \tag{3.1.2}$$

One such metric on L_D is given by

$$h_{n,z_D}(z) = \exp\left\{-\frac{2\pi}{\tau_y}\left[ny\left(y-\tau_y\right) - 2y_D y\right]\right\}.$$

This family of metrics has the same additivity properties as the functions f_{n,z_D}, and therefore, we can decompose the bundle, canonical section, and metric:

$$\begin{aligned} L_D &\cong \bigotimes_{j=1}^{J} \left(L_{z_j}\right)^{\otimes n_j}; \\ s_{\sum_{j=1}^{J} n_j z_j} &= \prod_{j=1}^{J} \left(s_{z_j}\right)^{n_j} = \prod_{j=1}^{J} \left(f_{1,z_j}(z)\sigma(z-z_j)\right)^{n_j}; \\ h_{n,z_D} &= \prod_{j=1}^{J} \left(h_{1,z_j}\right)^{n_j}. \end{aligned}$$

Now let $w : \Sigma \to \mathbb{P}^1$ be a holomorphic map of degree d; then we can express w as

$$w(\tilde{z}) = c\frac{\prod_{j=1}^{d} \sigma\left(z-a_j\right)}{\prod_{k=1}^{d} \sigma\left(z-b_k\right)},$$

where $\{\tilde{a}_1, \ldots, \tilde{a}_d\} \cap \left\{\tilde{b}_1, \ldots, \tilde{b}_d\right\} = \emptyset$, and $\sum_j a_j = \sum_k b_k$. $L_w := w^*\left(T\mathbb{P}^1\right)$ is the line bundle associated to $2D_\infty(w)$, where $D_\infty(w) = \sum \tilde{b}_j$ is the divisor of poles of w. Let $B := \sum b_j$. In addition to the metric $h_{2d,2B}$, we have another choice of metric on L_w given by the pullback of the Fubini-Study metric, which we denote as h_w. If 1_w is the canonical section of L_w determined by pulling back $\dfrac{\partial}{\partial z}$ via w, then $h_w(1_w, 1_w) = \left(1 + |w|^2\right)^{-2}$. Since $\pi^*(1_w) = \prod_{j=1}^d \left(f_{1,b_j}(z)\sigma\left(z - b_j\right)\right)^2 = \prod_{j=1}^d s_{b_j}(z)^2$, we see that h_w is represented by the function

$$h_w(z) = \left[\left(1 + |w|^2\right) \prod_{j=1}^{d} \left|s_{b_j}(z)\right|^2\right]^{-2}.$$

One can easily check that $h_w(z)$ is well-defined and extends to be nonzero on the complex plane, and satisfies the correct quasi-periodicity conditions (3.1.2).

3.2 Dependence upon Conformal Anomaly

In this section, we will compute the dependence of $\det \Delta_w$ upon the conformal anomaly in the metric h_w. Formally, of course, if $\lambda_1 \leq \lambda_2 \leq \cdots$ are the positive eigenvalues of Δ_w, we would like to define

$$\det \Delta_w = \prod_{j=1}^{\infty} \lambda_j^2.$$

We take the product of the squares of the eigenvalues because we wish to calculate the determinant over the reals, and fibers of the bundle are complex lines. To make sense of this product, we define:

$$\ln \det \Delta_w := -2\zeta'(0),$$

where $\zeta(s) := \sum_{j=1}^{\infty} \lambda^{-s}$ for $\Re(s) > 1$. It was first proven in [17] that ζ can be uniquely meromorphically extended to the entire complex plane, and this extension is regular at $s = 0$.

3.2.1 Variation of the Conformal Anomaly

The work in this subsection applies to line bundles of sufficiently high degree over any compact Riemann surface. In particular, the bundle must have degree at least $2G-1$, where G is the genus of the surface.

In order to compute the dependence upon the conformal anomaly, we need a base metric, which we take to be

$$h_0(z,\bar{z})$$

If $h := e^{2\sigma}h_0$ is any metric in the conformal class of h_0, then in terms of local coordinates,

$$\Delta_h := \bar{\partial}^{*_h}\bar{\partial} = -g^{-1}h^{-1}\partial_z h\partial_{\bar{z}}.$$

We will also require the operator

$$\tilde{\Delta}_h := -\partial_{\bar{z}}g^{-1}h^{-1}\partial_z h,$$

which can be thought of as the operator $\bar{\partial}\bar{\partial}^*$ acting on L-valued $(0,1)$-forms. Let ψ be a tangent vector to 0 in the space of smooth real functions on Σ. ψ can be naturally identified with a function, which we shall also denote by ψ. Let $\sigma(u)$ be a one-parameter family of functions such that $\sigma(0)=0$ and $\frac{d}{du}\big|_{u=0}\sigma(u)=\psi$. Then for any metric h, $h(u) := e^{2\sigma(u)}h$ is a one-parameter family of metrics satisfying $\frac{d}{du}\big|_{u=0}h(u) = 2\psi h$. Then,

$$\begin{aligned}
\left.\frac{d}{du}\right|_{u=0}\Delta_{h(u)} &= \left.\frac{d}{du}\right|_{u=0}\left(-g^{-1}h(u)^{-1}\partial_z\left(h(u)\partial_{\bar{z}}\right)\right)\\
&= 2\psi g^{-1}h^{-1}\partial_z\left(h\partial_{\bar{z}}\right) - g^{-1}h^{-1}\partial_z\left(2\psi h\partial_{\bar{z}}\right)\\
&= -2\psi\Delta_h - g^{-1}h^{-1}\partial_z\left(2\psi h\partial_{\bar{z}}\right).
\end{aligned}$$

Let ζ_u denote the zeta function associated to $\Delta_{h(u)}$. From the result in Appendix A, we know

$$\begin{aligned} 2\Gamma(s)\zeta_u(s) &= \int_0^1 \left(\mathrm{Tr}\left\{e^{-t\Delta_{h(u)}}\right\} - \alpha_0 - \frac{\alpha_{-1}}{t}\right) t^s \frac{dt}{t} + \frac{\alpha_{-1}}{s-1} + \frac{\alpha_0 - p_0}{s} \\ &\quad + \int_1^\infty \left(\mathrm{Tr}\left\{e^{-t\Delta_{h(u)}}\right\} - p_0\right) t^s \frac{dt}{t}, \end{aligned}$$

where p_0 is the dimension of the kernel of $\Delta_{h(u)}$ and α_{-1}, α_0 are the Seeley coefficients in the expansion of the heat kernel. In Appendix B, it is calculated that α_{-1} and α_0 depend only upon the geometry Σ and the topology of the bundle; they are independent of u. Therefore, taking advantage of the cyclic nature of Trace, we find

$$\begin{aligned} 2\left.\frac{d}{du}\right|_{u=0} \zeta_u(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\left\{-t\left[-2\psi\Delta_h e^{-t\Delta_h} + 2\psi\tilde{\Delta}_h e^{-t\tilde{\Delta}_h}\right]\right\} t^s \frac{dt}{t} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\left\{2\psi\left[\Delta_h e^{-t\Delta_h} - \tilde{\Delta}_h e^{-t\tilde{\Delta}_h}\right]\right\} t^s dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{d}{dt}\left[\mathrm{Tr}\left\{2\psi\left(-e^{-t\Delta_h} + e^{-t\tilde{\Delta}_h}\right)\right\}\right] t^s dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{d}{dt}\left[\mathrm{Tr}\left\{2\psi\left(-e^{-t\Delta_h} + e^{-t\tilde{\Delta}_h} + P\right)\right\}\right] t^s dt \\ &= \frac{-s}{\Gamma(s)} \int_0^\infty \mathrm{Tr}\left\{2\psi\left(-e^{-t\Delta_h} + e^{-t\tilde{\Delta}_h} + P\right)\right\} t^s \frac{dt}{t}, \end{aligned}$$

where P is the orthogonal projection onto the kernel of Δ_h. The integration by parts is valid for $\Re(s) > 0$ because the trace dies exponentially for large t and is $O(1)$ for small t (the t^{-1} terms from the heat kernels cancel each other). We also used the result from Appendix B.3 that $\tilde{\Delta}_h$ has no kernel for bundles of sufficiently high degree. This resulting integral only converges for $\Re(s) > 0$; however it does have a finite limit as s approaches 0.

In order to simplify the integral, we first split it up into 2 parts:

$$2\left.\frac{d}{du}\right|_{u=0} \zeta_u(s) = \frac{-s}{\Gamma(s)} \left(\int_0^1 + \int_1^\infty\right) \mathrm{Tr}\left\{2\psi\left(-e^{-t\Delta_h} + e^{-t\tilde{\Delta}_h} + P\right)\right\} t^s \frac{dt}{t}.$$

The integral from 1 to ∞ yields an entire function of s. To simplify the integral from 0 to 1, we use the Seeley expansions of the heat kernels (B.1.1,B.2.1). The t^{-1} terms cancel out, and the order t part will yield a function analytic in a neighborhood of 0. Therefore, using the fact that $\dfrac{-s}{\Gamma(s)} = -s^2 + O(s^3)$,

$$\begin{aligned} 2\left.\frac{d}{du}\right|_{u=0} \zeta_u(s) &= -s^2 \int_0^1 \left[\int_\Sigma 2\psi \left(\frac{1}{4\pi g}\partial_\mu\partial_\mu \log g + \frac{1}{2\pi g}\partial_\mu\partial_\mu \log h\right) d\mu_g\right] t^s \frac{dt}{t} \\ &\quad -s^2 \int_0^1 \operatorname{Tr}(2\psi P)\, t^s \frac{dt}{t} + \beta(s) \\ &= -s \left\{\int_\Sigma 2\psi \left(\frac{1}{4\pi g}\partial_\mu\partial_\mu \log g + \frac{1}{2\pi g}\partial_\mu\partial_\mu \log h\right) d\mu_g + \operatorname{Tr}(2\psi P)\right\} + \beta(s), \end{aligned}$$

where $\beta(s)$ is analytic in a neighborhood around 0, satisfying $\beta(0) = \beta'(0) = 0$. All the manipulations performed are valid in the region $\Re(s) > 0$, but the final result is once again valid in a neighborhood of 0. Here, $\partial_\mu\partial_\mu$ represents the usual Euclidean Laplacian: $\partial_x^2 + \partial_y^2 = 4\partial_z\partial_{\bar{z}}$. Therefore,

$$-2\left.\frac{d}{du}\right|_{u=0} \zeta_u'(s) = \int_\Sigma 2\psi \left(\frac{1}{4\pi g}\partial_\mu\partial_\mu \log g + \frac{1}{2\pi g}\partial_\mu\partial_\mu \log h\right) d\mu_g + \operatorname{Tr}(2\psi P)\,.$$

Let $\tilde{\Sigma}$ be a fundamental domain for Σ in $\mathbb{C}$. Then the variation in the determinant is:

$$\left.\frac{d}{du}\right|_{u=0} \ln\det{}'\Delta_{h(u)} = \int_{\tilde{\Sigma}} 2\psi \left(\frac{1}{4\pi}\partial_\mu\partial_\mu \log g + \frac{1}{2\pi}\partial_\mu\partial_\mu \log h\right) |dz|^2 + \operatorname{Tr}(2\psi P)\,,$$

where $|dz|^2 = \frac{i}{2} dz \wedge d\bar{z}$. In general, since $h(u) = e^{2\sigma u}h$, we can compute $\frac{d}{du}\ln\det{}'\Delta_{h(u)}$ at any value of u; the only change in the above formula being that $\log h$ becomes $\log h(u)$.

In order to integrate this from h_0 to $h = e^{2\sigma}h_0$, let $\sigma(u) := u\sigma$, so that $2\psi = 2\sigma = \log h - \log h_0$. Thus,

$$\log h(u) = 2\sigma(u) + \log h_0 = 2u\sigma + \log h_0 = u(\log h - \log h_0) + \log h_0.$$

We will use the shorthand $\log\left(\frac{h}{h_0}\right)$ for 2σ. Integrating the variation, we find that

$$\begin{aligned}
\ln\det{}'\Delta_h - \ln\det{}'\Delta_{h_0} &= \int_0^1 \int_{\tilde{\Sigma}} \left(\log\frac{h}{h_0}\right)\left(\frac{1}{4\pi}\partial_\mu\partial_\mu \log g + \frac{1}{2\pi}\partial_\mu\partial_\mu \log h(u)\right)|dz|^2\, du \\
&\quad + \int_0^1 \mathrm{Tr}\left(\left(\log\frac{h}{h_0}\right)P\right) du \\
&= \frac{1}{4\pi}\int_{\tilde{\Sigma}} \left(\log\frac{h}{h_0}\right)\partial_\mu\partial_\mu \log g\,|dz|^2 && (3.2.1) \\
&\quad + \frac{1}{2\pi}\int_{\tilde{\Sigma}} \left(\log\frac{h}{h_0}\right)\partial_\mu\partial_\mu \log h_0\,|dz|^2 && (3.2.2) \\
&\quad + \frac{1}{4\pi}\int_{\tilde{\Sigma}} \left(\log\frac{h}{h_0}\right)\partial_\mu\partial_\mu \left(\log\frac{h}{h_0}\right)|dz|^2 && (3.2.3) \\
&\quad + \mathrm{Tr}\left(\left(\log\frac{h}{h_0}\right)P\right). && (3.2.4)
\end{aligned}$$

Before concluding this subsection, let us say one more word about the trace that must be computed in the last line of this formula. Let $\{\phi_k(u)\}_{k=1}^{2n}$ be an orthonormal (with respect to $h(u)$) basis of the kernel of $\Delta_{h(u)}$, with $\phi_{n+k} = i\phi_k$. Let $\{\Phi_k\}$ be a basis independent of u satisfying the same criterion. Let the $n\times n$ matrix A be given by $\phi_j = \sum_k \Phi_k A_{kj}$ for $1 \le j,k \le n$. Let $Q(h(u))$ be the $n\times n$ matrix given by $Q(h(u)) = \langle\Phi_k,\Phi_l\rangle_{h(u)}$ for $1 \le k,l \le n$, so that $Q(h(u))^{-1} = AA^\dagger$. We can calculate

$$\begin{aligned}
\frac{d}{du}\ln\det Q(h(u)) &= \mathrm{Tr}_{\mathbb{C}}\left[Q'(h(u))Q^{-1}\right] \\
&= \sum_{k,l=1}^{n} Q'(h(u))_{\bar{k}l}\left(Q^{-1}\right)_{l\bar{k}} \\
&= \sum_{k,l=1}^{n}\left(\int_\Sigma \log\frac{h}{h_0} h(u)\overline{\Phi_k}\Phi_l d\mu_g\right)\left(Q^{-1}\right)_{l\bar{k}}.
\end{aligned}$$

Then

$$\int_0^1 \mathrm{Tr}_{\mathbb{R}}\left(\left(\log\frac{h}{h_0}\right)P\right)du = \int_0^1 2\sum_{j=1}^{n}\left\langle\left(\log\frac{h}{h_0}\right)\phi_j,\phi_j\right\rangle du$$

$$
\begin{aligned}
&= 2\int_0^1 \int_\Sigma \log \frac{h}{h_0} h(u) \left(\sum_{j=1}^n |\phi_j|^2 \right) d\mu_g du \\
&= 2\int_0^1 \int_\Sigma \log \frac{h}{h_0} h(u) \left(\sum_{j,k,l=1}^n \bar{\Phi}_k \Phi_l \bar{A}_{kj} A_{lj} \right) d\mu_g du \\
&= 2\int_0^1 \int_\Sigma \log \frac{h}{h_0} h(u) \left(\sum_{k,l=1}^n \bar{\Phi}_k \Phi_l \left(A A^\dagger \right)_{lk} \right) d\mu_g du \\
&= 2\int_0^1 \int_\Sigma \log \frac{h}{h_0} h(u) \left(\sum_{k,l=1}^n \left(Q^{-1} \right)_{lk} \bar{\Phi}_k \Phi_l \right) d\mu_g du \\
&= 2\int_0^1 \frac{d}{du} \ln \det Q(h(u)) du \\
&= 2 \ln \det Q(h(1)) - 2 \ln \det Q(h(0)). \qquad (3.2.5)
\end{aligned}
$$

3.2.2 Formula for the Dependence upon the Conformal Anomaly

We will now apply the result of the last section to the bundle L_w with metric h_w; we take $h_{2d,2B}$ as our base metric. From these formulae, we can write down the important quantity,

$$
\log \left(\frac{h_w}{h_{2d,2B}} \right) (z) = -2 \log \left(\left(1 + |w|^2\right) \prod_{j=1}^d h_{1,b_j}(z) \left| s_{1,b_j}(z) \right|^2 \right).
$$

This function is well-defined on the torus; it is doubly periodic and the argument of the logarithm has no zeroes or singularities. Before continuing, it is useful to have the asymptotic expansions of the parts of logarithm for reference.

Claim. Useful asymptotics: As $|z-b| \to 0$, we have

$$
\begin{aligned}
\log \left(h_{1,b}(z) \left| s_{1,b}(z) \right|^2 \right) &= \log |z-b|^2 + 2\Re \left[\eta b^2 + i\pi b \right] + \frac{2\pi}{\tau_y} \left(b_y^2 + \tau_y b_y \right) + O(|z-b|) \\
&= \log |z-b|^2 + 2\Re \left[\eta b^2 \right] + \frac{2\pi}{\tau_y} b_y^2 + O(|z-b|); \\
\partial_z \log \left(h_{1,b}(z) \left| s_{1,b}(z) \right|^2 \right) &= \frac{1}{z-b} + O(|z-b|).
\end{aligned}
$$

As $|z-b_k|\to 0$, we have

$$\begin{aligned}\partial_{\bar z}\log\left(1+|w|^2\right) &= -\left(\overline{z-b_k}\right)^{-1}+O(1)\\ \log\left(1+|w|^2\right) &= -\log\left(|z-b_k|^2\right)+\log\left(\frac{|c|^2\prod_{l=1}^{d}|\sigma(b_k-a_l)|^2}{\prod_{l\neq k}|\sigma(b_k-b_l)|^2}\right)+O\left(|z-b_k|\right).\end{aligned}$$

Finally, we introduce one final bit of notation. Let $\left\{S^0_{r,2B}\right\}_{r=0}^{4d-1}$ be the following set of holomorphic sections of the bundle L_w:

$$\begin{aligned}S^0_{r,2B}(z) &= \sum_{k\in\mathbb{Z}}\exp\left\{\frac{\pi i\tau}{2d}(2dk+r)^2+2\pi i(2dk+r)\left(z-\frac{B}{d}-\frac{1}{2}-\frac{\tau}{2}\right)\right\}\\ S^0_{r+2d,2B}(z) &= iS_{0,r}(z)\end{aligned}$$

for $0\le r\le 2d-1$, where . Later, it will be shown that this set is actually a basis (over the reals) for the space of holomorphic sections. In fact, for $0\le r\le 2d-1$, $S^0_{r,2B}(z)$ is simply a scaled version of the theta function with characteristics:

$$S^0_{r,2B}(z|\tau) = \theta_{\frac{r}{2d},0}\left(2d\left(z-\frac{B}{d}-\frac{1}{2}-\frac{\tau}{2}\right)\middle|\,2d\tau\right), \tag{3.2.6}$$

where $\theta_{a,b}(z|\tau)=\sum_{k\in\mathbb{Z}}\exp\{i\pi\tau(k+a)^2+2\pi i(k+a)(z+b)\}$, for $a,b\in\frac{1}{2d}\mathbb{Z}$.

Thus, for any metric h on L_w, $\left\{S^0_{r,2B}\right\}_r$ is a basis for the kernel of Δ_h. Define

$$Q_{\bar j k}(h)=\left\langle S^0_{j,2B},S^0_{k,2B}\right\rangle_h=\int_\Sigma h\overline{S^0_{j,2B}}S^0_{k,2B}\,d\mu_g.$$

We are now ready to compute the main calculation of this section.

Theorem. *The dependence on the conformal anomaly of the metric is given by*

$$\ln\det{}'\Delta_w-\ln\det{}'\Delta_{h_{2d,2B}} = 4d+4d\log|c|^2+4\log\left(\prod_{j,k=1}^{d}|\sigma(b_k-a_j)|^2\right)$$

$$
\begin{aligned}
&-4d\sum_{j=1}^{d}\left[2\Re(\eta b_j^2)+\frac{2\pi}{\tau_y}b_{j_y}^2\right]+8\left[2\Re(\eta B^2)+\frac{2\pi}{\tau_y}B_y^2\right]\\
&-\frac{2d}{\tau_y}\int_{\Sigma}\log\left(\frac{h_w}{h_{2d,2B}}\right)d\mu_g-\frac{4d}{\tau_y}\int_{\Sigma}\log\left(1+|w|^2\right)d\mu_g\\
&+2\ln\det Q(h_w)+4d\log 2-2d\log\left(\frac{\tau_y}{d}\right)\\
&-4\pi\tau_y\left(\frac{2B_y}{\tau_y}+d\right)^2.
\end{aligned}
$$

Proof. We must compute (3.2.1)-(3.2.4). (3.2.1) is the 0 since Σ is flat.

