Rational Curves in Low Degree Hypersurfaces in Grassmannian Varieties

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Abstract of the Dissertation

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We consider two properties of the Kontsevich moduli spaces of genus-0 stable maps to a variety X. The first, irreducibility, implies that certain genus-0 Gromov-Witten invariants are enumerative. The second, existence of very twisting families, implies the existence of sections for a two parameter family with vanishing elementary obstruction. Both of these properties are known to hold for homogeneous varieties, as well as low degree hypersurfaces in projective space.

Motivated by these results for projective space, we prove that the Kontsevich moduli spaces are irreducible when X is a low degree hypersurface in a Grassmannian variety. We conjecture a sharp inequality $kd^2 < n$ for when a two parameter family of degree d hypersurfaces in the Grassmannian $G(k, n)$ with vanishing elementary obstruction admits a rational section, and prove that a slightly weaker result holds.

Contents

Chapter 1

Introduction

In this dissertation, we prove two independent new results concerning the Kontsevich moduli space of rational stable maps to a general low degree hypersurface X in the Grassmannian variety of k planes in an n dimensional vector space. These results have immediate applications to the enumerativity of genus-0 Gromov-Witten invariants, and to the construction of rational sections over a surface.

Given a scheme X, the Kontsevich moduli space $\overline{\mathcal{M}}_{0,m}(X,\beta)$ parameterizes morphisms from an at worst nodal genus-0 curve C with m marked points to X, such that the image of C has homology class β. $\overline{\mathcal{M}}_{0,m}(X,\beta)$ is a compactification of the space of smooth embedded rational curves in X with homology class β. As such, it serves as a basis for techniques aimed at studying the target space X in terms of its rational curves.

The first application of Kontsevich moduli spaces was the definition of Gromov-Witten invariants. If we have a collection of cohomology classes $\gamma_1,\ldots,\gamma_m$ such that $\sum_i codim(\gamma_i)$ is equal to the expected dimension of $\overline{\mathcal{M}}_{0,m}(X,\beta)$, then if $\overline{\mathcal{M}}_{0,m}(X,\beta)$ is irreducible of the expected dimension, and if some transversality condition is satisfied, the Gromov-Witten invariant $I_\beta(\gamma_1,\ldots,\gamma_m)$ in some sense counts the number of rational curves with class β spanning general subschemes representing $\gamma_1,\ldots,\gamma_m$.

As another application, one can define a notion of *rational simple connectedness*. A scheme X is *rationally connected* if there is a rational curve spanning two general points of X. In applications, this notion has proven to be analogous to path connectedness, and so motivates us to find a definition of rational simple connectedness that serves a similar use to the corresponding topological notion. Ideally, such a definition would incorporate the rational connectedness of some moduli of rational curves, and have applications to the construction of sections over surfaces.

In fact, there are currently several similar notions of rational simple connectedness, each of them requiring some moduli $M \subset \overline{\mathcal{M}}_{0,2}(X, \beta)$ of rational curves spanning two fixed general points to be rationally connected. However, a common theme among all notions is the existence of a *very twisting* family of pointed rational curves. This is a morphism $\mathbb{P}^1 \to \overline{\mathcal{M}}_{0,1}(X, \beta)$ satisfying some positivity conditions necessary for glueing and smoothing arguments.

Thus, given a scheme X, two important questions we might ask about: when is $\overline{\mathcal{M}}_{0,m}(X, \beta)$ irreducible, and when does $\overline{\mathcal{M}}_{0,1}(X, \beta)$ contain very twisting family? It turns out that these are both difficult questions, and current results are limited to homogeneous varieties and hypersurfaces or complete intersections in projective space. For homogeneous varieties, irreducibility is first proved by Kim and Pandharipande in [KP01], and the existence of very twisting families of lines is later proved by de Jong, He, and Starr in [dJHS08]. For a general degree d hypersurface in $\mathbb{P}^n$, Harris, Roth, and Starr prove irreducibility when $d < (n+1)/2$ in [HRS04], and in [Sta06], Starr proves the existence of a very twisting family of lines when $d^2 \leq n$.

In Chapters 2 and 3, we prove results analogous to those in projective space, for X a general low degree hypersurface in the Grassmannian $G(k, V)$. Additionally, we show that X satisfies a form of rational simple connectedness necessary for the theorem of [dJHS08] to imply the existence of rational sections over a surface.

1.1 Outline and statement of results.

First, some brief notation. Throughout this dissertation all varieties will be assumed to be over $\mathbb{C}$. We will denote by V an n dimensional vector space, and k will be assumed to be less than or equal to $n/2$.

Throughout this thesis a *curve* will be a reduced, connected, proper scheme of dimension 1, possibly reducible. A *family* will be a flat family over a connected base.

Definition 1.1. An *m-pointed stable map to X* is a morphism g from a connected reduced curve C to X, together with a collection of points $p_1, \ldots, p_m \in C$ satisfying the following conditions.

(1) C has at worst nodal singlularities.

(2) The morphism g has no infinitesimal automorphisms, which is equivalent to the following

  (i) if E is a genus 0 component of C which is contracted by f, then E contains at least three nodes or marked points, and

  (ii) if E is a genus 1 component which is contracted, then E contains at least one node or marked point.

Definition 1.2. A *family of m-pointed stable maps to X* is a tuple

$$(C \to B, g; p_1, \ldots, p_m)$$

consisting of a flat morphism $\pi : C \to B$, a morphism $g : C \to X$, and sections $p_i : B \to C$ such that for every closed point $b \in B$, the fiber C_b is a connected reduced genus g curve, with n distinct points specified by $\{p_i(b), 1 \leq i \leq m\}$, and the morphism $f_b : C_b \to X$ induced by restriction is a stable map.

The *Kontsevich moduli space* $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is the moduli functor $\mathfrak{Sch} \to \mathfrak{Set}$ associating to each scheme B the set of isomorphism classes of families of stable maps $(C \to B, f)$, with C a genus g curve, such that for each closed point $b \in B$, $f_{b*}([C_b]) = \beta$, for a homology class β in $\mathrm{H}_2(X, \mathbb{Q})$. $\overline{\mathcal{M}}_{g,m}(X, \beta)$ is a stack, and is coarsely represented by a scheme $\overline{M}_{g,m}(X, \beta)$.

For the remainder of this thesis, we will be concerned only with the genus 0 case of $\overline{\mathcal{M}}_{0,m}(X, \beta)$.

$\overline{\mathcal{M}}_{0,m}(X, \beta)$ comes with a universal family $\mathcal{U} \to \overline{\mathcal{M}}_{0,m}(X, \beta)$ and a morphism $g : \mathcal{U} \to X$. Denote the sections of this family by $\sigma_i : \overline{\mathcal{M}}_{0,n} \to \mathcal{U}, i = 1 \ldots n$. Denote by $ev_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$, $1 \leq i \leq n$ the i^{th} evaluation map, which associates to every stable curve the image of it's i^{th} marked point. Denote by $\mathcal{M}_{g,n}(X, \beta)$ the open substack of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ parametrizing morphisms with an irreducible domain.

The following results is [FP97, Theorem 2] combined with the connectedness result of [KP01, Theorem 1].

Theorem 1.3. *If X is a projective, nonsingular, homogeneous variety, then $\overline{M}_{0,m}(X,\beta)$ is a normal, irreducibile, projective variety of pure dimension*

$$\dim(X) + \int_\beta c_1(T_X) + m - 3$$

with orbifold singularities, and $\overline{\mathcal{M}}_{0,m}(X,\beta)$ is a smooth stack.

In particular, this result holds for Grassmannian varieties.

In fact, the above dimension count holds more generally in many cases, and so we have the following definition.

Definition 1.4. The *expected dimension* of $\overline{\mathcal{M}}_{0,m}(X,\beta)$ is $\dim(X) + \int_\beta c_1(T_X) + m - 3$.

As hypersurfaces in $G(k,V)$ have picard number 1, every integral curve class is a multiple of the class of a line. Thus, if α is the class of a line, we will use the notation $\overline{\mathcal{M}}_{0,m}(X,e)$ to refer to the moduli space $\overline{\mathcal{M}}_{0,m}(X,e\cdot\alpha)$.

In chapter 2, we prove the following irreducibility result.

Theorem 1.5. *Let X be a degree d hypersurface in $G(k,V)$ with $d \leq n - k(n-k)/2$, and $(d,k,n) \neq (2,2,4)$. Then $\overline{\mathcal{M}}_{0,n}(X,e)$ is irreducible.*

This result relies on an induction argument laid out in [HRS04], but will require a new argument in the base case $e = 1$, and some slight optimization for the induction argument to work. To this end, we first prove the following via synthetic methods.

Theorem 1.6. *For X a general degree d hypersurface in $G(k,r)$ such that $0 < d \leq n-k-1$, the map $ev_1 : \overline{\mathcal{M}}_{0,1}(X,1) \to X$ is flat of the expected fiber dimension $n-d-1$.*

In the case of lines, $\overline{\mathcal{M}}_{0,1}(X,1)$ is just the classical Fano variety $F(0,1;X)$, and so we adopt this notation in section 2.1. Irreducibility of $\overline{\mathcal{M}}_{0,1}(X,1)$ is an easy corollary of Theorem 1.6.

When $e > 1$, $\overline{\mathcal{M}}_{0,m}(X,e)$ has a boundary locus consisting of stable maps with reducible domain. One of the features of the Kontsevich moduli spaces is that this boundary has a nice stratification, called the Behrend-Manin decomposition, corresponding to different configurations of reducible components, marked points, and degree. In the induction argument of [HRS04], this decomposition is used to understand how the properties of flatness and irreducibility relate among different components of the boundary. To this end, in Section 2.2 we introduce the Behrend-Manin decomposition and adapt some of the arguments of [HRS04] to the case of hypersurfaces in $G(k, V)$. After verifying certain additional threshold conditions, a bend-and-break argument allows us to deduce the irreducibility of $\overline{\mathcal{M}}_{0,m}(X, e)$ from the irreducibility of $\overline{\mathcal{M}}_{0,1}(X, 1)$, proving Theorem 1.5

It should be noted that there are a limited set of k, d for which Theorem 1.5 holds. In the case $k = 1$, we get the result of [HRS04]. If $k = 2$ we get irreducibility when $d = 1, 2$. However, in the case that $d \leq n - k - 1$, we will be able to use Theorem 1.6 and results from section 2.2 to deduce the existence of a *canonical irreducible component* of $\overline{\mathcal{M}}_{0,m}(X, e)$. Specifically, we have the following lemma.

Lemma 1.7. *[dJS, Lemma 3.5] Let $M_{\alpha,0}$ be an irreducible component of $\overline{\mathcal{M}}_{0,0}(X, \alpha)$ whose general point parameterizes a smooth, free curve. Denote by $M_{\alpha,1}$ the unique irreducible component of $\overline{\mathcal{M}}_{0,1}(X, \alpha)$ dominating $M_{\alpha,0}$. Assume the geometric generic fiber of the restriction*

$$ev|_M : M_{\alpha,1} \to X$$

is geometrically irreducible.

For every positive integer e there is a unique irreducible component $M_{e\cdot\alpha,0}$ of $\overline{\mathcal{M}}_{0,0}(X, e\alpha)$ parameterizing (among others) a reducible curve whose (non-contracted) components are all multiple covers of free curves parameterized by $M_{\alpha,0}$.

A general point of $M_{e\cdot\alpha,0}$ parameterizes a smooth, free curve. Denoting by $M_{e\cdot\alpha,1}$ the unique irreducible component of $\overline{\mathcal{M}}_{0,1}(X, e\alpha)$ dominating $M_{e\cdot\alpha,0}$, the restriction

$$ev|_M : M_{e\cdot\alpha,1} \to X$$

is dominant with irreducible geometric generic fiber.

In our case, α will be the class of a line, and by Theorem 1.6 we see that we have these canonical components whenever $d \leq n - k - 1$.

Next, in Chapter 3, we investigate the existence of very twisting families of lines. We defer the definition of *very twisting* to Chaper 3.

Theorem 1.8. *Let V be an n-dimensional vector space. Let $k, d,$ and n be such that*

$$(3k-1)d^2 - d + 4k + 2 \leq n. \tag{1.1}$$

Let X be a general degree d hypersurface in $G(k,V)$. Then there exists a very twisting family of pointed lines in X.

A very twisting family family of lines sweeps out a ruled surface $\Sigma \subset X$, and the property of being 'very twisting' can be reformulated in terms of Σ and the section $\sigma : \mathbb{P}^1 \to \Sigma$. In the case that Σ is a ruled surface corresponding to a very twisting family of lines, we say that Σ is a *very twisting surface.* The condition of containing a very twisting surface is an open condition on the space of hypersurfaces in $G(k,V)$, and so to prove Theorem 1.8 it suffices to exhibit one hypersurface containing a fixed very twisting surface in $G(k,V)$, and satisfying some transitivity conditions.

By better understanding the transitivity condition imposed on X, it is quite possible to improve inequality 1.1. Indeed, we make the following conjecture.

Conjecture 1.9. *If X is a general degree d hypersurface in $G(k,V)$ with $kd^2 < n$, then X contains a very twisting family of lines.*

This differs from the previously conjectured inequality $d^2 < n - k$.

Next, in Section 3.5 we prove that the space of chains of $k+1$ lines spanning two fixed general points is rationally connected for a general hypersurface X of degree d satisfying

$$kd^2 < \begin{cases} n, & d < k \\ n - 2k, & d \geq k \end{cases}$$

This conclusion, together with the existence of a very twisting family of lines, is sufficient for the existence of rational sections of two parameter families of low degree hypersurfaces in $G(k,V)$, provided the family has vanishing elementary obstruction. Thus, if Conjecture 1.9 holds, then any unobstructed family of degree d hypersurfaces with $kd^2 < n$ would admit a rational section. In Section 3.6 we discuss an example, due to Jason Starr of a family of degree d in $G(k,V)$ with $d^2 = n/k$, with vanishing elementary obstruction, and that admits no rational sections. This demonstrates that Conjecture 1.9 is sharp.

1.2 Facts about Grassmannians

Throughout this dissertation, we will denote the Grassmannian variety of k-planes in an n vector space V by $G(k,V)$. We recall here some necessary facts about $G(k,V)$, and fix some notation, providing proofs for the non-standard results. The experienced reader may skim this section for the notation and move on.

Here are some basic facts.

Proposition 1.10. *$G(k,V)$ is a smooth, projective variety of dimension $k(n-k)$. There is a Plücker embedding $pl : G(k,V) \to \mathbb{P}\bigwedge^k V$, and a very ample sheaf $\mathcal{O}(1)$ induced from this morphism. Denote by $N = \binom{n}{k} - 1$ the dimension of $\mathbb{P}\bigwedge^k V$.*

There is a natural action of $PGL(V)$ on $G(k,V)$, under which $G(k,V)$ is a homogeneous variety.

Recall that there is a tautological k-bundle S_k on $G(k,V)$ and an exact sequence

$$0 \to S_k \to V \otimes \mathcal{O}_{G(k,V)} \to Q_k \to 0$$

such that the the image in V of S_k at a point $\lambda \in G(k,V)$ is equal to λ. By the definition of the Plücker map, $\mathcal{O}(1) \cong \bigwedge^k S_k$.

Lemma 1.11. *The tangent bundle $T_{G(k,V)}$ to $G(k,V)$ is isomorphic to $\mathcal{H}om(S_k, Q_k)$. The space of global sections $\mathrm{H}^0(G(k,V), T_{G(k,V)})$ is isomorphic to $\mathrm{Hom}(V,V)$.*

Since we will be concerned with moduli of curves on $G(k,V)$, the following result is critical, and so a proof is provided.

Proposition 1.12. *Given a point p in $G(k,V) \subset \mathbb{P}\bigwedge^k V = \mathbb{P}^N$, the space of lines in $\mathbb{P}\bigwedge^k V$ contained in $G(k,V)$ and containing p is isomorphic to $\mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$. Specifically, if we let $\mathbb{P}^{N-1}$ be the space of lines through p in $\mathbb{P}^N$, there is an inclusion $\mathbb{P}^{k-1} \times \mathbb{P}^{r-k-1} \to \mathbb{P}^{N-1}$ that can be factored through $\mathbb{P}^{k(r-k)-1}$ as the Segre map followed by a linear embedding into $\mathbb{P}^{N-1}$ whose image is the space of lines in $\mathbb{P}^N$ contained in $G(k,r)$.*

Proof. Given a degree 1 map $f : \mathbb{P}^1 \to G(k,V)$, we get a k-bundle $f^*S_k^\vee$ on $\mathbb{P}^1$ and a surjective map $\mathcal{O}_{\mathbb{P}^1}^{\oplus n} \to f^*S_k^\vee$. By Grothendieck's lemma,

$$f^*S_k^\vee = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_k).$$

As $f^*S_k^\vee$ is a quotient of $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$, we must have that $a_i \geq 0$ for all i. Furthermore, we have that $\mathcal{O}_{\mathbb{P}^N}(1) \cong \bigwedge^k S_k^\vee$, and so

$$\deg f = \deg f^*\mathcal{O}_{\mathbb{P}\bigwedge^k V} = \deg f^* \bigwedge^k S_k^\vee = \deg \bigwedge^k f^*S_k^\vee = \sum a_i = 1.$$

Therefore

$$f^*S_k^\vee \cong \mathcal{O}(1) \oplus \mathcal{O} \oplus \cdots \oplus \mathcal{O}.$$

The splitting data therefore induces k distinct maps of $\mathbb{P}^1$ into V, one of degree 1, and $k-1$ of degree 0. Denote by E_{k-1} the $k-1$ plane spanned by the constant images of the constant morphisms. Denote by E_{k+1} the $k+1$ plane spanned by the images of all k morphisms. Then the set of k planes in the image of f is exactly those k planes containing E_{k-1} and contained in E_{k+1}. Conversely any such pair $(E_{k-1} \subset E_{k+1})$ determines a k bundle on $\mathbb{P}^1$ inducing a degree 1 morphism to $G(k,V)$. Given a point p in $G(k,V)$, we can identify the space of lines in $G(k,V)$ containing $S_k|_p$ with $G(k-1, S_k|_p) \times G(1, V/S_k|_p) \cong \mathbb{P}^{k-1} \times \mathbb{P}^{r-k-1}$.

