The local isometric embedding problem for 3-dimensional Riemannian manifolds with cleanly vanishing curvature

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Abstract of the Dissertation

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We prove the following result: Let (M, g) be a 3-dimensional C^∞ Riemannian manifold for which there exists a $p \in M$ and a $v \in T_pM$ such that

$$\mathbf{Riem}(p) = 0 \qquad \text{and} \qquad \nabla_v \mathbf{Riem}(p) \neq 0.$$

Then there exists a C^∞ local isometric embedding from a neighbourhood of p into $\mathbb{R}^6$.

To ΛEHA,

for her love and support

Contents

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Chapter 1

Introduction

Let (M, g) be an n-dimensional C^∞ Riemannian manifold. We say that (M, g) can be locally isometrically embedded into $\mathbb{R}^N$ if there exists an open set $\Omega \subset M$ and a C^∞ map $u : \Omega \to \mathbb{R}^N$ such that the induced metric on the image of u agrees with g. In local coordinates this is equivalent to u solving the following nonlinear system of partial differential equations

$$\Phi_{ij}(u) := \sum_{k=1}^{N} \frac{\partial u^k}{\partial x^i} \frac{\partial u^k}{\partial x^j} = g_{ij}, \quad 1 \le i, j \le n. \tag{1.1}$$

From (1.1) it is evident that the isometric embedding system is a collection of $n(n+1)/2$ equations in N unknowns. In this paper we will only consider the determined case when $N = n(n+1)/2$.

Since 1983 there has been much progress on the determined case in dimensions 2, 3 and 4. For example in dimension 2, a classical result says that if the Gaussian curvature does not vanish at a point then smooth embeddings exist. This result was generalized by Lin in [15] (a simplified proof was also provided by Han in [6]), where it was proven that if the Gaussian curvature vanishes cleanly at a point then sufficiently smooth embeddings exist. In [14] Lin also reached the same conclusion under the assumption that the Gaussian curvature is nonnegative. A detailed account of such results may be found in the book by Han and Hong [8]. Recent work by Han in [7] and Khuri in [12] has shown that if the Gaussian curvature vanishes to finite order along a curve, then a sufficiently smooth embedding exists. In the case where the Gaussian curvature is nonpositive, under suitable nondegeneracy assumptions on the gradient of the Gaussian curvature, it has been shown by Han, Hong and Lin in [9] that there exist smooth embeddings. For 3-dimensional Riemannian manifolds work by Bryant, Griffiths and Yang in [1], Goodman and Yang in [4] and Nakamura and Maeda in [16] has shown that if there exists a point p for which $\mathbf{Riem}(p) \neq 0$ then smooth embeddings exist. In 4-dimensions there exists a finite set of algebraic equations involving the curvature tensor and its covariant derivatives such that if these equations do not all vanish at a point, then there exist smooth embeddings. These results follow from the work of [1], together with the Moser estimates proved in [4] and [16]. To the best of author's knowledge there are no known local isometric embedding results for a generic Riemannian manifold in dimensions ≥ 5. Finally, if the Riemannian metric is analytic, a famous theorem of Cartan and Janet (c.f. [11]) tells us that we always have an analytic embedding.

The main result of this thesis is the following generalization of the work carried out in [1] and [15].

Theorem 1.1 (Main theorem). *Let (M, g) be a 3-dimensional C^∞ Riemannian manifold such that there exists a $p \in M$ and a vector $v \in T_pM$ for* *which*

$$\mathbf{Riem}(p) = 0 \qquad and \qquad \nabla_v \mathbf{Riem}(p) \neq 0.$$

Then there exists a smooth isometric embedding from a neighbourhood of p *into* $\mathbb{R}^6$*.*

From now on (M, g) will always denote a 3-dimensional Riemannian manifold. Without loss of generality we may assume that Ω is a neighbourhood of the origin in $\mathbb{R}^3$ and $u : \Omega \to \mathbb{R}^6$. Let u_0 denote a smooth embedding choosen such that the induced metric $\partial_i u_0 \cdot \partial_j u_0$ is very close to g in the appropriate norm. The first difficulty encountered in trying to solve (1.1) is that its linearization is characteristic in every direction. To overcome this difficulty we follow the strategy of [1], where it was shown that any solution of (1.1) corresponds to a solution of a 3×3 system of differential equations

$$L[u]v = A^i(u)\frac{\partial v}{\partial x^i} + B(u) \tag{1.2}$$

and vice versa. The advantage of this approach is that for a suitably chosen u, (1.2) admits noncharacteristic directions. Thus to construct a solution to (1.1) we need only to prove the local solvability of $L[u]$ for u close to u_0, with the solutions satisfying the estimates needed for the Nash-Moser implicit function theorem. Let $\sigma(x, \xi)$ denote the determinant of the symbol of $L[u_0]$ at x. If $\mathbf{Riem}(0) \neq 0$ then it was shown in [1] that for a suitable choice of approximate solution

$$|\nabla_\xi \sigma(0, \xi)| \neq 0, \qquad \text{for all } \xi \in \mathbb{R}^3 - \{0\}. \tag{1.3}$$

A consequence of this is that $L[u]$ is of real principal type for any u close to u_0. Local solvability for linear operators of real principal type was proven by

Duistermaat and Hörmander in [2]. In the work of [4] and [16] the solutions were also shown to satisfy the estimates needed for the Nash-Moser implicit function theorem. In this paper we are assuming that $\mathbf{Riem}(0) = 0$, therefore it is *not* possible to find an approximate solution satisfying (1.3), however, we show that with the help of the cleanly vanishing property $|\nabla\mathbf{Riem}(0)| \neq 0$, it is still possible to construct an approximate solution of real principal type.

This thesis is divided into five chapters. In Chapter §2 we prove that solving $\Phi'(u)$ is equivalent to solving $L[u]$ plus a system of algebraic equations. In Chapter §3 we prove the lemmas required to construct an approximate solution u_0. In Chapter §4 we show that if we choose u_0 with an appropriate second fundamental form, then we can construct a special normal coordinate system, such that with respect to these coordinates $L[u_0]$ is of real principal type. In Chapter §5 we use the results of [4] and [16] to prove the desired Moser estimates. Finally, in Chapter §6, we apply the Nash-Moser implicit function theorem to solve (1.1).

Remark. Throughout we will use the convention where pairs of repeated indices in a product are to be summed over.

Chapter 2

Linearization of the embedding system

Let Ω be any open neighbourhood around $0 \in \mathbb{R}^3$ and let $u : \Omega \to \mathbb{R}^6$ be a smooth embedding. Let $v \in C^\infty(\Omega, \mathbb{R}^6)$ then the linearization of Φ_{ij} about u is

$$\begin{aligned}\Phi'_{ij}(u)v &= \frac{d}{dh}\Phi_{ij}(u+hv)|_{h=0} \\ &= \frac{\partial u}{\partial x^i}\cdot\frac{\partial v}{\partial x^j} + \frac{\partial u}{\partial x^j}\cdot\frac{\partial v}{\partial x^i}.\end{aligned} \tag{2.1}$$

Here $X \cdot Y := \sum_{k=1}^{6} X^k Y^k$. Since u is an embedding, it follows that for every $x \in \Omega$ the vectors $\partial_{x^1}u(x), \partial_{x^2}u(x)$ and $\partial_{x^3}u(x)$ are linearly independent. For each $x \in \Omega$, let $N_4(x), N_5(x)$ and $N_6(x)$ be an orthonormal set of vectors perpendicular to $\{\partial_{x^i}u(x)\}_{i=1}^3$. It is clear that for every $x \in \Omega$ the vectors $\{\partial_{x^i}u(x)\}_{i=1}^3$ and $\{N_\lambda(x)\}_{\lambda=4}^6$ span $\mathbb{R}^6$, therefore there exist functions Γ_{ij}^k and h_{ij}^λ such that

$$\frac{\partial^2 u}{\partial x^i \partial x^j} = \Gamma_{ij}^k \frac{\partial u}{\partial x^k} + h_{ij}^\lambda N_\lambda. \tag{2.2}$$

This formula is the Gauss formula. The coefficients Γ^k_{ij} are the Christoffel symbols of the metric induced by u, and h^μ_{ij} are the coefficients of the second fundamental form for the embedding u. In addition to the Gauss formula we also have the closely related Weingarten formula

$$\frac{\partial N_\lambda}{\partial x^i} = A^k_{i\lambda}\frac{\partial u}{\partial x^k} + \kappa^\mu_{i\lambda} N_\mu. \tag{2.3}$$

In the above formula $\kappa^\mu_{i\lambda}$ are the coefficients of the connection form on the normal bundle for the embedding u.

Given any smooth function $v : \Omega \to \mathbb{R}^6$ we decompose it into its tangential and normal components with respect to the embedding u as follows

$$v = v^l \frac{\partial u}{\partial x^l} + v^\lambda N_\lambda.$$

Differentiating this equation we find

$$\frac{\partial v}{\partial x^j} = \frac{\partial v^l}{\partial x^j}\frac{\partial u}{\partial x^l} + v^l \frac{\partial^2 u}{\partial x^l \partial x^j} + \frac{\partial v^\lambda}{\partial x^j} N_\lambda + v^\lambda \frac{\partial N_\lambda}{\partial x^j}.$$

Now

$$\begin{aligned} \frac{\partial u}{\partial x^i} \cdot \frac{\partial u}{\partial x^l} &:= p_{il} \\ \frac{\partial u}{\partial x^i} \cdot N_\lambda &= 0 \\ \frac{\partial u}{\partial x^i} \cdot \frac{\partial^2 u}{\partial x^l \partial x^j} &= \Gamma^k_{lj} p_{ik} \\ \frac{\partial N_\lambda}{\partial x^j} \cdot \frac{\partial u}{\partial x^i} &= -N_\lambda \cdot \frac{\partial^2 u}{\partial x^i \partial x^j} = -h^\lambda_{ij} \end{aligned}$$

where the last two equations follow from (2.2). Therefore

$$\frac{\partial u}{\partial x^i} \cdot \frac{\partial v}{\partial x^j} = p_{il}\frac{\partial v^l}{\partial x^j} + v^l \Gamma^k_{lj} p_{ik} - v^\lambda h^\lambda_{ij}. \tag{2.4}$$