To calculate (3.2.2), we see that

$$
\partial_\mu\partial_\mu\log h_{2d,2B} \quad = \quad -\frac{8\pi d}{\tau_y},
$$

and so

$$
\frac{1}{2\pi}\int_{\tilde{\Sigma}}\left(\log\frac{h_w}{h_{2d,2B}}\right)\partial_\mu\partial_\mu\log h_{2d,2B}d^2z \quad = \quad -\frac{4d}{\tau_y}\int_{\Sigma}\log\frac{h_w}{h_{2d,2B}}d\mu_g.
$$

Similarly,

$$
\partial_\mu\partial_\mu\left(\log\frac{h_w}{h_{2d,2B}}\right)=-2\partial_\mu\partial_\mu\log\left(1+|w|^2\right)+\frac{8\pi d}{\tau_y}.
$$

Thus, (3.2.3) also simplifies, but there is still some work to be done at the end:

$$
\begin{aligned}
&\frac{1}{4\pi}\int_{\tilde{\Sigma}}\left(\log\frac{h_w}{h_{2d,2B}}\right)\partial_\mu\partial_\mu\left(\log\frac{h_w}{h_{2d,2B}}\right)d^2z\\
= \quad &-\frac{1}{2\pi}\int_{\tilde{\Sigma}}\log\frac{h}{h_{2d,2B}}\partial_\mu\partial_\mu\log\left(1+|w|^2\right)d^2z\\
&+\frac{2d}{\tau_y}\int_{\tilde{\Sigma}}\log\frac{h_w}{h_{2d,2B}}d^2z. \qquad (3.2.7)
\end{aligned}
$$

We are left trying to perform the integral

$$\frac{1}{\pi}\int_{\tilde{\Sigma}} \log\left(\left(1+|w|^2\right)\prod_{j=1}^{d} h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right)\partial_\mu\partial_\mu \log\left(1+|w|^2\right) d^2z,$$

which we interpret as follows: for $1 \le j \le d$, let D_j be a disc centered around $\tilde{b}_j$ of radius ϵ_j. Furthermore, let us assume that the ϵ_j are small enough so that the D_j are mutually disjoint. Let $A = \mathbb{T} \setminus \bigcup_{j=1}^{d} D_j$. We evaluate the above integral by first integrating over A and then taking the limit as each ϵ_j tends to 0. We first break up the integral into two main parts.

$$\frac{2i}{\pi}\int_A \log\left(1+|w|^2\right)\partial_z\partial_{\bar{z}} \log\left(1+|w|^2\right) dz \wedge d\bar{z}$$

$$+\frac{2i}{\pi}\sum_{j=1}^{d}\int_A \log\left(h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right)\partial_z\partial_{\bar{z}} \log\left(1+|w|^2\right) dz \wedge d\bar{z}$$

The first integrand only has logarithmic singularities at each b_j, and so it is integrable on the entire torus. Using the fact that w is a d-fold cover, we can then compute

$$\begin{aligned}
& \frac{2i}{\pi}\int_\Sigma \log\left(1+|w|^2\right)\partial_z\partial_{\bar{z}} \log\left(1+|w|^2\right) dz \wedge d\bar{z} \\
= \; & \frac{2i}{\pi}\int_\Sigma \log\left(1+|w|^2\right)\partial_w\partial_{\bar{w}} \log\left(1+|w|^2\right) \partial_z w dz \wedge \partial_{\bar{z}}\bar{w} d\bar{z} \\
= \; & \frac{2di}{\pi}\int_{\mathbb{P}^1} \log\left(1+|w|^2\right)\partial_w\partial_{\bar{w}} \log\left(1+|w|^2\right) dw \wedge d\bar{w} \\
= \; & \frac{4d}{\pi}\int_0^{2\pi}\int_0^\infty \frac{\log\left(1+r^2\right)}{\left(1+r^2\right)^2} r dr d\theta \\
= \; & 4d.
\end{aligned} \tag{3.2.8}$$

For fixed $1 \le j \le d$, we compute

$$\frac{2i}{\pi}\int_A \log\left(h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right)\partial_z\partial_{\bar{z}} \log\left(1+|w|^2\right) dz \wedge d\bar{z},$$

by integrating by parts twice:

$$\begin{aligned}
&\frac{2i}{\pi}\int_A \log\left(h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right)\partial_z\partial_{\bar z}\log\left(1+|w|^2\right)dz\wedge d\bar z\\
= &-\frac{2i}{\pi}\sum_{k=1}^{d}\oint_{\partial D_k}\log\left(h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right)\partial_{\bar z}\log\left(1+|w|^2\right)d\bar z\\
&-\frac{2i}{\pi}\sum_{k=1}^{d}\oint_{\partial D_k}\partial_z\log\left(h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right)\log\left(1+|w|^2\right)dz\\
&-\frac{2i}{\tau_y}\int_A\log\left(1+|w|^2\right)dz\wedge d\bar z.
\end{aligned}$$

In the case when $k\neq j$, we can use the asymptotics to evaluate

$$\begin{aligned}
&-\frac{2i}{\pi}\oint_{\partial D_k}\log\left(h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right)\partial_{\bar z}\log\left(1+|w|^2\right)d\bar z\\
= &-\frac{2i}{\pi}\int_0^{2\pi}\left[\frac{\log\left(h_{1,b_j}(b_k)\left|s_{1,b_j}\left(b_k\right)\right|^2\right)}{-\epsilon_k e^{-i\theta}}+O(1)\right]\left(-i\epsilon_k e^{-i\theta}d\theta\right)\\
= &\ \frac{2}{\pi}\int_0^{2\pi}\left[\log\left(h_{1,b_j}(b_k)\left|s_{1,b_j}\left(b_k\right)\right|^2\right)+O(\epsilon_k)\right]d\theta\\
\to &\ 4\log\left(h_{1,b_j}(b_k)\left|s_{1,b_j}\left(b_k\right)\right|^2\right)
\end{aligned}$$

as $\epsilon_k\to 0$. Also,

$$\begin{aligned}
&-\frac{2i}{\pi}\oint_{\partial D_k}\partial_z\log\left(h_{1,b_j}(z)\left|s_{1,b_j}\left(z\right)\right|^2\right)\log\left(1+|w|^2\right)dz\\
= &-\frac{2i}{\pi}\int_0^{2\pi}\left(K\log\left(\epsilon_k^2\right)+O(1)\right)i\epsilon_k e^{i\theta}d\theta\\
\to &\ 0,
\end{aligned}$$

where $K=\partial_z|_{z=b_k}\log\left(h_{1,b_j}(z)\left|s_{1,b_j}\left(z\right)\right|^2\right)$ is a constant. The $k=j$ terms are more complicated:

$$
\begin{aligned}
&-\frac{2i}{\pi}\oint_{\partial D_j}\log\left(h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right)\partial_{\bar z}\log\left(1+|w|^2\right)d\bar z\\
=\ & \frac{2}{\pi}\int_0^{2\pi}\left[\log\epsilon_j^2+2\Re\left[\eta b_j^2\right]+\frac{2\pi}{\tau_y}b_{j_y}^2+O\left(\epsilon_j\log\epsilon_j\right)\right]d\theta
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{2i}{\pi}\oint_{\partial D_j}\partial_z\log\left(h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right)\log\left(1+|w|^2\right)dz\\
=\ & \frac{2}{\pi}\int_0^{2\pi}\left[-\log\epsilon_j^2+\log\left(\frac{|c|^2\prod_{k=1}^d\left|\sigma\left(b_j-a_k\right)\right|^2}{\prod_{k\neq j}\left|\sigma\left(b_j-b_k\right)\right|^2}\right)+O(\epsilon_j)\right]d\theta.
\end{aligned}
$$

Adding the two together, we see that the pieces that are divergent as $\epsilon_j\to 0$ cancel, so we can take said limit and arrive at

$$
4\left[2\Re\left[\eta b_j^2\right]+\frac{2\pi}{\tau_y}b_{j_y}^2+\log\left(\frac{|c|^2\prod_{k=1}^q\left|\sigma\left(b_j-a_k\right)\right|^2}{\prod_{k\neq j}\left|\sigma\left(b_j-b_k\right)\right|^2}\right)\right].
$$

Therefore,

$$
\begin{aligned}
&\frac{2i}{\pi}\sum_{j=1}^d\int_\Sigma\log\left(h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right)\partial_z\partial_{\bar z}\log\left(1+|w|^2\right)dz\wedge d\bar z\\
=\ & 4\sum_{j=1}^d\left[2\Re\left[\eta b_j^2\right]+\frac{2\pi}{\tau_y}b_{j_y}^2+\log\left(\frac{|c|^2\prod_{k=1}^d\left|\sigma\left(b_j-a_k\right)\right|^2}{\prod_{k\neq j}\left|\sigma\left(b_j-b_k\right)\right|^2}\right)\right]\\
&+4\sum_{j=1}^d\sum_{k\neq j}\log\left(h_{1,b_j}(b_k)\left|s_{1,b_j}(b_k)\right|^2\right)-\frac{4d}{\tau_y}\int_\Sigma\log\left(1+|w|^2\right)d\mu_g\\
=\ & 4\sum_{j=1}^d\left[\log\left(|c|^2\prod_{k=1}^d\left|\sigma\left(b_j-a_k\right)\right|^2\right)\right]-\frac{4d}{\tau_y}\int_\Sigma\log\left(1+|w|^2\right)d\mu_g\\
&+4(2-d)\sum_{j=1}^d\left[2\Re(\eta b_j^2)+\frac{2\pi}{\tau_y}b_{j_y}^2\right]+4\sum_{j\neq k}\left[4\Re(\eta b_jb_k)+\frac{4\pi}{\tau_y}b_{j_y}b_{k_y}\right].
\end{aligned}
$$

$$= \quad 4\sum_{j=1}^{d}\left[\log\left(|c|^2\prod_{k=1}^{d}\left|\sigma\left(b_j-a_k\right)\right|^2\right)\right]-\frac{4d}{\tau_y}\int_\Sigma \log\left(1+|w|^2\right)d\mu_g \tag{3.2.9}$$

$$-4d\sum_{j=1}^{d}\left[2\Re(\eta b_j^2)+\frac{2\pi}{\tau_y}b_{j_y}^2\right]+8\left[2\Re(\eta B^2)+\frac{2\pi}{\tau_y}B_y^2\right]. \tag{3.2.10}$$

Putting (3.2.7) through (3.2.10) together yields the value of (3.2.3).

Finally, the trace in (3.2.4) has already been computed in (3.2.5). We only need to compute $\det Q(h_{2d,2B})$ to conclude the proof. We claim that $Q(h_{2d,2B})$ is actually diagonal. Suppose $0 \le j,k \le 2d-1$, and suppose $j \ne k$. Since the summation in S_r^0 converges absolutely and uniformly,

$$\begin{aligned}
Q\left(h_{2d,2B}\right)_{\bar{j}k} &= \int_\Sigma h_{2d,2B}(z)\overline{S_j^0(z)}S_k^0(z)d\mu_g \\
&= \sum_{m,n\in\mathbb{Z}}\int_\Sigma \exp\left\{-\frac{\pi i\bar{\tau}}{2d}\left(2dm+j\right)^2-2\pi i\left(2dm+j\right)\left(\bar{z}-\frac{\bar{B}}{d}-\frac{1}{2}-\frac{\bar{\tau}}{2}\right)\right\} \\
&\quad\cdot\exp\left\{\frac{\pi i\tau}{2d}\left(2dn+k\right)^2+2\pi i\left(2dn+k\right)\left(z-\frac{B}{d}-\frac{1}{2}-\frac{\tau}{2}\right)\right\}h_{2q,2B}(z)d\mu_g \\
&= \sum_{m,n\in\mathbb{Z}}\exp\left\{-\frac{\pi i\bar{\tau}}{2d}\left(2dm+j\right)^2+\frac{\pi i\tau}{2d}\left(2dn+k\right)^2\right\} \\
&\quad\cdot\exp\left\{2\pi i\left(2dm+j\right)\left(\frac{\bar{B}}{d}+\frac{1}{2}+\frac{\bar{\tau}}{2}\right)-2\pi i\left(2dn+k\right)\left(\frac{B}{d}+\frac{1}{2}+\frac{\tau}{2}\right)\right\} \\
&\quad\cdot\int_0^{\tau_y}h_{2d,2B}(y)\exp\left\{-2\pi\left[\left(2dn+k\right)+\left(2dm+j\right)\right]y\right\} \\
&\quad\cdot\int_{\frac{y}{\tau_y}\tau_x}^{\frac{y}{\tau_y}\tau_x+1}\exp\left\{2\pi i\left[\left(2dn+k\right)-\left(2dm+j\right)\right]x\right\}dxdy.
\end{aligned}$$

The quantity $(2dn+k)-(2dm+j)$ never vanishes, so this integral is 0, and Q is diagonal. As an aside, let us mention that this result demonstrates the linear independence of the set $\left\{S_{r,2B}^0\right\}_{r=0}^{2d-1}$ over $\mathbb{C}$. Thus, $\left\{S_{r,2B}^0\right\}_{r=0}^{4d-1}$ is linearly independent over $\mathbb{R}$, and therefore a basis, as desired. In computing the diagonal elements, a similar manipulation shows that all terms with $m \neq n$ vanish. Therefore,

$$
\begin{aligned}
Q_{\bar{j}j} &= \int_\Sigma h_{2d,2B}(z)\overline{S^0_{j,2B}(z)}S^0_{j,2B}(z)d\mu_g \\
&= \sum_{n\in\mathbb{Z}} \exp\left\{-\frac{\pi i\bar{\tau}}{2d}(2dn+j)^2 + \frac{\pi i\tau}{2d}(2dn+j)^2\right\} \\
&\quad \cdot \exp\left\{2\pi i(2dn+j)\left(\frac{\bar{B}}{d}+\frac{1}{2}+\frac{\bar{\tau}}{2}\right) - 2\pi i(2dn+j)\left(\frac{B}{d}+\frac{1}{2}+\frac{\tau}{2}\right)\right\} \\
&\quad \cdot \int_0^{\tau_y} h_{2d,2B}(y)\exp\{-4\pi(2dn+j)y\}\int_{\frac{y}{\tau_y}\tau_x}^{\frac{y}{\tau_y}\tau_x+1} dxdy. \\
&= \sum_{n\in\mathbb{Z}} \int_0^{\tau_y} \exp\left\{-\frac{\pi\tau_y}{d}(2dn+j)^2 - 2\pi(2dn+j)\left(2y-\frac{2B_y}{d}-\tau_y\right)\right\} \\
&\quad \cdot \exp\left\{-\frac{2\pi}{\tau_y}\left[2dy(y-\tau_y)-4B_y y\right]\right\}dy \\
&= \sum_{n\in\mathbb{Z}} \int_0^{\tau_y} \exp\left\{-\frac{\pi\tau_y}{d}\left[j+2d\left(n+\frac{y}{\tau_y}\right)-2\frac{B_y}{\tau_y}-d\right]^2 + \frac{\pi\tau_y}{d}\left(\frac{2B_y}{\tau_y}+d\right)^2\right\}dy \\
&= \exp\left\{\frac{\pi\tau_y}{d}\left(\frac{2B_y}{\tau_y}+d\right)^2\right\}\sum_{n\in\mathbb{Z}}\int_{n\tau_y}^{(n+1)\tau_y} \exp\left\{-\frac{\pi\tau_y}{d}\left(2d\frac{y}{\tau_y}-2\frac{B_y}{\tau_y}-d+j\right)^2\right\}dy \\
&= \exp\left\{\frac{\pi\tau_y}{d}\left(\frac{2B_y}{\tau_y}+d\right)^2\right\}\int_{-\infty}^{\infty} \exp\left\{-\frac{\pi\tau_y}{d}\left(2d\frac{y}{\tau_y}-2\frac{B_y}{\tau_y}-d+j\right)^2\right\}dy \\
&= \frac{\tau_y}{2d}\exp\left\{\frac{\pi\tau_y}{d}\left(\frac{2B_y}{\tau_y}+d\right)^2\right\}\int_{-\infty}^{\infty}\exp\left\{-\frac{\pi\tau_y}{d}v^2\right\}dv \\
&= \frac{1}{2}\sqrt{\frac{\tau_y}{d}}\exp\left\{\frac{\pi\tau_y}{d}\left(\frac{2B_y}{\tau_y}+d\right)^2\right\}.
\end{aligned}
$$

This is independent of j, so $\det Q = 2^{-2d}\left(\frac{\tau_y}{d}\right)^d \exp\left\{2\pi\tau_y\left(\frac{2B_y}{\tau_y}+d\right)^2\right\}$. This completes the proof. $\square$

3.3 Determinant of $\Delta_{h_{2d,2B}}$

In this section, we will finish the computation of $\det' \Delta_w$ by explicitly calculating $\det' \Delta_{h_{2d,2B}}$. We will prove the following more general result:

Theorem. *Let $D = \sum_{j=1}^{J} n_j \tilde{z}_j$ be an effective divisor of degree n on Σ. With notation as in section 3.1, let Δ be the $\bar{\partial}$-Laplace operator acting on sections of L_D with respect to the hermitian metric h_{n,z_D}. Δ has pure point spectrum; its eigenvalues are given by*

$$\lambda_m := \left(\frac{\pi n}{\tau_y}\right) m, \quad m \in \mathbb{Z}_{\geq 0},$$

each with multiplicity n. A basis of the eigenspace for λ_m is given by the functions:

$$S^m_{r,z_D}(z) := \sum_{k \in \mathbb{Z}} \exp\left\{\frac{\pi i \tau}{n}(nk+r)^2 + Z_{n,k,r,D}\right\} H_m\left(\sqrt{\frac{\pi}{2n\tau_y}} Y_{n,k,r,D}\right), \tag{3.3.1}$$

$$\begin{aligned} Z_{n,k,r,D} &= 2\pi i (nk+r)\left(z - \frac{z_D}{n} - \frac{1}{2} - \frac{\tau}{2}\right) \\ Y_{n,k,r,D} &= [(2k-1)n + 2r]\,\tau_y + 2ny - 2y_D \end{aligned}$$

where $r = 0, 1, \ldots, n-1$, and $H_m(y)$ is the mth Hermite polynomial, defined by

$$H_m(y) := (-1)^m e^{y^2} \left(\frac{d^m}{dy^m}\right) e^{-y^2}.$$

From this, we deduce the following corollary:

Corollary. $\ln \det' \Delta_{h_{2d,2B}} = 2d \log\left(\frac{\tau_y}{d}\right)$.

Proof. $\ln \det' \Delta_{h_{2d,2B}} = -2\zeta'(0)$, where $\zeta(s)$ is the zeta-function corresponding to $\Delta_{h_{2d,2B}}$.

The theorem tells us that

$$\zeta(s) = 2d\left(\frac{\tau_y}{2\pi d}\right)^s \zeta_R(s),$$

where $\zeta_R(s)$ is the Riemann zeta-function. The result follows from basic facts about $\zeta_R(s)$; see for example [8]. $\square$

Before proving the theorem in this section, let us say a few words about the derivation of this result, which differs greatly from its proof. In terms of the x and y coordinates on the plane, $\Delta_{h_{n,z_D}}$ is expressed as:

$$-\frac{1}{4}\left(\partial_x^2 + \partial_y^2\right) - \frac{\pi i}{2\tau_y}\left[n(2y - \tau_y) - 2y_D\right]\left(\partial_x + i\partial_y\right).$$

Since we are looking for eigenfunctions of this operator that have period 1, we suppose the ansatz:

$$\sum_{k\in\mathbb{Z}} g_k(y) e^{2\pi i k x},$$

where $g_k(y)$ are functions of the imaginary part of z satisfying the following relation coming from the action of $\mathbb{Z}\tau$:

$$e^{2\pi i k\tau_x} g_k(y + \tau_y) = (-1)^n e^{2\pi n y + 2\pi i z_D} g_{k+n}(y) \qquad \forall k \in \mathbb{Z} \tag{3.3.2}$$

Writing down the eigenvalue equation for $\Delta_{h_{n,z_D}}$ with this ansatz, we get the following ODE for g_k.

$$g_k'' - \frac{2\pi}{\tau_y}\left[n(2y - \tau_y) - 2y_D\right] g_k' - \left(4\pi^2 k^2 + \frac{4\pi^2 k}{\tau_y}\left[n(2y - \tau_y) - 2y_D\right] - 4\lambda\right) g_k = 0.$$

If we look for solutions to this equation such that the quasi-periodicity relation is satisfied and the Fourier series converges, we are led precisely to the functions S^m_{r,z_D}. Now we are ready to prove the theorem.

