Orbifold Degeneration of Conformally Compact Einstein Metrics

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Abstract of the Dissertation

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In his investigation of the Dirichlet problem for conformally compact Einstein metrics, Anderson showed that there are (at most) three possibilities for the behavior, under subsequences, of a sequence of conformally compact Einstein metrics, with controlled conformal infinities, on a four-manifold: convergence, orbifold degeneration, or cusp formation.

Motivated by this result, we study the phenomenon of orbifold degeneration of a curve of conformally compact Einstein metrics. We start by presenting some background material. After this, we survey the known results concerning the Dirichlet problem, and we address some open questions regarding orbifold degeneration. We then analyze a concrete example of orbifold degeneration, namely, the Taub-bolt family of conformally compact Einstein metrics on the tangent bundle of the two-sphere, and we show that the orbifold Taub-bolt metric is nondegenerate, that is, the kernel

of the Bianchi gauged Einstein operator is trivial for this metric. Finally, we obtain results related to a conjecture of Anderson about the boundary of the completion, in the pointed Gromov-Hausdorff topology, of the space of conformally compact Einstein metrics on a four-manifold. These last results give necessary conditions for orbifold degeneration to occur.

To my loving parents, Salma and Mário.

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Chapter 1

Introduction

Conformal compactifications of Einstein metrics were introduced by Penrose in [27] with the purpose of studying the behavior, at null infinity, of solutions to the vacuum Einstein equations. In the Riemannian setting, the study of conformally compact Einstein metrics began with the work of Fefferman and Graham [15], in connection with their investigation of conformal invariants of Riemannian metrics.

Let M be the interior of a compact manifold $\overline{M}$ with boundary ∂M. A defining function on $\overline{M}$ is a smooth, nonnegative function ρ on $\overline{M}$ with $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ on ∂M. A complete Riemannian metric g on M is said to be conformally compact if there is a defining function ρ on $\overline{M}$ such that the conformally equivalent metric

$$\bar{g} = \rho^2 g$$

extends to a metric on $\overline{M}$.

Given a conformally compact metric g on M, there is a conformal class $[\gamma]$ on ∂M associated with g. This conformal class, called the conformal infinity of g, is obtained by taking γ as the metric on ∂M induced by a compactification $\bar{g}$ of g. Since there are many possible defining functions, there are many conformal compactifications of the metric g. Notice, however, that the metrics induced on ∂M by two distinct compactifications of g are conformally equivalent, and so, the conformal class $[\gamma]$ is uniquely determined by g.

The study of conformally compact Einstein metrics (also called Poincaré-Einstein metrics) gained a lot of interest after the introduction, by Maldacena [22], of the AdS/CFT correspondence in string theory. This correspondence relates gravitational theories on M with conformal field theories on ∂M, cf. [13], [33] and references therein.

Mathematically, an important problem in this area is the Dirichlet problem for conformally compact Einstein metrics: given the topological data $(M, \partial M)$, and a conformal class $[\gamma]$ on ∂M, does there exist a conformally compact Einstein metric g on M whose conformal infinity is $[\gamma]$? If such a metric g exists, is it unique?

The answer for the uniqueness question is, in general, no. The first example of non-uniqueness was found by Hawking and Page [18] in their analysis of the AdS-Schwarzschild metric.

Let $E(M)$ denote the space of conformally compact Einstein metrics on M and let $\mathcal{D}_1$ denote the space of diffeomorphisms of $\overline{M}$ fixing the boundary. We denote by $\mathcal{E}$ the moduli space $E/\mathcal{D}_1$. Also, we let $\mathcal{C}$ denote the space of conformal classes on ∂M. There exists a natural map

$$\Pi : \mathcal{E} \to \mathcal{C}, \quad \Pi(g) = [\gamma],$$

that takes a conformally compact Einstein metric g on M to its conformal infinity $[\gamma]$ on ∂M. Notice that global existence for the Dirichlet problem corresponds to surjectivity of the map Π, and that uniqueness corresponds to injectivity of Π.

In [9], Anderson shows that when $\pi_1(M, \partial M) = 0$, then the space $\mathcal{E}$, if non-empty, is a smooth infinite dimensional manifold. Thus, if M carries some conformally compact Einstein metric, then it also carries a large set of them. Furthermore, he showed that the boundary map Π is a C^∞ smooth Fredholm map of index 0. This implies that Π is a local diffeomorphism in a neighborhood of each regular point.

With regard to the global surjectivity question, a basic property that one needs to understand is whether Π is a proper map. If Π is not proper, it is important to understand exactly what possible degenerations of Poincaré-Einstein metrics can occur with controlled conformal infinity.

It is shown in [8] that for a sequence $\{g_i\}$ of Poincaré-Einstein metrics on a fixed 4-manifold M, with conformal infinities $\gamma_i \subset \Gamma$, where Γ is a compact subset of $\mathcal{C}$, there are (at most) three possibilities for the behavior of $\{g_i\}$ under subsequences:

I. Convergence.

II. Orbifold degeneration.

III. Formation of cusps.

An important remark is that orbifold degeneration can, in fact, occur: the Taub-bolt family is a 1-parameter family of Poincaré-Einstein metrics on $\mathbf{TS}^2$, the tangent bundle of the 2-sphere, that degenerates to a Poincaré-Einstein metric on the orbifold $C(\mathbf{RP}^3)$, the cone $\mathbf{RP}^3$. Furthermore, with the exception of metrics obtained by a connected sum construction due to Mazzeo and Pacard [24] (see Section 2.6 and Section 3.5), this is the only known example of orbifold degeneration of a family of conformally compact Einstein metrics.

As observed in [7], orbifold degenerations of conformally compact Einstein metrics need to be better understood. The main purpose of this dissertation is to obtain a better understanding of this type of degeneration. This is accomplished here by proving some results related to two conjectures of Anderson, which we now explain.

An important differential operator for the study of conformally compact Einstein metrics is the so called Bianchi gauged Einstein operator (see [9, 11], for example). Let us denote by L_g the linearization of this operator at some conformally compact metric g. A Poincaré-Einstein metric g is said to be nondegenerate if the kernel of L_g is trivial. If the kernel of L_g isn't trivial, we say that g is degenerate.

A conjecture made by Anderson says that whenever orbifold degeneration of a sequence of conformally compact Einstein metrics on a 4-manifold occurs, the limit orbifold Poincaré-Einstein metric is degenerate. For a precise statement, see Conjecture 2.23 on Section 2.7. This conjecture is also discussed in [25].

Before we state the next conjecture, we need to introduce some terminology.

The trichotomy I-III above leads one to consider the following approach: instead of working with the space $\mathcal{E}$, one can work with an enlarged space that includes the orbifold and cusps limits. Let then $\bar{\mathcal{E}}$ be the completion of the moduli space $\mathcal{E}$ of Poincaré-Einstein metrics with respect to the pointed Gromov-Hausdorff topology. The map Π extends to a continuous map $\bar{\Pi} : \bar{\mathcal{E}} \to \mathcal{C}$, and $\bar{\Pi}$ is proper.

If $\bar{\mathcal{E}}$ has roughly the structure of a manifold, then one can define a degree $\deg\bar{\Pi}$ associated with each component of $\bar{\mathcal{E}}$. If it happened that $\deg\bar{\Pi} \neq 0$, one would conclude (at least) that almost every choice of conformal class in $\mathcal{C}$ is the conformal infinity of a smooth Poincaré-Einstein metric on M.

Regarding the (point set) topology of $\bar{\mathcal{E}}$, it's conjectured that for any component $\mathcal{E}_0$ of $\mathcal{E}$, $\bar{\Pi}(\partial\mathcal{E}_0)$ has empty interior in $\mathcal{C}$, where $\partial\mathcal{E}_0 = \bar{\mathcal{E}}_0 \setminus \mathcal{E}_0$. This conjecture is stated in [7] (Conjecture 4.2). We also give a precise statement in Section 2.7 (Conjecture 2.24).

Let us now explain how the dissertation is organized. In Chapter 2 we review the background material that will be needed in the later chapters. In Section 2.5 we give a survey of results regarding the Dirichlet problem for conformally compact Einstein metrics. In Section 2.7 we give a precise statement of the result in [8] about the possible behavior under subsequences of a sequence of conformally compact Einstein metrics on a 4-manifold. Also in Section 2.7, we give precise statements of the two conjectures mentioned above.

In Chapter 3 we analyze the orbifold Taub-bolt metric on $C(\mathbf{RP}^3)$. We show that this Poincaré-Einstein orbifold has negative sectional curvature. We then conclude, by invoking a theorem of Koiso [20], that the orbifold Taub-bolt is nondegenerate. This answers Conjecture 2.23 negatively. We finish that chapter by pointing out how to use the Taub-bolt family to obtain more examples of orbifold degeneration.

In Chapter 4 we prove some results related to Conjecture 2.24. Let V be a 4-dimensional orbifold such that there exists a smooth resolution $\pi : M \to V$. Let now h be a nondegenerate conformally compact Einstein metric on V, and suppose that there exists a continuous curve g_t, $t \in (0,1)$, of conformally compact Einstein metrics on M such that (M, g_t) degenerates to (V, h).

Let $[\gamma]$ be the conformal infinity of h. Since we are assuming h to be nondegenerate, there exists a neighborhood W of $[\gamma]$ in $\mathcal{C}$ such that each $[\theta] \in W$ is the conformal infinity of a unique Poincaré-Einstein metric $h^{[\theta]}$ on V, $h^{[\theta]}$ near h (see Theorem 2.15). Suppose it is possible to take W in such a way that for each $[\theta] \in W$, there exists a smooth curve $g_t^{[\theta(t)]}$, $t \in (0,1)$, of Poincaré-Einstein metrics degenerating to $(V, h^{[\theta]})$. Assuming this set up, we prove some results that give necessary conditions for orbifold degeneration to occur. Two of these results are Theorem 4.1 and Theorem 4.10.

Theorem 4.1 says that if there exists $t_0 \in (0,1)$ such that g_t is nondegenerate, for each $t \in (0, t_0)$, then there exist continuous curves $g_1, g_2 : (0,1) \to E(M)$ with

$$\Pi(g_1(s)) = \Pi(g_2(s)),$$

for each $s \in (0,1)$, and $g_1(s_1) \neq g_2(s_2)$, for all $s_1, s_2 \in (0,1)$. In particular, the boundary map Π is not injective.

Theorem 4.10 says the following: suppose there exists $t_0 \in (0,1)$ such that g_t is nondegenerate, for each $t \in (0, t_0)$. For each $t \in (0, t_0)$, consider the map $\Psi_t : \mathbb{S}_\delta^{m,\alpha} \to \mathbb{S}_\delta^{m-2,\alpha}$ (see Definition 2.5), $\delta = 2$, defined by $k \mapsto \Phi_{g_t}(g_t + k)$, where Φ_g denotes the Bianchi gauged Einstein operator with background metric g (see Section 2.4). If there exists $t_1 \in (0, t_0)$ such that the map $k \mapsto \Psi'(k)$ is μ-Lipschitz, for each $t \in (0, t_1)$ (the constant μ independent of t), then the set

$$\{||(L_{g_t})^{-1}||; t \in (0, t_0)\}$$

is unbounded, where L_{g_t} is the linearization of Φ_{g_t} at $k = 0$.

For each $t \in (0, t_0)$, denote by ξ_{g_t} the first eigenvalue of L_{g_t}. We have that

$||(L_{g_t})^{-1}||$ equals $\xi_{g_t}^{-1}$, and hence, ξ_{g_t} bounded below by some positive constant corresponds to $||(L_{g_t})^{-1}||$ bounded above. Therefore, under the assumptions of Theorem 4.10, we conclude that if (M, g_t) degenerates to (V, h), then $\xi_{g_{t_i}} \to 0$ for some subsequence $\{t_i\} \subset (0, 1)$.

Chapter 2

Background

2.1 Hölder spaces

Let U be an open subset of $\mathbf{R}^n$ and let $\alpha \in (0,1)$. Recall that a function $f : U \to \mathbf{R}$ is Hölder continuous with exponent α if there exists a constant C such that

$$|f(x) - f(y)| \leq C|x - y|^\alpha,$$

for any $x, y \in U$. We say that a function $f : U \to \mathbf{R}$ is of class $C^{m,\alpha}$ (written $f \in C^{m,\alpha}$) if all of its derivatives up to order m are continuous and if its m-th derivatives are Hölder continuous with exponent α.

Let M be a smooth manifold (without boundary). A $C^{m,\alpha}$ symmetric bilinear form on M is a symmetric bilinear form v on M satisfying the following: for each $p \in M$, there exists a coordinate chart $(u^1, \ldots, u^n)$ around p such that the components v_{ij} of v in this coordinate chart are of class $C^{m,\alpha}$. A $C^{m,\alpha}$ Riemannian metric on M is a positive definite $C^{m,\alpha}$ symmetric bilinear form.

It may happen that the components of a $C^{m,\alpha}$ Riemannian metric in a particular coordinate chart are not of class $C^{m,\alpha}$. There are examples, for instance, of $C^{m,\alpha}$ metrics whose components in normal coordinates are not of class $C^{m,\alpha}$. It is important, therefore, to work with coordinates which, besides being adapted to the geometry of (M, g), have good analytic properties. This leads us to the introduction of harmonic coordinates.

Let g be a Riemannian metric on M of class $C^{m,\alpha}$, $m \geq 1$. The coordinates $(u^1, \ldots, u^n)$ are said to be harmonic with respect to the metric g (or g-harmonic) if

$$\Delta_g u^i = 0$$

for each $i = 1, \ldots, n$, where Δ_g is the Laplacian of g.

We now state some results regarding harmonic coordinates. The first result concerns the existence of such coordinates, while the second states that, in some sense, harmonic coordinates have optimal regularity. These results are proved in [14].

Proposition 2.1. *Let the metric on a Riemannian manifold (M, g) be of class $C^{m,\alpha}$ in a local coordinate chart about some point p. Then there is a neighborhood of p in which harmonic coordinates exist, these new coordinates being $C^{m+1,\alpha}$ functions of the original coordinates. Moreover, all harmonic charts defined near p have this regularity.*

Theorem 2.2. *Let the metric g be of class $C^{m,\alpha}$ in the coordinates $(u^1, \ldots, u^n)$. If a tensor T is of class $C^{l,\beta}$, $l \geq m$ and $\beta \geq \alpha$, in the coordinates $(u^1, \ldots, u^n)$, then T is of class $C^{l,\beta}$ in harmonic g-coordinates.*

Corollary 2.3. *If a metric g is of class $C^{m,\alpha}$, $m \geq 1$, in some coordinate chart, then it is also of class $C^{m,\alpha}$ in harmonic coordinates.*

Given a Riemannian manifold (M, g), we denote by $\zeta^{m,\alpha}(x)$ the $C^{m,\alpha}$ harmonic radius at $x \in M$, cf. [3]. This is the largest radius such that, for any $r < \zeta^{m,\alpha}(x)$, the geodesic ball $B(x, r)$ has harmonic coordinates in which the metric components g_{ij} satisfy

$$Q^{-1} I \leq g \leq QI \tag{2.1}$$

and

$$\begin{aligned} &\sum_{1 \leq |\beta| \leq m} r^{|\beta|} \sup_{y \in B(x,r)} |\partial^\beta g_{ij}(y)| \\ &+ \sum_{|\beta| = m} r^{m+\alpha} \sup_{y_1, y_2 \in B(x,r)} \frac{|\partial^\beta g_{ij}(y_1) - \partial^\beta g_{ij}(y_2)|}{|y_1 - y_2|^\alpha} \leq Q - 1. \end{aligned} \tag{2.2}$$

Here $Q > 1$ is a constant (close to 1) fixed once and for all.