Let $\{v_l\}_{l=1}^3$ be the coordinates of the dual 1-form to the vector field $v^l\partial_{x^l}u$, e.g.

$$v_l = p_{lk}v^k \quad \text{and} \quad v^l = p^{lk}v_k.$$

Here $(p^{lk}) = (p_{lk})^{-1}$. Therefore

$$p_{il}\frac{\partial v^l}{\partial x^j} = \frac{\partial v_i}{\partial x^j} - v^l\frac{\partial p_{il}}{\partial x^j}. \tag{2.5}$$

Plugging (2.5) into (2.4) we get

$$\frac{\partial u}{\partial x^i}\cdot\frac{\partial v}{\partial x^j} = \frac{\partial v^i}{\partial x^j} - v^l\frac{\partial p_{il}}{\partial x^j} + v^l\Gamma_{lj}^k p_{ik} - v^\lambda h_{ij}^\lambda. \tag{2.6}$$

Recalling the formula for the Christoffel symbols in terms of the metric

$$\Gamma_{lj}^k = \frac{1}{2}p^{km}(\frac{\partial p_{ml}}{\partial x^j} + \frac{\partial p_{mj}}{\partial x^l} - \frac{\partial p_{lj}}{\partial x^m})$$

we see that

$$\begin{aligned} -v^l\frac{\partial p_{il}}{\partial x^j} + v^l\Gamma_{lj}^k p_{ik} &= v^l(-\frac{\partial p_{il}}{\partial x^j} + \frac{1}{2}\frac{\partial p_{il}}{\partial x^j} + \frac{1}{2}\frac{\partial p_{ij}}{\partial x^l} - \frac{1}{2}\frac{\partial p_{lj}}{\partial x^i}) \\ &= -v_k p^{lk}(\frac{\partial p_{il}}{\partial x^j} + \frac{\partial p_{lj}}{\partial x^i} - \frac{\partial p_{ij}}{\partial x^l}) \\ &= -v_k\Gamma_{ij}^k. \end{aligned}$$

Therefore (2.6) becomes

$$\frac{\partial u}{\partial x^i}\cdot\frac{\partial v}{\partial x^j} = \frac{\partial v^i}{\partial x^j} - \Gamma_{ij}^k v_k - v^\lambda h_{ij}^\lambda$$

and so

$$\begin{aligned} \Phi'_{ij}(u)v &= \frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} - 2\Gamma_{ij}^k v_k - 2v^\lambda h_{ij}^\lambda \\ &:= f_{ij}. \end{aligned} \tag{2.7}$$

From (2.7) we have

$$f_{12} = \frac{\partial v_1}{\partial x^2} + \frac{\partial v_2}{\partial x^1} - 2\Gamma^k_{12}v_k - 2v^\lambda h^\lambda_{12}$$
$$f_{13} = \frac{\partial v_1}{\partial x^3} + \frac{\partial v_3}{\partial x^1} - 2\Gamma^k_{13}v_k - 2v^\lambda h^\lambda_{13}$$
$$f_{23} = \frac{\partial v_2}{\partial x^3} + \frac{\partial v_3}{\partial x^2} - 2\Gamma^k_{23}v_k - 2v^\lambda h^\lambda_{23}.$$

Writing these three equations in matrix form

$$\mathbf{H}\begin{pmatrix} v^4 \\ v^5 \\ v^6 \end{pmatrix} := \begin{pmatrix} h^4_{12} & h^5_{12} & h^6_{12} \\ h^4_{13} & h^5_{13} & h^6_{13} \\ h^4_{23} & h^5_{23} & h^6_{23} \end{pmatrix}\begin{pmatrix} v^4 \\ v^5 \\ v^6 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} f_{12} - \partial_{x^2}v_1 - \partial_{x^1}v_2 + 2\Gamma^k_{12}v_k \\ f_{13} - \partial_{x^3}v_1 - \partial_{x^1}v_3 + 2\Gamma^k_{13}v_k \\ f_{23} - \partial_{x^3}v_2 - \partial_{x^2}v_3 + 2\Gamma^k_{23}v_k \end{pmatrix} \tag{2.8}$$

we see that if the vectors h_{12}, h_{13} and h_{23} are linearly independent, then given v_1, v_2 and v_3 we can solve the above algebraic equations to find v^4, v^5 and v^6. This leads us to the following definition (cf. Griffiths and Jensen [3] page 100).

Definition 2.1 (Nondegenerate embedding)**.** *Let $u : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^6$ be a smooth embedding. Let h_{ij} denote the second fundamental form of u. We say that u is a nondegenerate embedding if the vectors $h_{12}(x), h_{13}(x), h_{23}(x)$ are linearly independent for all $x \in \Omega$.*

Assuming that u is a nondegenerate embedding it follows that there exists $C^\mu_i \in C^\infty(\Omega)$, $1 \le i \le 3$ and $4 \le \mu \le 6$, such that

$$\begin{aligned} h_{11}(x) &= C^4_1(x)h_{12}(x) + C^5_1(x)h_{13}(x) + C^6_1(x)h_{23}(x) \\ h_{22}(x) &= C^4_2(x)h_{12}(x) + C^5_2(x)h_{13}(x) + C^6_2(x)h_{23}(x) \\ h_{33}(x) &= C^4_3(x)h_{12}(x) + C^5_3(x)h_{13}(x) + C^6_3(x)h_{23}(x) \end{aligned} \tag{2.9}$$

for all $x \in \Omega$. Now

$$\begin{aligned} f_{ii} &= 2\frac{\partial v_i}{\partial x^i} - 2\Gamma_{ii}^k v_k - 2v^\lambda h_{ii}^\lambda \\ &= 2\frac{\partial v_i}{\partial x^i} - 2\Gamma_{ii}^k v_k - 2v^\lambda (C_i^4 h_{12}^\lambda + C_i^5 h_{13}^\lambda + C_i^6 h_{23}^\lambda), \end{aligned}$$

by (2.7) we have

$$2h_{ij}^\lambda v^\lambda = \frac{\partial v_i}{\partial x^j} + \frac{\partial v_j}{\partial x^i} - 2\Gamma_{ij}^k v_k - f_{ij}.$$

Therefore for $i = 1, 2, 3$

$$\begin{aligned} f_{ii} = 2\frac{\partial v_i}{\partial x^i} - 2\Gamma_{ii}^k v_k - C_i^4 (\frac{\partial v_1}{\partial x^2} + \frac{\partial v_2}{\partial x^1} - 2\Gamma_{12}^k v_k - f_{12}) \\ - C_i^5 (\frac{\partial v_1}{\partial x^3} + \frac{\partial v_3}{\partial x^1} - 2\Gamma_{13}^k v_k - f_{13}) \\ - C_i^6 (\frac{\partial v_2}{\partial x^3} + \frac{\partial v_3}{\partial x^2} - 2\Gamma_{23}^k v_k - f_{23}). \end{aligned}$$

We write the above linear system as

$$L[u]y := A^i \frac{\partial y}{\partial x^i} + By = g \tag{2.10}$$

where

$$\begin{aligned} A^1 &= \begin{pmatrix} 2 & -C_1^4 & -C_1^5 \\ 0 & -C_2^4 & -C_2^5 \\ 0 & -C_3^4 & -C_3^5 \end{pmatrix} \\ A^2 &= \begin{pmatrix} -C_1^4 & 0 & -C_1^6 \\ -C_2^4 & 2 & -C_2^6 \\ -C_3^4 & 0 & -C_3^6 \end{pmatrix} \\ A^3 &= \begin{pmatrix} -C_1^5 & -C_1^6 & 0 \\ -C_2^5 & -C_2^6 & 0 \\ -C_3^5 & -C_3^6 & 2 \end{pmatrix} \end{aligned} \tag{2.11}$$

$B_{ik} = 2(C_i^4\Gamma_{12}^k + C_i^5\Gamma_{13}^k + C_i^6\Gamma_{23}^k - \Gamma_{ii}^k)$, $g_i = f_{ii} + C_i^4 f_{12} + C_i^5 f_{13} + C_i^6 f_{23}$ and $y = (v_1, v_2, v_3)^T$.

We have shown the following: Let $(v_i)_{i=1}^3$ be a solution to (2.10) and let $(v^\lambda)_{\lambda=4}^6$ be a solution to (2.8), then

$$v := p^{lk} v_k \frac{\partial u}{\partial x^l} + v^\lambda N_\lambda$$

is a solution to (2.1). Conversely if v is a solution to (2.1), decomposing v into its normal and tangential parts gives us solutions to (2.8) and (2.10). It should be pointed out that a similar reduction was performed for arbitrary dimensions by Han and Khuri in the recent preprint [10].

Chapter 3

Construction of u_0

In this Chapter we prove the Lemmas required to construct an approximate solution for the isometric embedding system (1.1). We will work with a slightly generalized version of normal coordinates.

Definition 3.1 (Normal Coordinates)**.** *Let (x^1, x^2, x^3) be a coordinate system centered at $p \in M$. Let g_{ij} denote the components of g in this coordinate system. We say that (x^i) is a normal coordinate system if $\partial_{x^k} g_{ij}(0) = 0$ for $1 \leq k \leq 3$. We do not assume that $g_{ij}(0) = \delta_{ij}$.*

From now on (x^1, x^2, x^3) will always denote a normal coordinate system centered at p. Let R_{ijkl} denote the components of the Riemann curvature tensor with respect to these coordinates. We now impose the condition that the curvature tensor vanishes at p, therefore

$$R_{ijkl}(0) = 0$$

for all $1 \leq i, j, k, l \leq 3$. The following Lemma enables us to extend a zeroth order solution of the Gauss equations to a first order solution.

