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Symplectic Geometry of Rationally Connected Threefolds

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Abstract of the Dissertation

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We study the symplectic geometry of rationally connected 3-folds. The first result shows that rational connectedness is a symplectic deformation invariant in dimension 3. If a rationally connected 3-fold X is Fano or has Picard number 2, we prove that there is a non-zero Gromov-Witten invariant with two insertions being the class of a point. Finally we prove that many other rationally connected 3-folds have birational models admitting a non-zero Gromov-Witten invariant with two point insertions.

Contents

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1 Introduction

In this thesis we study the symplectic geometry of smooth projective rationally connected 3-folds over the complex numbers. First recall the following definitions.

Definition 1.1. A variety X is called *rationally connected* if two general points in X can be connected by a rational curve.

A related notion is uniruledness.

Definition 1.2. A variety X is called *uniruled* if there exists a rational curve through a general point.

The motivation of this thesis is the following theorem, proved independently by Kollár and Ruan.

Theorem 1.3 ([Kol98], [Rua99])**.** *Let X be a smooth projective uniruled variety. Then there is a non-zero Gromov-Witten invariant of the form $\langle [pt], \ldots \rangle_{0,\beta}^X$.*

If X and X' are two smooth projective varieties, then they can also be considered as symplectic manifolds with symplectic form ω and ω' given by the polarizations. We say that X and X' are *symplectic deformation equivalent* if there is a family of symplectic manifolds (X_t, ω_t) diffeomorphic to each other such that (X_0, ω_0) (resp. (X_1, ω_1)) is isomorphic to (X, ω) (resp. (X', ω')) as a symplectic manifold. Since Gromov-Witten invariants are symplectic deformation invariants, Kollár and Ruan's result has the following immediate corollary.

Corollary 1.4 ([Kol98], [Rua99])**.** *Let X and X' be two smooth projective varieties which are symplectic deformation equivalent. Then X is uniruled if and only if X' is.*

The proof of the theorem and its corollary will be given at the end of this chapter. We will now discuss two generalizations of these results to the case of rationally connected varieties.

1.1 Symplectic topology of rationally connected varieties

In dimensions 1 and 2, rational connectedness is a topological property in the sense that a smooth projective variety of dimension at most 2 is rationally connected if and only if it is *diffeomorphic* to a rationally connected variety. However this fails in higher dimensions. For more details and many other interesting problems about rationally connected varieties, see [Kol98].

Motivated by Corollary 1.4, Kollár conjectured the following.

Conjecture 1.5 (Kollár, [Kol98])**.** Let X and X' be two smooth projective varieties which are symplectic deformation equivalent. Then X is rationally connected if and only if X' is.

The first evidence of this conjecture in higher dimensions is the following theorem of Voisin [Voi08].

Theorem 1.6 ([Voi08])**.** *Let X and X' be two smooth projective 3-folds which are symplectic deformation equivalent. If X is Fano or rationally connected with Picard number 2, then X' is also rationally connected.*

The idea of the proof in [Voi08] is the following. It suffices to show that the *maximal rationally connected quotient (MRC-quotient)* of X' is a point. By the result of Kollár and Ruan (Theorem 1.3), the MRC quotient is either a surface, a curve or a point. For topological reasons, it cannot be a curve. If it is a surface, then X is birational to a conic bundle over a rational surface. Then the condition of being Fano or having Picard number 2 enables one to show that X is actually a conic bundle over a surface. Then Voisin shows that there is a non-zero (higher genus) Gromov-Witten invariant of the form

$$\langle \underbrace{[C], \ldots, [C]}_{\geq (g+1)[C]}, [A]^2, \ldots, [A]^2 \rangle^X_{g,\beta},$$

where $[C]$ is the curve class of a general fiber and $[A]$ is the class of a very ample divisor. Thus the MRC quotient surface has to be uniruled, which is impossible by results in [GHS03].

In this thesis we attack this problem in dimension 3 using the same strategy as [Voi08] together with techniques of handling Gromov-Witten invariants under blow-ups and blow-downs. We first construct a smooth birational model Y which is "almost" a conic bundle over a *smooth* rational surface. On this new threefold, there is a non-zero *genus zero* Gromov-Witten invariant of the form $\langle [C], \dots \rangle^Y_{0,\beta}$. By weak factorization, we can factorize the birational map from Y to X by a number of blow-ups and blow-downs. Then using the ideas developed in [MP06] and [HLR08], we show that there is a similar non-zero descendant Gromov-Witten invariant on X, hence on X'. So the MRC quotient of X' cannot be a surface. In this way we have verified Conjecture 1.5 for 3-folds.

Theorem 1.7. *Let X and X' be two smooth projective 3-folds which are symplectic deformation equivalent. Then X is rationally connected if and only if X' is.*

1.2 Symplectic birational geometry

Next we would like to mention the so called "symplectic birational geometry program". The ultimate goal of this program is to carry out a "birational" classification of symplectic manifolds. In this thesis we will restrict ourselves to the study of a particular class of symplectic manifolds.

Definition 1.8. A symplectic manifold is *symplectic uniruled* (resp. *symplectic rationally connected*) if there is a non-zero Gromov-Witten invariant of the form $\langle [pt], \dots \rangle^X_{0,\beta}$ (resp. $\langle [pt], [pt], \dots \rangle^X_{0,\beta}$).

There are two basic questions about these definitions.

1. Are these conditions symplectic birational invariant?
2. Is a smooth projective uniruled (resp. rationally connected) variety symplectic uniruled (resp. symplectic rationally connected)?

For symplectic uniruledness, the answer is positive by the work of Kollár, Ruan, and Hu-Li-Ruan ([Kol98], [Rua99], [HLR08]). It is not known if symplectic rational connectedness is a (symplectic) birational invariant, although we do expect this to be true. And it is not known if rationally connected projective manifolds are symplectic rationally connected. Note that Kollár's conjecture would follow if we could show that rational connectedness implies symplectic rational connectedness.

Every one dimensional rationally connected variety is just $\mathbb{P}^1$, and thus symplectic rationally connected. It is also easy to prove rational connectedness is equivalent to symplectic rational connectedness in dimension 2 (c.f. Proposition 4.16).

In general, this question is very difficult since the moduli space of stable maps might be reducible and there might be components whose dimensions are higher than the expected dimension. Then one has to introduce the *virtual fundamental class* in order to define the Gromov-Witten invariants. The components of dimension higher than the expected dimension can contribute negatively, thus making the Gromov-Witten invariant zero, see Example 42 in [Kol09].

Our second theorem addresses this question in some special cases.

Theorem 1.9. *Let X be a smooth projective rationally connected 3-fold. If X is Fano or has Picard number 2, then there is a non-zero Gromov-Witten invariant of the form $\langle [pt], [pt], \dots \rangle^X_{0,\beta}$.*

Here is one possible way to prove that every rationally connected variety is symplectic rationally connected. First show that symplectic rational connectedness is a birational invariant and then find in each birational class a "good" representative which is symplectic rationally connected. In this thesis, we partly carry out the second part in dimension 3.

By the minimal model program (MMP) in dimension 3, every rationally connected variety is birational to one of the following:

1. a conic bundle over a rational surface,
2. a fibration over $\mathbb{P}^1$ with general fiber a Del Pezzo surface, or
3. a $\mathbb{Q}$-Fano threefold.

Here is our third theorem.

Theorem 1.10. *Let Y be a rationally connected 3-fold with at worst terminal singularities and $\pi : Y \to S$ be a fiber type contraction of some K_Y-negative extremal face (equivalently, Y is a Mori fiber space). Assume one of the following holds.*

1. $\dim S \geq 1$, *that is, Y is a conic bundle or a Del Pezzo fibration.*
2. $\dim S = 0$, *that is, Y is a $\mathbb{Q}$-Fano 3-fold, and the smooth locus of Y is rationally connected.*

Then there is a resolution of singularities $X \to Y$ such that X is symplectic rationally connected.

Note that we do not assume the relative Picard number $\rho(Y/S)$ is 1 in the above theorem.

In general, it is difficult to determine if the smooth locus of a singular variety is rationally connected. However, in this thesis we note that Gorenstein $\mathbb{Q}$-Fano 3-folds satisfy this condition.

1.3 Proof of Theorem 1.3 and Corollary 1.4

We conclude this introduction by giving the proof of Theorem 1.3 in [Kol98] and [Rua99] and by explaining why the same strategy does not work for the case of rationally connected varieties.

We need to introduce some terminology.

Definition 1.11. Let X be a smooth variety. A curve $f : \mathbb{P}^1 \to X$ is called *free* (resp. *very free*) if $f^*T_X \cong \oplus \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq 0$ (resp. $a_i \geq 1$).

A smooth projective variety is uniruled (resp. rationally connected) if and only if there is a free (resp. very free) curve.

Proof of Theorem 1.3. We first choose a polarization of X. Then there exists a free curve C of minimal degree with respect to the polarization. Note that every rational curve through a very general point p in X is free. So if we choose such a point and consider all the curves mapping to X of class $[C]$ and passing through p, then we get a proper family (by minimality) of expected dimension (since the deformation is unobstructed). Therefore the Gromov-Witten invariant $\langle [pt], [A]^2, \ldots [A]^2 \rangle^X_{0,[C]}$ is non-zero, where $[A]$ is the class of a very ample divisor. Clearly, this is the number of curves meeting all the constraints. □

Proof of Corollary 1.4. Assume X is uniruled. Then by Theorem 1.3, there is a non-zero Gromov-Witten invariant of the form $\langle [pt], \ldots \rangle^X_{0,\beta}$. Since Gromov-Witten invariants are symplectic deformation invariant, we have a similar non-vanishing Gromov-Witten invariant on X'. Then the moduli space of rational curves passing through a general point is non-empty, otherwise the Gromov-Witten invariant is zero. Thus X' is also uniruled. □

One may want to prove a similar result for rationally connected varieties by choosing the minimal curve class such that a curve in this curve class connects two general points. However it may happen that every such curve is reducible and disappears even after an algebraic deformation. For example, in a Hirzebruch surface $\mathbb{F}_n$ with n large, we can choose a suitable polarization such that the minimal such curve is the union of two general fibers and the section at infinity. If n is large, this curve disappears after an algebraic deformation. Thus, it cannot give a non-zero Gromov-Witten invariant.

If one insists in choosing a minimal very free curve, then there may be reducible curves in the same curve class that lie in a component whose dimension is higher than the expected dimension. For example, in the case of the Hirzebruch surface $\mathbb{F}_n$, the curve class of the minimal very free curve is $S_\infty + nF$, where S_∞ is the section at infinity and F is a general fiber. Then the union of the multi-covers of two general fibers and S_∞ gives a component whose dimension is higher than the expected dimension. There is no known general methods to determine the contribution of such components other than by an explicit computation.

2 Symplectic deformation invariance for rationally connected threefolds

In this chapter we will prove symplectic deformation invariance for rationally connected 3-folds. As indicated in the introduction, we need to study the change of Gromov-Witten invariants under blow-ups/blow-downs. We first review the degeneration formula and relative Gromov-Witten invariants needed in the proof. Then we prove a blow-up/blow-down correspondence similar to the one in [HLR08]. And finally we give the proof of the symplectic deformation invariance.

2.1 Descendant GW-invariants, Relative GW-invariants, and the Degeneration formula

In this section we recall some variants of Gromov-Witten invariants.

