Weighted L^2 interpolation on non-uniformly separated sequences

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Abstract of the Dissertation

Weighted L^2 interpolation on non-uniformly separated sequences

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We define several weighted ℓ^2-norms associated to a discrete sequence Γ in $\mathbb{C}$ and a weight function φ. We then give a sufficient condition which ensures that we can always extend weighted-ℓ^2 data to global holomorphic functions which are also weighted-L^2. The condition is that the so-called upper density of Γ is strictly less then one.

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> The woods are lovely, dark and deep,
> But I have promises to keep,
> And miles to go before I sleep,
> And miles to go before I sleep...
>
> ”Stopping By Woods on a Snowy Evening” by Robert Frost

Completing this thesis represents the culmination of a journey. As I stand at the end of the road, I want to thank the people who walked by my side, guiding me along through the rises and the falls, the sudden turns and the prolonged plateaus, the wrong ways and the dead ends.

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Chapter 1

Introduction

1.1 Motivation for the central problem

A fundamental problem in mathematics is the problem of interpolation: given a set of points, i.e., values of an independent variable, and a set of numbers associated to those points, i.e., values of the dependent variable, find a function that realizes the given values at the given points. Without any further restrictions, this problem is trivial to solve. But if one restricts the set of possible values and the class of interpolating functions, it is not clear whether one can find a function for a given set of values.

1.1.1 Lagrange Interpolation

One of the oldest versions of this problem, which we teach in school today, is the so-called Lagrange Interpolation Problem: given N points $\gamma_1, ..., \gamma_N$ in the complex plane, and N complex numbers $c_1, ..., c_N$, find a polynomial p of degree d such that

$$p(\gamma_j) = c_j, \quad 1 \leq j \leq N.$$

The problem always has a solution if and only if $d \geq N - 1$.

1.1.2 Extension of entire functions

A slightly less classical version of the problem is the following theorem, often taught in a first course in complex analysis at the graduate level.

THEOREM 1. *Let $\{z_j\}$ be a discrete sequence in $\mathbb{C}$ and let $\{a_j\}$ be an arbitrary sequence of complex numbers. Then there exists an entire function f such that $f(z_j) = a_j$.*

One way to interpret the above theorem is the following. Fix a discrete sequence Γ in $\mathbb{C}$. Any sequence of complex numbers can be considered as a function on Γ and vice versa. Then Theorem 1 says that any function on Γ can be extended to an entire function. Note that Theorem 1 is trivially an *if and only if* statement. We will come back to this point later.

Remark. The classical proof of Theorem 1 uses the Mittag-Leffler Theorem (see [15] for the statement of the Mittag-Leffler Theorem and the classical proof of Theorem 1). We will latter give a different proof using the so-called $\bar{\partial}$-technique.

1.1.3 The Fock Space

It is interesting to try to equip the sequences in the hypothesis of Theorem 1 with more structure and ask if there still exists an *interpolating* function which behaves well with respect to that structure.

In the late 1980's, the following problem arose in the study of laser physics (see [5]). Find two complex numbers σ and τ that are independent over $\mathbb{R}$, such that the lattice

$$\Gamma := \{a\sigma + b\tau \ ; \ a, b \in \mathbb{Z}\}$$

has the following properties.

1. For any set of complex numbers $\{c_\gamma\}_{\gamma\in\Gamma}$ such that

$$\sum_{\gamma\in\Gamma} |c_\gamma|^2 e^{-\pi|\gamma|^2} < +\infty$$

   there is a function f in the Fock space

$$\mathscr{F}(\mathbb{C}) := \left\{ f \in \mathcal{O}(\mathbb{C}) \ ; \ \int_{\mathbb{C}} |f(z)|^2 e^{-\pi|z|^2} dA(z) < +\infty \right\}$$

   such that $f(\gamma) = c_\gamma$ for all $\gamma \in \Gamma$.

2. There are positive constants C and D for every $f \in \mathscr{F}(\mathbb{C})$,

$$C \int_{\mathbb{C}} |f(z)|^2 e^{-\pi|z|^2} dA(z) \le \sum_{\gamma\in\Gamma} |f(\gamma)|^2 e^{-\pi|\gamma|^2} \le D \int_{\mathbb{C}} |f(z)|^2 e^{-\pi|z|^2} dA(z). \tag{1.1.1}$$

Note that 2 implies that if one can find the function in 1, then that function is unique.

Note that the use of a lattice in the statement of the problem just discussed is not necessary; one can ask the question for any discrete set of points in the complex plane. In 1992 K. Seip proved that no such set Γ, a lattice or not, can exist (see [17]).

The results of Seip and his collaborators have even had applications in the theory of sampling and interpolation— a theory that is central in the storage and retrieval of data. We refer the reader to [3, 4, 5] for further references on the applications of these ideas.

In this thesis, we will mostly be concerned with conditions under which a collection of values can be extended, i.e., generalizations on part 1 of the problem just stated.

1.2 Extension Problems in Generalized Fock Space

1.2.1 Extension in Generalized Fock space

Let Γ be a discrete sequence in $\mathbb{C}$ and let $\nu : \mathbb{C} \to \mathbb{R}$ be a continuous function. We will soon place further restrictions on ν, but for the purposes of the next definitions continuity is more than enough. Consider the following Hilbert spaces:

$$\mathscr{H}^2_\nu(\mathbb{C}) := \left\{ F \in \mathcal{O}(\mathbb{C}) \ : \ ||F||^2_\nu := \int_{\mathbb{C}} |F|^2 e^{-\nu} dA < +\infty \right\}$$

and

$$\ell^2_\nu(\Gamma) := \left\{ \{a_\gamma\}_{\gamma\in\Gamma} : \sum_{\gamma\in\Gamma} |a_\gamma|^2 e^{-\nu(\gamma)} < +\infty \right\}.$$

We would like to understand what conditions on Γ and ν will ensure that given any $\{a_\gamma\}_{\gamma\in\Gamma} \in \ell^2_\nu(\Gamma)$ there exists an $F \in \mathscr{H}^2_\nu(\mathbb{C})$ such that $F(\gamma) = a_\gamma$. Another way to phrase this requirement is the following. Consider the restriction mapping

$$R_\Gamma : \mathscr{H}^2_\nu(\mathbb{C}) \to \ell^2_\nu(\Gamma)$$

which sends F to its restriction to Γ. Then the question is, when is R_Γ surjective? Notice that the question of surjectivity is independent of whether R_Γ is well-defined on all of $\mathscr{H}^2_\nu(\mathbb{C})$. We will also consider the question of whether R_Γ is bounded.

REMARK. Note that in the lattice problem discussed above, Statement 2 is a stronger version of asking that R_Γ be injective; in fact, the inequality (1.1.1) is equivalent to requiring that R_Γ be bounded, injective, and with closed range.

1.2.2 Interpolation in the classical Fock space

The first positive results were obtained by Seip and Wallsten [17, 19], who considered the case of $\nu = \pi|z|^2$. (The constant π is irrelevant, being a normalization of sorts. It is only important that $\pi > 0$.) To state the results of Seip-Wallsten, we need to define a notion of density of a sequence— the analog of a concept that A. Beurling introduced in the case of the Hardy space.

DEFINITION 1.2.1. The *upper Beurling density* of Γ is the number

$$D^+(\Gamma) := \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{n_\Gamma(z,r)}{\pi r^2}.$$

In the last statement, $n_\Gamma(z,r)$ denotes the cardinality of the set $\Gamma_r(z) := B_r(z)\cap\Gamma$, where $B_r(z)$ is the disk of radius r centered around z. We shall employ this notation from here on.

REMARK. Since Γ is discrete, $n_\Gamma(z,r) < +\infty$ for any z and r.

We also need the following notion of separation for a sequence.