Lemma 3.1. *Let H^μ_{ij}, $1 \le i,j \le 3$ and $4 \le \mu \le 6$, be constants satisfying the Gauss equations at 0 and let the vectors H_{12}, H_{13} and H_{23} be linearly independent. Then there exists smooth functions h^μ_{ij} which satisfy the Gauss equations to 1^{st} order*

$$R_{ijkl}(0) = h^\mu_{ik}(0)h^\mu_{jl}(0) - h^\mu_{il}(0)h^\mu_{jk}(0)$$

and

$$\frac{\partial R_{ijkl}}{\partial x^a}(0) = \frac{\partial h^\mu_{ik}}{\partial x^a}(0)h^\mu_{jl}(0) + h^\mu_{ik}(0)\frac{\partial h^\mu_{jl}}{\partial x^a}(0) - \frac{\partial h^\mu_{il}(0)}{\partial x^a}h^\mu_{jk}(0) - h^\mu_{il}(0)\frac{\partial h^\mu_{jk}}{\partial x^a}(0).$$

Furthermore we may assume that

$$\frac{\partial h^\mu_{ij}}{\partial x^l}(0) = \frac{\partial h^\mu_{lj}}{\partial x^i}(0), \qquad 1 \le i,j,l \le 3 \text{ and } 4 \le \mu \le 6.$$

Proof. Define

$$h^\mu_{ij}(x) = H^\mu_{ij} + H^\mu_{ija}x^a \tag{3.1}$$

where $H^\mu_{ija} = H^\mu_{aji}$. By the hypothesis of the Lemma we know that the Gauss equations hold at 0. Differentiating both sides of the Gauss equations we find

$$\frac{\partial R_{ijkl}}{\partial x^a}(0) = H^\mu_{ika}H^\mu_{jl} + H^\mu_{ik}H^\mu_{jla} - H^\mu_{ila}H^\mu_{jk} - H^\mu_{il}H^\mu_{jka}. \tag{3.2}$$

The above equations are a linear system with H^μ_{ijk} as the unknowns. As $H^\mu_{ija} = H^\mu_{aji}$ we have 30 unknowns, furthermore by the second Bianchi identity it follows that (3.2) consists of 15 equations. From linear algebra we expect a 15 parameter solution space to (3.2). Provided H_{12}, H_{13} and H_{23} are linearly independent such a result is true. For a proof see [3], pp. 102-111. □

Using a first order solution of the Gauss equations we construct an approximate solution to (1.1).

Lemma 3.2. *Let h^μ_{ij}, $1 \leq i,j \leq 3$ and $4 \leq \mu \leq 6$, be smooth functions satisfying the Gauss equations*

$$R_{ijkl} = h^\mu_{ik} h^\mu_{jl} - h^\mu_{il} h^\mu_{jk} \tag{3.3}$$

to 1^{st} order, and satisfying the equations

$$\frac{\partial h^\mu_{ij}}{\partial x^l}(0) = \frac{\partial h^\mu_{lj}}{\partial x^i}(0), \qquad 1 \leq i,j,l \leq 3 \text{ and } 4 \leq \mu \leq 6.$$

Then there exists a smooth embedding $u : \mathbb{R}^3 \to \mathbb{R}^6$ such that

1.
$$\frac{\partial u}{\partial x^i} \cdot \frac{\partial u}{\partial x^j} = g_{ij} + O(|x|^4) \quad \text{as } |x| \to 0. \tag{3.4}$$

2. *For each x, there exists a basis $\{N_\lambda(x)\}_{\lambda=4}^6$ spanning the subspace perpendicular to $\{\partial_{x^i} u(x)\}_{i=1}^3$ such that, the coefficients of the second fundamental form for u with respect to N_λ agree to first order with h^μ_{ij}.*

3. *With $\{N_\lambda(x)\}_{\lambda=4}^6$ defined as above we have*
$$\frac{\partial N_\lambda}{\partial x^i}(0) \in \operatorname{span}\{\partial_{x^j} u(0)\}_{j=1}^3 \tag{3.5}$$
for all $1 \leq i \leq 3$ and $4 \leq \lambda \leq 6$.

Proof. Let $\kappa^\mu_{ij} : \mathbb{R}^3 \to \mathbb{R}$ denote the coefficients of the connection form on the normal bundle for the embedding u. A sufficient condition for solving (3.4) is that the Gauss-Codazzi-Ricci equations

$$\begin{aligned}
h^\mu_{ik} h^\mu_{jl} - h^\mu_{il} h^\mu_{jk} &= R_{ijkl} && \text{Gauss equations}\\
\frac{\partial h^\mu_{ij}}{\partial x^l} - \frac{\partial h^\mu_{lj}}{\partial x^i} + \Gamma^k_{ij} h^\mu_{lk} - \Gamma^k_{lj} h^\mu_{ik} + \kappa^\mu_{l\lambda} h^\lambda_{ij} - \kappa^\mu_{i\lambda} h^\lambda_{lj} &= 0 && \text{Codazzi equations}\\
\frac{\partial \kappa^\mu_{i\lambda}}{\partial x^j} - \frac{\partial \kappa^\mu_{j\lambda}}{\partial x^i} - g^{kl}(h^\mu_{ki} h^\lambda_{jl} - h^\mu_{kj} h^\lambda_{il}) + k^\mu_{j\epsilon} k^\epsilon_{i\lambda} - k^\mu_{i\epsilon} k^\epsilon_{j\lambda} &= 0 && \text{Ricci equations}
\end{aligned} \tag{3.6}$$

are solved up to first order in a neighbourhood of 0. Let

$$H^\mu_{ij} := h^\mu_{ij}(0)$$
$$H^\mu_{ija} := \frac{\partial h^\mu_{ij}}{\partial x^a}(0)$$
$$H^\mu_{ijab} := \frac{\partial h^\mu_{ij}}{\partial x^a \partial x^b}(0)$$

therefore

$$h^\mu_{ij}(x) = H^\mu_{ij} + H^\mu_{ija}x^a + \frac{1}{2!}H^\mu_{ijab}x^a x^b + O(|x|^3). \tag{3.7}$$

Define

$$\kappa^\mu_{ij}(x) = K^\mu_{ija}x^a + \frac{1}{2!}K^\mu_{ijab}x^a x^b. \tag{3.8}$$

From the hypothesis of the Lemma the Gauss equations are solved to first order at 0. As $R_{ijkl}(0) = 0$ it follows that

$$\partial_{x^l}\Gamma^k_{ij}(0) = 0. \tag{3.9}$$

Substituting (3.7),(3.8) and (3.9) into the Codazzi-Ricci equations we find

$$(H^\mu_{ijl} - H^\mu_{lji}) + (H^\mu_{ijla} - H^\mu_{ljia} + K^\mu_{l\lambda a}H^\lambda_{ij} - K^\mu_{i\lambda a}H^\lambda_{lj})x^a + O(|x|^2) = 0 \tag{3.10}$$

and

$$\begin{aligned} &K^\mu_{i\lambda j} - K^\mu_{j\lambda i} - g^{kl}(0)(H^\mu_{ki}H^\lambda_{jl} - H^\mu_{kj}H^\lambda_{il}) \\ &+ [K^\mu_{i\lambda ja} - K^\mu_{j\lambda ia} - g^{kl}(0)(H^\mu_{kia}H^\lambda_{jl} + H^\mu_{ki}H^\lambda_{jla} - H^\mu_{kja}H^\lambda_{il} - H^\mu_{kj}H^\lambda_{ila})]x^a + O(|x|^2) = 0. \end{aligned} \tag{3.11}$$

Thus to solve the Codazzi-Ricci equations to first order we must take

$$\begin{aligned} &H^\mu_{ijl} = H^\mu_{lji} \\ &K^\mu_{i\lambda j} = K^\mu_{j\lambda i} + g^{kl}(0)(H^\mu_{ki}H^\lambda_{jl} - H^\mu_{kj}H^\lambda_{il}) \\ &H^\mu_{ijla} = H^\mu_{ljia} - K^\mu_{l\lambda a}H^\lambda_{ij} + K^\mu_{i\lambda a}H^\lambda_{lj} \\ &K^\mu_{i\lambda ja} = K^\mu_{j\lambda ia} + g^{kl}(0)(H^\mu_{kia}H^\lambda_{jl} + H^\mu_{ki}H^\lambda_{jla} - H^\mu_{kja}H^\lambda_{il} - H^\mu_{kj}H^\lambda_{ila}). \end{aligned}$$

By the assumptions of the Lemma the first of these equations are satisfied. As the remaining equations only involve specifying the 2^{nd} derivatives of the second fundamental form, the Gauss equations are still satisfied up to 1^{st} order. Finally (3.5) follows from the Weingarten formula (2.3) and the fact that $\kappa^{\mu}_{ij}(0) = 0$. $\square$

Chapter 4

$L[u_O]$ is of real principal type

In this Chapter we will use Lemmas 3.1 and 3.2 to construct an approximate solution u_0 to (1.1), such that $L[u_0]$ is of real principal type at $x = 0$.

Let $(x_i)_{i=1}^3$ denote the standard coordinates on $\mathbb{R}^3$ and let L be a linear partial differential operator of the form

$$Lu = A^i \frac{\partial u}{\partial x^i} + Bu.$$

Recall that the symbol of L is the map

$$\begin{aligned} \sigma : \Omega \times (\mathbb{R}^3 - 0) &\to \mathbb{R} \\ (x, \xi) &\mapsto \det(A^i(x)\xi_i). \end{aligned}$$

Using σ we can define a vector field on $\Omega \times (\mathbb{R}^3 - 0)$ as follows

$$H_\sigma := \frac{\partial \sigma}{\partial x^i}\frac{\partial}{\partial \xi_i} - \frac{\partial \sigma}{\partial \xi_i}\frac{\partial}{\partial x^i}.$$

The vector field H_σ is called the Hamiltonian of σ. If γ is an integral curve for H_σ, then along γ the value of the symbol σ is constant. Let

$\gamma : (-s, s) \to \Omega \times (\mathbb{R}^3 - 0)$ be an integral curve for H_σ such that $\sigma(\gamma(0)) = 0$, then γ is called a null bicharacteristic. In particular $\sigma(\gamma(t)) = 0$ for all $t \in (-s, s)$.

We define $\pi : \Omega \times (\mathbb{R}^3 - 0) \to \Omega$ to be the projection map where $\pi(x, \xi) := x$. The following definition is taken from Goodman and Yang in [4].