Proof. First of all, it is easy to see that S^m_{r,z_D}, given in (3.3.1), converges absolutely and uniformly on compact sets. This is because the terms are dominated by a polynomial multiplied by a decaying Gaussian in k. One also quickly sees that this function transforms according to (3.1.1) under the action of the lattice, so that it is a well-defined section of L_D. Now we will show that this function is indeed an eigenfunction of Δ with the prescribed eigenvalues. In order to do this, we will make use of the following recurrence relations satisfied by the Hermite polynomials:

$$\begin{aligned} H_m(x) &= 2xH_{m-1}(x) - 2(m-1)H_{m-2}(x), \\ H'_m(x) &= 2mH_{m-1}(x). \end{aligned}$$

$\Delta = -\left(\partial_z + \frac{\pi i}{\tau_y}\left[n\left(2y-\tau_y\right)-2y_D\right]\right)\partial_{\bar{z}}$. Applying this operator to (3.3.1) yields:

$$\sum_{k\in\mathbb{Z}} \exp\left\{\frac{\pi i\tau}{n}\left(nk+r\right)^2 + 2\pi i\left(nk+r\right)\left(z-\frac{z_D}{n}-\frac{1}{2}-\frac{\tau}{2}\right)\right\}\cdot$$

$$\frac{n\pi}{\tau_y}\left[\sqrt{\frac{\pi}{2n\tau_y}}Y_{n,k,r,D}H'_m\left(\sqrt{\frac{\pi}{2n\tau_y}}Y_{n,k,r,D}\right) - \frac{1}{2}H''_m\left(\sqrt{\frac{\pi}{2n\tau_y}}Y_{n,k,r,D}\right)\right].$$

By applying the second recurrence relation, we can transform the bracketed quantity into

$$2m\left[\sqrt{\frac{\pi}{2n\tau_y}}Y_{n,k,r,D}H_{m-1}\left(\sqrt{\frac{\pi}{2n\tau_y}}Y_{n,k,r,D}\right) - (m-1)H_{m-2}\left(\sqrt{\frac{\pi}{2n\tau_y}}Y_{n,k,r,D}\right)\right].$$

We can now apply the first recurrence relation to obtain:

$$mH_m\left(\sqrt{\frac{\pi}{2n\tau_y}}Y_{n,k,r,D}\right).$$

Therefore, S^m_{r,z_D} defines an eigenfunction of Δ with eigenvalue $\frac{n\pi}{\tau_y}m$. Now we will show that the collection of these functions generates a dense subset of $L^2(L_D)$, thereby proving that our analysis of the spectrum is complete. At the end, we will also prove that the set is linearly independent, thus completing the proof.

To this end, let f be an L^2 section of L_D, with respect to the metric h_{n,z_D}, and assume that f is orthogonal to all the above series. We must show that $f \equiv 0$. Let us introduce new coordinates u and v so that the fundamental domain of Σ is given simply by $0 \leq u, v \leq 1$.

$$u := x - \frac{\tau_x}{\tau_y} y, \quad v := \frac{y}{\tau_y}$$

In short, u, v are real numbers such that $z = x + iy = u + \tau v$, and $z_D = u_D + \tau v_D$. In these coordinates, the quasi-periodicity conditions satisfied by the sections of L_D (3.1.1) read:

$$\begin{aligned} f(u+1, v) &= f(u, v) \\ f(u, v+1) &= (-1)^n \exp\left\{-2\pi i \left(nz - z_D\right)\right\} f(u, v). \end{aligned} \tag{3.3.3}$$

Expressing the eigenfunctions S^m_{r,z_D} in terms of u and v, we get:

$$\sum_{k \in \mathbb{Z}} \exp\left\{\frac{\pi i \tau}{n} \left(nk + r\right)^2 + Z_{n,k,r,D}\right\} H_m\left(\sqrt{\frac{\pi}{2n\tau_y}} Y_{n,k,r,D}\right),$$

$$\begin{aligned} Z_{n,k,r,D} &= 2\pi i \left(nk + r\right) \left(\left(u - \frac{1}{2} - \frac{u_D}{n}\right) + \tau \left(v - \frac{1}{2} - \frac{v_D}{n}\right)\right); \\ Y_{n,k,r,D} &= \left[(2k-1)n + 2r\right] \tau_y + 2ny - 2y_D. \end{aligned}$$

For each $k \in \mathbb{Z}$, we introduce a new variable v_k, which is simply a translate of v:

$$v_k = v - \frac{1}{2} + k + \frac{r}{n} - \frac{v_D}{n},$$

so that $Y_{n,k,r,D} = 2n\tau_y v_k$. In these coordinates, the eigensection is

$$\begin{aligned} S_{m,r} &= \sum_{k\in\mathbb{Z}} \exp\left\{\frac{\pi i\tau}{n}(nk+r)^2 + Z_{n,k,r,D}\right\} H_m\left(\sqrt{2\pi n\tau_y}v_k\right), \\ Z_{n,k,r,D} &= 2\pi i(nk+r)\left(\left(u-\frac{1}{2}-\frac{u_D}{n}\right)+\tau\left(v-\frac{1}{2}-\frac{v_D}{n}\right)\right) \end{aligned}$$

and the metric reads

$$\begin{aligned} h_{n,z_D}(z) &= \exp\left\{-2\pi\tau_y\left[nv(v-1)-2v_D v\right]\right\} \\ &= \exp\left\{-2\pi\tau_y\left[nv_k^2+\frac{(nk+r)^2}{n}-2v_k(nk+r)-v_D-\frac{v_D^2}{n}\right]\right\}. \end{aligned}$$

Finally, since f is a section of L_D, it transforms like (3.3.3) under the action of the lattice. Therefore,

$$\begin{aligned} f(u,v) &= f\left(u, v_k+\frac{1}{2}-k-\frac{r}{n}-\frac{v_D}{n}\right) \\ &= (-1)^{nk}\exp\left\{2\pi ik\left[nu-A_u+\tau\left(nv_k-r-\frac{kn}{2}\right)\right]\right\} f\left(u, v_k+\frac{1}{2}-\frac{r}{n}-\frac{v_D}{n}\right). \end{aligned}$$

Our assumption about f means that for all $m\in\mathbb{Z}$, $0\le r<n$,

$$\begin{aligned} 0 &= \int\int_{\Sigma} h_{n,z_D}\bar{f}S_{m,r}d^2z \\ &= \int_0^1\int_0^1 h_{n,z_D}(u,v)\overline{f(u,v)}S_{m,r}(u,v)dudv \\ &= \sum_{k\in\mathbb{Z}}\int_0^1\int_0^1 h_{n,z_D}\overline{f(u,v)}\exp\left\{\frac{\pi i\tau}{n}(nk+r)^2+Z_{n,k,r,D}\right\}H_m\left(\sqrt{\frac{\pi}{2n\tau_y}}Y_{n,k,r,D}\right)dudv \\ &= \sum_{k\in\mathbb{Z}}\int_{k-\frac{1}{2}+\frac{r}{n}-\frac{v_D}{n}}^{1+k-\frac{1}{2}+\frac{r}{n}-\frac{v_D}{n}}\int_0^1 \overline{f(u,v-k)}e^{2\pi iru}e^{2\pi ir\bar{\tau}v_k}\exp\left(-\pi n\tau_y v_k^2\right)\tilde{H}_m\left(\sqrt{2\pi n\tau_y}v_k\right)dudv_k \\ &= \int_{-\infty}^{\infty}\int_0^1 \overline{f\left(u,v+\frac{1}{2}-\frac{r}{n}-\frac{v_D}{n}\right)}e^{2\pi iru}e^{2\pi ir\bar{\tau}v}\exp\left(-\pi n\tau_y v^2\right)\tilde{H}_m\left(\sqrt{2\pi n\tau_y}v\right)dudv, \end{aligned}$$

where $\tilde{H}_m$ is the mth Hermite function $\tilde{H}_m(x) := H_m(x)e^{-x^2/2}$. Define

$$F(v) := \int_0^1 f\left(u, v + \frac{1}{2} - \frac{r}{n} - \frac{v_D}{n}\right) e^{-2\pi i r u} e^{-2\pi i r \tau v} du.$$

(3.3.3) implies that F grows at worst like $e^{2\pi\tau_y|v|}$. Therefore, $F(v)\exp\left(-\pi n \tau_y v^2\right) \in L^2(\mathbb{R})$. Since the Hermite functions $H_m(x)e^{-x^2/2}$ form an orthogonal basis of L^2, we must have $F(v)\exp\left(-\pi n\tau_y v^2\right) \equiv 0$. Therefore, $\int_0^1 f\left(u, v + \frac{1}{2} - \frac{r}{n} - \frac{v_D}{n}\right) e^{-2\pi i r u} du \equiv 0$ as a function of v.

Now that we know $\int_0^1 f\left(u, v\right) e^{-2\pi i r u} du$ is identically 0, we are ready to finish the proof. f is periodic in u with period 1, and so can be expanded in a Fourier series whose coefficients are functions of v:

$$f(u, v) = \sum_{l \in \mathbb{Z}} \hat{f}_l(v) e^{2\pi i l u}.$$

We have proven that $\hat{f}_r \equiv 0$. However, the Fourier coefficients of f are related to one another is in the formula (3.3.2). Therefore, $\hat{f}_l \equiv 0$ for all l in the residue class of r modulo n. Since r ranges over a complete set of residues modulo n, we have $f \equiv 0$ as desired.

The proof that the set $\left\{S^m_{r,z_D}\right\}_{m,r}$ is linearly independent is a direct generalization of the argument already made in the last section for $\left\{S^0_{r,z_D}\right\}$. In fact, for each $m \geq 0$, the set $\left\{S^m_{r,z_D} | 0 \leq r < n\right\}$ consists of orthogonal eigensections with respect to h_{n,z_D}. $\square$

Chapter 4

Measure on Critical Submanifolds

In this chapter, we will make frequent use of the usual quasi-periodicity relations of the Weierstrass elliptic functions, which can be found in any introductory text on the subject, for instance [1]. These relations include:

$$
\begin{aligned}
\sigma(z \pm 1) &= -e^{\eta(\pm 2z+1)}\sigma(z); \\
\sigma(z \pm \tau) &= -e^{\eta'(\pm 2z+\tau)}\sigma(z); \\
\zeta(z \pm 1) &= \zeta(z) \pm 2\eta; \\
\zeta(z \pm \tau) &= \zeta(z) \pm 2\eta';
\end{aligned}
$$

where $\zeta(z) = \partial_z \log \sigma(z)$, and η and η' are some τ-dependent complex numbers satisfying $\eta\tau - \eta' = i\pi$.

4.1 Form of the Measure

There is a dense open subset of N_d consisting of those holomorphic maps where all zeros and poles are simple; on this subset, $\tilde{a}_j \neq \tilde{a}_k$ and $\tilde{b}_j \neq \tilde{b}_k$ for $j \neq k$, and $\tilde{a}_m \neq \tilde{b}_n$

for any m, n. In particular, N_d has dimension $2d$. We choose the system of coordinates $(c, a_1, \ldots, a_{d-1}, b_1, \ldots, b_{d-1}, B)$ on N_d, where $c \in \mathbb{C}^*$, and $a_1, \ldots, a_{d-1}, b_1, \ldots, b_{d-1}, B$ lie in the fundamental domain of Σ with vertices at $0, 1, \tau, 1+\tau$. Defining $a_d = B - \sum_{j=1}^{d-1} a_j$ and $b_d = B - \sum_{j=1}^{d-1} b_j$,

$$w(z) := c\frac{\prod_{j=1}^{d} \sigma(z - a_j)}{\prod_{j=1}^{d} \sigma(z - b_j)}$$

defines a point of N_d. As the a_j's and the b_k's can be permuted amongst themselves separately without changing the instanton w, these coordinates form a $(d!)^2$-fold branched cover of N_d. For ease of notation, we denote the coordinates by ξ_j in the following fashion:

$$\xi_0 = c, \ \xi_{2d-1} = B$$

$$\xi_j = a_j, \ \xi_{d-1+j} = b_j \ \text{ for } 1 \leq j \leq d-1.$$

As we mentioned in Chapter 2, there is a natural isomorphism between the tangent space $T_w N_d$ and the space of holomorphic sections of $w^*(T\mathbb{P}^1)$. Under this isomorphism, the coordinate tangent vector $\partial/\partial\xi_j|_w$ is identified with the section $\omega_j := \dfrac{\partial w}{\partial \xi_j} \prod_{k=1}^{d} (s_{1,b_k}(z - b_k))^2$. Let us write out the formulas for these sections:

$$\begin{aligned}
\omega_0(z) &= f_{2d,2B}(z) \prod_{k=1}^{d} (\sigma(z - a_k)\sigma(z - b_k)) \\
\omega_j(z) &= c f_{2d,2B}(z) \prod_{k=1}^{d} (\sigma(z - a_k)\sigma(z - b_k)) \left(\zeta(z - a_d) - \zeta(z - a_j)\right) \\
\omega_{j+d-1}(z) &= c f_{2d,2B}(z) \prod_{k=1}^{d} (\sigma(z - a_k)\sigma(z - b_k)) \left(\zeta(z - b_j) - \zeta(z - b_d)\right) \\
\omega_{2d-1}(z) &= c f_{2d,2B}(z) \prod_{k=1}^{d} (\sigma(z - a_k)\sigma(z - b_k)) \left(\zeta(z - b_d) - \zeta(z - a_d)\right),
\end{aligned}$$

where $1 \le j \le d-1$. If $w \in N_d$, these sections form a basis of $Hol\,(w^*\,(T\mathbb{P}^1))$. To see this, we first note that dim $Hol\,(w^*\,(T\mathbb{P}^1)) = 2d$, so there are the correct number of tangent vectors. It is easy to show that they are also linearly independent. Suppose $\{x_j\}_{j=0}^{2d-1}$ are constants such that $\sum_j x_j\omega_j(z) = 0$. For each $1 \le j \le d-1$, the only section that doesn't vanish at a_j (resp. b_j) is ω_j (resp. ω_{j+d-1}). Therefore, $x_j = 0$ for all $1 \le j \le 2d-2$. Furthermore, ω_{2d-1} does not vanish at a_d while ω_0 does. So $x_{2d-1} = 0$. Since ω_0 is not identically zero, x_0 must now be zero as well. Thus, $\{\omega_j\}_{j=0}^{2d-1}$ is a maximally linearly independent set of sections.

In terms of these coordinates, the measure induced by the hermitian structure on N_d is given by

$$d\mu_d = (d!)^{-2}\left(\frac{i}{2}\right)^{2d}(\det W)\, d\xi_0 \wedge d\bar{\xi}_0 \wedge \cdots \wedge d\xi_{2d-1} \wedge d\bar{\xi}_{2d-1},$$

where W is the w-dependent matrix

$$W_{\bar{j}k} = \int_\Sigma h_w(z)\overline{\omega_j(z)}\omega_k(z)d^2z.$$

In order to compute $\det W$, we first change our basis of sections from $\{\omega_j\}_{j=0}^{2d-1}$ to the sections $\left\{S^0_{r,2B}\right\}_{r=0}^{2d-1}$ discussed in section 3.3. If U is the change of basis matrix, so that

$$\omega_j = \sum_r U_{rj}S^0_{r,2B}, \tag{4.1.1}$$

then $W = U^\dagger Q(h_w)U$, where $Q(h_w)$ is the matrix defined in section 3.3, and $\det W = |\det U|^2 \det Q(h_w)$. We are left with computing the determinant of U.

4.2 Dependence of $\det U$ upon the Coordinates

We first determine how the entries of U transform under the lattice action on the coordinates $\xi_1, \ldots, \xi_{2d-1}$. For all r, $S^0_{r,2B}(z)$ is invariant under the transformations $\xi_j \mapsto \xi_j + 1$ and

$\xi_j \mapsto \xi_j + \tau$ for $1 \leq j \leq 2d-2$. However, under the lattice action on $\xi_{2d-1} = B$, $S^0_{r,2B}(z)$ transforms in the following way:

$$S^0_{r,2B}(z) \mapsto e^{-\frac{2\pi i r}{d}} S^0_{r,2B}(z) \quad \text{when} \quad \xi_{2d-1} \mapsto \xi_{2d-1} + 1$$
$$S^0_{r,2B}(z) \mapsto e^{-\frac{2\pi i \tau}{d}} \exp\left\{4\pi i \left(z - \frac{B}{d} - \frac{1}{2} - \frac{\tau}{2}\right)\right\} S^0_{r-2,2B}(z) \quad \text{when} \quad \xi_{2d-1} \mapsto \xi_{2d-1} + \tau.$$

Here, $S^0_{r-2,2B}$ is interpreted as $S^0_{r-2+2d,2B}$ when $r = 0, 1$.

Furthermore, w is must be invariant under any of the above lattice actions; a meromorphic function on the torus should be independent of the choice of representatives of the zeroes and poles. Therefore, $\xi_0 = c$ transforms under each of the lattice actions in the following ways:

Table 1: Transformation of the coordinate c under the lattice actions.

	$c \mapsto$
$\xi_j \mapsto \xi_j + 1$ $1 \leq j \leq d-1$	$\exp\{-\eta(2\xi_j - 2a_d + 2)\} c$
$\xi_j \mapsto \xi_j + \tau$ $1 \leq j \leq d-1$	$\exp\{-\eta'(2\xi_j - 2a_d + 2\tau)\} c$
$\xi_j \mapsto \xi_j + 1$ $q \leq j \leq 2d-2$	$\exp\{\eta(2\xi_j - 2b_d + 2)\} c$
$\xi_j \mapsto \xi_j + \tau$ $q \leq j \leq 2d-2$	$\exp\{\eta'(2\xi_j - 2b_d + 2\tau)\} c$
$\xi_{2d-1} \mapsto \xi_{2d-1} + 1$	$\exp\{-\eta(2a_d - 2b_d)\} c$
$\xi_{2d-1} \mapsto \xi_{2d-1} + \tau$	$\exp\{-\eta'(2a_d - 2b_d)\} c$

Thus, each $\omega_k(z)$ transforms as follows:

Table 2a: Transformation of basis elements under lattice actions on ξ_j, $1 \leq j \leq d-1$:

	$\xi_j \mapsto \xi_j + 1$	$\xi_j \mapsto \xi_j + \tau$
$\omega_0 \mapsto$	$e^{\eta(2\xi_j - 2a_d + 2)}\omega_0$	$e^{\eta'(2\xi_j - 2a_d + 2\tau)}\omega_0$
$\omega_j \mapsto$	$\omega_j + 4\eta c\omega_0$	$\omega_j + 4\eta' c\omega_0$
$\omega_k \mapsto$ $_{1 \leq k \neq j \leq d-1}$	$(\omega + 2\eta c\omega_0)$	$\omega_k + 2\eta' c\omega_0$
$\omega_k \mapsto$ $_{q \leq k \leq 2d-2}$	ω_k	ω_k
$\omega_{2d-1} \mapsto$	$\omega_{2d-1} + 2\eta c\omega_0$	$\omega_{2d-1} + 2\eta' c\omega_0$

Table 2b: Transformation of basis elements under lattice actions on ξ_j, $d \leq j \leq 2d-2$:

	$\xi_j \mapsto \xi_j + 1$	$\xi_j \mapsto \xi_j + \tau$
$\omega_0 \mapsto$	$e^{\eta(2\xi_j - 2b_d + 2)}\omega_0$	$e^{\eta'(2\xi_j - 2b_d + 2\tau)}\omega_0$
$\omega_j \mapsto$	$e^{2\eta(2\xi_j - 2b_d + 2)}\left(\omega_j - 4\eta c\omega_0\right)$	$e^{2\eta'(2\xi_j - 2b_d + 2\tau)}\left(\omega_j - 4\eta' c\omega_0\right)$
$\omega_k \mapsto$ $_{1 \leq k \leq d-1}$	$e^{2\eta(2\xi_j - 2b_d + 2)}\omega_k$	$e^{2\eta'(2\xi_j - 2b_d + 2\tau)}\omega_k$
$\omega_k \mapsto$ $_{d \leq k \neq j \leq 2d-2}$	$e^{2\eta(2\xi_j - 2b_d + 2)}\left(\omega_k - 2\eta c\omega_0\right)$	$e^{2\eta'(2\xi_j - 2b_d + 2\tau)}\left(\omega_k - 2\eta' c\omega_0\right)$
$\omega_{2d-1} \mapsto$	$e^{2\eta(2\xi_j - 2b_d + 2)}\left(\omega_{2d-1} + 2\eta c\omega_0\right)$	$e^{2\eta'(2\xi_j - 2b_d + 2\tau)}\left(\omega_{2d-1} + 2\eta' c\omega_0\right)$