Definition 2.1. Let $\overline{\mathcal{M}}_{0,n}^{X,\beta}$ be the moduli stack of genus 0, n-pointed stable maps to X whose curve class is β. Let $\mathcal{L}_i$ be the line bundle on $\overline{\mathcal{M}}_{0,n}^{X,\beta}$ whose fiber over each point $(C, p_1, \ldots, p_n)$ is the restriction of the sheaf of differentials of the curve C to the point p_i. Let ψ_i be the first Chern class of $\mathcal{L}_i$. Then the descendant Gromov-Witten invariant is defined as

$$\langle \tau_{k_1}\gamma_1, \ldots, \tau_{k_n}\gamma_n \rangle_{0,\beta}^X = \int_{[\overline{\mathcal{M}}_{0,n}^{X,\beta}]^{\mathrm{virt}}} \prod_i \psi_i^{k_i} ev_i^* \gamma_i,$$

where ev_i is the evaluation map given by the i-th marked point, and $\gamma_i \in H^*(X, \mathbb{Q})$.

For $\langle \tau_{k_1}\gamma_1, \ldots, \tau_{k_n}\gamma_n \rangle_{0,\beta}^X$, we can associate a decorated graph Γ of one vertex decorated by β and a tail for each marked points, decorated by (k_i, γ_i). The resulting graph $\Gamma(\{(k_i, \gamma_i)\})$ is called a *decorated weighed graph.*

Next we discuss *relative* Gromov-Witten invariants, which were first introduced in the symplectic category by Li-Ruan [LR01] and in the algebraic category by Jun Li [Li01], [Li02]. We will not recall the precise definition here since it is not needed. The reader should refer to the above-mentioned papers for more details.

Intuitively, the relative Gromov-Witten invariants count the number of stable maps satisfying certain incidence constraints and having prescribed tangency condition with a given divisor. Let X be a smooth projective variety and $D \subset X$ be a smooth divisor. Fix a curve class β such that the intersection number $D \cdot \beta = m$ is non-negative. The relative Gromov-Witten invariants are not defined if the number $D \cdot \beta$ is negative. Also choose a partition $\{m_i, i = 1, 2, \ldots, s\}$ of m. Then the relative Gromov-Witten invariants count the number of stable maps $f : (C, p_1, \ldots, p_r, q_1, \ldots, q_s) \to X$ with $r + s$ marked points such that the first r points (absolute marked points) are mapped to cycles in X and the last s points (relative marked points) are mapped to some cycles in D and $f^*D = \sum m_i q_i$. We can also define descendant relative Gromov-Witten invariants. We write such invariants as

$$\langle \tau_{d_1}\gamma_1, \ldots, \tau_{d_r}\gamma_r | (m_1, \delta_1), \ldots, (m_s, \delta_s) \rangle_{0,\beta}^{(X,D)}$$

where $\gamma_i \in H^*(X, \mathbb{Q}), \delta_j \in H^*(D, \mathbb{Q})$. We also use the abbreviation

$$\langle \Gamma\{(d_i, \gamma_i)\} | \mathcal{T}_s \rangle_\beta^{X,D}$$

following [HLR08], where

$$\mathcal{T}_s = \{(m_1, \delta_1), \ldots, (m_s, \delta_s)\}$$

is called the *weighted partition*. In the degeneration formula, we have to consider stable maps from disconnected domains. The corresponding relative invariants are defined to be the product of those of stable maps from connected domains. Such invariants are denoted by

$$\langle \Gamma^\bullet\{(d_i, \gamma_i)\} | \mathcal{T}_s \rangle_\beta^{X,D}.$$

We use $\bullet$ to indicate that the invariant is for a disconnected curve as [HLR08] and [MP06]. Finally we note that we can represent these invariants by decorated weighted graphs (c.f. Section 3.2 in [HLR08]), which is the disjoint union of the graphs described in Definition 2.1.

Now we describe the degeneration formula. Let $W \to S$ be a projective morphism from a smooth variety to a pointed curve $(S, 0)$ such that a general fiber is smooth and connected and the fiber over 0 is the union of two smooth irreducible varieties (W^+, W^-) intersecting transversely at a smooth subvariety

Z. Let γ_i be cohomology classes in a general fiber. Assume that the specialization of γ_i in W_0 can be written as $\gamma_i(0) = \gamma_i^+ + \gamma_i^-$, where $\gamma_i^+ \in H^*(W^+, \mathbb{Q})$ and $\gamma_i^- \in H^*(W^-, \mathbb{Q})$.

We first specify a map from a curve of genus 0 to $W^+ \cup W^-$ with the following properties:

(i) Each connected component is mapped to either W^+ or W^- and carries a degree 2 homology class;

(ii) The marked points are not mapped to Z;

(iii) Each point mapped to Z carries a positive integer representing the order of the tangency.

The above data gives two graphs describing relative stable maps from possibly disconnected domains to (W^+, Z) and (W^-, Z), the graph of which are denoted by $\Gamma_+^\bullet$ and $\Gamma_-^\bullet$. From (iii) we get two partitions T_+ and T_-. Call the above data a *degenerate* (β, l) *graph* if the resulting pairs $(\Gamma_+^\bullet, T_+)$ and $(\Gamma_-^\bullet, T_-)$ satisfies the following: the total number of marked points is l, $T_+ = T_-$, and the identification of relative tails produces a connected graph of W with total homology class $[\beta]$ and genus 0.

Denote by $Aut(T_k)$ the automorphism group of such partitions. Let $\{\delta_i\}$ be a self-dual basis of $H^*(Z, \mathbb{Q})$. By (iii), we have a weighted partition $\mathcal{T}_k = \{(t_j, \delta_{a_j})\}$ and its dual partition $\check{\mathcal{T}}_k = \{(t_j, \check{\delta}_{a_j})\}$, where $\check{\delta}_{a_j}$ is the Poincaré dual of δ_{a_j}. Let β_+ (resp. β_-) be the total homology class of the curves mapped to W^+ (resp. W^-) in a degenerate (β, l) graph. Then $\beta = \beta_+ + \beta_-$. The degeneration formula expresses the Gromov-Witten invariants of a general fiber in terms of the relative Gromov-Witten invariants of the degeneration in the following way:

$$\langle \prod_i \tau_{d_i} \gamma_i \rangle_{0,[\beta]}^{W_t} = \sum \Delta(\mathcal{T}_k) \langle \Gamma^\bullet \{(d_i, \gamma_i^+)\} | \mathcal{T}_k \rangle_{0,\beta_+}^{W^+,Z} \langle \Gamma^\bullet \{(d_i, \gamma_i^-)\} | \check{\mathcal{T}}_k \rangle_{0,\beta_-}^{W^-,Z},$$

where the summation is taken over all possible degenerate (β, l) graphs, and

$$\Delta(\mathcal{T}_k) = |Aut(T_k)| \cdot \prod_j t_j.$$

By convention, if β_+ or β_- is the zero homology class, the relative invariant is defined to be 1.

In this paper we are mainly interested in the following special case of such degenerations: the deformation to the normal cone. Namely, let X be a smooth projective variety and $S \subset X$ be a smooth subvariety. Then we take W to be the blow-up of $X \times \mathbb{A}^1$ with blow-up center $S \times 0$. In this case, $W^- \cong \tilde{X}$, the blow-up of X along S, and $W^+ \cong \mathbb{P}_S(\mathcal{O} \oplus N_{S/X})$.

2.2 A partial ordering

Let X be a smooth projective 3-fold and $S \subset X$ be a smooth subvariety of codimension $k(=1,2,3)$. Denote by $\tilde{X}$ the blow-up of X along S and by E the exceptional divisor. Here we allow S to be a codimension 1 subvariety, i.e. a divisor. In this case $\tilde{X}$ is isomorphic to X and E is isomorphic to S.

In the following we will define a partial ordering on certain Gromov-Witten invariants. The partial ordering is basically the same as the one in [HLR08]. The major difference is that we choose a different self-dual basis of E (c.f. Remark 2.2).

Let $\theta_1, \theta_2, \ldots, \theta_{m_S} \in H^*(S, \mathbb{Q})$ be a self dual basis of S, where θ_1 (resp. θ_{m_S}) is the generator of the degree 0 (resp. $2(3-k)$) cohomology. We now describe a self dual basis of the exceptional divisor E. Note that $E = \mathbb{P}_S(N_{S/X})$ is a $\mathbb{P}^{k-1}$-bundle over S. Let $[E]$ be the first Chern class of the relative $\mathcal{O}(-1)$ bundle over $\mathbb{P}_S(N_{S/X})$. If k is 2, i.e. S is a smooth curve in X, then $\pi_S : E \to S$ is a ruled surface over S. In this case, define

$$\lambda = [E] - \frac{[E]\cdot[E]}{2}\pi_S^*\theta_{m_S},$$

so that $\lambda^2 = 0$. Otherwise just take λ to be $[E]$. Then the cohomology classes

$$\pi_S^*\theta_i \cup \lambda^j,\ 1 \le i \le m_S,\ 0 \le j \le k-1,$$

form a self-dual basis of E. Denote it by $\Theta = \{\delta_i\}$.

Remark 2.2. In [HLR08], the authors claim $\{\pi_S^*\theta_i \cup [E]^j\}$ to be self dual, which is not true if $N_{S/X}$ is not a trivial bundle over S. However, this has been fixed and the proof is essentially the same since only the degree of the $[E]$ part is important in the proof.

Definition 2.3. A *standard (relative) weighted partition* μ is a partition

$$\mu = \{(\mu_1, \delta_{d_1}), \ldots, (\mu_{l(\mu)}, \delta_{d_{l(\mu)}})\},$$

where μ_i and d_i are positive integers with $d_i \le k m_S$. $l(\mu)$ is called the *length* of the partition.

For $\delta = \pi_S^*\theta \cup \lambda^j \in H^*(E, \mathbb{Q})$, with $j \leq k-1$, define

$$\deg_S(\delta) = \deg \theta, \deg_f(\delta) = 2j.$$

For a standard weighted partition μ, define

$$\deg_S(\mu) = \sum_{i=1}^{l(\mu)} \deg_S(\delta_{d_i}), \deg_f(\mu) = \sum_{i=1}^{l(\mu)} \deg_f(\delta_{d_i}).$$

We define a partial ordering on the set of pairs $\mathbb{Z}^+ \times H^*(E, \mathbb{Q})$ by setting

$$(m, \delta) > (m', \delta'), \text{if}$$

1. $m > m'$, or
2. $m = m'$ and $\deg_S(\delta) > \deg_S(\delta')$, or
3. $m = m'$, $\deg_S(\delta) = \deg_S(\delta')$, and $\deg_f(\delta) > \deg_f(\delta')$.

We define a partial ordering on the set of weighted partitions by setting

$$\mu > \mu'$$

if after the pairs of μ and μ' are arranged in decreasing order, the first pair for which μ and μ' are not equal is larger for the μ, or if all the pairs of μ' appear in μ and μ has more pairs.

Let $\sigma_1, \ldots, \sigma_{m_X}$ be a basis of $H^*(X, \mathbb{Q})$. Then the set of cohomology classes

$$\gamma_j = \pi^*\sigma_j, 1 \leq j \leq m_X,$$

$$\gamma_{j+m_X} = \iota_*(\delta_j), 1 \leq j \leq km_S$$

generate a basis of $H^(\tilde{X}, \mathbb{Q})$, where $\pi : \tilde{X} \to X$ is the blow-up along S, $\iota : E \to \tilde{X}$ is the inclusion and ι_* is the induced Gysin map.

Notation 2.4. A *connected standard relative Gromov-Witten invariant* of $(\tilde{X}, E)$ is of the form

$$\langle \omega | \mu \rangle_{0,A}^{\tilde{X},E} = \langle \tau_{k_i}\gamma_{L_1}, \ldots, \tau_{k_n}\gamma_{L_n} | \mu \rangle_{0,A}^{\tilde{X},E},$$

where A is an effective curve class on $\tilde{X}$, μ is a standard weighted partition with $\sum \mu_j = E \cdot A$, and $\gamma_{L_i} = \pi^*\sigma_{L_i}$.