DEFINITION 1.2.2. A sequence Γ is said to be *uniformly separated* if

$$\rho := \inf_{\substack{\tilde{\gamma},\gamma\in\Gamma \\ \tilde{\gamma}\neq\gamma}} |\gamma - \tilde{\gamma}| > 0.$$

In this case ρ is called the *separation constant* of Γ.

We then have the following theorem.

THEOREM 2 (Seip, Wallsten). *Let $\nu = \pi|z|^2$. The following statements are equivalent.*

1. *For any $\{a_\gamma\}_{\gamma\in\Gamma} \in \ell^2_\nu(\Gamma)$ there exists $F \in \mathscr{H}^2_\nu(\mathbb{C})$ such that $F(\gamma) = a_\gamma$ for all $\gamma \in \Gamma$, i.e., the restriction map R_Γ is surjective.*

2. *The sequence Γ is uniformly separated and $D^+(\Gamma) < 1$.*

The original proof of Theorem 2 used classical results of a one-variable nature together with much unpublished work of Beurling, and in addition relied heavily on the translation invariance of the euclidean norm. As mentioned already, we will employ $\bar{\partial}$- techniques to prove theorems of the type given above.

1.2.3 Sampling in the classical Fock space

For the sake of completeness, we shall state Seip-Wallsten's Theorem on the injectivity of the restriction map, though we shall not return to the subject again.

DEFINITION 1.2.3. The lower Beurling density of a sequence Γ is the number

$$D^{-}(\Gamma) := \liminf_{r\to\infty} \inf_{z\in\mathbb{C}} \frac{n_\Gamma(z,r)}{\pi r^2}.$$

We then have the following theorem.

THEOREM 3 (Seip, Wallsten). *Let $\nu = \pi|z|^2$. The following statements are equivalent.*

1. *The sampling inequality* (1.1.1) *is satisfied.*
2. *The sequence Γ can be written as a finite union of uniformly separated sequences $\Gamma = \Gamma_1 \cup ... \cup \Gamma_N$ such that $D^{-}(\Gamma_1) > 1$.*

1.2.4 Interpolation in the generalized Fock Space

The next results that we discuss are due to Berndtsson, Ortega-Cerdà, and Seip [2, 13]. These results are direct generalizations of the results of Seip and Wallsten.

As before, we first need to make a few definitions. Let $C^2(\mathbb{C})$ denote the space of real valued twice-continuously differentiable functions on $\mathbb{C}$. From now on, fix $\varphi \in C^2(\mathbb{C})$ such that for some constants $M > m > 0$,

$$m \leq \Delta\varphi \leq M. \tag{1.2.1}$$

We have set $\Delta := \frac{1}{\pi}\frac{\partial^2}{\partial z \partial \bar{z}}$, which is off by a factor of 4π from the usual Laplacian. This normalization is convenient in the formulation of our results. We also need a notion of density that is adapted to the weight φ.

DEFINITION 1.2.4. The *upper Beurling density with respect to* φ is the number

$$D^+_\varphi(\Gamma) := \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{n_\Gamma(z,r)}{\int_{B_r(z)} \Delta\varphi dA}$$

THEOREM 4 (Berndtsson, Ortega-Cerdà, Seip). *The following statements are equivalent.*

1. *For any* $\{a_\gamma\}_{\gamma\in\Gamma} \in \ell^2_\varphi(\Gamma)$ *there exists* $F \in \mathscr{H}^2_\varphi(\mathbb{C})$ *such that* $F(\gamma) = a_\gamma$ *for all* $\gamma \in \Gamma$*, i.e., the restriction map* R_Γ *is surjective.*

2. *The sequence* Γ *is uniformly separated and* $D^+_\varphi(\Gamma) < 1$.

REMARK. Theorem 4 has a *sampling* companion analogous to Theorem 3.

The history of the subject of generalized Fock interpolation is rich and varied, and we shall not go into it beyond what we have already said. The interested reader can find more in the papers cited above.

1.2.5 Other underlying spaces

Some work in higher dimensions has been done [11, 14]. There is also an analogous set of problems that can be formulated in the unit disk. See for example [18] or the book [6], and [8] for the treatment of the Bergmann ball in higher dimensions. Finally, some work has been done in more general complex manifolds: the one dimensional case was first treated by Schuster and Varolin in [16], and later by Ortega-Cerdà in [12], and the higher dimensional case was touched upon by Forgàcs [7]

1.3 Main results

In this thesis our goal is to investigate what happens when the sequence Γ is no longer uniformly separated. Because the theorem of Berndtsson et al provides necessary and sufficient conditions for extension, something must be changed in the formulation of problem we consider. We have chosen to replace the space $\ell^2_\varphi(\Gamma)$ with another weighted-ℓ^2 space and seek a positive result.

We associate to every $\gamma \in \Gamma$ the following numbers:

$$\rho_\gamma := \min(1, \inf_{\tilde{\gamma}\in\Gamma\setminus\{\gamma\}} \frac{|\gamma - \tilde{\gamma}|}{2}) \quad \text{and} \quad n_\gamma := n_\Gamma(\gamma, 1).$$

DEFINITION 1.3.1. We define the Hilbert space

$$\mathfrak{H}^2_\varphi(\Gamma) := \left\{ \{a_\gamma\}_{\gamma\in\Gamma} : \ \|\{a_\gamma\}\|^2_\varphi := \sum_{\gamma\in\Gamma} |a_\gamma|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^{2n_\gamma}} < +\infty \right\}.$$

With this notation, we are ready to state our main result.

THEOREM A. *Let Γ and φ be as above and suppose that $D^+_\varphi(\Gamma) < 1$. Then the restriction map*

$$R_\Gamma : \mathscr{H}^2_\varphi(\mathbb{C}) \to \mathfrak{H}^2_\varphi(\Gamma),$$

which sends f to its restriction to Γ, is surjective.

Remark. Note that we do not claim the map R_Γ is well-defined; in fact Proposition 2.2.4 shows that it is defined and bounded on all of $\mathscr{H}^2_\varphi(\mathbb{C})$ if and only if the sequence Γ is *uniformly separated*, i.e., there exists $\varepsilon > 0$ such that

$$\rho_\gamma \geq \varepsilon \quad \text{for all } \gamma \in \Gamma.$$

Remark. If the restriction map is surjective, we say that Γ is *interpolating*. Note that the property of being interpolating is always relative to the norms of the Hilbert spaces under consideration.

Remark. Our theorem does not provide any necessary conditions for interpolation. It is not known at this time (and is likely not the case) wether the density condition is also necessary and not just sufficient.

For various reasons that originated in higher dimensional considerations, it is also desirable to understand another kind of ℓ^2-norm whose definition may seem rather unmotivated at first.

DEFINITION 1.3.2. Let

$$\lambda_r(z) := \exp(\sum_{\gamma\in\Gamma} \log|z-\gamma| - \frac{1}{\pi r^2}\int_{B(z,r)} \log|\xi-\gamma| dA_\xi).$$

DEFINITION 1.3.3. Let

$$\|\{a_\gamma\}\|_r^2 := \sum_{\gamma\in\Gamma} |a_\gamma|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^2 |\partial\lambda_r(\gamma)|^2}.$$

Then define

$$\mathfrak{H}^2_{\varphi,\lambda_r}(\Gamma) := \{\{a_\gamma\}_{\gamma\in\Gamma} \ : \ \|\{a_\gamma\}\|_r^2 < \infty\}.$$

We then have the following theorem.

THEOREM B. *Let Γ and φ be as above and suppose that $D^+_\varphi(\Gamma) < 1$. Then the restriction map*

$$\mathcal{R}_\Gamma : \mathscr{H}^2_\varphi(\mathbb{C}) \to \mathfrak{H}^2_{\varphi,\lambda_r}(\Gamma)$$

is surjective for r sufficiently large.

