Axioms for Differential Cohomology

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Abstract of the Dissertation

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A differential cohomology theory is a type of extension of a cohomology theory E^* restricted to smooth manifolds that encodes information that is not homotopy invariant. In particular, it takes values in graded abelian groups, and is equipped with natural transformations to both E^* and closed differential forms with values in the graded vector space $V = E^*(\text{point}) \otimes \mathbb{R}$. Differential cohomology theories for certain choices of an underlying cohomolgy theory have been conjectured by Freed to be the proper home for certain types of quantized B-fields in superstring theory.

In the case of ordinary integral cohomology, Simons and Sullivan showed they all were naturally isomorphic via a unique isomorphism. Bunke and Schick, under the assumptions that E^* is countably generated in each degree and rationally even (*i. e.*, $E^{2k+1}(\text{point}) \otimes \mathbb{Q} = 0$), arrive at the same result only when they also require the differential cohomology theories each have a degree -1 “integration” natural transformation that is compatible with the integration along the fiber map for forms and the suspension isomorphism for E^*. We also construct such a natural isomorphism, and our only requirement of the cohomology theory is that it be finitely generated in each degree. However, we also require that the differential cohomology theory be defined on a particular type of

category that is larger than the historical domain of the category of smooth manifolds with corners.

To my parents.

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Chapter 1

Preliminaries

1.1 Introduction

1.1.1 History and Motivations

The original notion of a differential cohomology theory is that of a contravariant functor from the category of smooth manifolds to graded abelian groups that refines the ordinary integral cohomology H^* of a manifold. Every element of the differential cohomology of a manifold M naturally determines both an element of $H^*(M)$ and a closed differential form whose de Rham class is the real reduction of the aforementioned integral cohomology class. The first example of such functor was the abelian group of differential characters $\hat{H}^*$ introduced by Cheeger and Simons in [CS85], defined as

$$\hat{H}^k(M) := \left\{ \varphi \in \mathrm{Hom}(Z_{k-1}(M),\, \mathbb{R}/\mathbb{Z}) \;\middle|\; \exists\, \omega \in \Omega^k(M) \text{ s. t. } f \circ \partial = \left[\int_{(\cdot)} \omega \right]_{\mathbb{Z}} \right\}.$$

This functor fits into a particular diagram of natural transformations:

$$\begin{tikzcd}
 & 0 \arrow[dr] & & & & 0 & \\
\ddots \arrow[dr] & & H^{*-1}(-;\,\mathbb{R}/\mathbb{Z}) \arrow[rr] \arrow[dr, dashed] & & H^*(-;\,\mathbb{Z}) \arrow[ur] \arrow[dr] & & \iddots \\
 & H^{*-1}(-;\,\mathbb{R}) \arrow[ur] \arrow[dr] & & \hat{H}^* \arrow[ur, dashed] \arrow[dr, dashed] & & H^*(-;\,\mathbb{R}) \arrow[ur] \arrow[dr] & \\
\iddots \arrow[ur] & & \frac{\Omega^{*-1}}{\Omega^{*-1}_{\mathbb{Z}}} \arrow[rr] \arrow[ur, dashed] & & \Omega^*_{\mathbb{Z}} \arrow[ur] \arrow[dr] & & \ddots \\
 & 0 \arrow[ur] & & & & 0 &
\end{tikzcd} \tag{1.1}$$

The coefficient short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0.$$

induces a long exact sequence in ordinary cohomology, which is the sequence running along the top of this diagram. Here, Ω^* are the de Rham differential forms and $\Omega^*_{\mathbb{Z}}$ are the closed forms with integral periods. The arrows along the bottom are from the long exact sequence in homology that results from the short exact sequence

$$0 \to \Omega^*_{\mathbb{Z}} \to \Omega^* \to \Omega^*/\Omega^*_{\mathbb{Z}} \to 0$$

of chain complexes (where $\Omega^*_{\mathbb{Z}}$ and $\Omega^*/\Omega^*_{\mathbb{Z}}$ are given the trivial differentials). The four natural transformations indicated with dashed arrows make the diagonal sequences exact.

A host of other examples of functors that also fit into this diagram were defined, and they were all eventually shown to be isomorphic [GS89, Har89, HLZ03, HL06]. This led to Simons and Sullivan showing in [SS08] that it is indeed the case that any functor equipped with four natural transformations indicated is naturally isomorphic to any other via a unique natural transformation that is compatible with the four given.

As part of an effort to build geometric refinements of intersection pairings, Hopkins and Singer show in [HS05] that, for an arbitrary generalized cohomology theory, there exists an analogue of differential cohomology that fits into a diagram analogous to Diag. (1.1). They state there that the direction of their work was inspired by a comment of Witten about describing brane partition functions in terms of these refinements. It is therefore perhaps not surprising that differential cohomology has in recent years become an object of interest in physics. Distler, Freed, and Moore postulate in [DFM10] that because of anomaly cancellation in superstring theory, B-field fluxes for the oriented superstring seem to be quantized by certain Postnikov truncations of K-theory rather than a product of ordinary integral cohomology, and that the corresponding differential cohomology groups are therefore the proper home for the Dirac quantized B-field of the theory. Differential K-theory itself has a geometrically appealing interpretation as the Grothendieck construction applied to equivalence classes of vector bundles with connection [SS10b].

A natural question is whether or not this diagram of natural transformations determines a generalized differential cohomology theory up to unique natural isomorphism. Bunke and Schick answer this in the negative in [BS10] by constructing an alternate abelian group structure for a given version of differential K-theory that is inequivalent to the original. This holds even though they are using a smaller diagram that doesn't involve the $\mathbb{R}/\mathbb{Z}$ cohomology[1]. In the same paper, however, they obtain a positive result by also requiring that each of the differential cohomology

[1]They show that the kernel of the natural transformation to the analogue of $\Omega^*_{\mathbb{Z}}$ is isomorphic to the $\mathbb{R}/\mathbb{Z}$ cohomology, but potentially non-uniquely.

theories in question have a “wrong-way” natural transformation

$$\int : \hat{E}^{*+1}(S^1 \times -,\ \{0\} \times -) \longrightarrow \hat{E}^*$$

(which they call an “integration structure”) that is compatible with the corresponding natural transformations for each of the other functors in the diagram. For the functors along the top sequence, this is the suspension isomorphism. For the bottom sequence, this is the integration along the fibers map of the projection

$$p_2 : (S^1 \times -) \to (-).$$

Under this condition and the condition that the starting cohomology theory is rationally even (the groups of a point are concentrated in even degrees after tensoring with $\mathbb{Q}$), there always exists a unique natural isomorphism between any two of these “differential cohomology theories with integration” that commutes with the wrong-way maps and which is compatible with the analogues of three of four natural transformations in Diag. (1.1).

1.1.2 Outline

The main technical thrust of [BS10] is approximating spectra by smooth manifolds. Our approach is to instead consider differential cohomology theories on a category that contains smooth manifolds, but which is also large enough to include all spectra. This category is equipped with a functor that restricts to differential forms on objects that are manifolds, and which enjoys many of the same properties, including the sheaf condition. We show that this, together with the half-exactness of a cohomology theory (which we call the Mayer-Vietoris property), implies that any such differential cohomology theory also satisfies the Mayer Vietoris property. This gives the existence (but not uniqueness) of differential cohomology elements on a space given a coherent collection of elements on the pieces of a decomposition of the space.

Following the lead of [BS10] with regard to addition structures and the axiom of an integration natural transformation, we define a slightly different version of an integration structure and use the Mayer-Vietoris property to show that any natural transformation between the underlying pointed sets of a pair of differential cohomology theories with integration that is compatible with (the integration map and three of the four other natural transformations) must be a homomorphism. For any cohomology theory E^*, we provide another model for differential cohomology with an integration map that takes values in pointed sets. We show that if $E^*(\text{point})$ is finitely generated in each degree, then the functor (that takes values in the underlying pointed set of any differential cohomology theory with integration) is naturally isomorphic to our model. Therefore any pair of differential cohomology theories with integration are naturally isomorphic as pointed sets, and thus as abelian groups as well.

1.2 Underlying Category

One of the main innovations of this dissertation is to consider differential cohomology theories defined on a larger category $\mathcal{C}$ than the traditional ones of smooth manifolds or smooth manifolds with corners. Since there are many possible categories for which these constructions work, we describe $\mathcal{C}$ axiomatically rather than picking a particular one.

1.2.1 Given Categories

In a nutshell, we desire a category that contains smooth manifolds with corners, retains several of the features of smooth manifolds with corners, but which also has all small colimits[2]. First we outline the properties of some categories that we wish to emulate.

We define the the following categories:

$$
\begin{aligned}
\mathsf{Set} &:= \text{sets.} \\
\mathsf{Ab} &:= \text{abelian groups.} \\
\mathsf{Ab}^{\bullet} &:= \mathbb{Z}\text{-graded abelian groups.} \\
\mathsf{Mfld} &:= \text{smooth manifolds with corners.} \\
\mathsf{Top} &:= \text{topological spaces that have the homotopy type of a CW complex.} \\
\mathsf{DGCA} &:= \text{differential graded-commutative } \mathbb{R}\text{-algebras.} \\
\mathrm{CC}(\mathsf{Vect}_{\mathbb{R}}) &:= \text{chain complexes of } \mathbb{R} \text{ vector spaces.} \\
\boldsymbol{\Delta} &:= \text{the simplex category.}
\end{aligned}
$$

Let $(-)^{\mathrm{op}}$ denote the opposite category of $(-)$. Between these categories we have the diagram of functors and natural transformations:

$$
\begin{tikzcd}
\boldsymbol{\Delta}^{\mathrm{op}} \arrow[r, "\sigma^{\mathrm{op}}"] & \mathsf{Mfld}^{\mathrm{op}} \arrow[r, "\Omega^*"] \arrow[d, "U^{\mathrm{op}}"'] \arrow[dr, "C^*_{\mathrm{sm}}"] & \mathsf{DGCA} \arrow[d, "F_{\mathrm{alg}}"] \arrow[dl, phantom, "\swarrow N_{\int}"] \\
& \mathsf{Top}^{\mathrm{op}} \arrow[r, "C^*_{\mathrm{sing}}"'] \arrow[ur, phantom, "\nearrow N_{\mathrm{sm}}"] & \mathrm{CC}(\mathsf{Vect}_{\mathbb{R}})
\end{tikzcd},
$$

where the functors are defined as follows:

[2] A colimit of (a diagram of objects and arrows in a category) is a universal target object for that diagram, which means it comes equipped with an arrow from each object in the diagram such that all the triangles formed from all the arrows in the diagram commute. Disjoint unions, identification spaces, direct sums, and direct limits are all examples.

$$
\begin{aligned}
U &:= \text{the underlying topological space.}\\
F_{\text{alg}} &:= \text{forgets the algebra structure.}\\
\Omega^* &:= \text{the smooth differential forms.}\\
C^*_{\text{sing}}(-;\,\mathbb{R}) &:= \text{the singular } \mathbb{R}\text{-valued cochains.}\\
C^*_{\text{sm}}(-;\,\mathbb{R}) &:= \text{the } \textit{smooth} \text{ singular } \mathbb{R}\text{-valued cochains.}\\
\sigma &:= \text{realizes each } n\text{-simplex, } \Delta^n\text{, as a smooth manifold with corners.}
\end{aligned}
$$

The functor $C^*_{\text{sm}}(-;\,\mathbb{R})$ can be built formally out of σ. For $M \in \mathsf{Mfld}$,

$$C^n_{\text{sm}}(M;\,\mathbb{R}) := \mathrm{Hom}_{\mathsf{Set}}(\mathrm{Hom}_{\mathsf{Mfld}}(\sigma(\Delta^n), M),\ \mathbb{R}).$$

$C^*_{\text{sing}}(-;\,\mathbb{R})$ can be similarly defined by replacing σ with $U \circ \sigma$. Because U induces a map

$$\mathrm{Hom}_{\mathsf{Mfld}}(\sigma(\Delta^n), M) \to \mathrm{Hom}_{\mathsf{Top}}(U(\sigma(\Delta^n)), U(M)),$$

we have the natural transformation $N_{\text{sm}} : C^*_{\text{sing}}(U(-);\,\mathbb{R}) \to C^*_{\text{sm}}(-;\,\mathbb{R})$. The natural transformation $N_{\int}$ is given by integration over simplices. The important point for Mfld is that both of these natural transformations are quasi-isomorphisms.

Let $\tilde{I} := \sigma(\Delta^1)$. Using this, we can express notions of homotopy in all of the above categories (except $\boldsymbol{\Delta}$) that is compatible with all the above functors . To wit,

- Mfld has $\mathrm{Hom}_{\mathsf{Mfld}}\left(\tilde{I} \times X,\ \ Y\right)$.
- Top has $\mathrm{Hom}_{\mathsf{Top}}\left(U\left(\tilde{I}\right) \times X,\ \ Y\right)$.
- $\mathrm{CC}(\mathsf{Vect}_{\mathbb{R}})$ has chain homotopies.
- DGCA has chain homotopies that respect the algebra structure.

The precise notion of compatibility for Ω^* is given by the wrong-way natural transformation

$$(p_2)_! \ : \ \Omega^{*+1}\left(\tilde{I} \times -\right) \ \longrightarrow\ \Omega^*(-)$$

associated to the projection $p_2 : \left(\tilde{I} \times -\right) \to (-)$ that is given by integration along the fibers. Then if $\iota_0, \iota_1 : (-) \to \left(\tilde{I} \times -\right)$ are the maps that include $(-)$ to $(\{0\} \times -)$ and $(\{1\} \times -)$ respectively,

$$\iota_1^* - \iota_0^* = d(p_2)_! + (p_2)_! d.$$

1.2.2 Axioms

Let $\mathcal{C}$ be a category that satisfies the following:

C1. $\mathcal{C}$ has all small colimits and all finite products, both of which are given functorially (*e. g.*, $\forall X \in \mathcal{C}$, $(-) \mapsto (X \times -)$ is an endofunctor of $\mathcal{C}$).

C2. $\mathcal{C}$ is equipped with the functors indicated by dotted arrows in the diagram

$$\begin{tikzcd} \mathsf{Mfld}^{\mathrm{op}} \arrow[r, "\Omega^*"] \arrow[d, "U^{\mathrm{op}}"'] \arrow[dr, dotted, "m^{\mathrm{op}}"] & \mathsf{DGCA} \\ \mathsf{Top}^{\mathrm{op}} & \mathcal{C}^{\mathrm{op}} \arrow[l, dotted, "T^{\mathrm{op}}"] \arrow[u, dotted, "\Lambda^*"'] \end{tikzcd} ,$$

such that the compositions on the boundaries of the both triangles are naturally isomorphic.

C3. $m : \mathsf{Mfld} \to \mathcal{C}$ is faithful, co-continuous (preserves all colimits) and continuous (preserves all limits[3].

C4. $T : \mathcal{C} \to \mathsf{Top}$ is co-continuous and finitely continuous[4].

C5. $\Lambda^* : \mathcal{C}^{\mathrm{op}} \to \mathsf{DGCA}$ is continuous and is non-trivial only in non-negative degrees.

With the above, we can define the analogues of both the singular and smooth singular $\mathbb{R}$-valued cochains, and there will be a natural transformation $N_{\mathrm{sm},\mathcal{C}}$ that is analogous to N_{sm}. Also, since $\Lambda^*(m(\sigma(\Delta^n))) \cong \Omega^*(\sigma(\Delta^n))$, we can define an integration-over-the-simplices natural transformation $N_{\int,\mathcal{C}}$ from Λ^* to the analogue of the smooth singular $\mathbb{R}$-valued cochains. We require that

C6. $N_{\mathrm{sm},\mathcal{C}}$ and $N_{\int,\mathcal{C}}$ are quasi-isomorphisms. In other words, the cohomology of our "forms" is the $\mathbb{R}$-valued ordinary cohomology of the underlying space.

We also require the functors in the above diagram be compatible with homotopies. Let $\tilde{\mathrm{pt}} \in \mathsf{Mfld}$ denote the point space, and $\mathrm{pt} := m\left(\tilde{\mathrm{pt}}\right)$. Let $I := m\left(\tilde{I}\right)$.

C7. There exists a natural transformation $(p_2)_* : \Lambda^{*+1}(I \times -) \to \Lambda^*(-)$

C8. $(p_2)_*$ is *strongly surjective.* This means the following. Let K be a closed proper subset of $\tilde{I}$ that consists of a finite number of closed intervals and points. Let $\varphi_K : K \hookrightarrow \tilde{I}$ denote the inclusion. We require that for all $X \in \mathcal{C}$, $(p_2)_*$ restricted to the kernel of $(m(\varphi_K) \times \mathrm{id}_X)^*$ is surjective.

[3]Limits are the categorical dual to the concept of a colimit. They are the universal domains for diagrams. Products, direct products, fiber products, and inverse limits are all examples.

[4]Preserves *finite limits*, *i. e.*, limits involving only finitely many objects.