It can be seen that this identification in fact corresponds to a linear embedding of the Segre variety. □

Chapter 2

Irreducibility

Let X be a degree d hypersurface in $G(k,V)$. The homology $\mathrm{H}_2(X,\mathbb{Q})$ is generated by the class of a line, and so we follow the convention of denoting the moduli space $\overline{\mathcal{M}}_{g,m}(X, e\cdot\alpha)$ by $\overline{\mathcal{M}}_{g,m}(X,e)$. From the splitting principle for Chern classes and the conormal exact sequence

$$0 \to I_X/I_X^2 \to TG \to TX \to 0,$$

we see that $c_1(TX) = (n-d)\cdot H$, where H is the hyperplane class. From Definition 1.4 the expected dimension of $\overline{\mathcal{M}}_{0,m}(X,e)$ is $\dim X + e(n-d) + m - 3$.

In [HRS04], the authors lay out a general induction argument for proving that $\overline{\mathcal{M}}_{0,m}(X,e)$ is irreducible. We adapt the argument for the case of hypersurfaces in $G(k,V)$. The general technique will be discussed more in Section 2.2, however in the next section we will first prove the base case for the induction, which is the following.

Theorem 2.1. *For X a general degree d hypersurface in $G(k,V)$ such that $d \le n-k-1$, the map $ev_1 : \overline{\mathcal{M}}_{0,1}(X,1) \to X$ is flat of the expected fiber dimension.*

From this we will conclude the irreducibility of $\overline{\mathcal{M}}_{0,1}(X,1)$.

2.1 Lines in a hypersurface

Since in the degree one case the Kontsevich moduli spaces have no boundary components, in this section we will use instead the notation of Fano varieties. Given a subscheme Z of $G(k,V)$, $\overline{\mathcal{M}}_{0,0}(Z,1)$ is a scheme and is isomorphic to the Fano subvariety $F_1(Z) \subset G(2,\bigwedge^k V)$ of lines in $\mathbb{P}\bigwedge^k V$ that are contained in Z. Similarly, $\overline{\mathcal{M}}_{0,1}(Z,1)$ is isomorphic to the incidence variety

$$F_{0,1}(Z) = \{(p, L) \in G(k, V) \times G(2, \bigwedge^k V) \,|\, p \in L; L \subset Z\}$$

Note that $F_{0,1}(G)$ is a projective bundle over $G(k, V)$, and so is nonsingular. The evaluation map $ev_G : F_{0,1}(G(k, V)) \to G(k, V)$ is just the restriction of the projection map to the incidence subscheme. Given a hypersurface X in $G(k, V)$, the inclusion $X \hookrightarrow G(k, V)$ induces an inclusion $F_{0,1}(X) \hookrightarrow F_{0,1}(G(k, V))$. Denote by $ev : F_{0,1}(X) \to X$ the restriction of ev_G to $F_{0,1}(G(k, V))$. The latter map factors through the Fano scheme $F_1(X) \subset G(2, N+1)$ of lines in X.

Alternatively, we could describe $F_{0,1}(G(k, V))$ as an incidence subscheme of $G(k, V) \times G(k-1, r) \times G(k+1, r)$ consisting of points (P, Q, R) such that $Q \subset P \subset R$. Corresponding to this description, we have the pull backs to $F_{0,1}(G)$ of the tautological bundles E_{k-1}, E_k, and E_{k+1} and their duals $E_{k-1}^\vee$, $E_k^\vee$, and $E_{k+1}^\vee$, which by abuse of notation we will refer to without reference to the projection morphisms.

Let $W = \mathrm{Sym}^d \bigwedge^k V^\vee$ be the space of degree d homogeneous polynomials in $\mathbb{P} \bigwedge^k V$. For each $w \in \mathbb{P}W$, let $X = X_w$ be the corresponding hypersurface $G(k, V) \cap \mathbb{V}(w)$. Now, consider the scheme $\mathbb{P}W \times F_{0,1}(G)$, again with the bundles E_{k-1}, E_k, and E_{k+1}. Denote by π_1 the projection onto $\mathbb{P}W$, and π_2 the projection onto $F_{0,1}(G)$. Let $\mathcal{F}_{0,1}$ be the incidence subscheme of $\mathbb{P}W \times F_{0,1}(G)$ consisting of (w, p, l) such that

$$p \in l \subset X_w \subset G(k, V).$$

Then $\pi_1^{-1}(w) = F_{0,1}(X_w)$. Let $\mathcal{X}$ be the incidence subscheme of $\mathbb{P}W \times G(k, V)$ consisting of points (w, p) such that $p \in X_w$. Denote by $\mathfrak{ev} : \mathcal{F}_{0,1} \to \mathcal{X}$ the morphism induced by the projection maps. With the maps ev, π_1, and π_2 defined as above, $\mathfrak{ev} = \pi_1 \times (ev_G \circ \pi_2)|_{\mathcal{F}_{0,1}}$.

Proposition 2.2. *The dimension of each nonempty fiber of* $\mathfrak{ev}$ *is at least* $n - d - 2$.

Proof. On $\mathbb{P}W \times F_{0,1}(G)$, form the projective bundle

$$\zeta : \mathbb{P}\pi_2^*(E_{k+1}/E_{k-1}) \to \mathbb{P}W \times F_{0,1}(G).$$

Then the fiber $\zeta^{-1}(w, p, l)$ in $\mathbb{P}W \times F_{0,1}(G)$ is isomorphic to the line l, and we have a morphism $\iota : P \to G(k, V)$. Denoting by S_k the tautological rank k subbundle of $V \otimes \mathcal{O}_{G(k,V)}$, there is an exact sequence

$$0 \to \zeta^* \pi_2^* E_{k-1} \to \iota^* S_k \to \mathcal{O}_{\mathbb{P}\pi_2^*(E_{k+1}/E_{k-1})}(-1) \to 0.$$

Dualizing, and taking top exterior powers, from [Har77, ex. II.5.16d] we get an identity

$$\iota^* \mathcal{O}_{G(k,V)}(1) \cong \zeta^* \bigwedge^{k-1} E_{k-1} \otimes \mathcal{O}_{\mathbb{P}\pi_2^*(E_{k+1}/E_{k-1})}(1) \to 0$$

Now, the fiber $\zeta_* \iota^*(\mathcal{O}_{G(k,V)}(1)$ is the bundle whose stalk at a point $(w, p, l) \in \mathbb{P}W \times F_{0,1}(G)$ is isomorphic to $\mathrm{H}^0(l, \mathcal{O}_{G(k,V)}(1))$. By the projection formula, $\zeta_* \iota^* \mathcal{O}_{G(k,V)}(1) \cong \bigwedge^{k-1} \pi_2^* E_{k-1}^\vee \otimes \pi_2^*(E_{k+1}/E_{k-1})^\vee$.

There is a tautological inclusion $\pi_1^* \mathcal{O}_{\mathbb{P}W}(-1) \subset \mathrm{H}^0(\mathbb{P} \bigwedge^k V, \mathcal{O}(d)) \otimes \mathcal{O}_{\mathbb{P}W \times F_{0,1}(G)}$. There is a map of sheaves

$$\pi_1^* \mathcal{O}_{\mathbb{P}W}(-1) \to \bigwedge^{k-1} \pi_2^* E_{k-1}^\vee \otimes \pi_2^*(E_{k+1}/E_{k-1})^\vee.$$

whose fiber at a point (w, p, l) is induced by restricting a section to the line l. By adjunction, we get a morphism

$$\pi_2^* \mathcal{O}_{\mathbb{P}W}(-1) \otimes \pi_2^* \bigwedge^{k-1} E_{k-1} \otimes (E_{k+1}/E_{k-1}) \to \mathcal{O}_{\mathbb{P}W \times F_{0,1}(G)} \tag{2.1}$$

whose image is the ideal sheaf of $\mathcal{F}_{0,1}$. As $\pi_2^* \mathcal{O}_{\mathbb{P}W}(-1) \otimes \pi_2^* \bigwedge^{k-1} E_{k-1} \otimes (E_{k+1}/E_{k-1})$ is a rank $d+1$ bundle, the codimension of $\mathcal{F}_{l,\infty}$ in $\mathbb{P}W \times F_{0,1}(G)$ is at most $d+1$. Now, the condition of containing a point imposes one linear condition on $\mathbb{P}W$, and so $\mathcal{X}$ is a $\mathbb{P}^{N-2}$ bundle over $G(k, V)$, hence of dimension $k(n-k)+N-2$. $F_{0,1}(G)$ is a $\mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$ bundle over $G(k, V)$, hence $\mathbb{P}W \times F_{0,1}(G)$ is dimension $k(n-k)+n+N-3$. The dimension of $\mathcal{F}_{0,1}$ is therefore at least $k(n-k)+n+N-d-2$, and so each nonempty fiber of $\mathfrak{ev}$ has dimension at least $n-d-2$. □

Denote by U the subset of $\mathcal{X}$ over which $\mathfrak{ev}$ has constant fiber dimension $n-d-2$.

Corollary 2.3. *The restriction of* $\mathfrak{ev}$ *to* $\mathfrak{ev}^{-1}(U)$ *is a flat morphism.*

Proof. On the subset $\mathfrak{ev}^{-1}(U)$, we form the Koszul complex corresponding to the presentation (2.1). By [Mat89, Theorem 17.4, (iii)], this complex is acyclic over U, and so we can use it to compute the Hilbert polynomial of each fiber. It follows that all fibers X_w have the same Hilbert polynomial, and therefore ([Har77, III.9.9]) $\mathfrak{ev}$ is flat. □

Now, let $Y \subset \mathcal{X}$ be the 'bad' subset. Specifically,

$$Y = \{(H,p) \subset \mathbb{P}W \times G(k,V) \,\big|\, \operatorname{codim} \mathfrak{ev}^{-1}(H,p) < d+1\}$$

By semicontinuity, Y is a closed subscheme of $\mathbb{P}W \times G(k,V)$. Let us denote by ϕ_1 and ϕ_2 the respective projections on $\mathbb{P}W \times G(k,V)$.

Proposition 2.4. *For all $p \in G(k,V)$, the codimension of $Y \cap \phi_2^{-1}(p)$ is greater than $k(n-k)$.*

The proof of the Proposition 2.4 may be reduced to the following lemma. By a *linear subvariety* of $\mathbb{P}^a \times \mathbb{P}^b$, we mean a subvariety V, together with an isomorphism $e : \mathbb{P}^\alpha \times \mathbb{P}^\beta \to V$, such that $e^*(\mathcal{O}_{\mathbb{P}^a \times \mathbb{P}^b}(1,1)) \cong \mathcal{O}_{\mathbb{P}^\alpha \times \mathbb{P}^\beta}(1,1)$.

Lemma 2.5. *Let X be a pure D-dimensional subvariety in $\mathbb{P}^a \times \mathbb{P}^b$. Then for all $i \geq 0$, the rank of the restriction map $r_X : \mathrm{H}^0(\mathbb{P}^a \times \mathbb{P}^b, \mathcal{O}_{\mathbb{P}^a \times \mathbb{P}^b}(i,i)) \to \mathrm{H}^0(X, \mathcal{O}(i,i)|_X)$ of is greater than or equal to the rank of the restriction map $r_P : \mathrm{H}^0(\mathbb{P}^a \times \mathbb{P}^b, \mathcal{O}_{\mathbb{P}^a \times \mathbb{P}^b}(i,i)) \to \mathrm{H}^0(P, \mathcal{O}(i,i)|_P)$, for a linear subvariety $P \cong \mathbb{P}^\alpha \times \mathbb{P}^\beta$, $\alpha + \beta = D$.*

Proof. Given a subscheme A of $\mathbb{P}^a \times \mathbb{P}^b$, denote by r_A the restriction map $r_A : \mathrm{H}^0(\mathbb{P}^a \times \mathbb{P}^b, \mathcal{O}_{\mathbb{P}^a \times \mathbb{P}^b}(i,i)) \to \mathrm{H}^0(A, \mathcal{O}_{\mathbb{P}^a \times \mathbb{P}^b}(i,i)|_A)$.

Given an irreducible subset Z of X, endowed with the reduced induced scheme structure, the embedding $Z \to \mathbb{P}^a \times \mathbb{P}^b$ factors through X, and so rank of the restriction map r_X is at least the rank of the restriction map r_Z. Therefore we may assume that X is integral.

Let $P_0 \cong \mathbb{P}^{\alpha_0} \times \mathbb{P}^{\beta_0}$ be a linear subvariety of $\mathbb{P}^a \times \mathbb{P}^b$ containing X as a set, and which is minimal in the sense that if X is contained in a linear embedding of $\mathbb{P}^{\alpha'} \times \mathbb{P}^{\beta'}$, then $\alpha' + \beta' \geq \alpha_0 + \beta_0$. If $\alpha_0 + \beta_0 = D$, then $X = P_0$ and we are done, so assume that $\alpha_0 + \beta_0 > D$. By way of induction, assume that the lemma holds for all D-dimensional subvarieties in $\mathbb{P}^a \times \mathbb{P}^b$ contained in a $\mathbb{P}_0^{\alpha'} \times \mathbb{P}^{\beta'}$ with $\alpha' + \beta' < \alpha_0 + \beta_0$. Let $Y = \mathbb{V}(f)$ be a hypersurface in P containing X. f is a bihomogeneous polynomial on $\mathbb{P}^{\alpha_0} \times \mathbb{P}^{\beta_0}$, and so denote its bidegree by (r,s). By a suitable choice of homogeneous coordinates $[x_0, ..., x_{\alpha_0}], [y_0, ..., y_{\beta_0}]$, we may assume $f = x_0^r y_0^s + f'$, where the degree of x_0 in each monomial of f' is less than r, and the degree of y_0 in each monomial of f' is less than s.

Now define a $\mathbb{G}_m$ action on P_0 by

$$t \cdot ([x_0, \ldots], [y_0, \ldots]) = ([tx_0, t^{-1}X_1, \ldots, t^{-1}x_{\alpha_0}], [ty_0, t^{-1}y_1, \ldots, t^{-1}y_{\beta_0}]).$$

This determines a family $p_Y : \mathfrak{Y} \to \mathbb{G}_m$ of hypersurfaces, and a family $p_X : \mathfrak{X} \to \mathbb{G}_m$ of subschemes of P_0 isomorphic to X, such that $X_t = p_X^{-1}(t) \subset p_Y^{-1}(t)$. As the Hilbert scheme of $\mathbb{P}^{\alpha_0} \times \mathbb{P}^{\beta_0}$ is proper, both of these families extend to families over $\mathbb{A}^1$. The flat limit Y_0 of p_Y will be $\mathbb{V}(x_0^r y_0^s)$, and will contain the flat limit X_0 of p_X.

The rank of the maps r_{X_t} is lower semicontinuous on $\mathbb{A}^1$ (possibly dropping on the vanishing locus of certain determinantal polynomials). Furthermore $X_t \cong X$ for $t \neq 0$, so we may conclude that the rank of r_X is at least the rank of r_{X_0}.

Let X_0' be an irreducible closed subset of X_0, endowed with the reduced induces subscheme structure. Then by the reasoning above, the rank of r_{X_0} is greater than or equal to the rank of r'_{X_0}. Furthermore, X_0', being integral, is contained in either $\mathbb{V}(X_0)$ or $\mathbb{V}(Y_0)$. It follows that X_0' is contained in a linear subvariety $P_1 \cong \mathbb{P}^{\alpha_1} \times \mathbb{P}^{\beta_1}$ with $\alpha_1 + \beta_1 < \alpha_0 + \beta_0$. By the inductive hypothesis, the lemma holds for X' and hence holds for X as well. □

Proof of Proposition 2.4. Working with local affine coordinates on a neighborhood U of $P \in \mathbb{P} \bigwedge^k V$, we may assume that $P = 0$, and so lines through P may be written $L = \{tQ\}$, for $Q \in \mathbb{A}^{N-1}$. We then have a splitting on W induced by the order of vanishing at P. More specifically, For $f \in W$, we have that

$$f(tQ) = f_0(Q) + tf_1(Q) + t^2 f_2(Q) + \cdots + t^d f_d(Q).$$

Let $W = E_0 \oplus E_i \oplus \cdots \oplus E_d$ be such a splitting. The line $L = \{tQ\}$ is contained in $\mathbb{V}(f)$ if and only if $Q \in \mathbb{V}(f_0, \ldots, f_d)$.

Now, by Lemma 1.12, the homogeneous degree i forms f_i pulls back to a form of bidegree (i, i) via the embedding $\mathbb{P}^{k-1} \times \mathbb{P}^{r-k-1} \hookrightarrow \mathbb{P}T_P$. The point (f, p) is in $Y \cap \pi_2^{-1}(p)$ if and only if the f_i do not form a regular sequence. Denote by $\overline{W} \subset W$ the space of forms f for which the f_i do not form a regular sequence. Denote by $\overline{W}_i \subset W$ the space of forms $f = \sum f_i$ for which $(f_1, \ldots, f_{i-1})$ forms a regular sequence, but $(f_1, \ldots, \mathbb{V}(f_i))$ does not. Then $\overline{W} = \cup_i \overline{W}_i$, and so it will suffice to show that the codimension of each $\overline{W}_i$ is at least $k(n-k)$

Thus, suppose that $\mathbb{V}(f_1) \cap \ldots \cap \mathbb{V}(f_{i-1})$ is a complete intersection $V \subset \mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$. It will suffice to show that $\dim \mathrm{H}^0(V, \mathcal{O}(i,i)) > k(n-k)$. Lemma 2.5 reduces this question to the case that $\mathbb{V}(f_1) \cap \cdots \cap \mathbb{V}(f_i)$ is a linear subspace of $\mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$. It is now sufficient to show, for a and b nonnegative integers with $a + b = i - 1$, and $V = \mathbb{P}^{k-a-1} \times \mathbb{P}^{n-k-b-1}$, that $\dim H^0(V, \mathcal{O}(i,i)) > k(n-k)$. Substituting, this is equivalent to the statement that, $\binom{k-a-1+i}{i}\binom{n-k+a}{i} > k(n-k)$ for all $0 \le a \le k-1$. We prove this by induction on n. As $d \le n-k-1$, we may assume that $i \le n-k-1$, and so $n \ge i+k+1$. Setting $n_0 = i+k+1$, we first show that $\binom{k-a-1+i}{i}\binom{n_0-k+a}{i} > k(n_0-k)$. Making substitutions, this reduces to

$$\binom{k+i-1-a}{i}\binom{i+1+a}{i} > k(i+1) \tag{2.2}$$

which holds as $0 \le a \le i-1$.