Throughout, the notation $f \lesssim g$ will be used to mean that there exists a constant $\tilde{C} > 0$ independent from f and g such that $f \le \tilde{C}g$ and $f \simeq g$ will mean that $f \lesssim g$ and $g \lesssim f$. Furthermore, C will be used to denote an arbitrary positive constant whose value could change from one occurrence to the next. We will sometimes write C_r when we wish to emphasize a dependence on some particular parameter r. Finally, we will use $\lesssim_r$ and $\simeq_r$ to emphasize that the constants involved might depend on the parameter r.

Chapter 2

Preliminary facts

2.1 Local estimates

The following Lemmas can be found, in one form or another, in the papers [2] and [13]. They are fundamental to much of what we do later.

LEMMA 2.1.1. *Let $\nu \in C^2(\mathbb{C})$ be a subharmonic function and suppose there exists a constant $M > 0$ such that $\Delta\nu \leq M$. Take any $z \in \mathbb{C}$ and any $r > 0$. Then there exists a holomorphic function H_z defined in $B_r(z)$, with $H_z(z) = 0$, and a constant C_r independent of z such that*

$$|\nu(z) - \nu(w) + 2\,\mathrm{Re}\,H_z(w)| \leq C_r$$

for all $w \in B_r(z)$. Moreover, there exists a constant C (independent of r) such that $C_r \leq C$ if $0 < r \leq 1$.

Proof. If we define h_z in $B_r(z)$ by

$$h_z(w) := \nu(w) - \nu(z) + \int_{B_r(z)} (\ln|z-\xi| - \ln|w-\xi|)\Delta\nu(\xi)dA_\xi \qquad (2.1.1)$$

then h_z is harmonic and $h_z(z) = 0$. Since $B_r(z)$ is simply connected there exists a holomorphic function H_z such that $2\,\mathrm{Re}\,H_z = h_z$ and $\mathrm{Im}\,H_z(z) = 0$. We also have the following estimates:

1. $-\frac{M\pi}{2} \leq \int_{B_r(z)} \ln|z-\xi|\Delta\nu(\xi)dA_\xi \leq M\pi r^2 \ln r$
2. $-M\pi r^2 \ln 2r \leq -\int_{B_r(z)} \ln|w-\xi|\Delta\nu(\xi)dA_\xi \leq \frac{\pi M}{2}$

To see the upper estimate in (1) we observe that since $\ln|z-\xi| \leq 0$ for $\xi \in B_1(z)$ and $0 \leq \ln|z-\xi| \leq \ln r$ for $\xi \in B_r(z) \setminus B_1(z)$ and $0 \leq \Delta\nu \leq M$ we have that

$$\int_{B_r(z)} \ln|z-\xi|\Delta\nu(\xi)dA_\xi \leq \int_{B_r(z)\setminus B_1(z)} \ln|z-\xi|\Delta\nu(\xi)dA_\xi \leq M\pi r^2 \ln r.$$

The lower estimate holds because

$$\begin{aligned}
\int_{B_r(z)} \ln|z-\xi|\Delta\nu(\xi)dA_\xi &\geq \int_{B_1(z)} \ln|z-\xi|\Delta\nu(\xi)dA_\xi \\
&\geq M\int_{B_1(z)} \ln|z-\xi|dA_\xi \\
&= 2\pi M\int_0^1 u\ln u\,du \\
&= -\frac{\pi M}{2}.
\end{aligned}$$

To see (2) let $\Delta_1 = B_r(z) \cap B_1(w)$ and $\Delta_2 = B_r(z) \setminus \Delta_1$. Then

$$-\int_{B_r(z)} \ln|w-\xi|\Delta\nu(\xi)dA_\xi = I_1 + I_2$$

where

$$I_1 = -\int_{\Delta_1} \ln|w-\xi|\Delta\nu(\xi)dA_\xi$$

and

$$I_2 = -\int_{\Delta_2} \ln|w-\xi|\Delta\nu(\xi)dA_\xi.$$

Note that $|w-\xi| \leq 1$ for $\xi \in \Delta_1$ and $1 \leq |w-\xi| \leq 2r$ for $\xi \in \Delta_2$. Then just as above we have the estimates

$$\begin{aligned} 0 \leq I_1 &\leq -\int_{B_1(w)} \ln|w-\xi|\Delta\nu(\xi)dA_\xi \\ &\leq -2\pi M \int_0^1 u \ln u \, du \\ &= \frac{\pi M}{2} \end{aligned}$$

and

$$\begin{aligned} 0 \geq I_2 &\geq -\int_{\Delta_2} \ln 2r \Delta\nu(\xi)dA_\xi \\ &\geq -M \ln 2r \int_{B_r(z)} dA_\xi \\ &= -M\pi r^2 \ln 2r. \end{aligned}$$

Hence

$$-M\pi r^2 \ln 2r \leq I_1 + I_2 \leq \frac{\pi M}{2},$$

which is what (2) claims. If we choose $C_r = M\pi(r^2 \ln 2r + \frac{1}{2})$, the first part of the claim follows. Since $C_r \to 0$ as $r \to 0$ we can choose a constant C such that $C_r \leq C$ for $0 < r \leq 1$. This completes the proof. $\square$

Remark. We point out the fact that we do not require a strictly positive lower bound on $\Delta\nu$ in Lemma 2.1.1. While we do not use this fact in our work it becomes important if one tries to relax the positivity assumptions on ν.

LEMMA 2.1.2. *Let ν be as in Lemma 2.1.1. Take any $z \in \mathbb{C}$ and any $0 < r \leq 1$. Then given any $F \in \mathscr{H}^2_\nu(\mathbb{C})$, the following estimate holds:*

$$|F(z)|^2 e^{-\nu(z)} \lesssim \frac{1}{\pi r^2} \int_{B_r(z)} |F(w)|^2 e^{-\nu(w)} dA_w \lesssim \frac{1}{\pi r^2} \|F\|^2_\nu.$$

Proof. Take any H_z that satisfies the conclusions of Lemma 2.1.1. Using Cauchy's Integral Formula and Lemma 2.1.1, we get the estimate

$$\begin{aligned} |F(z)|^2 = |F(z)e^{-H_z(z)}|^2 &\leq \frac{1}{\pi r^2} \int_{B_r(z)} |F(w)|^2 e^{-2\operatorname{Re} H_z(w)} dA_w \\ &\lesssim \frac{1}{\pi r^2} \int_{B_r(z)} |F(w)|^2 e^{-\nu(w)} dA_w e^{\nu(z)}, \end{aligned}$$

which implies the result. $\square$

2.2 Density and separation

We first make a few general comments regarding density. Note that, by definition, $D_\varphi^+(\Gamma) < \alpha$ if and only if there exists some $\delta > 0$ such that for all $z \in \mathbb{C}$ and all r sufficiently large, $n_\Gamma(z, r) < (\alpha - \delta) \int_{B_r(z)} \Delta\varphi dA$. Then an upper bound on $\Delta\varphi$ implies that if the density is finite, then the number of points of Γ contained in a disk of radius r can grow at most quadratically as a function of r. Also, we have defined density by counting the number of points of the sequence contained in disks (and dividing by their weighted area). In [10], Landau showed that if the sequence is uniformly separated, then one can measure the density by counting the number of points contained in translations and dilations of an arbitrary compact set of measure 1 and whose boundary has measure 0. In particular, we can use squares instead of disks to compute the density, i.e.

$$D_\varphi^+(\Gamma) := \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{n_\Gamma(z,r)}{\int_{B_r(z)} \Delta\varphi dA} = \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{\#\{S_r(z)\cap\Gamma\}}{\int_{S_r(z)} \Delta\varphi dA}$$

where $S_r(z)$ is a square centered at z with side length r and $\#\{S_r(z)\cap\Gamma\}$ is the number of points of Γ lying in $S_r(z)$. Strictly speaking, Landau's result is for $\varphi = |z|^2$ but the general case follows since $\Delta\varphi$ is uniformly bounded.