Let g be a $C^{m,\alpha}$ metric on the smooth manifold M, and suppose there exist positive real numbers Λ and i_0 such that the following bounds hold:

$$||\nabla^{m-1} \mathrm{Ric}||_{L^\infty} \leq \Lambda \text{ and } \mathrm{inj}_{(M,g)} \geq i_0,$$

where $\mathrm{inj}_{(M,g)}$ denotes the injectivity radius of (M, g). It is proved in [3] that for such a Riemannian manifold, there exists a lower bound $\zeta_0 > 0$ for $\zeta^{m,\alpha}(x)$, that is,

$$\zeta^{m,\alpha}(x) \geq \zeta_0$$

for each $x \in M$. Moreover, the bound ζ_0 depends only on the bounds Λ and i_0.

Given a smooth manifold M, we say that a family Ω_λ of open subsets of M is a uniformly locally finite covering of M if the following holds: Ω_λ is a covering of M and there exists a positive integer N such that each point $x \in M$ has a neighborhood which intersects at most N of the Ω_λ's. One then has the following result (for a proof, see for example [19]):

Lemma 2.4. *Let M be a smooth n-dimensional manifold and let g be a Riemannian metric of class $C^{m,\alpha}$, $m \geq 1$. Suppose (M, g) is complete with Ricci curvature bounded below by some real number κ, and let $r_0 > 0$ be given. There exists a sequence $\{x_i\}$ of points of M such that for any $r \geq r_0$, the following hold:*

(i) the family $(B(x_i, r))$ is a uniformly locally finite covering of M, and there is a upper bound for N in terms of n, r_0, r and κ;

(ii) for each $i \neq j$, $B(x_i, r_0/2)$ and $B(x_j, r_0/2)$ are disjoint.

Here, $B(x, r)$ denotes the geodesic ball of center $x \in M$ and radius r.

For a smooth manifold M, let $\mathbb{M}^{m,\alpha}$, $m \geq 1$, denote the space of $C^{m,\alpha}$ Riemannian metrics on M with $||\nabla^{m-1}\mathrm{Ric}||_{L^\infty}$ bounded above and $\mathrm{inj}_{(M,g)}$ bounded below (by some positive real numbers depending on g). We are now in a position to define the $C^{m,\alpha}$ topology on the space $\mathbb{M}^{m,\alpha}$.

Let then g be an element of $\mathbb{M}^{m,\alpha}$. By Lemma 2.4, it is possible to choose a uniformly locally finite covering of (M, g) consisting of a collection of geodesic balls $B(x_i, \zeta_0/2)$ such that the balls $B(x_i, \zeta_0/4)$ are pairwise disjoint. Now let g' be another element of $\mathbb{M}^{m,\alpha}$, so that $v = g - g'$ is a $C^{m,\alpha}$ symmetric bilinear form. We define

$$||g'||_{C^{m,\alpha}(g)} = ||v||_{C^{m,\alpha}(g)},$$

with

$$\begin{aligned} ||v||_{C^{m,\alpha}(g)} = \quad & \sup_{x_i} \left\{ \sum_{1\leq|\beta|\leq m} \zeta_0^{|\beta|} \sup_{y\in B(x_i,\zeta_0)} |\partial^\beta v_{ij}(y)| \right. \\ & \left. + \sum_{|\beta|=m} \zeta_0^{m+\alpha} \sup_{y_1,y_2\in B(x_i,\zeta_0)} \frac{|\partial^\beta v_{ij}(y_1) - \partial^\beta v_{ij}(y_2)|}{|y_1 - y_2|^\alpha} \right\}, \end{aligned}$$

where the components v_{ij} are taken in local g-harmonic coordinates satisfying the bounds (2.1) and (2.2).

The norms $|| \, ||_{C^{m,\alpha}(g)}$ define the $C^{m,\alpha}$ topology on the space $\mathbb{M}^{m,\alpha}$ by defining the open balls centered at g in the usual way.

So far, we were assuming that $\partial M = \emptyset$. Let us now briefly discuss the case of a compact manifold with boundary.

Let then $\overline{M}$ be a compact manifold with nonempty boundary ∂M, let g be a Riemannian metric on $\overline{M}$, and let γ be the metric that g induces on ∂M. Let $p \in \partial M$ and let $(u^1, \ldots, u^n)$ be coordinates, of class at least C^1, defined on a neighborhood U of p. We say that the coordinates $(u^1, \ldots, u^n)$ are boundary harmonic coordinates if $\Delta_g u^j = 0$, $j = 1, \ldots, n$; the coordinates $(\hat{u}^1, \ldots, \hat{u}^{(n-1)})$, where $\hat{u}^j = u^j|_{\partial M}$, are harmonic coordinates on $(\partial M, \gamma)$; and $u^n = 0$ on $U \cap \partial M$.

In the context of a compact manifold with boundary $(\overline{M}, g)$, the injectivity radius is defined as the supremum, among all positive numbers i for which

$$\exp_p : B(0,r) \to M,$$

where $B(0,r) = \{w \in T_pM;\ g(w,w) < r^2\}$, is a diffeomorphism for $r = i$ if $\mathrm{dist}(p, \partial M) \geq i$ and it is a diffeomorphism for $r = \mathrm{dist}(p, \partial M)$ if $\mathrm{dist}(p, \partial M) \leq i$.

Let us also use the notation $\mathbb{M}^{m,\alpha}$ to denote the space of $C^{m,\alpha}$ Riemannian metrics on $\overline{M}$ with $||\nabla^{m-1}\mathrm{Ric}||_{L^\infty}$ bounded above and with $\mathrm{inj}_{(\overline{M},g)}$ bounded below (by positive constants depending on g). In a way similar to the case of a manifold M without boundary, we can define the $C^{m,\alpha}(\overline{M})$ topology, $m \geq 1$, on the space $\mathbb{M}^{m,\alpha}$. For more details, we refer the reader to [3].

2.2 Conformally compact Einstein metrics

Let M be the interior of a compact $(n+1)$-dimensional manifold $\overline{M}$ with boundary ∂M. Denote by $C^{m,\alpha}(\overline{M})$ the space of functions on $\overline{M}$ with m derivatives that are Hölder continuous of degree α up to the boundary in each background coordinate chart.

Recall that a function ρ on $\overline{M}$ is called a defining function if ρ is a smooth, nonnegative function on $\overline{M}$ with $\rho^{-1}(0) = \partial M$ and $d\rho \neq 0$ on ∂M. A complete Riemannian metric g on M is $C^{m,\alpha}$ conformally compact if there is a defining function ρ on $\overline{M}$ such that the conformally equivalent metric

$$\bar{g} = \rho^2 g \tag{2.3}$$

extends to a $C^{m,\alpha}$ metric on the compactification $\overline{M}$. The induced metric $\gamma = \bar{g}|_{\partial M}$

is the boundary metric associated to the compactification $\bar{g}$. If ρ_1 and ρ_2 are distinct defining functions, then

$$\bar{g}_1 = \rho_1^2 g \text{ and } \bar{g}_2 = \rho_2^2 g$$

are distinct metrics on $\overline{M}$. Consider the function f on $\overline{M}$ given by

$$f = (\rho_2 \rho_1^{-1})^2.$$

The conditions $\rho_j^{-1}(0) = \partial M$ and $d\rho_j \neq 0$ on ∂M, $j = 1, 2$, imply that f is well defined and $f > 0$ on $\overline{M}$. Notice that

$$\bar{g}_2 = f \bar{g}_1.$$

Thus, γ_1 and γ_2, the boundary metrics associated with $\bar{g}_1$ and $\bar{g}_2$, respectively, satisfy

$$\gamma_2 = f|_{\partial M} \gamma_1,$$

that is, γ_1 and γ_2 are conformally equivalent. This shows that the conformal class $[\gamma]$ of γ on ∂M is uniquely determined by (M, g). The conformal class $[\gamma]$ is called the conformal infinity of g.

We denote by $\mathrm{Met}^{m,\alpha}(\partial M)$ the space of $C^{m,\alpha}$ metrics on ∂M, and we give this space the $C^{m,\alpha'}$ topology, for a fixed $\alpha' < \alpha$, so that bounded sequences in the $C^{m,\alpha'}$ norm have convergent subsequences. The corresponding space of pointwise conformal classes is denoted $\mathcal{C}^{m,\alpha}$. Next, let $\mathbb{S}^{k,\beta}(M)$ be the Banach space of $C^{k,\beta}$ symmetric bilinear forms on M, and let $\mathbb{S}^{k,\beta}(\overline{M})$ be the corresponding space of symmetric bilinear forms on the closure $\overline{M}$, again with the $C^{k,\beta'}$ topology, $\beta' < \beta$.

Fix a smooth defining function ρ and define a function $r = r(\rho)$ by

$$r = -\log\left(\frac{\rho}{2}\right).$$

Consider a complete Riemannian metric g of bounded geometry on M, i.e. g has bounded sectional curvature and injectivity radius bounded below on M.

Definition 2.5. *The weighted Hölder space $\mathbb{S}^{k,\beta}_\delta(M) = \mathbb{S}^{k,\beta}_\delta(M, g)$ is the Banach space of symmetric bilinear forms f on M such that*

$$f = e^{-\delta r} f_0,$$

where $f_0 \in \mathbb{S}^{k,\beta}(M)$ satisfies $||f_0||_{C^{k,\beta}(M,g)} \leq C$, for some constant $C < \infty$. The *weighted norm of f is defined by*

$$||f||_{C^{k,\beta}_{\delta}(M)} = ||f_0||_{C^{k,\beta}(M)}.$$

In this dissertation, we are mainly concerned with conformally compact Einstein metrics g, normalized so that

$$\mathrm{Ric}_g = -ng.$$

By relating the curvatures of g and $\bar{g}$, it is not hard to show that if the Einstein metric g is at least C^2 conformally compact, then the sectional curvature K_g of g satisfies

$$|K_g + 1| = O(\rho^2). \tag{2.4}$$

Thus, conformally compact Einstein metrics generalize the Poincaré model of hyperbolic space $\mathbf{H}^{n+1}$. For this reason, these metrics are frequently called asymptotically hyperbolic (AH), or also Poincaré-Einstein, and these three different names will be used interchangeably in this dissertation.

The decay (2.4) suggests that a natural choice for δ is

$$\delta = 2.$$

We fix this choice of δ for the rest of the dissertation. For more details about weighted Hölder spaces and this natural choice of δ, see the discussion in Section 4 of [9].

Let $E^{m,\alpha}(M)$ be the space of Poincaré-Einstein metrics on M which admit a C^2 conformal compactification as in (2.3), with $C^{m,\alpha}$ boundary metric γ on ∂M. Here, $0 < \alpha < 1$, $m \geq 3$, and we allow $m = \infty$. The space $E^{m,\alpha}$ is given the $C^{m,\alpha'}$ topology on metrics on $\overline{M}$, for any $\alpha' < \alpha$, via a fixed conformal compactification as in (2.3). Let

$$\mathcal{E}^{m,\alpha}(M) = E^{m,\alpha}(M)/\mathcal{D}_1^{m+1,\alpha}(\overline{M}),$$

where $\mathcal{D}_1^{m+1,\alpha}(\overline{M})$ is the group of $C^{m+1,\alpha}$ diffeomorphisms of $\overline{M}$ inducing the identity on ∂M, acting on $E^{m,\alpha}$ in the usual way by pullback. Similar spaces can by defined in the case of an orbifold V (our definition of orbifold is given in Section 2.3). We choose, however, to use the notation $F^{m,\alpha}(V)$ and $\mathcal{F}^{m,\alpha}(V)$ when working with an orbifold.

The natural map

$$\Pi_{\mathcal{E}} : \mathcal{E}^{m,\alpha} \to \mathcal{C}^{m,\alpha}, \quad \Pi[g] = [\gamma],$$

called Dirichlet boundary map, takes a Poincaré-Einstein metric g on M to its conformal infinity $[\gamma]$ on ∂M (when working with orbifolds, we denote the boundary map by $\Pi_{\mathcal{F}} : \mathcal{F}^{m,\alpha} \to \mathcal{C}^{m,\alpha}$). Regarding the boundary map Π, a natural question that arises is the Dirichlet Problem for Poincaré-Einstein metrics: given the topological data $(M, \partial M)$, and a conformal class $[\gamma]$ on ∂M, does there exist a conformally compact Einstein metric on M whose conformal infinity is $[\gamma]$? If such a metric exists, is it unique? Of course, the Dirichlet problem can be formulated in terms of the boundary map Π, with existence corresponding to surjectivity Π and uniqueness corresponding to injectivity of Π. We will discuss some results related to the Dirichlet problem in Section 2.5.

2.3 Orbifolds

We introduce now some concepts related to orbifolds. We remark that our definition of orbifolds (see [8], [9] or [6]) is more restrictive than the general definition due to Thurston [29].

Definition 2.6. *A topological space V (respectively, $\overline{V}$) is an $(n+1)$-dimensional orbifold (respectively, $\overline{V}$ is an $(n+1)$-dimensional orbifold with boundary) if V is an $(n+1)$-dimensional manifold (respectively, $\overline{V}$ is a $(n+1)$-dimensional manifold with boundary) away from finitely many singular points $\{q_1, \dots, q_k\}$ in the interior, each having a neighborhood homeomorphic to a cone on a spherical space form.*

The subset $V \setminus \{q_1, \dots, q_k\}$, that is, the complement of the singular set, is called the regular set of V, and is denoted by V_{reg}.

Definition 2.7. *A $C^{m,\alpha}$ metric on a $(n+1)$-dimensional orbifold V is a $C^{m,\alpha}$ metric h on V_{reg} such that, in a local uniformization $\mathbf{B}^{n+1} \setminus \{0\}$ of each cone, h extends to a $C^{m,\alpha}$ metric on the ball $\mathbf{B}^{n+1}$.*

The metric h is a $C^{m,\alpha}$ Einstein metric on V if $h|_{V_{\text{reg}}}$ is a $C^{m,\alpha}$ Einstein metric on the smooth manifold V_{reg}.

If V is the interior of a $(n+1)$-dimensional compact orbifold with boundary, there is the notion of $C^{m,\alpha}$ conformally compact metric on V, which is totally analogous to the case of a smooth manifold M. There are also spaces similar to the spaces $E^{m,\alpha}(M)$ and $\mathcal{E}^{m,\alpha}(M)$ defined previously for a smooth manifold M. We use the notation $F^{m,\alpha}(V)$ and $\mathcal{F}^{m,\alpha}(V)$ when referring to these spaces, and the natural boundary map is denoted by

$$\Pi_{\mathcal{F}} : \mathcal{F}^{m,\alpha}(V) \to \mathcal{C}^{m,\alpha}(\partial V).$$

For the next two definitions, we restrict ourselves to dimension four.

Definition 2.8. *A pair (M, π), where M is a 4-dimensional smooth manifold and $\pi : M \to V$ is a continuous map, is said to be a smooth resolution of V if*

$$\pi|_{\pi^{-1}(V_{\mathrm{reg}})} : \pi^{-1}(V_{\mathrm{reg}}) \to V_{\mathrm{reg}}$$

is a diffeomorphism and each $\pi^{-1}(q_i)$ is a connected 2-dimensional CW complex in M, where $\{q_1, \ldots, q_k\}$ is the singular set of V.

Definition 2.9. *An orbifold singular Einstein metric on a smooth 4-dimensional manifold M is a symmetric bilinear form of the form $\pi^*(h)$, where (M, π) is a smooth resolution of an orbifold V and h an Einstein metric on V.*

Before we move to the next section, we would like to consider some explicit examples of Poincaré-Einstein metrics defined on a orbifold V.

Let $\overline{D}$ be the unit disk in $\mathbf{R}^{n+1}$, that is,

$$\overline{D} = \{x \in \mathbf{R}^{n+1}; |x| \leq 1\}.$$

The group $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ acts on $\overline{D}$ in the usual way:

$$\bar{0} \cdot x = x \text{ and } \bar{1} \cdot x = -x.$$

By taking the quotient of $\overline{D}$ by this action, we get an orbifold with boundary. There is only one orbifold singularity, which corresponds to the origin in $\mathbf{R}^{n+1}$. The interior of this orbifold, that is, the points corresponding to the interior D of $\overline{D}$, is called cone $\mathbf{RP}^3$, and denoted $C(\mathbf{RP}^3)$.