Definition 4.1 (Differential operator of real principal type)**.** *The differential operator L is of real principal type at* 0 *if there exists a compact subset $K \subset \Omega$ containing 0, such that if $\gamma : (-s, s) \to \Omega \times (\mathbb{R}^3 - 0)$ is a null bicharacteristic satisfying $\pi(\gamma(0)) \in K$, then there exists a $T > 0$ such that $\pi(\gamma(\pm T)) \notin K$. That is every null bicharacteristic sitting over K leaves K going forwards and backwards in time.*

Remark. In the paper by Nakamura and Maeda [16] a slightly different definition of an operator of real principal type is given, there L is said to be of real principal type at 0 if there exists a 3×3 matrix valued symbol $p(x, \xi)$, a scalar symbol $q(x, \xi)$, an open set W containing 0 such that

$$p(x, \xi)(A^i(x)\xi_i) = q(x, \xi)\text{Id}, \qquad \text{for all } (x, \xi) \in W \times (\mathbb{R}^3 - 0)$$

and the principal symbol $q_1(x, \xi)$ of $q(x, \xi)$ satisfies the condition that dq_1 and $\theta := \xi_i dx^i$ are linearly independent on $W \times (\mathbb{R}^3 - 0) \cap q_1^{-1}(0)$.

We claim that Definition 4.1 implies the definition given in [16]: Let $a(x, \xi) = A^i(x)\xi_i$ and define

$$[p(x, \xi)]_{ij} = (-1)^{i+j}\det(a_{ij}(x, \xi)), \qquad 1 \leq i, j \leq 3$$

where $a_{ij}(x,\xi)$ denotes the ij cofactor of $a(x,\xi)$, obtained by deleting the i^{th} row and j^{th} column of $a(x,\xi)$. Therefore

$$p(x,\xi)(A^i(x)\xi_i) = \det(A^i(x)\xi_i)\mathrm{Id} = \sigma(x,\xi)\mathrm{Id}.$$

Suppose now $d\sigma$ and θ are linearly dependent. Then

$$\frac{\partial\sigma}{\partial\xi_i}(x,\xi) = 0, \qquad 1 \le i \le 3$$

for all $(x,\xi) \in W \times (\mathbb{R}^3 - 0) \cap \sigma^{-1}(0)$, therefore if $\gamma(t) = (x^i(t), \xi_i(t))$ is a null bicharacteristic of σ, it follows that

$$\frac{dx^i}{dt} = -\frac{\partial\sigma}{\partial\xi_i} = 0.$$

Therefore γ gets trapped over every fiber, contradicting Definition 4.1.

Let

$$\begin{aligned} H_{12} &= (1,0,0)\\ H_{13} &= (0,1,0)\\ H_{23} &= (0,0,1)\\ H_{11} &= H_{12} + H_{13}\\ H_{22} &= H_{12} + H_{23}\\ H_{33} &= H_{13} + H_{23}, \end{aligned} \tag{4.1}$$

therefore

$$0 = H^\mu_{ik}H^\mu_{jl} - H^\mu_{il}H^\mu_{jk}.$$

As H_{12}, H_{13} and H_{23} are linearly independent if follows from Lemma 3.1 and Lemma 3.2 that there exists a nondegenerate embedding $u_0 : \mathbb{R}^3 \to \mathbb{R}^6$ such that

$$\frac{\partial u_0}{\partial x^i} \cdot \frac{\partial u_0}{\partial x^j} = g_{ij} + O(|x|^4).$$

Let $\{N_\lambda(x)\}_{\lambda=4}^{6}$ denote a basis for the normal bundle of the embedding u_0, such that with respect to this basis the Gauss formula (equation 2.2) becomes

$$\frac{\partial^2 u}{\partial x^i \partial x^j} = \Gamma^k_{ij}\frac{\partial u}{\partial x^k} + h^\lambda_{ij} N_\lambda$$

where

$$h^\lambda_{ij}(0) = H^\lambda_{ij}(0) \quad \text{and} \quad \frac{\partial N_\lambda}{\partial x^i}(0) \in \operatorname{span}\{\partial_{x^j} u(0)\}_{j=1}^{3}. \tag{4.2}$$

Defining $h_{ij} = h^\lambda_{ij} N_\lambda$ it follows from (4.1) that

$$h_{12}(0) = N_4(0)$$
$$h_{13}(0) = N_5(0)$$
$$h_{23}(0) = N_6(0).$$

A consequence of (4.2) is that

$$\frac{\partial h^\mu_{ij}}{\partial x^k}(0) = \frac{\partial h_{ij}}{\partial x^k}(0)\cdot N_\mu(0). \tag{4.3}$$

From (2.9) and (4.1) it follows that

$$\begin{aligned} C^4_1(0) = C^5_1(0) = 1 \\ C^4_2(0) = C^6_2(0) = 1 \\ C^5_3(0) = C^6_3(0) = 1 \end{aligned} \tag{4.4}$$

with all other $C^\mu_j(0) = 0$. Subsituting (4.4) into (2.11) we obtain

$$A^1(0) = \begin{pmatrix} 2 & -1 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; A^2(0) = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 2 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \; A^3(0) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & 2 \end{pmatrix}.$$

We now change coordinates in the ξ variable. Let

$$\xi_1 = \eta_2 + \eta_3$$

$$\xi_2 = \eta_1 + \eta_3$$

$$\xi_3 = \eta_1 + \eta_2$$

then the symbol of $L[u_0]$ becomes

$$\sigma(x, \eta) = \det(\eta_1 \tilde{A}^1 + \eta_2 \tilde{A}^2 + \eta_3 \tilde{A}^3)$$

where

$$\tilde{A}^1 := A^2 + A^3, \quad \tilde{A}^2 := A^1 + A^3, \quad \tilde{A}^3 := A^1 + A^2.$$

A long but straightforward calculation shows that

$$\begin{aligned} \sigma(0, \eta) &= -16\eta_1\eta_2\eta_3 \\ \sigma(x, \eta) &= 2E^1(x)\eta_1^3 + 2E^2(x)\eta_2^3 + 2E^3(x)\eta_3^3 + \sum_{\substack{|\alpha|=3 \\ 0 \le \alpha_i \le 2}} F^\alpha \eta_1^{\alpha_1}\eta_2^{\alpha_2}\eta_3^{\alpha_3} \end{aligned} \tag{4.5}$$

where E^i and F^α are quadratic polynomials in C_i^μ. Explicitly

$$E^1 = (C_1^4 + C_1^5)(C_2^6 + C_3^6 - 2) - C_1^6(C_2^4 + C_3^4 + C_2^5 + C_3^5)$$

$$E^2 = (C_2^4 + C_2^6)(C_1^5 + C_3^5 - 2) - C_2^5(C_1^4 + C_3^4 + C_1^6 + C_3^6)$$

$$E^3 = (C_3^5 + C_3^6)(C_1^4 + C_2^4 - 2) - C_3^4(C_1^5 + C_2^5 + C_1^6 + C_2^6).$$

Using (4.4) we see that

$$\begin{aligned} \frac{1}{2}\frac{\partial E^1}{\partial x^k}(0) &= \frac{\partial C_2^6}{\partial x^k}(0) + \frac{\partial C_3^6}{\partial x^k}(0) - \frac{\partial C_1^6}{\partial x^k}(0) \\ \frac{1}{2}\frac{\partial E^2}{\partial x^k}(0) &= \frac{\partial C_1^5}{\partial x^k}(0) + \frac{\partial C_3^5}{\partial x^k}(0) - \frac{\partial C_2^5}{\partial x^k}(0) \\ \frac{1}{2}\frac{\partial E^3}{\partial x^k}(0) &= \frac{\partial C_1^4}{\partial x^k}(0) + \frac{\partial C_2^4}{\partial x^k}(0) - \frac{\partial C_3^4}{\partial x^k}(0) \end{aligned}$$

for all $1 \le k \le 3$.

Lemma 4.1. *For* $1 \leq k \leq 3$ *we have*

$$\frac{1}{2}\frac{\partial E^1}{\partial x^k}(0) = \frac{\partial R_{2323}}{\partial x^k}(0) + \frac{\partial R_{1223}}{\partial x^k}(0) - \frac{\partial R_{1213}}{\partial x^k}(0) - \frac{\partial R_{1323}}{\partial x^k}(0)$$
$$\frac{1}{2}\frac{\partial E^2}{\partial x^k}(0) = \frac{\partial R_{1313}}{\partial x^k}(0) + \frac{\partial R_{1223}}{\partial x^k}(0) - \frac{\partial R_{1213}}{\partial x^k}(0) - \frac{\partial R_{1323}}{\partial x^k}(0)$$
$$\frac{1}{2}\frac{\partial E^3}{\partial x^k}(0) = \frac{\partial R_{1212}}{\partial x^k}(0) + \frac{\partial R_{1223}}{\partial x^k}(0) - \frac{\partial R_{1213}}{\partial x^k}(0) - \frac{\partial R_{1323}}{\partial x^k}(0).$$

Proof. By (2.9) we have

$$h_{ii}(x) = C_i^4(x)h_{12}(x) + C_i^5(x)h_{13}(x) + C_i^6(x)h_{23}(x)$$

for all $x \in \Omega$ and $1 \leq i \leq 3$, therefore

$$\begin{aligned}\frac{\partial h_{ii}}{\partial x^k}(0) =& \frac{\partial C_i^4}{\partial x^k}(0)h_{12}(0) + C_i^4(0)\frac{\partial h_{12}}{\partial x^k}(0) \\ &+ \frac{\partial C_i^5}{\partial x^k}(0)h_{13}(0) + C_i^5(0)\frac{\partial h_{13}}{\partial x^k}(0) \\ &+ \frac{\partial C_i^6}{\partial x^k}(0)h_{23}(0) + C_i^6(0)\frac{\partial h_{23}}{\partial x^k}(0).\end{aligned}$$

Define $H^\mu_{ijk} = \partial_{x^k} h^\mu_{ij}(0)$, by (4.1) and (4.3) it follows that

$$\begin{aligned}\frac{\partial C_1^\mu}{\partial x^k}(0) &= H^\mu_{11k} - H^\mu_{12k} - H^\mu_{13k} \\ \frac{\partial C_2^\mu}{\partial x^k}(0) &= H^\mu_{22k} - H^\mu_{12k} - H^\mu_{23k} \\ \frac{\partial C_3^\mu}{\partial x^k}(0) &= H^\mu_{33k} - H^\mu_{13k} - H^\mu_{23k}\end{aligned} \tag{4.6}$$

for all $1 \leq k \leq 3$ and $4 \leq \mu \leq 6$.