C9. If $N_\Pi : (I \times -) \circ m \to m \circ \left(\tilde{I} \times -\right)$ and $N_{\Lambda^*} : \Lambda^* \circ m^{\mathrm{op}} \to \Omega^*$ are the natural isomorphisms described in Axiom C3 (m preserves finite products) and the upper triangular cell of the diagram of Axiom C2, then we require that

$$N_{\Lambda^*} \circ ((p_2)_* \circ_2 \mathrm{id}_{m^{\mathrm{op}}}) = (p_2)_! \circ \left(N_{\Lambda^{*+1}} \circ_2 N_\Pi^{\mathrm{op}}\right)$$

where "$\circ_2$" refers to horizontal composition of natural transformations. Modulo the identification of the products, this says that $(p_2)_* \circ N_{\Lambda^{*+1}} = N_{\Lambda^*} \circ (p_2)_!$. In other words, $(p_2)_*$ extends $(p_2)_!$.

Since m is co-continuous and $\tilde{\mathrm{pt}}$ is terminal in Mfld, pt is terminal in $\mathcal{C}$. Thus the endofunctor on $\mathcal{C}$ given by $(\mathrm{pt} \times -)$ is naturally isomorphic to the identity functor. Let $f_0, f_1 : \tilde{\mathrm{pt}} \to \tilde{I}$ be the face maps of $\tilde{I} = \sigma(\Delta^1)$ in Mfld. Then, for every $X \in \mathcal{C}$ and for $j = 0, 1$, we have the map $\iota_j : X \to I \times X$ that corresponds to $m(f_j) \times \mathrm{id}_X$.

C10. In terms of the above, we require that for all $X \in \mathcal{C}$,

$$\iota_1^* - \iota_0^* = d(p_2)_* + (p_2)_* d$$

To summarize: Axiom 1 states that we have a category that is complete in a certain sense. Axiom 2 states we have functors relating this category to familiar ones. Axioms 3-5 state that these functors preserve the completeness. Axioms 6 is a de Rham theorem. Axioms 7-10 state that the functors are also compatible with homotopies.

One can build a category that satisfies the above axioms by taking a particular co-complete subcategory of Chen spaces[5] where the $N_{\mathrm{sm},\mathcal{C}}$ and $N_{\int,\mathcal{C}}$ are quasi-isomorphisms. The condition that these are so is closed under taking pushouts and disjoint unions, which generate all colimits, so we can take the co-closure of the full subcategory of Chen spaces generated by Mfld.

1.2.3 Properties of $\mathcal{C}$

Let SSet denote the category of simplicial sets.

Lemma 1.1. *There exists a functor $|\cdot|_\mathcal{C} : \mathsf{SSet} \to \mathcal{C}$ that makes the diagram*

$$\begin{tikzcd} \mathsf{SSet} \arrow[r, "|\cdot|_\mathcal{C}"] \arrow[dr, "|\cdot|"'] & \mathcal{C} \arrow[d, "T"] \\ & \mathsf{Top} \end{tikzcd}$$

"commute", where "$|\cdot|$" denotes the geometric realization functor.

[5] See [Che73] for a definition of Chen spaces.

Proof. A simplicial set is a presheaf on $\mathbf{\Delta}$ (*i. e.*, a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathsf{Set}$). The category of presheaves (with natural transformations acting as morphisms) is the universal category with all small colimits that receives a functor from $\mathbf{\Delta}$. So the functor $m \circ \sigma : \mathbf{\Delta} \to \mathcal{C}$ extends to a functor $|\cdot|_{\mathcal{C}} : \mathsf{SSet} \to \mathcal{C}$. Note that

$$T \circ m \circ \sigma \cong U \circ \sigma \cong |\cdot|\Big|_{\mathbf{\Delta}}.$$

Because T is co-continuous, it must send the image of (every formal colimit added to $\mathbf{\Delta}$ to get SSet) under $|\cdot|_{\mathcal{C}}$ in $\mathcal{C}$ to the corresponding colimit in Top. Thus $T \circ |\cdot|_{\mathcal{C}} \cong |\cdot|$. ■

Intuitively, we can think of the geometric realization of simplicial set as taking the disjoint union of the sets of simplices, and then using the face and degeneracy maps to make all the identifications. This identification spaces is an example of a colimit. Since we have objects in $\mathcal{C}$ that correspond to each simplex ($m(\Delta^k)$ for each k), and since $\mathcal{C}$ has all small colimits, we can form the corresponding colimit in $\mathcal{C}$. Because T is co-continuous, it must send this colimit to the corresponding one in Top, which is precisely the geometric realization. In the sequel, all references to "identification spaces" made out of objects of $\mathcal{C}$ will actually mean the corresponding colimit.

An interesting point about $|\cdot|$ is that it has a right adjoint $\mathrm{Sing} : \mathsf{Top} \to \mathsf{SSet}$. For $X \in \mathsf{Top}$, the set n-simplices of $\mathrm{Sing}\, X$ is $\mathrm{Hom}_{\mathsf{Top}}(|\Delta^n|, X)$. A classical result of Quillen states that S and $|\cdot|$ give an equivalence of homotopy categories [Qui67]. So for $X \in \mathsf{Top}$, the adjoint map to $\mathrm{id} \in \mathrm{Hom}_{\mathsf{SSet}}(\mathrm{Sing}\, X, \mathrm{Sing}\, X)$ gives a homotopy equivalence $|\mathrm{Sing}\, X| \to X$. This means that

$$X \simeq |\mathrm{Sing}\, X| \cong T(|\mathrm{Sing}\, X|_{\mathcal{C}}).$$

So $|\mathrm{Sing}(-)|_{\mathcal{C}}$ is like a homotopy section of T. This suggests that it might be useful to define a notion of homotopy in $\mathcal{C}$. Let I_n be the image under $|\cdot|_{\mathcal{C}}$ of the simplicial set constructed from gluing n copies of Δ^1 end-to-end.

Definition 1.2. A *homotopy* in $\mathcal{C}$ is a morphism $h \in \mathrm{Hom}_{\mathcal{C}}(I_n \times X,\ Y)$ for some $n \in \mathbb{Z}_{>0}$.

If $r_j : I \to I_n$ is the inclusion of the j^{th} sub-interval, then we can extend $(p_2)_*$ to (Λ^{*+1} applied to objects in $\mathcal{C}$ of the form $I_n \times X$) by letting

$$(p_2)_* := \sum_j (p_2)_* \circ (r_j \times \mathrm{id}_X)^*$$

Thus, a homotopy in $\mathcal{C}$ is sent by Λ^* to a chain homotopy in DGCA via this extended $(p_2)_*$ and Axiom C10, and by T to a topological homotopy in Top.

1.2.4 Associated Categories

1.2.4.1 Pointed Categories

The categories Mfld, Top, SSet, and $\mathcal{C}$ all have a notion of a point object by looking at the image of $\Delta^0 \in \mathbf{\Delta}$ included into them. Let $\mathcal{A}$ denote any of these categories, and let $P \in \mathcal{A}$ denote the appropriate point object. We define the category $\mathcal{A}_0$ of pointed objects as the coslice category $(P \downarrow \mathcal{A})$ whose objects are pairs consisting of an object of $\mathcal{A}$ together with a morphism from P to the given object, which we refer to as the base point. Since all these categories have coproducts, we have the inclusion functor $(-)_+ : \mathcal{A} \to \mathcal{A}_0$ defined by adding a disjoint base point. There is also the functor $B_{\mathcal{A}} : \mathcal{A}_0 \to \mathcal{A}$ that forgets the base point. Since the notion of the point object in each of these categories is obtained by pushing the point object through the functors relating these categories, each of these functors induces a functor between the corresponding pointed categories.

Definition 1.3. For any abelian category $\mathcal{M}$ and any functor $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{M}$, the *reduced version of* F is the functor $\tilde{F} : \mathcal{A}_0^{\mathrm{op}} \to \mathcal{M}$ defined by $\tilde{F}(X, b) := \ker(F(b))$.

For all $X \in \mathcal{A}$, let c_X be the unique morphism $X \to P$. Then one can see by purely functorial arguments that for all $(X, b) \in \mathcal{A}_0$,

$$F(X) \cong \tilde{F}(X, b) \oplus \operatorname{im}(F(c_X)),$$

and that for any morphism f in $\mathcal{A}_0$, the induced morphism $B_{\mathcal{A}}(f)$ in $\mathcal{A}$ will preserve this splitting. Moreover, if F preserves finite products, then $\tilde{F}(-)_+$ is naturally isomorphic to F. In this way, one can think of $\tilde{F}$ as an extension of F. Since all of the functors from $\mathcal{C}^{\mathrm{op}}$ that we will be dealing with have this property, we will always be working with the reduced versions of all functors unless it is stated otherwise; if we apply such a functor to an object of $\mathcal{A}$, we are implicitly composing with the functor with $(-)_+$.

Remark. What we have done in creating the pointed category is promote P from a terminal object in $\mathcal{A}$ to a zero object in $\mathcal{A}_0$. Thus, any sufficiently continuous functor to an abelian category will send it to the zero object there.

1.2.4.2 Pair Categories

This generalizes the above pointed category.

Definition 1.4. The category of pairs of $\mathcal{A}$ is defined similarly to that of $\mathcal{A}_0$. Namely, it's the coslice category $(\mathcal{C} \downarrow \mathcal{C})$ whose objects are pairs of objects in $\mathcal{A}$ together with an (mono)morphism from the first to the second. Morphisms of pairs are two morphism between the corresponding objects of the pairs that make the obvious square commute.

We denote pairs by either writing the morphism explicitly, *e. g.*$L \xrightarrow{f} K$; or by just listing the objects (with an implied morphism) when the morphism is clear or not explicitly needed, *e. g.*(K, L). Since $\mathcal{A}$ has products, for any two pairs of objects in $\mathcal{A}$, (K, L) and (K', L'), let $(K, L) \times_{\mathrm{p}} (K', L')$ denote the product of pairs, $(K \times K', L \times K' \sqcup_{K \times K'} K \times L')$.

Definition 1.5. For a fixed pair (X, Y), let $G_{(X,Y)}$ denote the endofunctor on pairs of objects in $\mathcal{A}$ that takes (K, L) to $(K, L) \times_{\mathrm{p}} (X, Y)$.

Definition 1.6. An *non-latching pair* is an morphism $f : A \to X$ that has a retraction $r : X \to A$.

Note that G_X preserves non-latching pairs for any pair X.

The terminality of $P \in \mathcal{A}$ makes any pair $P \to X$ non-latching. This allowed us to define the reduced versions of our functors. Similarly, for general non-latching pairs and any of a functor $F : \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ that we are considering, it makes sense to define $F(X, A) := \ker(f)$, as this will isomorphic to the standard definitions for both cohomology (because the long exact sequence of the pair will have trivial coboundary) and differential forms (because the functor that defines them is continuous).

1.2.4.3 Stable Categories

Since Mfld isn't complete or co-complete, we now restrict to the case where $\mathcal{A} \in \{\mathsf{Top}, \mathsf{SSet}, \mathcal{C}\}$. One can define the analogue of the smash product using limits and colimits (first define the wedge sum as the coproduct in $\mathcal{A}_0$, then construct a map from it to the product, and then take the colimit that identifies its image to a point). This smash product is both commutative and associative in the sense that there are natural isomorphisms between all the different orders and parenthesizations of the factor objects. The analogue of base point preserving homotopies is given by maps not from the product with an interval object, but from the smash product of $(-)_+$ applied to an interval object (in the case of $\mathcal{C}$, there is potentially more than one interval object; namely all the I_n).

All of the above listed categories also have a pointed circle object in $\mathcal{A}_0$. We can thus define a functor S for each of these categories by smashing with this circle. Commutativity and associativity of the smash product guarantee that this functor is appropriately compatible with homotopies.

Definition 1.7. A *pre-spectrum* of $\mathcal{A}$ is a collection $\{(\mathbf{A}_n, \alpha_n) \mid n \in \mathbb{Z}_{\geq 0}\}$ where $\mathbf{A}_n \in \mathcal{A}$ and $\alpha_n \in \mathrm{Hom}_{\mathcal{A}}(S\mathbf{A}_n,\, \mathbf{A}_{n+1})$. The latter of these are called its *structure morphisms.*

Definition 1.8. If $\mathbf{A} = \{(\mathbf{A}_n, \alpha_n)\}$ and $\mathbf{B} = \{(\mathbf{B}_n, \beta_n)\}$ are two pre-spectra of $\mathcal{A}$, a *morphism*[6] $f : \mathbf{A} \to \mathbf{B}$ is a collection of morphisms $f_n \in \mathrm{Hom}_{\mathcal{A}}(\mathbf{A}_n, \mathbf{B}_n)$ such that $\beta_n \circ (Sf_n) = f_{n+1} \circ \alpha_n$.

[6]Note that this terminology differs from the classical definition of a morphism of spectra that involves cofinal subspectra.

Let $\mathrm{ps}\mathcal{A}$ denote the category of pre-spectra of $\mathcal{A}$. Thus, in particular, we can form the category $\mathrm{ps}\mathcal{C}_0$ of pre-spectra of $\mathcal{C}_0$. Since T_0 and $(|\cdot|_{\mathcal{C}})_0$ are continuous and co-continuous, they preserve all of the above operations. Thus, they induce functors between the corresponding categories of pre-spectra.

Theorem 1.9. *For any cohomology theory $E^* : \mathsf{Top}_0^{\mathrm{op}} \to \mathsf{Ab}^\bullet$ whose coefficients E^* are countably generated in each degree, we can find a classifying spectrum for $E^* \circ T_0$ in $\mathrm{ps}\mathcal{C}_0$.*

Proof. In §2 of [BS10], they construct approximations in to CW spectra for such cohomology theories by way of increasing unions of smooth manifolds. We can include this diagram of manifolds into $\mathcal{C}$ via m, and take its colimit there. Essentially, this construction first approximates with finite simplicial complexes, and then includes each finite simplicial complex into a Euclidean space, where it is thickened into a manifold. Since the structure maps restricted to (the suspension of) each finite piece will map into a a finite piece of the target, we can like-wise approximate them by first simplicial and then smooth maps. By taking the base points to be vertices in the original CW complexes, the above approximations will preserve them as well. ∎

Definition 1.10. An object $X \in \mathcal{C}$ (or $\mathcal{C}_0$) is of *finite type* if $T(X)$ (respectively, $T_0(X)$) has the homotopy type of a finite CW complex.

In the sequel, we will use the notation that if $\mathbf{E} \in \mathrm{ps}\mathcal{C}_0$, then $E^* : \mathcal{C}_0^{\mathrm{op}} \to \mathsf{Ab}^\bullet$ will be the corresponding additive reduced cohomology theory on $\mathcal{C}_0$ defined by

$$E^n := \mathrm{colim}_k \operatorname{Hom}_{\mathrm{h}\mathcal{C}_0}(S^k -, \mathbf{E}_{n+k})$$

for finite type objects in $\mathcal{C}$.

1.3 Cohomology

Let $\mathbf{E}$ be a spectrum of $\mathcal{C}_0$ such that $E^*(\mathrm{pt})$ is finitely generated in each degree, where E^* is the generalized reduced cohomology theory defined by $\mathbf{E}$. This cohomology theory is a fixed input of the remainder of this text.

1.3.1 Coefficients

For any abelian group G, we can define *E-cohomology with coefficients in G*, $E^*_G(\,\cdot\,)$ as the cohomology theory associated to the spectrum $\mathbf{E}^G := \mathbf{E} \wedge \mathbf{M}G$, where $\mathbf{M}G$ is a Moore spectrum for G. $\mathbf{E}^{\mathbb{Z}}$ is naturally equivalent to $\mathbf{E}$. The short exact sequence of groups $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$ yields the triangle of spectra

$$\mathbf{E} \to \mathbf{E}^{\mathbb{R}} \to \mathbf{E}^{\mathbb{R}/\mathbb{Z}} \to \mathbf{E}[1],$$

which yields the long exact sequence

$$\cdots \to E_{\mathbb{R}}^{*-1} \xrightarrow{N_\pi} E_{\mathbb{R}/\mathbb{Z}}^{*-1} \xrightarrow{N_B} E^* \xrightarrow{N_\iota} E_{\mathbb{R}}^* \to \cdots .$$

For any $\lambda \in \mathbb{R}$, the endomorphism of $\mathbb{R}$ given by multiplying λ induces an endomorphism of $\mathbf{E}^{\mathbb{R}}$, and thus an endomorphism of $E_{\mathbb{R}}^*$, endowing $E_{\mathbb{R}}^*$ with the structure of a real vector space. For any finite type space $X \in \mathcal{C}_0$, $E_{\mathbb{R}}^*(X)$ is naturally isomorphic to $E^*(X) \otimes \mathbb{R}$.