One can consider, for example, the "Euclidean" metric h_0 on the orbifold $C(\mathbf{RP}^3)$, that is, the metric induced from the Euclidean metric g_0 on D. An explicit expression for h_0 is

$$h_0 = dr^2 + r^2 g_{\mathbf{RP}^3}, \tag{2.5}$$

where, for $[x] \in C(\mathbf{RP}^3)$, r denotes the Euclidean distance from x to the center of D, and $g_{\mathbf{RP}^3}$ is the metric induced on $\mathbf{RP}^3$ by the round metric on the unit sphere $\mathbf{S}^n(1)$.

By considering conformally compact Einstein metrics on D which are symmetric with respect to the action of $\mathbb{Z}_2$, we can find conformally compact Einstein metrics on $C(\mathbf{RP}^3)$. One obvious example is then the metric obtained from the Poincaré model for the hyperbolic space (D, g_{-1}). We will denote this metric by h_{-1}, and will also use the terminology *hyperbolic metric* when referring to it. An explicit expression for this metric is

$$h_{-1} = \frac{4}{(1-r^2)^2} h_0,$$

where h_0 is the "Euclidean" metric (2.5). Another example of this type is the orbifold Taub-bolt metric, denoted by h_{TB}. This metric will be studied in detail in Chapter 3.

2.4 The Bianchi gauged Einstein operator

Consider the operator $E : \mathbb{M}^{m,\alpha} \to \mathbb{S}^{m-2,\alpha}$ given by

$$E(g) = \mathrm{Ric}_g + ng. \tag{2.6}$$

We call this operator the Einstein operator.

Suppose we have a $C^{m,\alpha}$ metric $\tilde{g}$ and we would like to find an Einstein metric g (that is, a metric g such that $E(g) = 0$), close to $\tilde{g}$ in the $C^{m,\alpha}$ topology, and such that g and $\tilde{g}$ have the same conformal infinity. This corresponds to finding $f \in \mathbb{S}^{m,\alpha}_{\delta}$ (see Definition 2.5), $\delta = 2$, with $||f||_{C^{m,\alpha}(\tilde{g})}$ sufficiently small (so that $\tilde{g}+f$ is a Riemannian metric) such that

$$E(\tilde{g} + f) = 0. \tag{2.7}$$

Equation (2.7) is a nonlinear, very complicated equation. It is natural then to consider the linearization of E at $\tilde{g}$.

Recall that if ω is a 1-form, then $(\delta^{\tilde{g}})^*\omega$ is its symmetrized covariant derivative. In coordinates,

$$((\delta^{\tilde{g}})^*\omega)_{ij} = \frac{1}{2}(\omega_{i;j} + \omega_{j;i}).$$

Its formal adjoint acting on symmetric 2-tensors is the divergence operator $\delta^{\tilde{g}}$,

$$\delta^{\tilde{g}} k_j = -k^{j}_{i;j}.$$

The linearization of the operator E at $\tilde{g}$ is given by

$$(DE)_{\tilde{g}} \cdot k = \frac{1}{2}\Delta^{\tilde{g}}_L k - (\delta^{\tilde{g}})^*\delta^{\tilde{g}} k - \frac{1}{2}\nabla^{\tilde{g}} d(\mathrm{tr}_{\tilde{g}} k) + nk, \tag{2.8}$$

where the Lichnerowicz Laplacian $\Delta_L^{\tilde g}$ is defined by

$$\Delta_L^{\tilde g} k = (\nabla^{\tilde g})^* \nabla^{\tilde g} k + \mathrm{Ric}^{\tilde g} \circ k + k \circ \mathrm{Ric}^{\tilde g} - 2\mathring{R}^{\tilde g} k.$$

Here,

$$(\mathring{R}^{\tilde g} k)_{ij} = \mathrm{Rm}_{ipjq} k^{pq},\ \ \mathrm{Ric}^{\tilde g} \circ k = \mathrm{Ric}_i{}^p k_{pj},\ \ k \circ \mathrm{Ric}^{\tilde g} = k_i{}^p \mathrm{Ric}_{pj},$$

and all curvatures are computed with respect to $\tilde g$.

Due to the diffeomorphism invariance of (2.6), the linearization (2.8) is not an elliptic operator. To remedy this problem, one must choose some gauge condition. A good choice in this situation is the so called Bianchi gauge. We will then look for a metric g, close to $\tilde g$, such that

$$\begin{aligned} \mathrm{Ric}_g + ng &= 0 \\ \delta^{\tilde g} g + \frac{1}{2} d(\mathrm{tr}_{\tilde g} g) &= 0. \end{aligned} \tag{2.9}$$

This system is elliptic in the sense of Agmon-Douglis-Nirenberg [1, 2], but it is more convenient to work with the single equation $\Phi_{\tilde g}(g) = 0$, where

$$\Phi_{\tilde g}(g) = \mathrm{Ric}_g + ng + (\delta^g)^* \left(\delta^{\tilde g} g + \frac{1}{2} d(\mathrm{tr}_{\tilde g} g) \right). \tag{2.10}$$

In the case of interest to us, system (2.9) is equivalent to equation (2.10). Indeed, (2.9) obviously implies $\Phi_{\tilde g}(g) = 0$. The converse is provided by the following lemma (for a proof, see [11]).

Lemma 2.10. *Suppose $\Phi_{\tilde g}(g) = 0$ and $\mathrm{Ric}_g < 0$. If $|\delta^{\tilde g} g + \frac{1}{2} d(\mathrm{tr}_{\tilde g} g)|(x) \to 0$ as $x \to \partial M$, then any solution of $\Phi_{\tilde g}(g) = 0$ is a solution of system (2.9).*

By considering the linearization of $2\Phi_{\tilde g}$ at $g = \tilde g$, we get an operator

$$L_{\tilde g} : \mathbb{S}_\delta^{m,\alpha} \to \mathbb{S}_\delta^{m-2,\alpha},$$

(recall that we are taking $\delta = 2$) which is given by the simple expression

$$L_{\tilde g}(k) = (\nabla^{\tilde g})^* \nabla^{\tilde g} k + \mathrm{Ric}^{\tilde g} \circ k + k \circ \mathrm{Ric}^{\tilde g} - 2\mathring{R}^{\tilde g} k. \tag{2.11}$$

This is an elliptic operator. By [16], it is Fredholm, and thus it has finite dimensional kernel and cokernel. If the metric $\tilde g$ is Einstein, that is, if $\mathrm{Ric}_{\tilde g} + n\tilde g = 0$, equation

(2.11) simplifies to

$$L_{\tilde{g}}(k) = (\nabla^{\tilde{g}})^* \nabla^{\tilde{g}} k - 2\mathring{R}^{\tilde{g}} k.$$

Definition 2.11. *We say that the Einstein metric g is nondegenerate if the kernel of the Fredholm operator*

$$L_g : \mathbb{S}_\delta^{m,\alpha} \longrightarrow \mathbb{S}_\delta^{m-2,\alpha},$$

(with $\delta = 2$) is trivial. If the kernel of L_g isn't trivial, we say that the Einstein metric g is degenerate.

We now state a theorem of Koiso [20] about the nondegeneracy of Einstein metrics of negative sectional curvature. The original theorem is stated for compact manifolds, but it is also true for conformally compact manifolds (see, for example, Chapter 8 of [21]).

Theorem 2.12. *Let (M,g) be a (conformally) compact Einstein manifold satisfying*

$$Ric_g = \lambda g.$$

Let β_0 be the largest eigenvalue of the operator $\mathring{R}$ on trace-free symmetric 2-tensor fields. If

$$\beta_0 < \max\left\{-\lambda, \frac{\lambda}{2}\right\},$$

then g is nondegenerate.

Corollary 2.13. *Let g be a $C^{m,\alpha}$ conformally compact Einstein metric on the smooth manifold M. If g has negative sectional curvature, then g is nondegenerate.*

Even though the concepts and statements in this section refer to a smooth manifold M, they also make sense, and are true, in the context of an orbifold V. This is due to the simple nature of the orbifold singularities.

2.5 Some results about the Dirichlet problem

In this section we survey some important results regarding the Dirichlet Problem for Poincaré-Einstein metrics. In [16], Graham and Lee proved the following perturbation result:

Theorem 2.14. *Let $\mathbf{B}^{n+1}$, $n \geq 3$, be the $(n+1)$-dimensional ball and let γ_1 be the standard metric on the sphere $\mathbf{S}^n$. For any smooth Riemannian metric γ on $\mathbf{S}^n$ which* *is sufficiently close to γ_1 in $C^{2,\alpha}$ norm if $n \geq 4$, or $C^{3,\alpha}$ norm if $n = 3$, for some $0 < \alpha < 1$, there exists a smooth metric g in $\mathbf{B}^{n+1}$ such that*

(i) $\mathrm{Ric}_g = -ng$,

(ii) g has $[\gamma]$ as conformal infinity.

In [11], Biquard generalizes Theorem 2.14 to the setting of asymptotically symmetric Einstein metrics. A generalization of Theorem 2.14 was also given by Lee in [21].

The following result proved by Anderson in [9] generalizes all the results mentioned on this section so far. This result will be very important for us later on in this dissertation. We will state here the version of the result for smooth manifolds, but as observed in [7] (Remark 4.1), the theorem is also valid for orbifolds.

Theorem 2.15. *Let M be a compact, oriented 4-manifold with boundary ∂M satisfying $\pi_1(M, \partial M) = 0$. If for a given (m, α), $m \geq 3$, $\mathcal{E}^{m,\alpha}$ is non-empty, then $\mathcal{E}^{m,\alpha}$ is a C^∞ smooth infinite dimensional separable Banach manifold. Furthermore, the boundary map*

$$\Pi : \mathcal{E}^{m,\alpha} \to \mathcal{C}^{m,\alpha}$$

is a C^∞ smooth Fredholm map of index 0.

Implicit in Theorem 2.15 is the boundary regularity statement that a conformally compact Einstein metric with $C^{m,\alpha}$ conformal infinity has a $C^{m,\alpha}$ compactification. Versions of Theorem 2.15 also hold in dimension $n > 4$; see Theorem 5.5 and Theorem 5.6 in [9] for precise statements.

Let $\mathcal{C}^o$ be the space of nonnegative conformal classes $[\gamma]$ on ∂M, in the sense that $[\gamma]$ has a non-flat representative γ of nonnegative scalar curvature. Let $\mathcal{E}^o = \Pi^{-1}(\mathcal{C}^o)$ be the space of AH Einstein metrics on M with conformal infinity in $\mathcal{C}^o$. Thus, one has the restricted boundary map

$$\Pi^o = \Pi|_{\mathcal{E}^{m,\alpha}} : \mathcal{E}^o \to \mathcal{C}^o.$$

Theorem 2.16. *Le M be a 4-manifold satisfying $\pi_1(M, \partial M) = 0$ and for which the inclusion $\iota : \partial M \to M$ induces a surjection*

$$H_2(\partial M, \mathbb{F}) \to H_2(\overline{M}, \mathbb{F}) \to 0,$$

for all fields $\mathbb{F}$. Then, for any (m,α), $m \geq 4$, the boundary map

$$\Pi^o : \mathcal{E}^o \to \mathcal{C}^o$$

is proper.

The theorem above is proved in [8]. Together with a result of Smale [28], it implies that Π^o has a well-defined mod 2 degree, $\deg_2 \Pi^o \in \mathbb{Z}_2$, on each component of $\mathcal{E}^o$. In fact, building on work of Tromba [30] and White [31, 32], Anderson shows (Theorem 6.1 of [8]) that Π^o has a $\mathbb{Z}$-valued degree

$$\deg \Pi^o \in \mathbb{Z},$$

again on components of $\mathcal{E}^o$. Of course, if $\deg \Pi^o \neq 0$, then Π^o is surjective.

Regarding the surjectivity of the boundary map Π, there is the following nice result, also due to Anderson [8].

Theorem 2.17. *Let $M = \mathbf{B}^4$ be the 4-ball, with $\partial M = \mathbf{S}^3$, and let $\mathcal{C}^o$ be the component of the non-negative $C^{m,\alpha}$ conformal classes containing the round metric on $\mathbf{S}^3$. Also, let $\mathcal{E}^o$ be the component of $\Pi^{-1}(\mathcal{C}^o)$ containing the Poincaré metric on $\mathbf{B}^4$. Then*

$$\deg_{\mathbf{B}^4} \Pi^o = 1.$$

In particular, for any (m,α), $m \geq 4$, any conformal class $[\gamma] \in \mathcal{C}^o$ on $\mathbf{S}^3$ is the conformal infinity of a AH Einstein metric on $\mathbf{B}^4$.

The previous result ties nicely with a recent result of Marques [23], which we now state.

Theorem 2.18. *The space of positive scalar curvature metrics on the 3-sphere is path-connected in the C^∞ topology.*

This theorem is actually a corollary of a more general result. We refer the reader to [23] for details.

Taken together, Theorem 2.17 and Theorem 2.18 imply that if $[\gamma] \in \mathcal{C}^\infty(\mathbf{S}^3)$ is a positive conformal class, that is, a class that has a representative γ of positive scalar curvature, then there exists a Poincaré-Einstein metric g on $\mathbf{B}^4$ such that the conformal infinity of g is $[\gamma]$. In other words, the boundary map

$$\Pi : \mathcal{E}^\infty(\mathbf{B}^4) \to \mathcal{C}^\infty(\mathbf{S}^3)$$

is surjective. It is still an open question to determine if this map is injective.

2.6 A boundary connected sum result

Another important result is the one obtained by Mazzeo and Pacard in [24]. We explain their result in this section.

Let M_j be the interior of a compact $(n+1)$-manifold $\overline{M}_j$ with boundary ∂M_j, $j = 1, 2$. Fix a point $p_j \in \partial M_j$ and excise a small open half-ball $B_+(p_j)$, $j = 1, 2$. The boundary connected sum $\overline{M}_1 \#_b \overline{M}_2$ is the compact manifold $\overline{M}$ obtained by identifying the hemispherical portions of the boundaries of $B_+(p_1)$ and $B_+(p_2)$. The interior M of $\overline{M}$ is denoted by $M_1 \#_b M_2$. Notice that the boundary of $\overline{M}_1 \#_b \overline{M}_2$ is the connected sum $\partial M_1 \# \partial M_2$. For simplicity, we let ρ denote a defining function for whichever manifold we are considering at a particular moment (i.e. M_1, M_2 or $M_1 \#_b M_2$).

In the statement of the main result in [24], it is assumed that the conformally compact Einstein metrics there considered are weakly nondegenerate (see the discussion on pages 392-394 of [24]). The weak nondegeneracy property corresponds to a unique continuation property at infinity for solutions of the linearized Einstein equations (see [9], Remark 3.2). Fortunately, the unique continuation property was proved to always hold [10, 12]. We then have the following version of main theorem in [24]:

Theorem 2.19. *If (M_1, g_1) and (M_2, g_2) are conformally compact Einstein manifolds, then the manifold $M = M_1 \#_b M_2$ carries a family of Poincaré-Einstein metrics g_ϵ with the following two properties:*

(a) the restriction of g_ϵ to $M_j - B_+(p_j)$ converges to g_j;

(b) the restriction of $\bar{g}_\epsilon = \rho^2 g_\epsilon$ to $\partial M_j - (B_+(p_j) \cap \partial M_j)$ converges to $\bar{g}_j = \rho^2 g_j$.

The convergence in either case is polynomial in a geometrically natural parameter ϵ.

The convergence in (a) is with respect to some (weighted scale invariant) Hölder space of order $(2, \alpha)$, and the convergence in (b) is with respect to the $C^{2,\alpha}(\partial M)$ topology.

It is worthwhile to say a few words about the proof of Theorem 2.19. The proof has two steps. The first step consists of constructing an approximate solution $\tilde{g}_\epsilon$. The second step, which is the technical one, consists of showing that it is possible to perturb the metric $\tilde{g}_\epsilon$ to obtain an exact solution g_ϵ of the Bianchi gauged Einstein operator.