To finish the induction, we need only show that

$$\frac{\binom{k-a-1+i}{i}\binom{n+1-k+a}{i}}{\binom{k-a-1+i}{i}\binom{n-k+a}{i}} \ge \frac{k(n+1-k)}{k(n-k)} \tag{2.3}$$

or rather that

$$\frac{n+1-k+a}{n+1-k+a-i} \ge \frac{n+1-k}{n-k} \tag{2.4}$$

which can be checked directly. □

Proof of Theorem 2.1. From Proposition 2.4 we see that the codimension of Y is at least $k(n-k)+1$, and so Y cannot intersect the general fiber of ϕ_1, and so the image of$Y$ in $\mathbb{P}W$ is a proper closed subscheme. For $w \in \mathbb{P}W \setminus \phi_1(Y)$ consider the inclusion $i_w : X_w \to \mathcal{X}$. Since $\phi_1^{-1}(w)$ does not intersect Y, this inclusion factors through U, and by base change of $\mathfrak{ev}$ with respect to i_w, we have a flat morphism $\mathfrak{ev}|_{\phi_1^{-1}(w)} : \phi_1^{-1}(w) \to \pi^{-1}(w)$. But this map is just $ev : F_{0,1}(X) \to X$. □

Corollary 2.6. *For a general degree $d \le n-k-1$ dimensional hypersurface, $F_{0,1}(X)$ is irreducible.*

Proof. By generality, we may assume that X is irreducible and that $ev : F_{0,1}(X) \to X$ is flat of the expected dimension. It will therefore be enough to show that a general fiber of $ev : F_{0,1}(X) \to X$ is irreducible. The incidence variety $\mathcal{F}_{0,1}$ is a projective bundle over $F_{0,1}(G)$, and so is nonsingular. Therefore, by generic smoothness, a general fiber of $\mathfrak{ev} : \mathcal{F}_{0,1} \to \mathcal{X}$ is nonsingular, and so a general fiber of ev is nonsingular for a general X. It remains only to show that a general fiber of ev is connected. But from the proof of Lemma 2.4 we know that every fiber of ev is an $n-d-2$ dimensional complete intersection of ample hypersurfaces in $\mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$, and so connectedness follows from repeated application of the Lefschetz hyperplane theorem. □

2.2 Moduli of higher degree curves

In order to prove Theorem 1.5, we will need to use a bend-and-break technique to reduce to the degree one case. For this to work, we will need to keep track of the way that a smooth curve can degenerate to a boundary component. This is the strategy laid out in [HRS04]. Many of the statements there work directly for hypersurfaces $G(k,V)$, and we provide proofs for those that do not. In order to formally discuss the boundary components of $\overline{\mathcal{M}}(X,e)$, it is necessary to introduce the language of stable A-graphs, and the Behrend-Manin decomposition.

2.2.1 Notation: The Behrend-Manin decomposition

A *graph* is a combinatorial tool that is useful for representing the configuration of different irreducible components of a curve.

Definition 2.7. A graph τ is a 4-tuple $(F_\tau, W_\tau, j_\tau, \delta_\tau)$ where

(1) F_τ is a finite set called the set of *flags*.

(2) W_τ is a finite set called the set of *vertices*.

(3) $j_\tau : F_\tau \to F_\tau$ is an involution.

(4) $\delta_\tau : F_\tau \to W_\tau$ is a map, traditionally called the evaluation map, which we will refer to here as the *incidence map* to avoid confusion with other evaluation maps.

Additionally, we will refer to the set of *tails* $S_\tau \subset F_\tau$ consisting of the fixed points of j_τ, and the set of *edges* E_τ which is the quotient of $F_\tau \setminus S_\tau$ by j_τ.

Each graph τ has a geometric realization as a CW-complex $|\tau|$, which we will be using in diagrams, where edges and tails correspond to 1-cells and vertices correspond to 0-cells, with glueing given by the incidence map δ_τ. As an example, in 2.1, vertices are represented by circular dots, with flags attached to vetices and terminated in a smaller rectangle. A *tree* is a graph such that $H_1(|\tau|, \mathbb{Z}) = 0$. All the graphs we will be concerned with will be trees.

Denote by $Vertex(\tau)$ the set of vertices in τ, by $Edge(\tau)$ the set of edges in τ, and by $Tail(\tau)$ the set of tails in τ.

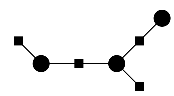


Figure 2.1: A CW-complex representing a graph.

Given a pointed curve C, there is a graph τ_C, called the *dual graph* to C, such that $Vertex(\tau_C)$ is the set of irreducible components of C, $Edge(\tau_C)$ is the set of intersection points between irreducible components, and $Tail(\tau_C)$ is the set of marked points on C. If C is rational, τ_C is a tree. Given a vertex $v \in Vertex(\tau)$, we will refer to the irreducible component of C corresponding to v by C_v.

Definition 2.8. An *A-graph* is a pair (τ, β), where τ is a tree and $\beta : Vertex(\tau) \to \mathbb{Z}_{\geq 0}$ is a map called the *A-structure*. There are two integers associated to an A-graph. Define

$$\beta(\tau) = \sum_{v \in Vertices(\tau)} \beta(v)$$

and

$$E(\tau) = \sup_{v \in Vertices(\tau)} \beta(v).$$

We will sometimes refer to $E(\tau)$ as the *maximum component degree* of τ.

When the context is clear, the pair (τ, β) may be denoted simply by τ.

A-graphs allow us to extend the combinatorial data of the dual curve to keep track of the degree data of a morphism. Specifically, if X is a scheme equipped with an ample line bundle L, and $h : C \to X$ is a morphism of a rational curve C, then there is an A-graph τ_h such that the underlying graph of τ is the dual graph to C, and such that for a vertex $v \in Vertex(\tau)$, $\beta(v)$ is the degree of $h^*(L)$ on C_v. If $h : C \to X$ is a morphism corresponding to an A-graph τ, we say that h is a *strict τ-map*. If a strict τ-map is a stable map, we say that τ is a *stable* A-graph. Note that this just means that for any vertex $v \in Vertex(\tau)$ with $\beta(v) = 0$, the valence of v in $|\tau|$ is at least 3.

Definition 2.9. A *family of strict τ-maps* over B is a triple (π, h, q), where

(1) π is a collection of maps $\pi_v : C_v \to B$ indexed by $Vertex(\tau)$ and such that each geometric fiber is an irreducible rational curve.

(2) h is a collection of maps $h_v : C_v \to X$ such that on each geometric fiber of π_v, $h_v^* L$ has degree $\beta(v)$.

(3) q is a collection of maps $q_f : B \to C_{\delta f}$ such that

  (1) $\pi_{\delta f} \circ q_f = id_B$.

  (2) If f and f' are distinct flags, q_f is a disjoint section from $q_{f'}$.

  (3) If $\overline{f} = j_\tau(f)$, then $h_{\delta f} \circ q_f = h_{\delta \overline{f}} \circ q_{\overline{f}}$.

We can glue the families C_v to form a family $(\pi : C \to B, h : C \to X, (q_s : B \to C)_{s \in Tail(\tau)})$ such that each fiber is a strict τ-map. The extra notation, however, is necessary to keep track of the irreducible components in each fiber of π.

Definition 2.10. If $\xi = (\pi, h, q)$ and $\xi' = (\pi', h', q')$ are two families of strict τ-maps, a *morphism* $\phi : \xi \to \xi'$ of families of strict τ-maps is a collection of isomorphisms $\phi_v : C_v \to C'_v$ indexed by $Vertex(\tau)$ and such that

(1) $h'_v \circ \phi_v = h_v$

(2) $\phi_{\delta f} \circ q_f = q'_f$.

As one might guess, when τ is stable, there is a nice moduli space of strict τ-maps to X, denoted by $\mathcal{M}(X, \tau)$, which is a stack. For the construction of these spaces, we refer the reader again to [BM96]. We recount here only the properties of these spaces that will be necessary in what follows.

As a solution to a moduli problem, $\mathcal{M}(X, \tau)$ is equipped with a universal family $(\pi : \mathcal{C} \to \mathcal{M}(X, \tau), h : \mathcal{C} \to X, (q_f : \mathcal{M}(X, \tau) \to \mathcal{C})_{f \in Flag(\tau)})$. In the case that X is projective and L is ample, we have (eg [HRS04, Theorem 3.10]) that each $\mathcal{M}(X, \tau)$ is a Deligne-Mumford stack. Furthermore, there is a stable compactification of $\mathcal{M}(X, \tau)$, denoted by $\overline{\mathcal{M}}(X, \tau)$, which is also a Deligne-Mumford stack, corresponding to allowing the components C_v to degenerate into reducible curves.

Definition 2.11. For each $f \in Flag(\tau)$, define a 1-morphism $ev_f : T \to X$ by $h_{\delta f} \circ q_f$.

To understand the relationships of the moduli spaces $\overline{\mathcal{M}}(X, \tau)$ for varying τ we will use the fact that there is a category whose objects are stable A-graphs, from which $\overline{\mathcal{M}}(X, -)$ is

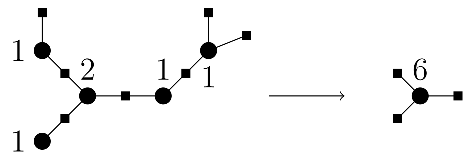


Figure 2.2: A contraction.

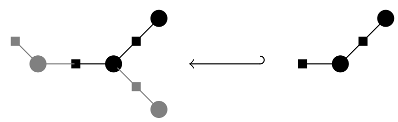


Figure 2.3: A combinatorial morphism.

a functor to the category of Deligne-Mumford stacks. There are two types of morphism in this category of stable A-graphs, contractions and combinatorial morphisms.

Roughly speaking, a contraction (Figure 2.2) $\alpha : \sigma \to \tau$ is a pair of maps $\alpha_V : Vertices(\sigma) \to Vertices(\tau)$, $\alpha_F : Flag(\tau)$ to $Flag(\sigma)$ such that α_V is surjective, and α_F is injective, and such that given $w \in Vertices(\tau)$,

$$\beta(w) = \sum_{\substack{v \in Vertices(\sigma) \\ \alpha_V(v) = w}} \beta(v)$$

The functor $\overline{\mathcal{M}}(X, -)$ takes a contraction $\alpha : \sigma \to \tau$ to a morphism of stacks $\overline{\mathcal{M}}(X, \alpha) : \overline{\mathcal{M}}(X, \sigma) \to \overline{\mathcal{M}}(X, \tau)$ which corresponds to the inclusion of a boundary component.

A *combinatorial morphism* $\phi : \tau \hookleftarrow \sigma$ is roughly the inclusion of σ as a subgraph of τ (Figure 2.3). Following the convention of [HRS04] we draw the arrow backwards, as $\overline{\mathcal{M}}(X, -)$ is contravariant on combinatorial morphisms. Often the complement of σ in τ will be a collection of tails and/or a subgraph consisting of degree 0 vertices. In this case, the corresponding morphism $\mathcal{M}(X, \phi) : \overline{\mathcal{M}}(X, \tau) \to \overline{\mathcal{M}}(X, \sigma)$ is just the forgetful morphism that forgets some marked points and stabilizes if necessary.

Given a contraction $\alpha : \sigma \to \tau$, we will consider the restriction of $\mathcal{M}(X, \alpha)$, which by abuse of notation we refer to as $\mathcal{M}(X, \alpha)$ as well.

Proposition 2.12. *$\overline{\mathcal{M}}(X, \tau) \setminus \mathcal{M}(X, \tau)$ is the union of the images of $\mathcal{M}(X, \sigma)$ under $\mathcal{M}(X, \alpha)$ as α ranges over contractions $\sigma \to \tau$ with $\sigma \neq \tau$.*

We will be especially interested in certain types of A-graphs. Denote by $\tau_m(e)$ the A-graph which has just one vertex v, with $\beta(v) = e$, and m tails attached to e. Denote by $\tau_{a,b}(i,j)$ the A-traph which has two vertices v_1 and v_2, with $\beta(v_1) = i$, $\beta(v_2) = j$ and a tails attached to v_1, b tails attached to v_2.

A stable $\tau_m(e)$-map is just a degree e stable map, and in fact by construction we have that $\overline{\mathcal{M}}(X, \tau_m(e) = \overline{\mathcal{M}}_{0,m}(X, e)$. Thus the moduli spaces of stable A-maps form a useful framework for the systematic study of boundary degenerations on $\overline{\mathcal{M}}_{0,m}(X, e)$, which we explain in the next section.

Definition 2.13. Given a projective scheme X, and a stable A-graph τ, define the *expected dimension* $\dim(X, \tau)$ to be

$$dim(X, \tau) = \dim X - 3 + \beta(\tau)c_1(T_X) + \#Tail(\tau) - \#Edge(\tau)$$

2.2.2 Irreducibility of moduli of higher degree maps

We now turn to the task of proving Theorem 1.5. We will need to understand how the property of irreducibility is 'transferred' under contractions and combinatorial morphisms. Properties invariant under base extension, flatness and geometrically irreducible fibers, will allow us to conclude irreducibility of moduli corresponding to complicated A-graphs from the irreducibility of simpler subgraphs, via combinatorial morphisms. A version of bend-and-break allows us to deduce irreducibility from irreducibility of boundary components via contractions.

There is a bit more subtlety involved, as we will see. The following proposition allows us to build complicated boundary components with flat evaluation maps, provided the evaluation maps are flat on moduli of subgraphs.

Proposition 2.14. *[HRS04, Proposition 4.8] Suppose that τ is a stable A-graph with $E(\tau) = E)$. If for each $e = 0, \ldots, E$ we have that $ev_{f_1} : \overline{\mathcal{M}}(X, \tau_1(e))$ is flat of the expected dimension, then for each flag $f \in Flag(\tau)$, ev_f is flat of the expected dimension.*

The following lemma allows us to much simplify the criterion for flatness.

Lemma 2.15. *If X is a complete intersection in $G(k,V)$, then for $f \in Tail(\tau)$, the evaluation map ev_f is flat if and only if every irreducible component of every geometric fiber has dimension equal to the expected dimension.*

Proof. The proof of [HRS04, Lemma 4.6] in fact works in this case as well, and so we give here only the general idea. As $G(k,V)$ is homogeneous, $\overline{\mathcal{M}}(G(k,V),\tau))$ is irreducible and of the expected dimension. And so, if $\pi : \mathcal{C} \to \overline{\mathcal{M}}(Y,\tau)$ is the universal family, with the morphism $h : \mathcal{C} \to G(k(V))$, a collection of defining equations cutting out X defines a section of $\pi_* h^*(\mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_r))$, whose vanishing locus $\overline{\mathcal{M}}(X,\tau))$. Thus $\overline{\mathcal{M}}(X,\tau)$ is Cohen Macaulay, and ev_f is a dominant morphism to a smooth scheme with constant fiber dimension, and so by [Har77, ex. III.10.8], ev_f is flat. □

Given two flags $f_1, f_2 \in Flag(\tau)$, denote by $ev_{f_1,f_2} : \overline{\mathcal{M}}(X,\tau) \to X \times X$ the product of ev_{f_1} and ev_{f_2}.

Lemma 2.16. *[HRS04, Proposition 5.1] There are no proper curves in a fiber of ev_{f_1,f_2} : $\mathcal{M}(X,\tau_2(e)) \to X \times X$.*

This allows us to prove a crucial step in our induction argument. The following is a small improvement on [HRS04, Proposition 5.3], first suggested in [CS09].

Proposition 2.17. *Denote by $ev^e_{f_1}$ the evaluation map corresponding to the unique tail of $\tau_1(e)$. Suppose that X is a complete intersection in $G(k,V)$. Suppose that ev_{f_1} is flat of the expected dimension for every $1 \leq e < E$, and suppose that the expected fiber dimension of $\mathcal{M}(X,\tau_1(E))$ is at least $dim(X) - 1$. Then $ev^E_{f_1}$ is also flat of the expected fiber dimension.*

Proof. By Lemma 2.15 it suffices to show that $ev^E_{f_1}$ has the expected fiber dimension. Suppose that this were not the case, and denote by $Y \subset X$ the subscheme of X over which ev^E_1 has greater than expected fiber dimension. Let $\Phi : \overline{\mathcal{M}}(X,\tau_2(E)) \to \overline{\mathcal{M}}(X,\tau_1(E))$ be the morphism corresponding to the combinatorial morphism forgetting the second tail f_2 of $\tau_2(E)$. Then the following diagram commutes.

$$\begin{array}{ccc} \overline{\mathcal{M}}(X,\tau_2(E)) & \xrightarrow{ev_{f_1,f_2}} & X\times X \\ \big\downarrow \Phi & & \big\downarrow \pi_1 \\ \overline{\mathcal{M}}(X,\tau_1(E)) & \xrightarrow{ev_{f_1}} & X \end{array}$$

Where $\pi_1 : X\times X \to X$ is projection onto the first factor. Denote by $Y\subset X$ the locus over which ev_{f_1} has greater than expected fiber dimension. Then it follows that $\dim ev_{f_1}^{-1}(Y) \geq \dim X + \dim Y$, and that $\dim \phi^{-1}(ev_{f_1}^{-1}(Y)) \geq \dim X + \dim Y + 1$. As $\dim \pi_1^{-1}(Y) = \dim Y + \dim X$, it there must be a point $p \in \pi_1^{-1}(Y)$ such that $\dim ev_{f_1,f_2}^{-1}(p) \geq 1$. By 2.16, we must have that $ev_{f_1,f_2}^{-1}(p)$ intersects the boundary of $\overline{\mathcal{M}}(X,\tau_2(E))$. As the morphism Φ restricts to a morphism $\Phi' : \mathcal{M}(X,\tau_2(E)) \to \mathcal{M}(X,\tau_1(E))$, it follows that $ev_1^{-1}(Y)$ intersects the boundary of $\overline{\mathcal{M}}(X,\tau_1(E))$. But then there is some contraction $\alpha : \sigma \to \tau$ such that $ev_{\alpha^{-1}(f_1)} : \overline{\mathcal{M}}(X,\sigma)$ to X is not of the expected fiber dimension. As any such contraction must have $E(\sigma) < E$, this contradicts Lemma 2.14. □

This motivates the following definition.