We write $\Gamma(\omega)|\mu$ for the decorated graph of such invariants.

Definition 2.5. Define $c(\Gamma)$ to be the number of connected components of the curve corresponding to the graph Γ and $\|\omega\|$ to be the number of insertions in ω.

Definition 2.6. For two effective curve classes β and β' in $H_2(\tilde{X}, \mathbb{Z})$, we say $\beta < \beta'$ if the difference $\pi_*(\beta') - \pi_*(\beta)$ is an effective curve class in X; $\beta \sim \beta'$ if $\pi_*(\beta) = \pi_*(\beta')$.

Definition 2.7. We define the partial ordering on the set of decorated graphs of the standard relative invariants of $\tilde{X}$ by setting

$$\Gamma(\omega)|\mu < \Gamma(\omega')|\mu', \text{if}$$

1. $\beta < \beta'$, or
2. $\beta \sim \beta'$ and $c(\Gamma) > c(\Gamma')$, or
3. $\beta \sim \beta', c(\Gamma) = c(\Gamma')$, $\|\omega\| < \|\omega'\|$, or
4. $\beta \sim \beta', c(\Gamma) = c(\Gamma'), \|\omega'\| = \|\omega\|$, $\deg_S(\mu) > \deg_S(\mu')$, or
5. $\beta \sim \beta', c(\Gamma) = c(\Gamma'), \|\omega'\| = \|\omega\|, \deg_S(\mu) = \deg_S(\mu')$, $\mu > \mu'$.

We have the following observation:

Lemma 2.8. *Given a standard relative invariants, there are only finitely many non-zero standard relative invariants smaller than it in the partial ordering defined above.*

Proof. For a curve class β of $\tilde{X}$, there are only finitely many curve classes of the form $\pi_*(\beta')$ in X such that $\pi_*(\beta) - \pi_*(\beta')$ is an effective curve class in X. Two different curve classes in $\tilde{X}$ gives the same curve class in X if and only if the difference is a multiple of L, where L is a line or a ruling in the exceptional divisor. And once β' is fixed, k is bounded below since $\beta' + kL$ has to be an effective curve class, and bounded above since $E \cdot (\beta' + kL)$ is non-negative and $E \cdot L$ is -1. Thus such classes are bounded. For every such curve class, there are only finitely many non-zero relative invariants. $\square$

2.3 From relative to absolute

We relate absolute invariants of X to relative invariants of $\tilde{X}$ in this section.

To a relative insertion (m,δ) with $\delta = \pi_S^*\theta_i \cup \lambda^j$, we associate the absolute insertion $\tau_{d(m,\delta)}(\tilde{\delta})$, where

$$\tilde{\delta} = \iota_*(\theta_i), d(m,\delta) = km - k + j.$$

Given a weighted partition $\mu = \{(\mu_i, \delta_{k_i})\}$, we define

$$d_i(\mu) = d(\mu_i, \delta_{k_i}) = k\mu_i - k + \frac{1}{2}\deg_f(\delta_{k_i}),$$

$$\tilde{\mu} = \{\tau_{d_1(\mu)}(\tilde{\delta}_{k_1}), \ldots, \tau_{d_{l(\mu)}(\mu)}(\tilde{\delta}_{k_{l(\mu)}})\}.$$

Given a standard relative invariant $\langle \Gamma^\bullet(\omega)|\mu\rangle^{\tilde{X},E}$, we define the absolute descendant invariant associated to the relative invariant to be

$$\langle \Gamma^\bullet(\omega, \tilde{\mu})\rangle^X$$

Here all the insertions ω in the relative invariants are of the form $\pi^*\sigma_i$; the corresponding insertions in the absolute invariants are just σ_i.

Definition 2.9. An absolute descendant invariant of X is called a *colored absolute descendant invariant relative to* S if its insertions are divided into two collections ω and $\tilde{\mu}$ such that each insertion in ω is of the form $\tau_{d_i}\sigma_i$ and each insertion in $\tilde{\mu}$ is of the form $\tau_{d_k}\tilde{\delta}_k$.

Remark 2.10. An absolute invariant may give different colored invariants depending on how one groups the insertions.

Definition 2.11. If $k = 1$, then a colored absolute descendant invariant of X relative to S (with curve class β) is called *admissible* if $\sum \mu_j = E \cdot \beta$.

The following lemma is essentially Lemma 5.14 in [HLR08]. Note that in their paper they only consider the case of primary Gromov-Witten invariants. But the proof is actually the same.

Lemma 2.12. *If $\mu \neq \mu'$, then $\tilde{\mu} \neq \tilde{\mu}'$. Therefore there is a natural bijection between the set of colored weighted absolute graphs relative to S and the set of weighted relative graphs in $\tilde{X}$ relative to E if $k > 1$. The same is true if we restrict to the admissible ones when $k = 1$.*

Remark 2.13. Notice that different relative invariants may give the same absolute invariants. But these absolute invariants are different as colored absolute invariants.

Finally, let C be a curve in $\tilde{X}$ which does not intersect E. Then the image of C under the map $\tilde{X} \to X$ is a curve in X, also denoted by C. Note that $[C]$ as an element of $H^4(\tilde{X})$ is just the pull back of $[C]$ in $H^4(X)$. Let I be the partially ordered set of standard weighted relative graph $\Gamma^\bullet([C], \omega)|\mu$.

Define $\mathbb{R}^I_{\tilde{X},E}$ to be an infinite dimensional vector space whose coordinates are ordered in the same way as the partial ordering in I. A standard weighted relative invariant $\langle \Gamma^\bullet([C], \omega)|\mu\rangle^{\tilde{X},E}$ gives a vector $v_{\tilde{X},E}$ in $\mathbb{R}^I_{\tilde{X},E}$. By Lemma 2.12, I is also the set of colored standard weighted absolute graphs relative to S. Thus we also have an infinite dimensional vector space $\mathbb{R}^I_{X,S}$ whose coordinates are also ordered by the partial ordering in I. Similarly, an absolute invariant $\langle \Gamma^\bullet([C], \omega, \tilde{\mu})\rangle^X$ gives a vector $v_{X,S}$ in this vector space.

Theorem 2.14. *Let $\pi : \tilde{X} \to X$ be the blow-up of a 3-fold along a smooth center S. Then there is an invertible lower triangular linear map*

$$A_S : \mathbb{R}^I_{\tilde{X},E} \to \mathbb{R}^I_{X,S},$$

given by the degeneration formula such that $A_S(v_{\tilde{X},E}) = v_{X,S}$ and A_S only depends on S and its normal bundle.

Proof. In the setup of the degeneration formula, we take W to be the blow-up of $X \times \mathbb{A}^1$ along the smooth subvariety $S \times 0$. Then $W^- \cong \tilde{X}, W^+ \cong \mathbb{P}_S(\mathcal{O} \oplus N_{S/X})$ and they intersect transversely at E. We will apply the degeneration formula in this setting.

We start with a connected standard weighted relative invariant

$$\langle \Gamma([C], \omega)|\mu\rangle^{(\tilde{X},E)}_{0,\beta}$$

with vertex decorated by β. Then the associated absolute invariant is

$$\langle \Gamma([C], \omega, \tilde{\mu})\rangle^X_{0,\pi_*(\beta)}.$$

In order to apply the degeneration formula, we have to specify the specialization of the cohomology classes. Since C does not intersect E, we may specialize C to lie entirely in W^-, i.e. we set $[C]^- = [C]$ and $[C]^+ = 0$. The Poincaré duals of the cohomology classes in $\tilde{\mu}$ are supported in S. So we specialize these cohomology classes to the W^+ side and set the cohomology classes in the W^- side to be zero. Finally, classes in ω are of the form σ_i. Therefore we set $\sigma_i^- = \gamma_i = \pi^*\sigma_i$ with appropriate classes σ^+ in W^+ side.

Then the degeneration formula in this case gives:

$$\begin{aligned}&\langle [C], \omega, \tilde{\mu}\rangle^X_{\pi_*(\beta)}\\ =&\sum \langle \Gamma^\bullet_-([C],\omega_1)|\mathcal{T}\rangle^{(\tilde{X},E)}\Delta(\mathcal{T})\langle \Gamma^\bullet_+(\omega_2,\tilde{\mu})|\check{\mathcal{T}}\rangle^{\mathbb{P}_S(\mathcal{O}\oplus N_{S/X}),E}.\end{aligned}$$

We view $\Delta(\eta)\langle \Gamma^\bullet_+(\omega_2,\tilde{\mu})|\check{\mathcal{T}}\rangle^{\mathbb{P}_S(\mathcal{O}\oplus N_{S/X}),E}$ as the coefficients of the linear map A_S.

We show that $\langle \Gamma^\bullet_-([C],\omega)|\mu\rangle^{(\tilde{X},E)}$ is the largest term with non-zero coefficient on the right hand side.

First we show that its coefficient $\langle \Gamma^\bullet_+(\tilde{\mu})|\check{\mu}\rangle^{\mathbb{P}_S(\mathcal{O}\oplus N_{S/X}),E}$ is non-zero. This is basically step II in the proof of Theorem 5.15 in [HLR08]. With our choice of the self-dual basis, the coefficient is the product of the relative invariants

$$\langle \tau_{nd-1-j}[pt]|H^j\rangle^{\mathbb{P}^k,\mathbb{P}^{k-1}}_{0,dL},$$

where $j = \deg_f(\delta_{k_i})$ and H is the hyperplane class. These invariants are computed in [HLR08] via virtual localization and are shown to be non-zero. So the diagonal of the linear map A_S is non-zero.

Notice that this is the only step in [HLR08] where the form of self dual basis matters since we need to know the diagonal is non-zero. For the rest part, the proof proceeds exactly as the proof of Theorem 5.15 in [HLR08]. □

2.4 Birational invariance

In this subsection we prove the following theorem.

Theorem 2.15. *Let $\pi : \tilde{X} \to X$ be the blow-up of a smooth projective 3-fold along a smooth subvariety S. Also let C be a curve in $\tilde{X}$ which does not intersect the exceptional divisor E. Then there is a non-zero descendant Gromov-Witten invariant on $\tilde{X}$ of the form*

$$\langle [C], \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n\rangle^{\tilde{X}}_{0,\tilde{\beta}}$$

if and only if there is a non-zero descendant Gromov-Witten invariant on X of the form

$$\langle [C], \tau_{d_1}\gamma_1, \ldots, \tau_{d_m}\gamma_m\rangle^{X}_{0,\beta}.$$

Here we use C to denote both the curve on $\tilde{X}$ and its image in X.

Proof. Suppose there is a non-zero descendant Gromov-Witten invariant on X of the form

$$\langle [C], \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle^X_{0,\beta}.$$

We may assume that all the γ_i are of the form σ_i. We degenerate X into $\tilde{X}$ and $\mathbb{P}_S(\mathcal{O} \oplus N_{S/X})$ and apply the degeneration formula. So there is a non-zero relative invariant:

$$\langle [C], \tau_{d_1}\gamma_1, \ldots, \tau_{d_k}\gamma_k | \mu \rangle^{\tilde{X},E}_{0,\beta}.$$

Then apply Theorem 2.14 to $(\mathrm{Bl}_E\tilde{X}, E)$ and $(\tilde{X}, E)$. Note that the blow-up of $\tilde{X}$ with center E is $\tilde{X}$ itself. So Theorem 2.14 gives a non-zero absolute invariant of $\tilde{X}$ of the desired form.