Let γ_j be a representative of the conformal infinity of (M_j, g_j), $j = 1, 2$. The metric γ_1 determines a unique geodesic defining function x. Fix normal coordinates y centered at $p_1 \in \partial M_1$ and consider the boundary normal coordinates $u = (x, y)$ around $p_1 \in \overline{M}_1$. Consider also boundary normal coordinates $u' = (x', y')$ around $p_2 \in \overline{M}_2$, constructed in a similar way.

Let A_ϵ and A'_ϵ denote the annuli $\{\epsilon/2 \le |u| \le 2\epsilon\}$ and $\{\epsilon/2 \le |u'| \le 2\epsilon\}$, respectively. Identifying these annuli by means of the inversion map $u' = I_\epsilon(u)$, where $I_\epsilon(u)$ is defined by $I_\epsilon(u) = \epsilon^2 u/|u|^2$, we define the smooth manifold with boundary

$$\overline{M}_\epsilon = \left(\overline{M}_1 - B_{\epsilon/2}(p_1)\right) \bigcup_{I_\epsilon} \left(\overline{M}_2 - B_{\epsilon/2}(p_2)\right).$$

It is convenient to use a rescaling of the coordinate systems u and u', so that we may regard the gluing region (also referred to as neck region or connected sum region) as a fixed annulus A. Consider then the dilation T_ϵ that sends u to ϵu (and u' to $\epsilon u'$). The annuli A and A' of inner and outer radii $1/2$ and 2 in the u, u' coordinates are mapped by T_ϵ to A_ϵ and A'_ϵ, respectively, and are identified by the fixed inversion $I(u) = u/|u|^2$.

The metrics $g_{j,\epsilon} = T_\epsilon^*(g_j)$ are defined on the half-ball of radius C/ϵ for some $C > 0$; these are just isometric forms of the initial metrics g_j. Let now χ be a nonnegative, smooth cutoff function which equals 1 for $r = |w| \ge 2$ and vanishes for $r \le 1/2$. Our approximate solution is then the metric $\tilde{g}_\epsilon$ defined on the interior M_ϵ of $\overline{M}_\epsilon$ by

$$\tilde{g}_\epsilon = \chi(r) g_{1,\epsilon} + (1 - \chi(r))\, I^*(g_{2,\epsilon}).$$

The conformal infinity of $\tilde{g}_\epsilon$ is represented by the metric γ_ϵ which is obtained by identifying the annuli $1/2 \le |y| \le 2$ and $1/2 \le |y'| \le 2$ in the rescaled normal coordinates on ∂M_1 and ∂M_2 using the inversion $J(y) = y/|y|^2$ (which is the restriction of the inversion I to the boundary) and using the cutoff function $\chi(|y|)$ to paste together the metrics γ_1 and γ_2.

Now that we have the approximate solution $\tilde{g}_\epsilon$, one can prove, using a contraction mapping principle argument, the existence of a symmetric bilinear form k_ϵ in a suitable weighted scale invariant Hölder space such that $g_\epsilon = \tilde{g}_\epsilon + k_\epsilon$ is a exact solution of the Bianchi gauged Einstein operator. The estimates needed are technical and we refer to [24] for details.

2.7 Orbifold degeneration of conformally compact Einstein metrics

In [8], Anderson studies degenerations of Einstein metrics on conformally compact 4-manifolds. There he proves that if $\{g_i\}$ is a sequence of Einstein metrics on a fixed conformally compact 4-manifold M, with the conformal infinities $\{[\gamma_i]\} \in \Gamma$, where Γ is a compact subset of $\mathcal{C}^{m,\alpha}$, then there are (at most) three possibilities for the behavior of $\{g_i\}$ in subsequences.

I. Convergence: A subsequence of $\{g_i\}$ converges, modulo diffeomorphisms, to a limit Poincaré-Einstein metric g on M, with boundary metric $\gamma \in \Gamma$. There is a compactification $\bar{g}_i = \rho^2 g_i$ of g_i such that the subsequence $\{\bar{g}_i\}$ converges in the $C^{m,\alpha}$ topology on $\overline{M}$.

II. Orbifolds: A subsequence of $\{g_i\}$ converges, modulo diffeomorphisms, to a limit Poincaré-Einstein orbifold-singular metric h on M, with boundary metric $\gamma \in \Gamma$. The singular metric h is a smooth metric on a orbifold V, and M is a smooth resolution of V. There are only a finite number of singularities, each the vertex of a cone on a spherical space form. Away from the singularities, the convergence is smooth, as in I. The subsequence (M, g_i) converges to (V, h) in the Gromov-Hausdorff topology.

III. Cusps: A subsequence of $\{g_i\}$ converges, modulo diffeomorphisms, to a limit Poincaré-Einstein metric with cusps g on a connected manifold N, with boundary metric $\gamma \in \Gamma$, possibly with a finite number of orbifold singularities.

Related to (II), we make the following definition:

Definition 2.20. *Let M be the interior of a compact, oriented 4-manifold with boundary, and let V be the interior of a 4-dimensional orbifold with boundary. We say that a sequence $\{g_i\}$ of $C^{m,\alpha}$ conformally compact metrics on M degenerates to a $C^{m,\alpha}$ conformally compact metric h on V if the following hold:*

(i) (M, g_i) converges to (V, h) in the Gromov-Hausdorff topology;

(ii) there exist a smooth resolution $\pi : M \to V$ and a defining function ρ of $\overline{M}$ satisfying the following: for any open subset U of $\overline{M}$ such that $U \subset \pi^{-1}(V_{\mathrm{reg}})$, the sequence of compactified metrics $\{\bar{g}_i = \rho^2 g_i\}$ converges, in the $C^{m,\alpha}(U)$ topology, to $\rho^2 \pi^* h$.

In Chapter 4, we will be working with curves g_t, $t \in (0, t')$, where t' is some number in $(0, 1)$, of $C^{m,\alpha}$ conformally compact metrics on M. We say that the curve g_t, $t \in (0, t')$ degenerates, as $t \to 0$, to a $C^{m,\alpha}$ conformally compact metric h on V if for any sequence $\{t_i\} \subset (0, t')$ with $t_i \to 0$, we have that $\{g_{t_i}\}$ degenerates to h in the sense of Definition 2.20.

We remark that the phenomenon of orbifold degeneration can, in fact, occur. The Taub-bolt family [17] is a 1-parameter family of Poincaré-Einstein metrics on $\mathbf{TS}^2$ that degenerates to the orbifold Taub-bolt metric h_{TB}, a Poincaré-Einstein metric on the orbifold $C(\mathbf{RP}^3)$. With the exception of metrics obtained by a connected sum construction due to Mazzeo and Pacard [24] (see Section 2.6 and Section 3.5), this is the only known example of orbifold degeneration of a family of Poincaré-Einstein metrics.

Based on the comments above, one naturally asks the following question:

Question 2.21. *Let h be a (smooth) Poincaré-Einstein metric on a 4-dimensional orbifold V, and let M be a smooth resolution of V. Does there exist a sequence of Poincaré-Einstein metrics $\{g_i\}$ on M such that (M, g_i) degenerates to (V, h)?*

Related to this question, Anderson made some conjectures. One of them is the following:

Conjecture 2.22. *There does not exist a sequence $\{g_i\}$ of conformally compact Einstein metrics on $\mathbf{TS}^2$ that degenerates to the hyperbolic metric on $C(\mathbf{RP}^3)$.*

We believe Conjecture 2.22 to be true. We were not able, however, to prove it. The following is another conjecture of Anderson. For some discussions related to this conjecture, see the work of Mazzeo and Singer [25].

Conjecture 2.23. *Let M be the interior of an oriented 4-dimensional compact manifold $\overline{M}$ with boundary ∂M, let V be an orbifold such that there exists a smooth resolution $\pi : M \to V$, and let h be a Poincaré-Einstein metric on V. If there exists a sequence $\{g_i\}$ of Poincaré-Einstein metrics on M such that (M, g_i) degenerates to (V, h), then h is degenerate (in the sense of Definition 2.11).*

In Chapter 3, we will show that the orbifold Taub-bolt metric h_{TB} on $C(\mathbf{RP}^3)$ is nondegenerate. This will give a negative answer to Conjecture 2.23.

There is one more conjecture made by Anderson that we would like to mention. Before stating this conjecture, we need to introduce some terminology.

The trichotomy I-III above leads one to consider the following approach: instead of working with the space $\mathcal{E}^{m,\alpha}$, one can work with an enlarged space that includes the orbifold and cusps limits. Let then $\bar{\mathcal{E}}^{m,\alpha}$ be the completion of the moduli space $\mathcal{E}^{m,\alpha}$ of Poincaré-Einstein metrics with respect to the pointed Gromov-Hausdorff topology; the base points x being chosen so that

$$\mathrm{dist}_{\bar{g}}(x, \partial M) = 1,$$

for example (here $\bar{g}$ denotes a compactification of g).

Now one has an extension $\bar{\Pi}$ of Π to $\bar{\mathcal{E}}^{m,\alpha}$, and $\bar{\Pi} : \bar{\mathcal{E}}^{m,\alpha} \to \mathcal{C}^{m,\alpha}$ is continuous, cf. [8]. Moreover, by construction, $\bar{\Pi}$ is proper. If $\bar{\mathcal{E}}^{m,\alpha}$ has roughly the structure of a manifold, then one can define a degree $\deg\bar{\Pi}$ associated with each component of $\bar{\mathcal{E}}^{m,\alpha}$ and

$$\deg\bar{\Pi} = \deg\Pi.$$

If it happened that $\deg\bar{\Pi} \neq 0$, one would conclude (at least) that almost every choice of conformal class in $\mathcal{C}^{m,\alpha}$ is the conformal infinity of a smooth Poincaré-Einstein metric on M.

We can finally state the following conjecture of Anderson, which is a conjecture about the point set topology of $\bar{\mathcal{E}}^{m,\alpha}$. This conjecture is stated in [7] (Conjecture 4.2).

Conjecture 2.24. *Let $\mathcal{E}_0$ be a component of $\mathcal{E}^{m,\alpha}$, and let $\partial\mathcal{E}_0 = \bar{\mathcal{E}}_0 \setminus \mathcal{E}_0$. The set $\bar{\Pi}(\partial\mathcal{E}_0)$ has empty interior in $\mathcal{C}^{m,\alpha}$.*

It is proved in [4] (see also [5]) that if the conformal infinity $[\gamma]$ has a representative of positive scalar curvature, then one can rule out cusp formation. Thus, in this case, the set $\partial\mathcal{E}_0$ corresponds to the orbifold singular Einstein metrics that are the limit, in the pointed Gromov-Hausdorff topology, of a sequence $\{g_i\}$ of smooth conformally compact Einstein metrics on M.

In Chapter 4, we obtain some results related to Conjecture 2.24.

Chapter 3

The Taub-bolt family

3.1 Berger spheres

Recall the Lie group $SU(2)$, which is defined as

$$SU(2) = \left\{ \begin{pmatrix} z & -w \\ \bar{w} & \bar{z} \end{pmatrix} ; z, w \in \mathbb{C} \text{ and } |z|^2 + |w|^2 = 1 \right\}.$$

Clearly, under the identification

$$\mathbb{C}^2 = \left\{ \begin{pmatrix} z & -w \\ \bar{w} & \bar{z} \end{pmatrix} ; z, w \in \mathbb{C} \right\},$$

$SU(2)$ corresponds to the unit sphere $\mathbf{S}^3$ in $\mathbb{C}^2$.

Consider the action of $\mathbf{S}^1$ on $\mathbf{S}^3$ given by multiplication on the left by the matrices

$$\begin{pmatrix} 0 & e^{i\theta} \\ e^{i\theta} & 0 \end{pmatrix} \in SU(2).$$

This action is smooth, free, and proper, and the orbit space $\mathbf{S}^3/\mathbf{S}^1$ is diffeomorphic to $\mathbf{S}^2$. The quotient map $\mathbf{S}^3 \to \mathbf{S}^2$ is known as the Hopf fibration.

The Lie algebra of $SU(2)$ is spanned by

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; X_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } X_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

These matrices give rise to left invariant vector fields on $SU(2)$. If we declare them to be orthonormal, then we get a left-invariant metric on $SU(2)$. If instead we declare them to be merely orthogonal, X_1 to have length a, and the other two to be unit vectors, we get a one parameter family of metrics γ_a on $\mathbf{S}^3$. These distorted spheres are called Berger spheres. Note that X_1 is tangent to the orbits of the circle action. The Berger spheres are, therefore, obtained from the canonical metric by multiplying the metric on the Hopf fiber by a^2. If we let $\{\sigma_1, \sigma_2, \sigma_3\}$ be the coframe dual to the frame $\{X_1, X_2, X_3\}$, then the Berger metric γ_a can be written as

$$\gamma_a = a^2(\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2.$$

In many situations it is convenient to consider "spherical coordinates" (ψ, θ, ϕ) on $\mathbf{S}^3$. In these coordinates we have

$$\begin{aligned}
\sigma_1 &= d\psi + \cos\theta d\phi, \\
\sigma_2 &= \sin\psi d\theta - \sin\theta\cos\psi d\phi, \\
\sigma_3 &= -\cos\psi d\theta - \sin\theta\sin\psi d\phi.
\end{aligned}$$

3.2 Some bundles over $\mathbf{S}^2$

Consider the unit disk bundle over $\mathbf{S}^2$. This is the manifold with boundary $\overline{M}$ given by

$$\overline{M} = \{(x, v_x) \in \mathbf{TS}^2;\ |v_x| \leq 1\}.$$

The boundary of this manifold is the unit circle bundle

$$\partial M = \{(x, v_x) \in \mathbf{TS}^2;\ |v_x| = 1\}.$$

Recall that ∂M is homeomorphic to $SO(3)$. One way to see this is to consider $\mathbf{S}^3$ as a subset of $\mathbf{R}^3$ in the usual way, that is,

$$\mathbf{S}^3 = \{p \in \mathbf{R}^3; |p| = 1\},$$

and to associate to each element (p, v_p) of ∂M the element of $SO(3)$ corresponding to the positively oriented frame $(p, v_p, p \times v_p)$, where $\times$ denotes the cross product in $\mathbf{R}^3$. The same argument shows that, for any $s > 0$, the manifold

$$\{(x, v_x) \in \mathbf{TS}^2;\ |v_x| = s\}$$

is homeomorphic to $\mathbf{RP}^3$. Therefore, if N is the manifold obtained by removing the zero section

$$\{(x, 0_x) \in \mathbf{TS}^2; x \in \mathbf{S}^2\}$$

from $\mathbf{TS}^2$, then N is homeomorphic to $(0, \infty) \times \mathbf{RP}^3$.

3.3 The Taub-bolt family

We describe now a family of conformally compact Einstein metrics on $\mathbf{TS}^2$, known as the Taub-bolt family [17]. To describe this family of metrics, we will use coordinates (r, τ, θ, ϕ), where (θ, ϕ) are "spherical coordinates" on the base $\mathbf{S}^2$, and (r, τ) are polar coordinates on each fiber $T_x\mathbf{S}^2$. In these coordinates, the metric is given by

$$\begin{aligned} g_{TB}(s) &= \frac{1}{4}E[\frac{4(r^2-1)}{F(r)}dr^2 + \frac{F(r)}{E(r^2-1)}(d\tau + E^{1/2}\cos\theta d\phi)^2 \\ &\quad +(r^2-1)(d\theta^2 + \sin^2\theta d\phi^2)], \end{aligned} \tag{3.1}$$

where $F = F_s(u)$ is given by

$$\begin{aligned} F(u) &= Eu^4 + (4-6E)u^2 \\ &\quad + \left[-Es^3 + (6E-4)s + \frac{3E-4}{s}\right]u + (4-3E), \end{aligned} \tag{3.2}$$

and $E = E_s$ is given by

$$E = \frac{4}{3}\frac{1}{s+1}.$$

The parameter τ has period $2\pi E^{1/2}$. In order to avoid curvature singularities, we must take $s > 1$ and $r > s$. As $s \to 1$, E_s tends to $2/3$, and the area of the bolt $\mathbf{S}^2$ at $\{r = s\}$ converges to 0, and vanishes at $\{r = 1\}$ when $s = 1$.