Definition 2.18. Let X be a complete intersection in $G(k,V)$ of multidegree $\mathbf{d} = (d_1,\ldots,d_r)$. Define the *flatness threshold degree* $E_{Flat}(X)$ to be

$$E_{Flat}(X) = \left\lceil \frac{k(n-k)+1-r}{n-(d_1+\cdots+d_r)} \right\rceil$$

Corollary 2.19. *If X is a complete intersection in $G(k,V)$ such that $ev_{f_1} : \overline{\mathcal{M}}(X,\tau_1(e)) \to X$ is flat of the expected dimension for all $1 \leq e < E_{Flat}(X)$, then for every stable A-graph τ, and every $f \in Flag(\tau)$, ev_f is flat of the expected dimension.*

Proof. Induction, using Proposition 2.17 and Proposition 2.14, as in [HRS04, Corollary 5.5]. □

Now, since we have already proven that $ev : \overline{\mathcal{M}}_{0,1}(X,1) \to X$ is flat of the expected fiber dimension, we have the following corollary.

Corollary 2.20. *Let X be a general hypersurface in $G(k,V)$ of degree $d \leq n - k(n-k)/2$, with $(d,k,n) \neq (2,2,4)$, and let τ be a stable A-graph. For each flag $f \in Flag(\tau)$, the evaluation morphism $ev_f : \overline{\mathcal{M}}(X,\tau) \to X$ is flat of the expected fiber dimension.*

Proof. This is just the condition for $E_{Flat}(X) \leq 2$. If $d \leq n - k(n-k)/2$, then except when $(d, k, n) = (2, 2, 4)$, we have $d \leq n - k - 1$, and so by Theorem 2.1, we know that flatness holds for $e = 1$, and we can apply Corollary 2.19. □

We will, however, need a slightly stronger inequality to be assured that we have a codimension one boundary in each irreducible component. To this end, we have the following.

Proposition 2.21. *Let X be a complete intersection in a homogeneous variety. Suppose that $ev_{f_1} : \overline{\mathcal{M}}(X, \tau_1(e)) \to X$ is flat of the expected fiber dimension for every $1 \leq e < E$ and suppose that every irreducible component of $\mathcal{M}(X, \tau_1(E))$ has dimension at least $2dim(X)$. Then for every irreducible component $M \subset \mathcal{M}(X, \tau_0(E))$ there is a graph $\tau_{0,0}(i, j)$, $0 < i, j$ and $i + j = E$, and an irreducible component $N \subset \mathcal{M}(X, \tau_{0,0}(i, j)$ such that $N \subset \overline{M}$ is a codimension 1 subvariety.*

Proof. The proof is identical to that of [HRS04, Proposition 5.7], using Lemma 2.15 to replace the hypothesis that X is a complete intersection in $\mathbb{P}^N$. □

Definition 2.22. Let X be a complete intersection in $G(k, V)$ of multidegree $\mathbf{d} = (d_1, \ldots, d_r)$. Define the *breaking threshold degree* $E_{Break}(X)$ to be

$$E_{Break}(X) = \left\lceil \frac{k(n-k) + 2 - r}{n - (d_1 + \cdots + d_r)} \right\rceil$$

Theorem 2.23. *Let X be a complete intersection in $G(k, V)$. Let τ be a stable A-graph and let M be an irreducible component of $\mathcal{M}(X, \tau)$. Suppose that $ev_{f_1} : \overline{\mathcal{M}}(X, \tau_1(e)) \to X$ is flat for each $1 \leq e < E_{Flat}(X)$. Then there exists a contraction $\alpha : \sigma \to \tau$ and an irreducible component $N \subset \mathcal{M}(X, \sigma)$ such that $E(\sigma) < E_{Break}(X)$ and such that $N \subset \overline{M}$.*

Proof. The proof is identical to that of [HRS04, Theorem 5.10], replacing references to [HRS04, Proposition 5.7] with Proposition 2.21, and references to [HRS04, Proposition 5.3] with Proposition 2.17. □

At this point we can almost run the argument in [HRS04], but we must first check that a few more base conditions hold. There is one more property necessary for induction, which we define here.

Definition 2.24. We say that $\mathcal{B}(X,\tau,f)$ holds if

(1) $ev_f : \overline{\mathcal{M}}(X,\tau) \to X$ is flat of the expected fiber dimension.

(2) The general fiber of ev_f is geometrically irreducible.

(3) There is a strict τ-map $h : C \to X$ which is free: i.e. h^*T_X is globally generated.

Corollary 2.25. *For a general degree d hypersurface X in $G(k,V)$ such that $d < n-k-1$, $\mathcal{B}(X,\tau_1(1),f_1)$ holds.*

Proof. 1 is just 2.1, and we have 2 from the proof of 2.6, so it remains to verify 3. But by as long as $d \leq n$, a general line is free as X is Fano. □

Proposition 2.26. *[HRS04, Proposition 6.5] Suppose $X \subset \mathbb{P}^N$ is a smooth subvariety which satisfies $\mathcal{B}(X,\tau_1(e),f_1)$ for $e = 1,\dots,E$. Let τ be an A-graph such that $E(\tau) \leq E$. Then we have the following:*

(1) For each $f \in Flag(\tau)$, we have $\mathcal{B}(X,\tau,f)$.

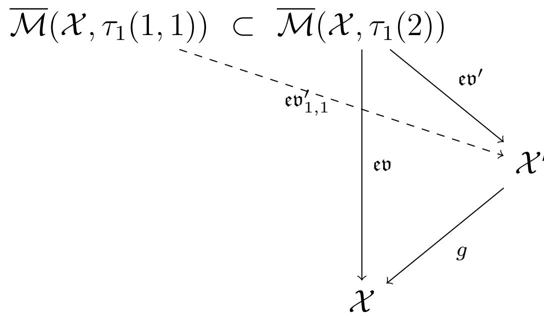
(2) $\mathcal{M}(X,\tau)$ is an irreducible stack.

Since in our case $0 < E_{Flat} - E_{Break} < 1$, we need to verify irreducibility for one more base case in order to apply the argument from [HRS04].

Corollary 2.27. *For X a general degree d hypersurface in $G(k,V)$ with $d < n-k(n-k)/2$, $\overline{\mathcal{M}}(X,\tau_1(2))$ is irreducible.*

Proof. Consider the Plücker embedding of $G(k,V)$ in $\mathbb{P}^N$. Denote by $\mathcal{X} \subset \mathbb{P}\mathrm{H}^0(\mathbb{P}^N,\mathcal{O}(2)) \times G(k,V)$ the incidence subscheme consisting of pairs (F,p) such that $p \in \mathbb{V}(F)$. We have a functorial morphism $\phi : \overline{\mathcal{M}}(\mathcal{X},\tau_1(2)) \to \overline{\mathcal{M}}(G(k,V),\tau_1(2))$. Given a stable map $i : C \to G(k,V)$ and a point $p_C \in \overline{\mathcal{M}}(G(k,V),\tau_1(2)$ corresponding to i, the fiber $\phi^{-1}(p_C)$ is the subspace of $\mathbb{P}\mathrm{H}^0(\mathbb{P}^N,\mathcal{O}(2))$ consisting of hypersurfaces containing the image of i. Thus fiber dimension of ϕ jumps when i is a double cover of a line, or C has two components each mapping to the same line. By semicontinuity, the locus Y on which the fiber dimension jumps is closed. Define $\overline{\mathcal{M}}^o(X,\tau_1(2)) \subset \overline{\mathcal{M}}(X,\tau_1(2))$ to be the compliment of $\phi^{-1}Y$. Then $\overline{\mathcal{M}}^o(\mathcal{X},\tau_1(2))$ is a $\mathbb{P}^N$ bundle over $\overline{\mathcal{M}}^o(G(k,V),\tau_1(2))$, and so is nonsingular. Now let $\rho : \overline{\mathcal{M}}(\mathcal{X},\tau_1(2)) \to \mathcal{X}$ be the evaluation map. By construction, the fiber of ρ over

(F, p) is $ev^{-1}(p) \subset \overline{\mathcal{M}}^o(\mathbb{V}(F), \tau_1(2))$. Denote by $\mathfrak{ev}_{1,1} : \overline{\mathcal{M}}(\mathcal{X}, \tau_1(1,1)) \to \mathcal{X}$ the evaluation map on the boundary of $\overline{\mathcal{M}}(\mathcal{X}, \tau_1(2)$, and consider the stein factorization in the following commutative diagram.



$\overline{\mathcal{M}}^o(\mathcal{X}, \tau_1(1,1))$ is irreducible, and $\mathfrak{ev} : \overline{\mathcal{M}}(\mathcal{X}, \tau_1(1,1)) \to \mathcal{X}$ is dominant, with connected fibers, therefore the image of $\overline{\mathcal{M}}(\mathcal{X}, \tau_1(1,1))$ in $\overline{\mathcal{M}}(\mathcal{X}, \tau_1(2))$ is an irreducible component on which g is bijective. But $\overline{\mathcal{M}}^o(\mathcal{X}, \tau_1(2))'$ is the image under ev' of $\overline{\mathcal{M}}^o(\mathcal{X}, \tau_1(2))$, and so is irreducible and therefore must be equal to the image of $\overline{\mathcal{M}}(\mathcal{X}, \tau_1(1,1))$. It follows that g is bijective and so $\mathfrak{ev}|_{\overline{\mathcal{M}}^o(X, \tau_1(2))}$ has connected fibers.

But since $\overline{\mathcal{M}}^o(\mathcal{X}, \tau_1(2)$ is nonsingular, by generic smoothness a general fiber of $\mathfrak{ev}|_{\overline{\mathcal{M}}^o(X, \tau_1(2))}$ is nonsingular and connected, hence irreducible.

Now, for a general choice of degree d hypersurface X the evaluation morphism ev_{f_1} is flat, by Corollary 2.20, with geometrically irreducible fibers, hence $\overline{\mathcal{M}}^o(\mathcal{X}, \tau_1(2))$ is irreducible. Denote by M^o the irreducible component of $\overline{\mathcal{M}}(X, \tau_1(2))$ that is the closure of $\overline{\mathcal{M}}^o(X, \tau_1(2))$.

For such an X, denote by $D \subset \overline{\mathcal{M}}(X, \tau_1(2))$ the locus of double lines and double covers of a line: i.e. $\phi^{-1}(Y) \cap \overline{\mathcal{M}}(X, \tau_1(2))$. Denote the fano variety of lines in X by $F_1(X)$, as in Section 2.1. Then there exists a fibration $p : D \to F_1(X)$. Given a line l in $F_1(X)$, there is an isomorphism $\psi : \mathrm{Sym}^2(l) \to p^{-1}(l)$. Given $P + Q \in \mathrm{Sym}^2(l)$, if $P \neq Q$ then $\psi(P + Q)$ is a double cover of l ramified over P and Q. If $P = Q$, then $\psi(P + Q)$ is the double line with P the image of the node in the domain.

For a free line l in X, the fiber $p^{-1}(l)$ consists of smooth points of $\overline{\mathcal{M}}(X, \tau_1(2))$, and therefore intersects at most one irreducible component M of $\overline{\mathcal{M}}(X, \tau_1(2))$. But $p^{-1}(l)$ contains a double line, and hence intersects the image of $\mathcal{M}(X, \tau_1(1,1))$, hence intersects the closure of $\overline{\mathcal{M}}^o(X, \tau_1(2))$, and so $M = M^o$.

As X is Fano, every irreducible component of D contains the fiber of p over a free line, and so $\overline{\mathcal{M}}(X, \tau_1(2))$ is irreducible. □

Proposition 2.28. *[HRS04, Corollary 6.7] Suppose $X \subset \mathbb{P}^N$ is a smooth subvariety which satisfies $\mathcal{B}(X, \tau_1(e), f_1)$ for all $e = 1, \ldots, E$. Let τ be an A-graph with $E(\tau) \leq E$ and suppose that $\alpha : \tau \to \sigma$ is a contraction. The morphism $\mathcal{M}(X, \alpha)$ maps a general point of $\mathcal{M}(X, \tau)$ to a smooth point of $\mathcal{M}(X, \sigma)$.*

It follows from Proposition 2.28 that given a contraction $\alpha : \tau \to \sigma$, for each irreducible component N of $M(X, \tau)$ there is at most one irreducible component $M(\alpha, N)$ of $\overline{\mathcal{M}}(X, \sigma)$ which contains the image of N. In particular, if $\mathcal{M}(X, \tau)$ is irreducible, there exactly one irreducible component $M(\alpha)$ which contains the image of $\mathcal{M}(X, \tau)$.

Proposition 2.29. *[HRS04, Proposition 6.8] Suppose $X \subset \mathbb{P}^N$ is a smooth variety satisfying*

(1) $\mathcal{B}(X, \tau_1, f_1)$ holds.
(2) $ev_{f_1} : \overline{\mathcal{M}}(X, \tau_1(e)) \to X$ is flat of the expected fiber dimension for $e = 1, \ldots, E$.
(3) $\mathcal{M}(X, \tau_0(e))$ is irreducible for $e = 1, \ldots, E$.

Then for each stable A-graph τ with $E(\tau) < E$, and for each flag $f \in Flag(\tau)$, $\mathcal{B}(X, \tau, f)$ holds and there is a contraction $\alpha : \sigma \to \tau$ such that $E(\sigma) \leq 1$ and such that $\mathcal{M}(X, \alpha)$ maps the general point of $\mathcal{M}(X, \sigma)$ to a smooth point of $\overline{\mathcal{M}}(X, \tau)$.

In the cases of hypersurfaces in $G(k, V)$, we will apply Proposition 2.29 with $E = 2$. We are now prepared to prove theorem 1.5. Condition (3) follows by Corollary 2.27, and the existence of the dominant forgetful morphism $\overline{\mathcal{M}}(X, \tau_1(2)) \to \overline{\mathcal{M}}(X, \tau_0(2))$.

Proof of Theorem 1.5. As in the hypothesis, let X be a degree d hypersurface in $G(k, V)$ such that $d < n - k(n-k)/2$, and let $\overline{e}$ be a positive integer. If $\overline{e} \leq 2$, we are done, so assume $\overline{e} > 2$. We seek to show that $\overline{\mathcal{M}}(X, \tau_1(\overline{e}))$ is irreducible. By Corollary 2.20, we can conclude that the evaluation maps $ev_{f_1} : \overline{\mathcal{M}}(X, \tau_1(e)) \to X$ are flat for all e. As $E_{Break} - E_{Flat} \leq 1$, we must have $E_{Break} \leq 2$. By Corollary 2.27, we conclude that for all $e \leq E_{Break}$, $\mathcal{M}(X, \tau_0(e))$ is irreducible. But then by Theorem 2.23 for each irreducible component M of $\overline{\mathcal{M}}(X, \tau_0(e))$, there exists a contraction $\alpha : \sigma \to \tau$ and such that M is

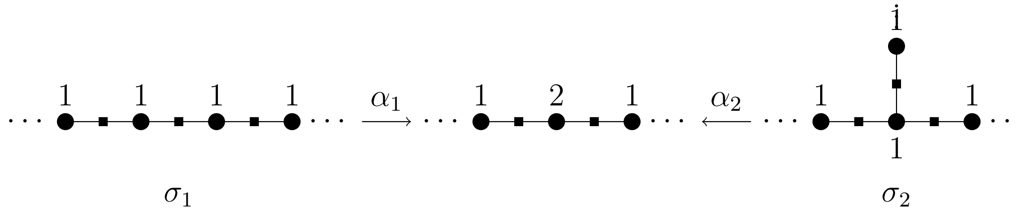


Figure 2.4: Resolving to a path.

the unique irreducible component of $\overline{\mathcal{M}}(X, \tau_0(e))$ containing the image of $\mathcal{M}(X, \sigma)$ under $\mathcal{M}(X, \alpha)$. It follows by Proposition 2.29 that the irreducible components of $\overline{\mathcal{M}}(X, \tau_0(e))$ are indexed by the contractions $\alpha : \sigma \to \tau$ with $E(\sigma) \leq 1$.

Now, to show that $\overline{\mathcal{M}}(X, \tau_0(e))$ is irreducible, it suffices to show that all irreducible components $M(\alpha)$ are in fact equal. This can be done, as in the proof of [HRS04, Proposition 7.2], by showing that all irreducible components are in fact equal to the irreducible component $M(\alpha')$, where $\alpha' : \sigma' \to \tau_0(\overline{e})$, is a 'path', which is to say that no vertex has more than two edges. (So that a curve with σ' as its dual graph is just a chain of lines). Note that α' is the unique contraction from σ' to $\tau_0(\overline{e})$. Suppose $\alpha_1 : \sigma_1 \to \tau$ is a contraction with $E(\sigma_1) = 1$). Given a vertex $v \in Vertices(\sigma)$, denote by $EV(v)$ the edge valence at v, and set $Excess(\sigma) = \sum_{v \in Vertices(\sigma)} \max(EV(v) - 2, 0)$. If σ_1 is not a path, then there is a vertex $v \in Vertices(\sigma_1)$ with $EV(v) > 2$. But then there is a contraction $\alpha_1' : \sigma \to \omega$ which contracts an edge at v, and there is another contraction $\sigma_2 \to \omega$, as indicated in Figure 2.4, such that $Excess(\sigma_2) < Excess(\sigma_1)$. Furthermore, as $\tau_1(\overline{e})$ contains only one vertex, there is a contraction $\rho_1 : \omega \to \tau_1(\overline{e})$ such that $\alpha_1 = \rho_1 \circ \alpha_1'$. It follows that $M(\rho_1) = M(\alpha_1) = M(\rho_1 \circ \alpha_2')$. Repeating this process at most $Excess(\sigma_2)$ times, we are able to conclude that $M(\alpha_1) = M(\alpha')$, and therefore $\overline{\mathcal{M}}(X, \tau_0(e))$ is irreducible for all positive integers e.