Table 2c: Transformation of basis elements under lattice actions on ξ_{2d-1}:

	$\xi_{2d-1} \mapsto \xi_{2d-1} + 1$	$\xi_{2d-1} \mapsto \xi_{2d-1} + \tau$
$\omega_0 \mapsto$	$e^{\eta(2a_d + 2b_d + 2)}\omega_0$	$e^{4\pi iz}e^{\eta'(2a_d + 2b_d + 2\tau)}\omega_0$
$\omega_k \mapsto$ $_{1 \leq k \leq q-1}$	$e^{\eta(4b_d + 2)}\left(\omega_k - 2\eta c\omega_0\right)$	$e^{4\pi iz}e^{\eta'(4b_d + 2\tau)}\left(\omega_k - 2\eta' c\omega_0\right)$
$\omega_k \mapsto$ $_{d \leq k \leq 2d-2}$	$e^{\eta(4b_d + 2)}\left(\omega_k + 2\eta c\omega_0\right)$	$e^{4\pi iz}e^{\eta'(4b_d + 2\tau)}\left(\omega_k + 2\eta' c\omega_0\right)$
$\omega_{2d-1} \mapsto$	$e^{\eta(4b_d + 2)}\omega_{2d-1}$	$e^{4\pi iz}e^{\eta'(4b_d + 2\tau)}\omega_{2d-1}$

Putting all of this information together, we can write down how each matrix entry U_{rk} changes under these transformations. For $1 \leq j \leq 2d-2$, U_{rk} transforms in the same way as ω_k under the lattice actions on ξ_j. (ω_0 is replaced by U_{r0} when it appears as an additive factor.) Under the lattice action on ξ_{2d-1}, U_{rk} transforms as follows:

Table 3: Transformation of matrix elements under the lattice action on ξ_{2d-1}:

	$\xi_{2d-1} \mapsto \xi_{2d-1} + 1$	$\xi_{2d-1} \mapsto \xi_{2d-1} + \tau$
$U_{r0} \mapsto$	$e^{2\pi ir/d} e^{\eta(2a_d+2b_d+2)} U_{r0}$	$e^{2\pi i\tau/d} e^{4\pi i\left(\frac{B}{d}+\frac{1}{2}+\frac{\tau}{2}\right)} e^{\eta'(2a_d+2b_d+2\tau)} U_{r-2,0}$
$U_{rk} \mapsto$ $1 \le k \le d-1$	$e^{2\pi ir/d} e^{\eta(4b_d+2)} (U_{rk} - 2\eta c U_{r0})$	$e^{2\pi i\tau/d} e^{4\pi i\left(\frac{B}{d}+\frac{1}{2}+\frac{\tau}{2}\right)} e^{\eta'(4b_d+2\tau)} (U_{r-2,k} - 2\eta' c U_{r-2,0})$
$U_{rk} \mapsto$ $d \le k \le 2d-2$	$e^{2\pi ir/d} e^{\eta(4b_d+2)} (U_{rk} + 2\eta c U_{r0})$	$e^{2\pi i\tau/d} e^{4\pi i\left(\frac{B}{d}+\frac{1}{2}+\frac{\tau}{2}\right)} e^{\eta'(4b_d+2\tau)} (U_{r-2,k} + 2\eta' c U_{r-2,0})$
$U_{r,2d-1} \mapsto$	$e^{2\pi ir/d} e^{\eta(4b_d+2)} U_{r,2d-1}$	$e^{2\pi i\tau/d} e^{4\pi i\left(\frac{B}{d}+\frac{1}{2}+\frac{\tau}{2}\right)} e^{\eta'(4b_d+2\tau)} U_{r-2,2d-1}$

Since $\det U$ remains unchanged when a multiple of one column is added to another, in calculating how $\det U$ transforms, we can ignore the additive factors of $U_{r,0}$. Under the transformation $\xi_{2q-1} \mapsto \xi_{2q-1} + \tau$, the rows are cycled by twice, which also does not affect the determinant. Therefore, the only aspect of the lattice actions that would change the determinant is the scaling factor that multiplies each column. Thus, multiplying these factors together tells us how $\det U$ transforms:

Table 4: Transformation of $\det U$ under the lattice actions.

	$\det U \mapsto$
$\xi_j \mapsto \xi_j + 1$ $1 \le j \le d-1$	$e^{\eta(2\xi_j - 2a_d + 2)} \det U$
$\xi_j \mapsto \xi_j + \tau$ $1 \le j \le d-1$	$e^{\eta'(2\xi_j - 2a_d + 2\tau)} \det U$
$\xi_j \mapsto \xi_j + 1$ $q \le j \le 2d-2$	$e^{(4d-1)\eta(2\xi_j - 2b_d + 2)} \det U$
$\xi_j \mapsto \xi_j + \tau$ $q \le j \le 2d-2$	$e^{(4d-1)\eta'(2\xi_j - 2b_d + 2\tau)} \det U$
$\xi_{2d-1} \mapsto \xi_{2d-1} + 1$	$e^{\eta(2a_d+2b_d+2)} e^{(2d-1)\eta(4b_d+2)} \det U$
$\xi_{2d-1} \mapsto \xi_{2d-1} + \tau$	$e^{4\pi i(2B+(d+1)\tau)} e^{\eta'(2a_d+2b_d+2\tau)} e^{(2d-1)\eta'(4b_d+2\tau)} \det U$

From this information, we can compute the dependence of $\det U$ on the coordinates. We note that if $a_j = a_k$ or $b_j = b_k$ for any $1 \leq j \neq k \leq d-1$, then $\det U = 0$. (Either $\omega_j = \omega_k$ or $\omega_{j+q-1} = \omega_{k+q-1}$.) Furthermore, if $a_j = a_d$ or $b_j = b_d$, then $\omega_j = 0$ or $\omega_{j+d-1} = 0$, so $\det U = 0$. If $a_l = b_m$ for any $1 \leq l, m \leq d-1$, then $\omega_l + \omega_{m+d-1} + \omega_{2d-1} = 0$. Finally, if $a_l = b_d$ or $b_m = a_d$, then $\omega_l + \omega_{2d-1} = 0$ or $\omega_{m+d-1} + \omega_{2d-1} = 0$, and if $a_d = b_d$, then $\omega_{2d-1} = 0$. Furthermore, there is a factor of c in every column of U aside from the first. Therefore, $\det U$ is divisible by

$$c^{2d-1} \prod_{1 \leq j < k \leq d} \sigma\left(a_j - a_k\right) \sigma\left(b_j - b_k\right) \prod_{1 \leq l, m \leq d} \sigma\left(a_l - b_m\right).$$

This quantity transforms in the correct way under the lattice actions on $\xi_1, \ldots, \xi_{2d-2}$. In order to make it transform correctly under the lattice action on ξ_{2d-1}, we need to add the exponential factor $\exp\left(4\eta B^2 + 4\pi i d B\right)$. Hence,

$$\det U = K_d[\tau] c^{2d-1} \exp\left(4\eta B^2 + 4\pi i d B\right) \prod_{1 \leq j < k \leq d} \sigma\left(a_j - a_k\right) \sigma\left(b_j - b_k\right) \prod_{1 \leq l, m \leq d} \sigma\left(a_l - b_m\right), \tag{4.2.1}$$

where $K_d[\tau]$ is independent of all the coordinates $\xi_0, \ldots, \xi_{2d-1}$.

4.3 Dependence of K_d on τ

To calculate $K_d[\tau]$, we first determine its τ-dependence by computing how it transforms under the modular group $PSL_2(\mathbb{Z})$. $PSL_2(\mathbb{Z})$ acts on the upper half plane by fractional linear transformations, and this action is generated by the transformations

$$S \cdot \tau = -\frac{1}{\tau}, \qquad T \cdot \tau = \tau + 1.$$

4.3.1 Transformation under $\tau \mapsto \tau + 1$

It is easy to see that $\sigma(z|\tau+1) = \sigma(z|\tau)$ and $\zeta(z|\tau+1) = \zeta(z|\tau)$, and thus $\eta[\tau+1] = \eta[\tau]$, where $\eta[\tau] = \zeta\left(\frac{1}{2}|\tau\right)$. Therefore, ω_j is invariant under T for each j. However, for any $a, b \in \frac{1}{2d}\mathbb{Z}$,

$$
\begin{aligned}
\theta_{a,b}\left(2dz|2d\left(\tau+1\right)\right) &= \sum_{k\in\mathbb{Z}} \exp\left\{i\pi 2d\left(\tau+1\right)\left(k+a\right)^2 + 2\pi i\left(k+a\right)\left(2dz+b\right)\right\} \\
&= \sum_{k\in\mathbb{Z}} e^{2\pi i d\left(k^2+2ka+a^2\right)} \exp\left\{i\pi 2d\tau\left(k+a\right)^2 + 2\pi i\left(k+a\right)\left(2dz+b\right)\right\} \\
&= \exp\left\{2d\pi i a^2\right\} \theta_{a,b}\left(2dz|2d\tau\right).
\end{aligned}
$$

Therefore, from (3.2.6), we see that $S^0_{r,2B}(z|\tau)$ transforms as:

$$
\begin{aligned}
S^0_{r,2B}(z|\tau+1) &= \theta_{\frac{r}{2d},0}\left(2d\left(z - \frac{B}{d} - \frac{1}{2} - \frac{\tau+1}{2}\right)\middle| 2d\left(\tau+1\right)\right) \\
&= \exp\left\{\frac{\pi i}{2d} r^2\right\} \theta_{\frac{r}{2d},0}\left(2d\left(z - \frac{B}{d} - \frac{1}{2} - \frac{\tau}{2}\right) - d\middle| 2d\tau\right) \\
&= \exp\left\{\frac{\pi i}{2d} r^2\right\} \exp\left\{\pi i r\right\} \theta_{\frac{r}{2d},0}\left(2d\left(z - \frac{B}{d} - \frac{1}{2} - \frac{\tau}{2}\right)\middle| 2d\tau\right) \\
&= \exp\left\{\frac{\pi i}{2d}\left(r^2 - 2dr\right)\right\} S^0_{r,2B}(z|\tau).
\end{aligned}
$$

We conclude that $U_{rj}\left[\tau+1\right] = \exp\left\{-\frac{\pi i}{2d}\left(r^2 - 2dr\right)\right\} U_{rj}\left[\tau\right]$ (4.1.1). Therefore,

$$
\begin{aligned}
\det U[\tau+1] &= \exp\left\{-\frac{\pi i}{2d} \sum_{r=0}^{2d-1}\left(r^2 - 2dr\right)\right\} \det U[\tau] \\
&= \exp\left\{-\pi i\left(\frac{\left(2d-1\right)\left(4d-1\right)}{6} - \frac{\left(2d-1\right)\left(2d\right)}{2}\right)\right\} \det U[\tau] \\
&= \exp\left\{\frac{\pi i}{6}\left(4d^2 - 1\right)\right\} \det U[\tau]
\end{aligned}
$$

Finally, by (4.2.1),

$$K_d[\tau+1] = \exp\left\{\frac{\pi i}{6}\left(4d^2-1\right)\right\} K_d[\tau]. \tag{4.3.1}$$

4.3.2 Transformation under $\tau \mapsto -\tau^{-1}$

The transformation under the flip, S, is trickier to calculate. First,

$$\begin{aligned} \sigma\left(z\Big| -\frac{1}{\tau}\right) &= \frac{1}{\tau}\sigma\left(\tau z|\tau\right) \\ \zeta\left(z\Big| -\frac{1}{\tau}\right) &= \tau\zeta\left(\tau z|\tau\right) \\ \eta\left[-\frac{1}{\tau}\right] &= \tau\eta'\left[\tau\right] = \eta\left[\tau\right]\tau^2 - i\pi\tau, \end{aligned}$$

where $\eta'[\tau] = \zeta\left(\frac{\tau}{2}|\tau\right)$. Therefore,

$$\begin{aligned} f_{1,z_D}\left(z\Big| -\frac{1}{\tau}\right)\sigma\left(z-z_D\Big| -\frac{1}{\tau}\right) &= \frac{1}{\tau}e^{i\pi\tau\left(z^2-2z_Dz-z\right)+i\pi z}f_{1,\tau z_D}\left(\tau z|\tau\right)\sigma\left(\tau(z-z_D)|\tau\right) \\ \omega_0\left(z\Big| -\frac{1}{\tau}\right) &= \tau^{-2d}e^{i\pi\tau\left(2dz^2-4Bz-2dz\right)+2d\pi i z}\omega_0\left(\tau z|\tau\right) \\ \omega_j\left(z\Big| -\frac{1}{\tau}\right) &= \tau^{-2d+1}e^{i\pi\tau\left(2dz^2-4Bz-2dz\right)+2d\pi i z}\omega_j\left(\tau z|\tau\right) \end{aligned}$$

for $1 \le j \le 2d-1$. Therefore,

$$\sum_{r=0}^{2d-1} U_{r0}\left[-\frac{1}{\tau}\right] S^0_{r,2B}\left(z\,\middle|\,-\frac{1}{\tau}\right) = \tau^{-2d}e^{i\pi\tau\left(2dz^2-4Bz-2dz\right)+2d\pi i z}\sum_{s=0}^{2d-1} U_{s0}\left[\tau\right] S^0_{s,2B\tau}\left(\tau z|\tau\right). \tag{4.3.2}$$

Let C be the matrix defined by $C_{rs} = \exp\left\{\frac{i\pi}{2d}rs\right\}$ for $0 \le r, s \le 2d-1$. C is basically the character table for the group $\mathbb{Z}/2d\mathbb{Z}$, and so it is invertible. We have the following result:

Lemma. *Given any* $0 \leq r \leq 2d-1$*,*

$$\theta_{\frac{r}{2d},0}\left(2dz\left|-\frac{2d}{\tau}\right.\right) = \sqrt{-2di\tau}e^{2d\pi i\tau z^2}\sum_{s=0}^{2d-1}\left(C^{-1}\right)_{sr}\theta_{\frac{s}{2d},0}\left(2d\tau z|2d\tau\right).$$

Proof. This relies on two relations involving theta functions. The first is that

$$\theta_{0,\frac{s}{2d}}\left(z\left|-\frac{1}{2d\tau}\right.\right) = \sum_{r=0}^{2d-1} C_{rs}\theta_{\frac{r}{2d},0}\left(2dz\left|-\frac{2d}{\tau}\right.\right)$$

for any $0 \leq s \leq 2d-1$. To show this, we merely compare Fourier coefficients on both sides, since $\theta_{\frac{r}{2d},0}\left(2dz\left|-\frac{2d}{\tau}\right.\right)$ only have Fourier coefficients equivalent to r modulo $2d$.

$$\theta_{0,\frac{s}{2d}}\left(z\left|\frac{\tau}{2d}\right.\right) = \sum_{k\in\mathbb{Z}}\exp\left\{i\pi\frac{\tau}{2d}k^2 + 2\pi ik\left(z+\frac{s}{2d}\right)\right\};$$

so its rth Fourier coefficient is $\exp\left\{i\pi\frac{\tau}{2d}r^2 + \frac{\pi i}{d}rs\right\} = \exp\left\{i\pi\frac{\tau}{2d}r^2\right\}C_{rs}$. Also, the rth Fourier coefficient of $\theta_{\frac{r}{2d},0}$ is $\exp\left\{i\pi\frac{\tau}{2d}r^2\right\}$. This proves the formula. Thus,

$$\theta_{\frac{r}{2d},0}\left(2dz\left|-\frac{2d}{\tau}\right.\right) = \sum_{s=0}^{2d-1}\left(C^{-1}\right)_{sr}\theta_{0,\frac{s}{2d}}\left(z\left|-\frac{1}{2d\tau}\right.\right)$$

The second relation is the well-known action on the theta function by the involution in the modular group:

$$\theta_{0,0}\left(z\left|-\frac{1}{\tau}\right.\right) = \sqrt{-i\tau}e^{\pi i\tau z^2}\theta_{0,0}\left(\tau z|\tau\right).$$

Replacing τ by $2d\tau$ and z by $z+\frac{s}{2d}$, we see:

$$\begin{aligned}\theta_{0,\frac{s}{2d}}\left(z\left|-\frac{1}{2d\tau}\right.\right) &= \theta_{0,0}\left(z+\frac{s}{2d}\left|-\frac{1}{2d\tau}\right.\right) = \sqrt{-2di\tau}e^{2d\pi i\tau\left(z+\frac{s}{2d}\right)^2}\theta_{0,0}\left(2d\tau z+\tau s\right|2d\tau) \\ &= \sqrt{-2di\tau}e^{2d\pi i\tau\left(z+\frac{s}{2d}\right)^2}\sum_{k\in\mathbb{Z}}\exp\left\{\pi i\left(2d\tau\right)k^2 + 2\pi ik\left(2d\tau z\right) + 2\pi i\tau ks\right\}\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{-2di\tau}e^{2d\pi i\tau\left(z+\frac{s}{2d}\right)^2}\sum_{k\in\mathbb{Z}}\exp\left\{\pi i\left(2d\tau\right)\left(k^2+2k\frac{s}{2d}\right)+2\pi ik\left(2d\tau z\right)\right\}\\
&= \sqrt{-2di\tau}e^{2d\pi i\tau\left(z+\frac{s}{2d}\right)^2}e^{-\pi i\tau s\left(2z+\frac{s}{2d}\right)}\sum_{k\in\mathbb{Z}}e^{\pi i(2d\tau)\left(k+\frac{s}{2d}\right)^2+2\pi i\left(k+\frac{s}{2d}\right)(2d\tau z)}\\
&= \sqrt{-2di\tau}e^{2d\pi i\tau z^2}\theta_{\frac{s}{2d},0}\left(2d\tau z|2d\tau\right).
\end{aligned}
$$

This finishes the lemma. $\square$

Using (3.2.6), we can apply this lemma to $S^0_{r,2B}\left(z\left|-\frac{1}{\tau}\right.\right)$:

$$
\begin{aligned}
S^0_{r,2B}\left(z\left|-\frac{1}{\tau}\right.\right) &= \theta_{\frac{r}{2d},0}\left(2d\left(z-\frac{B}{d}-\frac{1}{2}+\frac{1}{2\tau}\right)\left|-\frac{2d}{\tau}\right.\right)\\
&= \sqrt{-2di\tau}e^{2d\pi i\tau Z^2}\sum_{s=0}^{2d-1}\left(C^{-1}\right)_{sr}\theta_{\frac{s}{2d},0}\left(2d\tau Z|\,2d\tau\right)\\
&= \sqrt{-2di\tau}e^{2d\pi i\tau Z^2}\sum_{s=0}^{2d-1}\left(C^{-1}\right)_{sr}\theta_{\frac{s}{2d},0}\left(2d\left(\tau Z-1\right)+2d|\,2d\tau\right)\\
&= \sqrt{-2di\tau}e^{2d\pi i\tau Z^2}\sum_{s=0}^{2d-1}\left(C^{-1}\right)_{sr}\theta_{\frac{s}{2d},0}\left(2d\left(\tau Z-1\right)|\,2d\tau\right)\\
&= \sqrt{-2di\tau}e^{2d\pi i\tau Z^2}\sum_{s=0}^{2d-1}\left(C^{-1}\right)_{sr}\theta_{\frac{s}{2d},0}\left(2d\left(\tau z-\frac{\tau B}{d}-\frac{\tau}{2}-\frac{1}{2}\right)\left|\,2d\tau\right.\right)\\
&= \sqrt{-2di\tau}e^{2d\pi i\tau Z^2}\sum_{s=0}^{2d-1}\left(C^{-1}\right)_{sr}S^0_{s,2B\tau}\left(\tau z|\tau\right),
\end{aligned}
$$

where $Z = z-\frac{B}{d}-\frac{1}{2}+\frac{1}{2\tau}$. Now we can calculate how the matrix entries transform. From (4.3.2),

$$
\begin{aligned}
&\sqrt{-2di\tau}e^{2d\pi i\tau Z^2}\sum_{s=0}^{2d-1}\left(\sum_{r=0}^{2d-1}\left(C^{-1}\right)_{sr}U_{r0}\left[-\frac{1}{\tau}\right]\right)S^0_{s,2B\tau}\left(\tau z|\tau\right)\\
= {}& \tau^{-2d}e^{i\pi\tau\left(2dz^2-4Bz-2dz\right)+2d\pi iz}\sum_{s=0}^{2d-1}U_{s0}\left[\tau\right]S^0_{s,2B\tau}\left(\tau z|\tau\right)
\end{aligned}
$$