When $s = 1$ and $E = 2/3$,

$$\begin{aligned} F(u) &= (2/3)(u^4 - 4u + 3) \\ &= (2/3)(u^2 + 2u + 3)(u-1)^2, \end{aligned}$$

and we get the following expression for $h_{TB} = g_{TB}(1)$:

$$\begin{aligned} g_{TB}(1) &= \frac{(r+1)}{(r-1)(r^2+2r+3)}dr^2 + \frac{(r-1)(r^2+2r+3)}{6(r+1)}\left(d\tau + (2/3)^{\frac{1}{2}}\cos\theta d\phi\right)^2 \\ &\quad + \frac{(r-1)(r+1)}{6}\left(d\theta^2 + \sin^2\theta d\phi^2\right). \end{aligned}$$

This expression gives a conformally compact Einstein metric on the orbifold $C(\mathbf{RP}^3)$, called the orbifold Taub-bolt metric, and it is not hard to verify that the metrics $g_{TB}(s)$ on $\mathbf{TS}^2$ degenerate, as $s \to 1$, to the orbifold metric $h_{TB} = g_{TB}(1)$ (in the sense of Definition 2.20).

Let us now show that the metrics $g_{TB}(s)$ are, in fact, conformally compact. To do this, we consider the change of variables

$$\vartheta = 1 - r^{-1}.$$

In the coordinates $(\vartheta, \tau, \theta, \phi)$, we have the following expression for $g_{TB}(s)$:

$$\begin{aligned} g_{TB}(s) &= \frac{E}{4(1-\vartheta)^2}[\frac{4(2\vartheta-\vartheta^2)}{G(1-\vartheta)}d\vartheta^2 + \frac{G(1-\vartheta)}{E(2\vartheta-\vartheta^2)}(d\tau + E^{1/2}\cos\theta d\phi)^2 \\ &\quad +(2\vartheta-\vartheta^2)(d\theta^2 + \sin^2\theta d\phi^2)], \end{aligned}$$

where $G = G_s(u)$ is given by

$$\begin{aligned} G(u) &= (4E-3)u^4 + \left[-Es^3 + (6E-4)s + \frac{3E-4}{s}\right]u^3 \\ &\quad +(4-6E)u^2 + E. \end{aligned}$$

Thus, if we take

$$\rho = 1 - \vartheta$$

as our boundary defining function, we have that

$$\rho^2 g_{TB}(s)$$

extends (at least in the region where r is large), to a smooth metric on the disk bundle over $\mathbf{S}^2$, and hence, $g_{TB}(s)$ is conformally compact. As we will see below, $g_{TB}(s)$ is an Einstein metric. Therefore, $g_{TB}(s)$ is a Poincaré-Einstein metric on $\mathbf{TS}^2$.

To describe the conformal infinity of $g_{TB}(s)$, we first consider the change of variables $\tau = E^{1/2}\psi$. After this change of variables, we get

$$\begin{aligned} g_{TB}(s) &= \frac{1}{4}E[\frac{4(r^2-1)}{F(r)}dr^2 + \frac{F(r)}{(r^2-1)}(d\psi + \cos\theta d\phi)^2 \\ &\quad +(r^2-1)(d\theta^2 + \sin^2\theta d\phi^2)]. \end{aligned} \tag{3.3}$$

Using the above expression, we find that the conformal infinity of $g_{TB}(s)$ is given by a squashed (Berger) metric on $\mathbf{RP}^3$.

The metrics that we have just described are part of a more general family (also called Taub-bolt family) of conformally compact metrics on $\mathbf{R}^2$ bundles over $\mathbf{S}^2$. We remark that each member of this family can be obtained as a special case of the complex metrics given in [26] . Next, we give a description of this more general Taub-bolt family. For a study of the thermodynamics of these spaces, see [17].

It is a known fact that $\mathbf{R}^2$ bundles over $\mathbf{S}^2$ are classified by their Chern number k. Let then N_k, $k \geq 1$, be the $\mathbf{R}^2$ bundle over $\mathbf{S}^2$ of Chern number k. To describe this family of metrics, we will use coordinates (r, τ, θ, ϕ) as before, that is, (θ, ϕ) are "spherical coordinates" on the base $\mathbf{S}^2$, and (r, τ) are polar coordinates on each fiber $\mathbf{R}^2_x$. In these coordinates, the metric is given by the same expression given in (3.1), that is,

$$\begin{aligned} g_{TB}(k,s) &= \frac{1}{4}E[\frac{4(r^2-1)}{F(r)}dr^2 + \frac{F(r)}{E(r^2-1)}(d\tau + E^{1/2}\cos\theta d\phi)^2 \\ &\quad +(r^2-1)(d\theta^2 + \sin^2\theta d\phi^2)], \end{aligned}$$

where $F = F_s(u)$ has the same expression as the one given in (3.2), that is,

$$\begin{aligned} F(u) &= Eu^4 + (4-6E)u^2 \\ &\quad + \left[-Es^3 + (6E-4)s + \frac{3E-4}{s}\right]u + 4 - 3E. \end{aligned}$$

The parameter E, however, depends on k. More precisely,

$$E_s = \frac{2ks-4}{3(s^2-1)}.$$

The parameter τ has period $\beta = 4\pi E^{1/2}/k$. In order to avoid curvature singularities, we must take $s > 1$, $s > 2/k$ and $r > s$.

The metric $g_{TB}(k, s)$ is a conformally compact Einstein metric on N_k. The conformal infinity is given by the conformal class of a Berger (or squashed) metric on the lens space $\mathbf{S}^3/\mathbb{Z}_k$. When $k = 2$, the manifold N_k is diffeomorphic to $\mathbf{TS}^2$ and the metric $g_{TB}(k, s)$ coincides with $g_{TB}(s)$. Moreover, $k = 2$ is the only value of k for which orbifold degeneration occurs.

3.4 The curvatures of the Taub-bolt orbifold

We would like now to compute the curvatures of $g_{TB}(s)$. We will actually compute the curvatures of a more general family of metrics on $C(\mathbf{RP^3})$.

Consider coordinates (r, ψ, θ, ϕ) on $(0, \infty) \times \mathbf{RP^3}$, where r parametrizes $(0, \infty)$ and (ψ, θ, ϕ) are "spherical coordinates" on $\mathbf{RP^3}$. We then consider metrics of the following form:

$$ds^2 = A^2 dr^2 + \frac{1}{A^2}(d\psi + \cos\theta d\phi)^2 + C^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.4}$$

where A and C are functions of r only. We would like to compute the curvatures of (N_k, ds^2). For this, we take the orthonormal frame given by

$$\begin{aligned}
X_1 &= \frac{1}{A}\frac{\partial}{\partial r} \\
X_2 &= A\frac{\partial}{\partial \psi} \\
X_3 &= \frac{1}{C}(\sin\psi\frac{\partial}{\partial\theta} - \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi} + \frac{\cos\psi\cos\theta}{\sin\theta}\frac{\partial}{\partial\psi}) \\
X_4 &= \frac{1}{C}(-\cos\psi\frac{\partial}{\partial\theta} - \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi} + \frac{\sin\psi\cos\theta}{\sin\theta}\frac{\partial}{\partial\psi}).
\end{aligned}$$

The coframe dual to the above frame is given by

$$\begin{aligned}
\omega_1 &= Adr \\
\omega_2 &= \frac{1}{A}(d\psi + \cos\theta d\phi) \\
\omega_3 &= C(\sin\psi d\theta - \sin\theta\cos\psi d\phi) \\
\omega_4 &= C(-\cos\psi d\theta - \sin\theta\sin\psi d\phi).
\end{aligned}$$

After some calculations, we get

$$\begin{aligned}
d\omega_1 &= 0 \\
d\omega_2 &= \left(\frac{1}{A}\right)'\omega_1\wedge\omega_2 + \left(\frac{1}{AC^2}\right)\omega_3\wedge\omega_4 \\
d\omega_3 &= \left(\frac{C'}{AC}\right)\omega_1\wedge\omega_3 - A\omega_2\wedge\omega_4 \\
d\omega_4 &= \left(\frac{C'}{AC}\right)\omega_1\wedge\omega_4 + A\omega_2\wedge\omega_3,
\end{aligned}$$

where prime denotes derivative with respect to r. The connection forms are given by

$$\begin{aligned}
\omega_{12} &= \left(\frac{1}{A}\right)'\omega_2 \qquad & \omega_{13} &= \left(\frac{C'}{AC}\right)\omega_3 \\
\omega_{14} &= \left(\frac{C'}{AC}\right)\omega_4 \qquad & \omega_{23} &= \left(-\frac{1}{2AC^2}\right)\omega_4 \\
\omega_{24} &= \left(\frac{1}{2AC^2}\right)\omega_3 \qquad & \omega_{34} &= \left(\frac{1}{2AC^2}-A\right)\omega_2.
\end{aligned}$$

Finally, the curvature forms are given by

$$\begin{aligned}
\Omega_{12} &= \left[\frac{1}{2}\left(\frac{1}{A}\right)^2\right]''\omega_1\wedge\omega_2 + \left[\left(\frac{1}{A}\right)'\left(\frac{1}{AC^2}\right) - \left(\frac{1}{AC^2}\right)\left(\frac{C'}{AC}\right)\right]\omega_3\wedge\omega_4 \\
\Omega_{13} &= \left[\left(\frac{1}{A}\right)\left(\frac{C'}{AC}\right)' + \left(\frac{C'}{AC}\right)^2\right]\omega_1\wedge\omega_3 \\
&\quad + \left[\left(\frac{1}{A}\right)'\left(\frac{1}{2AC^2}\right) - \left(\frac{1}{2AC^2}\right)\left(\frac{C'}{AC}\right)\right]\omega_2\wedge\omega_4 \\
\Omega_{14} &= \left[\left(\frac{1}{A}\right)\left(\frac{C'}{AC}\right)' + \left(\frac{C'}{AC}\right)^2\right]\omega_1\wedge\omega_4 \\
&\quad - \left[\left(\frac{1}{A}\right)'\left(\frac{1}{2AC^2}\right) - \left(\frac{1}{2AC^2}\right)\left(\frac{C'}{AC}\right)\right]\omega_2\wedge\omega_3 \\
\Omega_{23} &= -\left[\left(\frac{1}{A}\right)\left(\frac{1}{2AC^2}\right)' + \left(\frac{1}{2AC^2}\right)\left(\frac{C'}{AC}\right)\right]\omega_1\wedge\omega_4 \\
&\quad + \left[\left(\frac{1}{A}\right)'\left(\frac{C'}{AC}\right) - \left(\frac{1}{2AC^2}\right)^2\right]\omega_2\wedge\omega_3 \\
\Omega_{24} &= \left[\left(\frac{1}{A}\right)\left(\frac{1}{2AC^2}\right)' + \left(\frac{1}{2AC^2}\right)\left(\frac{C'}{AC}\right)\right]\omega_1\wedge\omega_3 \\
&\quad + \left[\left(\frac{1}{A}\right)'\left(\frac{C'}{AC}\right) - \left(\frac{1}{2AC^2}\right)^2\right]\omega_2\wedge\omega_4 \\
\Omega_{34} &= \left[\left(\frac{1}{A}\right)\left(\frac{1}{2AC^2}-A\right)\right]'\omega_1\wedge\omega_2 \\
&\quad + \left[\left(\frac{C'}{AC}\right)^2 + \left(\frac{1}{2AC^2}\right)^2 + \left(\frac{1}{2AC^2}-A\right)\left(\frac{1}{AC^2}\right)\right]\omega_3\wedge\omega_4.
\end{aligned}$$

We would like to write the metric (3.1) in the form

$$ds^2 = b^2E[A^2dr^2 + \frac{1}{A^2}(d\psi+\cos\theta d\phi)^2 + C^2(d\theta^2+\sin^2\theta d\phi^2)], \tag{3.5}$$

where $A = A(r)$ and $C = C(r)$. To accomplish this, we make the following change of variables: $\tau = E^{1/2}\psi$, which gives $d\tau = E^{1/2}d\psi$; and $r = \tilde{r}/2$, which gives $2dr = d\tilde{r}$. After these change of variables (and dropping the tilde for convenience), we get that the metric is in the form (3.5), with

$$A^2(r) = \frac{6(r+2)}{(r^3 + 2r^2 + 4r - 24)}$$

and

$$C^2(r) = \frac{r^2 - 4}{4}.$$

To compute the curvatures, it will be convenient to drop the factor b^2E. We then have a metric in the form (3.4).

Let $f(r) = 1/(A^2)$, $g(r) = C'/C$ and $h(r) = 1/(C^2)$, where prime denotes derivative with respect to r. The following are expressions for some of the components of the Riemann curvature tensor Rm:

$$\begin{aligned}
-\mathrm{Rm}_{1212} &= \frac{1}{2}f''(r) \\
&= \frac{r^3 + 6r^2 + 12r - 24}{6(r+2)^3} \\
-\mathrm{Rm}_{1313} = -\mathrm{Rm}_{1414} &= \frac{1}{2}f'(r)g(r) + f(r)g'(r) + f(r)(g(r))^2 \\
&= \frac{r^3 + 6r^2 + 12r + 24}{6(r+2)^3} \\
-\mathrm{Rm}_{2323} = -\mathrm{Rm}_{2424} &= \frac{1}{2}f'(r)g(r) - \frac{1}{4}f(r)(h(r))^2 \\
&= \frac{r^3 + 6r^2 + 12r + 24}{6(r+2)^3} \\
-\mathrm{Rm}_{3434} &= f(r)(g(r))^2 + \frac{3}{4}f(r)(h(r))^2 - h(r) \\
&= \frac{r^3 + 6r^2 + 12r - 24}{6(r+2)^3} \\
-\mathrm{Rm}_{1234} &= \frac{1}{2}f'(r)h(r) - f(r)g(r)h(r) \\
&= \frac{16}{3(r+2)^3} \\
-\mathrm{Rm}_{1324} = -\mathrm{Rm}_{1432} &= \frac{1}{4}f'(r)h(r) - \frac{1}{2}f(r)g(r)h(r) \\
&= \frac{8}{3(r+2)^3}.
\end{aligned}$$

All the other components of Rm are either zero or can be found from the above using the symmetries of the Riemann curvature tensor. It will be used later on that if $\#\{i,j,k,l\} = 3$, then $\mathrm{Rm}_{ijkl} = 0$, where $\#$ denotes the number of elements.

We now prove a very simple (but helpful) algebraic lemma.

Lemma 3.1. *Let a, b, c be real numbers, a and c positive, and consider the function $P : \mathbf{R}^2 \to \mathbf{R}$ given by*

$$P(x,y) = ax^2 + bxy + cy^2.$$

If $4ac - b^2 \geq 0$, then $P(x,y) \geq 0$, for all $x, y \in \mathbf{R}$. Moreover, if $4ac - b^2 > 0$, then $P(x,y) = 0$ only when $(x,y) = (0,0)$.

Proof. Just notice that

$$P(x,y) = a\left[\left(x + \frac{by}{2a}\right)^2 + (4ac - b^2)\left(\frac{y}{2a}\right)^2\right].$$

If $4ac - b^2 > 0$ and $P(x,y) = 0$, then necessarily

$$\left(x + \frac{by}{2a}\right) = 0 \text{ and } \left(\frac{y}{2a}\right) = 0,$$

which gives $x = y = 0$. □

Proposition 3.2. *The Taub-bolt orbifold metric on $C(\mathbf{RP}^3)$ has negative sectional curvature.*

Proof. Let z be a point of $C(\mathbf{RP}^3)$ that is not the orbifold singularity, and let v, w be orthonormal vectors in the tangent space of $C(\mathbf{RP}^3)$ at z. Write

$$v = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4$$

and

$$w = b_1X_1 + b_2X_2 + b_3X_3 + b_4X_4,$$

with $a_i, b_i \in \mathbf{R}$, $i = 1,2,3,4$.

Since v and w are orthonormal, the sectional curvature of the plane generated by v and w is given by $-\mathrm{Rm}(v,w,v,w)$.