But now by Proposition 2.29, for each stable A-graph τ and each flag $f \in Flag(\tau)$, $\mathcal{B}(X, \tau, f)$ holds. In particular, $\mathcal{B}(X, \tau_1(e), f_1)$ holds for all e, and so by Proposition 2.26, $\overline{\mathcal{M}}(X, \tau_1(e))$ is irreducible for all positive integers e. $\square$

Chapter 3

Very Twisting Surfaces and Existence of Rational Sections

In the introduction, the various notions of rational simple connectedness are described as corresponding to the rational connectedness of a subvariety M of the general fiber of the evaluation map $\overline{\mathcal{M}}_{0,2}(X,\beta)$. Taking M to be all of $ev^{-1}(p,q) \subset \overline{\mathcal{M}}_{0,2}(X,\beta)$, we say that X is *irreducibly rationally simply connected* if for all β sufficiently positive, $ev : \overline{\mathcal{M}}_{0,2}(X,e) \to X \times X$ is dominant with rationally connected general fiber. In the case of hypersurfaces in $G(k,V)$, where we have a generating class for the homology of curves, there is no ambiguity in the phrase 'sufficiently positive'. To be precise, there must exist some positive integer E such that for all $e \geq E$, $ev : \overline{\mathcal{M}}_{0,2}(X,e) \to X \times X$ is dominant, and a general fiber is rationally connected.

In fact, as we have seen, under the much less restrictive hypothesis $d \leq n-k-1$, there is a canonical irreducible component whose general point parameterizes smooth, free curves. Taking M to be this canonical irreducible component, we make the following definition.

Definition 3.1. A scheme X is *rationally simply connected* if for some positive integer E, for all $e \geq E$ there exists a canonically defined irreducible component $M_{e,2} \subset \overline{\mathcal{M}}_{0,2}(X,e)$ such that the restriction of the evaluation map $ev : \overline{\mathcal{M}}_{0,2}(X,e) \to X \times X$ to $M_{e,2}$ is dominant, with rationally connected general fiber.

A stronger condition than rational simple connectedness is to impose that for any $m \geq 2$ for every e sufficiently positive there exists a canonical irreducible component $M_{e,m} \subset \overline{\mathcal{M}}_{0,m}(X,e)$ such that the restriction of the evaluation map $ev : \overline{\mathcal{M}}_{0,m}(X,e) \to X^m$ is dominant, with rationally connected general fiber. In this case, we say that X is *strongly rationally simply connected*.

A smooth, projective morphism $p : X \to B$ between a smooth, projective, complex variety X and a smooth, projective complex curve B satisfies *weak approximation* if, given a collection of points $b_1, \dots, b_r$ and a collection of jets $s_i \in X(\hat{\mathcal{O}}_{B,b_i})$, then for every positive integer M there exist sections of p congruent to each s_i modulo $\mathfrak{m}^M_{B,b_i}$. In the case that X is strongly rationally simply connected, Hassett has shown that weak approximation holds over the function field of a curve ([HT09]).

A second application of the principles of rational simple connectedness, as mentioned in the introduction, is the construction of sections over the function field of a surface. For this application, we will need a form of rational simple connectedness corresponding to the locus M parameterizing chains of lines. The main result here is the following from [dJHS08]:

Theorem 3.2. *[dJHS08, Theorem 1.1] Let $f : X \to S$ be a morphism of nonsingular projective varieties over k with S a surface. If*

(1) there exists a Zariski open subset U of S whose compliment has codimension 2 such that X_u is irreducible for $u \in U(k)$.

(2) there exists an invertible sheaf $\mathcal{L}$ on $f^{-1}(U)$ which is f-relatively ample.

(3) the geometric generic fiber $(X_{\overline{\eta}}, \mathcal{L}_{\overline{\eta}})$ of f is rationally simply connected by chains of free lines and has a very twisting family of lines.

then there exists a rational section of f.

Some explanation is in order. Item (2) is equivalent to the vanishing of the elementary obstruction of the morphism f, to be explained in more detail in Section 3.6. The existence of a very twisting family of lines is Theorem 1.8, the main result of this chapter. We say a scheme X is *rationally simply connected by chains of free lines* if

(1) the space of lines through a general point of X is nonempty, irreducible, and rationally connected, and

(2) there exists a positive integer r such that the space of chains of r chains of free lines connecting two general points is nonempty, irreducible, and birationally rationally connected.

If the general chain of r lines is a chain of r free lines, then *birational rational connectedness* is implied by the rational connectedness of the moduli of r chains of lines (free or not). We consider this property in Section 3.5.

3.1 Very twisting surfaces

In this section, we define the notion of very twisting families of rational curves. The hardest condition to verify will be positivity of the relative tangent sheaf T_{ev} along a morphism $\varphi : B \to \overline{\mathcal{M}}_{0,1}(X, 1)$, in the following sense.

Definition 3.3. Let Y and Z be finite type Deligne-Mumford stacks over $\operatorname{Spec} K$. Denote by $Z^o \subset Z$ the smooth locus of Z over K. Let $g : Y \to Z$ be a morphism, and denote by $Y^o \subset f^{-1}(Z^o)$ the maximal open subset on which g is smooth. Denote by T_g the vertical tangent bundle to the morphism g. Then we say that a morphism $f : \mathbb{P}^1 \to Y$ is *g-relatively free*, resp. *g-relatively very free* if

(1) $f(\mathbb{P}^1) \subset Y^o$,

(2) $g \circ f$ is free, and

(3) f^*T_g is generated by global sections, resp. ample.

Let B be isomorphic to $\mathbb{P}^1$. By pulling back the universal family, a morphism $\varphi : B \to \overline{\mathcal{M}}_{0,1}(X, e)$ determines a family $\pi : \Sigma \to B$ of stable maps, together with a section $\sigma : B \to \Sigma$ and a morphism $g : \Sigma \to X$ such that $ev \circ \phi = g \circ \sigma$. Denote by $\psi^\vee$ the sheaf $\sigma^*(\mathcal{O}_\Sigma(\sigma(B)))$.

Definition 3.4. We say a morphism $\varphi : B \to \mathcal{M}_{0,1}(X, e)$ is *very twisting* if the following conditions are satisfied.

(1) $ev : \overline{\mathcal{M}}_{0,m}(X, e) \to X$ is unobstructed at every geometric point of the image of φ.

(2) The morphism φ is ev-relatively very free.

(3) The degree of $\psi^\vee$ is nonnegative.

(4) The image under φ of the geometric generic point of $\mathbb{P}^1$ is a stable map with irreducible domain.

Remark 3.5. To prove Theorem 1.8, we will be concerned with finding very twisting morphisms $\varphi : B \to \mathcal{M}_{0,1}(X, 1)$, for X a low degree hypersurface in $G(k, V)$. In this case, $\mathcal{M}_{0,1}(X, 1)$ will be irreducible, the smooth locus of ev is equal to the unobstructed locus of ev, and condition 4 is trivial. Additionally, condition (1) is in this case equivalent to the composition $ev \circ \varphi : \mathbb{P}^1 \to X^r$ defining a free morphism.

In case that $\pi : \Sigma \to \mathbb{P}^1$ is a family determined by a very twisting morphism $\varphi : B \to \overline{\mathcal{M}}_{0,1}(X, e)$, we say that Σ is a *very twisting surface* in X. Denote by $p : \overline{\mathcal{M}}_{0,1}(X, \beta) \to \overline{\mathcal{M}}_{0,0}(X, \beta)$ the combinatorial morphism corresponding to forgetting the marked point, and by $\overline{p}$ the induced morphism on universal curves, as in figure 3.1. Denote by $N_{\mathcal{C}_1/\overline{\mathcal{M}}_{0,1}\times X}$ the normal bundle to the embedding $\mathcal{C}_1 \xrightarrow{\pi_1 \times \overline{p}} \overline{\mathcal{M}}_{0,1}(X, \beta) \times \mathcal{X}_0$.

Figure 3.1: Universal Curves

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{\overline{p}} & \mathcal{C}_0 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} \\ \overline{\mathcal{M}}^o_{0,1}(X, \beta) & \xrightarrow{p} & \overline{\mathcal{M}}^o_{0,0}(X, \beta) \end{array}$$

Proposition 3.6. *Over the unobstructed locus* $\mathcal{M}^o_{0,1}(X, e) \subset \overline{\mathcal{M}}_{0,1}(X, e)$, T_{ev} *is isomorphic to* $\pi_*(N_{\mathcal{C}_1/\overline{\mathcal{M}}_{0,1}\times X}(-\sigma(\overline{\mathcal{M}}_{0,1})))$.

Proof. Given a smooth degree d curve C, denote by $N_{C/X}$ the normal bundle to C in X. The stalk of the tangent bundle to $\mathcal{M}_{0,0}(X, d)$ at the closed point corresponding to C may be canonically identified with $\mathrm{H}^0(C, N_{C/X})$. Now, given a closed point $b \in \overline{\mathcal{M}}_{0,0}(X, d)$, $N_{\pi_0^{-1}(b)/X}$ may be identified with $N_{\mathcal{C}_0/\overline{\mathcal{M}}_{0,0}\times X}|_{\pi^{-1}(b)}$, and furthermore we have the identification $T_{\overline{\mathcal{M}}_{0,0}} \cong \pi_{0*}(\mathbb{N}_{\mathcal{C}_0/\overline{\mathcal{M}}_{0,0}\times X})$. As figure (3.1) is cartesian, we can also identify $p^*(T_{\overline{\mathcal{M}}_{0,0}})$ with $\pi_{1*}(N_{\mathcal{C}_0/\overline{\mathcal{M}}_{0,1}\times X})$.

Note that we have the following exact sequence of sheaves

$$0 \to T_p \to ev^* T_X \to \sigma^* N_{\mathcal{C}_1/\overline{\mathcal{M}}_{0,1}\times X} \to 0$$

which fits into the following diagram.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & 0 & \longrightarrow & T_{ev} & \longrightarrow & T_{ev} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_p & \longrightarrow & T_{\overline{\mathcal{M}}_{0,1}} & \longrightarrow & p^*T_{\overline{\mathcal{M}}_{0,0}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \vdots & & \\
0 & \longrightarrow & T_p & \longrightarrow & ev^*T_X & \longrightarrow & \sigma^*N_{\mathcal{C}_1/\overline{\mathcal{M}}_{0,1}\times X} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Here the solid arrows commute, and so the dashed arrow is induced. Furthermore, some diagram chasing shows that the rightmost column is exact: ie we have an exact sequence

$$0 \to T_{ev} \to p^*T_{\overline{\mathcal{M}}_{0,0}} \to \sigma^*N_{\mathcal{C}_1/\overline{\mathcal{M}}_{0.1}\times X} \to 0.$$

Finally, consider the exact sequence

$$0 \to N_{\mathcal{C}_1/\overline{\mathcal{M}}_{0,1}\times X}(-\sigma(\overline{\mathcal{M}}_{0,1})) \to N_{\mathcal{C}_1/\overline{\mathcal{M}}_{0,1}\times X} \to N_{\mathcal{C}/\overline{\mathcal{M}}_{0,1}\times X}|_{\sigma(\overline{\mathcal{M}}_{0,1})} \to 0.$$

As the normal bundle to a line in X is globally generated, pushing forward to $\overline{\mathcal{M}}_{0,1}(X,\beta)$ yields the exact sequence

$$0 \to \pi_{1*}N_{\mathcal{C}_1/\overline{\mathcal{M}}_{0,1}\times X}(-\sigma(\overline{\mathcal{M}}_{0,1})) \to T_{\overline{\mathcal{M}}_{0,0}} \to \sigma^*N_{\mathcal{C}_1/\overline{\mathcal{M}}_{0,1}\times X} \to 0$$

It follows that $T_{ev} \cong \pi_{1*}N_{\mathcal{C}_1/\overline{\mathcal{M}}_{0,1}\times X}(-\sigma(\overline{\mathcal{M}}_{0,1}(X,\beta)))$. $\square$

Note that given a family $\Sigma \subset \mathcal{C}_1$, the sheaf $N_{\mathcal{C}_1/\overline{\mathcal{M}}_{0,1}\times X}|_\Sigma = N_{\Sigma/B\times X}$, and so if ϕ lies entirely within the unobstructed locus of ev, we have an identification $\phi^*T_{ev} \cong \pi_*N_{\Sigma/B\times X}$. When the context is clear, we denote ϕ^*T_{ev} simply by T_{ev}.

In the case that Σ_0 is contained in a hypersurface $X \subset G(k,V)$, denote by T_{ev}^G the relative tangent sheaf to the evaluation map $ev_G : \overline{\mathcal{M}}_{0,1}(G(k,V),e)$, and by T_{ev}^X the relative tangent sheaf to the evaluation map $ev : \overline{\mathcal{M}}_{0,1}(X,e)$.

3.2 A very twisting family in $G(k, V)$

As we saw in chapter 2, a family of pointed lines in $G(k, r)$ over $\mathbb{P}^1$ is specified by choosing bundles E_{k-1}, E_k and E_{k+1} of rank $k-1$, k, and $k+1$ respectively, with $E_{k-1} \subset E_k \subset E_{k+1}$. The surface $\mathbb{P}(E_{k+1}/E_{k-1})$ embeds in $G(k, V)$, and we have the following diagram of maps.

$$\begin{array}{ccc} \Sigma = \mathbb{P}_B\left(\frac{E_{k+1}}{E_{k-1}}\right) & \xhookrightarrow{\;g\;} & G(k, V) \\ \sigma \uparrow \quad \downarrow \pi & & \\ B & & \end{array}$$

Proposition 3.7. *For a family of pointed lines over $B \cong \mathbb{P}^1$ specified by bundles $E_{k-1} \subset E_k \subset E_{k+1}$, we have the following isomorphisms:*

$$T_{ev} \cong [(E_{k+1}/E_k)^\vee \otimes ((V \otimes \mathcal{O}_\Sigma)/E_{k+1})] \oplus [(E_k/E_{k-1}) \otimes E_{k-1}^\vee]$$
$$\psi^\vee \cong (E_{k+1}/E_k) \otimes (E_k/E_{k-1})^\vee$$

Proof. This is [dJS06] Proposition 2.3. □

Proposition 3.8. *Let $B \cong \mathbb{P}^1$, and let Σ be a family of lines over B determined by the triple of bundles (E_{k-1}, E_k, E_{k+1}). Then Σ is very twisting in $G(k, V)$ if and only if:*

(1) $\deg \left[\left(\frac{E_{k+1}}{E_k} \right) \otimes \left(\frac{E_k}{E_{k-1}} \right)^\vee \right] \geq 0$

(2) $h^1(B, T_{ev}(-2)) = 0$

Proof. $\overline{\mathcal{M}}_{0,1}(G(k, V), 1)$ is smooth and ev is unobstructed, so the induced morphism $\varphi : B \to \overline{\mathcal{M}}_{0,1}(G(k, V), 1)$ trivially lies in the unobstructed locus. As $G(k, V)$ is convex, every rational curve is free, and thus $ev \circ \varphi$ is free. Furthermore condition (2) is equivalent to T_{ev}^G being ample, and so φ is ev-relatively very free.

Condition (1) is equivalent to $\psi^\vee$ being globally generated, and so $\varphi^*\psi^\vee$ is nonnegative.

As $\overline{\mathcal{M}}_{0,1}(G(k, V), 1)$ has no boundary component, all stable maps in the image of φ have irreducible domain. Therefore the conditions of Definition 3.4 are satisfied. □

Pushing forward to B, we can reinterpret condition (2) as

$$h^1(B, \pi_*(N(-\sigma(B))) \otimes \omega_B = 0$$

But then condition (2) is equivalent to

$$h^1\left(B, \mathcal{H}om\left(\frac{E_{k+1}}{E_k}, \frac{V\otimes\mathcal{O}_B}{E_{k+1}}\right)\otimes_{\mathcal{O}_B}\omega_\pi\right) = 0$$

and

$$h^1\left(B, \mathcal{H}om\left(E_{k-1}, \frac{E_k}{E_{k-1}}\right)\otimes_{\mathcal{O}_B}\omega_\pi\right) = 0.$$

For the next part, we will fix one very twisting surface in $G(k, V)$, determined by bundles

$$\begin{aligned} E_{k-1} &\cong \mathcal{O}(-3)^{\oplus k-1}, \\ E_k &\cong \mathcal{O}(-3)^{\oplus k-1}\oplus\mathcal{O}(-2) \\ E_{k+1} &\cong \mathcal{O}(-3)^{\oplus k-1}\oplus\mathcal{O}(-2)\oplus\mathcal{O}(-2) \end{aligned}$$

and surjective morphisms

$$\begin{aligned} V\otimes\mathcal{O}_B &\to E_{k+1}^\vee \\ E_{k+1}^\vee &\to E_k^\vee \\ E_k^\vee &\to E_{k-1}^\vee \end{aligned}$$

Denote this family by $\pi : \Sigma_0 \to B$, and denote by $g : \Sigma_0 \to G(k, V)$ the embedding in $G(k, V)$.

Proposition 3.9. *The surface Σ_0 is very twisting in $G(k, V)$.*

Proof. This follows directly from Proposition (3.8). □

The rank two bundle E_{k+1}/E_{k-1} splits as $L_1 \oplus L_2$, with $L_i \cong \mathcal{O}(-2)$. We have three distinguished rank one summands, coming from L_1, L_2, and the diagonal. Together, they determine a projective frame for $\mathbb{P}(E_{k+1}/E_{k-1})$, and so give an isomorphism $\Sigma_0 \cong \mathbb{P}^1\times\mathbb{P}^1$. Denote by $\mathcal{O}_{\Sigma_0}(i, j)$ the invertible sheaf determined by the isomorphism $Pic(\mathbb{P}^1\times\mathbb{P}^1) \cong \mathbb{Z}\times\mathbb{Z}$. Under this isomorphism, the sheaf $\mathcal{O}_{\Sigma_0}(\sigma(B))$ is isomorphic to $\mathcal{O}_{\Sigma_0}(1, 0)$. Given $p \in B$, $\mathcal{O}_{\Sigma_0}(\pi^{-1}(p)) \cong \mathcal{O}_{\Sigma_0}(0, 1)$.

Let L be the divisor on Σ_0 corresponding to a fiber of π. Denote by $\mathcal{L}$ the divisor $\sigma(B) + 2L$.