1.3.2 Differential Forms

We refer to elements of the functor Λ^* described in Axiom C2 as "differential forms". Because of Axiom C6, $H(\Lambda^*) =: H_{\mathrm{dR}}^* \cong H_{\mathrm{sing}}^*(T_0(-); \mathbb{R})$. We define the graded vector space V by

$$V^* := E^*(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{R} \cong E_{\mathbb{R}}^*(\mathrm{pt}) .$$

Let

$$\Lambda^*(\,\cdot\,; V) := \prod_{i=0}^{\infty} \left(\Lambda^i \otimes_{\mathbb{R}} V^{*-i}\right)$$

be the degree-$*$ differential forms with values in V. Note that when this functor is composed with $m : \mathsf{Mfld}_0 \to \mathcal{C}_0$, the resulting definition is naturally isomorphic to the standard one of differential forms with values in a graded vector space V. The natural differential splits with respect to this product, and thus the cohomology of this complex will be

$$H_{\mathrm{dR}}^*(\,\cdot\,; V) := \prod_{i=0}^{\infty} H_{\mathrm{dR}}^i(\,\cdot\,; V^{*-i}).$$

Henceforth, unless otherwise indicated, all forms will have values in V, and we will denote $\Lambda^*(\,\cdot\,; V)$ by Λ^*.

First note that

$$\pi_i(\mathbf{E}_k) \cong [S^i, \mathbf{E}_k] \cong [S^0, \mathbf{E}_{k-i}] \cong E^{k-i}(\mathrm{pt}).$$

Then because $\mathbf{E}^{\mathbb{R}}$ splits into a product of Eilenberg-Maclane spaces[7], for any $X \in \mathcal{C}_0$,

$$\begin{aligned} E_{\mathbb{R}}^k(X) &\cong \left[X,\ \mathbf{E}_k^{\mathbb{R}}\right] \cong \left[X,\ \prod_{i=0}^{\infty} K(\pi_i(\mathbf{E}_k) \otimes \mathbb{R}, i)\right] \cong \left[X,\ \prod_{i=0}^{\infty} K\Big(V^{k-i}, i\Big)\right] \\ &\cong \prod_{i=0}^{\infty} \left[X,\ K\Big(V^{k-i}, i\Big)\right] \cong \prod_{i=0}^{\infty} H_{\mathrm{dR}}^i\left(X; V^{k-i}\right) = H_{\mathrm{dR}}^k(X; V). \end{aligned}$$

This establishes a natural isomorphism $\mathrm{dR} : E_{\mathbb{R}}^* \to H_{\mathrm{dR}}^*(\,\cdot\,; V)$. Henceforth, unless

[7] As an H-space, the k-invariants of $\mathbf{E}^{\mathbb{R}}$ vanish when tensored with $\mathbb{Q}$; but tensoring a real vector space with $\mathbb{Q}$ doesn't kill anything.

otherwise stated, $H^*_{\mathrm{dR}}(\,\cdot\,;V)$ will be denoted by H^*_{dR}.

Definition 1.11. *Special forms* $\Lambda^*_{\mathbb{Z}}$ are closed forms whose de Rham class is in the image of $\mathrm{dR} \circ N_\iota : E^* \to H^*_{\mathrm{dR}}$.

Note that this includes all exact forms. This will be a functor because it was defined in terms of natural transformations. The inclusion $\Lambda^*_{\mathbb{Z}} \hookrightarrow \Lambda^*$ is a natural transformation, and thus the quotient $\Lambda^*/\Lambda^*_{\mathbb{Z}}$ is a functor too. Thus we have the following short exact sequence of natural transformations:

$$0 \to \Lambda^*_{\mathbb{Z}} \to \Lambda^* \to \frac{\Lambda^*}{\Lambda^*_{\mathbb{Z}}} \to 0$$

If we give the first and last functors the zero differential, then these are chain maps. Thus, by the snake lemma, we get the long exact sequence

$$\cdots \to H^{*-1}_{\mathrm{dR}} \to \frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}} \xrightarrow{d} \Lambda^*_{\mathbb{Z}} \to H^*_{\mathrm{dR}} \to \cdots .$$

Chapter 2

Differential Cohomology

All of §1.3.1 and §1.3.2 can be summarized with the following diagram of natural transformations:

$$
\begin{tikzcd}
\cdots \arrow[r] & E_{\mathbb{R}}^{*-1} \arrow[r, "N_\pi"] \arrow[d, leftrightarrow, "\cong"'] & E_{\mathbb{R}/\mathbb{Z}}^{*-1} \arrow[r, "N_B"] & E^* \arrow[r, "N_\iota"] & E_{\mathbb{R}}^{*} \arrow[r] \arrow[d, leftrightarrow, "\cong"'] & \cdots \\
\cdots \arrow[r] & H_{\mathrm{dR}}^{*-1} \arrow[r] & \frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}} \arrow[r, "d"] & \Lambda_{\mathbb{Z}}^{*} \arrow[r] & H_{\mathrm{dR}}^{*} \arrow[r] & \cdots
\end{tikzcd}
\tag{2.1}
$$

Definition 2.1. A *differential cohomology theory for E^** is a functor $W_E^* : \mathcal{C}_0^{\mathrm{op}} \to \mathsf{Ab}^\bullet$, together with four natural transformations (indicated with dotted arrows) such that the following diagram commutes and the diagonal sequences are exact:

$$
\begin{tikzcd}
 & & 0 \arrow[dr] & & & & 0 & & \\
\cdots \arrow[r] & E_{\mathbb{R}}^{*-1} \arrow[rr, "N_\pi"] \arrow[dd, leftrightarrow, "\cong"'] & & E_{\mathbb{R}/\mathbb{Z}}^{*-1} \arrow[rr, "N_B"] \arrow[dr, dotted, "i_1"] & & E^* \arrow[rr, "N_\iota"] \arrow[ur] & & E_{\mathbb{R}}^{*} \arrow[r] \arrow[dd, leftrightarrow, "\cong"'] & \cdots \\
 & & & & W_E^* \arrow[ur, dotted, "\delta_2"'] \arrow[dr, dotted, "\delta_1"] & & & & \\
\cdots \arrow[r] & H_{\mathrm{dR}}^{*-1} \arrow[rr] & & \frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}} \arrow[rr, "d"] \arrow[ur, dotted, "i_2"] & & \Lambda_{\mathbb{Z}}^{*} \arrow[rr] \arrow[dr] & & H_{\mathrm{dR}}^{*} \arrow[r] & \cdots \\
 & & 0 \arrow[ur] & & & & 0 & &
\end{tikzcd}
\quad . \tag{2.2}
$$

For the remainder of this section, let W_E^* be any differential cohomology theory for E^* and i_1, i_2, δ_1, δ_2 be the indicated natural transformations.

Definition 2.2. A $\left({}^{3}\!/\!{}_{4}\right)$*-morphism of differential cohomology theories* is a natural transformation $\Psi : W_E^* \to {W'}_E^*$ such that $\Psi \circ i_2 = i_2'$, $\delta_1 \circ \Psi = \delta_1'$, and $\delta_2 \circ \Psi = \delta_2'$.

In other words, it's compatible with three of the four natural transformations.

Proposition 2.3. *Every $\left(3\!/\!4\right)$-morphism is an isomorphism.*

Proof. Apply the Five Lemma to the diagonal exact sequence of Diag. (2.2) that involves $\frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}$ and E^*. ■

Definition 2.4. A *morphism of differential cohomology theories* is a $\left(3\!/\!4\right)$-morphism $\Psi : W^*_E \to W'^*_E$ such that $\Psi \circ i_1 = i'_1$ as well.

2.1 Properties

Lemma 2.5. *Suppose that $u \in W^*_E(I_+ \wedge X)$, and $\iota_0, \iota_1 : X \to I_+ \wedge X$ are induced by the inclusions of the endpoints into I. Then*

$$(\iota_1^* - \iota_0^*)u = i_2[(p_2)_* \delta_1 u]. \tag{2.3}$$

Proof. (This is essentially the same as given the first page of [BS10].) The formula holds when u is in the image of i_2, and the general case can be reduced to this by subtracting off $p_2^* \iota_0^* u$. ■

2.1.1 Mayer-Vietoris

Suppose that X is the homotopy colimit of the diagram

$$\widetilde{A} \xleftarrow{\widetilde{\rho_A}} D \xrightarrow{\widetilde{\rho_B}} \widetilde{B}$$

in $\mathcal{C}_0$. By this, we mean that if we define

$$C := \widetilde{C} \coprod_{\widetilde{\rho_C}} (I \times D)$$

for $C = A, B$, we have the following commutative diagram of pointed spaces

$$\begin{tikzcd} D \arrow[r, "\rho_A"] \arrow[d, "\rho_B"'] & A \arrow[d, "\iota_A"] \\ B \arrow[r, "\iota_B"] & X \end{tikzcd} .$$

where ρ_A and ρ_B are the respective inclusions of D into the unglued ends of $I \times D$. If we apply any contravariant functor F, we get the diagram

$$\begin{tikzcd} F(D) & F(A) \arrow[l, "F(\rho_A)"'] \\ F(B) \arrow[u, "F(\rho_B)"] & F(X) \arrow[l, "F(\iota_B)"'] \arrow[u, "F(\iota_A)"'] \end{tikzcd} .$$

If the target category has limits, then we get a morphism

$$F(X) \xrightarrow{MV_F} F(A) \prod_{F(D)} F(B). \tag{2.4}$$

Remark. We also have a map

$$\gamma : X \to SD$$

defined by compressing $\widetilde{A} \subseteq X$ to the vertex of one cone in SD, and $\widetilde{B} \subseteq X$ to the other. Up to homotopy, this is the mapping cone of $\widetilde{A} \vee \widetilde{B} \hookrightarrow X$. Thus, if F is a cohomology theory, then the kernel of MV_F will be the image of $F(\gamma)$.

Definition 2.6. We say that F has the *Mayer-Vietoris property* (or *MV property*) if for all spaces as given above, the corresponding MV_F given in Eq. (2.4) is an epimorphism.

Definition 2.7. We say that F has the *strong Mayer-Vietoris property* (or *strong MV property*) if for all spaces as given above, the corresponding MV_F given in Eq. (2.4) is an isomorphism.

A contravariant functor F having the strong MV property is the same as F being a sheaf. The condition in Axiom C5 that Λ^* is continuous means precisely that Λ^* satisfies the strong MV property.

Theorem 2.8. *Any differential cohomology theory W_E^* for a cohomology theory E^* has the MV property.*

See Appendix A1 for proof. It closely follows [SS10a], which proves this theorem for differential cohomology theories on the category of smooth manifolds with corners.

2.1.2 Homotopy Wedge Sums

Given the setup of spaces from the previous section, suppose that $x \in \ker(MV_{W_E^*})$. Then $\delta_1 x = 0$ because Λ^* has the strong MV property, and hence $MV_{\Lambda_{\mathbb{Z}}^*}$ is injective. So $x = i_1\theta$ for some $\theta \in E_{\mathbb{R}/\mathbb{Z}}^{*-1}(X)$. Since i_1 is injective, we also know that $\theta \in \ker MV_{E_{\mathbb{R}/\mathbb{Z}}^{*-1}}$. Because $E_{\mathbb{R}/\mathbb{Z}}^*$ is a cohomology theory, we know that $\ker MV_{E_{\mathbb{R}/\mathbb{Z}}^{*-1}} = \operatorname{im} E_{\mathbb{R}/\mathbb{Z}}^{*-1}(\gamma)$. To summarize, we have the diagram of horizontal exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & E_{\mathbb{R}/\mathbb{Z}}^{*-2}(A \sqcup B) \arrow[r] \arrow[d, equal] & E_{\mathbb{R}/\mathbb{Z}}^{*-2}(D) \arrow[r] \arrow[d, equal] & E_{\mathbb{R}/\mathbb{Z}}^{*-1}(X) \arrow[r] \arrow[d, hook, "i_1"'] & E_{\mathbb{R}/\mathbb{Z}}^{*-1}(A \sqcup B) \arrow[r] \arrow[d, hook, "i_1"'] & E_{\mathbb{R}/\mathbb{Z}}^{*-1}(D) \arrow[d, hook, "i_1"'] \\
\cdots \arrow[r] & E_{\mathbb{R}/\mathbb{Z}}^{*-2}(A \sqcup B) \arrow[r] & E_{\mathbb{R}/\mathbb{Z}}^{*-2}(D) \arrow[r] & W_E^*(X) \arrow[r] & W_E^*(A \sqcup B) \arrow[r] & W_E^*(D)
\end{tikzcd}$$

From this, it is easy to see the following.

Lemma 2.9. *If the coboundary map $E^{*-1}_{\mathbb{R}/\mathbb{Z}}(D) \to E^*_{\mathbb{R}/\mathbb{Z}}(X)$ is trivial, then*

$$W^*_E(X) \xrightarrow{MV_{W^*_E}} W^*_E(A) \prod_{W^*_E(D)} W^*_E(B).$$

is an isomorphism.

In particular, this is true if D is contractible, or if either $D \to A$ or $D \to B$ are non-latching pairs.

Corollary 2.10. *If $D = (\mathrm{pt}, \mathrm{id}_{\mathrm{pt}})$, then*

$$W^*_E(X) \cong W^*_E(A) \oplus W^*_E(B).$$

2.2 Integration

2.2.1 As an additional axiom

For all the functors F in Diag. (2.1), we have an integration natural transformation

$$\int_F : F^{*+1}(SX) \to F^*(X). \tag{2.5}$$

For the upper long exact sequence, it's the suspension isomorphism. For the lower long exact sequence, it's pulling back to $m(I) \times X$ and then applying the natural transformation given in Axiom C7. Because the integration map for Λ^* realizes the suspension isomorphism for its de Rham theory, the integral of a special form is special. Furthermore, all the natural transformations in Diag. (2.1) graded-commute with the integration maps. Thus, a natural definition to make is

Definition 2.11. A *differential cohomology theory with integration* is a differential cohomology theory W^*_E that has an integration natural transformation, $\int_{W_E}$, as given in Eq. (2.5) that graded-commutes[1] with all the natural transformations in Diag. (2.2).

Remarks. (1) This is not an unreasonable request to make of a differential cohomology theory, as the general construction of a differential cohomology theory for an arbitrary cohomology theory given by Hopkins and Singer in [HS05] has an integration natural transformation.

(2) Because $(S^1, \{0\})$ is non-latching, $G_X(S^1, \{0\})$ is as well. Since the map of pairs $G_X(S^1, \{0\}) \to SX$ induces an isomorphism for $E^{*-1}_{\mathbb{R}/\mathbb{Z}}$ (via a sequence of applications of excision and homotopy invariance) and Λ^*_Z (by continuity), we can

[1] This integration natural transformation is an odd degree operation. Therefore it anticommutes with i_1 and i_2, both of which are odd degree operations, just as it does with the "coboundary" maps (the Bockstein and d) from Diag. (2.1).

conclude via the diagonal short exact sequence involving i_1 and δ_1 of Diag. (2.2) that it induces an isomorphism for W_E^*. Therefore, we can replace SX in the above integration natural transformation with $G_X(S^1,\{0\})$ without loss of generality.

Definition 2.12. A *morphism* (respectively, $(\frac{3}{4})$-*morphism*) between differential cohomology theories with integration is a morphism (respectively, $(\frac{3}{4})$-morphism) between the underlying differential cohomology theories that also commutes with the integration transformation.

Let DC_E and $\mathsf{DC}_E^{(3/4)}$ denote the category of differential cohomology theories with integration for E^* with the respective notions of morphisms.

2.2.2 Surjectivity

Definition 2.13. If $X \in \mathcal{C}_0$, then a map of pairs $f : G_X(S^1,\{0\}) \to Y$ is *stably surjective* if the compositions

$$\left(\int_{\Lambda/\Lambda_{\mathbb{Z}}} \circ f^*\right) : \frac{\Lambda^*}{\Lambda^*_{\mathbb{Z}}}(Y) \longrightarrow \frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}(X)$$

and

$$\left(\int_{E} \circ f^*\right) : E^{*+1}(Y) \longrightarrow E^*(X)$$

are both surjective.[2]

Lemma 2.14. *If $f : G_X(S^1,\{0\}) \to Y$ is a stably surjective map of pairs, and $W_E^* \in \mathsf{DC}_E^{(3/4)}$, then the composition*

$$\int_{W_E} \circ f^* : W_E^{*+1}(Y) \to W_E^*(X)$$

is surjective.

Proof. Stable surjectivity of f gives the surjectivity of the first and third vertical arrows of the diagram

$$\begin{tikzcd}
0 \arrow[r] & \frac{\Lambda^*}{\Lambda^*_{\mathbb{Z}}}(Y) \arrow[r, "i_2"] \arrow[d, "-\int_{\Lambda/\Lambda_{\mathbb{Z}}} \circ f^*"'] & W_E^{*+1}(Y) \arrow[r, "\delta_2"] \arrow[d, "\int_{W_E} \circ f^*"'] & E^{*+1}(Y) \arrow[r] \arrow[d, "\int_E \circ f^*"'] & 0 \\
0 \arrow[r] & \frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}(X) \arrow[r, "i_2"] \arrow[d] & W_E^*(X) \arrow[r, "\delta_2"] & E^*(X) \arrow[r] \arrow[d] & 0 \\
& 0 & & 0 &
\end{tikzcd}.$$

[2]Recall Definition 1.5 for G.