Claim 1

$$\frac{\partial C_2^6}{\partial x^k}(0) + \frac{\partial C_3^6}{\partial x^k}(0) - \frac{\partial C_1^6}{\partial x^k}(0) = \frac{\partial R_{2323}}{\partial x^k}(0) + \frac{\partial R_{1223}}{\partial x^k}(0) - \frac{\partial R_{1213}}{\partial x^k}(0) - \frac{\partial R_{1323}}{\partial x^k}(0).$$

Proof of Claim 1:

We prove *Claim 1* only for $k = 1$. The proofs for $k = 2, 3$ are identical. From equations (4.6) we have

$$\frac{\partial C_2^6}{\partial x^1}(0) + \frac{\partial C_3^6}{\partial x^1}(0) - \frac{\partial C_1^6}{\partial x^1}(0) = H_{122}^6 + H_{133}^6 - 2H_{123}^6 - H_{111}^6.$$

Now using the derivatives of the Gauss equations (3.2) we see that

$$\begin{aligned}
\frac{\partial R_{2323}}{\partial x^1}(0) &= H_{122}^\mu H_{33}^\mu + H_{133}^\mu H_{22}^\mu - 2H_{123}^\mu H_{23}^\mu \\
\frac{\partial R_{1223}}{\partial x^1}(0) &= H_{112}^\mu H_{23}^\mu + H_{123}^\mu H_{12}^\mu - H_{113}^\mu H_{22}^\mu - H_{122}^\mu H_{13}^\mu \\
\frac{\partial R_{1213}}{\partial x^1}(0) &= H_{111}^\mu H_{23}^\mu + H_{123}^\mu H_{11}^\mu - H_{113}^\mu H_{12}^\mu - H_{112}^\mu H_{13}^\mu \\
\frac{\partial R_{1323}}{\partial x^1}(0) &= H_{112}^\mu H_{33}^\mu + H_{133}^\mu H_{12}^\mu - H_{113}^\mu H_{23}^\mu - H_{123}^\mu H_{13}^\mu.
\end{aligned} \tag{4.7}$$

Therefore

$$\begin{aligned}
&\frac{\partial R_{2323}}{\partial x^1}(0) + \frac{\partial R_{1223}}{\partial x^1}(0) - \frac{\partial R_{1213}}{\partial x^1}(0) - \frac{\partial R_{1323}}{\partial x^1}(0) \\
&= (H_{122}^\mu H_{33}^\mu + H_{133}^\mu H_{22}^\mu - 2H_{123}^\mu H_{23}^\mu) \\
&+ (H_{112}^\mu H_{23}^\mu + H_{123}^\mu H_{12}^\mu - H_{113}^\mu H_{22}^\mu - H_{122}^\mu H_{13}^\mu) \\
&- (H_{111}^\mu H_{23}^\mu + H_{123}^\mu H_{11}^\mu - H_{113}^\mu H_{12}^\mu - H_{112}^\mu H_{13}^\mu) \\
&- (H_{112}^\mu H_{33}^\mu + H_{133}^\mu H_{12}^\mu - H_{113}^\mu H_{23}^\mu - H_{123}^\mu H_{13}^\mu).
\end{aligned} \tag{4.8}$$

From (4.1) it follows that for all $1 \leq i, j, k \leq 3$

$$\begin{aligned}
H_{ijk}^\mu H_{11}^\mu &= H_{ijk}^\mu H_{12}^\mu + H_{ijk}^\mu H_{13}^\mu = H_{ijk}^4 + H_{ijk}^5 \\
H_{ijk}^\mu H_{22}^\mu &= H_{ijk}^\mu H_{12}^\mu + H_{ijk}^\mu H_{23}^\mu = H_{ijk}^4 + H_{ijk}^6 \\
H_{ijk}^\mu H_{33}^\mu &= H_{ijk}^\mu H_{13}^\mu + H_{ijk}^\mu H_{23}^\mu = H_{ijk}^5 + H_{ijk}^6.
\end{aligned}$$

Using the above equations we rewrite (4.8) as

$$\frac{\partial R_{2323}}{\partial x^1}(0)+\frac{\partial R_{1223}}{\partial x^1}(0)-\frac{\partial R_{1213}}{\partial x^1}(0)-\frac{\partial R_{1323}}{\partial x^1}(0)$$
$$=(H^6_{112}+H^4_{123}-H^4_{113}-H^6_{113}-H^5_{122})+(H^5_{122}+H^6_{122}+H^4_{133}+H^6_{133}-2H^6_{123})$$
$$-(H^6_{111}+H^4_{123}+H^5_{123}-H^4_{113}-H^5_{112})-(H^5_{112}+H^6_{112}+H^4_{133}-H^6_{113}-H^5_{123})$$
$$=H^6_{122}+H^6_{133}-H^6_{111}-2H^6_{123}$$
$$=\frac{\partial C^6_2}{\partial x^1}(0)+\frac{\partial C^6_3}{\partial x^1}(0)-\frac{\partial C^6_1}{\partial x^1}(0).$$

Thereby proving *Claim 1.*

Claim 2

$$\frac{\partial C^5_1}{\partial x^k}(0)+\frac{\partial C^5_3}{\partial x^k}(0)-\frac{\partial C^5_2}{\partial x^k}(0)=\frac{\partial R_{1313}}{\partial x^k}(0)+\frac{\partial R_{1223}}{\partial x^k}(0)-\frac{\partial R_{1213}}{\partial x^k}(0)-\frac{\partial R_{1323}}{\partial x^k}(0).$$

Proof of Claim 2:

Again we will prove the above claim only for $k=1$, as $k=2,3$ are identical. From equations (4.6) we have

$$\frac{\partial C^5_1}{\partial x^1}(0)+\frac{\partial C^5_3}{\partial x^1}(0)-\frac{\partial C^5_2}{\partial x^1}(0)=H^5_{111}+H^5_{133}-2H^5_{113}-H^5_{122}.$$

Equations (4.7) provide us with formulas for $\partial_{x^1}R_{1223}(0), \partial_{x^1}R_{1213}(0)$ and $\partial_{x^1}R_{1323}(0)$ therefore we need only the formula for $\partial_{x^1}R_{1313}(0)$ which by equations (3.2) is

$$\frac{\partial R_{1313}}{\partial x^1}(0)=H^\mu_{111}H^\mu_{33}+H^\mu_{133}H^\mu_{11}-2H^\mu_{113}H^\mu_{13}.$$

Therefore

$$
\begin{aligned}
&\frac{\partial R_{1313}}{\partial x^1}(0)+\frac{\partial R_{1223}}{\partial x^1}(0)-\frac{\partial R_{1213}}{\partial x^1}(0)-\frac{\partial R_{1323}}{\partial x^1}(0)\\
&=(H^5_{111}+H^6_{111}+H^4_{133}+H^5_{133}-2H^5_{113})+(H^6_{112}+H^4_{123}-H^4_{113}-H^6_{113}-H^5_{122})\\
&-(H^6_{111}+H^4_{123}+H^5_{123}-H^4_{113}-H^5_{112})-(H^5_{112}+H^6_{112}+H^4_{133}-H^6_{113}-H^5_{123})\\
&=H^5_{111}+H^5_{133}-2H^5_{113}-H^5_{122}\\
&=\frac{\partial C^5_1}{\partial x^1}(0)+\frac{\partial C^5_3}{\partial x^1}(0)-\frac{\partial C^5_2}{\partial x^1}(0).
\end{aligned}
$$

Thereby proving *Claim 2.*

Claim 3

$$\frac{\partial C^4_1}{\partial x^k}(0)+\frac{\partial C^4_2}{\partial x^k}(0)-\frac{\partial C^4_3}{\partial x^k}(0)=\frac{\partial R_{1212}}{\partial x^k}(0)+\frac{\partial R_{1223}}{\partial x^k}(0)-\frac{\partial R_{1213}}{\partial x^k}(0)-\frac{\partial R_{1323}}{\partial x^k}(0).$$

Proof of Claim 3:

Again we prove the above claim only for $k=1$. From equations (4.6) we have

$$\frac{\partial C^4_1}{\partial x^1}(0)+\frac{\partial C^4_2}{\partial x^1}(0)-\frac{\partial C^4_3}{\partial x^1}(0)=H^4_{111}+H^4_{122}-2H^4_{112}-H^4_{133}.$$

We already have formulas for $\partial_{x^1}R_{1223}(0),\partial_{x^1}R_{1213}(0)$ and $\partial_{x^1}R_{1323}(0)$ therefore we need only the formula for $\partial_{x^1}R_{1212}(0)$ which by equations (3.2) is

$$\frac{\partial R_{1212}}{\partial x^1}(0)=H^\mu_{111}H^\mu_{22}+H^\mu_{122}H^\mu_{11}-2H^\mu_{112}H^\mu_{12}.$$

Therefore

$$\frac{\partial R_{1212}}{\partial x^1}(0) + \frac{\partial R_{1223}}{\partial x^1}(0) - \frac{\partial R_{1213}}{\partial x^1}(0) - \frac{\partial R_{1323}}{\partial x^1}(0)$$
$$= (H^4_{111} + H^6_{111} + H^4_{122} + H^5_{122} - 2H^4_{112}) + (H^6_{112} + H^4_{123} - H^4_{113} - H^6_{113} - H^5_{122})$$
$$- (H^6_{111} + H^4_{123} + H^5_{123} - H^4_{113} - H^5_{112}) - (H^5_{112} + H^6_{112} + H^4_{133} - H^6_{113} - H^5_{123})$$
$$= H^4_{111} + H^4_{122} - 2H^4_{112} - H^4_{133}$$
$$= \frac{\partial C^4_1}{\partial x^1}(0) + \frac{\partial C^4_2}{\partial x^1}(0) - \frac{\partial C^4_3}{\partial x^1}(0).$$

Thereby proving *Claim 3* and hence the Lemma. □

We have not yet used the existence of a $v \in T_pM$ such that $\nabla_v \mathbf{Riem}(p) \neq 0$. Using this fact we now show that we can choose our normal coordinate system (x^1, x^2, x^3) such that

$$\frac{\partial E^i}{\partial x^1}(0) \quad \text{and} \quad \frac{\partial E^i}{\partial x^2}(0) \quad \text{are both non-zero}$$

for $i = 1, 2, 3$. This will imply that $L[u_0]$ is of real principal type at 0.