Lemma 3.10. *Let d be any positive integer. Let X be a degree d hypersurface containing Σ_0. Then $\mathbb{N}_{X/G}(-\mathcal{L})$ is globally generated.*

Proof. Recall that $g \circ \sigma = ev$. By construction, $ev^*(\mathcal{O}_{\bigwedge^k V}(1)) \cong \mathcal{O}(3k-1)$, and so $\mathcal{O}_{\bigwedge^k V}(1)|_{\Sigma_0}.\sigma(B) = 3k-1$. As fibers of π map to lines in $\mathbb{P}\bigwedge^k V$, for any $p \in B$, $\mathcal{O}_{\bigwedge^k V}(1)|_{\Sigma_0}.\pi^{-1}(-p) = 1$. Therefore $\mathcal{O}_{\bigwedge^k V}(1)|_{\Sigma_0} \cong \mathcal{O}(3k-1,1)$. The normal bundle of X in $G(k,V)$ is isomorphic to $\mathcal{O}_{\mathbb{P}\bigwedge^k V}(d)|_{\mathcal{G}(k,V)}$, and so $N_{X/G}|_{\Sigma_0} \cong \mathcal{O}_{\Sigma_0}(d(3k-1),d)$. Twisting down by $\mathcal{L}$, we have

$$N_{X/G}(-\mathcal{L}) \cong \mathcal{O}_{\Sigma_0}(d(3k-1)-2, d-1)$$

which is globally generated as $k, d \geq 1$. □

The following corollary will be needed in the proof of Thorem 1.8.

Corollary 3.11. *For all integers $d \geq 1$, the cup product map*

$$\kappa : \mathcal{O}_{\mathbb{P}\bigwedge^k V}(1)|_{\Sigma_0}(-\mathcal{L}) \otimes \mathcal{O}_{\mathbb{P}\bigwedge^k V}(d-1)|(-\mathcal{L}) \to \mathcal{O}_{\mathbb{P}\bigwedge^k V}(d)|_{\Sigma_0}$$

is surjective on global sections.

Proof. This is just the cup product map

$$\mathrm{H}^0(\Sigma_0, \mathcal{O}_{\Sigma_0}(3k-3,0)) \otimes_{\mathbb{C}} \mathrm{H}^0(\Sigma_0, \mathcal{O}_{\Sigma_0}((d-1)(3k-1), d-1)) \to \mathrm{H}^0(\Sigma_0, \mathcal{O}_{\Sigma_0}(d(3k-1)-1, d)),$$

which is surjective by the corresponding result for bihomogeneous polynomials. □

3.3 The derivative pairing

As containing a very twisting surface is an open condition, given a very twisting surface in $G(k,V)$, we need only find one degree d hypersurface $X \subset G(k,V)$ such that X contains Σ_0, and such that Σ_0 is very twisting in X. In fact, we need to check even less.

Proposition 3.12. *If X is a hypersurface containing Σ_0 such that T^X_{ev} is ample on B, then there is a very twisting family of lines in X.*

Proof. Let $\varphi : B \to \overline{\mathcal{M}}_{0,1}(X,1)$ be the morphism induced by the embedding $g_X : \Sigma_0 \to X$. The normal bundle to the section σ is independent is independent of g_X, and so $\varphi^*\psi^\vee$ is still nonnegative. Furthermore, all stable maps in the image of φ have irreducible domain.

Two conditions remain,

(1) $ev \circ \varphi$ is a free morphism, and

(2) the image of φ lies entirely within the unobstructed locus to ev.

In fact, if $\overline{\mathcal{M}}_{0,1}(X,1)$ is irreducible, then by Remark 3.5 these conditions are equivalent. However, in any case these conditions are both nonempty open conditions on the component containing φ of the Hilbert scheme of morphisms $\mathrm{Hom}(B, \overline{\mathcal{M}}_{0,1}(X,1))$. The positivity of T_{ev} and nonnegativity of $\psi^\vee$ are both open conditions as well, and so there exists a very twisting morphism in the component of $\mathrm{Hom}(B, \overline{\mathcal{M}}_{0,1}(X,1)$. ☐

As before, denote by $\pi : \Sigma_0 \to B$ the projection to B, and by $g : \Sigma_0 \to G(k,V)$, the embedding in $G(k,V)$. Denote by $N_{\Sigma_0/B\times G}$ the normal bundle to the embedding $\Sigma_0 \to B \times G$. Let $X = \mathbb{V}(F)$ be a hypersurface in $G(k,V)$, and denote by $N_{X/G}$ the normal bundle to X in $G(k,V)$. If X contains Σ_0, then the differential

$$dF : T_G \to N_{X/G}$$

factors through $N_{\Sigma_0/G}$. A warning about notation: this is the first of many 'differential' maps that are induced by an equation F. Composing with the differential

$$dg : N_{\Sigma_0/B\times G} \to N_{g(\Sigma_0)/G}$$

yields a morphism

$$dF' : N_{\Sigma_0/B\times G} \to i^* N_{X/G}.$$

We will use the convention of using an upper case D when refering to differential maps on global sections. In this case, taking global sections we have a morphism

$$DF' : \mathrm{H}^0(\Sigma_0, N_{\Sigma_0/B\times G}) \to \mathrm{H}^0(\Sigma_0, i^* N_{X/G}).$$

The main tool for proving Theorem 1.8 will be the following proposition.

Proposition 3.13. *Let $\pi : \Sigma \to B$ be a family of pointed lines in a $G(k,V)$ such that T^G_{ev} is ample on B. Let X be a hypersurface in projective space $\mathbb{P}W$ containing Σ. Let L be a divisor on Σ corresponding to a fiber of π, and let $\mathcal{L}$ be the divisor $\sigma(B) + 2L$. Suppose that $N_{X/G}(-\mathcal{L})$ globally generated. then T^X_{ev} is ample on B if and only if*

$$DF'' : H^0(\Sigma, N_{\Sigma/B\times G}(-\mathcal{L}) \to H^0(\Sigma, i^* N_{X/G}(-\mathcal{L}))$$

is surjective.

Proof. From the exact sequence

$$0 \to T_X \to T_G \xrightarrow{dF} N_{X/G} \to 0$$

we get an exact sequence

$$0 \to N_{\Sigma/B\times X} \to N_{\Sigma/B\times G} \xrightarrow{dF'} g^* N_{X/G} \to 0.$$

Twisting down by $-\mathcal{L}$ and applying π_* gives the exact sequence

$$\begin{aligned} 0 \to \pi_* N_{\Sigma/X}(-\mathcal{L}) \to T^G_{ev}(-2) \xrightarrow{\Delta} \pi_* g^* N_{X/G}(-\mathcal{L}) \to \cdots \\ \cdots R^1\pi_* N_{\Sigma/B\times X}(-\mathcal{L}) \to R^1\pi_* N_{\Sigma/B\times G}(-\mathcal{L}) \end{aligned}$$

As $G(k,V)$ is homogeneous, lines are free and $N_{\Sigma/B\times G}$ is globally generated on fibers of π. By the projection formula, it follows that $R^1\pi_* N_{\Sigma/B\times G}(-\sigma(B)) = 0$ If $N_{X/G}(-\mathcal{L})$ is globally generated, and dF'' is surjective, then the sheaf map Δ is surjective. It follows that $R^1\pi_* N_{\Sigma/B\times X} = 0$, and therefore by applying Proposition 3.6 we see that $\pi_* N_{\Sigma/X}(-\mathcal{L}) = T^X_{ev}(-2)$. We now have an exact sequence

$$0 \to T^X_{ev}(-2) \to T^G_{ev}(-2) \to \pi_* g^* N_{X/G}(-\mathcal{L}) \to 0.$$

By the long exact sequence in cohomology, and the fact that $\mathrm{H}^1(B, T^G_{ev}(-2)) = 0$, we have an exact sequence

$$\mathrm{H}^0(B, T^G_{ev}(-2)) \xrightarrow{\pi_* dF''} \mathrm{H}^0(B, \pi_* g^* N_{X/G}(-\mathcal{L})) \to \mathrm{H}^1(B, T_{ev}(-2)) \to 0$$

As dF'' is surjective, $\mathrm{H}^1(B, T^X_{ev}(-2)) = 0$, hence T^X_{ev} is ample. □

Remark 3.14. In the case of Σ_0 defined above, by applying Lemma 3.10 we see that for any degree d hypersurface X containing Σ_0, if the map, T^X_{ev} is ample if and only if DF'' is surjective.

Denote by I the subspace of $\mathrm{H}^0(\mathbb{P}\bigwedge^k V, \mathcal{O}(d))$ consisting of degree d forms containing Σ_0. Then the derivative maps induce a pairing

$$D'_d : I \otimes_{\mathbb{C}} \mathrm{H}^0(\Sigma_0, N_{\Sigma_0/B\times G}) \to \mathrm{H}^0(\Sigma_0, \mathcal{O}(d))$$

defined by $D_d(F \otimes s) = DF'(s)$. Twisting down by $\mathcal{L} = \mathcal{O}(\sigma + 2L)$, we get a pairing

$$D''_d : I \otimes_{\mathbb{C}} \mathrm{H}^0(\Sigma_0, N_{\Sigma_0/B\times G}(-\mathcal{L})) \to \mathrm{H}^0(\Sigma_0, \mathcal{O}(d)(-\mathcal{L}))$$

Given a form $F \in I$, denote by $\tau_F : \mathrm{H}^0(\Sigma_0, N_{\Sigma_0/G}(-\mathcal{L})) \to I \otimes \mathrm{H}^0(\Sigma_0, N_{\Sigma_0/G}(-\mathcal{L}))$ the morphism given by $s \mapsto F \otimes s$. In order to prove Theorem 1.8 it suffices to show that there is a form $F \in I$ such that $D''_d \circ \tau_F$ is surjective.

Let $I_1 \subset \mathrm{H}^0(\mathbb{P}\bigwedge^k V, \mathcal{O}(1))$ be the subspace of linear forms vanishing on Σ_0.

Lemma 3.15. *On $I_1 \otimes \mathrm{H}^0(\mathbb{P}\bigwedge^k V, \mathcal{O}_{\mathbb{P}\bigwedge^k V}(d-1))$, the differential pairing D'_d decomposes as follows.*

$$\begin{array}{c} I_1 \otimes \mathrm{H}^0(\Sigma_0, N_{\Sigma_0/\mathbb{P}\bigwedge^k V}) \otimes \mathrm{H}^0(\mathbb{P}\bigwedge^k V, \mathcal{O}_{\mathbb{P}\bigwedge^k V}(d-1)) \\ \Big\downarrow {\scriptstyle D'_1|_{I_1 \otimes \mathrm{H}^0(\Sigma_0, N_{\Sigma_0/\mathbb{P}\bigwedge^k V})} \otimes r_{d-1}} \\ \mathrm{H}^0(\Sigma_0, \mathcal{O}_{\mathbb{P}\bigwedge^k V}(1)) \otimes \mathrm{H}^0(\Sigma_0, \mathcal{O}_{\mathbb{P}\bigwedge^k V}(d-1)) \\ \Big\downarrow {\scriptstyle \kappa} \\ \mathrm{H}^0(\Sigma_0, \mathcal{O}_{\mathbb{P}\bigwedge^k V}(d)) \end{array}$$

Proof. This lemma amounts to Leibniz's rule. Briefly, given a polynomial $F = \kappa(F' \otimes F''$, with $F' \in I_1$, $F'' \in \mathrm{H}^0(\mathbb{P}\bigwedge^k V, \mathcal{O}_{\mathbb{P}\bigwedge^k V}(d-1))$, the Jacobian $J(F)$ is equal to $F''J(F') + F'J(F'')$. But F' vanished on Σ_0, so $J(F)|_{\Sigma_0} = F''J(F')|_{\Sigma_0}$. Extending linearly yields the lemma. ∎

3.4 A very twisting family in a hypersurface

Let $U \subset V$ be the subspace spanned by k-planes in Σ_0. Furthermore, choose an isomorphism $U^\vee \cong \mathrm{H}^0(\Sigma_0, E_{k+1}^\vee)$. Let W be the cokernel of the injection $U \to V$. Then

$$\dim U = h^0(B, E_{k+1}) = 4k + 2$$

and so

$$\dim W = n - 4k - 2$$

We have an inclusion $G(k,U) \to G(k,V)$. Furthermore, we have an exact sequence

$$0 \to \mathcal{H}om(S_k, U \otimes_{\mathbb{C}} \mathcal{O}_{G(k,U)}/S_K) \to \mathcal{H}om(S_k, V \otimes_{\mathbb{C}} \mathcal{O}_{G(k,U)}/S_k) \to \mathcal{H}om(S_k, W \otimes_{\mathbb{C}} \mathcal{O}_{G(k,U)}) \to 0$$

that commutes with the isomorphisms $T_{G(k,U)} \cong \mathcal{H}om(S_k, U \otimes_{\mathbb{C}} \mathcal{O}_{G(k,U)}/S_k)$ and $T_{G(k,V)}|_{G(k,U} \cong \mathcal{H}om(S_k, V \otimes_{\mathbb{C}} \mathcal{O}_{G(k,U)}/S_k)$, and so we get an isomorphism $N_{G(k,U)/G(k,V)} \cong \mathcal{H}om(S_k, W \otimes_{\mathbb{C}} \mathcal{O}_{G(k,U)})$. It follows that $\mathrm{H}^0(G(k,U), N_{G(k,U)/G(k,V)}) \cong \mathrm{Hom}(U, W)$.

By linearity, I_1 vanishes on the linear span of $G(k,U)$ in $\mathbb{P}\bigwedge^k V$, which is $\mathbb{P}\bigwedge^k U$. Then $I_1 \cong W^\vee \otimes_{\mathbb{C}} \bigwedge^{k-1} V^\vee$. As Σ_0 embeds in $G(k,U)$, the bundle $N_{G(k,U)/G(k,V)}|_{\Sigma_0}$ is a subbundle of $N_{\Sigma_0/B\times G}$. From above,

$$\mathrm{H}^0(G(k,U), N_{G(k,U)/G(k,V)}) \cong U^\vee \otimes W,$$

and so the differential pairing $D_1'|_{I_1 \otimes \mathrm{H}^0(\Sigma_0, N_{\Sigma_0/\mathbb{P}\bigwedge^k V})}$ induces a pairing

$$W^\vee \otimes_{\mathbb{C}} \bigwedge^{k-1} V^\vee \otimes_{\mathbb{C}} U^\vee \otimes_{\mathbb{C}} W \to \mathrm{H}^0(\Sigma_0, \mathcal{O}_{\mathbb{P}\bigwedge^k V}(1)).$$

A linear form $F \in I_1$ is degenerate for this pairing if and only if every section of the normal bundle $N_{G(k,U)/G(k,V)}$ lies entirely within the tangent bundle to $\mathbb{V}(F)$. We can represent a section of $N_{G(k,U)/G(k,V)}$ by a section $\phi \in \mathrm{Hom}(U, V)$ of the tangent bundle to $G(k,V)$ along $G(k,U)$. At a point $u_1 \wedge \cdots \wedge u_k$ in $G(k,U)$, we get a tangent vector $u_1 + \varepsilon\phi(u_1) \wedge \cdots \wedge u_k + \varepsilon\phi(u_k)$. Expanding, we see that a linear form $F \in I_1$ is degenerate for the pairing Γ if and only if it vanishes on all products of the form $v \wedge u_2 \wedge \cdots \wedge u_k$, for $u_2, \ldots, u_k \in U$, $v \in V$. The ideal I_2 of such degenerate forms is thus isomorphic to $\bigwedge^2 W^\vee \otimes_{\mathbb{C}} \bigwedge^{k-2} V$. Then I_1/I_2 is isomorphic to $\bigwedge^{k-1} U^\vee \otimes_{\mathcal{O}_B} W^\vee$. Restricting the pairing D_1' gives a pairing.

$$\Gamma : (\bigwedge^{k-1} U^\vee \otimes_{\mathcal{O}_B} W^\vee) \otimes_{\mathcal{O}_B} (W \otimes_{\mathcal{O}_B} U^\vee) \to \bigwedge^k U^\vee.$$

A straightforward computation yields the following proposition.

Proposition 3.16. *Denote by* $\xi : (\bigwedge^{k-1} U^\vee \otimes W^\vee) \otimes (W \otimes U^\vee) \to \bigwedge^{k-1} U^\vee \otimes U^\vee$ *the morphism corresponding to the pairing on the factors* $W^\vee$ *and* W*. Denote by* $\rho : \bigwedge^{k-1} U^\vee \otimes U^\vee \to \bigwedge^k U^\vee$ *the projection onto the exterior product. Then* Γ *is equal to* $\rho \circ \xi$*.*

As before, let $\mathcal{L}$ be the divisor $\sigma(B) + 2L$. The surface Σ_0 is abstractly isomorphic to $\mathbb{P}_B(E_{k+1}/E_{k-1})$, so let us denote by $\mathcal{O}_{\mathbb{P}E_{k+1}/E_{k-1}}(-1)$ the tautological line subbundle of $\mathcal{O}_{\Sigma_0} \otimes_{\mathbb{C}} E_{k+1}/E_{k-1}$ coming from this isomorphism.

Lemma 3.17. *The sheaf* $\mathcal{O}(-\mathcal{L})$ *is isomorphic to* $\mathcal{O}_{\mathbb{P}E_{k+1}/E_{k-1}}(-1)$*. Furthermore,* $\mathcal{O}_{\Sigma_0}(\mathcal{L})$ *is a subsheaf of* $S_k^\vee$*.*

Proof. Since $\mathcal{O}_{\Sigma_0}(\sigma) \otimes_{\mathcal{O}_B} \mathcal{O}_{\mathbb{P}E_{k+1}/E_{k-1}}(-1)$ is trivial on fibers, it follows that $\mathcal{O}_{\Sigma_0}(-\sigma) \cong \mathcal{O}_{\mathbb{P}E_{k+1}/E_{k-1}}(-1) \otimes \pi^*(\mathcal{G})$ for some invertible sheaf $\mathcal{G}$ on B. But $\sigma^*(\mathcal{O}_\Sigma(\sigma(B))) \cong (E_k/E_{k-1})^\vee \otimes_{\mathcal{O}_B} (E_{k+1}/E_k)$, and $\sigma^*(\mathcal{O}_{\mathbb{P}E_{k+1}/E_{k-1}}(-1)) \cong (E_k/E_{k-1})$, so we see that $\mathcal{G} \cong E_{k+1}/E_k$, which by our choice of E_{k-1}, E_k, and E_{k+1} is $\mathcal{O}(-2)$. As also $\mathcal{O}_{\Sigma_0}(-\mathcal{L}) \cong \mathcal{O}_{\Sigma_0}(-\sigma(B)) \otimes_{\mathcal{O}_{\Sigma_0}} \pi^*(\mathcal{O}_B(-2))$, it follows that $\mathcal{O}_{\Sigma_0}(-\mathcal{L}) \cong \mathcal{O}_{\mathbb{P}E_{k+1}/E_{k-1}}(-1)$.