Conversely, suppose there is a non-zero descendant Gromov-Witten invariant on $\tilde{X}$ of the form

$$\langle [C], \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle^{\tilde{X}}_{0,\tilde{\beta}}.$$

We may assume that $\gamma_i, 1 \leq i \leq m$ are of the form $\pi^*\sigma_{j_i}$ and $\gamma_i, m+1 \leq i \leq n$ are of the form $\iota_*(\delta_{j_i})$. Then we degenerate $\tilde{X}$ into $\tilde{X}$ and $\mathbb{P}_E(\mathcal{O} \oplus N_{E/X})$ and specialize γ_i, for $m+1 \leq i \leq n$, to the projective bundle side. Then there is a non-zero relative invariant of the form:

$$\langle [C], \tau_{d_1}\gamma_1, \ldots, \tau_{d_k}\gamma_k | \mu \rangle^{\tilde{X},E}_{0,\beta},$$

with $k \leq m$. In particular, all the $\gamma_i, 1 \leq i \leq k$ are of the form $\pi^*\sigma_{j_i}$. Again apply Theorem 2.14 to $(\tilde{X}, E)$ and (X, S). We get a non-zero absolute descendant invariant of desired form. □

Remark 2.16. We may get a non-vanishing Gromov-Witten invariant of a *disconnected* curve in using the invertible map A_S. However this is sufficient since this invariant is the product of the Gromov-Witten invariants for the connected components and we only need to keep track of one insertion. This is also the reason that the same argument cannot prove the birational invariance of rational connectedness since then we need to keep track of two insertions.

2.5 Proof of Theorem 1.7

Assume X is rationally connected. We show that X' is also rationally connected.

By Theorem 1.3, X' is uniruled. Let $X' \dashrightarrow S$ be the *maximal rationally connected (MRC) quotient* of X'. This is a rational map such that the closure of a general fiber (in X') is the equivalence class of points in X' that are connected by a chain of rational curves. X' is rationally connected if and only if S is point. In our case, the dimension of S is at most 2.

We will use proof by contradiction. So assume S is not a point. By Corollary 1.4 in [GHS03], S is not uniruled.

If S is a curve, it cannot be a rational curve. A non-zero section of $H^0(S, \Omega_S)$ pulls back to a non-zero section of $H^0(X', \Omega_{X'})$. But X' is simply connected since X and X' are diffeomorphic and X is simply connected. Then by Hodge decomposition, $H^0(X', \Omega_{X'}) = 0$. This is a contradiction.

So S has to be a non-uniruled surface. The closure of a general fiber of the rational map $X' \dashrightarrow S$ is a rational curve passing through a general point in X', and thus free. The rational map can be extended to the complement of a codimension 2 locus. Therefore it is actually well-defined on the closure of a general fiber since a free rational curve can be moved away from any codimension 2 locus.

Furthermore, a general fiber C has to be the minimal free curve of X'; otherwise S is uniruled. So,

$$\langle [pt] \rangle^{X'}_{0,[C]} = 1, -K_{X'} \cdot C = 2.$$

Both of these conditions are symplectic deformation invariant. Thus,

$$\langle [pt] \rangle^{X}_{0,[C]} = 1, -K_{X} \cdot C = 2.$$

Clearly $[C]$ is also the minimal free curve class on X otherwise the minimal free curves in X give rise to rational curves in X' not in the fiber of $X' \dashrightarrow S$. We have seen that this Gromov-Witten invariant is enumerative. So there is exactly one minimal free curve passing through a general point in X. Let

$$\pi : \mathcal{C} \to \Sigma$$

be the universal family of the minimal free curves and

$$f : \mathcal{C} \to X$$

be the universal map. Then the morphism f is birational. Thus there is a rational map $X \dashrightarrow \Sigma$. That is, X is birational to a conic bundle over a rational surface.

Let $\Gamma \subset X \times \Sigma$ be the closure of the rational map $X \dashrightarrow \Sigma$. By the same argument proving the map $X' \dashrightarrow S$ is well-defined along a general fiber, the map $X \dashrightarrow \Sigma$ is also well-defined along a general fiber. Then the exceptional divisors of $\Gamma \to X$ do not dominate Σ. So there is an open subset U of X and a smooth open subset V of Σ such that $U \to V$ is a well defined proper morphism and a general fiber is $\mathbb{P}^1$. we can choose smooth projective compactifications of U and V, denoted by Y and Σ', together with a morphism $Y \to \Sigma'$ such that a general fiber is $\mathbb{P}^1$. By Theorem 0.1.1 in [AKMW02], we can factorize the birational map $X \dashrightarrow Y$ by blow-ups and blow-downs

$$X = X_0 \dashrightarrow X_1 \ldots \dashrightarrow X_n = Y$$

such that every birational map is an isomorphism over U. In particular, there is a free curve C in every X_i away from every exceptional divisor.

By Theorem 4.15, there is a non-zero Gromov-Witten invariant on Y of the form $\langle [C], \ldots, [C], [A]^2, \ldots, [A]^2 \rangle^Y_{0,\beta}$ with $[A] \in H^2(Y, \mathbb{Q})$ being the class of a very ample divisor of Y. Then by the proof of Theorem 2.15, there is a descendant Gromov-Witten invariant on X of the form $\langle [C], \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle^X_{0,\beta'}$ with $\gamma_j \in H^{\geq 4}(X, \mathbb{Q})$. Since X' is symplectic deformation equivalent to X, $\langle [C], \tau_{d_1}\gamma_1, \ldots, \tau_{d_n}\gamma_n \rangle^{X'}_{0,\beta'} \neq 0$. If a curve of class $[\beta']$ in X' were supported in a general fiber of $X' \dashrightarrow S$, we would have that $[\beta']$ is a multiple of $[C]$ and $-K_{X'} \cdot \beta' \geq 2$. So there are other insertions in the descendant invariant. But we can choose representatives of the cycles γ_i disjoint from a general fiber C. Then the invariant should be zero since a curve supported in a fiber cannot meet the cycles representing γ_i. Thus the curves with curve class $[\beta']$ are not supported in the fibers of the rational map $X' \dashrightarrow S$ and S is uniruled by their images. This is a contradiction.

3 Fano Varieties

The main result in this chapter is the following theorem.

Theorem 3.1. *If X is a Fano threefold, then there is a non-zero Gromov-Witten invariant of the form $\langle [pt], [pt], [A]^2, \ldots, [A]^2 \rangle_{0,\beta}^X$, where $[A]$ is the class of a very ample divisor.*

We begin by reviewing some results from birational geometry which allows us to construct low degree very free curves in a Fano 3-fold. Then we prove that these low degree curves give non-zero Gromov-Witten invariants we want. The key observation is that bend-and-break give some control on the deformations of low degree curves in a Fano variety (Lemma 3.7).

Throughout the chapter we will use terminology in birational geometry freely without defining them. The reader is referred to Section 1.3 of [KM98] for precise definitions.

3.1 Some results from birational geometry

In this subsection we collect some results on the classification of K_X-negative extremal contractions on a smooth projective 3-fold.

Theorem 3.2 ([KM98] [Kol91])**.** *Let X be a smooth threefold and contr : $X \to Y$ be the contraction of a K_X-negative extremal ray. Then one of the followings holds.*

(E1) Y is smooth and X is the blow-up of Y along a smooth curve;

(E2) Y is smooth and X is the blow-up of Y along a point;

(E3) Y is singular and locally analytically isomorphic to $x^2+y^2+z^2+w^2=0$, and X is the blow-up at the singular point;

(E4) Y is singular and locally analytically isomorphic to $x^2+y^2+z^2+w^3=0$, and X is the blow-up at the singular point.

(E5) contr contracts a smooth $\mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$ to a point of multiplicity 4 in Y;

(C) Y is a smooth surface and X is a conic bundle over Y;

(D) Y is a smooth curve and X is a fibration of Del Pezzo surfaces;

(F) X is a Fano 3-fold with Picard number one and Y is a point.

It is easy to workout what the exceptional divisors are in the cases of exceptional contractions and we have the following corollary.

Corollary 3.3. *In the case of (E2)-(E5), the exceptional divisor is rationally connected and the following is the list of very free curves of minimal degree in the exceptional divisor and their normal bundles in X.*

(E2) A line L in $\mathbb{P}^2$, $N_{L/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(-1)$;

(E3) A conic C in a smooth quadric hypersurface. $N_{C/X} \cong \mathcal{O}(2) \oplus \mathcal{O}(-2)$;

(E4) A conic C in a quadric cone. $N_{C/X} \cong \mathcal{O}(2) \oplus \mathcal{O}(-2)$;

(E5) A line L in $\mathbb{P}^2$, $N_{L/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(-2)$.

In particular, the $-K_X$ degree of these curves is at most 2.

We also need the following result from [MM82] and [MM86].

Proposition 3.4. *Let X be a Fano threefold and $contr : X \to Y$ be the blow-up along a smooth curve in Y. Then Y is Fano unless X is the blow-up along a smooth $\mathbb{P}^1$ whose normal bundle in Y is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In this case, the exceptional divisor E is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and the normal bundle of the curve of bi-degree $(1, 1)$ is $\mathcal{O}(2) \oplus \mathcal{O}(-2)$.*

Definition 3.5. Let X be a Fano 3-fold. We say that X is *primitive* if it is not the blow-up along a smooth curve of another smooth Fano 3-fold.

3.2 Construction of low degree very free curves

We first show the following theorem.

Theorem 3.6. *Let X be a Fano threefold. Then there is very free curve whose $-K_X$ degree is at most 6.*

Proof. We may assume that X is a primitive Fano 3-fold. Otherwise there is a birational morphism $X \to Y$ such that Y is a primitive Fano 3-fold. Then the very free curve in Y can be moved away from the blow-up centers and gives a very free curve in X with the same anticanonical degree.

We first consider the case X has Picard number 1. Fix a polarization on X. Let C be a general minimal free rational curve. We know $2 \leq -K_X \cdot C \leq 4$ by Theorem 2.10 in Chapter IV, [Kol96].

If $-K_X \cdot C = 4$, then $N_{C/X} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$ (c.f. Theorem 2.10 in Chapter IV, [Kol96]). We are done in this case.

If $-K_X \cdot C = 3$, then we can fix a general point in X and the deformation of C fixing that point sweeps out a surface S. Since $\rho(X) = 1$, the divisor S is ample. If we take a minimal free curve through another general point, S intersects that curve in a finite number of points. So we have a reducible curve, which is the union of two free curves passing through two general points in X. A general deformation of this curve is an irreducible curve passing through two general points, thus very free. The anti-canonical degree of the very free curve is 6.

If $-K_X \cdot C = 2$, then it is proved in Corollary 4.14, Chapter IV and Proposition 2.6, Chapter V of [Kol96] that there is a chain of free curves of length at most $3(= \dim X)$ connecting two general points. Thus a deformation of this chain gives a very free curve whose anti-canonical degree is at most 6.

Next assume $\rho(X) \geq 2$. Under the assumption that X is primitive, every exceptional divisor is rationally connected by Corollary 3.3 and Proposition 3.4.

It is proved in [MM86] that there is an extremal ray R1 corresponding to a contraction of type (C) (c.f. Theorem 7.1.6 in [Sha99]). Let F be a general fiber. Then $-K_X \cdot F = 2$ and F is a free curve. In the following we will try to construct a very free curve using F and some other curves coming from contractions of other extremal rays.

The discussion is divided into three parts according to the type of contractions given by other extremal rays.

If there is an extremal ray corresponding to a divisorial contraction with exceptional divisor E, then $E \cdot F > 0$. In fact, we may find a possibly reducible curve of class $[F]$ which intersects E by specializing a general curve F. But then $E \cdot F > 0$ since $[F]$ spans an extremal ray and the class of every irreducible component of the specialization of F lies in that extremal ray. So we can assemble a chain of rational curves in the following way. Take two fiber curves of class F, each through a general point and a minimal very free curve in E

connecting the two curves. This chain of rational curves deforms to a very free rational curve with $-K_X$ degree at most 6 (c.f. Corollary 3.3 and Proposition 3.4).