Chasing this diagram then yields that the middle vertical arrow is also surjective. Note the minus sign on the first horizontal arrow for $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$ which makes the square of which it is an edge commute. ■

Let PP denote the "pair of pants" manifold with boundary. Let $L_1, L_2 \subset PP$ denote the the two cylinder "legs" whose union is all of PP, arranged such that $L_1 \cap L_2$ is isomorphic to a closed disk. Let $\varphi : S^1 \to PP$ be a smooth embedding that is isotopic to the "input"boundary, *i. e.*, not either of the unglued boundaries of L_1 or L_2. Note that this means that it will have some point in it's image that is in $L_1 \cap L_2$. Lift PP, L_1, and L_2 to objects in Mfld_0 by letting this point be the base point $b_0 \in PP$ (and re-parameterize S^1 such that φ is a base point preserving map). Let ι_1 and ι_2 denote the respective inclusions of L_1 and L_2 into PP.

We also have a base point preserving map $\rho_1 : PP \to L_1$ that, after identifying PP and L_1 with the respective homotopy equivalent spaces of a wedge of two circles and a circle, is equivalent to the maps that collapse the circle corresponding to L_2 to point. Thus $\rho_1 \circ \varphi$ is a homotopy equivalence, and we chose ρ_1 such that this composition is still a smooth embedding. We have an analogous map $\rho_2 : PP \to L_2$ where the roles of L_1 and L_2 are reversed. The compositions $\rho_1 \circ \iota_1$ and $\rho_2 \circ \iota_2$ are homotopic to the identity maps on L_1 and L_2 respectively.

We summarize the above with the commutative diagram

$$\begin{tikzcd}
L_1 \arrow[r, "\iota_1"] \arrow[d, "\simeq"'] & PP \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & L_2 \arrow[l, hook', "\iota_2"'] \arrow[d, "\simeq"] \\
L_1 & S^1 \arrow[l, hook', "\simeq"] \arrow[r, hook, "\simeq"'] \arrow[u, hook, "\varphi"'] & L_2
\end{tikzcd} . \tag{2.6}$$

Also, $\rho_1 \circ \iota_2$ and $\rho_2 \circ \iota_1$ are null-homotopic.

Lemma 2.15. *For any $W_E^* \in \mathsf{DC}_E^{(3/4)}$, and for any $X \in \mathcal{C}_0$, $G_X(\rho_j \circ \varphi)$ is stably surjective for $j = 1, 2$.*

Proof. Because $\rho_j \circ \varphi$ is a smooth embedding, $(\rho_j \circ \varphi)^* : \Lambda^*(L_j) \to \Lambda^*(S^1)$ is surjective, and thus

$$(G_X(\rho_j \circ \varphi))^* : \Lambda^*(G_X(L_j)) \to \Lambda^*(G_X(S^1, \{0\}))$$

is surjective. Since the integration along the fibers map

$$\int_\Lambda : \Lambda^{*+1}(G_X(S^1, \{0\})) \to \Lambda^*(X)$$

is surjective, the composition $\int_\Lambda \circ (G_X(\rho_j \circ \varphi))^*$ is surjective. Therefore the corresponding composition for $\frac{\Lambda^*}{\Lambda^*_{\mathbb{Z}}}$ is surjective.

In the initial setup of the spaces in Diag. (2.6), it was noted that $\rho_j \circ \varphi$ is a homotopy equivalence. Therefore, $G_X(\rho_j \circ \varphi)$ will be as well. And since the integration map for E^* is the suspension isomorphism, the composition

$$\int_E \circ (G_X(\rho_j \circ \varphi))^*$$

will be an isomorphism. ■

2.3 Addition Structure

Let $U : \mathsf{Ab}^\bullet \to (\mathsf{Set}_0)^\bullet$ be the forgetful functor from the category of ($\mathbb{Z}$-graded) abelian groups to the category of ($\mathbb{Z}$-graded) pointed sets. If W_E^* is a differential cohomology theory for E^*, then $U \circ W_E^*$ still retains most of the original structure. We still have a diagram analogous to Diag. (2.2) by horizontally obtained composing all the natural transformations with id_U. If W_E^* has an integration natural transformation, then so does $U \circ W_E^*$. The notions of a morphism and and $\binom{3}{4}$-morphism of these "pointed-set-valued differential cohomology theories with strongly surjective integration" are analogous to those given earlier in this chapter; namely, a natural transformation that is compatible with the four natural transformations coming into or going out of $U \circ W_E^*$ which also graded-commutes with the integration map.

But now consider the forgetful functor that doesn't completely forget the addition structure as above, but that instead retains the structure of a pointed $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$-set. In other words, if a is the addition map, then we only remember the map $a \circ (i_2 \times \mathrm{id})$. The four natural transformations δ_1, δ_2, i_1, i_2, and $\int_{W_E}$ will now not just be maps of pointed sets.

- i_2 will be a map of pointed $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$-sets, where $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$ acts on itself by addition.
- $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$ acts on Λ^* by addition of its image under d. Then δ_1 will be a map of pointed $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$-sets.
- If we give E^* the trivial action, then δ_2 will be a map of pointed $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$-sets.
- H_{dR}^{*-1} acts on both $E_{\mathbb{R}/\mathbb{Z}}^{*-1}$ and $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$ via addition of its image in each. This turns a differential cohomology theory whose addition structure has been partially forgotten into a pointed H_{dR}^{*-1}-set. Then i_1 will be a map of pointed H_{dR}^{*-1}-sets.
- $\int_{W_E}$ will be a map of pointed $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$-sets.

All of these will be natural in the following sense. If $f : X \to Y$ is a morphism of $\mathcal{C}$, then all of the above functors applied to X will have the appropriate group action structures over the functors applied to Y via f^*. The induced morphism for each functor will be equivariant with respect to the functors applied to Y. Moreover, because the correction terms added when one changes a map by a homotopy are always elements of $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$ (see Lemma 2.5), we can still compute those changes.

Definition 2.16. Let DC_E^Λ denote the following category.

- The objects are (contravariant functors $\mathcal{W}_E^*$ from $\mathcal{C}$ to $\mathbb{Z}$-graded pointed sets) that are equipped with four natural transformations as in Diag. (2.2) that have the above algebraic properties that turn $\mathcal{W}_E^*$ into a $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$-set. We also require an integration natural transformation, that the diagonal sequences analogous to those in Diag. (2.2) are exact[3], and that induced morphisms of $\mathcal{W}_E^*$ transform under homotopies by the formula given in Eq. (2.3).

- The morphisms are natural transformations preserve all of the above structure, much as in the definition of DC_E.

Definition 2.17. Let $\mathsf{DC}_E^{\Lambda(3/4)}$ denote the category whose objects are the same as DC_E^Λ, and whose morphisms satisfy all the same properties except for compatibility with i_1.

Let $U_\Lambda : \mathsf{DC}_E \to \mathsf{DC}_E^\Lambda$ and $U_\Lambda^{(3/4)} : \mathsf{DC}_E^{(3/4)} \to \mathsf{DC}_E^{\Lambda(3/4)}$ denote the above described forgetful functors.

Theorem 2.18. *Any object in the image of $U_\Lambda^{(3/4)}$ has a uniquely defined addition structure that makes it into a functor into $\mathsf{Ab}^\bullet$ such all five of its natural transformations in (pointed sets with the above described algebraic structures) become natural transformations in Ab.*

Proof. Let $W_E^* \in \mathsf{DC}_E$. Let $X \in \mathcal{C}_0$. Supposed $x_1, x_2 \in W_E^*(X)$. Our goal is to reconstruct their sum without ever explicitly adding them. If we apply G_X to Diag. (2.6) we obtain the diagram

$$\begin{tikzcd}
G_X(L_1) \arrow[r, "G_X(\iota_1)"] \arrow[d, "\simeq"'] & G_X(PP) \arrow[dl, "G_X(\rho_1)"'] \arrow[dr, "G_X(\rho_2)"] & G_X(L_2) \arrow[l, "G_X(\iota_2)"'] \arrow[d, "\simeq"] \\
G_X(L_1) & G_X(S^1) \arrow[l, hook', "\simeq"] \arrow[r, hook, "\simeq"'] \arrow[u, hook, "G_X(\varphi)"] & G_X(L_2)
\end{tikzcd}.$$

Because of Lemma 2.15, there exists $\tilde{x}_j \in W_E^{*+1}(G_X(L_j))$ such that

$$\left(\int_{W_E} \circ \, (G_X(\varphi))^* \circ (G_X(\rho_j))^* \right) (\tilde{x}_j) = x_j$$

for $j = 1, 2$. Clearly, if we could reconstruct $y := (G_X(\rho_1))^*(\tilde{x}_1) + (G_X(\rho_2))^*(\tilde{x}_2)$, we could then apply $\int_{W_E} \circ (G_X(\varphi))^*$ and have $x_1 + x_2$.

The inclusion $L_1 \cap L_2 \hookrightarrow L_j$ is non-latching. This remains true after taking a product with X. Therefore by examining the Mayer-Vietoris sequence for the

[3] Here, since the category of pointed sets has a zero object, we can define kernels and images.

decomposition of $PP = L_1 \cup L_2$, we see that

$$E^{*-1}_{\mathbb{R}/\mathbb{Z}}(G_X(PP)) \xrightarrow{\cong} E^{*-1}_{\mathbb{R}/\mathbb{Z}}(G_X(L_1)) \oplus E^{*-1}_{\mathbb{R}/\mathbb{Z}}(G_X(L_2)),$$

which is induced by the maps $G_X(\iota_1)$ and $G_X(\iota_2)$. In particular, we can apply Lemma 2.9 and conclude that the corresponding map for W^*_E,

$$W^*_E(G_X(PP)) \to W^*_E(G_X(L_1)) \oplus W^*_E(G_X(L_2)), \tag{2.7}$$

is injective. Thus, the secret value of $y \in W^{*+1}_E(G_X(PP))$ that we are trying to reconstruct is uniquely determined by its restrictions to $G_X(L_j)$ via $G_X(\iota_j)$ for $j = 1, 2$. Because $\rho_1 \circ \iota_1$ is homotopic to the identity, and $\rho_2 \circ \iota_1$ is null-homotopic, we compute that

$$\begin{aligned}
(G_X(\iota_1))^*(y) &= (G_X(\iota_1))^* \left((G_X(\rho_1))^*(\tilde{x}_1) + (G_X(\rho_2))^*(\tilde{x}_2)\right) \\
&= (G_X(\rho_1 \circ \iota_1))^*(\tilde{x}_1) \quad + \quad (G_X(\rho_2 \circ \iota_1))^*(\tilde{x}_2) \\
&= \tilde{x}_1 + i_2[\eta_1]
\end{aligned}$$

where $[\eta_1] \in \frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}(G_X(L_1))$ is the sum of the correction terms coming from G_X applied to the above mentioned homotopies. Note that this last line doesn't need the full addition structure of W^*_E and can be computed using only its pointed $\frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}$-set structure. A completely symmetric computation shows that we can compute $(G_X(\iota_2))^*(y)$ using only this structure as well.

Thus we can construct the appropriate values to which y should pull back in $G_X(L_1)$ and $G_X(L_2)$. By the injectivity in Eq. (2.7), there is a unique element in $W^{*+1}_E(G_X(PP))$ that pulls back thusly, so we can reconstruct y, and thus x_1+x_2. ■

Remarks. (1) This doesn't prove that any object in $\mathsf{DC}^{\Lambda}_E{}^{(\frac{3}{4})}$ has a unique addition structure. Even if we assumed the contents of Lemma 2.9 (the proof that this property holds uses the addition structure), this definition of the sum of two elements depends on the choice of $\tilde{x}_1$ and $\tilde{x}_2$, as well as the fact that there exists an element y that restricts to the appropriate values under $G_X(\iota_j)$ for $j = 1, 2$. Because we are assuming that our functor came from an actual differential cohomology theory that takes values in $\mathsf{Ab}^\bullet$, we know that $x_1 + x_2$ will be independent of these choices. So the addition structure is encoded in a "good" choice of integration natural transformation.

(2) This entire proof works for differential cohomology theories that are only defined on Mfld, or even smooth manifolds if one replaces the pair of pants PP with its interior. Thus the above and the following theorems hold in those contexts.

Theorem 2.19. *The functor* $U^{(\frac{3}{4})}_{\Lambda} : \mathsf{DC}^{(\frac{3}{4})}_E \to \mathsf{DC}^{\Lambda}_E{}^{(\frac{3}{4})}$ *is full.*

This should not be a surprise. The previous theorem states that we can reconstruct the sum of two elements using in a differential cohomology theory using only its pointed $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$-set structure, so any natural transformation that preserves said structure should preserve the sum.

Proof. Let $A_E^*, B_E^* \in \mathsf{DC}_E$ and $\Psi : U_\Lambda^{(3/4)} A_E^* \to U_\Lambda^{(3/4)} B_E^*$ be a morphism in $\mathsf{DC}^{\Lambda\,(3/4)}$. The only question is whether or not Ψ is an additive homomorphism. We assume the setup and notation from the proof of Theorem 2.18 applied to A_E^* and let

$$y := (G_X(\rho_1))^*(\tilde{x}_1) + (G_X(\rho_2))^*(\tilde{x}_2) \in A_E^{*+1}(G_X(PP)).$$

Note that

$$\begin{aligned}(G_X(\iota_1))^*\Psi(y) &= \Psi(G_X(\iota_1))^*(y)\\ &= \Psi(G_X(\iota_1))^*((G_X(\rho_1))^*\tilde{x}_1 + (G_X(\rho_2))^*\tilde{x}_2)\\ &= \Psi(\tilde{x}_1 + i_{2,A}[\eta_1]) = \Psi(\tilde{x}_1) + i_{2,B}[\eta_1],\end{aligned}$$

and similarly, $(G_X(\iota_2))^*\Psi(y) = \Psi(\tilde{x}_2) + i_{2,B}[\eta_2]$. Let $h_{(1,1)}$ and $h_{(2,2)}$ be the homotopies to the identity of $\rho_1 \circ \iota_1$ and $\rho_2 \circ \iota_2$ respectively. Let $h_{(1,2)}$ and $h_{(2,1)}$ be the null homotopies of $\rho_2 \circ \iota_1$ and $\rho_1 \circ \iota_2$ respectively (note the reversal). These homotopies are used to compute the correction forms $[\eta_1]$ and $[\eta_2]$:

$$\begin{aligned}[\eta_1] &= \left[(p_2)_*\delta_{1,A}\left(h_{(1,1)}^*\tilde{x}_1 + h_{(1,2)}^*\tilde{x}_2\right)\right]\\ &= \left[(p_2)_*\left(h_{(1,1)}^*\delta_{1,A}\tilde{x}_1 + h_{(1,2)}^*\delta_{1,A}\tilde{x}_2\right)\right]\\ &= \left[(p_2)_*\left(h_{(1,1)}^*\delta_{1,B}\Psi(\tilde{x}_1) + h_{(1,2)}^*\delta_{1,A}\Psi(\tilde{x}_2)\right)\right],\end{aligned}$$

and similarly for $[\eta_2]$. In other words, the correction forms are independent of the differential cohomology theory. Thus

$$(G_X(\iota_1))^*\left((G_X(\rho_1))^*\Psi(\tilde{x}_1) + (G_X(\rho_2))^*\Psi(\tilde{x}_2)\right) = \Psi(\tilde{x}_1) + i_{2,B}[\eta_1]$$

and

$$(G_X(\iota_2))^*\left((G_X(\rho_1))^*\Psi(\tilde{x}_1) + (G_X(\rho_2))^*\Psi(\tilde{x}_2)\right) = \Psi(\tilde{x}_2) + i_{2,B}[\eta_2].$$

By comparing this to Eq. (2.3) (and its analogue with 1 and 2 exchanged), we can see that

$$\Psi(y) = \Psi\left((G_X(\rho_1))^*\tilde{x}_1 + (G_X(\rho_2))^*\tilde{x}_2\right)$$

and

$$G_X(\rho_1))^*\Psi(\tilde{x}_1) + (G_X(\rho_2))^*\Psi(\tilde{x}_2)$$

have the sames images when restricted to $G_X(L_j)$ via $G_X(\iota_j)$ for $j = 1$ and $j = 2$. As pointed out in the proof of Theorem 2.18, these images uniquely characterize an element of $B_E^{*+1}(G_X(PP))$, and thus they must be equal. So

$$\begin{aligned}
\Psi(x_1+x_2) &= \left(\Psi\circ\int_{A_E}\circ(G_X(\varphi))^*\right)(y) = \left(\int_{B_E}\circ(G_X(\varphi))^*\circ\Psi\right)(y) \\
&= \left(\int_{B_E}\circ(G_X(\varphi))^*\right)(G_X(\rho_1))^*\Psi(\tilde{x}_1)+(G_X(\rho_2))^*\Psi(\tilde{x}_2)) \\
&= \left(\int_{B_E}\circ(G_X(\rho_1\circ\varphi))^*\Psi(\tilde{x}_1)\right)+\left(\int_{B_E}\circ(G_X(\rho_2\circ\varphi))^*\Psi(\tilde{x}_2)\right) \\
&= \left(\Psi\int_{A_E}\circ(G_X(\rho_1\circ\varphi))^*\tilde{x}_1\right)+\left(\Psi\int_{A_E}\circ(G_X(\rho_2\circ\varphi))^*\tilde{x}_2\right) \\
&= \Psi(X_1)+\Psi(x_2)
\end{aligned}$$