We see from the exact sequence

$$0 \to \bigwedge^{k-1} E_{k-1} \to S_k \to \mathcal{O}_{\mathbb{P}E_{k+1}/E_{k-1}}(-1) \to 0$$

that $\mathcal{O}_{\Sigma_0}(\mathcal{L})^\vee$ is a subsheaf of $S_k^\vee$. □

Remark 3.18. As $\mathcal{L}$ is isomorphic to $\mathcal{O}_{\mathbb{P}E_{k+1}/E_{k-1}}(-1)$, $O(1) \otimes \mathcal{L}$ is isomorphic to $\bigwedge^{k-1} E_{k-1}$. Furthermore, as $\mathcal{O}_{\Sigma_0}(-\mathcal{L})$ is a subsheaf of $S_k^\vee$ by Lemma 3.17, tensoring the inclusion $\mathcal{L}^\vee \hookrightarrow S_k^\vee$ with $W \otimes \mathcal{L}$ gives an inclusion $W \hookrightarrow N_{G(k,U)/G(k,V)}(-\mathcal{L})$. By restricting the differential pairing D''_d to $I_1/I_2 \otimes W$, we then get a pairing

$$\Gamma' : \bigwedge^{k-1} U^\vee \otimes_{\mathbb{C}} W^\vee \otimes_{\mathbb{C}} W \to \mathrm{H}^0(B, \bigwedge^{k-1} E_{k-1})$$

Lemma 3.19. *The restriction maps* $r_d : \mathrm{H}^0(\mathbb{P}\bigwedge^k V, \mathcal{O}(d)) \to \mathrm{H}^0(\Sigma_0, i^*\mathcal{O}(d))$ *are surjective, for all* $d \geq 1$*.*

Proof. Because $i^*\mathcal{O}(d)$ is globally generated, the natural morphism $\mathrm{Sym}^d \mathrm{H}^0(\Sigma_0, i^*\mathcal{O}(1)) \to \mathrm{H}^0(\Sigma_0, i^*\mathcal{O}(d))$ is surjective. Furthermore, the following diagram commutes

$$\begin{array}{ccc} \operatorname{Sym}^d \mathrm{H}^0(\mathbb{P}\bigwedge^k V, \mathcal{O}(1)) & \longrightarrow & \mathrm{H}^0(\mathbb{P}\bigwedge^k V, \mathcal{O}(d)) \\ \downarrow{\scriptstyle \operatorname{Sym}^d r_1} & & \downarrow{\scriptstyle r_d} \\ \operatorname{Sym}^d \mathrm{H}^0(\Sigma_0, \mathcal{O}(1)) & \longrightarrow & \mathrm{H}^0(\Sigma_0, \mathcal{O}(d)) \end{array}$$

and so it suffices to prove the claim in the case that $d = 1$.

We can factor $i : \Sigma_0 \to \mathbb{P}\bigwedge^k V$ as follows.

$$\Sigma_0 \xrightarrow{\alpha} G(k, E_{k+1}) \xrightarrow{\beta} G(k, U) \xrightarrow{\gamma} G(k, V) \xrightarrow{\delta} \mathbb{P}\bigwedge^k V \qquad (i = \delta\circ\gamma\circ\beta\circ\alpha)$$

Label the restriction maps as follows:

$$\begin{aligned} r_1^\alpha &: \mathrm{H}^0(G(k, E_{k+1}), (\delta\circ\gamma\circ\beta)^*\mathcal{O}(1)) \to \mathrm{H}^0(\Sigma_0, i^*\mathcal{O}(1)) \\ r_1^\beta &: H^0(G(k, U), (\delta\circ\gamma)^*\mathcal{O}(1)) \to \mathrm{H}^0(G(k, E_{k+1}), (\delta\circ\gamma\circ\beta)^*\mathcal{O}(1)) \\ r_1^\gamma &: \mathrm{H}^0(G(k, V), \delta^*\mathcal{O}(1)) \to \mathrm{H}^0(G(k, U), (\delta\circ\gamma)^*\mathcal{O}(1)) \\ r_1^\delta &: \mathrm{H}^0(\mathbb{P}\bigwedge^k V, \mathcal{O}(1)) \to \mathrm{H}^0(G(k, V), \delta^*\mathcal{O}(1)) \end{aligned}$$

Now, by the exact sequence

$$0 \to \pi^* E_{k-1} \to i^* S_k \to \mathcal{O}_{\mathbb{P}E_{k+1}/E_{k-1}}(-1) \to 0$$

we see that $\bigwedge^k i^* S_k^\vee = \mathcal{O}_{\mathbb{P}\bigwedge^k V}(1)|_{\Sigma_0}$ is isomorphic to $\pi^* \bigwedge^{k-1} E_{k-1}^\vee \otimes \mathcal{O}_{\mathbb{P}E_{k+1}/E_{k-1}}(1)$, and so by the projection formula $\pi_* \mathcal{O}_{\mathbb{P}\bigwedge^k V}(1)$ is isomorphic to $\bigwedge^{k-1} E_{k-1}^\vee \otimes_{\mathcal{O}_B} (E_{k+1}/E_{k-1})^\vee$.

The exact sequence

$$0 \to E_{k-1} \to E_{k+1} \to E_{k+1}/E_{k-1} \to 0$$

induces a filtration on exterior powers

$$\bigwedge^k E_{k+1} = F^0 \supseteq F^1 \supseteq F^2 \supseteq \cdots \supseteq F^{k+1} = 0$$

such that $F^p/F^{p+1} \cong \bigwedge^p E_{k-1} \otimes_{\mathcal{O}_B} \bigwedge^{k-p}(E_{k+1}/E_{k-1})$. (See [Har77] II.5 ex. 16d). In particular, we have an inclusion $\bigwedge^{k-1} E_{k-1} \otimes_{\mathcal{O}_B} (E_{k+1}/E_{k-1}) \hookrightarrow \bigwedge^k E_{k+1}$. Dualizing, we get a surjection $\bigwedge^k E_{k+1}^\vee \to \bigwedge^{k-1} E_{k-1}^\vee \otimes_{O_B} (E_{k+1}/E_{k-1})^\vee$ which is globally split. Therefore, taking global sections, we get a surjective homomorphism $\mathrm{H}^0(B, \bigwedge^k E_{k+1}^\vee) \to \mathrm{H}^0(B, \bigwedge^{k-1} E_{k-1}^\vee \otimes_{\mathcal{O}_B} (E_{k+1}/E_{k-1})^\vee)$. Recalling the identity $\mathcal{O}_{\mathbb{P}\bigwedge^k V}(1)|_{\Sigma_0} \cong \bigwedge^{k-1} E_{k-1} \otimes \mathcal{O}_{\mathbb{P}E_{k+1}/E_{k-1}}(1)$, this surjection is just the homomorphism

$$r_1^\alpha : \mathrm{H}^0(B, \bigwedge^k E_{k+1}^\vee) \to \mathrm{H}^0(\Sigma_0, \mathcal{O}_{\mathbb{P}\bigwedge^k V}(1)).$$

By construction $U^\vee \cong \mathrm{H}^0(B, E_{k+1}^\vee)$, and so $\bigwedge^k U^\vee \cong \bigwedge^k \mathrm{H}^0(B, E_{k+1}^\vee) \cong \bigwedge^k \mathrm{H}^0(\Sigma_0, \pi^* E_{k+1}^\vee)$. As $E_{k+1}^\vee$ is globally generated, the map $\bigwedge^k \mathrm{H}^0(B, E_{k+1}^\vee) \to \mathrm{H}^0(B, \bigwedge^k E_{k+1}^\vee)$ induced by the cup product map on tensor powers is surjective. Composing with the surjection $\bigwedge^k V^\vee \to \bigwedge^k U^\vee$ and the isomorphism $\bigwedge^k U^\vee \cong \bigwedge^k \mathrm{H}^0(B, E_{k+1}^\vee)$, we have a surjection

$$r_1^\beta \circ r_1^\gamma \circ r_1^\delta : \bigwedge^k V^\vee \to \mathrm{H}^0(B, \bigwedge^k E_{k+1}^\vee).$$

Thus, we see that $r_1^\alpha \circ r_1^\beta \circ r_1^\gamma \circ r_1^\delta : \bigwedge^k V^\vee \to \mathrm{H}^0(\Sigma_0, \mathcal{O}_{\mathbb{P}\bigwedge^k V})$ is surjective. Therefore i^* is surjective □

We are now ready to prove Theorem (1.8).

Proof of Theorem (1.8). Let Γ' be the pairing from Remark 3.18. As $E_{k+1}^\vee$ and E_{k-1} are globally generated, the surjection $E_{k+1}^\vee \to E_{k-1}^\vee$ determines a surjection $\mathfrak{r} : \bigwedge^{k-1} \mathrm{H}^0(B, E_{k+1}) = \bigwedge^{k-1} U^\vee \to \mathrm{H}^0(B, \bigwedge^{k-1} E_{k-1})$. By Proposition (3.16), the following diagram commutes.

$$\begin{array}{ccc} \bigwedge^{k-1} U^\vee \otimes_{\mathbb{C}} W^\vee \otimes_{\mathbb{C}} W & \xrightarrow{\Gamma'} & \mathrm{H}^0(B, \bigwedge^{k-1} E_{k-1}) \\ & \searrow^{\xi} & \uparrow \mathfrak{r} \\ & & \bigwedge^{k-1} U^\vee \end{array}$$

Let $A = \dim \mathrm{H}^0(\Sigma_0, \mathcal{O}_{\mathbb{P}\bigwedge^k V}(d)|_{\Sigma_0}(-\mathcal{L}))$. By Lemma (3.19) and Corollary 3.11, we can choose collections

$$\begin{aligned} &(b_i)_{1 \le i \le A}, \qquad b_i \in \mathrm{H}^0(\mathbb{P}\bigwedge^k V, \mathcal{O}(d-1)) \\ &(a_i)_{1 \le i \le A}, \qquad a_i \in \bigwedge^{k-1} U^\vee \end{aligned}$$

such that $\kappa(\mathfrak{r}(a_i) \otimes r_{d-1}(b_i))$ form a basis for $\mathrm{H}^0(\Sigma_0, \mathcal{O}(d)(-\mathcal{L}))$. By Lemma 3.10,

$$\begin{aligned} h^0(\Sigma_0, \mathcal{O}_{\mathbb{P}\bigwedge^k v}(d)) &= h^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d(3k-1)-2, d-1)) \\ &= d(d(3k-1)-1) \\ &= 3kd^2 - d^2 - d, \end{aligned}$$

and so $A = 3kd^2 - d^2 - d$. By the inequality 1.1, $A \leq n - 4k - 2 = \dim W$, and so we can choose an injection of sets

$$J : \{1, \ldots, 3kd^2 - d^2 - d\} \to \{1, \ldots, n - 4k - 2\}.$$

Now define F as follows

$$F = \sum_{i=1}^{3kd^2-d^2-d} a_i \otimes x_{J(i)} \otimes b_i.$$

Denote by $\mathfrak{D}$ the restriction of the differential pairing D''_d to the subspace

$$\bigwedge^{k-1} U^\vee \otimes_{\mathbb{C}} W^\vee \otimes_{\mathbb{C}} W \otimes H^0(\mathbb{P}\bigwedge^k V, \mathcal{O}(d-1))$$

of

$$\mathrm{H}^0(\mathbb{P}\bigwedge^k, \mathcal{O}(1)) \otimes \mathrm{H}^0(\Sigma_0, N_{\Sigma_0/B\times G}) \otimes H^0(\mathbb{P}\bigwedge^k V, \mathcal{O}(d-1))$$

By Lemma 3.15, $\mathfrak{D}$ factors as $\kappa(\Gamma' \otimes r_{d-1})$. By our choice of (a_i) and (b_i), $\mathfrak{D}$ restricted to $F \otimes W$ is surjective onto $\mathrm{H}^0(\Sigma_0, \mathcal{O}(d)(-\mathcal{L}))$.

Let X be the hypersurface $\mathbb{V}(F)$. By Proposition 3.13, T^X_{ev} is ample on B, and so by Proposition 3.12, there exists a very twisting surface on $\mathbb{V}(F)$. □

3.5 Rational simple connectedness of chains of lines

Given two general points $p, q \subset G(k, V)$, denote by λ_p, λ_q the corresponding k-planes in V. The k planes in a line $l \subset G(k, V)$ span a $k+1$ dimensional subspace of V, and more generally as long as $r \leq n - k$, the k planes in a general chain of r lines span a $k + r$ dimensional subspace of V. In this case we say that the r-chain is *linearly nondegenerate*. A chain of lines in $G(k, V)$ connecting p to q must span $\lambda_p + \lambda_q$, and so the minumum length of a chain of lines connecting p to q is k.

Let υ_r denote the stable A-graph corresponding to a chain of lines between two marked points. That is to say, υ_r has r vertices, $\beta(v) = 1$ for each vertex, $r - 1$ edges, no vertex incident upon more than two edges, and with two tails f_1 and f_2 attached at the two vertices that are each incident upon only one edge, as in Figure 3.2.

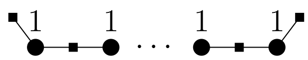


Figure 3.2: A $k+1$ chain

For a general degree d hypersurface $X \subset G(k, V)$ with $d \leq n-k-1$, by Theorem 2.1 and Proposition 2.14, $\overline{\mathcal{M}}(X, \upsilon_r)$ is $k(n-k)-1+r(n-d-1)$-dimensional, and so the fiber dimension of the morphism

$$ev = ev_{f_1} \times ev_{f_2} : \overline{\mathcal{M}}(X, \upsilon_r) \to X \times X$$

is $r(n-d-1) - k(n-k) + 1$. Thus we expect that the general fiber is nonempty whenever

$$r \geq k + \frac{k(d+1-k)-1}{n-d-1}$$

When $d \geq k$ and $kd < n$, the minimal length of a chain of lines connecting two general points is a $k+1$ chain, and we say that $k+1$-chains are *minimal* for X. Furthermore, given $p, q \in X$ general, the fiber $ev^{-1}((p,q))$ consists entirely of linearly nondegenerate $k+1$ chains.

In this case, we have a nice parameterization of $N = ev^{-1}((p,q))$. To avoid confusion, denote by $ev_G : \overline{\mathcal{M}}(G(k,V), \upsilon_{k+1}) \to X \times X$ the evaluation at both marked points on $k+1$ chains in $G(k,V)$. Denote by $M^{nd} \subset ev_G^{-1}((p,q))$ the open locus parameterizing linearly nondegenerate $k+1$-chains in $G(k,V)$ spanning p and q. N is a closed subset of M^{nd}.

Given a linearly nondegenerate $k+1$ chain $l_0, \ldots, l_k$ in M^{nd}, l_{i-1} intersects l_i in a point r_i, and $l_i \cap l_j = \emptyset$ for $|i-j| \geq 2$. Denote by λ_i the k-plane corresponding to r_i. λ_i intersects λ_{i+j} in a $k-j$ plane, and λ_i and λ_{i+j} span a $k+j$ plane.

Let v be the one-dimensional subspace $\lambda_1 \cap \lambda_k \subset V$. Denote by E_i the i-plane $\lambda_p \cap \lambda_{k-i-1}$, and by F_i the i-plane $\lambda_q \cap \lambda_{i+1}$. Then v, together with complete flags $(E_1, E_2, \ldots, E_{k-1}) \subset Flag(1, 2, \ldots, k-1; \lambda_p)$ and $(F_1, F_2, \ldots, F_{k-1}) \subset Flag(1, 2, \ldots, k-1; \lambda_q)$ completely determine the chain $l_0, \ldots, l_k$. Indeed, for $1 \leq i \leq k-1$, the line l_i is determined by the $k-1$ plane $span\,\langle E_{k-i-1}, v, F_{i-1}\rangle$ and the $k+1$ plane $span\,\langle E_{k-i}, v, F_i\rangle$. The line l_0 is determined by E_{k-1} and $span\,\langle \lambda_p, v\rangle$, and the line l_k is determined by F_{k-1} and $span\,\langle \lambda_q, v\rangle$. This determines an isomorphism

$$\Lambda : Flag(1, \ldots, k-1; \lambda_p) \times Flag(1, \ldots, k-1; \lambda_q) \times \mathbb{P}V \setminus \{\lambda_p + \lambda_q\} \to M^{nd}$$

And so M^{nd} is a nonsingular quasi-projective variety. To ease notation denote by $Flag(\lambda_p)$ or $Flag(\lambda_q)$, the complete flag variety of subspaces of λ_p or λ_q.

Lemma 3.20. *For a general degree d hypersurface X with $k \leq d < n/k$ and $d \leq n-k-1$, for general points $p, q \in X$, $ev^{-1}((p,q)) \subset \overline{\mathcal{M}}(X, \upsilon_{k+1})$ is smooth of the expected dimension.*

Proof. A $k+1$ chain in $G(k, V)$ is linearly nondegenerate if and only if its image in $\mathbb{P}\bigwedge^k V$ spans a $\mathbb{P}^{k+1}$. Furthermore, the $k+1$ chain is determined by $k+2$ linearly general points (the points p, q, and the $k-1$ nodes) in this $\mathbb{P}^{k+1}$. As it takes $k+3$ linearly general points to form a projective frame for $\mathbb{P}^{k+1}$, any two $k+1$ chains spanning the same $\mathbb{P}^{k+1}$ are projectively equivalent.

For p, q general, denote as above $M^{nd} \subset ev_G^{-1}((p,q))$ the smooth, quasi-projective locus of linearly nondegenerate $k+1$ chains in $G(k, V)$. Denote by $\mathfrak{X}$ the incidence correspondence

$$\mathfrak{X} \subset \mathbb{P}\bigwedge^k V^\vee \times M^{nd}$$

consisting of pairs $(w, (l_0, \ldots, l_k))$ such that $\mathbb{V}(w)$ contains $l_0 \cup \cdots \cup l_k$.

As any two linearly nondegenerate $k+1$ chains are projectively equivalent, they have the same Hilbert function. Therefore, under the morphism $\rho : \mathfrak{X} \to M^{nd}$, $\mathfrak{X}$ is a projective bundle, and hence nonsingular. By generic smoothness, for general X the fiber $ev^{-1}((p,q))$ is smooth, and of the expected dimension by Theorem 2.1 and Proposition 2.14. □

Theorem 3.21. *Let X be a degree d hypersurface in $G(k, V)$ with $kd^2 < n$ and $d \geq k$. Then for a general point $(p,q) \in X \times X$, the fiber $N = ev^{-1}((p,q)) \subset \overline{\mathcal{M}}(X, \upsilon_{k+1})$ is rationally connected.*

Furthermore, if $d < k$, the same result hold under the stronger inequality $kd^2 < n - 2k$.