If there is an extremal contraction of type (D), X is a Del Pezzo fibration over $\mathbb{P}^1$. We can assemble a connected reducible curve by gluing a free curve F to a minimal very free curve in a general fiber. It is easy to see that a general deformation of this curve is very free with $-K_X$ degree at most 6.

Finally, assume all the extremal contractions are of type (C). Then there are at least two free curves C_1 and C_2 through a general point (one coming from this contraction and one in class F). A general deformation of the union of these two curves is a free curve of $-K_X$ degree 4. If it is not very free, we may deform this curve fixing a point and get a divisor H. We are done if H intersects either C_1 or C_2. If $H \cdot C_1 = H \cdot C_2 = 0$, then $\rho(X) \geq 3$. Note that H is nef since we can move H by changing the point so that it does not contain any pre-specified curve. Thus some multiple of H is base-point-free and defines a morphism $\pi : X \to \mathbb{P}^1$. This morphism contracts both C_1 and C_2 and a general fiber is a Del Pezzo surface. There is a third extremal ray, necessarily of type (C) by our assumption. Then we get a very free curve with anticanonical degree no more than 6 by the same argument as above. □

We will need the following observation later.

Lemma 3.7. *Let X be a Fano 3-fold. Let $C = C_1 \cup C_2 \cup C_3$ be a chain of $\mathbb{P}^1$s and $f : C \to X$ be a stable map. Assume that $-f^*K_X \cdot C_2 = 1$ and C_i passes through a very general point for $i = 1, 3$. Then a general point of any irreducible component of the Kontsevich moduli space containing (C, f) corresponds to an irreducible very free curve.*

Proof. Notice that C_2 only deforms in a surface S. Both C_1 and C_3 intersect the surface S at finitely many points and C_2 has to pass through at least two of them. Then C_2 does not move once we make the choice of the two points. Otherwise we can deform C_2 fixing two points and, by bend-and-break, C_2 breaks into a reducible or non-reduced curve. But this cannot happen since $-K_X \cdot C_2 = 1$ and $-K_X$ is ample.

We claim that a general deformation of the stable map $f : C \to X$ smooths at least one of the nodes. If not, then a general deformation is given by the deformation of the two free curves and the deformation of the degree 1 curve. By the above argument, up to finitely many choices, the deformation of the degree 1 curve is determined by the deformation of the two free curves. So the deformation space has dimension equal to the sum of the dimension of deformation space of C_1 and C_3, which is $(-f^*K_X)\cdot C_1 + (-f^*K_X)\cdot C_3$. But every irreducible component containing the point $(f : C \to X)$ has dimension at least $-f^*K_X \cdot C$, which is greater than $(-f^*K_X)\cdot C_1 + (-f^*K_X)\cdot C_3$. So at least one of the nodes can be smoothed out and we get a reducible curve, which is the union of two irreducible curves each passing through a very general point. Then it is easy to see that a general deformation of this new curve is very free. □

3.3 Proof of Theorem 3.1

Let C be a very free curve with minimal $-K_X$ degree. Since C is very free, $-K_X \cdot C \geq 4$. By Theorem 3.6, $-K_X \cdot C \leq 6$. We consider Gromov-Witten invariants of the form $\langle [pt], [pt], [A]^2, \ldots, [A]^2 \rangle^X_{0,C}$, where $[A]$ is the class of a very ample divisor. The number of $[A]^2$-insertions is $-K_X \cdot C - 4$. Note that in any case, the components of the Kontsevich moduli space $\overline{\mathcal{M}}^{X,[C]}_{0,n}$ whose general points parametrize very free curves contribute positively to this Gromov-Witten invariant.

We first choose two very general points such that any irreducible rational curve through them is very free and any irreducible rational curve through one of them is free. In particular, the anti-canonical degree of any irreducible rational curve through one of them is at least 2.

We show that if the constraints are general enough, then no reducible curve meets all the constraints. We only need to consider reducible curves that pass through the two general points.

If a reducible curve passes through the two general points and is smoothable, then it lies in a irreducible component of $\overline{\mathcal{M}}^{X,[C]}_{0,n}$, whose general points parametrize irreducible very free curves passing through 2 general points. Thus, this irreducible component has expected dimension and the constraints can be chosen so that such reducible curves do not meet all of the constraints.

We will use this observation in the following two situations. If the reducible curve is a union of free curves, then it is certainly smoothable. Another case is the curve in Lemma 3.7.

Note that the two points are not connected by a single irreducible component of the reducible curve by minimality of the very free curve.

We discuss 3 different cases according to the anti-canonical degree of C.

1. $-K_X \cdot C = 4$. Then the only reducible curve that can pass through 2 general points is the union of two free curves. We are done in this case.

2. $-K_X \cdot C = 5$. There is one additional constraint, a curve which is complete intersection of two very ample divisors. Notice that for degree reasons, every possible reducible curve is either a union of free curves or a curve as in Lemma 3.7. So we are done in this case.

3. $-K_X \cdot C = 6$. In this case we add two more constraints, both of which are complete intersections of pairs of very ample divisors. By Lemma 3.7, we only need to consider 2 cases: a curve with four irreducible components whose $-K_X$ degrees are $2, 2, 1, 1$, and a curve with 3 irreducible components with $-K_X$ degree 2 each.

   First consider the former case. Denote the two degree 1 curve by D_1 and D_2 and the degree 2 curves by C_1 and C_2. Note that by Lemma 3.7, neither D_1 or D_2 connects both C_1 and C_2, otherwise we have a very free curve with smaller $-K_X$ degree. Assume that C_i is connected to D_i for $i = 1, 2$ and D_1 and D_2 are also connected to each other.

   Assume that D_1 and D_2 deform in two different surfaces. Denote the corresponding surface by S_1 and S_2. We first choose one of our constraint curves to avoid the degree 2 curves. Then one of the degree one curves, say D_1, has to pass through the intersection point of this constraint curve with the surface S_1. Notice that there are only finitely many such choices by bend-and-break (c.f. the first paragraph of the proof of Lemma 3.7). Also once we make the choice for D_1, there are only finitely many choices for D_2 since it has to pass through the intersection of C_2 and D_1 with the surface S_2. So we can choose the last constraint to avoid all of four components.

   Now assume the two degree one curves deform in the same surface S. We may choose the constraint curves to intersect the surface S at general points. Note that the degree 2 curves cannot deform once the two points are fixed. Thus they cannot meet the other constraints. If the two degree 1 curve meet all the other constraints, then one of them, say D_1, has to pass through both the intersection of C_1 with the surface and a general point in the surface (i.e. the intersection of one constraint curve with the surface). Thus we get a smoothable chain of rational curves consisting of C_1, D_1, and C_2 as in Lemma 3.7. Then we get a very free curve with $-K_X$ degree 5. This contradicts our choice of C.

   We now consider the case where the curve consists of 3 irreducible components with $-K_X$ degree 2 and one of them is not free.

If the non-free curve is the specialization of a free curve, then it lies in a component of dimension 2. But if we choose the two points to be general, the two free curve of $-K_X$ degree 2 cannot both meet this non-free curve. So the non-free curve only deforms in a surface.

Once the two points are chosen, the two free curves cannot meet any more constraints. Thus, if this reducible curve meets all the constraints, then the non-free curve passes through at least 4 general points of the surface coming from the intersections of the two free curves and the constraints with the surface. So after we fix three general points, the curve deforms in a positive dimensional family. Then by bend-and-break it breaks into two irreducible components or a non-reduced curve. In this way we get a rational curve with $-K_X$ degree 1 and passing through 2 general points in the surface. In particular, there is a chain of curves of $-K_X$ degree 5 as in Lemma 3.7. Then we can smooth them and get a very free curve of $-K_X$ degree 5. This is a contradiction.

Remark 3.8. For another proof, see Section 4.1.

Corollary 3.9. *Let X be a Fano threefold and $f : Y \to X$ a birational morphism. Then Y is symplectic rationally connected.*

Proof. The images under f of the exceptional divisors have codimension at least 2 in X. Thus the minimal very free curve in X can be deformed away from them. We can also choose the constraints in Theorem 3.1 to be away from the blow-up centers. Then the very free curves in X meeting all the constraints are all away from the images of the exceptional divisors. We choose the constraints in Y to be the inverse image of the constraints in X. Observe that the image of any curve satisfying the constraints in Y also satisfies the constraints in X. Thus the images of such curves are irreducible curves not intersecting the exceptional locus. Then it follows that no components are contracted by the map $f : Y \to X$ and the curves in Y that meet these constraints are again irreducible very free curves. □

4 Mori fiber spaces and rationally connected 3-folds of Picard number 2

In this chapter we first discuss rationally connected threefolds which are Mori fiber spaces. Then we finish the proof Theorem 1.9 in the Picard number 2 case.

Throughout the chapter, we will use X to denote a smooth projective rationally connected 3-fold. Usually X is a resolution of singularities of a singular projective variety Y with some special properties. The precise relation of X and Y should be clear from the context in each section.

4.1 $\mathbb{Q}$-Fano varieties

The proof of symplectic rational connectedness of Fano 3-folds could be greatly simplified if we could choose the constraints to be 3 points in the case of $-K_X$ degree 6. Indeed, if we could, then the delicate bend-and-break argument is not necessary since the no irreducible components meet the third point. But in order to have positive contributions from some component, we need to know that a general very free curve constructed in the proof has normal bundle $\mathcal{O}(2) \oplus \mathcal{O}(2)$. This idea works in a more general context as below.

We begin with some preliminary definitions and observations.

Definition 4.1. A normal projective variety Y is a $\mathbb{Q}$-*Fano variety* if Y has terminal singularities, and $-K_Y$ is ample.

Lemma 4.2. *Let Y be a projective variety of dimension n with terminal singularities. Then for any irreducible rational curve C passing through r very general points, the intersection number $-K_Y \cdot C$ is at least $(n-1)(r-1)+2$. If this lower bound is achieved, then C is a curve contained in the smooth locus of Y.*

Proof. Let $f : X \to Y$ be a resolution of singularities such that X and Y are isomorphic over the smooth locus of Y. by definition of terminal singularities, we have $K_X = f^*K_Y + \sum b_i E_i$ where E_i's are exceptional divisors of f and b_i are positive rational numbers. The normal bundle of an irreducible rational curve C through r very general points is

$$N_{C/X} \cong \oplus_i \mathcal{O}(a_i), a_i \geq r-1.$$

So

$$-K_Y \cdot C \geq -K_X \cdot C \geq (n-1)(r-1)+2.$$

The first inequality follows from the fact that $E_i \cdot C \geq 0$ and equality holds if and only if C does not intersect E_i, or equivalently, $f(C)$ is contained in the smooth locus of Y. □

The importance of these very free curves is clear from the following observation.

Proposition 4.3. *Let Y be $\mathbb{Q}$-Fano 3-fold and let $f : X \to Y$ be a resolution of singularities. Assume that there is a very free curve in the smooth locus of Y whose normal bundle is $\mathcal{O}(a) \oplus \mathcal{O}(a), a \geq 1$. Then X is symplectic rationally connected.*

Proof. Let C be such a very free curve in the smooth locus of Y. We can move C away from the locus where f is not an isomorphism. So we get a very free curve C in X with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a)$, for $a \geq 1$.