Let $\mathbf{Riem}(p)$ denote the Riemannian curvature tensor at a point p. Then $\nabla\mathbf{Riem}(p)$ is the linear map

$$\nabla\mathbf{Riem}(p) : T_pM \times \bigwedge^2 T_pM \times \bigwedge^2 T_pM \to \mathbb{R}$$

where $\nabla\mathbf{Riem}(p)(X, Y, Z) = \nabla\mathbf{Riem}(p)(X, Z, Y)$ for all $X \in T_pM$ and $Y, Z \in \bigwedge^2 T_pM$. Let $\{E_i\}_{i=1}^3$ be a basis for T_pM and let $\{E_i \wedge E_j\}_{1 \leq i < j \leq 3}$ be a basis for $\bigwedge^2 T_pM$. We define

$$\nabla_a \mathbf{Riem}_{\alpha\beta} := \nabla\mathbf{Riem}(p)(E_a, E_i \wedge E_j, E_k \wedge E_l)$$

where

$$\tau_{ij} = \alpha \quad \text{and} \quad \tau_{kl} = \beta.$$

Here $\tau_{12} := \tau_{21} := 1, \tau_{13} := \tau_{31} := 2, \tau_{23} := \tau_{32} := 3$ and $\tau_{11} := \tau_{22} := \tau_{33} := 0$. Note that $\nabla_a \mathbf{Riem}_{\alpha\beta} = \nabla_a \mathbf{Riem}_{\beta\alpha}$. Therefore for a fixed a we can consider $\nabla_a \mathbf{Riem}$ as a symmetric matrix. The following Lemma is a straightforward consequence of the transformation laws for the covariant derivative of the Riemann curvature tensor.

Lemma 4.2. *Let $\{E_i\}$ and $\{F_j\}$ be two bases for T_pM and let $L : T_pM \to T_pM$ be a linear map where $F_i = L_i^j E_j$. If $\nabla_a \mathbf{Riem}$ denotes the components of $\nabla \mathbf{Riem}(p)$ with respect to the basis $\{E_i\}$ and $\overline{\nabla_a \mathbf{Riem}}$ denotes the components of $\nabla \mathbf{Riem}(p)$ with respect to the basis $\{F_j\}$ then*

$$\overline{\nabla_a \mathbf{Riem}} = L_a^{a'} (X \cdot \nabla_{a'} \mathbf{Riem} \cdot X^T)$$

where

$$X = \begin{pmatrix} L_1^1 \wedge L_2^2 & L_1^1 \wedge L_3^2 & L_2^1 \wedge L_3^2 \\ L_1^1 \wedge L_2^3 & L_1^1 \wedge L_3^3 & L_2^1 \wedge L_3^3 \\ L_1^2 \wedge L_2^3 & L_1^2 \wedge L_3^3 & L_2^2 \wedge L_3^3 \end{pmatrix}. \tag{4.9}$$

Here $L_i^\alpha \wedge L_j^\beta := L_i^\alpha L_j^\beta - L_i^\beta L_j^\alpha$.

Lemma 4.3. *Suppose there exists a vector $v \in T_pM$ such that $\nabla_v \mathbf{Riem}(p) \neq 0$. Then there exists a basis $\{F_i\}$ for T_pM such that with respect to this basis the quantities*

$$S_k^1 := \nabla_k \mathbf{Riem}_{33} + \nabla_k \mathbf{Riem}_{13} - \nabla_k \mathbf{Riem}_{12} - \nabla_k \mathbf{Riem}_{23}$$
$$S_k^2 := \nabla_k \mathbf{Riem}_{22} + \nabla_k \mathbf{Riem}_{13} - \nabla_k \mathbf{Riem}_{12} - \nabla_k \mathbf{Riem}_{23}$$
$$S_k^3 := \nabla_k \mathbf{Riem}_{11} + \nabla_k \mathbf{Riem}_{13} - \nabla_k \mathbf{Riem}_{12} - \nabla_k \mathbf{Riem}_{23}$$

where $k = 1, 2$ are all non-zero.

Proof. We first prove that there exists a basis $\{E_i\}_{i=1}^3$ for T_pM such that with respect to this basis

$$(\nabla_1 \mathbf{Riem}_{\alpha\beta}) = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}, \qquad a \neq 0. \tag{4.10}$$

By assumption there exists a $v \in T_pM$ for which $\nabla_v \mathbf{Riem}(p) \neq 0$, therefore we can choose a basis $\{E_i\}_{i=1}^3$ for T_pM such that

$$(\nabla_v \mathbf{Riem}_{\alpha\beta}) = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

where $A \neq 0$ and $A+B \neq 0$ (note that if $A+B=0, B+C=0$ and $A+C=0$ then $A=B=C=0$). Now let $\{F_i\}_{i=1}^3$ be a new basis for T_pM where

$$F_1 = v = k^1E_1 + k^2E_2 + k^3E_3$$
$$F_2 = E_2 + E_3$$
$$F_3 = E_2 - E_3.$$

We have assumed that $v \notin \text{span}\{E_2, E_3\}$ and so $k^1 \neq 0$. Therefore $F_j = L^i_j E_i$ where

$$L = \begin{pmatrix} k^1 & 0 & 0 \\ k^2 & 1 & 1 \\ k^3 & 1 & -1 \end{pmatrix}.$$

Now by Lemma 4.2 it follows that

$$\begin{aligned} \overline{\nabla_1 \mathbf{Riem}_{\alpha\beta}} &= L^a_1 (X \cdot \nabla_a \mathbf{Riem} \cdot X^T)_{\alpha\beta} \\ &= (X \cdot k^i \nabla_i \mathbf{Riem} \cdot X^T)_{\alpha\beta} \\ &= (X \cdot \nabla_v \mathbf{Riem} \cdot X^T)_{\alpha\beta} \end{aligned}$$

where

$$X = \begin{pmatrix} k^1 & k^1 & 0 \\ k^1 & -k^1 & 0 \\ k^2 - k^3 & -(k^2 + k^3) & -2 \end{pmatrix}.$$

Therefore

$$\overline{\nabla_1 \mathbf{Riem}_{11}} = (k^1)^2(A + B) \neq 0$$

and thus we may assume that (4.10) holds.

Define

$$L = \begin{pmatrix} 1/\epsilon & 1/\epsilon & 0 \\ 0 & -\epsilon & \epsilon^2 \\ 0 & \epsilon & \epsilon^2 \end{pmatrix}$$

where $\epsilon > 0$ is a small number to be chosen later. Note that $\det L = -2\epsilon^2 \neq 0$ and so $L \in GL(\mathbb{R}^3)$. Substituting L into the matrix (4.9) we get

$$X = \begin{pmatrix} -1 & \epsilon & \epsilon \\ 1 & \epsilon & \epsilon \\ 0 & 0 & -2\epsilon^3 \end{pmatrix}.$$

By Lemma 4.2 we know that

$$\overline{\nabla_a \mathbf{Riem}_{\alpha\beta}} = L_a^{a'} (X \cdot \nabla_{a'} \mathbf{Riem} \cdot X^T)_{\alpha\beta}.$$

Now since L_1^1 and $L_2^1 = 1/\epsilon \gg L_i^{2,3} = O(\epsilon)$ for $i = 1, 2, 3$ we have

$$\overline{\nabla_1 \mathbf{Riem}_{\alpha\beta}} = \frac{1}{\epsilon}(X \cdot \nabla_1 \mathbf{Riem} \cdot X^T)_{\alpha\beta}$$
$$\overline{\nabla_2 \mathbf{Riem}_{\alpha\beta}} = \frac{1}{\epsilon}(X \cdot \nabla_1 \mathbf{Riem} \cdot X^T)_{\alpha\beta} + O(\epsilon).$$

Therefore for $k = 1, 2$ we have

$$\overline{\nabla_k \mathbf{Riem}} = \frac{1}{\epsilon} \begin{pmatrix} a & -a & 0 \\ -a & a & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(\epsilon),$$

where $a = (k^1)^2(A + B)$. It follows that for $k = 1, 2$

$$S^1_k = \frac{a}{\epsilon} + O(\epsilon)$$
$$S^2_k = \frac{2a}{\epsilon} + O(\epsilon)$$
$$S^3_k = \frac{2a}{\epsilon} + O(\epsilon).$$

Since $a \neq 0$ by taking ϵ small enough we can ensure that $S^i_k \neq 0$ for $k = 1, 2$. □

Combining Lemma 4.1 and Lemma 4.3 together we obtain the following Corollary.

Corollary 4.1. *There exists a normal coordinate system (see Definition 3.1) centered at p, (x^1, x^2, x^3) such that for $k = 1, 2$ the quantities*

$$\frac{1}{2}\frac{\partial E^1}{\partial x^k}(0), \quad \frac{1}{2}\frac{\partial E^2}{\partial x^k}(0), \quad \frac{1}{2}\frac{\partial E^3}{\partial x^k}(0)$$

are all non-zero.

To show that $L[u_0]$ is an operator of real principal type we will use the following Lemma.

Lemma 4.4. *Let $p \in \Omega$. If no null bicharacteristic of σ is contained in the fiber $\{p\} \times (\mathbb{R}^3 - 0)$ then there is a compact neighbourhood K of p such that no null bicharacteristic of σ remains in $\pi^{-1}(K)$.*

Proof. See [17] Lemma 7.8 on page 344. □

Theorem 4.1. *$L[u_0]$ is of real principal type at* 0.