Therefore,

$$\begin{aligned}
\sqrt{-2di\tau}e^{2d\pi i\tau Z^2}\left(\sum_{r=0}^{2d-1}\left(C^{-1}\right)_{sr}U_{r0}\left[-\frac{1}{\tau}\right]\right) &= \tau^{-2d}e^{i\pi\tau\left(2dz^2-4Bz-2dz\right)+2d\pi iz}U_{s0}\left[\tau\right] \\
\sum_{r=0}^{2d-1}\left(C^{-1}\right)_{sr}U_{r0}\left[-\frac{1}{\tau}\right] &= \frac{\tau^{-2d}}{\sqrt{-2di\tau}}e^{-2d\pi i\tau\left(-\frac{B}{d}-\frac{1}{2}+\frac{1}{2\tau}\right)^2}U_{s0}\left[\tau\right] \\
U_{r0}\left[-\frac{1}{\tau}\right] &= \frac{\tau^{-2d}}{\sqrt{-2di\tau}}e^{-2d\pi i\tau\left(-\frac{B}{d}-\frac{1}{2}+\frac{1}{2\tau}\right)^2}\sum_{s=0}^{2d-1}C_{rs}U_{s0}\left[\tau\right].
\end{aligned}$$

For $1 \leq j \leq 2d-1$, the only difference is that ω_j contains a difference of Weierstrass ζ-functions, which adds a power of τ to the numerator of the computation:

$$U_{rj}\left[-\frac{1}{\tau}\right] = \frac{\tau^{-2d+1}}{\sqrt{-2di\tau}}e^{-2d\pi i\tau\left(-\frac{B}{d}-\frac{1}{2}+\frac{1}{2\tau}\right)^2}\sum_{s=0}^{2d-1}C_{rs}U_{sj}\left[\tau\right]$$

Therefore, $U\left[-\frac{1}{\tau}\right] = \frac{\tau^{-2d+1}}{\sqrt{-2di\tau}}e^{-2d\pi i\tau\left(-\frac{B}{d}-\frac{1}{2}+\frac{1}{2\tau}\right)^2}DCU\left[\tau\right]$, where D is the diagonal matrix with $D_{jj} = \tau^{-1}$ if $j=0$, and 1 otherwise. Therefore,

$$\det U\left[-\frac{1}{\tau}\right] = \tau^{-4d^2+2d-1}\left(-2di\tau\right)^{-d}e^{-4d^2\pi i\tau\left(-\frac{B}{d}-\frac{1}{2}+\frac{1}{2\tau}\right)^2}\det C\det U\left[\tau\right].$$

Since C is the character table of the group $\mathbb{Z}/(2d)\mathbb{Z}$. The character orthogonality relations imply $C\bar{C}^t = (2d)\,I$, and therefore, $\det C = \gamma\,(2d)^d$, for some root of unity γ. Hence,

$$\det U\left[-\frac{1}{\tau}\right] = \gamma\tau^{-4d^2+2d-1}\left(-i\tau\right)^{-d}e^{-4d^2\pi i\tau\left(-\frac{B}{d}-\frac{1}{2}+\frac{1}{2\tau}\right)^2}\det U\left[\tau\right]. \tag{4.3.3}$$

The exponential factor on the other side of (4.2.1) transforms as follows:

$$\begin{aligned}
e^{4\eta\left[-\tau^{-1}\right]B^2+4\pi idB} &= e^{4\eta[\tau]\tau^2B^2-i\pi\tau B+4\pi idB} \\
&= e^{4\eta[\tau](\tau B)^2+4\pi id(\tau B)}e^{-4\pi id^2\tau\left(\frac{B^2}{d^2}+\frac{B}{d\tau}-\frac{B}{d}\right)}.
\end{aligned}$$

Each σ-function transforms like $\sigma\left(a_i - a_j \left| -\frac{1}{\tau}\right.\right) = \frac{1}{\tau}\sigma\left(\tau a_i - \tau a_j|\tau\right)$. Furthermore, there are $2d^2 - d$ σ-factors on the right-hand side of (4.2.1). Therefore, if we plug in the appropriate expressions for $\det U$ in both sides of (4.3.3) and cancel corresponding factors, we find

$$\begin{aligned}\tau^{-2d^2+d} K_d\left[-\frac{1}{\tau}\right] &= \gamma\tau^{-4d^2+2d-1}\left(-i\tau\right)^{-d} e^{-d^2\pi i\left(\tau+\tau^{-1}\right)} K_d\left[\tau\right] \\ K_d\left[-\frac{1}{\tau}\right] &= \gamma i^{-2d^2+d-1}\left(-i\tau\right)^{-d} e^{-d^2\pi i\left(\tau+\tau^{-1}\right)} K_d\left[\tau\right].\end{aligned}$$

If we plug in $\tau = i$, we can compute the value of γ:$1 = \gamma i^{-2d^2+d-1}$; $\gamma = (-i)^{-2d^2+d-1}$. Therefore,

$$K_d\left[-\frac{1}{\tau}\right] = \left(-i\tau\right)^{-2d^2-1} e^{-d^2\pi i\left(\tau+\tau^{-1}\right)} K_d\left[\tau\right]. \tag{4.3.4}$$

If we set $\tilde{K}_d[\tau] := e^{-d^2\pi i\tau} K_d[\tau]$, then we can rewrite (4.3.1) and (4.3.4) as:

$$\begin{aligned}\tilde{K}_d[\tau+1] &= \exp\left\{\frac{\pi i}{6}\left(-2d^2-1\right)\right\}\tilde{K}_d[\tau] \\ \tilde{K}_d\left[-\tau^{-1}\right] &= \left(-i\tau\right)^{-2d^2-1}\tilde{K}_d[\tau].\end{aligned}$$

Therefore, $\tilde{K}[\tau]$ transforms in the same way under the modular group as the $(-4d^2-2)$th power of Dedekind's η-function:

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}}\prod_{n=1}^{\infty}\left(1-q^n\right); \quad q = e^{2\pi i\tau}.$$

Thus $\tilde{K}_d[\tau]\left(\eta(\tau)\right)^{4d^2+2}$ is invariant under the modular group. Furthermore, either $\eta\left(\tau\right)$ or $\tilde{K}_d[\tau]$ have any zeroes or poles in the upper half plane. (For η, this fact is evident from the product expansion above. $\tilde{K}_d$ cannot have zeroes or poles because the invertibility of U is independent of the complex structure on the torus.) Therefore, $\tilde{K}_d[\tau]\left(\eta(\tau)\right)^{4d^2+2}$ must be constant. Thus,

$$K_d[\tau] = A_d e^{d^2\pi i\tau}\eta\left(\tau\right)^{-4d^2-2} \tag{4.3.5}$$

for some constant A_d.

4.4 Asymptotics as $\tau \to i\infty$

From [1], we have the following asymptotics as $\tau \to i\infty$:

$$
\begin{aligned}
f_{1,z_D}(z)\,\sigma(z-z_D|\tau) &= \exp\left\{-\eta\left(z^2-2z_Dz\right)+i\pi z\right\}\sigma(z-z_D|\tau) \\
&= \exp\left\{\eta z_D^2\right\}\exp\left\{-\eta(z-z_D)^2\right\}\sigma(z-z_D|\tau)\,e^{i\pi z} \\
&= \exp\left\{\eta z_D^2\right\}\frac{\theta_{11}(z-z_D)}{\theta_{11}'(0)}e^{i\pi z} \\
&= \exp\left\{\frac{\pi^2}{6}\left(1+O\left(e^{2\pi i\tau}\right)\right)z_D^2\right\}\frac{\sin(\pi(z-z_D))+O\left(e^{2\pi i\tau}\right)}{\pi\left(1+O\left(e^{2\pi i\tau}\right)\right)}e^{i\pi z} \\
&= \exp\left\{\frac{\pi^2}{6}z_D^2\right\}e^{i\pi z}\frac{\sin(\pi(z-z_D))}{\pi}+O\left(e^{2\pi i\tau}\right).
\end{aligned}
$$

Also,

$$
\begin{aligned}
\zeta(z-z_D) &= \frac{\pi^2}{3}(z-z_D)+\pi\cot(\pi(z-z_D))+O\left(e^{2\pi i\tau}\right). \\
\zeta(z-a)-\zeta(z-b) &= \frac{\pi^2}{3}(b-a)+\pi\frac{\sin(\pi(a-b))}{\sin(\pi(z-a))\sin(\pi(z-b))}.
\end{aligned}
$$

Therefore, for $1 \le j \le d-1$

$$
\begin{aligned}
\omega_0(z) &= \frac{1}{\pi^{2d}}e^{\frac{\pi^2}{6}\sum_{k=1}^d a_k^2+b_k^2}e^{2di\pi z}\prod_{k=1}^d\sin(\pi(z-a_k))\sin(\pi(z-b_k))+O\left(e^{2\pi i\tau}\right) \\
\omega_j(z) &= \frac{c}{\pi^{2d}}\left[\frac{\pi^2}{3}(a_j-a_d)+\pi\frac{\sin(\pi(a_d-a_j))}{\sin(\pi(z-a_j))\sin(\pi(z-a_d))}\right]\exp\left\{\frac{\pi^2}{6}\sum_{k=1}^d a_k^2+b_k^2\right\} \\
&\quad\cdot e^{2di\pi z}\prod_{k=1}^d\sin(\pi(z-a_k))\sin(\pi(z-b_k))+O\left(e^{2\pi i\tau}\right) \\
\omega_{j+d-1}(z) &= \frac{c}{\pi^{2d}}\left[\frac{\pi^2}{3}(b_d-b_j)+\pi\frac{\sin(\pi(b_j-b_d))}{\sin(\pi(z-b_j))\sin(\pi(z-b_d))}\right]\exp\left\{\frac{\pi^2}{6}\sum_{k=1}^d a_k^2+b_k^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot e^{2di\pi z}\prod_{k=1}^{d}\sin\left(\pi\left(z-a_k\right)\right)\sin\left(\pi\left(z-b_k\right)\right)+O\left(e^{2\pi i\tau}\right)\\
\omega_{2d-1}(z) \;\; = \;\; &\frac{c}{\pi^{2d}}\left[\frac{\pi^2}{3}\left(a_d-b_d\right)+\pi\frac{\sin\left(\pi\left(b_d-a_d\right)\right)}{\sin\left(\pi\left(z-a_d\right)\right)\sin\left(\pi\left(z-b_d\right)\right)}\right]\exp\left\{\frac{\pi^2}{6}\sum_{k=1}^{d}a_k^2+b_k^2\right\}\\
&\cdot e^{2di\pi z}\prod_{k=1}^{d}\sin\left(\pi\left(z-a_k\right)\right)\sin\left(\pi\left(z-b_k\right)\right)+O\left(e^{2\pi i\tau}\right).
\end{aligned}
$$

The same argument that shows that the set $\{\omega_j\}_{j=0}^{2d-1}$ is linearly independent demonstrates that their leading terms are independent, as well. In addition, the leading terms span the subspace of polynomials in $e^{2\pi iz}$ of degree $2d$ satisfying the condition:

$$
c_{2d}=e^{-4\pi iB}c_0,
$$

where c_0 is the constant term and c_{2d} is the coefficient of $e^{2\pi i(2d)z}$. Let $\tilde{\omega}_j$ be defined as follows:

$$
\begin{aligned}
\tilde{\omega}_0(z) &= e^{2d\pi iz}\prod_{k=1}^{d}\sin\left(\pi\left(z-a_k\right)\right)\sin\left(\pi\left(z-b_k\right)\right)\\
\tilde{\omega}_j(z) &= \frac{e^{2d\pi iz}}{\sin\left(\pi\left(z-a_j\right)\right)\sin\left(\pi\left(z-a_d\right)\right)}\prod_{k=1}^{d}\sin\left(\pi\left(z-a_k\right)\right)\sin\left(\pi\left(z-b_k\right)\right)\\
\tilde{\omega}_{j+d-1}(z) &= \frac{e^{2d\pi iz}}{\sin\left(\pi\left(z-b_j\right)\right)\sin\left(\pi\left(z-b_d\right)\right)}\prod_{k=1}^{d}\sin\left(\pi\left(z-a_k\right)\right)\sin\left(\pi\left(z-b_k\right)\right)\\
\tilde{\omega}_{2d-1}(z) &= \frac{e^{2d\pi iz}}{\sin\left(\pi\left(z-b_d\right)\right)\sin\left(\pi\left(z-a_d\right)\right)}\prod_{k=1}^{d}\sin\left(\pi\left(z-a_k\right)\right)\sin\left(\pi\left(z-b_k\right)\right)
\end{aligned}
$$

for $1\le j\le d-1$. Furthermore, for $0\le k,r\le 2d-1$, define $\tilde{U}_{rk}$ to be the Fourier coefficients of $\tilde{\omega}_j$:

$$
\tilde{\omega}_k(z)=\tilde{U}_{0k}\left(1+e^{-4\pi iB}e^{4d\pi iz}\right)+\sum_{r=1}^{2d-1}\tilde{U}_{rk}e^{2\pi irz}.
$$

In particular, $\tilde{U}_{0k}=0$ for $k>0$.

On the other hand, for any r,

$$\begin{aligned} S^0_{r,2B}(z|\tau) &= \sum_{k\in\mathbb{Z}} \exp\left\{\frac{\pi i\tau}{2d}(2dk+r)^2 + 2\pi i(2dk+r)\left(z-\frac{B}{d}-\frac{1}{2}-\frac{\tau}{2}\right)\right\} \\ &= \sum_{k\in\mathbb{Z}} \exp\left\{\frac{\pi i\tau}{2d}\left[(2dk+r)^2 - 2d(2dk+r)\right] + 2\pi i(2dk+r)\left(z-\frac{B}{d}-\frac{1}{2}\right)\right\} \\ &= \sum_{k\in\mathbb{Z}} \exp\left\{\frac{\pi i\tau}{2d}\left[(2dk+r-d)^2 - d^2\right] + 2\pi i(2dk+r)\left(z-\frac{B}{d}-\frac{1}{2}\right)\right\} \\ &= e^{-\frac{d\pi i\tau}{2}} \sum_{k\in\mathbb{Z}} \exp\left\{\frac{\pi i\tau}{2d}(2dk+r-d)^2 + 2\pi i(2dk+r)\left(z-\frac{B}{d}-\frac{1}{2}\right)\right\}. \end{aligned}$$

Therefore, for any $0 \le r \le 2d-1$, the leading term in $S^0_{r,2B}(z|\tau)$ grows like $e^{\frac{\pi i\tau}{2d}\left(r^2-2dr\right)}$, while the rest of the terms die off exponentially as $\tau \to i\infty$. More explicitly, for $1 \le r \le d-1$,

$$\begin{aligned} S^0_{0,2B}(z|\tau) &= \left(1+e^{4d\pi iz}e^{-4\pi iB}\right) + O\left(e^{4d\pi i\tau}\right) \\ S^0_{r,2B}(z|\tau) &= e^{\frac{\pi i\tau}{2d}\left(r^2-2dr\right)} \exp\left\{2\pi ir\left(z-\frac{B}{d}-\frac{1}{2}\right)\right\} + O\left(e^{\frac{\pi i\tau}{2d}\left(2dr+r^2\right)}\right) \\ S^0_{d,2B}(z|\tau) &= e^{-\frac{d\pi i\tau}{2}} \exp\left\{2\pi id\left(z-\frac{B}{d}-\frac{1}{2}\right)\right\} + O\left(e^{\frac{3}{2}d\pi i\tau}\right) \\ S^0_{d+r,2B}(z|\tau) &= e^{-\frac{d\pi i\tau}{2}} e^{\frac{\pi i\tau}{2d}r^2} \exp\left\{2\pi i(d+r)\left(z-\frac{B}{d}-\frac{1}{2}\right)\right\} + O\left(e^{\frac{\pi i\tau}{2d}\left(3d^2-4dr+r^2\right)}\right). \end{aligned}$$

As functions of z, the leading terms of $S^0_{r,2B}(z|\tau)$ are clearly a basis for the same space as those of the ω_j. Expanding the matrix entries asymptotically from (4.1.1), we find that the leading term in the asymptotic expansion of U_{rk} is:

Table 5: Leading Terms in the Asymptotic Expansion of Matrix Entries

	Leading Term
U_{r0}	$\frac{1}{\pi^{2d}} \exp\left\{\frac{\pi^2}{6}\sum_{j=1}^{d}\left(a_j^2+b_j^2\right)\right\} e^{\frac{\pi i\tau}{2d}\left(2dr-r^2\right)} e^{2\pi i r\left(\frac{B}{d}+\frac{1}{2}\right)} \tilde{U}_{r0}$
U_{rk} $1 \le k \le d-1$	$\frac{c}{\pi^{2d}} \exp\left\{\frac{\pi^2}{6}\sum_{j=1}^{d}\left(a_j^2+b_j^2\right)\right\} e^{\frac{\pi i\tau}{2d}\left(2dr-r^2\right)} e^{2\pi i r\left(\frac{B}{d}+\frac{1}{2}\right)}$ $\cdot\left[\frac{\pi^2}{3}\left(a_k-a_d\right)\tilde{U}_{r0}+\pi\sin\left(\pi\left(a_d-a_k\right)\right)\tilde{U}_{rk}\right]$
U_{rk} $d \le k \le 2d-2$	$\frac{c}{\pi^{2d}} \exp\left\{\frac{\pi^2}{6}\sum_{j=1}^{d}\left(a_j^2+b_j^2\right)\right\} e^{\frac{\pi i\tau}{2d}\left(2dr-r^2\right)} e^{2\pi i r\left(\frac{B}{d}+\frac{1}{2}\right)}$ $\cdot\left[\frac{\pi^2}{3}\left(b_d-b_k\right)\tilde{U}_{r0}+\pi\sin\left(\pi\left(b_k-b_d\right)\right)\tilde{U}_{rk}\right]$
$U_{r,2d-1}$	$\frac{c}{\pi^{2d}} \exp\left\{\frac{\pi^2}{6}\sum_{j=1}^{d}\left(a_j^2+b_j^2\right)\right\} e^{\frac{\pi i\tau}{2d}\left(2dr-r^2\right)} e^{2\pi i r\left(\frac{B}{d}+\frac{1}{2}\right)}$ $\cdot\left[\frac{\pi^2}{3}\left(a_d-b_d\right)\tilde{U}_{r0}+\pi\sin\left(\pi\left(b_d-a_d\right)\right)\tilde{U}_{r,2d-1}\right]$

Therefore, the leading term of $\det U$ is

$$\frac{c^{2d-1}}{\pi^{4d^2}} \exp\left\{2d\frac{\pi^2}{6}\sum_{j=1}^{d}\left(a_j^2+b_j^2\right)\right\} \exp\left\{\frac{\pi i\tau}{6}\left(4d^2-1\right)\right\} e^{\pi i(4d-2)B}\left(-1\right)^d \pi^{2d-1}$$
$$\cdot\sin\left(\pi\left(b_d-a_d\right)\right)\left[\prod_{j=1}^{d-1}\sin\left(\pi\left(a_d-a_k\right)\right)\sin\left(\pi\left(b_k-b_d\right)\right)\right]\det\tilde{U}.$$

Furthermore, the asymptotics of K_d can be determined from (4.3.5):

$$\begin{aligned} K_d[\tau] &= A_d e^{d^2\pi i\tau}\left(e^{\frac{\pi i\tau}{12}}\left(1+O\left(e^{2\pi i\tau}\right)\right)\right)^{-4d^2-2} \\ &= A_d e^{d^2\pi i\tau}\exp\left\{\frac{\pi i\tau}{6}\left(-2d^2-1\right)\right\}\left(1+O\left(e^{2\pi i\tau}\right)\right) \\ &= A_d\exp\left\{\frac{\pi i\tau}{6}\left(4d^2-1\right)\right\}\left(1+O\left(e^{2\pi i\tau}\right)\right). \end{aligned}$$

Therefore, the right-hand side of (4.2.1) has the form

$$A_d\exp\left\{\frac{\pi i\tau}{6}\left(4d^2-1\right)\right\}c^{2d-1}\exp\left(4\frac{\pi^2}{6}B^2+4\pi i dB\right)\pi^{-2d^2+d}$$

$$\prod_{1\le j<k\le d} e^{\pi^2\left(a_j-a_k\right)^2/6}\sin\left(\pi\left(a_j-a_k\right)\right)e^{\pi^2\left(b_j-b_k\right)^2/6}\sin\left(\pi\left(b_j-b_k\right)\right)$$

$$\prod_{1\leq l,m\leq d} e^{\pi^2(a_l-b_m)^2/6}\sin\left(\pi\left(a_l-b_m\right)\right)+O\left(e^{2\pi i\tau}\right).$$

Cancelling common factors from the dominant terms yields

$$\det\tilde{U}=A_d e^{2\pi iB}\pi^{2d^2-d+1}\prod_{1\leq j<k\leq d-1}\sin\left(\pi\left(a_j-a_k\right)\right)\sin\left(\pi\left(b_j-b_k\right)\right)\prod_{\substack{1\leq l,m\leq d\\(l,m)\neq(d,d)}}\sin\left(\pi\left(a_l-b_m\right)\right).$$

But $\det\tilde{U}=\tilde{U}_{00}\det\tilde{U}'=(2i)^{-2d}\,e^{2\pi iB}\det\tilde{U}'$, where $\tilde{U}'$ is the lower right $(2d-1)\times(2d-1)$ block of $\tilde{U}$. Thus,

$$\det\tilde{U}'=A_d\left(2i\right)^{2d}\pi^{2d^2-d+1}\prod_{j<k\leq d-1}\sin\left(\pi\left(a_j-a_k\right)\right)\sin\left(\pi\left(b_j-b_k\right)\right)\prod_{\substack{1\leq l,m\leq d\\(l,m)\neq(d,d)}}\sin\left(\pi\left(a_l-b_m\right)\right).$$

We can compute $\det\tilde{U}'^{(d)}$, and thus A_d, inductively. First, for $d=2$,

$$\begin{aligned}\det\tilde{U}'^{(2)} &= \begin{vmatrix} -\frac{1}{4}e^{\pi iB} & -\frac{1}{4}e^{\pi iB} & -\frac{1}{4}e^{\pi i(a_1+b_1)} \\ \frac{1}{2}\cos\left(\pi\left(b_2-b_1\right)\right) & \frac{1}{2}\cos\left(\pi\left(a_2-a_1\right)\right) & \frac{1}{2}\cos\left(\pi\left(b_1-a_1\right)\right) \\ -\frac{1}{4}e^{-\pi iB} & -\frac{1}{4}e^{-\pi iB} & -\frac{1}{4}e^{-\pi i(a_1+b_1)} \end{vmatrix} \\ &= -\frac{i}{8}\sin\left(\pi\left(a_1-b_1\right)\right)\sin\left(\pi\left(a_1-b_2\right)\right)\sin\left(\pi\left(a_2-b_1\right)\right).\end{aligned}$$

Therefore, $-\frac{i}{8}=A_2\left(2i\right)^4\pi^7$. So $A_2=-\left(2\pi i\right)^{-7}$.