Denote the set $\{i,j,k,l\}$ by I. Using that $\mathrm{Rm}_{ijkl} = 0$ if $\#I = 3$, we have

$$\mathrm{Rm}(v,w,v,w) = \sum_I a_ib_ja_kb_l\mathrm{Rm}_{ijkl}$$

$$
\begin{aligned}
&= \sum_{\#I=4} a_i b_j a_k b_l \mathrm{Rm}_{ijkl} + \sum_{\#I=2} a_i b_j a_k b_l \mathrm{Rm}_{ijkl} \\
&= \sum_{\#I=4} \left((a_i b_j)^2 \mathrm{Rm}_{ijij} + a_i b_j a_k b_l \mathrm{Rm}_{ijkl} + (a_k b_l)^2 \mathrm{Rm}_{klkl} \right).
\end{aligned}
$$

By Lemma 3.1, if it happens that

$$
4\mathrm{Rm}_{ijij}\mathrm{Rm}_{klkl} - (\mathrm{Rm}_{ijkl})^2 > 0, \tag{3.6}
$$

then $\mathrm{Rm}(v, w, v, w)$ will also be nonnegative. To verify the set of inequalities (3.6), one just needs to consider the following cases: $(i, j, k, l) = (1, 2, 3, 4)$, $(i, k, j, l) = (1, 3, 2, 4)$, and $(i, k, j, l) = (1, 4, 2, 3)$.

If $(i, j, k, l) = (1, 2, 3, 4)$, then the left hand side of (3.6) equals

$$
\frac{1}{9}\left[\frac{(r^3 + 6r^2 + 12r - 24)^2 - 256}{(r+2)^2}\right],
$$

which is positive for $r \geq 2$. If $(i, j, k, l) = (1, 3, 2, 4)$ or $(i, j, k, l) = (1, 4, 2, 3)$, then the left hand side of (3.6) equals

$$
\frac{1}{9}\left[\frac{(r^3 + 6r^2 + 12r + 24)^2 - 64}{(r+2)^2}\right],
$$

which is also positive for $r \geq 2$. This implies $\mathrm{Rm}(v, w, v, w) \geq 0$.

Now, since v and w are linearly independent, it is possible to choose $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, such that $a_i b_j \neq 0$. Lemma 3.1 then gives that, for this particular choice of i and j, we have

$$
(a_i b_j)^2 \mathrm{Rm}_{ijij} + a_i b_j a_k b_l \mathrm{Rm}_{ijkl} + (a_k b_l)^2 \mathrm{Rm}_{klkl} > 0.
$$

Therefore, $\mathrm{Rm}(v, w, v, w) > 0$. □

Now that we know that the Taub-bolt orbifold metric has negative sectional curvature, we can invoke Theorem 2.13 and conclude that this orbifold metric is nondegenerate.

3.5 More examples of orbifold degeneration

We can use the boundary connected sum result of Mazzeo and Pacard to produce other examples of curves of conformally compact Einstein metrics degenerating to a

Poincaré-Einstein metric with orbifold singularities.

Denote by M the smooth manifold $\ell\#_b\mathbf{TS}^2$ (the boundary connected sum construction is performed ℓ times). The boundary of M is $\partial M = \ell\#\mathbf{RP}^3$. Consider the Taub-bolt family $g_{TB}(s)$ on $\mathbf{TS}^2$, $s > 1$. Theorem 2.19 guarantees the existence of a Poincaré-Einstein metric $g_\ell(s)$, on M, and this metric can be taken so that, outside the "connected sum regions" (see Section 2.6), it satisfies

$$||g_\ell(s) - g_{TB}(s)|| < s^{-1}, \tag{3.7}$$

where the norm is with respect to some (weighted scale invariant) Hölder space of order $(2, \alpha)$. This implies that $(M, g_\ell(s))$ degenerates to a Poincaré-Einstein metric h_ℓ on the orbifold $V_\ell = \ell\#_b C(\mathbf{RP}^3)$, which has ℓ orbifold singularities.

Regarding the conformal infinity of $g_\ell(s)$, we have, changing the metrics g_ℓ if necessary (in a way that (3.7) still holds), that outside the "connected sum regions", the inequality

$$||\gamma_\ell(s) - \gamma_{TB}(s)||_{C^{2,\alpha}} < s^{-1}$$

holds, where $\gamma_\ell(s)$ and $\gamma_{TB}(s)$ are suitable representatives of the conformal infinities of $g_\ell(s)$ and $g_{TB}(s)$, respectively.

Obviously, the construction above, which was done for the Taub-bolt curve, works for any curve of conformally compact Einstein manifolds (M, g_t) degenerating to a conformally compact Einstein orbifold (V, h). However, the Taub-bolt curve (and the metrics described above) are the only known examples of orbifold degeneration of a curve (or sequence) of Poincaré-Einstein metrics. It would be very interesting to find other examples of orbifold degeneration, especially on manifolds topologically distinct from the ones described above.

Chapter 4

Orbifold degeneration

4.1 Orbifold degeneration I

Throughout this chapter, M denotes the interior of a compact, oriented, 4-manifold $\overline{M}$ with boundary ∂M, and V is a 4-orbifold such that there exists a smooth resolution $\pi : M \to V$. We also assume that the conclusions of Theorem 2.15 hold for M and V.

Consider a nondegenerate $C^{m,\alpha}$ conformally compact Einstein metric h on V, and let $(\partial M, [\gamma])$ be the conformal infinity of (V, h). Since (V, h) is nondegenerate, Theorem 2.15 (which is also valid for orbifolds) says that there exist a neighborhood W of $[\gamma]$ in $\mathcal{C}^{m,\alpha}$ and a neighborhood Z of h in $\mathcal{F}^{m,\alpha}$ such that $\Pi_{\mathcal{F}} : Z \to W$ is a diffeomorphism. For $[\theta] \in W$, we denote by $h^{[\theta]}$ the unique element of Z such that $\Pi_{\mathcal{F}}(h^{[\theta]}) = [\theta]$.

The following theorem implies, in particular, that if a conformally compact Einstein metric h on V is such that some smooth curve g_t of nondegenerate Poincaré-Einstein metrics on M degenerates to h, then the boundary map $\Pi_{\mathcal{E}(M)}$ is not injective. We suggest the reader look at Figure 4.1 a few times when reading the proof of this theorem.

Theorem 4.1. *Let $h = h^{[\gamma]}$ be a nondegenerate $C^{m,\alpha}$ conformally compact Einstein metric on the 4-orbifold V. Suppose that, for each $[\theta] \in W$, there exists a smooth curve $g_t^{[\theta(t)]}$ in $E^{m,\alpha}(M)$ that degenerates to $h^{[\theta]}$, as $t \to 0$. If there exists $t_0 \in (0,1)$ such that $g_t = g_t^{[\gamma(t)]}$ is nondegenerate, for each $t \in (0, t_0)$, then there exist smooth curves $g_1, g_2 : (0,1) \to E^{m,\alpha}(M)$ satisfying the following: $g_1(s)$ is nondegenerate, for each $s \in (0,1)$; $\Pi(g_1(s)) = \Pi(g_2(s))$, for each $s \in (0,1)$; and $g_1(s_1) \neq g_2(s_2)$ for all $s_1, s_2 \in (0,1)$.*

Proof. We can assume, without loss of generality, that the curve $g_t = g_t^{[\gamma(t)]}$ is parametrized in such way that

$$\sup_{x\in M} |\mathrm{Rm}_{g_t}|(x) = t^{-1}.$$

Since g_t is nondegenerate, for each $t \in (0, t_0)$, Theorem 2.15 implies that there exist neighborhoods Y_t of g_t in $\mathcal{E}^{m,\alpha}$ and X_t of $[\gamma(t)]$ in $\mathcal{C}^{m,\alpha}$ such that $\Pi : Y_t \to X_t$ is a diffeomorphism, for each $t \in (0, t_0)$. By shrinking the neighborhoods Y_t and X_t if necessary, we can assume that for any $\breve{g} \in Y_t$ the inequality

$$\sup_{x\in M} |\mathrm{Rm}_{\breve{g}}|(x) > \frac{1}{2} t^{-1}$$

holds, for each $t \in (0, t_0)$.

Let $t_1 \in (0, t_0)$ be such that $W \cap X_{t_1}$ isn't empty. Pick any $[\theta] \in W \cap X_{t_1}$ and consider the curve $g_t^{[\theta(t)]}$, $t \in (0, 1)$. Since $[\theta(t)] \to [\theta]$, there exists $t_2 \in (0, t_1/2)$ such that $[\theta(t)] \in W \cap X_{t_1}$, for each $t \in (0, t_2)$.

Let now t_3 and t_4 be such that $0 < t_3 < t_4 < t_2$, and consider the restriction of the curve $[\theta(t)]$ to the interval (t_3, t_4). Theorem 2.15 gives us a curve $\breve{g}_t$, $t \in (t_3, t_4)$, with conformal infinity $[\theta(t)]$, for each $t \in (t_3, t_4)$. Notice that, since $t_2 < t_1/2$, the sets $\{g_t^{[\theta(t)]};\ t \in (t_3, t_4)\}$ and $\{\breve{g}_t;\ t \in (t_3, t_4)\}$ are disjoint. Thus, after performing the change of variables $s = (t - t_3)/(t_4 - t_3)$, we obtain curves $g_1, g_2 : (0, 1) \to E^{m,\alpha}(M)$ satisfying

$$\Pi(g_1(s)) = \Pi(g_2(s)),$$

for each $s \in (0, 1)$, and such that $g_1(s_1) \neq g_2(s_2)$ for all $s_1, s_2 \in (0, 1)$. $\square$

Corollary 4.2. *Let $h = h^{[\gamma]}$ be a nondegenerate $C^{m,\alpha}$ conformally compact Einstein metric on the 4-orbifold V. Suppose that, for each $[\theta] \in W$, there exists a smooth curve $g_t^{[\theta(t)]}$ in $E^{m,\alpha}(M)$ that degenerates to $h^{[\theta]}$, as $t \to 0$. If W can be taken so that, for each $[\theta] \in W$, there exists $t_{[\theta]} \in (0, 1)$ with $g_t^{[\theta(t)]}$ nondegenerate, for each $t \in (0, t_{[\theta]})$, then, for any $k \in \mathbb{N}$, it is possible to find k smooth curves $g_1, \ldots, g_k : (0, 1) \to E^{m,\alpha}$ such that, for any $i, j \in \{1, \ldots, k\}$, $i \neq j$, the following hold:*

$$\{g_i(s);\ s \in (0, 1)\} \text{ and } \{g_j(s);\ s \in (0, 1)\}$$

are disjoint and

$$\Pi(g_i(s)) = \Pi(g_j(s)),$$

for each $s \in (0, 1)$.

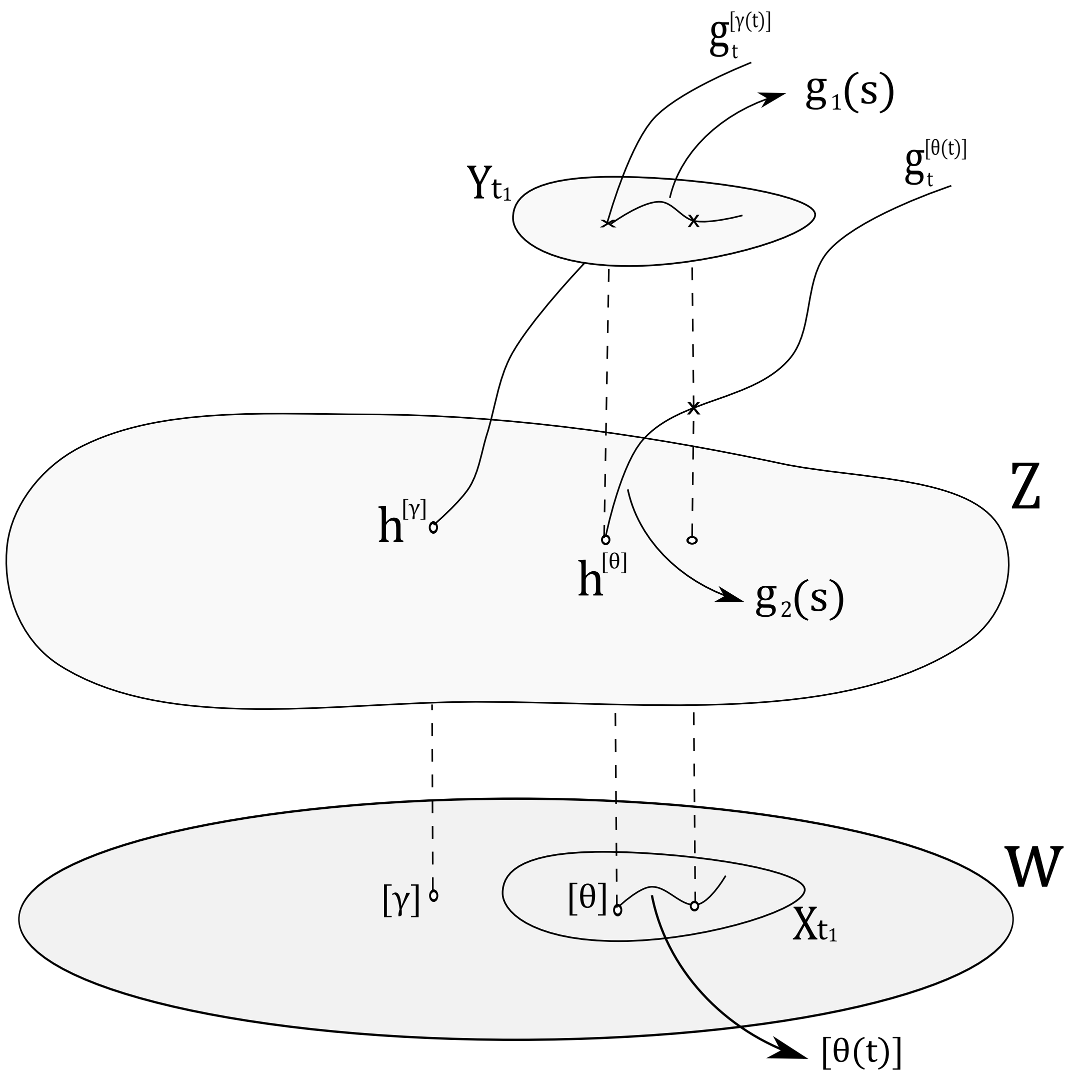

Figure 4.1: Visualization of the proof of Theorem 4.1.

Corollary 4.2 implies that if there exists a conformally compact Einstein metric h on V satisfying its hypothesis, then the boundary map Π has a very strange behavior.

Theorem 4.1 and Corollary 4.2 lead us to make the following conjecture:

Conjecture 4.3. *Let $h = h^{[\gamma]}$ be a nondegenerate $C^{m,\alpha}$ conformally compact Einstein metric on the 4-orbifold V. If there exists a smooth curve $g_t^{[\gamma(t)]}$ in $E^{m,\alpha}(M)$ that degenerates to h, as $t \to 0$, then there exists a sequence $\{t_i\} \subset (0,1)$, $t_i \to 0$, such that each g_{t_i} is degenerate.*

It would be very interesting if one could prove (or find a counterexample to) Conjecture 4.3. A proof would be interesting even if it requires the additional hypothesis that every orbifold metric $h^{[\theta]}$ near h is such that some smooth curve $g_t^{[\theta(t)]}$ in $E^{m,\alpha}(M)$ degenerates to $h^{[\theta]}$.

4.2 Some functional analysis

Let $(X, || \, ||_X)$ and $(Y, || \, ||_Y)$ be Banach spaces. As usual, we denote by $\mathcal{L}(X, Y)$ the space of bounded linear maps from X to Y.

Let $\Psi : X \to Y$ be a map of class C^1, and let $x \in X$. We denote by Q_x the remainder obtained when considering the first order expansion of Ψ around x. Thus, for any $k \in X$, $Q_x(k)$ is defined by the equation

$$\Psi(x + k) = \Psi(x) + \Psi'(x) \cdot k + Q_x(k).$$

We have that $Q_x : X \to Y$ is of class C^1, $Q_x(0) = 0$ and $(Q_x)'(k) = \Psi'(x+k) - \Psi'(x)$. In particular, $(Q_x)'(0) = 0 \in \mathcal{L}(X, Y)$.