We choose the constraints to be $a+1$ general points. Any irreducible curve in X through k general points has $-f^*K_Y$ degree at least $2k$ by Lemma 4.2 and equality holds if and only if its image in Y is contained in the locus where X and Y are isomorphic, which is contained in the smooth locus of Y. Note that $-f^*K_Y$ is nef and the $-f^*K_Y$ degree of C is $2a$. This forces every irreducible component passing through one of the general points to lie in the locus where X and Y are isomorphic. Then no components can be contracted by the map $X \to Y$. Thus every irreducible component contains one of the chosen point and is free. Now the proposition follows immediately. □

Therefore the problem is reduced to the existence of such curves. Fortunately there is a way to show the existence in dimension 3.

Definition 4.4. Let $C_i \subset X_i$ be a curve on a variety X_i, $i = 1, 2$. We say (X_1, C_1) is *equivalent to* (X_2, C_2) if there is an open neighborhood V_i of C_i in X_i and an isomorphism $f : V_1 \to V_2$ such that $f|_{C_1} : C_1 \to C_2$ is also an isomorphism.

Theorem 4.5. *Let Y be a $\mathbb{Q}$-Fano 3-fold. Assume the smooth locus of Y is rationally connected. Then there is a very free curve in the smooth locus with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a), a \geq 1$.*

We start with the following construction in [She10] for smooth rationally connected 3-folds.

Let $f : X \to Y$ be a strong resolution of singularities which is isomorphic over the smooth locus of X. Let C be a very free curve in the smooth locus and general in an irreducible component of the moduli space of very free curves. We may assume that $-K_Y \cdot C$ is an even number (otherwise take a two-fold cover and a general deformation). Assume the normal bundle of C is $\mathcal{O}(a+2b) \oplus \mathcal{O}(a), a, b \geq 1$. In the following we will not distinguish between $-K_Y$ and $-f^*K_Y$.

Choose $a+1$ points in C and deform C with these points fixed. Then the deformation of C sweeps out a surface Σ in X. Let Σ' be the normalization and $\tilde{\Sigma}$ be the minimal resolution of Σ'.

The following results are proved in Section 2.2, 2.3 of [She10].

Proposition 4.6 ([She10])**.** *Keep the same notation as above.*

1. *Σ is independent of the choice of the points. Σ' is smooth along C and $N_{C/\Sigma'} \cong \mathcal{O}(a+2b)$.*

2. *There is a neighborhood U of C in $\tilde{\Sigma}$ such that the map $\phi : \tilde{\Sigma} \to X$ has injective tangent map. And the normal sheaf $N_{\tilde{\Sigma}/X}$ is locally free along C and $N_{\tilde{\Sigma}/X}|_C \cong \mathcal{O}(a)$.*

3. *The pair $(\tilde{\Sigma}, C)$ is equivalent to $(\mathbb{P}^2, conic)$ or $(\mathbb{F}_n, \sigma)$, where $\mathbb{F}_n$ is the n-th Hirzebruch surface and σ is a section of $\mathbb{F}_n \to \mathbb{P}^1$.*

4. *If the pair is equivalent to $(\mathbb{F}_n, \sigma)$, where σ is a section of $\mathbb{F}_n \to \mathbb{P}^1$, then there is a (reducible) curve D in $\tilde{\Sigma}$ such that $D^2 = -n$ and $D \cdot F = 1$, where F is a fiber of $\mathbb{F}_n \to \mathbb{P}^1$. If $C \cdot D > 0$, then a general fiber F lies in U. And the sheaf $N_{F/X}$ cannot be $\mathcal{O} \oplus \mathcal{O}(1)$.*

Our goal is to start with this curve C and produce another very free curve with balanced normal bundle. The case where the pair $(\tilde{\Sigma}, C)$ is equivalent to $(\mathbb{P}^2, conic)$ is straightforward.

Lemma 4.7. *If the pair $(\tilde{\Sigma}, C)$ is equivalent to $(\mathbb{P}^2, conic)$, then there is a very free rational curve C in the smooth locus of Y such that $N_{C/Y} \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$.*

Proof. In this case, $N_{C/X} \cong \mathcal{O}(4) \oplus \mathcal{O}(2)$. So $-K_X \cdot C = -K_Y \cdot C = 8$. C may degenerate in $\tilde{\Sigma}$ into two "lines" $C_i, i = 1, 2$ which can pass through 2 very general points in $\tilde{\Sigma}$, thus 2 general points in X. Hence they are very free with normal bundle (in X) $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Note that a general such "line" is necessarily contained in the smooth locus since the intersection number with $-K_X$ is 4, the same as the intersection number with $-K_Y$. □

Now assume that the pair is equivalent to $(\mathbb{F}_n, \sigma)$, where σ is a section. Let D in $\tilde{\Sigma}$ be the (reducible) curve such that $D^2 = -n$ and $D \cdot F = 1$ as in Proposition 4.6. Then $C = D + cF$.

Lemma 4.8. *Keep notations as above. Then*

$$n \leq c \leq a + b, c - n = a + 2b - c \geq b.$$

Proof. First note that $0 \leq C \cdot D = c - n$.

We know that $N_{C/\tilde{\Sigma}} \cong \mathcal{O}(a + 2b)$ and $N_{\tilde{\Sigma}/X}|_C \cong \mathcal{O}(a)$. Thus,

$$C \cdot C = 2c - n = a + 2b,$$

$$c(-K_Y \cdot F) = 2 + 2a + 2b - (-K_Y \cdot D) \leq 2 + 2a + 2b.$$

Since a general fiber F is free, $-K_Y \cdot F \geq -K_X \cdot F \geq 2$. Thus,

$$n \leq c \leq 1 + a + b.$$

If $c = 1 + a + b$, then

$$-K_Y \cdot F = 2, -K_Y \cdot D = 0.$$

The first equality implies that a general fiber F is mapped to a free curve in the smooth locus. The second one implies that D is either mapped to a point in X or into the exceptional divisors of $f : X \to Y$. But F and D intersect in $\tilde{\Sigma}$. So do their images in Y. Thus, D is contracted to a point in the smooth locus of Y. Then there are two free curves (i.e. images of two general fibers) meet in the smooth locus of Y, each having $-K_Y$ degree 2 and passing through a very general point. But if we choose the two points to be general enough, any two irreducible curve of $-K_Y$ degree 2 through these points cannot not meet each other. Therefore

$$c \leq a + b$$

and thus

$$c - n = a + 2b - c \geq b.$$

□

Lemma 4.9. *Keep notation as above. Then $-K_X \cdot F = 2$ and $N_{F/X} = \mathcal{O} \oplus \mathcal{O}$ for a general fiber F.*

Proof. Note that

$$-K_X \cdot F \leq -K_Y \cdot F \leq \frac{2+2a+2b}{c} = \frac{2(2+2a+2b)}{a+2b+n} \leq \frac{4(a+b+1)}{a+b+1} = 4.$$

Thus, $-K_X \cdot F$ is at most 4. On the other hand, it is at least 2 since a general fiber F passes through a very general point.

If $-K_X \cdot F = 4$, then every inequality above is an equality. Thus,

$$n = 0, b = 1, -K_Y \cdot F = -K_X \cdot F, -K_Y \cdot D = 0.$$

So F is a free curve in the smooth locus of Y and D is contracted to a point in the smooth locus. Furthermore, the pair $(\tilde{\Sigma}, \sigma)$ is equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$ with one ruling. Then D is a moving curve in $\tilde{\Sigma}$ and $-K_Y \cdot D > 0$ since some deformation of D is not contracted. This is a contradiction.

Since $C \cdot D > 0$, a general fiber F is contained in U and the normal bundle $N_{F/X}$ is not $\mathcal{O} \oplus \mathcal{O}(1)$ by Proposition 4.6. Thus $-K_X \cdot F$ is not 3. □

Now we are ready to finish the proof of Theorem 4.5.

Proof. It remains to consider the case where the pair $(\tilde{\Sigma}, C)$ is equivalent to $(\mathbb{F}_n, \sigma)$. By Lemma 4.8, $c - n \geq b$. Thus C specializes in $\tilde{\Sigma}$ to the union of a section C' whose curve class is $D + (c-b)F$ and b general fibers. Also note that C' passes through $a+1$ general points in $\tilde{\Sigma}$ since its normal bundle in $\tilde{\Sigma}$ is $\mathcal{O}(a)$. Since C passes through $a+1$ general points in X, the same is true for C'. Also notice that

$$-K_Y \cdot C' = -K_Y \cdot C - b(-K_Y) \cdot F \leq -K_Y \cdot C - b(-K_X) \cdot F = 2 + 2a.$$

For the last equality, we use Lemma 4.9. Then the equality has to hold by Lemma 4.2 and C' is a very free curve in the smooth locus with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a)$. □

Corollary 4.10. *On every smooth Fano 3-fold, there is an embedded very free curve with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a), a \geq 1$.*

Combining Proposition 4.3 and Corollary 4.10, we get a new proof that every smooth Fano 3-fold is symplectic rationally connected.

In general, it is not an easy task to determine if the smooth locus of a $\mathbb{Q}$-Fano variety is rationally connected. By the following Lemma, this is true for a large class of $\mathbb{Q}$-Fano varieties we are interested in.

Lemma 4.11. *Let Y be a Gorenstein $\mathbb{Q}$-Fano 3-fold. Then the smooth locus of Y is rationally connected.*

Proof. By a result of Namikawa [Nam97], there is a smoothing of Y, $\pi : \mathcal{Y} \to S$ such that a general fiber is a smooth Fano 3-fold and the central fiber $\mathcal{X}_0$ is X.

By Corollary 4.10, there is a very free curve D in a general fiber whose relative normal bundle is $\mathcal{O}(a) \oplus \mathcal{O}(a)$, for $a \geq 1$. So this curve can pass through $a+1$ general points in a general fiber. Choose $a+1$ general points in Y. We can find $a+1$ sections of $\mathcal{Y} \to S$ passing through these points in Y, possibly after a base change. Then consider the specialization of the curve D passing through these sections in the relative Kontsevich moduli space. We get a stable map to Y whose image contains the chosen $a+1$ general points. But as observed in the proof of Proposition 4.3, the domain of this stable map is irreducible and its image is contained in the smooth locus. □

Remark 4.12. A 3-fold Gorenstein terminal singularity is an isolated hypersurface singularity, in particular, a locally complete intersection singularity. Then Lemma 4.11 can be proved easily by comparing the deformation space of a very free curve in a resolution and that of its image in Y. But the proof presented here gives more information about minimal very free curves in the smooth locus. For example, we know that their anticanonical degree is no more than 8 by looking at the construction of low degree very free curves in the proof of Theorem 3.6.

4.2 Del Pezzo fibrations

Let Y be a normal projective threefold with at worst terminal singularities. And let $\pi : Y \to \mathbb{P}^1$ be a contraction of some K_X-negative extremal face. Then a general fiber of f is a smooth Del Pezzo surface. Let $f : X \to Y$ be a resolution of singularities that is isomorphic near a general fiber. Note that all the exceptional divisors are supported in special fibers of $\pi \circ f : X \to Y \to \mathbb{P}^1$. The main result in this section is the following.

Theorem 4.13. *There is a non-zero Gromov-Witten invariant $\langle [pt], [pt], \ldots \rangle^X_{0,\beta}$ for some class β which is a section of the fibration $X \to \mathbb{P}^1$.*

Proof. We proceed in two steps. First we construct a section satisfying certain properties and then show that this section give rise to the non-zero Gromov-Witten invariant.

Step 1. Construction

By Theorem 1.1 in [GHS03], there exists a section of $X \to \mathbb{P}^1$. By definition of terminal singularities, we have $K_X \sim_{\mathbb{Q}} f^*K_Y + \sum a_i E_i, a_i > 0$.