We now give several examples of density calculations. We will calculate density using squares instead of disks which we can do by the comments made above.

EXAMPLE 2.2.1. Let $\varphi = \pi|z|^2$ and $\Gamma = a\mathbb{Z} \times b\mathbb{Z}$ a rectangular lattice. Then

$D^+_\varphi(\Gamma) = \frac{1}{ab}$. To see this observe that for $r > 0$ we have the estimates

$$\left(\frac{r}{a} - 1\right)\left(\frac{r}{b} - 1\right) \leq \#\{S_r(z) \cap \Gamma\} \leq \left(\frac{r}{a} + 1\right)\left(\frac{r}{b} + 1\right). \tag{2.2.1}$$

Dividing (2.2.1) by $\int_{S_r(z)} \Delta\varphi dA = r^2$ and letting r go to infinity we get that $D^+_\varphi(\Gamma) = \frac{1}{ab}$. We also point out that for $\varepsilon > 0$ small enough, the density of a sequence obtained by perturbing the lattice Γ by ε is equal to the density of Γ, i.e.,

$$D^+_\varphi(\Gamma + \varepsilon) = D^+_\varphi(\Gamma).$$

EXAMPLE 2.2.2. Using an idea of Ortega and Seip from [13] we now show how to construct a sequence Γ such that $D^+_\varphi(\Gamma) = \alpha$ for any $0 < \alpha < +\infty$ for a general weight φ. First partition the plane into horizontal strips S_j defined by $j \leq \operatorname{Im} z \leq j + 1$ with $j \in \mathbb{Z}$. Then subdivide each S_j into rectangles R_{jk} such that $\int_{R_{jk}} \Delta\varphi dA = \alpha$. The length of the rectangles will be bounded above and bellow by some constants since $\Delta\varphi$ is bounded from above and bellow by constants. We now make a uniformly separated sequence Γ by placing a point in the center of each rectangle. It is easy to verify that $D^+_\varphi(\Gamma) = \alpha$. We can also make a sequence $\tilde{\Gamma}$ which is a union of n uniformly separated sequences by placing n points arbitrarily close together clustered around the center of each rectangle R_{jk}. It follows from the above construction and the additivity of density that $D^+_\varphi(\tilde{\Gamma}) = n\alpha$.

The next well known proposition shows that while a priori we do not make any restrictions on the separation properties of our sequence Γ in our main theorems, the assumption that the density of the sequence is finite (less then 1 in our case) already places a strong restriction on the separation properties of Γ.

PROPOSITION 2.2.3. *A sequence Γ is a finite union of uniformly separated sequences if and only if $D_\varphi^+(\Gamma) < +\infty$.*

Proof. First suppose that Γ is not a finite union of uniformly separated sequences. Then for any $r > 0$ and any integer m there exists a point $z_m \in \mathbb{C}$ such that $n_\Gamma(z_m, r) > m$. But then

$$\sup_{z\in\mathbb{C}} \frac{n_\Gamma(z,r)}{\int_{B_r(z)} \Delta\varphi} = +\infty,$$

which in turn implies that $D_\varphi^+(\Gamma) = +\infty$.

To see the other direction suppose that Γ is a finite union of uniformly separated sequences. Then for any $\delta > 0$ there exists some integer N_δ such that any disk of radius δ contains at most N_δ points of Γ, i.e. $n_\Gamma(z,\delta) \le N_\delta$ for all $z \in \mathbb{C}$. Now, any disk of radius r can be covered by a union of $\lceil 2r \rceil^2$ disks of radius $\frac{1}{\sqrt{2}}$, where $\lceil x \rceil$ denotes the ceiling function. If we let $\delta = \frac{1}{\sqrt{2}}$ we have the estimates

$$D_\varphi^+(\Gamma) \le \limsup_{r\to\infty} \sup_{z\in\mathbb{C}} \frac{N_\delta \lceil 2r \rceil^2}{\int_{B_r(z)} \Delta\varphi} \le \limsup_{r\to\infty} \frac{N_\delta (2r+1)^2}{m\pi r^2} = \frac{2N_\delta}{m\pi} < +\infty.$$

□

The following Theorem says that the so-called *one point interpolation problem* can always be solved with the weight φ.

THEOREM 5. *There exists a constant $C > 0$ with the following property. For any $u, a \in \mathbb{C}$ there exists an $F \in \mathscr{H}^2_\varphi(\mathbb{C})$ such that $F(u) = a$ and*

$$\int_{\mathbb{C}} |F|^2 e^{-\varphi} dA \leq C|a|^2 e^{-\varphi(u)}.$$

Theorem 5 follows from Theorem A since any point $u \in \mathbb{C}$ is trivially a sequence with density zero and $\rho_u = 1$. The existence of the estimate can be seen from the proof of Theorem A. However, one can also prove Theorem 5 directly. We will do so in Section 3.1 since the proof demonstrates some of the techniques we will use to prove our main theorems but in a simplified setting.

Using Lemma 2.1.2 and Theorem 5 we can prove

PROPOSITION 2.2.4. *The map R_Γ is defined and bounded on $\mathscr{H}^2_\varphi(\mathbb{C})$ if and only if the sequence Γ is uniformly separated.*

Proof. First suppose that Γ is uniformly separated, i.e., there exists $\varepsilon > 0$ such that $\rho_\gamma \geq \varepsilon$ for all $\gamma \in \Gamma$. Then there exists an integer N such $n_\gamma \leq N$ for all $\gamma \in \Gamma$. Using Lemma 2.1.2 and the fact that $B_\varepsilon(\gamma)$ are disjoint for all $\gamma \in \Gamma$, we have that for any $F \in \mathscr{H}^2_\varphi(\mathbb{C})$

$$\begin{aligned} R_\Gamma(F) = \sum_\gamma |F(\gamma)|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^{2n_\gamma}} &\leq \frac{1}{\varepsilon^{2N}} \sum_\gamma |F(\gamma)|^2 e^{-\varphi(\gamma)} \\ &\lesssim \sum_\gamma \int_{B_\varepsilon(\gamma)} |F(w)|^2 e^{-\varphi(w)} dA_w \\ &\lesssim \|F\|^2_\varphi. \end{aligned}$$

It then follows that R_Γ is defined and bounded on $\mathscr{H}^2_\varphi(\mathbb{C})$.

Now suppose that R_Γ is defined and bounded on $\mathscr{H}^2_\varphi(\mathbb{C})$. Take any $\gamma \in \Gamma$. By Theorem 5 there exist $F \in \mathscr{H}^2_\varphi(\mathbb{C})$ such that $F(\gamma) = e^{\frac{\varphi(\gamma)}{2}}$ and $\|F\|_\varphi \leq C$, where the constant is independent of γ. Since $\rho_\gamma \leq 1$, the following estimate holds :

$$\frac{1}{\rho_\gamma} \leq \sum_\gamma |F(\gamma)|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^{2n_\gamma}} \leq C.$$

Thus Γ is uniformly separated. $\square$

2.3 Distributions

We assume that the reader is familiar with the basic notions of distributions. In this section we introduce the notion of a *positive distribution* which is not common to distribution theory but is prevalent in the theory of currents. Let $\mathscr{D}(\mathbb{C})$ denote the space of test functions on $\mathbb{C}$, i.e., smooth functions with compact support and let $\mathscr{D}'(\mathbb{C})$ denote the space of distributions on $\mathbb{C}$, i.e., continuous, real valued, linear functionals on $\mathscr{D}(\mathbb{C})$.