Furthermore, for $d\geq 2$,

$$\begin{aligned}&\lim_{a_{d+1}\to b_{d+1}}\frac{\det\tilde{U}'^{(d+1)}}{\prod_{l=1}^{d}\sin\left(\pi\left(a_l-b_{d+1}\right)\right)\sin\left(\pi\left(a_{d+1}-b_l\right)\right)}\\ =\ &\frac{A_{d+1}}{A_d}(2i)^2\pi^{4d+1}\sin\left(\pi\left(a_d-b_d\right)\right)\left(\prod_{j=1}^{d-1}\sin\left(\pi\left(a_j-a_d\right)\right)\sin\left(\pi\left(b_j-b_d\right)\right)\right)\det\tilde{U}'^{(d)}.\end{aligned}$$

However, $\det\tilde{U}'^{(d+1)}=\det\begin{pmatrix}\tilde{\omega}_1^{(d+1)} & \cdots & \tilde{\omega}_{2d+1}^{(d+1)}\end{pmatrix}$, where $\tilde{\omega}_j^{(d+1)}$ has been equated with its column vector $\left(\tilde{U}'^{(d+1)}{}_{rj}\right)_r$. Furthermore, the angle addition formula for sine lets us rewrite $\sin\left(\pi\left(z-b_{d+1}\right)\right)$ as $\sin\left(\pi\left(z-a_j\right)\right)\cos\left(\pi\left(a_j-b_d\right)\right)+\cos\left(\pi\left(z-a_j\right)\right)\sin\left(\pi\left(a_j-b_d\right)\right)$. Therefore, for $1 \leq j \leq d$,

$$\tilde{\omega}_j^{(d+1)}(z) = \cos\left(\pi\left(a_j-b_{d+1}\right)\right)\tilde{\omega}_{2d+1}^{(d+1)}(z)+\sin\left(\pi\left(a_j-b_{d+1}\right)\right)\cos\left(\pi\left(z-a_j\right)\right)\frac{\tilde{\omega}_j^{(d+1)}(z)}{\sin\left(\pi\left(z-b_{d+1}\right)\right)}.$$

A similar expansion of $\sin\left(\pi\left(z-a_{d+1}\right)\right)$ yields the identity

$$\tilde{\omega}_{d+j}^{(d+1)}(z) = \cos\left(\pi\left(b_j-a_{d+1}\right)\right)\tilde{\omega}_{2d+1}^{(d+1)}(z)+\sin\left(\pi\left(b_j-a_{d+1}\right)\right)\cos\left(\pi\left(z-b_j\right)\right)\frac{\tilde{\omega}_j^{(d+1)}(z)}{\sin\left(\pi\left(z-a_{d+1}\right)\right)}.$$

In taking the determinant, the extra multiples of $\tilde{\omega}_{2d+1}^{(d+1)}$ do not contribute, and therefore,

$$\begin{aligned}\det\tilde{U}'^{(d+1)} &= (-1)^d\prod_{l=1}^{d}\sin\left(\pi\left(a_l-b_{d+1}\right)\right)\sin\left(\pi\left(a_{d+1}-b_l\right)\right)\\ &\cdot\det\begin{pmatrix}\frac{\cos(\pi(z-a_1))\tilde{\omega}_1^{(d+1)}(z)}{\sin(\pi(z-b_{d+1}))} & \cdots & \frac{\cos(\pi(z-b_d))\tilde{\omega}_{2d}^{(d+1)}(z)}{\sin(\pi(z-a_{d+1}))} & \tilde{\omega}_{2d+1}^{(d+1)}(z)\end{pmatrix}.\end{aligned}$$

None of the columns in the remaining determinant have any dependence upon a_{d+1} or b_{d+1}. In the limit as a_{d+1} approaches b_{d+1}, then, does not have any overt effect on the determinant, but it does ensure that $\sum_{j=1}^{d} a_j = B - a_{d+1} = B - b_{d+1} = \sum_{j=1}^{d} b_j$. Therefore, in this limit,

$$\frac{\cos\left(\pi\left(z-a_j\right)\right)\tilde{\omega}_j^{(d+1)}(z)}{\sin\left(\pi\left(z-b_{d+1}\right)\right)} = e^{2\pi iz}\cos\left(\pi\left(z-a_j\right)\right)\sin\left(\pi\left(z-a_d\right)\right)\tilde{\omega}_j^{(d)}(z)$$

and

$$\frac{\cos\left(\pi\left(z-a_d\right)\right)\tilde{\omega}_d^{(d+1)}(z)}{\sin\left(\pi\left(z-b_{d+1}\right)\right)} = e^{2\pi iz}\cos\left(\pi\left(z-a_d\right)\right)\sin\left(\pi\left(z-a_j\right)\right)\tilde{\omega}_j^{(d)}(z)$$

for each $1 \leq j \leq d-1$. Therefore, subtracting the dth column from the jth column transforms the jth column into

$$e^{2\pi iz}\sin\left(\pi\left(a_j-a_d\right)\right)\tilde{\omega}_j^{(d)}(z).$$

Similarly, subtracting the $2d$th column from the $(d+j)$th column transforms the jth column into

$$e^{2\pi iz}\sin\left(\pi\left(b_j-b_d\right)\right)\tilde{\omega}^{(d)}_{j+d-1}(z),$$

and subtracting the $2d$th column from the dth column changes the dth column to

$$e^{2\pi iz}\sin\left(\pi\left(a_d-b_d\right)\right)\tilde{\omega}^{(d)}_{2d-1}(z).$$

Since the functions $\tilde{\omega}^{(d)}$ have two fewer sinusoidal factors than $\tilde{\omega}^{(d+1)}$, the Fourier coefficients of $e^{2\pi iz}$ and $e^{(4d+2)\pi iz}$ in the expansion of $e^{2\pi iz}\tilde{\omega}^{(d)}$ both vanish. Hence,

$$\begin{aligned}
&\lim_{a_{d+1}\to b_{d+1}}\frac{\det\tilde{U}'^{(d+1)}}{\prod_{l=1}^{d}\sin\left(\pi\left(a_l-b_{d+1}\right)\right)\sin\left(\pi\left(a_{d+1}-b_l\right)\right)}\\
=\;&(-1)^d\sin\left(\pi\left(a_d-b_d\right)\right)\left(\prod_{j=1}^{d-1}\sin\left(\pi\left(a_j-a_d\right)\right)\sin\left(\pi\left(b_j-b_d\right)\right)\right)\\
&\cdot\det\begin{pmatrix} 0 & & 0 & 0 & 0 & & 0 & & \\ \tilde{\omega}^{(d)}_1 & \cdots & \tilde{\omega}^{(d)}_{d-1} & \tilde{\omega}^{(d)}_{2d-1} & \tilde{\omega}^{(d)}_{d} & \cdots & \tilde{\omega}^{(d)}_{2d-2} & \frac{\cos(\pi(z-b_d))\tilde{\omega}^{(d+1)}_{2d}(z)}{\sin(\pi(z-a_{d+1}))} & \tilde{\omega}^{(d+1)}_{2d+1}(z) \\ 0 & & 0 & 0 & 0 & & 0 & & \end{pmatrix}\\
=\;&\sin\left(\pi\left(a_d-b_d\right)\right)\left(\prod_{j=1}^{d-1}\sin\left(\pi\left(a_j-a_d\right)\right)\sin\left(\pi\left(b_j-b_d\right)\right)\right)\\
&\cdot\det\begin{pmatrix} 0 & & \\ \tilde{U}'^{(d)} & \frac{\cos(\pi(z-b_d))\tilde{\omega}^{(d+1)}_{2d}(z)}{\sin(\pi(z-a_{d+1}))} & \tilde{\omega}^{(d+1)}_{2d+1}(z) \\ 0 & & \end{pmatrix}.
\end{aligned}$$

By expressing $\tilde{\omega}^{(d+1)}_{2d}(z)$ and $\tilde{\omega}^{(d+1)}_{2d+1}(z)$ has a product of sines, cosines, and exponentials, it is easy to read off the coefficients of $e^{2\pi iz}$ and $e^{(4d+2)\pi iz}$. The coefficient of $e^{2\pi iz}$ in $\frac{\cos(\pi(z-b_d))\tilde{\omega}^{(d+1)}_{2d}(z)}{\sin(\pi(z-a_{d+1}))}$ is $-\frac{1}{2}\left(2i\right)^{-2d+1}e^{\pi i\sum_{j=1}^{d}a_j+b_j}$, and the coefficient of $e^{(4d+2)\pi iz}$ is $\frac{1}{2}\left(2i\right)^{-2d+1}\cdot e^{-\pi i\sum_{j=1}^{d}a_j+b_j}$. Similarly, the corresponding coefficients in the expansion of $\tilde{\omega}^{(d+1)}_{2d+1}(z)$ are

$(2i)^{-2d}\, e^{\pi i \sum_{j=1}^{d} a_j + b_j}$ and $(2i)^{-2d}\, e^{-\pi i \sum_{j=1}^{d} a_j + b_j}$. Ergo,

$$\begin{aligned}
& \lim_{a_{d+1}\to b_{d+1}} \frac{\det \tilde{U}'^{(d+1)}}{\prod_{l=1}^{d} \sin\left(\pi\left(a_l - b_{d+1}\right)\right) \sin\left(\pi\left(a_{d+1} - b_l\right)\right)} \\
= \;& \sin\left(\pi\left(a_d - b_d\right)\right) \left(\prod_{j=1}^{d-1} \sin\left(\pi\left(a_j - a_d\right)\right) \sin\left(\pi\left(b_j - b_d\right)\right)\right) \\
& \cdot \det \begin{pmatrix} 0 & -\frac{1}{2}\,(2i)^{-2d+1}\, e^{\pi i \sum_{j=1}^{d} a_j + b_j} & (2i)^{-2d}\, e^{\pi i \sum_{j=1}^{d} a_j + b_j} \\ \tilde{U}'^{(d)} & * & * \\ 0 & \frac{1}{2}\,(2i)^{-2d+1}\, e^{-\pi i \sum_{j=1}^{d} a_j + b_j} & (2i)^{-2d}\, e^{-\pi i \sum_{j=1}^{d} a_j + b_j} \end{pmatrix} \\
= \;& \sin\left(\pi\left(a_d - b_d\right)\right) \left(\prod_{j=1}^{d-1} \sin\left(\pi\left(a_j - a_d\right)\right) \sin\left(\pi\left(b_j - b_d\right)\right)\right) \\
& \cdot (2i)^{-4d+1} \det\left(\tilde{U}'^{(d)}\right).
\end{aligned}$$

Therefore, $\frac{A_{d+1}}{A_d}(2i)^2\pi^{4d+1} = (2i)^{-4d+1}$. We have proven the following:

Lemma. $A_2 = -\left(2\pi i\right)^{-7}$ *and for* $d \geq 2$, $A_{d+1} = \left(2\pi i\right)^{-4d-1} A_d$.

It follows that $A_d = -\left(2\pi i\right)^{-2d^2+d-1}$ for all $d \geq 2$. This finishes the computation of the determinant of U. Using this fact in conjunction with (4.3.5) and (4.2.1) yields

Theorem.

$$\begin{aligned}
\det U \;=\; & -\left(2\pi i\right)^{-2d^2+d-1} e^{d^2\pi i\tau} \eta\left(\tau\right)^{-4d^2-2} c^{2d-1} \exp\left(4\eta B^2 + 4\pi i d B\right) \\
& \prod_{1\leq j<k\leq d} \sigma\left(a_j - a_k\right) \sigma\left(b_j - b_k\right) \prod_{1\leq l,m\leq d} \sigma\left(a_l - b_m\right).
\end{aligned}$$

Chapter 5

Partition Function

5.1 The $d = 0$ Case

As stated in chapter 2, N_0 consists of constant maps; $N_0 = \{w(z) \equiv c | c \in \mathbb{P}^1\}$. By ignoring the point at ∞, we restrict ourselves to a dense open subset of N_0 with holomorphic coordinate c. If $w \in N_0$, then $w^*(T\mathbb{P}^1)$ is the trivial bundle. Furthermore, the space $T_w N_0$ is identified with the holomorphic, whence constant, sections of this bundle. Recall a Riemannian metric was placed on N_0 by the following formula, valid in our dense open coordinate chart:

$$\langle v_1, v_2 \rangle = \frac{1}{2} \int_\Sigma \frac{v_1 \bar{v}_2 + \bar{v}_1 v_2}{\left(1 + |w|^2\right)^2} d\mu_g,$$

where v_1, v_2 are now restricted to be elements of $T_w N_0$. We can easily express this metric in terms of the coordinates c and $\bar{c}$:

$$\left[\int_\Sigma \left(1 + |c|^2\right)^{-2} d\mu_g\right] dc d\bar{c} = \frac{Vol(\Sigma)}{\left(1 + |c|^2\right)^2} dc d\bar{c}.$$

The associated volume form, and hence the measure on N_0 is given by

$$\begin{aligned} d\mu_0 &= \frac{i}{2}\frac{Vol(\Sigma)}{\left(1+|c|^2\right)^2} dc \wedge d\bar{c}. \\ &= \frac{i\tau_y}{2}\left(1+|c|^2\right)^{-2} dc \wedge d\bar{c}. \end{aligned}$$

Now we must discuss the regularized determinant of the Laplace operator:

$$-\frac{1}{\det|g|}\left(1+|w|^2\right)^2 \partial_z\left(\frac{\partial_{\bar{z}}}{(1+|w|^2)^2}\right) = -\partial_z\partial_{\bar{z}} = -\frac{1}{4}\left(\partial_x^2+\partial_y^2\right).$$

It is well-known that this operator is diagonalized in the Fourier basis. Let $\Lambda^* = \mathbb{Z}\left(\frac{i}{\tau_y}\right) \oplus \mathbb{Z}\left(\frac{i\tau}{\tau_y}\right)$ be the dual lattice to Λ.

$$\{f_w(z) = \exp\left[\pi i\left(z\bar{w}+w\bar{z}\right)\right] | w \in \Lambda^*\}$$

is an orthogonal basis satisfying $-\partial_z\partial_{\bar{z}} f_w(z) = \pi^2 |w|^2 f_w(z)$. Therefore, the regularized zeta function for this Laplacian is defined by

$$\begin{aligned} \zeta(s) &= \sum_{w\in\Lambda^*\setminus\{0\}} \left(\pi^2|w|^2\right)^{-s} \\ &= \tau_y^s \pi^{-2s} \sum_{(n,m)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{\tau_y^s}{|n+m\tau|^{2s}} \end{aligned}$$

for $\Re(s) > 1$. The sum is an Eisenstein series $E(\tau, s)$, studied extensively by Kronecker [14]. The constant term of the expansion around $s = 1$ of $E(\tau, s)$ is the content of his first limit formula:

$$E(\tau,s) = \frac{\pi}{s-1} - \pi\log\left\{\frac{4\tau_y|\eta(\tau)|^4}{\exp\left(2\Gamma'(1)\right)}\right\} + O(s).$$

$E(\tau, s)$ also satisfies the functional equation:

$$\pi^{-s}\Gamma(s)E(\tau, s) = \pi^{s-1}\Gamma(1-s)E(\tau, 1-s),$$

which, when combined with the limit formula and the expansion of $\Gamma(s)$, yields an expansion of $E(\tau, s)$ around $s = 0$:

$$\pi^{-2s}E(\tau, s) = -1 - \log\left\{4\tau_y\left|\eta(\tau)\right|^4\right\}s + O(s^2).$$

Proofs of these formulas can be found in Chapter 8 of [25] or Chapter 20 of [15]. It follows that

$$\begin{aligned} \zeta'(0) &= -\log\tau_y - \log\left\{4\tau_y\left|\eta(\tau)\right|^4\right\} \\ &= -\log\left\{4\tau_y^2\left|\eta(\tau)\right|^4\right\}. \end{aligned}$$

Therefore, $\det'\Delta = \exp\left\{-2\zeta'(0)\right\} = 16\tau_y^4\left|\eta(\tau)\right|^8$. This computation has been performed many times with slight variations in different contexts: see for example Appendix 1.1 of [22], or [20, 16] for a mathematical perspective; or [19, 13] for an independent physics derivation.