Let

$$d_E = \min\{e \mid (\sum a_i E_i) \cdot s = e, s \text{ is a section}\}.$$

Once we have a section, we can attach very free curves in general fibers to it and deform the reducible curve to get a section which is a free curve. This operation will not change the intersection numbers with the exceptional divisors as long as the very free curves are disjoint from the exceptional divisors. So there is a free section s such that $(\sum a_i E_i) \cdot s = d_E$.

Define

$$B_1 = \min\{b \geq 0 : s \text{ is a section}, s \cdot (\sum a_i E_i) = d_E, \\ N_{s/X} \cong \mathcal{O}(a) \oplus \mathcal{O}(a+b), a, b \geq 0\}.$$

A general fiber of $X \to \mathbb{P}^1$ is a Del Pezzo surface. So it is either $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ or the blow-up of $\mathbb{P}^2$ at $d(1 \leq d \leq 8)$ general points.

Proposition 4.14. *If a general fiber is not* $\mathbb{P}^1 \times \mathbb{P}^1$*, then there is a section whose normal bundle is* $\mathcal{O}(a) \oplus \mathcal{O}(a)$ *with* a *arbitrarily large.*

If a general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$*, and* $B_1 > 0$*, then there is a very free section whose normal bundle is* $\mathcal{O}(a) \oplus \mathcal{O}(a + B_1)$ *with* a *arbitrarily large.*

If a general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$*, and* $B_1 = 0$*, then there is a very free section whose normal bundle is* $\mathcal{O}(a) \oplus \mathcal{O}(a + b)$ *with* a *arbitrarily large and* b *is at most* 2*.*

Proof. In the first case, take a section s with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a+b)$ for some $a \geq 0, b > 0$. We attach (the strict transform of) a line L in a general fiber to s along a general direction. Let $\mathcal{N}$ be the normal sheaf of this reducible curve in X. Choose a point p in the line L and a divisor $D = q_1 + \ldots + q_{a+2}$ in s. Let $\mathcal{E} = \mathcal{N}(-p - D)$. Then

$$\mathcal{E}|_L \cong \mathcal{O} \oplus \mathcal{O}, \mathcal{E}|_s \cong \mathcal{O}(-1) \oplus \mathcal{O}(b-2).$$

We have the short exact sequence of sheaves

$$0 \to \mathcal{E}|_L(-n) \to \mathcal{E} \to \mathcal{E}|_s \to 0,$$

where n is the node of $L \cup s$. Thus, $H^1(\mathcal{E}) = 0$. The same is true for a general deformation of $s \cup L$ by semi-continuity. Thus a general deformation is again a free section s' with $N_{s'/X} \cong \mathcal{O}(a') \oplus \mathcal{O}(a'+b'), a' \geq a+2, b' < b$. Continuing with this process, we obtain free sections whose normal bundle is $\mathcal{O}(a) \oplus \mathcal{O}(a)$ with a arbitrarily large.

When a general fiber is $\mathbb{P}^1 \times \mathbb{P}^1$, we run a similar argument as above. We start with a free section s whose normal bundle is $\mathcal{O}(a) \oplus \mathcal{O}(a+B_1)$ and attach curves of bi-degree $(1,0)$ or $(0,1)$ in a general fiber to s along a general normal direction. The tangent directions of these two types of curves at a point span the tangent space of $\mathbb{P}^1 \times \mathbb{P}^1$ at that point. So the above analysis of the normal bundle is still valid provided that we choose the appropriate type of curve. If $B_1 > 0$, in the end we can find very free sections whose normal bundle is $\mathcal{O}(a) \oplus \mathcal{O}(a+B_1)$ with a arbitrarily large. However, if $B_1 = 0$, we can only guarantee a very free curve with normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(a+b)$ with $b \leq 2$. ☐

In every case, there is a number B_2 such that there is a very free section s whose normal bundle is $\mathcal{O}(a) \oplus \mathcal{O}(a+b)$ with a arbitrarily large and b bounded above by B_2.

There exist a positive rational number ϵ such that $\pi^*\mathcal{O}(1) - \epsilon K_Y$ is ample on Y. So there are positive rational numbers b_i such that

$$\begin{aligned} H =& f^*(\pi^*\mathcal{O}(1) - \epsilon K_Y) - \sum b_i E_i \\ =& f^*\pi^*\mathcal{O}(1) - \epsilon K_X + \sum (\epsilon a_i - b_i) E_i \end{aligned}$$

is ample on X.

Let $\tilde{s}$ be a section such that $(-K_X) \cdot \tilde{s}$ is at most B_2. Then

$$H \cdot \tilde{s} \leq 1 + \epsilon B_2 + \sum |\epsilon a_i - b_i|$$

since $E_i \cdot \tilde{s}$ is either 0 or 1. So there are only finitely many such curve classes. Then there is an integer M such that any such section meets at most M general free rational curves with $-K_X$ degree 2 and there are only finitely many such sections. We also have a lower bound N of $-K_X \cdot \tilde{s}$ for all such sections.

The above discussion shows that there is a very free section s such that

$$(\sum a_i E_i) \cdot s = d_E, \tag{1}$$

$$N_{s/X} \cong \mathcal{O}(a) \oplus \mathcal{O}(a+b), a > 0, b \geq 0, \tag{2}$$

$$2M + 3(a+1-M) + N > 2 + 2a + B_2. \tag{3}$$

$$b \leq B_2. \tag{4}$$

Define

$$B = \min\{b \mid \text{ there is a section which satisfies (1), (2) and (3) above}\}.$$

Clearly $B \leq B_2$.

Denote by s a section of minimal degree with respect to some polarization in the set of all the very free sections with the above properties and whose normal bundle is of the form $\mathcal{O}(a) \oplus \mathcal{O}(a+B)$.

Step 2. Analyzing reducible curves in the curve class $[s]$.

We will prove that this curve class $[s]$ gives a non-zero Gromov-Witten invariant with two point insertions.

A general such section passes through $a+1$ general points. So we add $a+1$ point constraints. If $B > 0$, then we need to add B curve constraints. We take these curves to be general curves in a general fiber which is also an ample divisor in the fiber. They all lie in general fibers not containing any exceptional divisors. Then a general section will meet all of these constraints and contribute positively to the Gromov-Witten invariant

$$\langle \underbrace{[pt], \ldots, [pt]}_{a+1}, \underbrace{[\text{Curve}], \ldots, [\text{Curve}]}_{B} \rangle^X_{0,s}.$$

Now we show that no reducible curve in this curve class meets all of the constraints. Write the reducible curve as $C \cup C_e \cup C_g$, where C is a section, C_e the vertical components supported in the fibers containing exceptional divisors, and C_g all the other vertical components.

Since $E_i \cdot C_g = 0$,

$$\begin{aligned} -K_X \cdot C_e &= (-f^*K_Y - \sum a_i E_i) \cdot C_e \\ &= -f^*K_Y \cdot C_e - (\sum a_i E_i) \cdot (s - C - C_g) \\ &= -f^*K_Y \cdot C_e - (\sum a_i E_i) \cdot (s - C) \geq 0, \end{aligned}$$

with equality if and only if $(\sum a_i E_i) \cdot C = d_E$ and every irreducible component of C_e is mapped to a point in Y. There are three different cases according to what kind of curve C is.

Case 1. The section C is a free curve.

Suppose C meets a' general points and $b' (0 \leq b' \leq B)$ general curves in the fiber. Then

$$-K_X \cdot C = \dim \overline{\mathcal{M}}(X, [C]) \geq 2a' + b'.$$

We also have

$$-K_X \cdot C_g \geq 2(a+1-a') + (B-b')$$

and

$$-K_X \cdot C_e \geq 0.$$

Adding all of these together we get

$$-K_X \cdot (C + C_g + C_e) \geq 2a + 2 + b' + (B - b') = 2a + 2 + B.$$

So equality has to hold in each inequality. This implies that $(\sum a_i E_i) \cdot C = A$ and C_g is the union of $(a+1-a')$ free curves with $-K_X$ degree 2 and $(B-b')$ curves of $-K_X$ degree 1. Then the reducible curve consisting of C and $(a+1-a')$ free curves in C_g deforms to an irreducible section curve $\tilde{C}$, which passes through all $a+1$ general points.

Therefore the normal bundle of the new section curve $\tilde{C}$ is $\mathcal{O}(a'') \oplus (a''+b'')$, for some $a'' \geq a$. Since $(\sum a_i E_i) \cdot \tilde{C} = A$, $\tilde{C}$ satisfies (1), (2) and (3) above. So $b'' \geq B$. But we also have the reverse inequality, since

$$2a + B + 2 = -K_X \cdot C \geq -K_X \cdot \tilde{C} = 2a'' + b'' + 2.$$

Hence equality holds and $b' = b'' = B$. Then by the minimality of $[s]$, $C_e = \emptyset$. Thus, $C \cup C_e \cup C_g = C \cup C_g$ is in the boundary of an irreducible component of expected dimension. So we can choose the constraints to miss such configurations.

Case 2. The section C is not a free curve and $-K_X \cdot C > B$.

We may choose the $a+1$ points to lie in different general fibers and any irreducible curve through them is free. Then neither C or C_e passes through any of them. So

$$-K_X \cdot C_g \geq 2(a+1), \quad -K_X \cdot (C_e + C_g + C) > 0 + 2a + 2 + B.$$

This is impossible.

Case 3. The section C is not free and $-K_X \cdot C \leq B$.

Again C does not meet any point constraints. So C_g has at least $a+1$ curves D_i in different fibers and $-K_X \cdot D_i \geq 2$. If $-K_X \cdot D_i = 2$, D_i is an irreducible free curve. There are at most M such curves and only finitely many sections meet all of these curves. So if we choose the other points to be general, then every curve through those points with $-K_X$ degree 2 will not meet these sections. Thus, for all the other D_i's (which are possibly reducible),

$$-K_X \cdot D_i \geq 3.$$

But then

$$\begin{aligned}-K_X\cdot(C_e+C_g+C) &\geq 0+2M+3(a+1-M)+N\\ &> 2+2a+B_2\geq 2+2a+B=-K_Y\cdot s.\end{aligned}$$

This is impossible. □

4.3 Conic bundles

In this section we prove the following theorem.

Theorem 4.15. *Let $X\to\Sigma$ be a surjective morphism from a smooth projective rationally connected 3-fold X to a smooth projective surface Σ such that a general fiber is isomorphic to $\mathbb{P}^1$. Then X is symplectic rationally connected. There is also a non-zero Gromov-Witten invariant of the form $\langle [C],\dots,[C],[A]^2,\dots,[A]^2\rangle^X_{0,\beta}$, where $[C]$ is the class of a general fiber and $[A]$ is the class of a very ample divisor.*

As a preparation, we prove the following proposition. There are easier ways to prove the result. But here we present a proof which only depends on MMP on surfaces and requires no further knowledge about the classification of rational surfaces. This proof actually motivates the results in Section 4.2.

Proposition 4.16. *Let Σ be a rationally connected surface. Then there is a non-zero Gromov-Witten invariant of the form $\langle [pt],[pt],\dots,[pt]\rangle^\Sigma_{0,\beta}$.*

Proof. We can run the MMP for Σ. Then we have a sequence of contractions of (-1)-curves:

$$\Sigma = X_0\to X_1\to\ldots\to X_n,$$

where X_n is either a geometrically ruled surface over $\mathbb{P}^1$ or a Fano surface of Picard number 1 (thus is $\mathbb{P}^2$, but we do not need this).

In the former case, we may choose a section s_0 of the ruled surface and take the curve class to be s_0+kF, where F is a fiber class. If we take k large enough, a general curve in this class is an embedded very free curve passing through $m(\geq 3)$ general points. Now one can use a similar argument as in Step 2 of the proof of Theorem 4.13 to show that this section curve gives a non-zero enumerative Gromov-Witten invariant on X_n.