Proof. Let $\gamma(t) = (x(t), \eta(t))$ be an integral curve for the Hamiltonian vector field associated to the symbol σ

$$H_\sigma := \frac{\partial \sigma}{\partial x^i}\frac{\partial}{\partial \eta_i} - \frac{\partial \sigma}{\partial \eta_i}\frac{\partial}{\partial x^i}$$

where $x(0) = 0$ and $\sigma(\gamma(0)) = 0$. That is, γ is a null bicharacteristic of σ passing through $x = 0$. Let us now assume that γ gets trapped over the fibre $\{0\} \times (\mathbb{R}^3 - 0)$. Therefore there exists a $K > 0$ such that $x(t) = 0$ for $-K < t < K$. Using equation (4.5) it follows that

$$\begin{aligned} 0 &= \frac{dx^1}{dt}(t) = \frac{\partial \sigma}{\partial \eta_1}(\gamma(t)) = -16\eta_2(t)\eta_3(t) \\ 0 &= \frac{dx^2}{dt}(t) = \frac{\partial \sigma}{\partial \eta_2}(\gamma(t)) = -16\eta_1(t)\eta_3(t) \\ 0 &= \frac{dx^3}{dt}(t) = \frac{\partial \sigma}{\partial \eta_3}(\gamma(t)) = -16\eta_1(t)\eta_2(t) \end{aligned} \tag{4.11}$$

for all $-K < t < K$. Because of this we can find an $\epsilon > 0$ such that for all $-\epsilon < t < \epsilon$ only one of the three conditions holds

$$\begin{aligned} &\textbf{(i)} \quad \eta_1(t) = \eta_2(t) = 0, \quad \eta_3(t) \neq 0 \\ &\textbf{(ii)} \quad \eta_1(t) = \eta_3(t) = 0, \quad \eta_2(t) \neq 0 \\ &\textbf{(iii)} \quad \eta_2(t) = \eta_3(t) = 0, \quad \eta_1(t) \neq 0. \end{aligned}$$

Note that by assumption $\eta(t) \neq 0$. Suppose condition (**i**) were true. Then $d\eta_1(0)/dt = 0$. But by (4.5) and Corollary 4.1 we have

$$\frac{d\eta_1}{dt}(0) = -\frac{\partial \sigma}{\partial x^1}(0) = -2\frac{\partial E^3}{\partial x^1}(0)\eta_3^3(0) \neq 0.$$

Therefore we have a contradiction and so condition (**i**) cannot hold. Likewise conditions (**ii**) and (**iii**) also cannot be true. Therefore there exists a $K > 0$ such that $x(t) \neq 0$ for some $t = \pm K$.

By Lemma 4.4 it follows that $L[u_0]$ is of real principal type at $x = 0$. $\square$

Chapter 5

Moser Estimates for Φ'

We now use the results of [4] and [16] to prove the Moser estimates for the linearization of (1.1).

Let X be a bounded open subset of $\mathbb{R}^n$ and let $\bar{X}$ denote its closure. Define

$$C^\infty(\bar{X}, \mathbb{R}^N) = \{f \mid f = \varphi|_X \text{ for some } \varphi \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)\}.$$

Given any $f \in C^\infty(\bar{X}, \mathbb{R}^N)$ let

$$||f||_k := (\sum_{|\alpha|=0}^{k} \int_X |\partial^\alpha f|^2)^{1/2}.$$

Let $H^k(\bar{X}, \mathbb{R}^N)$ denote the completion of $C^\infty(\bar{X}, \mathbb{R}^N)$ with respect to $|| \cdot ||_k$. For any $f \in H^k(\bar{X}, \mathbb{R}^N)$ and $\epsilon > 0$ we define

$$B_\epsilon^k(f) = \{g \in H^k(\bar{X}, \mathbb{R}^N) \mid ||g - f||_k < \epsilon\}.$$

To prove Moser estimates for Φ' we use the following result of [4](see also [16] Theorem 3.2).

Theorem 5.1. *Let X be an open bounded subset of 0 in $\mathbb{R}^n$ and let $u \in C^\infty(X, \mathbb{R}^N)$. Let $L[u]$ be a first order operator of the form*

$$L[u]v := A^i(u)\frac{\partial v}{\partial x^i} + B(u)v$$

where A^i and B are $N \times N$ matrices depending smoothly on u and its first and second derivatives. Suppose there exists a u_0 such that $L[u_0]$ is of real principal type at 0, then there exists an open neighbourhood of $0 \in W \subset X$, an $\epsilon > 0$ and a $J, \alpha, \beta \in \mathbb{N}$ such that for all $u \in B^J_\epsilon(u_0) \cap C^\infty(\bar{W}, \mathbb{R}^N)$

1. *$L[u] : C^\infty(\bar{W}, \mathbb{R}^N) \to C^\infty(\bar{W}, \mathbb{R}^N)$ is surjective.*

2. *If $v, f \in C^\infty(\bar{W}, \mathbb{R}^N)$ such that $L[u]v = f$ then*

$$||v||_l \leq C(||f||_{l+\alpha} + ||u||_{l+\beta}||f||_\alpha) \tag{5.1}$$

for all $l \geq J$ where C is a constant which does not depend on u or f.

We also need to make repeated use of the following inequality.

Lemma 5.1 (Gagliardo-Nirenberg inequality)**.** *Let $u, v \in L^\infty(\Omega) \cap H^l(\Omega, \mathbb{R})$ and let α, β be multi-indices such that $|\alpha| + |\beta| = l$. Then*

$$||uv||_l \leq C(||u||_0||v||_l + ||u||_l||v||_0)$$

where C is a constant which depends on l, but does not depend on u or v.

Proof. See [18]. □

Let (Ω, x^1, x^2, x^3) be the normal coordinate system constructed in Corollary 4.1. Let $u_0 : \Omega \to \mathbb{R}^6$ be the smooth nondegenerate embedding constructed in Chapter §4. By Theorem 4.1 we know that $L[u_0]$ is of real principal type at 0.

Theorem 5.2. *There exists an open neighbourhood of 0, $W \subset \Omega$, an $\epsilon > 0$ and a $J, \alpha, \beta \in \mathbb{N}$ such that for all $u \in C^\infty(\bar{W}, \mathbb{R}^6) \cap B_\epsilon^J(u_0)$ the following hold*

1. *The linear map $\Phi'(u) : C^\infty(\bar{W}, \mathbb{R}^6) \to C^\infty(\bar{W}, \mathbb{R}^6)$ is surjective.*

2. *If $v, f \in C^\infty(\bar{W}, \mathbb{R}^6)$ such that $\Phi'(u)v = f$ then*

$$||v||_l \leq C(||f||_{l+\alpha} + ||u||_{l+\beta}||f||_\alpha)$$

for all $l \geq J$. Here C does not depend upon the functions u or f.

Proof. Since $L[u_0]$ is of real principal type we can apply Theorem 5.1. Therefore there exists an open neighbourhood of 0, $W \subset \Omega$, $J \in \mathbb{N}$ and $\epsilon > 0$ such that if $u \in B_\epsilon^J(u_0) \cap C^\infty(\bar{W}, \mathbb{R}^6)$, $L[u]$ is invertible and the Moser estimates, inequality (5.1) hold. From now on we assume that $u \in B_\epsilon^J(u_0) \cap C^\infty(\bar{W}, \mathbb{R}^6)$ and that for any such u, $||u||_{\tilde{J}} \leq 1$ for some $\tilde{J}$ much larger than J.

We first prove that $\Phi'(u) : C^\infty(\bar{W}, \mathbb{R}^6) \to C^\infty(\bar{W}, \mathbb{R}^6)$ is surjective: Let $f \in C^\infty(\bar{W}, \mathbb{R}^6)$ and let $y := (v_1, v_2, v_3)^T$ be a solution to the differential equations

$$L[u]y = \begin{pmatrix} f_{11} + C_1^4 f_{12} + C_1^5 f_{13} + C_1^6 f_{23} \\ f_{22} + C_2^4 f_{12} + C_2^5 f_{13} + C_2^6 f_{23} \\ f_{33} + C_3^4 f_{12} + C_3^5 f_{13} + C_3^6 f_{23} \end{pmatrix} := g(u). \tag{5.2}$$

Such a y exists since $L[u]$ is surjective. Let $\tilde{y} := (v^4, v^5, v^6)^T$ be given by the equations

$$\tilde{y} = -\frac{1}{2}\mathbf{H}^{-1} \begin{pmatrix} f_{12} - \partial_{x^2} v_1 - \partial_{x^1} v_2 + 2\Gamma_{12}^k v_k \\ f_{13} - \partial_{x^3} v_1 - \partial_{x^1} v_3 + 2\Gamma_{13}^k v_k \\ f_{23} - \partial_{x^3} v_2 - \partial_{x^2} v_3 + 2\Gamma_{23}^k v_k \end{pmatrix}. \tag{5.3}$$

That $\mathbf{H}^{-1}$ exists follows from the fact that u_0 is nondegenerate and any embedding u close to u_0 is also nondegenerate. In Chapter §2 it was shown that if we define

$$v = p^{lk} v_k \frac{\partial u}{\partial x^l} + v^\lambda N_\lambda$$

where $(p^{lk}) = (\partial_{x^i} u \cdot \partial_{x^j} u)^{-1}$ and N_λ are vectors perpendicular to $\partial_{x^l} u$, then

$$\Phi'(u)v = f.$$

Therefore Φ' is surjective. We now prove the Moser estimates.