Thus, the degree zero measure is given by:

$$\begin{aligned} &\left(2^{-1}\pi f\right)^{-1}\left(16\tau_y^4\left|\eta(\tau)\right|^8\right)^{-1/2}\tau_y\left(1+|c|^2\right)^{-2}|dc|^2 \\ = &\left(2\pi f\tau_y\left|\eta\left(\tau\right)\right|^4\right)^{-1}\left(1+|c|^2\right)^{-2}|dc|^2. \end{aligned}$$

5.2 The $d \geq 2$ Case

Combining the theorem from 3.2.2 and the corollary from 3.3, we get the following formula for $\det'\Delta_w$, for any $w \in N_d$, in terms of the coordinates on N_d:

$$
\begin{aligned}
\det{}'\Delta_w &= (2e)^{4d}\,|c|^{8d}\left(\prod_{j,k=1}^{d}|\sigma\,(b_k-a_j)|^8\right)\exp\left\{16\Re(\eta B^2)-4\pi\tau_y\left(\frac{4B_y d}{\tau_y}+d^2\right)\right\}\\
&\quad\cdot\exp\left\{-4d\sum_{j=1}^{d}\left[2\Re(\eta b_j^2)+\frac{2\pi}{\tau_y}b_{j_y}^2\right]-\frac{2d}{\tau_y}\int_\Sigma\log\left(\frac{h_w\left(1+|w|^2\right)^2}{h_{2d,2B}}\right)d\mu_g\right\}\\
&\quad\cdot\left(\det Q(h_w)\right)^2
\end{aligned}
$$

Also, the measure $d\mu_d(w)$ can be expressed as

$$
(d!)^{-2}\det Q(h_w)\,|\det U|^2\,|dc|^2\,|da_1|^2\cdots|da_{d-1}|^2\,|db_1|^2\cdots|db_{d-1}|^2\,|dB|^2\,,
$$

where $\det U$ is the result of the calculation in chapter 4:

$$
\begin{aligned}
|\det U|^2 &= (2\pi)^{-4d^2+2d-2}\,e^{-2d^2\pi\tau_y}\,|\eta\,(\tau)|^{-8d^2-4}\,|c|^{4d-2}\exp\left(8\Re\left(\eta B^2\right)-8\pi d B_y\right)\\
&\quad\cdot\prod_{1\le j<k\le d}|\sigma\,(a_j-a_k)\,\sigma\,(b_j-b_k)|^2\prod_{1\le l,m\le d}|\sigma\,(a_l-b_m)|^2\,.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
(\det{}'\Delta_w)^{-\frac{1}{2}}\,d\mu_d(w) &= (d!)^{-2}\,(2e)^{-2d}\,(2\pi)^{-4d^2+2d-2}\,|\eta\,(\tau)|^{-8d^2-4}\\
&\quad\prod_{1\le j<k\le d}|\sigma\,(a_j-a_k)\,\sigma\,(b_j-b_k)|^2\prod_{1\le l,m\le d}|\sigma\,(a_l-b_m)|^{-2} \qquad (5.2.1)\\
&\quad\exp\left\{2d\sum_{j=1}^{d}\left[2\Re(\eta b_j^2)+\frac{2\pi}{\tau_y}b_{j_y}^2\right]+\frac{d}{\tau_y}\int_\Sigma\log\left(\frac{h_w\left(1+|w|^2\right)^2}{h_{2d,2B}}\right)d\mu_g\right\} \qquad (5.2.2)\\
&\quad\left|\frac{dc}{c}\right|^2|da_1|^2\cdots|da_{d-1}|^2\,|db_1|^2\cdots|db_{d-1}|^2\,|dB|^2\,. \qquad (5.2.3)
\end{aligned}
$$

One should check that this is a well-defined measure on N_d; that it is invariant under the lattice actions on $\xi_1, \ldots, \xi_{2d-1}$, and so is independent of the choice of representatives of the zeroes and poles of w. Indeed, the lattice acts by translation on the representatives $a_1, \ldots, a_{d-1}, b_1, \ldots, b_{d-1}, B$, and scales c. Therefore, (5.2.3) is invariant under the lattice actions. It is an easy computation to see that (5.2.1) is also invariant, using the transformation rules of σ and the definition of a_d and b_d. It is more difficult to see the invariance of (5.2.2), but it is easier if we rewrite the integral as

$$\frac{d}{\tau_y} \int_\Sigma \log\left(\frac{h_w\left(1+|w|^2\right)^2}{h_{2d,2B}}\right) d\mu_g = \sum_{j=1}^{d} \frac{-2d}{\tau_y} \int_\Sigma \log\left(h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right) d^2 z.$$

From the definitions of $h_{1,b_j}(z)$ and $s_{1,b_j}(z)$, we see that

$$h_{1,b_j+1}(z)\left|s_{1,b_j+1}(z)\right|^2 = \exp\left\{2\Re\left[\eta\left(2b_j+1\right)\right]\right\} h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2.$$

Thus, under the transformation $b_j \mapsto b_j + 1$,

$$\frac{-2d}{\tau_y} \int_\Sigma \log\left(h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right) d^2 z \mapsto \frac{-2d}{\tau_y} \int_\Sigma \log\left(h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right) d^2 z - 4d\Re\left\{\eta\left(2b_j+1\right)\right\}.$$

But this is equal and opposite to the transformation undergone by $2d\left[2\Re(\eta b_j^2) + \frac{2\pi}{\tau_y} b_{j_y}^2\right]$. A similar computation can be performed for the transformation $b_j \mapsto b_j + \tau$. This proves the invariance of (5.2.2). As a side note, we observe that the first term in (5.2.2) can be brought inside the integral as follows:

$$\exp\left\{\frac{-2d}{\tau_y} \sum_{j=1}^{d} \int_\Sigma \left[\log\left(h_{1,b_j}(z)\left|s_{1,b_j}(z)\right|^2\right) - 2\Re(\eta b_j^2) - \frac{2\pi}{\tau_y} b_{j_y}^2\right] d^2 z\right\}.$$

Comparing this expression with the asymptotics given in Section 3.2.2, we see that the extra terms precisely cancel the z-independent terms of the expansion around $z = b_j$.

The above shows that $(\det{}'\Delta_w)^{-\frac{1}{2}}\, d\mu_d(w)$ is well-defined on our dense open subset of N_d. Furthermore, it extends continuously to all of N_d: the remainder of N_d consists precisely of degree-d meromorphic functions whose zeroes and poles are not all simple. The formula given for $(\det{}'\Delta_w)^{-\frac{1}{2}}\, d\mu_d(w)$ vanishes identically on this set.

5.3 The $d \leq -2$ Case

As mentioned in Chapter 2, in this case, the partition function integral localizes to the space of anti-instantons (anti-holomorphic maps.) Any and every anti-holomorphic map may be obtained through complex conjugation of a holomorphic map with the same zeroes and poles. Therefore, this case completely reduces to the positive d case discussed above, if w is replaced with $\bar{w}$. $\bar{w}^*\left(T\mathbb{P}^1\right)$ is the anti-holomorphic bundle determined by the multiplier conjugate to that of $w^*\left(T\mathbb{P}^1\right)$. Hence, metrics on $\bar{w}^*\left(T\mathbb{P}^1\right)$ and $w^*\left(T\mathbb{P}^1\right)$ obey the same quasi-periodicity relations, and furthermore, the pullback of the Fubini-Study metric by $\bar{w}$ is represented by the same function as the pullback by w. Thus, despite that $\Delta_{\bar{w}} = \partial^*\partial$, it follows that $\det'\Delta_{\bar{w}} = \det'\Delta_w$.

Similarly, upon taking conjugates when computing the measure, the matrix U is replaced by $\bar{U}$, leaving $|\det U|^2$ unchanged. Hence, as would be expected, the measure on N_d is the same as that on N_{-d}. Thus, for $d \leq -2$, we have

$$(\det{}'\Delta_{\bar{w}})^{-\frac{1}{2}}\, d\mu_d(\bar{w}) = (\det{}'\Delta_w)^{-\frac{1}{2}}\, d\mu_{|d|}(w).$$

5.4 Conclusions and Avenues for Future Research

We have successfully computed the approximation to the partition measure for all values of d.

$$Z[\Phi] \;\approx\; \frac{\sum_d \left(2^{-1}\pi f\right)^{-\dim N_d/2} e^{-\frac{\pi}{f}|d|} \int_{N_d} \Phi[w] \left(\det \Delta_w\right)^{-1/2} d\mu_d(w)}{\sum_d \left(2^{-1}\pi f\right)^{-\dim N_d/2} e^{-\frac{\pi}{f}|d|} \int_{N_d} \left(\det \Delta_w\right)^{-1/2} d\mu_d(w)}.$$

However, the only convergent term in the denominator is $d = 0$, which has the value

$$\left(2f\tau_y \left|\eta\left(\tau\right)\right|^8\right)^{-1}.$$

For $|d| \geq 2$, $\int_{N_d} \left(\det{}'\Delta_w\right)^{-\frac{1}{2}} d\mu_d(w)$ diverges due to the behavior of the measure near the boundary of N_d. One cause of divergence occurs when zeroes and poles become arbitrarily close: if $\tilde{a}_l$ and $\tilde{b}_m$ tend toward each other, for some l and m, then by (5.2.1), the integrand grows like $C\left|a_l - b_m\right|^{-2}$, which is not integrable in two dimensions. This is not entirely unexpected, however. Physical theories are rarely expected to hold to all length scales. The procedure to resolve this issue is to introduce an ultraviolet cutoff, for example by treating each zero and pole as having a "hard core"–enforcing that the distance between any zero and pole is greater than some small fixed length ϵ, and then taking the principal part as $\epsilon \to 0$. This is precisely what Fateev et al did in the genus 0 case, where analogous divergences occurred. This was part of their analogy with the classical Coulomb system at a temperature $T = 1$.

There is one other source of divergence, however: the integral over the scaling variable $\int_{\mathbb{C}^*} \left|\frac{dc}{c}\right|^2$. This did not occur in the genus zero case, but an examination of the work in Chapters 3 and 4 leads to a conjecture that similar divergences will occur if Σ is of higher genus G, provided that degree d is large enough. Indeed, if w is a map of degree d, then $\left(\det{}' \Delta_w\right)^{-1/2}$ contributes a factor of $|c|^{-4d}$, however, the measure $d\mu_d(w)$ contributes a factor of $|c|^{2n}$, where $n = h^0\left(\Sigma, w^*\left(T\mathbb{P}^1\right)\right) - 1$. Therefore, by Riemann-Roch, $|c|^{-2}$ will appear in the partition function measure with a power of

$$\deg w^*\left(T\mathbb{P}\right)^1 - h^0\left(\Sigma, w^*\left(T\mathbb{P}^1\right)\right) + 1 = h^0\left(\Sigma, w^*\left(T\mathbb{P}^1\right)^{-1} \otimes K\right) + G,$$

where K, the canonical bundle of Σ, has degree $2G-2$ [18]. Thus, in particular, if $d > G-1$, then $h^0\left(\Sigma, w^*\left(T\mathbb{P}^1\right)^{-1} \otimes K\right) = 0$ independently of the map w, and a factor of $\left|\frac{dc}{c^g}\right|^2$ will appear in the measure.

In the genus zero case, the quantity c had physical significance as the vacuum expectation value (the value at ∞) of the field. However, this variable still had to be integrated out before a comparison with either the classical Coulomb model or the sine-Gordon model. In our case, c offers no physical interpretation. Therefore, one possible solution to this problem is to define physically relevant observables as those for which the factors of c can be decoupled from the other variables and formally cancelled out of the equation. Future work is required to see if such a procedure can be carried out. Alternatively, it may be possible to introduce a cutoff to the domain of $|c|$ in some way, such as demanding $\frac{1}{R} \leq |c| \leq R$, and then finding a coherent way to take the remove the cutoff as $R \to \infty$. Some consistent method must be found if any analogy is to be drawn between this model and either the classical Coulomb model or the sine-Gordon model.

The methods in this thesis may be able to be generalized to $\mathbb{CP}^1$-instantons on higher genus surfaces, but only in the case where the degree of the instanton is large. Eells proved that, in the case of large degree, every harmonic map is holomorphic, and every cohomology class has a holomorphic representative [7, 6]. In general, it is a much more difficult question to describe low-degree harmonic and holomorphic maps on higher-genus Riemann surfaces, and different methods would be required to pursue this topic in that setting.

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Appendix A

Analytic Continuation of the Zeta Function

Suppose Δ_h is the $\bar{\partial}$-Laplacian on a complex line bundle $\pi : L \to \Sigma$ with hermitian metric h over a compact Riemann surface Σ. If $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ are the eigenvalues of Δ_h. We define $\zeta(s) := \sum_{j=1}^{\infty} \lambda_j^{-s}$. We continue $\zeta(s)$ by examining $\Gamma(s)\zeta(s)$. Since $\zeta(s)$ converges absolutely for $\Re(s) > 1$ [22], we have:

$$\begin{aligned}
\Gamma(s)\zeta(s) &= \left(\int_0^{\infty} e^{-t} t^s \frac{dt}{t}\right) \sum_{j=1}^{\infty} \lambda_j^{-s} \\
&= \sum_{j=1}^{\infty} \int_0^{\infty} e^{-t} \left(\frac{t}{\lambda_j}\right)^s \frac{dt}{t} \\
&= \sum_{j=1}^{\infty} \int_0^{\infty} e^{-\lambda_j t} t^s \frac{dt}{t} \\
&= \int_0^{\infty} \sum_{j=1}^{\infty} e^{-\lambda_j t} t^s \frac{dt}{t}.
\end{aligned}$$

For finite t, $2\sum_{j=0}^{\infty} e^{-\lambda_j t}$ is the trace of the heat operator $e^{-t\Delta_h}$. Therefore, if p_0 is the dimension of the kernel of Δ_h, then we obtain

$$2\Gamma(s)\zeta(s) = \int_0^\infty \left(\mathrm{Tr}\left\{e^{-t\Delta_h}\right\} - p_0\right) t^s \frac{dt}{t}.$$

This is not defined in a neighborhood of 0, so we expand the trace of the heat kernel. For small t,

$$\mathrm{Tr}\left\{e^{-t\Delta_h}\right\} = \frac{\alpha_{-1}}{t} + \alpha_0 + O(t),$$

where α_j are the Seeley coefficients for this Laplacian. Also, for large t, $\mathrm{Tr}(e^{-t\Delta_h}) - p_0$ decays exponentially, and so

$$\varphi(s) := \int_1^\infty \left(\mathrm{Tr}\left\{e^{-t\Delta_h}\right\} - p_0\right) t^s \frac{dt}{t}$$

is an entire function in s. Therefore, for $\Re(s) > 1$,

$$\begin{aligned}
2\Gamma(s)\zeta(s) : &= \int_0^1 \left(\mathrm{Tr}(e^{-t\Delta_h}) - p_0\right) t^s \frac{dt}{t} + \varphi(s) \\
&= \int_0^1 \left(\mathrm{Tr}(e^{-t\Delta_h}) - \alpha_0 - \frac{\alpha_{-1}}{t}\right) t^s \frac{dt}{t} + \frac{\alpha_{-1}}{s-1} + \frac{\alpha_0 - p_0}{s} + \varphi(s) \\
&= \int_0^1 O(t^s) dt + \frac{\alpha_{-1}}{s-1} + \frac{\alpha_0 - p_0}{s} + \varphi(s).
\end{aligned}$$

The remaining integral converges for $\Re(s) > -1$, and so we have meromorphically continued $2\Gamma(s)\zeta(s)$ to a neighborhood of 0. So

$$2\zeta(s) = \frac{\alpha_0 - p_0}{s\Gamma(s)} + \Phi(s)$$

where Φ is analytic in a neighborhood of 0, and 0 at 0. $2\zeta(s)$ is thus analytic at 0 as $s\Gamma(s)$ analytically continues to be 1 at $s = 0$, and we can define

$$\det \Delta_h = e^{-2\zeta'(0)}.$$

Appendix B

Asymptotics of Green's Functions

In this section, let Σ be a genus n Riemann surface with Riemannian metric g, and let $\pi : L \to \Sigma$ be a holomorphic line bundle with hermitian metric h. Also assume the degree of L is at least $2n-1$. We will be working in a local coordinate system such that the metric g has the form:

$$\begin{aligned} g_{zz} &= g_{\bar{z}\bar{z}} = 0, \\ g_{z\bar{z}} &= g_{\bar{z}z}. \end{aligned}$$

We also abuse notation, using g to denote the function $2g_{z\bar{z}}$. Let $\Delta_h = \bar{\partial}^*\bar{\partial} = -g^{-1}h^{-1}\partial_z h \partial_{\bar{z}}$ be the $\bar{\partial}$-laplacian on the space of sections L. We also define $\tilde{\Delta}_h := \bar{\partial}\bar{\partial}^* = -\partial_{\bar{z}}\left(g^{-1}h^{-1}\partial_z h\right)$, acting on E-valued $(0,1)$-forms. We will compute the large-t and small-t asymptotics for $e^{-t\Delta_h}$ and $e^{-t\tilde{\Delta}_h}$.

B.1 Seeley Coefficients of $\exp(-t\Delta_h)$

Let G be the Green's function for Δ_h. $G(z,z';t)$ satisfies

$$\begin{aligned} \frac{\partial G}{\partial t} &= -\Delta_h^z G; \\ G(z,z';0) &= \delta(z-z'). \end{aligned}$$

Seeley tells us that G has an asymptotic expansion in t near $t=0$ [22]:

$$\begin{aligned} G(z,z';t) &= \langle z'|\exp(-t\Delta_h)|z\rangle \\ &= \exp\left(\frac{-S(z,z')}{t}\right)\left(A_{-1}(z,z')t^{-1} + A_0(z,z') + A_1(z,z')t + O(t^2)\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial G}{\partial t} &= \frac{S}{t^2}\exp\left(\frac{-S}{t}\right)\left(A_{-1}t^{-1} + A_0 + A_1 t + O(t^2)\right) \\ &\quad + \exp\left(\frac{-S}{t}\right)\left(-A_{-1}t^{-2} + A_1 + O(t)\right) \\ &= \exp\left(\frac{-S}{t}\right)\left(SA_{-1}t^{-3} + \left[SA_0(x,y) - A_{-1}\right]t^{-2} + SA_1 t^{-1} + O(1)\right); \\ -\Delta_h G &= -\exp\left(\frac{-S}{t}\right)\left(\Delta_h A_{-1}t^{-1} + \Delta_h A_0 + \Delta_h A_1 t + O(t^2)\right) \\ &\quad + \exp\left(\frac{-S}{t}\right)\left(\Delta_h S A_{-1}t^{-2} + \Delta_h S A_0 t^{-1} + \Delta_h S A_1 + O(t)\right) \\ &\quad + g^{-1}\exp\left(\frac{-S}{t}\right)\left(\partial_z S\partial_{\bar z} S A_{-1}t^{-3} + \partial_z S\partial_{\bar z} S A_0 t^{-2} + \partial_z S\partial_{\bar z} S A_1 t^{-1} + O(1)\right) \\ &\quad - g^{-1}\exp\left(\frac{-S}{t}\right)\left(\partial_z S\partial_{\bar z} A_{-1}t^{-2} + \partial_z S\partial_{\bar z} A_0 t^{-1} + \partial_z S\partial_{\bar z} A_1 + O(t)\right) \\ &\quad - g^{-1}\exp\left(\frac{-S}{t}\right)\left(\partial_{\bar z} S\partial_z A_{-1}t^{-2} + \partial_{\bar z} S\partial_z A_0 t^{-1} + \partial_{\bar z} S\partial_z A_1 + O(t)\right). \end{aligned}$$

Comparing the coefficients for t^{-3}, we see that we must have

$$S = g^{-1}\partial_{\bar{z}} S \partial_z S.$$

A solution to this is given by $S(z, z') = d(z, z')^2$, where $d(z, z')$ is the distance between z and z' in the metric g. Thus, expanding around z', we have

$$S(z, z') = g(z')\, |z - z'|^2 + O\left(|z - z'|^3\right)$$

This is most easily seen in polar coordinates around z'. Plugging this into G we examine the initial condition to find the value of A_{-1} on the diagonal. We want

$$\lim_{t \to 0^+} \frac{1}{t} \int_\Sigma A_{-1}(z, z') \exp\left(-\frac{d(z, z')^2}{t}\right) d\mu_g(z) = 1.$$

But the integrand has a single critical point (and a maximum) at $z = z'$, so we can now evaluate the integral via steepest descent. The determinant of the Hessian of S at z' is easily seen to be $(2g(z'))^2$. Therefore, we must compute the Gaussian integral

$$\int_{\mathbb{R}^2} \exp\left(-\frac{g(z')(z - z')^2}{t}\right) d^2 z = \frac{t}{g(z')} \int_{\mathbb{R}^2} \exp\left(-u^2\right) d^2 u = \frac{t\pi}{g(z')}$$

Hence, we get

$$\lim_{t \to 0^+} \frac{A_{-1}(z', z') g(z')}{t} \frac{\pi t}{g(z')} = 1.$$

So, $A_{-1}(z', z') = \dfrac{1}{\pi}$. Note that this differs by a factor of 4 from the result in [9]. This is due to a reparameterization of t.