DEFINITION 2.3.1. A distribution $f \in \mathscr{D}'(\mathbb{C})$ is said to be *positive* if $\langle f, \alpha\rangle \geq 0$ for any $\alpha \in \mathscr{D}(\mathbb{C})$ such that $\alpha \geq 0$. We use the notation $f \geq 0$ to denote a positive distribution and $f \geq g$ means that $f - g \geq 0$ where g is also a distribution.

EXAMPLE 2.3.2. Any real valued function $f \in L^1_{loc}(\mathbb{C})$ defines a distribution by $\langle f, \alpha\rangle = \int_{\mathbb{C}} f\alpha dA$. This distribution is positive if f is positive almost everywhere.

EXAMPLE 2.3.3. The point-mass distribution centered as some point $z \in \mathbb{C}$, denoted by δ_z and defined by $\langle \delta_z, \alpha \rangle = \alpha(z)$ is a positive distribution.

The following two Lemmas will be used to establish the positivity of certain "singular" weights which we will introduce later.

LEMMA 2.3.4. $\Delta_z \log|z-w|^2 = \delta_w(z)$.

Proof. We wish to prove that given any $\alpha \in \mathscr{D}(\mathbb{C})$ the following equality holds:

$$\int_{\mathbb{C}} \log|z-w|^2 \Delta_z \alpha(z) dA_z = \alpha(w).$$

We will use the following version of Green's Theorem in the plane (see [20] for a proof) . Let $\Omega \subset\subset \mathbb{C}$ be a smoothly bounded domain and let u and v be two C^2-smooth real valued functions defined in a neighborhood of $\bar{\Omega}$. Then

$$\int_{\Omega} u\Delta v - v\Delta u dA = \frac{1}{4\pi} \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} dS \tag{2.3.1}$$

where $\frac{\partial}{\partial n}$ denotes the derivative in the direction normal to the boundary. The constant $\frac{1}{4\pi}$ arises because of our definition of the Laplacian.

Now take some $\alpha \in \mathscr{D}(\mathbb{C})$ choose r large enough so that the support of α lies in the disk $B_r(w)$. Since $\log|z-w|^2$ is locally integrable at w and harmonic in $\mathbb{C} \setminus \{w\}$ it holds that

$$\int_{\mathbb{C}} \log|z-w|^2 \Delta_z \alpha(z) dA_z \quad = \quad \lim_{\varepsilon \to 0} I_\varepsilon$$

where

$$I_\varepsilon = \int_{B_r(w)\setminus B_\varepsilon(w)} \log|z-w|^2 \Delta\alpha(z) - \Delta \log|z-w|^2 \alpha(z) dA_z.$$

For any given $\varepsilon > 0$, (2.3.1) along with the fact that the support of α lies in $B_r(0)$ gives that

$$\begin{aligned} I_\varepsilon &= -\frac{1}{4\pi}\int_0^{2\pi} \log \varepsilon^2 \frac{\partial \alpha(\rho e^{i\theta})}{\partial \rho}\bigg|_{\rho=\varepsilon} \varepsilon d\theta + \frac{1}{4\pi}\int_0^{2\pi} \frac{2}{\varepsilon}\alpha(\varepsilon e^{i\theta})\varepsilon d\theta \\ &= -\frac{1}{4\pi}\varepsilon \log \varepsilon^2 \int_0^{2\pi} \frac{\partial \alpha(\rho e^{i\theta})}{\partial \rho}\bigg|_{\rho=\varepsilon} d\theta + \frac{1}{2\pi}\int_0^{2\pi} \alpha(\varepsilon e^{i\theta}) d\theta. \end{aligned}$$

Since $\varepsilon \log \varepsilon^2 \to 0$ as $\varepsilon \to 0$ and α is C^2-smooth we have that

$$\lim_{\varepsilon \to 0} I_\varepsilon = \alpha(0).$$

□

LEMMA 2.3.5. *$\Delta_z \int_{B_r(z)} \log|\xi - w|^2 dA_\xi = 1$ for $w \in B_r(z)$ and $\Delta_z \int_{B_r(z)} \log|\xi - w|^2 dA_\xi = 0$ for $w \notin \overline{B_r(z)}$ in the sense of distributions.*

Proof. The lemma follows if we show that for all $\alpha \in \mathscr{D}(\mathbb{C})$,

$$\int_{\mathbb{C}} \left(\int_{B_r(z)} \log|\xi - w|^2 dA_\xi \right) \Delta_z \alpha(z) dA_z = \int_{B_r(w)} \alpha(\xi) dA(\xi).$$

Indeed, the latter statement says that as a distribution, the function $\Delta_z \int_{B_r(z)} \log|\xi - w|^2 dA_\xi$ is equal to the function $\mathbf{1}_{B_r(w)}(z)$. (Since $\partial B_r(w)$ has measure zero, the two functions $\mathbf{1}_{B_r(w)}(z)$ and $\mathbf{1}_{\overline{B_r(w)}}(z)$ induce the same distribution.) To this end, we have

$$\begin{aligned}
& \int_{\mathbb{C}} \left(\int_{B_r(z)} \log|\xi - w|^2 dA_\xi \right) \Delta_z \alpha(z) dA_z \\
= \ & \int_{\mathbb{C}} \left(\int_{B_r(0)} \log|z + u - w|^2 dA_u \right) \Delta_z \alpha(z) dA_z \\
= \ & \int_{B_r(0)} \left(\int_{\mathbb{C}} \log|z - (w-u)|^2 \Delta_z \alpha(z) dA_z \right) dA_u \\
= \ & \int_{B_r(0)} \alpha(w-u) dA_u \\
= \ & \int_{B_r(w)} \alpha(\xi) dA_\xi .
\end{aligned}$$

The first equality follows from a change of variables, the second from Fubini's Theorem (see [15]), the third from Lemma 2.3.4 and the fourth from another change of variables. The proof is finished. $\square$

2.4 Singularization of the weight

We will want to modify the weight φ to introduce singularities at the points of Γ. Toward this end, we introduce the function $s_r : \mathbb{C} \to [-\infty, +\infty)$ defined by

$$s_r(z) := \sum_{\gamma \in \Gamma} (\log|z - \gamma|^2 - \fint_{B_r(z)} \log|\xi - \gamma|^2 dA_\xi)$$

where

$$\fint_{B_r(z)} f dA := \frac{1}{\pi r^2} \int_{B_r(z)} f dA.$$

First note that s_r is well defined since $\log|\xi - \gamma|^2$ is locally integrable and harmonic for $\xi \neq \gamma$ and so by the mean value property for harmonic functions

$$s_r(z) = \sum_{\gamma \in \Gamma_r(z)} (\log|z-\gamma|^2 - \fint_{B_r(z)} \log|\xi - \gamma|^2 dA_\xi).$$

Recall that $\Gamma_r(z) = \Gamma \cap B_r(z)$. Similarly, since $\log|z - \gamma|^2$ is subharmonic for all $z \in \mathbb{C}$, we have by the sub-mean value property for subharmonic functions that

$$s_r(z) \leq 0.$$

Note that $e^{-s_r(z)}$ is not locally integrable at any $\gamma \in \Gamma$ since $\frac{1}{|z-\gamma|^2}$ is not locally integrable at γ. Furthermore, from Lemma 2.3.4 and Lemma 2.3.5 it follows that

$$\Delta_z s_r(z) = \sum_{\gamma \in \Gamma_r(z)} \delta_\gamma - \frac{n_\Gamma(z,r)}{\pi r^2} \geq -\frac{n_\Gamma(z,r)}{\pi r^2}$$

where δ_γ is the point mass distribution centered at γ and the inequality is meant in the sense of positive distributions. Now let

$$T_\gamma := \left\{ z \in \mathbb{C} \ : \frac{\rho_\gamma}{4} \leq |z - \gamma| \leq \ \frac{3\rho_\gamma}{4} \right\}.$$