By Theorem 5.1 we know that

$$||y||_l \leq C(||g||_{l+\alpha} + ||u||_{l+\beta}||g||_\alpha) \tag{5.4}$$

for all $l \geq J$. Now from (5.2) we have

$$\begin{aligned} ||g_i||_l &= ||f_{ii} + C_i^4 f_{12} + C_i^5 f_{13} + C_i^6 f_{23}||_l \\ &\leq ||f_{ii}||_l + ||C_i^4 f_{12}||_l + ||C_i^5 f_{13}||_l + ||C_i^6 f_{23}||_l. \end{aligned}$$

By Lemma 5.1 we have

$$||C_i^4 f_{12}||_l \leq C(||C_i^4||_0||f_{12}||_l + ||C_i^4||_l||f_{12}||_0).$$

Since C_j^μ depends only on the second derivatives of u it follows that

$$||C_j^\mu||_l \leq C||u||_{l+2}.$$

Therefore

$$\begin{aligned} ||C_i^4 f_{12}||_l &\leq C(||u||_2||f||_l + ||f||_0||u||_{l+2}) \\ &\leq C(||f||_l + ||f||_0||u||_{l+2}). \end{aligned}$$

Identical estimates also hold for $C_i^5 f_{13}$ and $C_i^6 f_{23}$. Therefore

$$||g||_l \leq C(||f||_l + ||f||_0||u||_{l+2}). \tag{5.5}$$

Substituting estimate (5.5) back into (5.4) gives us

$$||y||_l \leq C(||f||_{l+\alpha} + ||f||_0||u||_{l+\alpha+2} + ||u||_{l+\beta}(||f||_\alpha + ||f||_0||u||_{\alpha+2})).$$

Let $\tilde{\alpha} = \alpha$ and $\tilde{\beta} = \max\{\alpha + 2, \beta\}$ then

$$||y||_l \leq C(||f||_{l+\tilde{\alpha}} + ||f||_{\tilde{\alpha}}||u||_{l+\tilde{\beta}}), \qquad l \geq J. \tag{5.6}$$

We now derive similar estimates for $\tilde{y}$. By equations (5.3) we know that

$$\tilde{y}^\mu = -\frac{1}{2}\sum_{1\leq i<j\leq 3} h_\mu^{ij}(f_{ij} - \frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i} + 2\Gamma_{ij}^k v_k) \tag{5.7}$$

where h_μ^{ij} are smooth functions of $\partial_{x^i x^j} u$. Recall that Γ_{ij}^k are also smooth functions of the second partial derivatives of u. Therefore

$$||h_\mu^{ij}||_l + ||\Gamma_{ij}^k||_l \leq C||u||_{l+2}.$$

Applying Lemma 5.1 to (5.7) we obtain the following inequalities

$$\begin{aligned} ||\tilde{y}^\mu||_l \leq C \sum_{1\leq i<j\leq 3} (||h_\mu^{ij}||_0||f_{ij} - \frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i} + 2\Gamma_{ij}^k v_k||_l \\ +||h_\mu^{ij}||_l||f_{ij} - \frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i} + 2\Gamma_{ij}^k v_k||_0). \end{aligned} \tag{5.8}$$

Using the triangle inequality and Lemma 5.1 we obtain

$$\begin{aligned} ||f_{ij} - \frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i} + 2\Gamma_{ij}^k v_k||_l &\leq C(||f||_l + ||y||_{l+1} + ||\Gamma_{ij}^k v_k||_l) \\ &\leq C(||f||_l + ||y||_{l+1} + ||\Gamma_{ij}^k||_0||v_k||_l + ||\Gamma_{ij}^k||_l||v_k||_0) \\ &\leq C(||f||_l + ||y||_{l+1} + ||u||_2||y||_l + ||u||_{l+2}||y||_0) \\ &\leq C(||f||_l + ||y||_{l+1} + ||u||_{l+2}||y||_0) \end{aligned} \tag{5.9}$$

By estimate (5.6) it follows that

$$||y||_0 \leq C(||f||_{\tilde{\alpha}+J} + ||f||_{\tilde{\alpha}}||u||_{\tilde{\beta}+J}) \leq C||f||_{\tilde{\alpha}+J}.$$

Therefore inequality (5.9) becomes

$$||f_{ij} - \frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i} + 2\Gamma^k_{ij} v_k||_l \leq C(||f||_l + ||y||_{l+1} + ||u||_{l+2}||f||_{\tilde{\alpha}+J}) \quad (5.10)$$

and so setting $l = 0$ and using estimate (5.6) again gives us the following L^2 bound

$$\begin{aligned} ||f_{ij} - \frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i} + 2\Gamma^k_{ij} v_k||_0 &\leq C(||f||_0 + ||y||_1 + ||u||_2||f||_{\tilde{\alpha}+J}) \\ &\leq C||f||_{\tilde{\alpha}+J}. \end{aligned} \quad (5.11)$$

Substituting inequalities (5.10) and (5.11) into inequality (5.8) we find that

$$\begin{aligned} ||\tilde{y}^{\mu}||_l &\leq C(||f||_l + ||y||_{l+1} + ||u||_{l+2}||f||_{\tilde{\alpha}+J} + ||u||_{l+2}||f||_{\tilde{\alpha}+J}) \\ &\leq C(||f||_l + ||y||_{l+1} + ||u||_{l+2}||f||_{\tilde{\alpha}+J}). \end{aligned}$$

Finally using inequality (5.6) we obtain

$$\begin{aligned} ||\tilde{y}||_l &\leq C(||f||_l + ||f||_{l+\tilde{\alpha}+1} + ||f||_{\tilde{\alpha}}||u||_{l+\tilde{\beta}+1} + ||f||_{\tilde{\alpha}+J}||u||_{l+2}) \\ &\leq C(||f||_{l+\tilde{\alpha}+1} + ||f||_{\tilde{\alpha}+J}||u||_{l+\tilde{\beta}+1}), \qquad l \geq J. \end{aligned} \quad (5.12)$$

The final step of the proof is to use inequalities (5.6) and (5.12) to estimate v. Recall that in terms of v_k and v^{λ}, v is given by the formula

$$v = p^{jk} v_k \frac{\partial u}{\partial x^j} + v^{\lambda} N_{\lambda}.$$

Since p^{jk} and N_{λ} depend smoothly on the first derivatives of u it follows that

$$||p^{jk} \frac{\partial u}{\partial x^j}||_l + ||N_{\lambda}||_l \leq C||u||_{l+1}.$$

Using the above estimate and Lemma 5.1 we see that

$$\begin{aligned}
||v||_l &\leq ||p^{jk}v_k\frac{\partial u}{\partial x^j}||_l + ||v^\lambda N_\lambda||_l \\
&\leq C(||p^{jk}\frac{\partial u}{\partial x^j}||_0||v_k||_l + ||p^{jk}\frac{\partial u}{\partial x^j}||_l||v_k||_0 + ||N_\lambda||_l||v^\lambda||_0 + ||N_\lambda||_0||v^\lambda||_l) \\
&\leq C(||u||_1||y||_l + ||u||_{l+1}||y||_0 + ||u||_1||\tilde{y}||_l + ||u||_{l+1}||\tilde{y}||_0) \\
&\leq C(||y||_l + ||\tilde{y}||_l + ||u||_{l+1}(||y||_0 + ||\tilde{y}||_0)).
\end{aligned}$$

Applying inequalities (5.6) and (5.12) to this estimate gives us

$$\begin{aligned}
||v||_l &\leq C(||f||_{l+\tilde{\alpha}+1} + ||f||_{\tilde{\alpha}+J}||u||_{l+\tilde{\beta}+1} + ||u||_{l+1}||f||_{\tilde{\alpha}+J}) \\
&\leq C(||f||_{l+\tilde{\alpha}+J} + ||f||_{\tilde{\alpha}+J}||u||_{l+\tilde{\beta}+1}).
\end{aligned}$$

□

Remark. A result almost identical to Theorem 5.2 was proved in Section VI of [1].

Chapter 6

Proof of main theorem

Theorem 6.1 (Nash-Moser)**.** *Let N be a compact manifold with boundary and let U be an open subset of $C^\infty(N, \mathbb{R}^k)$. If $P : U \subset C^\infty(N, \mathbb{R}^k) \to C^\infty(N, \mathbb{R}^p)$ is a nonlinear partial differential operator such that*

1. *For all $u \in U$ and $f \in C^\infty(N, \mathbb{R}^p)$ there exists a unique $v \in C^\infty(N, \mathbb{R}^k)$ such that*

$$P'(u)v = f.$$

2. *There exists $\alpha, \beta, J \in \mathbb{N}$ such that if $P'(u)v = f$ then*

$$||v||_l \leq C(||f||_{l+\alpha} + ||u||_{l+\beta}||f||_\alpha)$$

for all $l \geq J$, where C is a constant which does not depend on u or f.

Then P is locally invertible.

Proof. See part III of the survey article by Hamilton [5]. □

Proof of main theorem. By Theorem 5.2 there exists an open neighbourhood W of 0 in $\mathbb{R}^3$, $J \in \mathbb{N}$ and $\delta > 0$ such that if we define

$$\begin{aligned} &P := \Phi \\ &U := C^\infty(\bar{W}, \mathbb{R}^6) \cap B^J_\delta(u_0) \\ &N := \bar{W} \\ &k, p := 6 \end{aligned}$$

then the assumptions of the Nash-Moser theorem are satisfied. Thus there exists an $\epsilon > 0$ and $K \in \mathbb{N}$ such that for all $f \in C^\infty(\bar{W}, \mathbb{R}^6) \cap B^K_\epsilon(\Phi(u_0))$ there exists a $u \in C^\infty(\bar{W}, \mathbb{R}^6)$ such that $\Phi(u) = f$.

Unfortunately, we are not done yet as the metric g_{ij} which we want to isometrically embed may not be in $B^K_\epsilon(\Phi(u_0))$. To overcome this problem we use the following trick of [1]. Let ρ be a smooth compactly supported function on $\mathbb{R}^6$ which is identically 1 in a neighbourhood of the origin and let g be our Riemannian metric on $\bar{W}$. Given $\delta > 0$ let

$$g_\delta(x) = \rho(\delta^{-1}x)g(x) + [1 - \rho(\delta^{-1}x)]\Phi(u_0)(x).$$

Using the formal part of the Cauchy-Kovalevskaya theorem (ignoring convergence) and the Borel theorem, we may extend u_0 such that $\Phi(u_0)$ agrees with g up to infinite order. It then follows that for δ sufficiently small, $g_\delta \in B^K_\epsilon(\Phi(u_0))$ (for a proof of this result see Proposition (6.b.1) in [1]). Therefore there exists a $u \in C^\infty(\bar{W}, \mathbb{R}^6)$ such that $\Phi(u) = g_\delta$. Since there exists an open neighbourhood of 0, $Y \subset W$, such that $g_\delta|_Y = g$, it follows that on Y, $\Phi(u) = g$. $\square$

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