Proof. As X is general, by Lemma 3.20 we may assume that N is smooth of the expected dimension. Then N is a proper subvariety of M^{nd}, and we have a dominant morphism $\rho : N \to Flag(\lambda_p) \times Flag(\lambda_q)$. By generic smoothness a general fiber of ρ is smooth.

Furthermore, given a triple $(E_*, F_*, v) \subset Flag(\lambda_p) \times Flag(\lambda_q) \times \mathbb{P}V \setminus \{\lambda_p + \lambda_q\}$, we may explicitly describe the morphism Λ. Choose a basis $(\mathbf{e_i})$ for λ_p and a basis $\mathbf{f_i}$ for λ_q such that $E_i = span\langle \mathbf{e}_1, \ldots, \mathbf{e}_i \rangle$ and $F_j = span\langle \mathbf{f}_1, \ldots, \mathbf{f}_k \rangle$. Then we can define a $k+1$-chain of lines

$l_0, \ldots, l_k$ by setting.

$$l_0([s,t]) = span\,\langle \mathbf{e}_1, \ldots, \mathbf{e}_{k-1}, s_0 v + t_0 \mathbf{e}_k \rangle$$
$$l_i([s,t]) = span\,\langle \mathbf{e}_1, \ldots, \mathbf{e}_{k-i-1}, v, \mathbf{f}_1, \ldots, \mathbf{f}_{i-1}, s_i \mathbf{f}_i + t_i \mathbf{e}_{k-i} \rangle \qquad 1 \leq i \leq k-1$$
$$l_k([s,t]) = span\,\langle \mathbf{f}_1, \ldots, \mathbf{f}_{k-1}, s_k \mathbf{f}_k + t_k v \rangle$$

Let us denote the $k-1$ plane corresponding to the intersection of k-planes in l_i by ε^i_{k-1}. Then there is a bundle $\mathcal{E}^i_{k-1}$ on $\overline{\mathcal{M}}(G(k,V), \upsilon_{k+1})$ whose fiber over the $k+1$ chain $l_0, \ldots, l_k$ is ε^i_{k-1}.

Fix two general complete flags E_* and F_* such that $\rho^{-1}((E_*, F_*))$ is smooth, and given by a basis $\mathbf{e}_1, \ldots, \mathbf{e}_k$ of λ_p and a basis $\mathbf{f}_f, \ldots, \mathbf{f}_k$ of λ_q. Under the morphism $\iota : \mathbb{P}V \to \overline{\mathcal{M}}(G(k,V), \upsilon_{k+1})$, $\iota^*(\bigwedge^{k-1} \mathcal{E}^i_{k-1})$ is a line bundle on $\mathbb{P}V$. By construction $\iota^*(\bigwedge^k \mathcal{E}^0_{k-1})$ and $\iota^*(\bigwedge^k \mathcal{E}^k_{k-1})$ are degree 0, $\iota^*(\bigwedge^k \mathcal{E}^i_{k-1})$ is degree 1, $1 \leq i \leq k-1$.

Recall that for a line l in $G(k,V)$ given by a $k-1$ plane E_{k-1} contained in a $k+1$ plane E_{k+1}, we have that $O_{\mathbb{P}\bigwedge^k V}(d)|_l \cong \mathrm{Sym}^d \bigwedge^{k-1} E^\vee_{k-1} \otimes \mathcal{O}_{\mathbb{P}E_{k+1}/E_{k-1}}(1)$. Thus, if $F \in \mathrm{H}^0(\mathbb{P}\bigwedge^k V, \mathcal{O}(1))$ is a defining equation for X, we have the following splitting.

$$F|_{l_i} = \sum_{j=0}^{d} c_{i,j} s_i^j t_i^{d-j}$$

Where the $c_{i,j}$ are polynomials on $\mathbb{P}V$. Note that as all lines contain $[1,0] \subset l_0$ and $[0,1] \subset l_k$, $c_{0,0} = c_{k,d} = 0$. Additionally, as $[1,0] \in l_i$ equals $[0,1] \in l_{i+1}$, $c_{i,0} = c_{i+1,k}$, for $1 \leq i \leq k-1$. $\rho^{-1}((E_*, F_*)) = \mathbb{V}((c_{i,j})_{0 \leq i \leq k, 0 \leq j \leq d})$. If $i = 0$, $c_{i,j}$ is degree j. If $i = k$, $c_{i,j}$ is degree $d-j$, and for $1 \leq i \leq k-1$, $c_{i,j}$ is degree d. Furthermore, by assumption $\rho^{-1}((E_*, F_*))$ is smooth, and as N is of the expected dimension, $\rho^{-1}((E_*, F_*))$ is a complete intersection. $\sum deg(c_{i,j}) = (k-1)d^2 + d(d+1)/2 + d(d-1)/2 = kd^2$, and so by adjunction, whenever $kd^2 < n$, $\rho^{-1}((E_*, F_*))$ is Fano, hence rationally connected. Thus ρ is a dominant morphism over a rationally connected base with rationally connected geometric generic fiber. By [GHS03], N is rationally connected.

In the case that $d < k$, N need not be smooth. However, if we choose an orthogonal complement W to $\lambda_p + \lambda_q$ in V, then by restricting the morphism Λ to $Flag(\lambda_p) \times Flag(\lambda_q) \times$

$\mathbb{P}W$, when $kd(d+1) < n - 2k$ we may run the above argument to produce a rationally connected subvariety Y of N. Furthermore, the normal bundle to Y in N is just the normal bundle to $\mathbb{P}W$ in $\mathbb{P}V$, and so we may produce a very free curve in N, hence N is rationally connected. □

3.6 Sections over a surface.

Let (d, k, n) satisfy condition 1.1. Consider a fibration by degree d hypersurfaces in $G(k, V)$ over a smooth surface, such that a general fiber is 'general' in the sense that it is irreducible, contains a very twisting surface by Theorem 1.8, and is rationally simply connected by chains of free lines by Theorem 3.21. Then conditions 1 and 3 are of Theorem 3.2 are satisfied, and so the existence of rational sections is implied by the vanishing of the *elementary obstruction*.

Similarly, conjecture 1.9, in combination with Theorem 3.21, would imply that there is a rational point of the generic fiber of a two parameter family of general degree d hypersurfaces in $G(k, V)$ with vanishing elementary obstruction, whenever $kd^2 < n$

When $n/k = d^2$, there is an example, due to Jason Starr, of a two parameter family of degree d hypersurfaces in $G(k, V)$ with vanishing elementary obstruction such that there is no rational point of the generic fiber. Before discussing this example, we briefly introduce the Brauer group, to understand exactly when the elementary obstruction vanishes.

Definition 3.22. An Azumaya algebra is a locally free $\mathcal{O}_S$ sheaf $\mathcal{A}$, étale locally isomorphic to a matrix algebra, together with a group operation given by morphisms

$$m : \mathcal{A} \otimes_{\mathcal{O}_S} \mathcal{A} \to \mathcal{A}$$
$$1 : \mathcal{O}_S \to \mathcal{A}.$$

The set of Azumaya algebras form a group under tensor powers. Two Azumaya algebras $\mathcal{A}$ and $\mathcal{A}'$ are *Morita equivalent* if there exist nonzero locally free sheaves $\mathcal{E}, \mathcal{E}'$ such that $\mathcal{A} \otimes_{\mathcal{O}_S} \mathrm{End}(\mathcal{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_S} \mathrm{End}(\mathcal{E}')$.

Denote by $Br(B)$, the *Brauer group of* B, the group of Azumaya algebras over S modulo morita equivalence.

Given a point $0 \in S$ and a central fiber $X_0 \subset X$, we have an exact sequence

$$Pic(X) \to Pic(X_0)^{\pi_1(B,0)} \xrightarrow{\delta} Br(S)$$

where $Br(S)$ is the Brauer group of S. The elementary obstruction of the fibration $f : X \to S$ is $\delta(\mathcal{O}(1))$.

Up to shrinking S, an Azumaya algebra $\mathcal{A}$ is isomorphic to $\mathrm{End}(\mathcal{E}) \otimes \mathcal{D}$, for some vector bundle $\mathcal{E}$ and some central division algebra $\mathcal{D}$ over $K(S)$, therefore we have $\mathcal{A} \sim \mathcal{D}$. Under some Galois extension K' of $K(S)$, $\mathcal{D} \cong \mathrm{End}(\mathcal{F})$, for some locally free sheaf $\mathcal{F}$, and so $\mathcal{A} \otimes_K K' \cong \mathrm{End}(\mathcal{E} \otimes \mathcal{F})$. In this case, the rank of $\mathcal{F}$ is called the *index* of $\mathcal{A}$. The order of $\mathcal{A}$ in $Br(S)$ is called the *period* of $\mathcal{A}$. In [dJ04], it is proved that when S is a surface, given an element $\mathcal{A} \in Br(S)$, the period of $\mathcal{A}$ equals the index of $\mathcal{A}$.

Now, there is a natural correspondence between PGL_n bundles over S, Azumaya algebras, and $G(k, V)$ bundles over S. Given a principal PGL_n bundle $T \to S$, we can form the $G(k, V)$ bundle $G(k, V) \times T/\Delta(PGL_n)$, where $\Delta(PGL_n)$ is the diagonal action. Alternatively, we can form the Azumaya algebra $Mat_{n\times n} \times T/\Delta(PGL_n)$. In fact, given a rank n^2 Azumaya algebra $\mathcal{A}$, there is a scheme $G_{k,\mathcal{A}}$ whose closed points parameterize rank nk subsheaves I of $\mathcal{A}$ which are right ideals and whose cokernel is locally free. If $\mathcal{A}$ is the Azumaya algebra corresponding to a PGL_n bundle $T \to S$, and G is the $G(k, V)$ bundle corresponding to to the same PGL_n bundle, then $G \cong G_{k,\mathcal{A}}$. By extending our base field to K', this isomorphism is explicit at the geometric generic point: $\mathcal{A} \otimes_K K' \cong \mathrm{End}(\mathcal{E} \otimes \mathcal{F}$, and the points of $G_{k,A} \otimes_K K'$ correspond to $\mathrm{Hom}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G})$ for some rank k locally free subsheaf $\mathcal{G} \subset \mathcal{E} \otimes \mathcal{F}$.

If $\mathcal{A}$ is an Azumaya algebra correponding to a $G(k, V)$ bundle $\pi : G \to S$, then the elementary obstruction of π is precisely $k\mathcal{A}$, as an element of $Br(S)$. Thus, for a contradiction we will seek to construct a rank n^2 Azumaya algebra $\mathcal{A}$ of period k, such that $G_{k,\mathcal{A}}$ has no rational sections. It will suffice to construct an Azumaya algebra $\mathcal{A}$ over the function field of a surface, such that the corresponding variety $G_{k,\mathcal{A}}$ has no rational points.

To do this, we will first construct a central division algebra of rank k^2 over the function field of a surface. Let $K = \mathbb{C}(\sigma, \tau)$, be a purely transcendental extension of $\mathbb{C}$. Let $K' = C(s, t)$ be a degree k^2 extension of K given by $s^k = \sigma$, $t^k = \tau$. Define matrices A and B as follows.

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \zeta & 0 & \cdots & 0 \\ 0 & 0 & \zeta^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \zeta^{k-1} \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Now let D be the subalgebra of $Mat_{k\times k}(K')$ generated over K by $x = sA$ and $y = tB$. Observe that $xy = \zeta yx$. Furthermore, it is not hard to see that D is a central simple algebra over K of rank k^2. The following proposition shows that D is indeed a division algebra.

Proposition 3.23. *Let M be a nonzero matrix in the span of x and y above, say $M = \sum \alpha_{i,j} x^i y^j$. Then M is invertible. Moreover, $\det M$ is an invertible element of $\mathbb{C}(\sigma, \tau)$.*

Proof. For readability, we will use the notation $[\cdots]_k$ to denote the class of $\cdots$ modulo k expressed in the range $0 \ldots k-1$. Since B is a shift matrix, and A acts diagonally, it is not hard to see that $M_{ij} = t^{[i-j]_k} \sum_{l=0}^{k-1} \alpha_{l,[i-j]_k} s^l \zeta^{[il]_k}$. Using the Leibniz formula for the determinant, we can thus express the determinant of M as

$$\det M = \sum_{\rho \in S_n} sgn(\rho) \prod_{i=0}^{k-1} t^{[i-\rho(i)]_k} \sum_{l=0}^{k-1} \alpha_{l,[i-\rho(i)]_k} s^l \zeta^{[il]_k}. \tag{3.1}$$

We will seek to show that this expression is zero if and only if all $\alpha_{i,j}$ are zero. Note that given a permutation ρ, the sum $\sum_{i=0}^{k-1} [i-\rho(i)]_k$ is divisible by k. Denote this sum by $w(\rho)$. Using this notation,

$$\det M = \sum_{\rho \in S_n} sgn(\rho) \tau^{w(\rho)} \prod_{i=0}^{k-1} \sum_{l=0}^{k-1} \alpha_{l,[i-\rho(i)]_k} s^l \zeta^{[il]_k}. \tag{3.2}$$

By using the notation (a_i) to denote a k-tuple $a_0, \ldots, a_k$ of integers with $0 \le a_i \le k-1$, we can expand the term for each permutation, and we get

$$\det M = \sum_{\rho \in S_n} sgn(\rho) \tau^{w(\rho)} \sum_{(a_i)} s^{\sum a_i} \zeta^{\sum [i a_i]_k} \prod_{i=0}^{k-1} \alpha_{a_i,[i-\rho(i)]_k}. \tag{3.3}$$

$$= \sum_{a,b} \tau^b s^a \sum_{\substack{\rho \in S_n \\ w(\rho)=b \\ \sum a_i = a}} sgn(\rho) \zeta^{ia} \prod_{i=0}^{k-1} \alpha_{a_i,[i-\rho(i)]_k} \tag{3.4}$$

A somewhat tedious combinatorial argument shows that in fact we may factor out $(1 + \zeta^{\sum a_i} + \zeta^{2\sum a_i} + \cdots + \zeta^{(k-1)\sum a_i})$ from the coefficient of $s^{\sum a_i}\tau^b$ in the expression for $\det M$, if $\sum a_i \neq 0$. Since this sum is zero whenever $[\sum a_i]_k \neq 0$, this justifies that $\det M$ is in fact contained in K. Since this is secondary to the conclusion that M is invertible, and is not necessary for what follows, we leave this to the reader.

Now define a total ordering on the coefficients $\alpha_{i,j}$ by $\alpha_{i,j} < \alpha_{l,m}$ if $ik + j < lk + m$, and assume $\det M = 0$. Note that complex component of $\det M$ is $\alpha_{0,0}^k$, and so $\alpha_{0,0} = 0$. By way of induction, assume that $\alpha_{i,j} = 0$ for $\alpha_{i,j} < \alpha_{i_0,j_0}$. Now consider the coefficient C_{i_0,j_0} of $\sigma^{i_0}\tau^{j_0}$ in equation 3.3. Since only one permutation-term, for $\rho(i) = [i + j_0]_k$, contains the term $(-1)^{k-1}\alpha_{i_0,j_0}^k$, we may write $C_{i_0,j_0} = (-1)^{k-1}\alpha_{i_0,j_0}^k + C'_{i_0,j_0}$, where each term in C'_{i_0,j_0} has a factor distinct from α_{i_0,j_0}. But then each term in C'_{i_0,j_0} must have a factor which is less than α_{i_0,j_0}. By the inductive hypothesis, it follows that C'_{i_0,j_0} is zero. As $\sigma^{i_0}\tau^{j_0}$ is independent from all other terms, if must be the case that $\alpha_{i_0,j_0} = 0$. It follows that all $\alpha_{i,j}$ are zero. Therefore, if M is nonzero it must be invertible. □

We can now construct the counterexample.

Proposition 3.24 (Starr). *Now, let k divide n and let $d = n/k$. Let $\mathcal{E}$ be an m dimensional vector space over K, and define $\mathcal{A}$ be the Azumaya algebra* $\mathrm{End}(\mathcal{E}) \otimes D$. *Then there exists a section $F \in \mathrm{H}^0(G_{k,\mathcal{A}}, \mathcal{O}(d))$ for which the variety $\mathbb{V}(F) \subset G_{k,\mathcal{A}}$ has no rational points.*

Proof. With K' defined as above, $\mathcal{A} \otimes_K K' \cong \mathrm{End}(\mathcal{E} \otimes \mathcal{F})$, for some rank k vector space $\mathcal{F}$ over k'. A point of $G_{k,\mathcal{A}}$ is right ideal $\mathcal{I}$ of $\mathrm{End}(\mathcal{E}) \otimes D$. As D is simple, $\mathcal{I}$ must be of the form $\mathcal{I}' \otimes D$. Furthermore, as above $G_{k,\mathcal{A}} \otimes_K K' \cong G(k, \mathcal{E} \otimes \mathcal{F})$. Let $< x_i >$ form a basis for $\mathcal{E}^\vee$, and define $T_i = \bigwedge^k (x_i \otimes \mathcal{F}^\vee) \subset \bigwedge^k (\mathcal{E} \otimes \mathcal{F})^\vee$. Then T_i defines a section of $\mathrm{H}^0(G_{k,\mathcal{A}} \otimes_K K', \mathcal{O}(1))$ which is invariant under the action of $Gal_{K'/K}$, and so descends to a section T'_i of $\mathrm{H}^0(G_{k,\mathcal{A}}, \mathcal{O}(1))$. Furthermore, there are no nontrivial ideals of the form $I \otimes D$ on which all T'_i vanish, and so $\mathbb{V}(T'_1, \ldots, T'_{d^2}) = \emptyset$.

Now, define a degree d homogeneous polynomial on $g_{k,\mathcal{A}}$ by

$$F(T'_1, \ldots, T'_m) = \sum_{\substack{0 \le i < d \\ 0 \le j < d}} \sigma^i \tau^j T'^d_{ik+j+1}.$$

As σ and τ are algebraically independent, $\mathbb{V}(F) = \mathbb{V}(T'_1, \ldots, T'_m)$, and so $\mathbb{V}(F) \subset G_{k,\mathcal{A}}$ has no rational points. $\square$

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