Proposition 4.4. *If D is a positive real number, then there exists $r > 0$ such that*

$$||Q_x(u) - Q_x(v)|| \leq D||u - v||, \tag{4.1}$$

for any $u, v \in X$ such that $||u||, ||v|| \leq r$. Moreover, if for some $\mu > 0$ the map $x \mapsto \Psi'(x)$ is μ-Lipschitz, then we can take r to be any number in the interval $(0, D\mu^{-1}]$.

Proof. Let $D > 0$ be given. Since Q_x is of class C^1, there exists $r > 0$ such that

$$||(Q_x)'(k)|| \leq D,$$

for any $k \in X$ with $||k|| \leq r$. Inequality (4.1) then follows from the mean value inequality.

If Ψ' is μ-Lipschitz, we can take r to be any number in $(0, D\mu^{-1}]$. In fact, for $||k|| \leq D\mu^{-1}$ we have

$$\begin{aligned} ||Q'_x(k)|| &= ||\Psi'(x+k) - \Psi'(x)|| \\ &\leq \mu||k|| \leq D. \end{aligned}$$

□

Suppose that $\Psi'(x) \in \mathcal{L}(X, Y)$ is an isomorphism. Let $C > 0$ be such that

$$||(\Psi'(x))^{-1}|| \leq C, \tag{4.2}$$

and let $D, r > 0$ be such that (4.1) holds for any $u, v \in X$ with $||u||, ||v|| \leq r$.

Suppose one would like to find $k \in X$ such that

$$\Psi(x+k) = 0.$$

This corresponds to finding a fixed point of the map $F : X \to X$ given by

$$F(k) = -\left(\Psi'(x)\right)^{-1} \cdot \left(\Psi(x) + Q_x(k)\right).$$

Proposition 4.5. *Let $\lambda \in (0,1)$ and let $r > 0$ be such that*

$$||Q_x(u) - Q_x(v)|| \leq \frac{\lambda}{C}||u - v||, \tag{4.3}$$

for any $u, v \in \bar{B}(r)$, where

$$\bar{B}(r) = \{k \in X; ||k|| \leq r\}.$$

If the inequality

$$||\Psi(x)|| \leq \frac{(1-\lambda)}{C} r \tag{4.4}$$

holds, then the image of $\bar{B}(r)$ under F is contained in $\bar{B}(r)$. Moreover,

$$F : \bar{B}(r) \to \bar{B}(r)$$

is a λ-contraction.

Proof. Let $k \in \bar{B}(r)$. We have

$$\begin{aligned} ||F(k)|| &\leq || \left(\Psi'(x)\right)^{-1} \cdot \Psi(x)|| + || \left(\Psi'(x)\right)^{-1} \cdot Q_x(k)|| \\ &\leq || \left(\Psi'(x)\right)^{-1} || \cdot ||\Psi(x)|| + || \left(\Psi'(x)\right)^{-1} || \cdot ||Q_x(k)||. \end{aligned}$$

Using inequalities (4.2), (4.3) and (4.4), we find that

$$\begin{aligned} ||F(k)|| &\leq C \cdot \frac{(1-\lambda)}{C} r + C \cdot \frac{\lambda}{C} ||k|| \\ &\leq (1-\lambda)r + \lambda r = r. \end{aligned}$$

Thus, $F(\bar{B}(r)) \subset \bar{B}(r)$.

Let us now show that F is a λ-contraction. For $u, v \in \bar{B}(r)$ we have

$$\begin{aligned} ||F(u) - F(v)|| &= ||(\Psi'(x) \cdot (Q_x(u) - Q_x(v))\,|| \\ &\leq ||(\Psi'(x)|| \cdot ||\,(Q_x(u) - Q_x(v))\,||. \end{aligned}$$

Now, using (4.2) and (4.3), we find

$$||F(u) - F(v)|| \leq \lambda ||u - v||.$$

□

4.3 Orbifold degeneration II

Theorem 4.6. *Let $g_t = g_t^{[\gamma(t)]}$, $t \in (0,1)$, be a smooth curve in $E^{m,\alpha}(M)$, where $\gamma(t) = \bar{g}_t|_{\partial M}$ and $\bar{g}_t = \rho^2 g_t$ for some fixed defining function ρ. Suppose g_t degenerates, as $t \to 0$, to a nondegenerate $C^{m,\alpha}$ conformally compact Einstein metric h on the 4-orbifold V. If there exists $t_0 \in (0,1)$ such that g_t is nondegenerate for each $t \in (0, t_0)$, then there exists a family U_t, $t \in (0, t_0)$, each U_t a neighborhood of $\gamma(t)$ in $C^{m,\alpha}(\partial M)$, satisfying the following: given any smooth curve $\theta : (0, t_0) \to \mathrm{Met}^{m,\alpha}(\partial M)$ with $\theta(t) \in U_t$, for each $t \in (0, t_0)$, there exists a smooth curve $\breve{g} : (0, t_0) \to E^{m,\alpha}(M)$ such that the conformal infinity of $\breve{g}_t$ is $[\theta(t)]$, for each $t \in (0, t_0)$.*

Remark 4.7. This theorem is, obviously, a direct consequence of Theorem 2.15. However, the proof below gives us control of the size of the neighborhoods U_t, which will be important later on.

Proof. Let $g_t = g_t^{[\gamma(t)]}$, $t \in (0,1)$, be a smooth curve in $E^{m,\alpha}(M)$, with g_t nondegenerate for $t \in (0,t_0)$, such that (M,g_t) degenerates to (V,h) as $t \to 0$, where h is a nondegenerate $C^{m,\alpha}$ conformally compact Einstein metric on the 4-orbifold V. Here, $\gamma(t) = \bar{g}_t|_{\partial M}$ and $\bar{g}_t = \rho^2 g_t$ for some fixed defining function ρ.

For a background metric g, we let Φ_g denote the Einstein operator in the Bianchi gauge and we let L_g denote its linearization. For $k \in \mathbb{S}^{m,\alpha}_\delta$ ($\delta = 2$) such that $g+k$ is a Riemannian metric (this is guaranteed, for example, if $||k||_{C^{m,\alpha}_\delta(g)} < 1$), we write

$$\Phi_g(g+k) = \Phi_g(g) + L_g \cdot k + Q_g(k).$$

This defines, for each background metric g, the remainder Q_g.

For the rest of the proof, unless otherwise stated, we use $||\ ||_{C^{m,\alpha}}$, $||\ ||_{C^{m,\alpha}_\delta}$ and $||\ ||_{C^{m-2,\alpha}_\delta}$ to denote the norms $||\ ||_{C^{m,\alpha}(g_t)}$, $||\ ||_{C^{m,\alpha}_\delta(g_t)}$ and $||\ ||_{C^{m-2,\alpha}_\delta(g_t)}$, respectively.

Since g_t is nondegenerate for each $t \in (0,t_0)$, there exists a smooth function $C : (0,t_0) \to (0,\infty)$ such that

$$||(L_{g_t})^{-1}|| \leq \frac{1}{2}C(t),$$

for each $t \in (0,t_0)$. Also, given a smooth function $D : (0,t_0) \to (0,\infty)$, Proposition 4.4 guarantees the existence of a smooth function $r : (0,t_0) \to (0,\infty)$ such that

$$||Q_{g_t}(u) - Q_{g_t}(v)||_{C^{m-2,\alpha}_\delta} \leq \frac{1}{2}D(t)||u-v||_{C^{m,\alpha}_\delta},$$

for all $u,v \in \{k \in \mathbb{S}^{m,\alpha}_\delta; ||k||_{C^{m,\alpha}_\delta} < 2r(t)\}$ and each $t \in (0,t_0)$. Changing the function r if necessary, we can suppose

$$r(t) < \frac{1}{2}, \tag{4.5}$$

for each $t \in (0,t_0)$.

We consider now a smooth function $b : (0,t_0) \to (0,\infty)$ such that for any curve $\hat{g}_t$ in $\mathrm{Met}^{m,\alpha}(M)$ with $||g_t - \hat{g}_t||_{C^{m,\alpha}(g_t)} < b(t)$, the following hold:

$$||k||_{C^{m,\alpha}_\delta(\hat{g}_t)} < 1,$$

whenever $k \in \mathbb{S}^{m,\alpha}_\delta$ is such that $||k||_{C^{m,\alpha}_\delta(g_t)} < 1/2$;

$$||(L_{\hat{g}_t})^{-1}|| \leq C(t), \tag{4.6}$$

for each $t \in (0, t_0)$; and

$$||Q_{\hat{g}_t}(u) - Q_{\hat{g}_t}(v)||_{C^{m-2,\alpha}_\delta} \leq D(t)||u - v||_{C^{m,\alpha}_\delta},$$

for all $u, v \in \{k \in \mathbb{S}^{m,\alpha}_\delta; ||k||_{C^{m,\alpha}_\delta} < r(t)\}$ and each $t \in (0, t_0)$.

Lemma 4.8. *For $\lambda \in (0, 1)$ fixed, it is possible to find a family U_t, $t \in (0, t_0)$, of neighborhoods of $\gamma(t)$ in $C^{m,\alpha}(\partial M)$ satisfying the following: for any smooth function $\theta : (0, t_0) \to \mathrm{Met}^{m,\alpha}(\partial M)$ with $\theta(t) \in U_t$, there exists a smooth curve $\tilde{g}_t = \tilde{g}_t^{[\theta(t)]}$ in $\mathrm{Met}^{m,\alpha}(M)$, $t \in (0, t_0)$, such that the inequalities*

$$||g_t - \tilde{g}_t||_{C^{m,\alpha}} < b(t) \tag{4.7}$$

and

$$||\Phi_{\tilde{g}_t}(\tilde{g}_t)||_{C^{m-2,\alpha}_\delta} < \frac{(1-\lambda)}{C(t)} r(t) \tag{4.8}$$

hold, for each $t \in (0, t_0)$. Moreover, if there exist $b_0, C_0, r_0 > 0$ such that $b(t) \geq b_0$, $C(t) \leq C_0$, and $r(t) \geq r_0$ for each $t \in (0, t_0)$, then there exists $\varepsilon_0 > 0$ such that the ball

$$B(\gamma(t), \varepsilon_0) = \{\theta; ||\theta - \gamma(t)||_{C^{m,\alpha}(\gamma(t))} < \varepsilon_0\}$$

is contained in U_t, for each $t \in (0, t_0)$.

We use the terminology *approximate solutions* to refer to the curves $\tilde{g}_t$ given by Lemma 4.8.

The lemma above will be proved in Section 4.4. Assuming it is proved, let us continue the proof of Theorem 4.6.

Let $\lambda \in (0, 1)$. Choose $D(t) = \lambda C(t)^{-1}$, for each $t \in (0, t_0)$, and let U_t, $t \in (0, t_0)$, be a family of open sets of $C^{m,\alpha}(\partial M)$ satisfying the conditions of Lemma 4.8.

Let $\theta : (0, t_0) \to \mathrm{Met}^{m,\alpha}(\partial M)$ be a smooth curve such that $\theta(t) \in U_t$, for each $t \in (0, t_0)$, and let $\tilde{g}_t = \tilde{g}_t^{[\theta(t)]}$, $t \in (0, t_0)$, be a smooth curve on $\mathrm{Met}^{m,\alpha}(M)$ satisfying inequalities (4.7) and (4.8). For each $t \in (0, t_0)$, we consider the map $F_t : \bar{B}_t \to \bar{B}_t$ defined by

$$F_t(k) = -(L_{\tilde{g}_t})^{-1} \left(\Phi_{\tilde{g}_t}(\tilde{g}_t) + Q_{\tilde{g}_t}(k)\right),$$

where $\bar{B}_t$ is the ball

$$\{k \in \mathbb{S}^{m,\alpha}_\delta; ||k||_{C^{m,\alpha}_\delta} \leq r(t)\}.$$

Notice that (4.5) implies $||k||_{C^{m,\alpha}(g_t)} < 1/2$ for $k \in \bar{B}_t$. Therefore, $||k||_{C^{m,\alpha}(\tilde{g}_t)} < 1$, and hence, $Q_{\tilde{g}_t}(k)$ makes sense for $k \in \bar{B}_t$.

Now, by Proposition 4.5, F_t is well defined and it is a λ-contraction. Thus, for each $t \in (0, t_0)$, there exists $f_t \in \mathbb{S}_\delta^{m,\alpha}$ with

$$||f_t||_{C_\delta^{m,\alpha}} < r(t)$$

and

$$\Phi_{\tilde{g}_t}(\tilde{g}_t + f_t) = 0.$$

Notice that, since $f_t \in \mathbb{S}_\delta^{m,\alpha}$ (and we are taking $\delta = 2$), the metric $\tilde{g}_t + f_t$ also has conformal infinity $[\theta(t)]$. Therefore, by taking $\breve{g}_t = \tilde{g}_t + f_t$, for each $t \in (0, t_0)$, we have a smooth function in $E^{m,\alpha}(M)$, $t \in (0, t_0)$, whose conformal infinity is $[\theta(t)]$, $t \in (0, t_0)$, as wished. $\square$

4.4 Construction of the approximate solutions

In this section we prove Lemma 4.8, that is, we explain the construction of the approximate solutions $\tilde{g}_t$.

Proof of Lemma 4.8. Let $\eta : \mathbf{R} \to \mathbf{R}$ be a smooth function such that $0 \leq \eta(s) \leq 1$, for all $s \in \mathbf{R}$; $\eta(s) = 0$, if $s \leq 1$; and $\eta(s) = 1$ if $s \geq 2$. For each $a > 0$, we consider the function $\eta_a : \mathbf{R} \to \mathbf{R}$ defined by $\eta_a(s) = \eta(s/a)$, for each $s \in \mathbf{R}$. The function η_a satisfies the following: $0 \leq \eta_a(s) \leq 1$, for all $s \in \mathbf{R}$; $\eta_a(s) = 0$, if $s \leq a$; and $\eta_a(s) = 1$, if $s \geq 2a$. Furthermore, it is worth noticing that $||\eta_a||_{C^{m,\alpha}} \to 1$ as $a \to \infty$.

In order to make the discussion a little more clear, we make the following assumption about (V, h):

$(*)$ The orbifold V has only one singular point, which we denote by p.

Once the proof is complete under this hypothesis, we will show how to remove it.

Fix a smooth resolution $\pi : M \to V$ (as an example, in the particular case in which $M = \mathbf{TS}^2$ and $V = C(\mathbf{RP}^3)$, this resolution can be taken to be the map that collapses the zero section of $\mathbf{TS}^2$ to the vertex p of the orbifold), and consider the function $d : M \to [0, \infty)$ defined, for each $x \in M$, as the distance, in the metric h, from $\pi(x)$ to the singularity p.

Let $\ell : (0, t_0) \to (0, \infty)$ be a smooth function. It is possible to pick a smooth function $\varrho : (0, t_0) \to (0, \infty)$ such that the region

$$M_t = \{x \in M;\ d(\pi(x)) > \varrho(t)\}$$

is homeomorphic to $(0,\infty)\times\partial M$ and such that, on this region, the inequality

$$||g_t - \pi^* h^{[\gamma(t)]}||_{C^{m,\alpha}(g_t)} < \ell(t) \tag{4.9}$$

holds, for each $t\in(0,t_0)$. Also, there exists a family U_t, $t\in(0,t_0)$, each U_t a neighborhood of $\gamma(t)$ in $C^{m,\alpha}(\partial M)$, such that if $\theta:(0,t_0)\to \mathrm{Met}^{m,\alpha}(\partial M)$ is a smooth function with $\theta(t)\in U_t$, then, on the region M_t,

$$||\pi^* h^{[\theta(t)]} - \pi^* h^{[\gamma(t)]}||_{C^{m,\alpha}(g_t)} < \ell(t), \tag{4.10}$$

for each $t\in(0,t_0)$. Moreover, if there exists $\ell_0>0$ such that $\ell(t)\geq \ell_0$, then the U_t's can be taken so that there exists $\varepsilon_0>0$ for which the ball

$$B(\gamma(t),\varepsilon_0) = \{\theta; ||\theta-\gamma(t)||_{C^{m,\alpha}(\gamma(t))} < \varepsilon_0\}$$

is contained in U_t, for each $t\in(0,t_0)$.