Then for $z \in T_\gamma$ we have

$$\begin{aligned}
\sum_{\tilde{\gamma}\in\Gamma_r(z)} \log|z-\tilde{\gamma}|^2 &= \sum_{\tilde{\gamma}\in\Gamma_r(z)\cap\Gamma_1(\gamma)} \log|z-\tilde{\gamma}|^2 \\
&+ \sum_{\tilde{\gamma}\in\Gamma_r(z)\setminus\Gamma_1(\gamma)} \log|z-\tilde{\gamma}|^2 \\
&\geq n_\gamma \log\left(\frac{\rho_\gamma}{4}\right)^2 + \log\left(\frac{1}{16}\right)(n_\Gamma(z,r)-n_\gamma) \\
&\geq n_\gamma \log \rho_\gamma^2 - Cn_\Gamma(z,r)
\end{aligned}$$

and

$$\begin{aligned}
-\sum_{\gamma\in\Gamma_r(z)} \fint_{B_r(z)} \log|\xi-\gamma|^2 dA_\xi &\geq -\sum_{\gamma\in\Gamma_r(z)} \frac{1}{\pi r^2}\int_{B_r(z)\setminus B_1(\gamma)} \log|\xi-\gamma|^2 dA_\xi \\
&\geq -\sum_{\gamma\in\Gamma_r(z)} \frac{1}{\pi r^2}\int_{B_r(z)\setminus B_1(\gamma)} \log(2r)^2 dA_\xi \\
&\geq -2n_\Gamma(z,r)\log 2r.
\end{aligned}$$

Thus we have that for $z \in T_\gamma$

$$s_r(z) \geq n_\gamma \log \rho_\gamma^2 - C_r n_\Gamma(z,r).$$

It turns out that rather than working with the weight φ itself we will want to average φ over disks and so we define

$$\varphi_r(z) := \fint_{B_r(z)} \varphi(\xi) dA_\xi.$$

The following Lemma says that we can use the weights φ and φ_r interchangeably.

LEMMA 2.4.1. $|\varphi - \varphi_r| \leq C_r$.

Proof. Let

$$I_z(w) = \int_{B_r(z)} (\ln|z-\xi| - \ln|w-\xi|)\Delta\varphi(\xi)dA_\xi.$$

Then the decomposition in (2.1.1) gives

$$\begin{aligned} |\varphi(z) - \varphi_r(z)| &= |\fint_{B_r(z)} \varphi(z) - \varphi(w)dA_w| \\ &= |\fint_{B_r(z)} h_z(w)dA_w - \fint_{B_r(z)} I_z(w)dA_w| \\ &= |\fint_{B_r(z)} I_z(w)dA_w| \end{aligned}$$

where the last equality holds because h_z is harmonic and $h_z(z) = 0$. The result now follows from the estimates done in the proof of Lemma 2.1.1. □

A different way to state Lemma 2.4.1 is that $e^{-\varphi} \simeq_r e^{\varphi_r}$. This in particular implies that the spaces $\mathscr{H}^2_\varphi(\mathbb{C})$ and $\mathscr{H}^2_{\varphi_r}(\mathbb{C})$ are the same with equivalent norms. The same is true of $\mathfrak{H}^2_\varphi(\Gamma)$ and $\mathfrak{H}^2_{\varphi_r}(\Gamma)$.

Also, since

$$\Delta_z\varphi_r(z) = \Delta_z \fint_{B_r(z)} \varphi(\xi)dA = \fint_{B_r(z)} \Delta_\xi\varphi(\xi)dA$$

we have that

$$m \leq \Delta\varphi_r \leq M.$$

We now define a new (singular) weight

$$\psi_r := \varphi_r + s_r.$$

What we have shown is:

LEMMA 2.4.2. *The functions ψ_r and φ_r have the following properties:*

(1) $e^{-\varphi_r(z)} \leq e^{-\psi_r(z)}$ *for all* $z \in \mathbb{C}$*.*

(2) for $z \in T_\gamma$ *we have that* $e^{-\psi_r(z)} \leq \frac{1}{\rho_\gamma^{2n_\gamma}} e^{C_r n_\Gamma(z,r)} e^{-\varphi_r(z)}$*.*

(3) $e^{-\psi_r(z)}$ *is not locally integrable at any* $\gamma \in \Gamma$*.*

2.5 Basic $\bar{\partial}$-technique

We now give a proof of Theorem 1 using the solvability of the $\bar{\partial}$-equation. The method of proof will be a prototype for some of the techniques we will use in the proof of our main theorems. The starting point is the following result which states that given smooth data we can always solve the $\bar{\partial}$-equation in the plane. The proof may be found in the first chapter of [9].

THEOREM 6. *Given any smooth function f defined on $\mathbb{C}$ there exists a smooth function u defined on $\mathbb{C}$ such that $\frac{\partial}{\partial \bar{z}} u = f$.*

Proof of Theorem 1. Let $\{z_j\}$ be a discrete sequence in $\mathbb{C}$ and let $\{a_j\}$ be an arbitrary sequence of complex numbers. Since the sequence $\{z_j\}$ is discrete there exist $\varepsilon_j > 0$ such that the disks $B_{\varepsilon_j}(z_j)$ are mutually disjoint. Let $\eta : [0, \infty) \to [0, 1]$ be a smooth function which is identically 1 on $[0, \frac{1}{4}]$ and identically 0 on $[\frac{3}{4}, \infty)$. Consider the function

$$F(z) := \sum_j a_j \eta_j(z)$$

where $\eta_j(z) := \eta(\frac{|z-z_j|^2}{\varepsilon_j^2})$. The function F is well-defined and smooth. Moreover, F is supported in $B_{\varepsilon_j}(z_j)$, holomorphic in $B_{\frac{\varepsilon_j}{4}}(z_j)$, and $F(z_j) = a_j$ for all j. By the Weierstrass Theorem (see [9]) the exits an entire function h which vanishes precisely on the sequence $\{z_j\}$ and all of the zeroes are simple. It then follows that the function

$$\alpha := \frac{1}{h} \frac{\partial F}{\partial \bar{z}}$$

is smooth everywhere (we define $\alpha(z_j) = 0$). By Theorem 6 there exists a smooth function u such that $\frac{\partial}{\partial \bar{z}} u = \alpha$. It follows that the function $f = F - uh$ is entire and $f(z_j) = a_j$. $\square$

Let us try to briefly summarize what happened in the proof of Theorem 6. We were looking for a holomorphic function with prescribed values on a sequence in the plane. We first constructed a smooth function which attained the correct values on the sequence. This was done by patching together local holomorphic functions and so the smooth function was holomorphic in a small neighborhood of each point in the sequence. We then found another function which vanished at every point of the sequence and the difference of the two functions was holomorphic and had the correct values on the sequence. This second function was found by solving the inhomogeneous $\bar{\partial}$-equation. This approach to finding global holomorphic functions with prescribed local properties is know as the $\bar{\partial}$-technique and we will come back to it several times including in the proof of our main theorems.

2.6 Hörmander's Theorem

The main technical tool that we will use in the proof of our main theorems is known as Hörmander's Theorem. We state and prove of a version of Hörmander's Theorem since it is hard to find in the literature in the exact form we require. We claim absolutely no originality here. The author learned this proof from a wonderful set of notes by Bo Berndtsson [1] which are available online but are unfortunately unpublished at this time.