■

Corollary 2.20. *The corresponding functor $U_\Lambda : \mathsf{DC}_E \to \mathsf{DC}_E^\Lambda$ is full.*

Proof. Since both of these categories are subcategories of the corresponding categories in Theorem 2.19, we can forget the fact that the natural transformation is compatible with i_1, and get a morphism in $\mathsf{DC}_E^{\Lambda^{(3/4)}}$. But because the underlying natural transformation of pointed $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$-sets commutes with i_1, this morphisms in $\mathsf{DC}_E^{\Lambda^{(3/4)}}$ will be a morphism in DC_E^Λ. ■

Chapter 3

Obstructions

3.1 $\lim^1$ in an Abelian Category

Let $\mathcal{M}$ be an concrete abelian category[1] that has all small products. Then we also have a category of "inverse systems" of objects in $\mathcal{M}$, which are diagrams in $\mathcal{M}$ of the form

$$\cdots \to A_3 \xrightarrow{\phi_3} A_2 \xrightarrow{\phi_2} A_1 \xrightarrow{\phi_1} 0. \tag{3.1}$$

We denote this category $\mathcal{M}^{\mathbb{N}}$. Taking the inverse limit of such a system gives a functor $\lim : \mathcal{M}^{\mathbb{N}} \to \mathcal{M}$. In general, $\lim$ is only left-exact, meaning that if we have a short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{M}^{\mathbb{N}}$, then we only have the exact sequence

$$0 \to \lim A \to \lim B \to \lim C$$

in $\mathcal{M}$ (*i. e.*, the second morphism can fail to be an epimorphism). However, one can define the right-derived functors of $\lim$, denoted $\lim^i$, so that the above short exact sequence in $\mathcal{M}^{\mathbb{N}}$ yields the long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \lim^0 A \arrow[r] & \lim^0 B \arrow[r] & \lim^0 C \arrow[dll, rounded corners] \\
& \lim^1 A \arrow[r] & \lim^1 B \arrow[r] & \lim^1 C \arrow[dll, rounded corners] \\
& \lim^2 A \arrow[r] & \lim^2 B \arrow[r] & \lim^2 C \arrow[dll, rounded corners] \\
& \cdots
\end{tikzcd}$$

[1] The only cases I'm interested in are Ab, topological abelian groups, and functor categories built out of these where the domain category is small.

where, $\lim^0$ is naturally isomorphic to lim. As is the case for all derived functors, they are computed by replacing your objects with chain complexes of "nice" objects whose homology is isomorphic to the original object. Here "nice" means that the functor in question is exact when restricted to such objects. It turns out that $\lim^i = 0$ for $i \geq 2$ for this particular diagram shape [Mit73].

Suppose we have an object $A \in \mathcal{M}^{\mathbb{N}}$ as in Diag. (3.1). Then all the ϕ_is in can be assembled into a morphism $\phi := \prod_i \phi_i : \prod_i A_i \to \prod_i A_i$.

Theorem 3.1. $\lim^1 A$ *fits into the exact the exact sequence*

$$0 \to \lim A \to \prod_i A_i \xrightarrow{\phi - \mathrm{id}} \prod_i A_i \to \lim^1 A \to 0. \tag{3.2}$$

Proof. See Ch. III, §2 of [Rud98]. ∎

Remark. The content of this section is also true with inverse systems of the form

$$\cdots \to A_2 \to A_1 \to A_0 \to A_{-1} \to A_{-2} \to \cdots,$$

the category of which we denote by $\mathcal{M}^{\mathbb{Z}}$, with essentially no modifications.

3.2 Extending to pre-spectra

Suppose that F is a functor into a graded abelian groups with integration and $\mathbf{A}$ is any pre-spectrum. Then we get the following diagram:

$$F^{*+k+1}(\mathbf{A}_{k+1}) \xrightarrow{F^{*+k+1}(\alpha_k)} F^{*+k+1}(S\mathbf{A}_k) \xrightarrow{\int_F} F^{*+k}(\mathbf{A}_k),$$

where α_k is the k^{th} structure map of $\mathbf{A}$. We define

$$F^{*+k}_{\text{stab}}(\alpha) := \int_F \circ F^{*+k+1}(\alpha_k).$$

This promotes F to (a functor from pre-spectra to $(\mathsf{Ab}^{\bullet})^{\mathbb{Z}}$). Namely, we define $F^*(\mathbf{A})$ to be the following diagram:

$$\cdots \to F^{*+k+1}(\mathbf{A}_{k+1}) \xrightarrow{F^{*+k}_{\text{stab}}(\alpha)} F^k(\mathbf{A}) \xrightarrow{F^{*+k-1}_{\text{stab}}(\alpha)} F^{k-1}(\mathbf{A}_{k-1}) \to \cdots$$

We can regain a functor into graded abelian groups by composing with the lim functor.

3.3 Equivalent Diagram

Using the functors and natural transformations of Diag. (2.1), we define

$$L^* := \ker\left(E^*_{\mathbb{R}} \to E^*_{\mathbb{R}/\mathbb{Z}}\right), \qquad T^* := \ker\left(E^*_{\mathbb{R}/\mathbb{Z}} \to E^{*+1}\right),$$

$$P^* := (\Lambda^*_{\mathbb{Z}}) \prod_{E^*_{\mathbb{R}}} (E^*), \qquad cP^* := \left(\frac{\Lambda^*}{\Lambda^*_{\mathbb{Z}}}\right) \coprod_{E^*_{\mathbb{R}}} E^*_{\mathbb{R}/\mathbb{Z}},$$

and

$$A^* := \Lambda^*_{\text{exact}} \oplus E^*_{\text{tor}}.$$

Then we get the following short exact sequences of natural transformations:

$$0 \to T^{*-1} \xrightarrow{\iota_T} cP^{*-1} \xrightarrow{\pi_T} A^* \to 0 \tag{3.3}$$

and

$$0 \to A^* \xrightarrow{\iota_L} P^* \xrightarrow{\pi_L} L^* \to 0 \tag{3.4}$$

If W^*_E is a differential cohomology theory for E^*, it will fit into the following diagram of natural transformations:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & T^{*-1} \arrow[r, equal] \arrow[d, "\iota_T"] & T^{*-1} \arrow[r] \arrow[d, dotted, "\iota"] & 0 \arrow[d] & \\
0 \arrow[r] & cP^{*-1} \arrow[r, dotted, "i"] \arrow[d, "\pi_T"] & W^*_E \arrow[r, dotted, "\pi"] \arrow[d, dotted, "\delta"] & L^* \arrow[r] \arrow[d, equal] & 0 \,, \\
0 \arrow[r] & A^* \arrow[r, "\iota_L"] \arrow[d] & P^* \arrow[r, "\pi_L"] \arrow[d] & L^* \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{3.5}$$

where all the columns and rows are exact. The four dotted arrows here are determined by the dotted arrows in Diag. (2.2). In fact, a functor that fits into this diagram is equivalent to one fitting into Diag. (2.2). Additionally, all the functors in Diag. (3.3) and Diag. (3.4) have integration maps. If W^*_E is differential cohomology theory with integration, then all the natural transformations in Diag. (3.5) will be compatible with it. So differential cohomology theories with integration defined with this diagram are all equivalent to the original ones.

Remark. Diag. (3.5) shows the importance of of the functors T^* and L^*. The middle vertical short exact sequence tells us that because of T^{*-1}, a differential cohomology theory W^*_E is measuring something more than just an E-cohomology class and differential form which satisfy a coherence relation (which is what P^* does)[2]. Similarly,

[2]For instance, in the case when E^* is K-theory and using the geometric model given in [SS10b],

the middle horizontal short exact sequence tells us that because of L^*, neither is W_E^* just a quotient of $E^{*-1}_{\mathbb{R}/\mathbb{Z}} \oplus \frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}$ (which is what cP^{*-1} is).

3.4 An Invariant of Differential Cohomology Theories

Suppose $W_E^* \in \mathsf{DC}_E$. Then by promoting the three functors in the middle vertical short exact sequence of Diag. (3.5) to functors on pre-spectra as described above, and then applying them to the spectrum $\mathbf{E}$ which represents E^*, we get the following short exact sequence of inverse systems:

$$0 \to T^{-1}(\mathbf{E}) \xrightarrow{\iota} W_E^0(\mathbf{E}) \xrightarrow{\delta} P^0(\mathbf{E}) \to 0 \tag{3.6}$$

This defines an element $\Theta(W_E^*) \in \mathrm{Ext}^1(P^0(\mathbf{E}), T^{-1}(\mathbf{E}))$, in the sense of the isomorphism class of this short exact sequence[3]. An isomorphism between differential cohomology theories gives a morphism of the short exact sequence in Diag. (3.6), and thus they will have the same Θ.

By taking limits of these inverse systems, we get the following long exact sequence:

$$0 \to \lim T^{-1}(\mathbf{E}) \xrightarrow{\lim \iota} \lim W_E^0(\mathbf{E}) \xrightarrow{\lim \delta} \lim P^0(\mathbf{E}) \xrightarrow{\Delta_\Theta} \lim{}^1 T^{-1}(\mathbf{E}) \to \cdots . \tag{3.7}$$

Once again, if we have isomorphic differential cohomology theories, then they will induce the same coboundary map Δ_Θ in the above sequence. In the §4.3.1, the non-vanishing of the image of a particular chosen element in $\lim P^*(\mathbf{E})$ under the above coboundary map will be an obstruction to finding a morphism between two differential cohomology theories.

3.5 Eliminating the Obstruction

3.5.1 Additional constraints

In the following, to make our definition of our functor and establish an isomorphism between it and an arbitrary differential cohomology theory, we need to assume some further restraints on the pre-spectra used to represent underlying cohomology theories E^*, $E^*_{\mathbb{R}}$ and $E^*_{\mathbb{R}/\mathbb{Z}}$. We need

1. all the structure maps of the spectra involved to induce surjections for Λ^*, and
2. the adjoints of all the structure maps to be homotopy equivalences.

the functor T^* gives the isomorphism classes of trivializable bundles with flat connection, which is non-trivial itself in general.

[3]Even though, for all k, $T^{k-1}(\mathbf{E}_k)$ is an injective object in Ab, $T^{-1}(\mathbf{E})$ as a whole might not be an injective object in $\mathsf{Ab}^{\mathbb{Z}}$, so Θ is not trivially trivial.

By using iterated mapping cones, the adjunction between the suspension and loop space functors, and CW models for the loop space; any pre-spectrum in Top can be modified to a homotopy equivalent one that satisfies these conditions. We start with such spectra before plugging them into Theorem 1.9. Since the structure maps were transformed into simplicial inclusions, their smooth approximations will be smooth embeddings, and thus induce surjections for Λ^*.

3.5.2 Topologizing cohomology and associated functors

Proposition 3.2. *In the case of $\mathcal{M} = \mathsf{TAb}$, the continuous homomorphism $\phi - \mathrm{id}$ from Diag.* (3.2) *will have dense image.*

Lemma 3.3. *For an inverse system $A \in \mathsf{TAb}^{\mathbb{N}}$, if each A_i is compact and Hausdorff, then $\lim^1 A_i = 0$.*

Proof. Since the image of of $\phi - \mathrm{id}$ in Diag. (3.2) is the continuous image of compact space, it's compact. Since the target is Hausdorff, it's therefore closed. And by the above proposition, it's dense as well. Therefore $\phi - \mathrm{id}$ is surjective. ■

Recall from [Mil62] that if A^* is any additive cohomology functor, then for any increasing union of finite-type spaces

$$X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots, \tag{3.8}$$

we have the following short exact sequence:

$$0 \to \lim_i{}^1 A^{*-1}(X_i) \to A^*(X) \to \lim_i A^*(X_i) \to 0$$

where $X := \operatorname{colim}_i X_i$. The $\lim^1$ term can be computed from Diag. (3.2) using the inverse system

$$\cdots \to A^*(X_3) \to A^*(X_2) \to A^*(X_1),$$

where the homomorphisms are induced from the inclusions $X_i \hookrightarrow X_{i+1}$.

Lemma 3.4. *For every $k \in \mathbb{Z}$, $E^k_{\mathbb{R}/\mathbb{Z}}$ restricted to (the full subcategory of finite-type spaces and countable colimits (increasing unions) thereof) has a lift to a functor into compact Hausdorff topological abelian groups.*

Proof. Note that for a space X of finite type, for every $k \in \mathbb{Z}$, $T^k(X)$ will be canonically isomorphic to the quotient of the finite dimensional $\mathbb{R}$ vector space $E^k(X) \otimes \mathbb{R}$ by the image of $E^k(X)$ inside of it. Thus it will be a finite dimensional torus. If Y is also a space of finite type and $f : X \to Y$, then the induced homomorphism $E^k(Y) \otimes \mathbb{R} \to E^k(X) \otimes \mathbb{R}$ will be continuous with respect to the canonical topology on a finite dimensional $\mathbb{R}$ vector space. Therefore, the induced homomorphism on T^k will also be continuous.

Consider the short exact sequence

$$0 \to T^k \to E^k_{\mathbb{R}/\mathbb{Z}} \to E^{k+1}_{\mathrm{tor}} \to 0$$

of natural transformations. Since T^k is a divisible group, this (non-canonically) splits. For spaces of finite type, E^{k+1}_{tor} will be a finite abelian group, which we can give the discrete topology. We use this bijection between $E^k_{\mathbb{R}/\mathbb{Z}}$ and $T^k \times E^{k+1}_{\mathrm{tor}}$ to define a topology on the former, which will be independent of the choice of splitting. Thus, we have topologized $E^k_{\mathbb{R}/\mathbb{Z}}$ of every finite type space as a compact Hausdorff abelian group.

Now suppose we have an increasing union of finite type spaces, as in Diag. (3.8). Let $X := \operatorname{colim}_i X_i$. Our choice of topologies makes $E^k_{\mathbb{R}/\mathbb{Z}}$ applied to each factor compact and Hausdorff, and thus we can apply Lemma 3.3 in the case of $A^* = E^*_{\mathbb{R}/\mathbb{Z}}$ and conclude that $\lim^1_i E^{k-1}_{\mathbb{R}/\mathbb{Z}}(X_i) = 0$. Thus we can identify $E^k_{\mathbb{R}/\mathbb{Z}}(X)$ with $\lim_i E^k_{\mathbb{R}/\mathbb{Z}}(X_i)$. This can in turn be identified with the pre-image of a closed set (namely $\{0\}$) under a continuous map from a compact Hausdorff space (see Diag. (3.2)), and thus is a compact Hausdorff subspace. We use this series of identifications to topologize $E^k_{\mathbb{R}/\mathbb{Z}}(X)$. ■

We can try to carry the same construction through with E^*. Since E^k of a finite type space will be a finitely generated abelian group, we give it the discrete topology. The density of $\phi\,\mathrm{id}$ still holds, but the surjectivity does not in general. One can then see that the natural topology that one could put on E^k (which will make the Bockstein continuous) will be non-Hausdorff if the $\lim^1$ term is non-trivial. This term corresponds to the well-known phenomenon of phantom maps, and is indeed non-trivial in many cases (see Ch. III of [Rud98]). For our purposes, we make the following definition.

Definition 3.5. An element of some contravariant functor into abelian groups applied to a space is a *phantom* if it vanishes when pulled back to any finite-type space.

The above lemma proves the non-existence of non-trivial phantoms for the $\mathbb{R}/\mathbb{Z}$ cohomology of a certain class of spaces. The closure of $\{0\} \subseteq E^k$ with respect to the topology described above will be the set of phantom classes.

Definition 3.6. If $(B_k)_* : E^k_{\mathbb{R}/\mathbb{Z}} \to E^{k+1}$ is the Bockstein,

$$\overline{T}^k(X) := \left\{ c \in E^k_{\mathbb{R}/\mathbb{Z}}(X) \ : \ (B_k)_* c \text{ is a phantom.} \right\}.$$

Lemma 3.7. *For a space X which is a colimit of finite-type spaces, $\overline{T}^k(X)$ is a closed subspace of $E^k_{\mathbb{R}/\mathbb{Z}}(X)$.*

Proof. Recall that for a finite-type space X, we topologized $E^k_{\mathbb{R}/\mathbb{Z}}(X)$ such that the Bockstein would be continuous if we endowed $E^{k+1}(X)$ with the discrete topology.