Let now $a:(0,t_0)\to(0,\infty)$ be a smooth function (which will be chosen later). For $t\in(0,t_0)$, let $\chi_t : M\to[0,\infty)$ be the function given by

$$\chi_t(x) = \eta_{a(t)}(d(x)-\varrho(t)),$$

for any $x\in M$. We clearly have that $0\leq\chi_t\leq 1$, $\chi_t(x)=0$ if $d(x)\leq\varrho(t)$, and $\chi_t(x)=1$ if $d(x)\geq\varrho(t)+a(t)$.

We can now define the approximate solutions $\tilde g_t = \tilde g_t^{[\theta(t)]}$ by

$$\tilde g_t = \chi_t h^{[\theta(t)]} + (1-\chi_t) g_t,$$

for each $t\in(0,t_0)$. We have, using $||\ ||_{C^{m,\alpha}}$ to denote $||\ ||_{C^{m,\alpha}(g_t)}$,

$$\begin{aligned} ||g_t-\tilde g_t||_{C^{m,\alpha}} &= ||g_t - \left(\chi_t h^{[\theta(t)]} + (1-\chi_t)g_t\right)||_{C^{m,\alpha}} \\ &= ||\chi_t(g_t - \pi^* h^{[\theta(t)]})||_{C^{m,\alpha}} \\ &\leq ||\chi_t(g_t-\pi^* h^{[\gamma(t)]})||_{C^{m,\alpha}} + ||\chi_t(\pi^* h^{[\gamma(t)]} - \pi^* h^{[\theta(t)]})||_{C^{m,\alpha}}. \end{aligned}$$

The function a can be chosen in such a way that

$$||\chi_t(g_t-\pi^* h^{[\gamma(t)]})||_{C^{m,\alpha}} \leq \Theta||g_t - \pi^* h^{[\gamma(t)]})||_{C^{m,\alpha}(M_t,g_t)}$$

and

$$||\chi_t(\pi^* h^{[\gamma(t)]} - \pi^* h^{[\theta(t)]})||_{C^{m,\alpha}} \leq \Theta ||(\pi^* h^{[\gamma(t)]} - \pi^* h^{[\theta(t)]})||_{C^{m,\alpha}(M_t,g_t)},$$

for each $t \in (0, t_0)$, where Θ is a constant not depending on t. Using inequalities (4.9) and (4.10), we get

$$||g_t - \tilde{g}_t||_{C^{m,\alpha}} \leq 2\Theta\ell(t), \tag{4.11}$$

for each $t \in (0, t_0)$.

Finally, it is possible to choose the function ℓ in a way that inequality (4.11) implies inequalities (4.7) and (4.8). Furthermore, if for some $b_0, C_0, r_0 > 0$ the inequalities $b(t) \geq b_0$, $C(t) \leq C_0$ and $r(t) \geq r_0$ hold, for each $t \in (0, t_0)$, it is possible to take the function ℓ so that for some $\ell_0 > 0$, the inequality $\ell(t) \geq \ell_0$ holds, for each $t \in (0, t_0)$. This finishes the proof under the condition $(*)$. Let us now show how to remove this condition.

Suppose then that $\{p_1, \ldots, p_k\}$, $k \geq 2$, are the orbifold singularities of V. For $i \in \{1, \ldots, k\}$, we define the function $d_i : M \to [0, \infty)$ as follows: for each $x \in M$, $d_i(x)$ is the distance, in the metric h, between the points $\pi(x)$ and p_i. Let $d : M \to [0, \infty)$ be defined by

$$d(x) = \min_{i \in \{1,\ldots,k\}} d_i(x),$$

for each $x \in M$.

It is not hard to see that the same proof given above goes through. □

4.5 Orbifold degeneration III

For all of this section, $h^{[\gamma]}$ will denote a nondegenerate $C^{m,\alpha}$ conformally compact Einstein metric on V with conformal infinity $[\gamma]$, and W will denote a neighborhood of $[\gamma]$ in $\mathcal{C}^{m,\alpha}$ as the one described in the beginning of Section 4.1.

Lemma 4.9. *Let $g_t = g_t^{[\gamma(t)]}$, $t \in (0, 1)$, be a smooth curve in $E^{m,\alpha}(M)$ such that g_t degenerates, as $t \to 0$, to a nondegenerate $C^{m,\alpha}$ conformally compact Einstein metric $h^{[\gamma]}$ on V. Suppose there exists $t_0 \in (0, 1)$ such that g_t is nondegenerate for each $t \in (0, t_0)$. If Y_t and X_t are neighborhoods of g_t in $\mathcal{E}^{m,\alpha}$ and $[\gamma(t)]$ in $\mathcal{C}^{m,\alpha}$, respectively, such that $\Pi : Y_t \to X_t$ is a diffeomorphism, then the set*

$$\{[\gamma(s)]; s \in (0, t]\} \cup \{[\gamma]\}$$

is not contained in X_t, for each $t \in (0, t_0)$.

Proof. First, extend the curve γ to a continuous curve defined on $[0,1)$ by setting $\gamma(0)=\gamma$. We then have

$$\{[\gamma(s)]; s\in(0,t]\}\cup\{[\gamma]\}=\{[\gamma(s)]; s\in[0,t]\}.$$

Suppose $t_1\in(0,t_0)$ is such that X_{t_1} contains $\{[\gamma(t)];\ t\in[0,t_1]\}$ (see Figure 4.2). We then get, via the diffeomorphism $\Pi : Y_{t_1}\to X_{t_1}$, a continuous curve $\breve{g}:[0,t_1]\to Y_{t_1}$ such that the conformal infinity of $\breve{g}_t$ is $[\gamma(t)]$, for each $t\in[0,t_1]$.

Consider now the set

$$A=\{t\in(0,t_1]; g_t=\breve{g}_t\}.$$

Using that Π is a local diffeomorphism near the nondegenerate metrics g_t, one clearly finds that the set A is open and closed in $(0,t_1]$. Also, since Π is injective on Y_{t_2}, A is not empty. Since $(0,t_1]$ is connected, we have $A=(0,t_1]$. Thus, $g_t=\breve{g}_t$ for each $t\in(0,t_1]$.

Now, since $\{[\gamma(t)],t\in[0,t_1]\}$ is a compact subset of $\mathcal{C}^{m,\alpha}$ and $\Pi^{-1}:X_{t_1}\to Y_{t_1}$ is continuous, the set $\{\breve{g}_t;t\in[0,t_1]\}\subset Y_{t_1}$ is a compact subset of $\mathcal{E}^{m,\alpha}$. This implies, on one hand, that

$$\{||\breve{g}_t||_{C^{m,\alpha}(g_t)};t\in(0,1]\}$$

is bounded. On the other hand, since g_t degenerates to the orbifold metric h, $||g_t||_{C^{m,\alpha}(g_t)}\to\infty$ as $t\to0$. This contradicts the fact that $g_t=\breve{g}_t$, for $t\in(0,t_1]$. $\square$

Theorem 4.10. *Let $g_t=g_t^{[\gamma(t)]}$, $t\in(0,1)$, be a smooth curve in $E^{m,\alpha}(M)$ such that g_t degenerates, as $t\to0$, to a nondegenerate $C^{m,\alpha}$ conformally compact Einstein metric $h^{[\gamma]}$ on V. For each $t\in(0,1)$, consider the map*

$$\begin{array}{rcl}\Psi_t:\mathbb{S}^{m,\alpha}_\delta & \to & \mathbb{S}^{m-2,\alpha}_\delta\\ k & \mapsto & \Phi_{g_t}(g_t+k)\end{array}$$

where $\delta=2$ and Φ_g denotes the Bianchi gauged Einstein operator with background metric g. Suppose there exists $t_0\in(0,1)$ such that g_t is nondegenerate, for each $t\in(0,t_0)$. If there exist $t_1\in(0,t_0)$ and a constant μ (independent of t) such that the map $(\Psi_t)'$ is μ-Lipschitz, for each $t\in(0,t_1)$, then the set

$$\{||(L_{g_t})^{-1}||;t\in(0,t_0)\}$$

is unbounded.

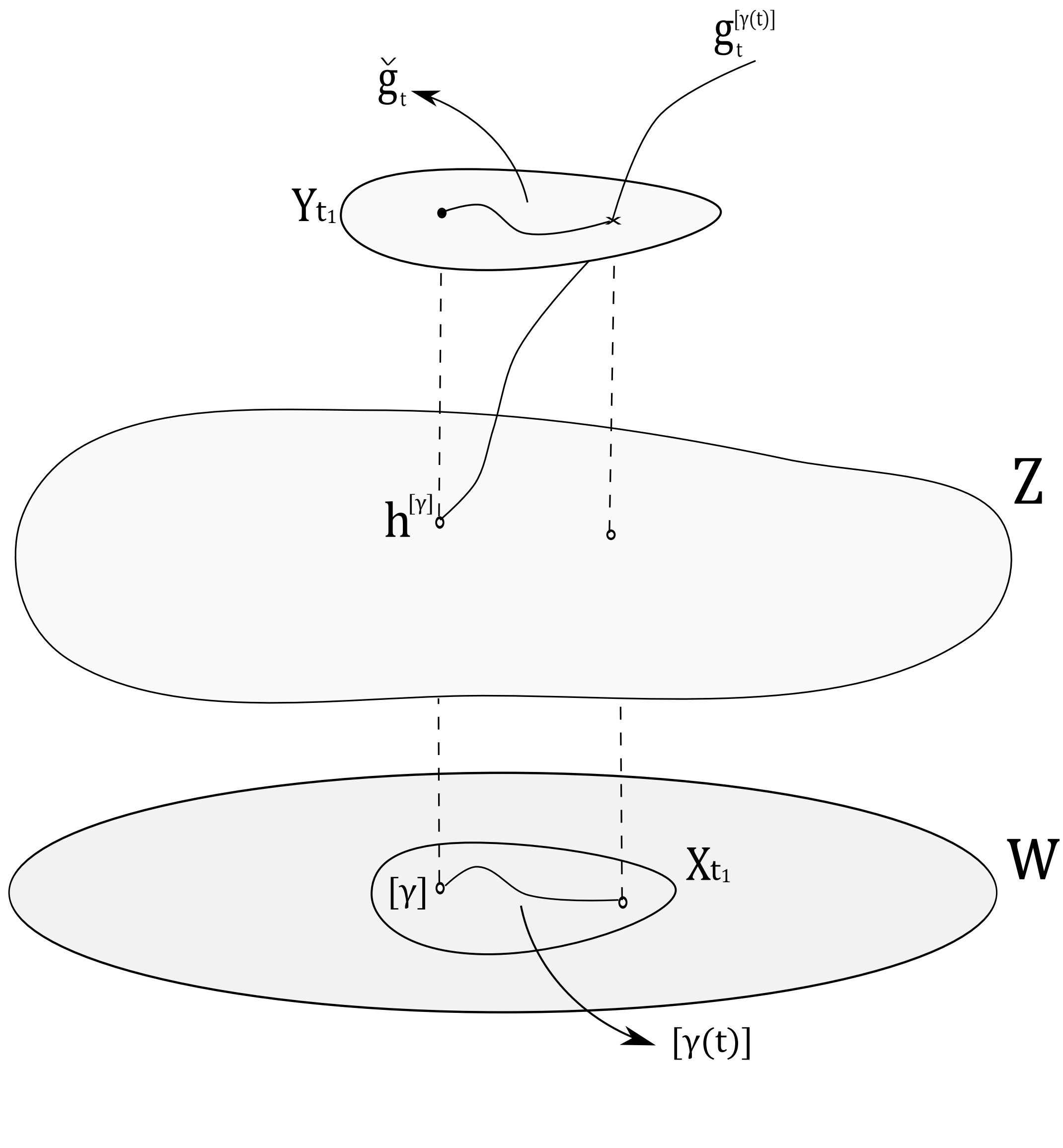

Figure 4.2: Visualization of the proof of Lemma 4.9.

Proof. Suppose, seeking a contradiction, that there exists $C > 0$ such that

$$||(L_{g_t})^{-1}|| \leq C, \tag{4.12}$$

for each $t \in (0, t_0)$. By Proposition 4.4, there exists r_0 independent of t such that the inequality

$$||Q_{g_t}(u) - Q_{g_t}(v)|| \leq \frac{\lambda}{C}||u - v|| \tag{4.13}$$

holds, for any u, v such that $||u||_{C^{m,\alpha}_\delta}, ||v||_{C^{m,\alpha}_\delta} \leq r_0$. Here, $\lambda \in (0,1)$ is fixed and Q_{g_t} is the remainder defined by the equation

$$\Psi_t(k) = \Psi(0) + \Psi'(0) \cdot k + Q_{g_t}(k),$$

or equivalently,

$$\Phi_{g_t}(g_t + k) = \Phi_{g_t}(g_t) + L_{g_t} \cdot k + Q_{g_t}(k).$$

Consider a function $b : (0, t_1) \to (0, \infty)$ as in the proof of Theorem 4.6, that is, the function b is such that for any curve $\hat{g}_t$ in $\mathrm{Met}^{m,\alpha}(M)$ with $||g_t - \hat{g}_t||_{C^{m,\alpha}(g_t)} < b(t)$, the following hold:

$$||k||_{C^{m,\alpha}_\delta(\hat{g}_t)} < 1,$$

whenever $k \in \mathbb{S}^{m,\alpha}_\delta$ is such that $||k||_{C^{m,\alpha}_\delta(g_t)} < 1/2$;

$$||(L_{\hat{g}_t})^{-1}|| \leq 2||(L_{g_t})^{-1}||,$$

for each $t \in (0, t_1)$; and

$$||Q_{\hat{g}_t}(u) - Q_{\hat{g}_t}(v)||_{C^{m-2,\alpha}_\delta} \leq 2||Q_{g_t}(u) - Q_{g_t}(v)||,$$

for all $u, v \in \{k \in \mathbb{S}^{m,\alpha}_\delta; ||k||_{C^{m,\alpha}_\delta} < r_0/2\}$ and each $t \in (0, t_1)$.

Now, inequalities (4.12) and (4.13) imply that the function b can be taken so that $b(t) \geq b_0$, for each $t \in (0, t_1)$, for some $b_0 > 0$ independent of t. By Lemma 4.8, the neighborhoods U_t given by Theorem 4.6 can be taken so that for some $\varepsilon_0 > 0$, U_t contains the ball

$$B(\gamma(t), \varepsilon_0) = \{\theta; ||\theta - \gamma(t)||_{C^{m,\alpha}(\gamma(t))} < \varepsilon_0\}.$$

If X_t is the neighborhood in $\mathcal{C}^{m,\alpha}$ corresponding to U_t, we have that some $t_2 \in (0, 1)$

is such that X_{t_2} contains

$$\{[\gamma(s)]; s \in (0, t_2]\} \cup \{[\gamma]\},$$

which cannot happen due to Lemma 4.9. $\square$

For each $t \in (0, t_0)$, denote by ξ_{g_t} the first eigenvalue of L_{g_t}. We have that $||(L_{g_t})^{-1}||$ equals $\xi_{g_t}^{-1}$, and hence, ξ_{g_t} bounded below by some positive constant corresponds to $||(L_{g_t})^{-1}||$ bounded above. Hence, under the assumptions of Theorem 4.10, we conclude that if (M, g_t) degenerates to (V, h), then $\xi_{g_{t_i}} \to 0$ for some subsequence $\{t_i\} \subset (0, 1)$.

One expects, therefore, the first eigenvalue ξ_h of L_h to be zero, that is, one expects Conjecture 2.23 to be true. Surprisingly, as we've seen in Chapter 3, this conjecture is false.

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