Now, we compare t^{-2} terms to get

$$SA_0 - A_{-1} = g^{-1}\partial_z S \partial_{\bar{z}} S A_0 + \Delta_h S A_{-1} - g^{-1}\partial_z S \partial_{\bar{z}} A_{-1} - g^{-1}\partial_{\bar{z}} S \partial_z A_{-1}.$$

The equation for S cancels the A_0 terms in this expression, leaving us with an equation that we can solve for higher coefficients in the Taylor expansion of A_1 around the diagonal, we will denote $z - z'$ simply by Z so the formulas appear less cluttered.

$$-A_{-1} = \Delta_h S A_{-1} - g^{-1}\partial_z S \partial_{\bar{z}} A_{-1} - g^{-1}\partial_{\bar{z}} S \partial_z A_{-1}.$$

Let us solve for A_{-1} up to the quadratic level:

$$A_{-1}(z, z') = \frac{1}{\pi} + a_{-1,1}(Z) + a_{-1,2}(Z, Z) + O\left(|Z|^3\right).$$

To do this, we will need more terms in the approximation for S above.

$$g^{-1}(z) = g^{-1}(z')\left(1 - (\log g)' \cdot (Z) - \frac{1}{2}(\log g)''(Z, Z) + \frac{1}{2}\left((\log g)' \cdot (Z)\right)^2 + O(|Z|^3)\right)$$

$$\begin{aligned}
S &= g(z')\,|Z|^2 + S_3(Z, Z, Z) + S_4(Z, Z, Z, Z) + O\left(|Z|^5\right). \\
S_3(Z, Z, Z) &= \frac{1}{2} g(z')\,(\log g)' \cdot (Z)\,|Z|^2 \\
S_4(Z, Z, Z, Z) &= g(z')\left(\frac{1}{6}(\log g)''(Z, Z) - \frac{1}{48}\left|(\log g)'\right|^2 |Z|^2 + \frac{1}{6}\left((\log g)' \cdot (Z)\right)^2\right)|Z|^2
\end{aligned}$$

From this, we can easily calculate the first few derivatives of S:

$$g^{-1}(z)\partial_z \partial_{\bar{z}} S = 1 + \frac{1}{12}\left(\partial_\mu \partial_\mu \log g\right)(z')\,|Z|^2 + O\left(|Z|^3\right)$$

$$\begin{aligned}
g^{-1}(z)\partial_z \log h(z) \partial_{\bar{z}} S &= \left(\partial_z \log h\right) Z + \left((\partial_z \log h)' \cdot Z\right) Z \\
&\quad - \left(\partial_z \log h\right)\left((\log g)' \cdot Z\right) Z + \frac{1}{2}\left(\partial_z \log h\right)\left(\partial_{\bar{z}} \log g\right)|Z|^2
\end{aligned}$$

$$+\frac{1}{2}\left(\partial_z \log h\right)\left(\left(\log g\right)' \cdot Z\right) Z + O\left(\left|Z\right|^3\right)$$

Now we can find $a_{-1,1}$ and $a_{-1,2}$ by expanding the equation for A_{-1} in powers of Z :

$$\begin{aligned}
A_{-1} &= -\Delta_h S A_{-1} + g^{-1}\partial_z S \partial_{\bar{z}} A_{-1} + g^{-1}\partial_{\bar{z}} S \partial_z A_{-1} \\
a_{-1,1}(Z) &= a_{-1,1}(Z) + \frac{1}{\pi}\left(\partial_z \log h\right) Z + \bar{Z}\partial_{\bar{z}}\left(a_{-1,1}(Z)\right) + Z\partial_z\left(a_{-1,1}(Z)\right) \\
a_{-1,1}(Z) &= -\frac{1}{\pi}\left(\partial_z \log h\right)(z')Z \\
a_{-1,2}(Z,Z) &= a_{-1,2}(Z,Z) + \frac{1}{12}\frac{1}{\pi}\left(\partial_\mu\partial_\mu \log g\right)\left|Z\right|^2 + Z\left(\partial_z \log h\right) a_{-1,1}(Z) \\
&\quad + \frac{1}{\pi}\left(\left(\partial_z \log h\right)' \cdot Z\right) Z - \frac{1}{\pi}\partial_z \log h\left(\left(\log g\right)' \cdot Z\right) Z \\
&\quad + \frac{1}{\pi}\frac{1}{2}\partial_z \log h\left(\partial_{\bar{z}} \log g\right)\left|Z\right|^2 + \frac{1}{\pi}\frac{1}{2}\left(\partial_z \log h\right)\left(\left(\log g\right)' \cdot Z\right) Z \\
&\quad + \sum_{j=1,2}\left(\bar{Z}\partial_{\bar{z},j} a_{-1,2} + Z\partial_{z,j} a_{-1,2}\right) \\
&\quad - \left(\left(\log g\right)' \cdot Z\right)\left(\bar{Z}\partial_{\bar{z}} a_{-1,1} + Z\partial_z a_{-1,1}\right) \\
&\quad + \frac{1}{2}\bar{Z}\left(\left(\log g\right)' \cdot Z\right)\partial_{\bar{z}} a_{-1,1} + \frac{1}{2}Z\left(\left(\log g\right)' \cdot Z\right)\partial_z a_{-1,1} \\
&\quad + \frac{1}{2}\left(\partial_z \log g\right)\left|Z\right|^2 \partial_{\bar{z}} a_{-1,1} + \frac{1}{2}\left(\partial_{\bar{z}} \log g\right)\left|Z\right|^2 \partial_z a_{-1,1} \\
-2a_{-1,2}(Z,Z) &= \frac{1}{12}\frac{1}{\pi}\left(\partial_\mu\partial_\mu \log g\right)\left|Z\right|^2 - \frac{1}{\pi}\left(\partial_z \log h\right)^2 Z^2 + \frac{1}{\pi}\left(\left(\partial_z \log h\right)' \cdot Z\right) Z
\end{aligned}$$

$$a_{-1,2}(Z,Z) = -\frac{1}{24}\frac{1}{\pi}\left(\partial_\mu\partial_\mu \log g\right)\left|Z\right|^2 + \frac{1}{2\pi}\left(\partial_z \log h\right)^2 Z^2 - \frac{1}{2\pi}Z(\partial_z \log h)' \cdot Z$$

where $\partial_\mu\partial_\mu$ is shorthand for $\partial_x^2 + \partial_y^2 = 4\partial_z\partial_{\bar{z}}$. The t^{-1} term in the expansion of the Green's function and heat equation gives:

$$\left(-\Delta_h S\right) A_0 = -\Delta_h A_{-1} - g^{-1}(z)\left(\partial_z S \partial_{\bar{z}} A_0 + \partial_{\bar{z}} S \partial_z A_0\right)$$

Expanding to lowest order in $|Z|$ gives

$$\begin{aligned} A_0 &= a_{0,0} + O(|Z|) \\ (-\Delta_h S)\, A_0 &= a_{0,0} + O(|Z|) \\ -\Delta_h A_{-1} &= -\frac{g^{-1}}{24\pi}\partial_\mu\partial_\mu \log g - \frac{g^{-1}}{8\pi}\partial_\mu\partial_\mu \log h + O(|Z|) \\ A_0(z', z') &= -\frac{g^{-1}}{24\pi}\partial_\mu\partial_\mu \log g - \frac{g^{-1}}{8\pi}\partial_\mu\partial_\mu \log h \end{aligned}$$

Thus, the Seeley expansion for small t is

$$\langle z'|\exp(-t\Delta_h)|z'\rangle = \frac{2}{\pi}t^{-1} - \frac{g^{-1}(z')}{12\pi}\partial_\mu\partial_\mu \log g(z') - \frac{g^{-1}(z')}{4\pi}\partial_\mu\partial_\mu \log h(z') + O(t). \quad \text{(B.1.1)}$$

We multiplied the result by 2 because we wanted to take the real trace of the heat kernel. Additionally, we note that $-\frac{1}{2g(z)}\partial_\mu\partial_\mu \log g(z) = K(z)$, where K is the Gaussian curvature of Σ. Furthermore,

$$\begin{aligned} \left(g^{-1}\partial_\mu\partial_\mu \log h\right) d\mu_g &= \left(\partial_\mu\partial_\mu \log h\right) \frac{dz \wedge d\bar{z}}{-2i} \\ &= 2i\partial\bar{\partial} \log h, \end{aligned}$$

which is $2i$ times the curvature of L. Thus, for small t,

$$\begin{aligned} \mathrm{Tr}\left(e^{-t\Delta_h}\right) &= \int_\Sigma \langle z\left|\exp\left(-t\Delta_h\right)\right| z\rangle\, d\mu_g(z) \\ &= \frac{2}{\pi} Area\left(\Sigma\right) t^{-1} + \frac{1}{6\pi}\int_\Sigma K d\mu_g - \frac{i}{4\pi}\int_\Sigma \partial\bar{\partial}\log h + O(t) \\ &= \frac{2}{\pi} Area\left(\Sigma\right) t^{-1} + \frac{1}{3}\chi\left(\Sigma\right) - \frac{1}{2}\deg(L) + O(t). \end{aligned}$$

Here, we have used the Gauss-Bonnet theorem to compute the first integral. The second used the fact that $\frac{i}{2\pi}\partial\bar{\partial}\log h$ is a representation of the Chern class of the bundle.

B.2 Seeley Coefficients of $\exp\left(-t\tilde{\Delta}_h\right)$

Now we must do the same for $\exp(-t\tilde{\Delta}_h)$.

$$\tilde{\Delta}_h = -g^{-1}\partial_z\partial_{\bar{z}} - g^{-1}\left(\partial_z \log h\right)\partial_{\bar{z}} - \left(\partial_{\bar{z}} g^{-1}\right)\partial_z - \left(\partial_{\bar{z}}\left(g^{-1}\partial_z \log h\right)\right)$$

We will denote the zeroth order term by C. Let $\tilde{G}$ be the Green's function for $\tilde{\Delta}_h$. $\tilde{G}(z, z'; t)$ satisfies

$$\begin{aligned}\frac{\partial \tilde{G}}{\partial t} &= -\tilde{\Delta}_h \tilde{G}; \\ \tilde{G}(z, z'; 0) &= \delta(z - z').\end{aligned}$$

For small t, the Seeley expansion of $\tilde{G}$ is given by:

$$\begin{aligned}\tilde{G}(z, z'; t) &= \langle z'|\exp(-t\tilde{\Delta}_h)|z\rangle \\ &= \exp\left(\frac{-\tilde{S}(z, z')}{t}\right)\left(\tilde{A}_{-1}(z, z')t^{-1} + \tilde{A}_0(z, z') + \tilde{A}_1(z, z')t + O(t^2)\right).\end{aligned}$$

Therefore, as before:

$$\begin{aligned}\frac{\partial \tilde{G}}{\partial t} &= \frac{\tilde{S}}{t^2}\exp\left(\frac{-\tilde{S}}{t}\right)\left(\tilde{A}_{-1}t^{-1} + \tilde{A}_0 + \tilde{A}_1 t + O(t^2)\right) \\ &\quad + \exp\left(\frac{-\tilde{S}}{t}\right)\left(-\tilde{A}_{-1}t^{-2} + \tilde{A}_1 + O(t)\right) \\ &= \exp\left(\frac{-\tilde{S}}{t}\right)\left(\tilde{S}\tilde{A}_{-1}t^{-3} + \left[\tilde{S}\tilde{A}_0(x, y) - \tilde{A}_{-1}\right]t^{-2} + \tilde{S}\tilde{A}_1 t^{-1} + O(1)\right); \\ -\tilde{\Delta}_h\tilde{G} &= -\exp\left(\frac{-\tilde{S}}{t}\right)\left(\tilde{\Delta}_h\tilde{A}_{-1}t^{-1} + \tilde{\Delta}_h\tilde{A}_0 + \tilde{\Delta}_h\tilde{A}_1 + O(t^2)\right) \\ &\quad + \exp\left(\frac{-\tilde{S}}{t}\right)\left(\left(\tilde{\Delta}_h - C\right)\tilde{S}\tilde{A}_{-1}t^{-2} + \left(\tilde{\Delta}_h - C\right)\tilde{S}\tilde{A}_0 t^{-1} + \left(\tilde{\Delta}_h - C\right)\tilde{S}\tilde{A}_1 + O(t)\right)\end{aligned}$$

$$+g^{-1}\exp\left(\frac{-\tilde{S}}{t}\right)\left(\partial_z\tilde{S}\partial_{\bar{z}}\tilde{S}\tilde{A}_{-1}t^{-3}+\partial_z\tilde{S}\partial_{\bar{z}}\tilde{S}\tilde{A}_0t^{-2}+\partial_z\tilde{S}\partial_{\bar{z}}\tilde{S}\tilde{A}_1t^{-1}+O(1)\right)$$
$$-g^{-1}\exp\left(\frac{-\tilde{S}}{t}\right)\left(\partial_z\tilde{S}\partial_{\bar{z}}\tilde{A}_{-1}t^{-2}+\partial_z\tilde{S}\partial_{\bar{z}}\tilde{A}_0t^{-1}+\partial_z\tilde{S}\partial_{\bar{z}}\tilde{A}_1+O(t)\right)$$
$$-g^{-1}\exp\left(\frac{-\tilde{S}}{t}\right)\left(\partial_{\bar{z}}\tilde{S}\partial_z\tilde{A}_{-1}t^{-2}+\partial_{\bar{z}}\tilde{S}\partial_z\tilde{A}_0t^{-1}+\partial_{\bar{z}}\tilde{S}\partial_z\tilde{A}_1+O(t)\right).$$

The t^{-3} term and initial condition have not changed. Therefore, $\tilde{S}=S$ and

$$\tilde{A}_{-1}=\frac{1}{\pi}+\tilde{a}_{-1,1}(Z)+\tilde{a}_{-1,2}\left(Z,Z\right)+O\left(|Z|^3\right).$$

To find $\tilde{a}_{-1,1}$ and $\tilde{a}_{-1,2}$, we look at the t^{-2} term:

$$\begin{aligned}
\tilde{A}_{-1} &= -\left(\tilde{\Delta}_h-C\right)S\tilde{A}_{-1}+g^{-1}\partial_zS\partial_{\bar{z}}\tilde{A}_{-1}+g^{-1}\partial_{\bar{z}}S\partial_z\tilde{A}_{-1}.\\
\tilde{a}_{-1,1}(Z) &= \tilde{a}_{-1,1}(Z)+\frac{1}{\pi}Z\partial_z\log h-\frac{1}{\pi}\bar{Z}\left(\partial_{\bar{z}}\log g\right)\\
&\quad+\bar{Z}\partial_{\bar{z}}\tilde{a}_{-1,1}+Z\partial_z\tilde{a}_{-1,1};\\
\tilde{a}_{-1,1}(Z) &= -\frac{1}{\pi}Z\partial_z\log h+\frac{1}{\pi}\bar{Z}\partial_{\bar{z}}\log g.\\
\tilde{a}_{-1,2}(Z,Z) &= \tilde{a}_{-1,2}(Z,Z)+\frac{1}{12}\frac{1}{\pi}\left(\partial_\mu\partial_\mu\log g\right)|Z|^2+Z\partial_z\log h\tilde{a}_{-1,1}(Z)\\
&\quad+\frac{1}{\pi}Z\left(\partial_z\log h\right)'\cdot Z-\frac{1}{\pi}Z\partial_z\log h\left(\log g\right)'\cdot Z\\
&\quad+\frac{1}{\pi}\frac{1}{2}\partial_z\log h\left(\partial_{\bar{z}}\log g\right)|Z|^2+\frac{1}{\pi}\frac{1}{2}Z\partial_z\log h(\log g)'\cdot Z\\
&\quad+\sum_{j=1,2}\left(\bar{Z}\partial_{\bar{z},j}\tilde{a}_{-1,2}+Z\partial_{z,j}\tilde{a}_{-1,2}\right)\\
&\quad-\left((\log g)'\cdot Z\right)\left(\bar{Z}\partial_{\bar{z}}\tilde{a}_{-1,1}+Z\partial_z\tilde{a}_{-1,1}\right)\\
&\quad+\frac{1}{2}\bar{Z}\left((\log g)'\cdot Z\right)\partial_{\bar{z}}\tilde{a}_{-1,1}+\frac{1}{2}Z\left((\log g)'\cdot Z\right)\partial_z\tilde{a}_{-1,1}\\
&\quad+\frac{1}{2}\left(\partial_z\log g\right)|Z|^2\partial_{\bar{z}}\tilde{a}_{-1,1}+\frac{1}{2}\left(\partial_{\bar{z}}\log g\right)|Z|^2\partial_z\tilde{a}_{-1,1}\\
&\quad-\frac{1}{\pi}\bar{Z}\left((\partial_{\bar{z}}\log g)'\cdot Z\right)+\frac{1}{\pi}\bar{Z}\left((\log g)'\cdot Z\right)\partial_{\bar{z}}\log g
\end{aligned}$$

$$
\begin{aligned}
&\quad -\frac{1}{\pi}\frac{1}{2}\left(\partial_{\bar z}\log g\right)\left(|Z|^2\left(\partial_z\log g\right)+\bar Z\left(\partial_z\log g\cdot Z\right)\right)-\bar Z\partial_{\bar z}\log g\tilde a_{-1,1}(Z);\\
-2\tilde a_{-1,2}\left(Z,Z\right) &= \frac{1}{12}\frac{1}{\pi}\left(\partial_\mu\partial_\mu\log g\right)|Z|^2-\frac{1}{\pi}\left(\partial_z\log h\right)^2Z^2\\
&\quad +\frac{1}{\pi}\left(\left(\partial_z\log h\right)'\cdot Z\right)Z+\frac{2}{\pi}\partial_z\log h\partial_{\bar z}\log g\,|Z|^2\\
&\quad -\frac{1}{\pi}\bar Z\left(\left(\partial_{\bar z}\log g\right)'\cdot Z\right)-\frac{1}{\pi}\left(\partial_{\bar z}\log g\right)^2\bar Z^2;
\end{aligned}
$$

$$
\begin{aligned}
\tilde\alpha_2(Z,Z) &= -\frac{1}{24}\frac{1}{\pi}\left(\partial_\mu\partial_\mu\log g\right)|Z|^2+\frac{1}{2\pi}\left(\partial_z\log h\right)^2Z^2-\frac{1}{2\pi}Z(\partial_z\log h)'\cdot Z\\
&\quad +\frac{1}{2\pi}\bar Z\left(\left(\partial_{\bar z}\log g\right)'\cdot Z\right)-\frac{1}{\pi}\partial_z\log h\partial_{\bar z}\log g\,|Z|^2+\frac{1}{2\pi}\left(\partial_{\bar z}\log g\right)^2\bar Z^2.
\end{aligned}
$$

The t^{-1} term in the expansion of the Green's function and heat equation gives:

$$
\left(-\tilde\Delta_h S\right)\tilde A_0=-\tilde\Delta_h\tilde A_{-1}-g^{-1}(z)\left(\partial_z S\partial_{\bar z}\tilde A_0+\partial_{\bar z}S\partial_z\tilde A_0\right).
$$

Expanding to lowest order in $(z-z')$ gives

$$
\begin{aligned}
\tilde A_0 &= \tilde a_{0,0}+O(|Z|);\\
\left(-\tilde\Delta_h S\right)\tilde A_0 &= \tilde a_{0,0}+O(|Z|);\\
-\tilde\Delta_h\tilde A_1 &= \frac{g^{-1}}{12\pi}\partial_\mu\partial_\mu\log g-\frac{g^{-1}}{8\pi}\partial_\mu\partial_\mu\log h-\frac{g^{-1}}{\pi}\partial_z\log h\partial_{\bar z}\log g\\
&\quad +\frac{g^{-1}}{\pi}\partial_{\bar z}\log g\partial_z\log h+\frac{g^{-1}}{\pi}\partial_z\log h\partial_{\bar z}\log g\\
&\quad +\frac{1}{\pi}\left(g^{-1}\partial_{\bar z}\partial_z\log h-g^{-1}\partial_{\bar z}\log g\partial_z\log h\right)+O(|Z|);\\
\tilde A_0(z',z') &= \frac{g^{-1}}{12\pi}\partial_\mu\partial_\mu\log g(z')+\frac{g^{-1}}{8\pi}\partial_\mu\partial_\mu\log h.
\end{aligned}
$$

Thus, the Seeley expansion for small t is

$$
\langle z'|\exp(-t\tilde\Delta_h)|z'\rangle=\frac{1}{\pi}t^{-1}+\frac{g^{-1}}{12\pi}\partial_\mu\partial_\mu\log g(z')+\frac{g^{-1}}{8\pi}\partial_\mu\partial_\mu\log h(z')+O(t).
$$

Again, we multiply by two to get the real trace:

$$\langle z'|\exp(-t\tilde{\Delta}_h)|z'\rangle = \frac{2}{\pi t} + \frac{g^{-1}(z')}{6\pi}\partial_\mu\partial_\mu \log g(z') + \frac{g^{-1}(z')}{4\pi}\partial_\mu\partial_\mu \log h(z') + O(t). \qquad \text{(B.2.1)}$$

Thus, for small t,

$$\begin{aligned}
\mathrm{Tr}\left(e^{-t\tilde{\Delta}_h}\right) &= \int_\Sigma \langle z\left|\exp\left(-t\Delta_h\right)\right| z\rangle \, d\mu_g(z) \\
&= \frac{2}{\pi} Area\left(\Sigma\right) t^{-1} + \frac{1}{3\pi}\int_\Sigma K d\mu_g + \frac{i}{4\pi}\int_\Sigma \partial\bar{\partial}\log h + O(t) \\
&= \frac{2}{\pi} Area\left(\Sigma\right) t^{-1} + \frac{2}{3}\chi\left(\Sigma\right) + \frac{1}{2}\deg(L) + O(t).
\end{aligned}$$

B.3 Expansion at large t

It is obvious that when t gets large, $\exp\left(-t\Delta_h\right)$ approaches P, the orthogonal projection operator onto the subspace of holomorphic sections of L. Our claim is that under our assumption about the degree of L, $\exp\left(-t\tilde{\Delta}_h\right)$ approaches 0; i.e. $\tilde{\Delta}_h$ has no kernel. Indeed, by basic Hodge theory and Serre duality, the space of harmonic L-valued $(0,1)$-forms is isomorphic to $H^1\left(\Sigma, L\right) \cong H^0\left(\Sigma, K\otimes L^*\right)^*$. Since K has degree $2n-2$, where n is the genus of Σ, $H^0\left(\Sigma, K\otimes L^*\right)$ is guaranteed to be trivial if L has degree at least $2n-1$.