Also note that in this case there can be no non-trivial phantoms in $E^{k+1}(X)$, and $\overline{T}^k(X) = T^k(X)$ is obviously closed.

Now suppose that $X = \operatorname{colim}_i X_i$ for an increasing union of finite-type spaces

$$X_1 \hookrightarrow X_2 \hookrightarrow X_3 \hookrightarrow \cdots .$$

Then all the arrows in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \lim_i E^k_{\mathbb{R}/\mathbb{Z}}(X_i) \arrow[r] \arrow[d, dotted] & \prod_i E^k_{\mathbb{R}/\mathbb{Z}}(X_i) \arrow[r] \arrow[d] & \prod_i E^k_{\mathbb{R}/\mathbb{Z}}(X_i) \arrow[rr] \arrow[d] & & 0 \\
0 \arrow[r] & \lim_i E^{k+1}(X_i) \arrow[r] & \prod_i E^{k+1}(X_i) \arrow[r] & \prod_i E^{k+1}(X_i) \arrow[r] & \lim_i{}^1 E^{k+1}(X_i) \arrow[r] & 0
\end{tikzcd}$$

are continuous if we topologize $\lim_i E^{k+1}(X_i)$ as a subspace of $\prod_i E^{k+1}(X_i)$. Most importantly, the dotted arrow in the above diagram is continuous. If we let $r_i : X_i \to X$ denote the inclusions of the finite pieces into X, then we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & E^k_{\mathbb{R}/\mathbb{Z}}(X) \arrow[r, "\prod_i r_i^*"] \arrow[d, "(B_k)_*"] & \lim_i E^k_{\mathbb{R}/\mathbb{Z}}(X_i) \arrow[r] \arrow[d, "\lim(B_k)_*"] & 0 \\
0 \arrow[r] & \lim_i{}^1 E^k(X) \arrow[r] & E^{k+1}(X) \arrow[r, "\prod_i r_i^*"] & \lim_i E^{k+1}(X_i) \arrow[r] & 0
\end{tikzcd} .$$

Then $\overline{T}^k(X)$ must be closed because

$$\overline{T}^k(X) = \ker\left(\prod_i r_i^* \circ (B_k)_*\right) = \ker\left(\lim(B_k)_* \circ \prod_i r_i^*\right)$$

and because $\lim(B_k)_* \circ \prod_i r_i^*$ is a continuous map into a Hausdorff space, and hence the pre-image of the closed set $\{0\}$ must be closed. ■

If we then apply the three previous lemmas, we get

Corollary 3.8. *For any spectrum* $\mathbf{A}$ *such that each* $\mathbf{A}_k$ *is a countable colimit of finite-type spaces,* $\lim^1 \overline{T}^*(\mathbf{A}) = 0$.

Chapter 4

Universal Model

The construction of our model was implicit in [BS10], and the natural transformation Φ together with the proof that it's an isomorphism are essentially direct lifts from the same.

4.1 Universal forms

For all $k \in \mathbb{Z}_{\geq 0}$, choose a closed form $\widetilde{\Omega}_k \in \Lambda^k(\mathbf{E}_k^{\mathbb{R}})$ whose de Rham class represents the universal element for $E_{\mathbb{R}}^k$. Let $\Omega_0 := \widetilde{\Omega}_0$. Let $\sigma^{\mathbb{R}}$ denotes the structure maps of $\mathbf{E}^{\mathbb{R}}$. Since $\Lambda_{\text{stab}}^k(\sigma^{\mathbb{R}})\widetilde{\Omega}_{k+1}$ is in the same de Rham class as $\widetilde{\Omega}_k$, their difference is an exact form $d\eta$. Since the integration map for Λ^* is strongly surjective and the structure maps are assumed induce surjections under Λ^*, $\Lambda_{\text{stab}}^k(\sigma^{\mathbb{R}})$ is surjective. Therefore, there exists $\tilde{\eta} \in \Lambda^{k+1}(\mathbf{E}_{k+1}^{\mathbb{R}})$ such that $\Lambda_{\text{stab}}^k(\sigma^{\mathbb{R}})(\tilde{\eta}) = \eta$. Thus

$$\Lambda_{\text{stab}}^k(\sigma^{\mathbb{R}})\left(\widetilde{\Omega}_{k+1} + d\tilde{\eta}\right) = \widetilde{\Omega}_k.$$

By starting with $k = 0$ and iterating the previous procedure, we can thus find $\left\{\Omega_k \ \in \Lambda^k(\mathbf{E}_k^{\mathbb{R}})\right\}_{k\in\mathbb{Z}_{\geq 0}}$ such that for all $k \in \mathbb{Z}_{\geq 0}$,

$$\Lambda_{\text{stab}}^k(\sigma^{\mathbb{R}})\left(\Omega_{k+1}\right) = \Omega_k$$

and Ω_k is in the same de Rham class of $\widetilde{\Omega}_k$ *i. e.*, we can find $\Omega \in \lim \Lambda^0(\mathbf{E}^{\mathbb{R}})$. Now let $\omega_k := \iota_k^* \Omega_k$ where $\iota : \mathbf{E} \to \mathbf{E}^{\mathbb{R}}$ is the map of spectra induced by the homomorphism $\mathbb{Z} \to \mathbb{R}$. These will satisfy an analogous stability property, and thus define an element $\omega \in \lim \Lambda_{\mathbb{Z}}^0(\mathbf{E})$.

4.2 Definition of model

4.2.1 The functor

For all $X \in \mathcal{C}$, we define the pointed set:

$$\widetilde{M}^k(X) := \left(\operatorname*{colim}_{\ell} \operatorname{Maps}\left(S^\ell X,\ \mathbf{E}_{k+\ell}\right)\right) \times \frac{\Lambda^{k-1}}{\Lambda^{k-1}_{\text{exact}}}(X). \tag{4.1}$$

Then, with our choice of $\{\omega_k \in \mathbf{E}_k\}_{k \in \mathbb{Z}_{\geq 0}}$, we define $\sim$ to be the equivalence relation given by

$$(\{h_1\}_\ell, \eta) \sim \left(\{h_0\}_\ell \ \ , \ \ \eta + (p_2)_* \left(\int_\Lambda \right)^\ell h^* \omega_{k+\ell} \right) \tag{4.2}$$

where "$\{\ \}_\ell$" denotes the stable class of a map from the ℓ^{th} stratum of the colimit in Eq. (4.1), $h : I_+ \wedge S^\ell X \to E_{k+\ell}$ is a base point preserving homotopy for some $\ell \geq 0$, $(p_2)_*$ is natural transformation given in Axiom C7, and $\int_\Lambda$ is the suspension map for forms. Note that this really is an equivalence relation. The correction form that gets added when one changes a map by a homotopy is independent of which stratum of the colimit in which one takes the homotopy as living.

$$\begin{aligned}
\left(\int_\Lambda \right)^\ell h^* \omega_{k+\ell} &= \left(\int_\Lambda \right)^\ell h^* \int_\Lambda \sigma^*_{k+\ell}\, \omega_{k+\ell+1} \\
&= \left(\int_\Lambda \right)^\ell \int_\Lambda (Sh)^* \sigma^*_{k+\ell}\, \omega_{k+\ell+1} \\
&= \left(\int_\Lambda \right)^{\ell+1} (\sigma_{k+\ell} \circ Sh)^*\, \omega_{k+\ell+1}.
\end{aligned}$$

If $h : I_+ \wedge S^\ell X \to \mathbf{E}_{k+\ell}$ is a homotopy between h_0 and h_1, then $\sigma_{k+\ell} \circ Sh$ is a homotopy between $\sigma_{k+\ell} \circ Sh_0$ and $\sigma_{k+\ell} \circ Sh_1$.

Theorem 4.1. *The pointed set*

$$M^k := \widetilde{M}^k / \sim$$

defines an object in $\mathsf{DC}_E^{\Lambda^{\text{thf}}}$.

Proof that M^* is a functor $\mathcal{C}^{\text{op}} \to \mathsf{Set}_0$. If $\varphi : X \to Y$ is a map of spaces, then we have an induced map $\varphi^* : M^k(Y) \to M^k(X)$ defined by

$$\varphi^*[\{f\}_\ell, \eta] = \left[\left\{ f \circ S^\ell \varphi \right\}_\ell, \varphi^* \eta \right]$$

Thus M^* is contravariant functor into graded pointed sets. ■

4.2.2 The natural transformations

Note that we have a natural transformation

$$\int_{\widetilde{M}} : \widetilde{M}^{k+1} \circ S \longrightarrow \widetilde{M}^k$$

which is the Cartesian product of the underlying operation of the suspension isomorphism for cohomology and the negative of integration along the fiber for forms, *i. e.*

$$\int_{\widetilde{M}} (\{f\}_\ell, \ \eta) = \left(\{f\}_{\ell+1}, \ -\int_\Lambda \eta \right).$$

This natural transformation descends to an integration natural transformation

$$\int_M : M^{k+1} \circ S \longrightarrow M^k.$$

The natural transformation $\delta_2 : M^k \to E^k$ is given by letting $\delta_2[f, \eta]$ be the stable homotopy class of f. Since our equivalence relation starts with stable classes of map, and then only allows them to change by a homotopy, this is obviously well-defined. Also, it is clear that this is compatible with the suspension isomorphism for E^*.

Let $c_{X,Y} : X \to Y$ denote the constant map from X to the base point of Y.

Lemma 4.2. *A well-defined natural transformation $i_2 : \frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}} \to M^k$ that is compatible with the integration natural transformations is given by the formula $i_2[\eta] = [\{c_{X,\mathbf{E}_k}\}_0, \eta]$.*

Proof. Suppose $\eta \in \Lambda^{k-1}_{\mathbb{Z}}(X)$. Then, $\eta = d\epsilon + \left(\int_\Lambda\right)^\ell g^*\omega_{k+\ell-1}$ for some map $g : S^\ell X \to \mathbf{E}_{k-1+\ell}$ and for some $\epsilon \in \Lambda^{k-2}(X)$. Since $\omega_{k+\ell-1} = \int_\Lambda \sigma^*_{k+\ell-1}\omega_{k+\ell}$, we have that

$$\begin{aligned}
\eta - d\epsilon &= \left(\int_\Lambda\right)^\ell g^*\omega_{k+\ell-1} = \left(\int_\Lambda\right)^\ell g^* \int_\Lambda \sigma^*_{k+\ell-1}\omega_{k+\ell} \\
&= \left(\int_\Lambda\right)^{\ell+1} (Sg)^*\sigma^*_{k+\ell-1}\omega_{k+\ell} = \left(\int_\Lambda\right)^{\ell+1} (\sigma_{k+\ell-1} \circ Sg)^*\omega_{k+\ell}.
\end{aligned}$$

If we think of $\sigma_{k+\ell-1} \circ Sg$ as a homotopy h between the constant map $c_{S^\ell X,\mathbf{E}_{k+\ell}}$ and itself, then

$$\eta - d\epsilon = (p_2)_* \left(\int_\Lambda\right)^\ell h^*\omega_{k+\ell}$$

which means that, by applying Eq. (4.2),

$$[\{c_{X,\mathbf{E}_k}\}_0, \eta] = [\{c_{S^\ell X,\mathbf{E}_{k+\ell}}\}_\ell, \eta] = [\{c_{S^\ell X,\mathbf{E}_{k+\ell}}\}_\ell, d\epsilon] = [\{c_{X,\mathbf{E}_k}\}_0, 0].$$

Therefore i_2 is well-defined. Compatibility with integration is verified by direct computation:

$$\int_M i_2[\eta] = \int_M [\{c_{SX,\mathbf{E}_{k+1}}\}_0, \eta] = \left[\{c_{SX,\mathbf{E}_{k+1}}\}_1, -\int_\Lambda \eta\right]$$
$$= \left[\{c_{X,\mathbf{E}_k}\}_0, -\int_\Lambda \eta\right] = i_2\left[-\int_\Lambda \eta\right] = i_2\left(-\int_{\Lambda/\Lambda_{\mathbb{Z}}} [\eta]\right).$$

■

This means that we have a well-defined an action of $\frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}$ on M^* via

$$[\eta'] \cdot [\{f\}_\ell, \eta] := [\{f\}_\ell, \eta + \eta'].$$

Here we are using $[-]$ to mean two different equivalence classes on two different sets: $[\eta'] \in \frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}$ is the equivalence class of $\eta' \in \frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\text{exact}}}$; and $[\{f\}_\ell, \eta] \in M^*$ is the equivalence class of $(\{f\}_\ell, \eta) \in \widetilde{M}^*$.

Lemma 4.3. *A well-defined natural transformation $\delta_1 : M^k \to \Lambda^k_{\mathbb{Z}}$ that is compatible with integration is given by the formula*

$$\delta_1[F, \eta] = d\eta + \left(\int_\Lambda\right)^\ell f^* \omega_{k+\ell}$$

where $f : S^\ell X \to \mathbf{E}_{k+\ell}$ is a representative for the stable class of F.

Proof. First we show this formula is independent of the choice of f. Note that

$$\begin{aligned}
\delta_1[\{\sigma_{k+\ell} \circ Sf\}_{\ell+1}, \eta] &= d\eta + \left(\int_\Lambda\right)^{\ell+1} (\sigma_{k+\ell} \circ Sf)^* \,\omega_{k+\ell+1} \\
&= d\eta + \left(\int_\Lambda\right)^{\ell+1} (Sf)^* \sigma^*_{k+\ell}\, \omega_{k+\ell+1} \\
&= d\eta + \left(\int_\Lambda\right)^{\ell} f^* \left(\int_\Lambda \sigma^*_{k+\ell}\, \omega_{k+\ell+1}\right) \\
&= d\eta + \left(\int_\Lambda\right)^{\ell} f^*\, \omega_{k+\ell} \\
&= \delta_1[\{f\}_\ell, \eta]
\end{aligned}$$

Second, we show that this formula is compatible with the equivalence relation given in Eq. (4.2). Suppose $h : I_+ \wedge S^\ell X \to \mathbf{E}_{k+\ell}$. Because d and $(p_2)_*$ both anticommute with $\int_\Lambda$, their composition commutes. So

$$\begin{aligned}
\delta_1[\{h_1\}_\ell, \eta] &= d\eta + \left(\int_\Lambda\right)^\ell h_1^*\,\omega_{k+\ell} \\
&= d\eta + \left(\int_\Lambda\right)^\ell (h_0^* + (d(p_2)_* + (p_2)_* d)h^*)\,\omega_{k+\ell} \\
&= d\eta + \left(\int_\Lambda\right)^\ell (h_0^* + d(p_2)_* h^*)\,\omega_{k+\ell} \\
&= \delta_1[\{h_0\}_\ell, \eta] + \left(\int_\Lambda\right)^\ell d(p_2)_* h^* \omega_{k+\ell} \\
&= \delta_1[\{h_0\}_\ell, \eta] + d(p_2)_* \left(\int_\Lambda\right)^\ell h^* \omega_{k+\ell} \\
&= \delta_1 \left[\{h_0\}_\ell \ \ , \ \ \eta + (p_2)_* \left(\int_\Lambda\right)^\ell h^* \omega_{k+\ell}\right].
\end{aligned}$$

Therefore δ_1 is well-defined. Compatibility with integration is once again straightforward. Let $F : S^\ell SX \to \mathbf{E}_{k+1+\ell}$ and $\eta \in \Lambda^k(SX)$. Then

$$\begin{aligned}
\int_\Lambda \delta_1[\{f\}_\ell, \eta] &= \int_\Lambda \left(\left(\int_\Lambda\right)^\ell f^* \omega_{k+1+\ell} + d\eta\right) \\
&= \left(\int_\Lambda\right)^{\ell+1} f^* \omega_{k+1+\ell} - d\int_\Lambda \eta \\
&= \delta_1 \left[\{f\}_{\ell+1}, -\int_\Lambda \eta\right] \\
&= \delta_1 \int_M [\{f\}_\ell, \eta].
\end{aligned}$$

■

The construction of i_1 is more involved, and actually unnecessary for our particular use for M^*. Therefore, we relegate it to Appendix A2. It is completely straightforward check that each of the above natural transformations are compatible with the action of $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}$ in the appropriate way. Thus, we have proved Theorem 4.1.