THEOREM 7. *Take $\psi \in C^2(\mathbb{C})$ such that there exists a $\delta > 0$ so that $\Delta\psi \geq \delta$. Then given any function $f \in L^2_\psi(\mathbb{C})$ there exist a function $u \in L^2_\psi(\mathbb{C})$ such that $\bar{\partial}u = f$ in the sense of distributions and*

$$\int_{\mathbb{C}} |u|^2 e^{-\psi} dA \lesssim \int_{\mathbb{C}} |f|^2 e^{-\psi} dA.$$

Recall that $\mathscr{D}(\mathbb{C})$ denote the space of test functions on $\mathbb{C}$, i.e., smooth functions with compact support. We remind the reader that given two functions $f, u \in$

$L^2(\mathbb{C})$ the distributional equation $\bar{\partial} u = f$ means that given $\alpha \in \mathscr{D}(\mathbb{C})$

$$-\int_{\mathbb{C}} u \frac{\partial}{\partial \bar{z}} \alpha dA = \int_{\mathbb{C}} f \alpha dA. \tag{2.6.1}$$

We now introduce an inner product structure on $L^2_\psi(\mathbb{C})$ by defining

$$\langle u, v \rangle_\psi := \int_{\mathbb{C}} u \bar{v} e^{-\psi} dA$$

for all $u, v \in L^2_\psi(\mathbb{C})$. If we define

$$\bar{\partial}^*_\psi \alpha := -e^{\psi} \frac{\partial}{\partial z}[\alpha e^{-\psi}]$$

for $\alpha \in \mathscr{D}(\mathbb{C})$ then (2.6.1) becomes

$$\int_{\mathbb{C}} u \overline{\bar{\partial}^*_\psi \alpha} e^{-\psi} dA = \int_{\mathbb{C}} f \bar{\alpha} e^{-\psi} dA. \tag{2.6.2}$$

If fact we have that

LEMMA 2.6.1. $\bar{\partial}^*_\psi$ *is the* formal adjoint *of* $\bar{\partial}$*, i.e.*

$$\langle \bar{\partial} \alpha, \beta \rangle_\psi = \langle \alpha, \bar{\partial}^*_\psi \beta \rangle_\psi \tag{2.6.3}$$

for all $\alpha, \beta \in \mathscr{D}(\mathbb{C})$.

Proof. This is an exercise in integration by parts. Unwinding the definitions, the statement that we need to prove is that

$$\int_{\mathbb{C}} \frac{\partial \alpha}{\partial \bar{z}} \bar{\beta} e^{-\psi} dA = -\int_{\mathbb{C}} \alpha \frac{\partial}{\partial \bar{z}} [\bar{\beta} e^{-\psi}] dA$$

for all $\alpha, \beta \in \mathscr{D}(\mathbb{C})$. Consider the $(1,0)$-form $h = \alpha \bar{\beta} e^{-\psi} dz$. If d denotes the usual exterior derivative on 1-forms and $\bar{\partial}$ is its (0,1)-component then

$$\begin{aligned} dh = \bar{\partial} h &= \left(\frac{\partial \alpha}{\partial \bar{z}} \bar{\beta} e^{-\psi} + \alpha \frac{\partial}{\partial \bar{z}} [\bar{\beta} e^{-\psi}] \right) d\bar{z} \wedge dz \\ &= 2i \left(\frac{\partial \alpha}{\partial \bar{z}} \bar{\beta} e^{-\psi} + \alpha \frac{\partial}{\partial \bar{z}} [\bar{\beta} e^{-\psi}] \right) dA. \end{aligned}$$

Since α and β have compact support, by Stokes Theorem (see [20]) we have that

$$0 = \int_{\mathbb{C}} dh = \int_{\mathbb{C}} \left(\frac{\partial \alpha}{\partial \bar{z}} \bar{\beta} e^{-\psi} + \alpha \frac{\partial}{\partial \bar{z}} [\bar{\beta} e^{-\psi}] \right) dA.$$

□

The following proposition reduces the proof of Hörmander's Theorem to proving an inequality.

PROPOSITION 2.6.2. *Take $\psi \in C^2(\mathbb{C})$ such that $\Delta \psi \geq \delta$ for some constant $\delta > 0$. Suppose that the estimate*

$$\int_{\mathbb{C}} |\alpha|^2 e^{-\psi} dA \lesssim \int_{\mathbb{C}} |\bar{\partial}_{\psi}^{*} \alpha|^2 e^{-\psi} dA \tag{2.6.4}$$

holds for all $\alpha \in \mathscr{D}(\mathbb{C})$. Then given any function $f \in L^2_{\psi}(\mathbb{C})$ there exist a function

$u \in L^2_\psi(\mathbb{C})$ *such that* $\bar{\partial}u = f$ *(in the sense of distributions) and*

$$\int_{\mathbb{C}} |u|^2 e^{-\psi} dA \lesssim \int_{\mathbb{C}} |f|^2 e^{-\psi} dA$$

Proof. Define the following subspace $W \subset L^2_\varphi(\mathbb{C})$:

$$W := \{\bar{\partial}^*_\psi \alpha : \alpha \in \mathscr{D}(\mathbb{C})\}.$$

Take any function $f \in L^2_\varphi(\mathbb{C})$ and define an anti-linear functional $\mathscr{L}_f : W \to \mathbb{C}$ by

$$\mathscr{L}_f(\bar{\partial}^*_\psi \alpha) := \int_{\mathbb{C}} f \bar{\alpha} e^{-\psi} dA.$$

Using (2.6.4) and the Cauchy-Schwarz Inequality we have the estimates

$$|\mathscr{L}_f(\bar{\partial}^*_\psi \alpha)| = |\langle f, \alpha \rangle_\psi| \leq ||f||_\psi ||\alpha||_\psi \lesssim ||f||_\psi ||\bar{\partial}^*_\psi \alpha||_\psi$$

for all $\bar{\partial}^*_\psi \alpha \in W$. So we see that on W, the functional $\mathscr{L}_f$ is well defined and of norm $||\mathscr{L}_f|| \lesssim ||f||_\psi$. The Hahn-Banach Theorem (see [15]) then implies that we can extend $\mathscr{L}_f$ to all of $L^2_\psi(\mathbb{C})$ and the extension will have the same norm. Then the Reisz Representation Theorem (see [15]) says that there exists some $u \in L^2_\psi(\mathbb{C})$ with $||u||_\psi \lesssim ||f||_\psi$ such that

$$\mathscr{L}_f(v) = \langle u, v \rangle_\psi \tag{2.6.5}$$

for all $v \in L^2_\psi(\mathbb{C})$. If we choose $v = \bar{\partial}^*_\psi \alpha$ then (2.6.5) is exactly the equality in

(2.6.2). $\square$

To finish the proof of Theorem 7 we need to establish an estimate of the type in (2.6.4) which is accomplished by establishing the following integral identity.