In the latter case, first choose a minimal free rational curve. If it is already very free, then we are done (here we pretend that we do not know X_n is $\mathbb{P}^2$). If not, then notice that we can take the union of two such general curves and a general deformation is a very free curve of $-K_X$ degree 4. It is easy to see that we get a non-zero enumerative Gromov-Witten invariant in this case using a similar but much easier argument as the proof of Theorem 3.1 for the case of smooth Fano 3-folds.

Then the proposition follows by comparing Gromov-Witten invariants on X and X_n as in Corollary 3.9. □

The proof of Theorem 4.15 is similar to the proof of Theorem 2.4 in [Voi08]. We only point out the necessary changes.

Proof of Theorem 4.15. By Proposition 4.16, there is a non-zero enumerative Gromov-Witten invariant of the form

$$\langle \underbrace{[pt], [pt], \ldots, [pt]}_{r\ [pt]} \rangle^{\Sigma}_{0,\beta}.$$

We can choose $[\beta]$ such that a general curve of class $[\beta]$ is an embedded curve. Here we need to know that a Del Pezzo surface of Picard number one is $\mathbb{P}^2$. The curve class corresponds to a free linear system. So we may choose the constraints in Σ to be general such that if Γ is the curve through these points, then $Z = \pi^{-1}(\Gamma)$ is a smooth surface.

Let s_0 be a section of $Z \to \Gamma$. Choose k large enough. Then it is easy to see that any curve in X in the class $s_0 + kC$ which meets 2 general points and $r-2$ general fibers (or r general fiber C) has to be mapped to an irreducible curve in Σ through r general points. Thus the curve lies in Z. We may take other constraints to be curves meeting the surface Z at finitely many points. There are some positive contributions to the Gromov-Witten invariant coming from these irreducible section curves.

The issure here is that the map $i_* : H_2(Z, \mathbb{Z}) \to H_2(X, \mathbb{Z})$ is not injective. So we have to consider the contributions coming from curves classes in Z whose image under i_* is also $s_0 + kC$, but are different as a curve class in $H_2(Z, \mathbb{Z})$. This has been done in Lemma 2.8 and Lemma 2.9 of [Voi08]. By deforming Z to be the blow-up of some Hirzebruch surface at *distinct* points in *distinct* fibers, it is shown there that all such contributions are non-negative. □

Remark 4.17. We need this theorem in three situations, in the proof of Theorem 1.7, the Picard number 2 case of Theorem 1.9, and the conic bundle case of Theorem 1.10. For our purpose, we can avoid doing such a deformation as in [Voi08].

In the first and third case, we are allowed to make a birational modification. And we can use relative MMP to get a "better" birational model in the following way. In the first case, first run a relative MMP to get a conic bundle over the base (which is smooth). In the third case, first take a resolution of singularities of the base, then a base change and a resolution of singularities of the total space, and finally a relative MMP over the base. Thus we are in the following situation: Y is a rationally connected 3-fold with terminal singularities and $Y \to \Sigma$ is a conic bundle coming from the contraction of a K_Y-negative extremal ray and Σ is smooth. The 3-fold Y has only isolated singularities. Take a strong resolution of singularities $X \to Y$ which is isomorphic over the smooth locus of Y. The exceptional divisors of $X \to Y$ are mapped to isolated points in Σ. Then Z is the blow-up of some Hirzebruch surface at distinct point in distinct fibers.

In the second case X is a smooth 3-fold and $X \to \Sigma$ is a contraction of type (C), then Σ is smooth. In this case Z is also the blow-up of some Hirzebruch surface at *distinct* point.

4.4 Rationally connected 3-folds with Picard number 2

In this section we prove the following theorem.

Theorem 4.18. *Let X be a smooth projective rationally connected 3-fold X with Picard number 2. Then X is symplectic rationally connected.*

There is at least one K_X-negative extremal ray of X since X is rationally connected. Let $f : X \to Y$ be the corresponding contraction. Then the case of contractions of type $(E1)$ and $(E2)$ are coved by Corollary 3.9, of type $(E3)$ and $(E4)$ by Theorem 4.5, Proposition 4.3, and Lemma 4.11, of type (C) by Theorem 4.15, and of type (D) by Theorem 4.13. The only remaining case is the $(E5)$ contraction, where the exceptional divisor E is a smooth $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-2)$. In this case the variety Y is a non-Gorenstein $\mathbb{Q}$-Fano 3-fold of Picard number 1.

We introduce some notation first. Let $A = -f^*K_Y - \epsilon E$, with $0 < \epsilon \ll 1$ be an ample $\mathbb{Q}$-divisor. Let C be a minimal free curve first with respect to $-f^*K_Y$ and then with respect to A. By abuse of notation we write K_Y instead of f^*K_Y below.

If $E \cdot C = 0$, then C give rise to a free curve in the smooth locus of Y. Thus, the smooth locus of Y is rationally connected and the result follows from Proposition 4.3 and Theorem 4.5.

In the following we assume $E \cdot C > 0$. First we have the following observation.

Lemma 4.19. *If $E \cdot C > 0$, then $-K_X \cdot C = 2$.*

Proof. If $-K_X \cdot C \geq 3$, then we can deform such a curve with one point fixed. The deformation of the curve C in X also gives a deformation of its image in Y. Since $E \cdot C \geq 1$, the deformation in Y fixes the chosen point and the unique singular point of Y. So by bend-and-break, we get a curve through the fixed point with smaller $-K_Y$ degree. If the fixed point is chosen to be a very general point, then this curve with smaller $-K_Y$ degree gives a free curve in X. This is a contradiction to our choice of C. □

Take a reducible curve Γ to be a union of two such free curves passing through two very general points and a line in $E \cong \mathbb{P}^2$. We can smooth the nodes of Γ and get a very free curve Γ' with $-K_X \cdot \Gamma' = 5$.

Lemma 4.20. *Let D be a reducible curve passing through 2 general points such that the two points are not connected by an irreducible component of the curve D. If $-K_Y \cdot D \leq -K_Y \cdot \Gamma'$ and $A \cdot D \leq A \cdot \Gamma'$, then one of the following holds:*

1. *the curve D consists of 3 irreducible components, two of which are the free curves of class $[C]$ and one of which is a line in the exceptional divisor E;*
2. *the curve D consists of 2 irreducible components, both of which are free curves with $-K_X$ degree 2 and 3.*

Furthermore, $[D] = [\Gamma']$.

Proof. Write $D = D_1 \cup D_2 \cup D_3$, where D_1 and D_2 are the irreducible components through the two general points and D_3 (possibly empty) is the rest of the irreducible components. Then

$$-K_Y \cdot (D_1 + D_2) \leq -K_Y \cdot D \leq -K_Y \cdot \Gamma' = 2(-K_Y \cdot C).$$

We also have the reverse inequality by our choice of C. Thus

$$-K_Y \cdot D_1 = -K_Y \cdot D_2 = -K_Y \cdot C$$

and all the irreducible components of D_3 are supported in the exceptional divisor E.

Write $D_1 = C + \lambda_1 L$ and $D_2 = C + \lambda_2 L$, where L is the class of a line in the exceptional divisor E. Note that λ_1 and λ_2 are non-negative rational numbers. Since $-K_X \cdot L = 1$, they are actually integers. Since $A \cdot D \leq A \cdot \Gamma'$, $\lambda_1 + \lambda_2 \leq 1$.

If $\lambda_1 + \lambda_2 = 0$, then D_1 and D_2 have the same curve class as C and we can choose the two points so that D_1 and D_2 do not intersect each other. In this case it is easy to see that D_3 is a line in the exceptional divisor E.

If $\lambda_1 + \lambda_2 = 1$, then D_1 and D_2 are the only components of D.

In both cases, $-K_Y \cdot D = -K_Y \cdot \Gamma'$ and $A \cdot D = A \cdot \Gamma'$. Since X has Picard number 2, $[D] = [\Gamma']$. □

Proposition 4.21. *If there is a very free irreducible curve C' such that $-K_X \cdot C' = 4$, $-K_Y \cdot C' \leq -K_Y \cdot \Gamma'$ and $A \cdot C' \leq A \cdot \Gamma'$, then there is a non-zero Gromov-Witten invariant of the form $\langle [pt], [pt] \rangle^X_{0,D}$.*

Proof. Choose a very free curve D whose $-f^*K_Y$ degree is minimal among all very free curves C' satisfying the conditions

$$-K_X \cdot C' = 4, -K_Y \cdot C' \leq -K_Y \cdot \Gamma', A \cdot C' \leq A \cdot \Gamma'.$$

Any curve satisfying these conditions and having minimal A-degree has the same curve class as D, since $b_2(X) = 2$. We show that $\langle [pt], [pt] \rangle^X_{0,D} \neq 0$.

First notice that there exists no reducible curve F passing through 2 general points such that

(1) the two points are not connected by an irreducible component of F,

(2) $-K_Y \cdot F \leq -K_Y \cdot D$, and

(3) $A \cdot F \leq A \cdot D$.

If there were such a curve, we could apply Lemma 4.20 since $-K_Y \cdot F \leq -K_Y \cdot \Gamma'$ and $A \cdot F \leq A \cdot \Gamma'$. Then the curve classes $[F]$, $[D]$, and $[\Gamma']$ would all be the same. This is impossible since $-K_X \cdot D = 4$ but $-K_X \cdot \Gamma' = 5$.

We claim that if F is an irreducible curve through the two points satisfying (2) and (3) above, then $[F] = [D]$. If $-K_X \cdot F = 4$, then this follows from our choice of D. If $-K_X \cdot F \geq 5$, then we may deform this curve with two points fixed to a reducible curve. By the previous paragraph, there is again an irreducible curve F_1 passing through 2 general points; it still satisfies (2) and (3) above. If $-K_X \cdot F_1 \geq 5$, deform F_1 with two points fixed. This process will terminate at some point, and we get an irreducible curve F_n passing through the two very general points (thus very free) such that $-K_X \cdot F_n = 4$.

Then,

$$-K_Y \cdot F_n \leq -K_Y \cdot D \leq -K_Y \cdot F_n.$$

Here we get the first inequality by the construction of F_n and the second inequality by the choice of D. Thus, $F_n = D$. This is a contradiction since by construction $A \cdot F_n < A \cdot D$.

The above discussion shows that no reducible curve of class $[D]$ meets all of the constraints. Thus, we are done. □

Next assume that there is no irreducible curve C' such that

$$-K_X \cdot C' = 4, -K_Y \cdot C' \leq -K_Y \cdot \Gamma', \text{and } A \cdot C' \leq A \cdot \Gamma'.$$

Choose a very free curve D with $-K_X$ degree 5 and minimal with respect to A. Choose the constraints to be two very general points and a moving curve G which meets very divisor but E (e.g. the strict transform of the intersection of two very ample divisors in Y).

The following lemma will conclude the proof of Theorem 4.18.

Lemma 4.22. *The Gromov-Witten invariant* $\langle [pt], [pt], G \rangle^X_{0,D}$ *is non-zero.*

Proof. The proof is similar to that of Proposition 4.21.

First we claim that there is no irreducible curve D' connecting the two general points with smaller A-degree and $-K_Y$-degree. If D' were such a curve, then $-K_X \cdot D'$ would be at least 5 by assumption. So we could deform D' with the points fixed and break D' into a reducible or non-reduced curve. Continue this process until we end up with a reducible/non-reduced curve with no irreducible components containing the two chosen points. Thus the curve has two components which are free curves and each passes through one chosen point. Then Lemma 4.20 gives a contradiction.

Thus, we only need to consider the case where there are two irreducible components, which are free curves passing through the two chosen general points. We are then in the situation of Lemma 4.20. Clearly we can choose G to avoid all such configuration of curves. □

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