4.3 Universality of M^*

4.3.1 Stable universal elements

Suppose $W_E^* \in \mathsf{DC}_E$. Then we get the middle vertical short exact sequence from Diag. (3.5):

$$0 \to T^{*-1} \xrightarrow{\iota} W_E^* \xrightarrow{\delta} P^* \to 0.$$

The previous choices of $\{\omega_k\}_{k\in\mathbb{Z}_{\geq 0}}$ give elements $(\omega_k, [\mathrm{id}_{\mathbf{E}_k}]) \in P^k(\mathbf{E}_k)$. For all $k \in \mathbb{Z}_{\geq 0}$, choose $\tilde{\mathcal{U}}_k \in W^k_E(\mathbf{E}_k)$ such that $\delta(\tilde{\mathcal{U}}_k) = (\omega_k, [\mathrm{id}_{\mathbf{E}_k}])$. Then

$$\delta\left((W^k_{E\mathrm{stab}}\tilde{\mathcal{U}}_{k+1}) - \tilde{\mathcal{U}}_k\right) = \left(\Lambda^k_{\mathrm{stab}}\omega_{k+1} - \omega_k\ ,\ E^k_{\mathrm{stab}}[\mathrm{id}_{\mathbf{E}_{k+1}}] - [\mathrm{id}_{\mathbf{E}_k}]\right) = 0,$$

so

$$(W^k_{E\mathrm{stab}}\tilde{\mathcal{U}}_{k+1}) - \tilde{\mathcal{U}}_k = \iota(t_k)$$

for a unique $t_k \in T^{k-1}(\mathbf{E}_k)$. The collection of these $\{t_k \in T^{k-1}(\mathbf{E}_k)\}_{k\in\mathbb{Z}}$ determines an element in $\lim^1 T^{*-1}(\mathbf{E})$ (*cf.*Diag. (3.2)). In fact, the previous construction is precisely how one constructs the coboundary map Δ_Θ from Diag. (3.7). The fact that

$$P^k_{\mathrm{stab}}(\omega_{k+1}, [\mathrm{id}_{\mathbf{E}_{k+1}}]) = (\omega_k, [\mathrm{id}_{\mathbf{E}_k}])$$

means that $\{(\omega_k, [\mathrm{id}_{\mathbf{E}_k}])\}_{k\in\mathbb{Z}_{\geq 0}} \in \lim P^0(\mathbf{E})$. $\Delta_{\Theta(W^*_E)}$ of this element is the aforementioned element of $\lim^1 T^{-1}(\mathbf{E})$.

If we assume that $\forall\, k \in \mathbb{Z}_{\geq 0}$, $\mathbf{E}_k$ is a countable colimit of finite-type spaces, then we can apply Corollary 3.8 and conclude that $\lim^1 \overline{T}^{*-1}(\mathbf{E}) = 0$. This means that there exists $\{\tau_k \in \overline{T}^{k-1}(\mathbf{E}_k)\}_{k\in\mathbb{Z}_{\geq 0}}$ such that $\overline{T}^{k-1}_{\mathrm{stab}}\tau_{k+1} - \tau_k = t_k$. So if we recall that
$overline T^{k-1} \subseteq E^{k-1}_{\mathbb{R}/\mathbb{Z}}$ and define $\mathcal{U}_k := \tilde{\mathcal{U}}_k - i_1(\tau_k)$, then

$$W^k_{E\mathrm{stab}}\mathcal{U}_{k+1} = \mathcal{U}_k,$$

i. e., we have an element $\mathcal{U} \in \lim W^0_E(\mathbf{E})$. However, it is not the case that $\lim \delta(\mathcal{U}) = \{(\omega_k, [\mathrm{id}_{\mathbf{E}_k}])\}_{k\in\mathbb{Z}_{\geq 0}}$. We pay for this stability with a change in the value of δ_2:

$$\delta_2\mathcal{U}_k = \delta_2(\tilde{\mathcal{U}}_k - i_1\tau_k) = [\mathrm{id}_{\mathbf{E}_k}] - (B_{k-1})_*\tau_k.$$

However, $(B_{k-1})_*\tau_k$ is a phantom by the definition of $\overline{T}^{k-1}$.

4.3.2 The natural transformation Φ

Let $U : \mathsf{Ab}^\bullet \to \mathsf{Set}^\bullet_0$ be the forgetful functor from graded abelian groups to graded pointed sets.

Theorem 4.4. *A natural transformation $\Phi : M^* \to U \circ W^*_E$ is given by the formula*

$$\Phi_k[\{f\}_\ell, \eta] = \left(\int_{W_E}\right)^\ell f^*\mathcal{U}_{k+\ell} + i_{2,W_E}[\eta].$$

Proof. Notice that because W^*_E satisfies the homotopy formula in Eq. (2.3), Φ is well-defined. To wit, if $h : I_+ \wedge S^\ell X \to \mathbf{E}_{k+\ell}$, and $\eta \in \Lambda^{k-1}(X)$, then

$$\begin{aligned}
\Phi_k[\{h_1\}_\ell, \eta] &= \left(\int_{W_E}\right)^\ell h_1^* \mathcal{U}_{k+\ell} + i_2[\eta] \\
&= \left(\int_{W_E}\right)^\ell (h_0^* \mathcal{U}_{k+\ell} + i_2[(p_2)_* h^* \delta_1 \mathcal{U}_{k+\ell}]) + i_2[\eta] \\
&= \left(\int_{W_E}\right)^\ell (h_0^* \mathcal{U}_{k+\ell} + i_2[(p_2)_* h^* \omega_{k+\ell}]) + i_2[\eta] \\
&= \left(\int_{W_E}\right)^\ell h_0^* \mathcal{U}_{k+\ell} + i_2 \left[\eta + \left(-\int_\Lambda\right)^\ell (p_2)_* h^* \omega_{k+\ell}\right] \\
&= \left(\int_{W_E}\right)^\ell h_0^* \mathcal{U}_{k+\ell} + i_2 \left[\eta + (p_2)_* \left(\int_\Lambda\right)^\ell h^* \omega_{k+\ell}\right] \\
&= \Phi_k \left[\{h_0\}_\ell\,,\ \eta + (p_2)_* \left(\int_\Lambda\right)^\ell h^* \omega_{k+\ell}\right].
\end{aligned}$$

Next we prove naturality. Suppose $\varphi : X \to Y$. If $[\{f\}_\ell, \eta] \in M^k(Y)$, then

$$\begin{aligned}
\Phi_k \varphi^*[\{f\}_\ell, \eta] &= \Phi_k \left[\left\{f \circ S^\ell \varphi\right\}_\ell\,,\ \varphi^* \eta\right] \\
&= \left(\int_{W_E}\right)^\ell \left(f \circ S^\ell \varphi\right)^* \mathcal{U}_{k+\ell} + i_2[\varphi^* \eta] \\
&= \varphi^* \left(\int_{W_E}\right)^\ell f^* + \varphi^* i_2[\eta] \\
&= \varphi^* \Phi_k[\{f\}_\ell, \eta].
\end{aligned}$$

Thus, Φ is natural. ■

Theorem 4.5. *Φ is compatible with δ_1, i_2, and the integration natural transformation; it is compatible with δ_2 when restricted to finite-type spaces. I.e,*

- $\delta_1 \Phi = \delta_{1,M}$,
- $\Phi i_{2,M} = i_2$,
- $\Phi_k \int_M = \int_{W_E} \Phi_{k+1}$,
- $\delta_2 \Phi = \delta_{2,M}$ *for finite-type spaces, and*
- Φ *is equivariant with respect to the action of* $\frac{\Lambda^{*-}}{\Lambda^{*-}_{\mathbb{Z}}}$.

The additional "M" subscript refers to the natural transformations defined for M^*.

Proof. It's compatible with δ_1 because

$$\begin{aligned}\delta_1\Phi_k[\{f\}_\ell,\eta] &= \delta_1\left(\left(\int_{W_E}\right)^\ell f^*\mathfrak{U}_{k+\ell} + i_2[\eta]\right) = \left(\int_\Lambda\right)^\ell \delta_1 f^*\mathfrak{U}_{k+\ell} + \delta_1 i_2[\eta] \\ &= \left(\int_\Lambda\right)^\ell f^*\omega_{k+\ell} + d\eta = \delta_{1,M}[\{f\}_\ell,\eta].\end{aligned}$$

Φ is compatible with i_2 because we're working with the reduced versions of all our functors, so every element vanishes when restricted to the base point:

$$\Phi_k i_{2,M}[\eta] = \Phi_k[\{c_{X,\mathbf{E}_k}\}_0,\eta] = c^*_{X,\mathbf{E}_k}\mathfrak{U}_k + i_2[\eta] = i_2[\eta].$$

Note that in general, if $[\{f\}_\ell,\eta]\in M^k(X)$ then

$$\begin{aligned}\delta_2\Phi_k[\{f\}_\ell,\eta] &= \delta_2\left(\left(\int_{W_E}\right)^\ell f^*\mathfrak{U}_{k+\ell} + i_2[\eta]\right) = \left(\int_E\right)^\ell \delta_2 f^*\mathfrak{U}_{k+\ell} + \delta_2 i_2[\eta] \\ &= \left(\int_E\right)^\ell f^*([\mathrm{id}_{\mathbf{E}_{k+\ell}}] - (B_{k+\ell-1})_*\tau_{k+\ell}) \\ &= \left(\int_E\right)^\ell ([f] - f^*(B_{k+\ell-1})_*\tau_{k+\ell}) \\ &\neq \left(\int_E\right)^\ell [f] = \delta_{2,M}[\{f\}_\ell,\eta].\end{aligned}$$

But if X is of finite type, then the fact that $(B_{k+\ell-1})_*\tau_{k+\ell}$ is a phantom means that $f^*(B_{k+\ell-1})_*\tau_{k+\ell} = 0$, and thus $\delta_2\Phi_k = \delta_{2,M}$. Φ is compatible with the integration homomorphism because

$$\begin{aligned}\Phi_k\int_M[\{f\}_\ell,\eta] &= \Phi_k\left[\{f\}_{\ell+1}\,,\,-\int_\Lambda\eta\right] = \left(\int_{W_E}\right)^{\ell+1} f^*\mathfrak{U}_{k+\ell+1} + i_2\left[-\int_\Lambda\eta\right] \\ &= \left(\int_{W_E}\right)^{\ell+1} f^*\mathfrak{U}_{k+\ell+1} + \int_{W_E} i_2[\eta] \\ &= \int_{W_E}\left(\left(\int_{W_E}\right)^\ell f^*\mathfrak{U}_{k+\ell+1} + i_2[\eta]\right) \\ &= \int_{W_E}\Phi_{k+1}[\{f\}_\ell,\eta].\end{aligned}$$

Note that because

$$\begin{aligned} i_2[\eta'] + \Phi[\{f\}_\ell, \eta] &= \left(\int_{W_E}\right)^\ell f^*\mathfrak{U}_{k+\ell} + i_2[\eta] + i_2[\eta'] \\ &= \Phi[\{f\}_\ell, \eta + \eta'] = \Phi\left([\eta'] \cdot [\{f\}_\ell, \eta]\right), \end{aligned}$$

Φ is equivariant. ■

Theorem 4.6. *Φ is an isomorphism of graded pointed sets when the functors are restricted to the category of finite-type spaces.*

Proof of surjectivity. For any element $a \in W_E^k(X)$, $\delta_2(a) = \left(\int_E\right)^\ell [f]$ for some $f : S^\ell X \to \mathbf{E}_{k+\ell}$. Then

$$\begin{aligned} \delta_2\left(a - \left(\int_{W_E}\right)^\ell f^*\mathfrak{U}_{k+\ell}\right) = 0 \quad &\Longrightarrow \quad a - \left(\int_{W_E}\right)^\ell f^*\mathfrak{U}_{k+\ell} = i_2[\eta] \\ &\Longrightarrow \quad a = \left(\int_{W_E}\right)^\ell f^*\mathfrak{U}_{k+\ell} + i_2[\eta] = \Phi_k[\{f\}_\ell, \eta] \end{aligned}$$

So Φ is surjective. ■

Proof of injectivity. First consider the special case when that

$$\Phi_k[\{f\}_\ell, \eta] = \Phi_k[\{f\}_\ell, \xi].$$

Then $i_2[\eta - \xi] = 0 \iff \eta - \xi \in \Lambda_{\mathbb{Z}}^{k-1}(X) \iff \eta - \xi = d\alpha + \left(\int_\Lambda\right)^n \varphi^*\omega_{k-1+n}$ for some $\varphi : S^n X \to \mathbf{E}_{k-1+n}$. Since $n > 0$ without loss of generality, we can redefine ℓ to be the $\max(\ell, n-1)$, and stabilize both φ and f so that $f : S^\ell X \to \mathbf{E}_{k+\ell}$ and $\varphi : S^{\ell+1}X \to \mathbf{E}_{k+\ell}$ for this new value of ℓ.

Think of φ as a homotopy between the constant map $c_{S^\ell X, \mathbf{E}_{k+\ell}}$ and itself. Then let $h : I_+ \wedge S^\ell X \to \mathbf{E}_{k+\ell}$ be the homotopy obtained by taking the point-wise product of φ and the constant homotopy between f and itself, which results in a non-trivial homotopy between f and itself.[1] Then

$$(p_2)_* \left(\int_\Lambda\right)^\ell h^*\omega_{k+\ell} = \left(\int_\Lambda\right)^{\ell+1} \varphi^*\omega_{k+\ell} = \eta - \xi - d\alpha$$

which implies that

$$[\{f\}_\ell, \eta] = \left[\{f\}_\ell\,,\ \xi + (p_2)_*\left(\int_\Lambda\right)^\ell h^*\omega_{k+\ell}\right] = [\{f\}_\ell, \xi].$$

[1]Here we are using the fact that $\mathbf{E}_{k+\ell}$ can be taken to be a topological group.

Now consider the general case where

$$\Phi_k[\{f\}_\ell, \eta] = \Phi_k[\{g\}_\ell, \xi].$$

Then

$$\begin{aligned} \left(\int_{W_E}\right)^\ell f^*\mathcal{U}_{k+\ell} + i_2[\eta] &= \left(\int_{W_E}\right)^\ell g^*\mathcal{U}_{k+\ell} + i_2[\xi] \\ \Longrightarrow \left(\int_{W_E}\right)^\ell (f^* - g^*)\mathcal{U}_{k+\ell} &= i_2[\xi - \eta] \\ \Longrightarrow 0 = \delta_2 \left(\int_{W_E}\right)^\ell (f^* - g^*)\mathcal{U}_{k+\ell} &= \left(\int_E\right)^\ell ([f] - [g]) \end{aligned}$$

which implies that there is a homotopy $h : I_+ \wedge S^m X \to \mathbf{E}_{k+m}$ such that $\{h_1\}_m = \{f\}_\ell$ and $\{h_0\}_m = \{g\}_\ell$. Therefore,

$$\begin{aligned} [\{f\}_\ell, \eta] = [\{h_1\}_m, \eta] &= \left[\{h_0\}_m\,,\ \eta + (p_2)_* \left(\int_\Lambda\right)^m h^*\omega_{k+m}\right] \\ &= \left[\{g\}_\ell\,,\ \eta + (p_2)_* \left(\int_\Lambda\right)^m h^*\omega_{k+m}\right], \end{aligned}$$

and by the previous case, $\left[\{g\}_\ell\,,\ \eta + (p_2)_* \left(\int_\Lambda\right)^m h^*\omega_{k+m}\right] = [\{g\}_\ell, \xi]$. Thus Φ is injective. ∎

Theorem 4.7. *Suppose E^* is a cohomology theory such that $E^*(\mathrm{pt})$ is finitely generated in each degree. Then, given two differential cohomology theories $A^*_E, B^*_E \in \mathsf{DC}_E$, there is a natural isomorphism between their restrictions to the full subcategory of finite-type spaces that is compatible with i_2, δ_1, δ_2, and their integration natural transformations.*

Proof. Recall the forgetful functor U_Λ from §2.3. We have a natural isomorphism $U_\Lambda(A^*_E) \to U_\Lambda(B^*_E)$ given by appropriately composing the isomorphisms obtained from the previous theorem. Because of Theorem 2.19, it must actually be a homomorphism. ∎

Theorem 4.8. *The set of all such natural isomorphisms is a torsor for* $\lim \overline{T}^{-1}(\mathbf{E})$.

Proof. This group is precisely where the difference between any two choices of the stable universal elements $\mathcal{U} \in \lim A^*_E(\mathbf{E})$ will lie. And the choice of this element was the only choice made in constructing Φ. ∎

Remark. We also made a choice of $\Omega \in \Lambda^0_{\mathrm{stab}}\left(\mathbf{E}^{\mathbb{R}}\right)$, but this choice was made independently of any given differential cohomology theory and is analogous to a choice of a basis that controls the formula for an isomorphism. Technically speaking, a difference choice of Ω leads to a different M^* and a different Φ. But as Theorem 4.7 shows, this new M^* will be isomorphic to the original.

4.4 Future Directions

In retrospect, it seems clear that the complexity of the axioms for $\mathcal{C}$ could be reduced by using the language of 2-categories. Since both cohomology and forms have higher degree integration maps, it might be fruitful to consider differential cohomology as functor of $(\infty, 1)$-categories[2], where we take into account all higher homotopies. Co-completeness would then be replaced with homotopy co-completeness. Hopkins has conjectured that differential cohomology is the homotopy fiber product of E^* and $\Lambda^*_{\mathbb{Z}}$ in a category large enough such that *both* of these functors are representable. The "homotopy" would then take care of the "torus" T^{*-1} that obstructed differential cohomology from just being the naïve fiber product. A guess for such a category would be the category simplicial sheaves on a small Grothendieck site.

On the other hand, one also wants $\mathcal{C}$ to be small enough that any differential cohomology theory defined on Mfld could be extended to $\mathcal{C}$. For a simplicial complex, we could use the trick of taking a small neighborhood in a large enough dimensional Euclidean space (*cf.* Theorem 1.9). The inclusion of the complex has a deformation retract in Top, and so if one can enlarge Ω^* to a forms functor Λ^* that can handle such non-smoothness at the end of the homotopy, one could use the homotopy formula in Eq. (2.3) as a guide to extending a differential cohomology theory on Mfld to SSet.

[2]See [Lur09] for an encyclopedic introduction to higher categories.

Bibliography

[BS10] Ulrich Bunke and Thomas Schick. Uniqueness of smooth extensions of generalized cohomology theories. *J. Topol.*, 3(1):110–156, 2010, http://dx.doi.org/10.1112/jtopol/jtq002.

[Che73] Kuo-Tsai Chen. Iterated integrals of differential forms and loop space homology. *Ann. of Math. (2)*, 97:217–246, 1973.

[CS85] Jeff Cheeger and James Simons. Differential characters and geometric invariants. In *Geometry and topology (College Park, Md., 1983/84)*, volume 1167 of *Lecture Notes in Math.*, pages 50–80. Springer, Berlin, 1985.

[DFM10] Jacqeus Distler, Daniel S. Freed, and Gregory W. Moore. Spin structures and superstrings. *ArXiv e-prints*, 2010, arXiv:1007.4581 [hep-th].

[GS89] Henri Gillet and Christophe Soulé. Arithmetic Chow groups and differential characters. In *Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987)*, volume 279 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 29–68. Kluwer Acad. Publ., Dordrecht, 1989.

[Har89] Bruno Harris. Differential characters and the Abel-Jacobi map. In *Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987)*, volume 279 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 69–86. Kluwer Acad. Publ., Dordrecht, 1989.

[HL06] Reese Harvey and Blaine Lawson. From sparks to grundles—differential characters. *Comm. Anal. Geom.*, 14(1):25–58, 2006, http://projecteuclid.org/getRecord?id=euclid.cag/1154442128.

[HLZ03] Reese Harvey, Blaine Lawson, and John Zweck. The de Rham-Federer theory of differential characters and character duality. *Amer. J. Math.*, 125(4):791–847, 2003, http://muse.jhu.edu/journals/american_journal_of_mathematics/v125/125.4harvey.pdf.

[HS05] M. J. Hopkins and I. M. Singer. Quadratic functions in geometry, topology, and M-theory. *J. Differential Geom.*, 70(3):329–452, 2005, http://projecteuclid.org/getRecord?id=euclid.jdg/1143642908.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Mil62] J. Milnor. On axiomatic homology theory. *Pacific J. Math.*, 12:337–341, 1962.

[Mit73] Barry Mitchell. The cohomological dimension of a directed set. *Canad. J. Math.*, 25:233–238, 1973.

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.

[Rud98] Yuli B. Rudyak. *On Thom spectra, orientability, and cobordism*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. With a foreword by Haynes Miller.

[SS08] James Simons and Dennis Sullivan. Axiomatic characterization of ordinary differential cohomology. *J. Topol.*, 1(1):45–56, 2008, `http://dx.doi.org/10.1112/jtopol/jtm006`.

[SS10a] J. Simons and D. Sullivan. The Mayer-Vietoris Property in Differential Cohomology. *ArXiv e-prints*, October 2010, `arXiv:1010.5269 [math.AT]`.

[SS10b] James Simons and Dennis Sullivan. Structured vector bundles define differential K-theory. In *Quanta of maths*, volume 11 of *Clay Math. Proc.*, pages 579–599. Amer. Math. Soc., Providence, RI, 2010.

Appendix

A1 Proof of Mayer-Vietoris Property

See §2.1.1 for the setup of the problem.

Note that $\gamma \circ \iota_A$ and $\gamma \circ \iota_B$ are null-homotopic, so $W_E^*(\gamma \circ \iota_A)$ and $W_E^*(\gamma \circ \iota_B)$ lift to homomorphisms from $W_E^*(SD)$ to $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}(A)$ and $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}(B)$ respectively. This defines a homomorphism

$$W_E^*(SD) \xrightarrow{\quad\kappa\quad} \frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}(A) \prod_{\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}(D)} \frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}(B)$$

where the target is the fiber product of $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}(A)$ and $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}(B)$ over $\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}(D)$, *i. e.*, pairs of quotient forms on A and B which agree when restricted to D. Given such a pair, one could take representatives for each quotient form, restrict the representatives to D, and then take their difference. Since this difference is zero as a quotient form, the result must always be an element of $\Lambda_{\mathbb{Z}}^{*-1}(D)$. This is well-defined up to adding elements of $\Lambda_{\mathbb{Z}}^{*-1}(A)$ and $\Lambda_{\mathbb{Z}}^{*-1}(B)$ to the respective original representatives. So we have a well-defined homomorphism

$$\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}(A) \prod_{\frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}(D)} \frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}(B) \xrightarrow{\quad\phi\quad} \frac{\Lambda_{\mathbb{Z}}^{*-1}(D)}{\operatorname{im}\left(\Lambda_{\mathbb{Z}}^{*-1}(\rho_A) + \Lambda_{\mathbb{Z}}^{*-1}(\rho_B)\right)}.$$

We have the natural map $r_D : S^1 \times D \to SD$ which collapses $(\{0\} \times D) \sqcup (S^1 \times \{*_D\})$ to a point (where $*_D$ is the base point of D). In terms of all of the above maps we have

Lemma A.1. *The following diagram commutes:*

$$\begin{tikzcd}
W^*_E(SD) \arrow[r, "\kappa"] \arrow[d, "\delta_1"] & \frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}(A) \prod\limits_{\frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}(D)} \frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}(B) \arrow[r, "\phi"] & \frac{\Lambda^{*-1}_{\mathbb{Z}}(D)}{\mathrm{im}\left(\Lambda^{*-1}_{\mathbb{Z}}(\rho_A)+\Lambda^{*-1}_{\mathbb{Z}}(\rho_B)\right)} \\
\Lambda^*_{\mathbb{Z}}(SD) \arrow[r, "r^*_D"] & \Lambda^*_{\mathbb{Z}}(S^1 \times D) \arrow[r, "(p_2)_*"] & \Lambda^{*-1}_{\mathbb{Z}}(D) \arrow[u]
\end{tikzcd} \tag{A.1}$$

Note that the map $(p_2)_* : \Lambda^*_{\text{closed}}(S^1 \times D) \to \Lambda^{*-1}_{\text{closed}}(D)$ doesn't necessarily send special forms to special forms, but that $(p_2)_* \circ r^*_D$ does. See §2.2.

Proof. Note that the image of $\gamma \circ \iota_A$ is in the top half of SD, and thus $\gamma \circ \iota_A$ can be homotoped to the constant map to the top vertex by using the standard deformation retract of a cone onto its vertex. Then after doing the same thing with $\gamma \circ \iota_B$ and the bottom half of SD, we can use the homotopy property to provide a lift K of κ:

$$\begin{tikzcd}
 & \Lambda^{*-1}(A) \oplus \Lambda^{*-1}(B) \arrow[d] \\
W^*_E(SD) \arrow[ur, "K"] \arrow[r, "\kappa"] & \frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}(A) \prod\limits_{\frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}(D)} \frac{\Lambda^{*-1}}{\Lambda^{*-1}_{\mathbb{Z}}}(B)
\end{tikzcd}$$

by letting

$$K \ := \ (p_2)_* \, (h^*_A \oplus h^*_B) \, \delta_1$$

where h_A and h_B are the two above described homotopies. From this, it is straightforward to see that $\forall\, x \in W^*_E(SD)$,

$$\phi\, \kappa(x) = \left\{ \big((p_2)_* h^*_A \ - \ (p_2)_* h^*_B \big) \delta_1(x) \right\} = \{ (p_2)_* r^*_D \delta_1(x) \}$$

because integrating out the contractions h_A and h_B is the same as integrating along the suspension directions in SD. Integrating over the top cone of SD is taken care of by h_A, and the bottom by h_B (though backwards, which takes care of the minus sign). ■

Lemma A.2. $(p_2)_* r^*_D$ *from Diag.* (A.1) *is surjective.*

Corollary A.3. *$\phi\, \kappa$ is surjective. Or equivalently, for any $\xi \in \Lambda^{*-1}_{\mathbb{Z}}(D)$, $\exists\, y \in W^*_E(SD)$ such that $\xi = \rho^*_A \alpha - \rho^*_B \beta$, where $i_2[\alpha] = (\gamma \circ \iota_A)^* y$ and $i_2[\beta] = (\gamma \circ \iota_B)^* y$.*

Proof of Mayer Vietoris Theorem *(Thm. 2.8).* Let $a \in W^*_E(A)$ and $b \in W^*_E(B)$ such that $\rho^*_A a = \rho^*_B b$. We need to show that $\exists\, x \in W^*_E(X)$ such that $\iota^*_A x = a$ and $\iota^*_B x = b$. Note that

$$\rho^*_A \delta_2 a = \delta_2 \rho^*_A a = \delta_2 \rho^*_B b = \rho^*_B \delta_2 b,$$

so because E^* has the MV property, $\exists\, \chi \in E^*(X)$ such that $\iota_A^*\chi = \delta_2 a$ and $\iota_B^*\chi = \delta_2 b$. Because δ_2 is surjective, $\exists\, \tilde{\tilde{x}} \in W_E^*(X)$ such that $\delta_2\tilde{\tilde{x}} = \chi$. Then

$$\delta_2\iota_A^*\tilde{\tilde{x}} = \iota_A^*\delta_2\tilde{\tilde{x}} = \iota_A^*\chi = \delta_2 a.$$

So

$$\delta_2(\iota_A^*\tilde{\tilde{x}} - a) = 0 \qquad \Longrightarrow \qquad \iota_A^*\tilde{x} - a = i_2[\eta_A]$$

for some $[\eta_A] \in \frac{\Lambda^{*-1}}{\Lambda_{\mathbb{Z}}^{*-1}}(A)$; and similarly, $\iota_B^*\tilde{\tilde{x}} - b = i_2[\eta_B]$. Then since

$$i_2(\rho_A^*[\eta_A] - \rho_B^*[\eta_B]) = \rho_A^*(\iota_A^*\tilde{\tilde{x}} - a) - \rho_B^*(\iota_B^*\tilde{\tilde{x}} - b) = \rho_B^* b - \rho_A^* a = 0,$$

and i_2 is injective, $\rho_A^*[\eta_A] - \rho_B^*[\eta_B] = 0$. This implies that $\rho_A^*\eta_A - \rho_B^*\eta_B =: \xi \in \Lambda_{\mathbb{Z}}^{*-1}(D)$ (note that now we're dealing with actual forms, not quotient forms).

By Corollary A.3, we can find an element $y \in W_E^*(SD)$ such that $(\gamma\circ\iota_A)^*y = i_2[\alpha]$ and $(\gamma \circ \iota_B)^*y = i_2[\beta]$ where $\xi = \rho_A^*\alpha - \rho_B^*\beta$. Let $\tilde{x} := \tilde{\tilde{x}} - \gamma^*y$. Then

$$\iota_A^*\tilde{x} - a = \iota_A^*\tilde{\tilde{x}} - (\gamma \circ \iota_A)^*y - a = i_2[\eta_A] - (\gamma \circ \iota_A)^*y = i_2[\eta_A - \alpha]$$

and similarly, $\iota_B^*\tilde{x} - b = i_2[\eta_B - \beta]$. Because

$$\rho_A^*(\eta_A - \alpha) - \rho_B^*(\eta_B - \beta) = (\rho_A^*\eta_A - \rho_B^*\eta_B) - (\rho_A^*\alpha - \rho_B^*\beta) = \xi - \xi = 0,$$

and because Λ^* has the strong MV property, $\exists!\, \eta \in \Lambda^{*-1}(X)$ such that $\iota_A^*\eta = \eta_A - \alpha$ and $\iota_B^*\eta = \eta_B - \beta$. So let $x := \tilde{x} - i_2[\eta]$. Then

$$\iota_A^*x = \iota_A^*(\tilde{x} - i_2[\eta]) = \iota_A^*\tilde{x} - i_2\iota_A^*\eta = \iota_A^*\tilde{x} - i_2[\eta_A - \alpha] = a$$

and similarly, $\iota_B^*x = b$. ■

A2 The natural transformation i_1 for M^*

If we use the mapping cone model for $\mathbf{E}^{\mathbb{R}/\mathbb{Z}}$, then we have the following commutative diagram of spectrum maps:

$$\begin{tikzcd} \mathbf{E}^{\mathbb{R}/\mathbb{Z}} \arrow[r, "\gamma"] \arrow[d] & S_{\text{flip}}\mathbf{E} \arrow[d, "S_{\text{flip}}(\iota)"] \\ C\mathbf{E}^{\mathbb{R}/\mathbb{Z}} \arrow[r, "H"] & S_{\text{flip}}\mathbf{E}^{\mathbb{R}} \end{tikzcd} \tag{A.2}$$

where $(S_{\text{flip}}\iota)_k = S\iota_k$. Then we define $\nu_k \in \Lambda^k\left(\mathbf{E}_k^{\mathbb{R}/\mathbb{Z}}\right)$ by

$$\nu_k := (p_2)_*H_k^*(\sigma_k^{\mathbb{R}})^*\Omega_{k+1}$$

where

- $\sigma_k^{\mathbb{R}} : S_{\text{flip}}\mathbf{E}_k^{\mathbb{R}} = S\mathbf{E}_k^{\mathbb{R}} \to \mathbf{E}_{k+1}^{\mathbb{R}}$ is the structure map for $\mathbf{E}^{\mathbb{R}}$.
- We think of the cone point of $C\mathbf{E}_k^{\mathbb{R}/\mathbb{Z}}$ as being at $t = 1$. This matters for the direction of the integration performed by $(p_2)_*$.

Because the H in Diag. (A.2) is a spectrum map,

$$H_k \circ \left(\mathrm{id}_I \wedge \sigma_{k-1}^{\mathbb{R}/\mathbb{Z}}\right) = (S_{\text{flip}}\sigma^{\mathbb{R}})_{k-1} \circ SH_{k-1} = S\sigma_{k-1}^{\mathbb{R}} \circ \left(\varphi \wedge \mathrm{id}_{\mathbf{E}_{k-1}^{\mathbb{R}}}\right) \circ SH_{k-1}$$

By using that the Bockstein map $B_k : \mathbf{E}_k^{\mathbb{R}/\mathbb{Z}} \to \mathbf{E}_{k+1}$ is $\sigma_k \circ \gamma_k$,

$$\begin{aligned}
B_k^*\omega_{k+1} &= B_k^*\iota_{k+1}^*\Omega_{k+1} = \gamma_k^*\sigma_k^*\iota_{k+1}^*\Omega_{k+1} = \gamma_k^*(S\iota_k)^*(\sigma_k^{\mathbb{R}})^*\Omega_{k+1} \\
&= \gamma_k^*(S_{\text{flip}}\iota)_k^*(\sigma_k^{\mathbb{R}})^*\Omega_{k+1} = (H_k)_0^*(\sigma_k^{\mathbb{R}})^*\Omega_{k+1} \\
&= (H_k)_1^*(\sigma_k^{\mathbb{R}})^*\Omega_{k+1} - \left(d(p_2)_* + (p_2)_*d\right)(H_k)^*(\sigma_k^{\mathbb{R}})^*\Omega_{k+1} \\
&= 0 - d(p_2)_*(H_k)^*(\sigma_k^{\mathbb{R}})^*\Omega_{k+1} = -d\nu_k.
\end{aligned}$$

This shows that for $\theta_k := [\{B_k\}_0, \nu_k] \in M^{k+1}\left(\mathbf{E}_k^{\mathbb{R}/\mathbb{Z}}\right)$,

$$\delta_1\theta_k = B_k^*\omega_{k+1} + d\nu_k = 0$$

Therefore any pullback of θ_k depends only on the homotopy class of the map. So we can define a natural transformation $i_1 : E_{\mathbb{R}/\mathbb{Z}}^{k-1} \to M^k$ by

$$i_1\left[\mathrm{id}_{\mathbf{E}_{k-1}^{\mathbb{R}/\mathbb{Z}}}\right] := \theta_{k-1}$$

and extend by naturality.

Lemma A.4. *The natural transformation i_1 as defined above is compatible with integration.*

Proof. A long, tedious, but straightforward computation shows that

$$\theta_{k-1} = \int_M \left(\sigma_{k-1}^{\mathbb{R}/\mathbb{Z}} \circ \left(\mathrm{inv} \wedge \mathrm{id}_{\mathbf{E}_{k-1}^{\mathbb{R}/\mathbb{Z}}}\right)\right)^* \theta_k,$$

which implies that

$$\int_M \circ\, i_1 = i_1\left(-\int_{E_{\mathbb{R}/\mathbb{Z}}}\right).$$

■