PROPOSITION 2.6.3. *Take $\psi \in C^2(\mathbb{C})$ and $\alpha \in \mathscr{D}(\mathbb{C})$. Then*

$$\int_{\mathbb{C}} \pi\Delta\psi|\alpha|^2 e^{-\psi} dA + \int_{\mathbb{C}} |\frac{\partial}{\partial\bar{z}}\alpha|^2 e^{-\psi} dA = \int_{\mathbb{C}} |\bar{\partial}_\psi^*\alpha|^2 e^{-\psi} dA$$

Proof. First observe that Lemma 2.6.3 implies that

$$\int_{\mathbb{C}} |\bar{\partial}_\psi^*\alpha|^2 e^{-\psi} dA = \int_{\mathbb{C}} \bar{\partial}\bar{\partial}_\psi^*\alpha \cdot \bar{\alpha} e^{-\psi} dA \tag{2.6.6}$$

where $\cdot$ denotes the usual Euclidean inner product. A calculations shows that

$$\bar{\partial}_\psi^*\alpha = -\frac{\partial\alpha}{\partial z} + \alpha\frac{\partial\psi}{\partial z}$$

and so

$$\bar{\partial}\bar{\partial}_\psi^*\alpha = -\frac{\partial}{\partial z}\left[\frac{\partial\alpha}{\partial\bar{z}}\right] + \frac{\partial\psi}{\partial z}\frac{\partial\alpha}{\partial\bar{z}} + \pi\alpha\Delta\psi = \bar{\partial}_\psi^*\left[\frac{\partial\alpha}{\partial\bar{z}}\right] + \pi\alpha\Delta\psi. \tag{2.6.7}$$

Plugging in (2.6.7) into (2.6.6) gives

$$\begin{aligned} \int_{\mathbb{C}} |\bar{\partial}_\psi^*\alpha|^2 e^{-\psi} dA &= \int_{\mathbb{C}} \pi\Delta\psi|\alpha|^2 e^{-\psi} dA + \langle\bar{\partial}_\psi^*\left[\frac{\partial\alpha}{\partial\bar{z}}\right], \alpha\rangle_\psi \\ &= \int_{\mathbb{C}} \pi\Delta\psi|\alpha|^2 e^{-\psi} dA + \int_{\mathbb{C}} |\frac{\partial}{\partial\bar{z}}\alpha|^2 e^{-\psi} dA \end{aligned}$$

where the last equality follows from Lemma 2.6.6. $\square$

In many applications of Hörmander's theorem (including ours) it is important to be able to relax the regularity assumption on the weight function ψ.

THEOREM 8. *Let ψ be a L^1_{loc} real valued function on $\mathbb{C}$ such that there exists a $\delta > 0$ so that $\Delta\psi \geq \delta$ in the sense of positive distributions. Then given any function $f \in L^2_\psi(\mathbb{C})$ there exist a function $u \in L^2_\psi(\mathbb{C})$ such that $\bar{\partial}u = f$ in the sense of distributions and we have the following estimate on its norm:*

$$\int_{\mathbb{C}} |u|^2 e^{-\psi} dA \lesssim \int_{\mathbb{C}} |f|^2 e^{-\psi} dA.$$

Proof. We will deduce the singular version of Hörmander's theorem by applying the smooth version plus a smoothing procedure for the weight ψ. Given a ψ satisfying the assumptions of the theorem and any $\varepsilon > 0$ there exists a smooth function ψ_ε such that $\Delta\psi_\varepsilon \geq \delta$ and $\psi_\varepsilon \searrow \psi$ as $\varepsilon \searrow 0$. In fact, ψ_ε is a convolution of ψ with a positive radial bump function (see [9]). Then Theorem 7 asserts that there exists a family of functions $\{u_\varepsilon\}$ such that $\bar{\partial}u_\varepsilon = f$ and

$$\int_{\mathbb{C}} |u_\varepsilon|^2 e^{-\psi_\varepsilon} dA \lesssim \int_{\mathbb{C}} |f|^2 e^{-\psi_\varepsilon} dA \leq \int_{\mathbb{C}} |f|^2 e^{-\psi} dA. \tag{2.6.8}$$

If we let $U_\varepsilon := u_\varepsilon e^{-\frac{1}{2}\psi_\varepsilon}$, then (2.6.8) says that $\{U_\varepsilon\} \subset L^2(\mathbb{C})$ is a uniformly bounded family and so has a subsequence which converges weakly to some function U. It now follows by construction that $u = Ue^{-\frac{1}{2}\psi}$ solves $\bar{\partial}u = f$ (in the sense of distributions) and

$$\int_{\mathbb{C}} |u|^2 e^{-\psi} dA \lesssim \int_{\mathbb{C}} |f|^2 e^{-\psi} dA.$$

$\square$

Chapter 3

Interpolation

3.1 One-point interpolation

We now turn to the proof of Theorem 5. As mentioned already, the proof will demonstrate the main ideas involved in the proofs of our main theorems but in a simpler situation.

Proof of Theorem 5. Take any $a, u \in \mathbb{C}$. We seek an $F \in \mathscr{H}^2_\varphi(\mathbb{C})$ such that

$$F(u) = a \quad \text{and} \quad \int_{\mathbb{C}} |F|^2 e^{-\varphi} dA \leq C|a|^2 e^{-\varphi(u)}$$

for some constant $C > 0$ independent of u and a.

To begin, let H_u be the function given by Lemma 2.1.1 (set $r = 1$). Then the function $\tilde{F} : B_1(u) \to \mathbb{C}$ defined by

$$\tilde{F}(z) := a e^{H_u(z)} \quad \text{for} \quad z \in B_1(u)$$

is holomorphic and $\tilde{F}(u) = a$. Furthermore, we have the estimates

$$\begin{aligned}\int_{B_1(u)} |\tilde{F}(z)|^2 e^{-\varphi(z)} dA_z &= |a_\gamma|^2 \int_{B_1(u)} \exp(2\operatorname{Re} H_u(z) - \varphi(z)) dA_z \\ &\leq C|a_\gamma|^2 e^{-\varphi(\gamma)}\end{aligned}$$

Let $\eta : [0, \infty) \to [0, 1]$ be a smooth function which is identically 1 on $[0, \frac{1}{4}]$ and identically 0 on $[\frac{3}{4}, \infty)$. Then the function $\hat{F} := \eta \tilde{F}$ is a globally defined smooth function such that

$$\hat{F}(u) = a \quad \text{and} \quad \int_{\mathbb{C}} |\hat{F}|^2 e^{-\varphi} dA \leq C|a|^2 e^{-\varphi(u)}.$$

Thus $\hat{F}$ is a smooth solution to our problem. We now want to correct $\hat{F}$ in some controlled manner in order to produce a holomorphic solution. There is a standard way to do this which we now describe.

Consider the function

$$\psi_r(z) := \varphi(z) + \log|z - u|^2 - \fint_{B_r(z)} \log|\zeta - u|^2 dA_\zeta.$$

The arguments from section 2.4 show that $\psi_r \leq \varphi$ and on the annulus $B_1(u) \setminus B_{\frac{1}{4}}(u)$ we have the lower bound

$$\psi_r \geq \varphi + C_r \tag{3.1.1}$$

where the constant C_r is independent of u. Moreover, Lemma 2.3.4 and Lemma

2.3.5 imply that

$$\Delta\psi_r = \Delta\varphi + \delta_u - \frac{1}{\pi r^2} \geq \varepsilon$$

for some $\varepsilon > 0$, so long as r is large enough. The inequality is meant in the sense of positive distributions. It follows that

$$\begin{aligned} \int_{\mathbb{C}} |\bar{\partial}\hat{F}(z)|^2 e^{-\psi_r(z)} dA(z) &= \int_{B_1(u)} |\tilde{F}(z)\bar{\partial}\eta(z)|^2 e^{-\psi_r(z)} dA_z \\ &\lesssim \int_{B_1(u)} |\tilde{F}(z)|^2 e^{-\psi_r(z)} dA_z \\ &\lesssim_r |a|^2 e^{-\varphi(u)}. \end{aligned}$$

By Hörmander's Theorem (Theorem 8) there is a function h such that

$$\bar{\partial}h = \bar{\partial}\hat{F} \quad \text{and} \int_{\mathbb{C}} |U|^2 e^{-\psi_r} dA \lesssim_r |a|^2 e^{-\varphi(u)}.$$

Moreover, we have that $h(u) = 0$ since $e^{-\psi_r}$ is not locally integrable at u. Finally, since $\psi_r \leq \varphi$ we have the estimates

$$\int_{\mathbb{C}} |U|^2 e^{-\varphi} dA \leq \int_{\mathbb{C}} |U|^2 e^{-\psi_r} dA \lesssim_r |a|^2 e^{-\varphi(u)}.$$

It follows then that the function $F = \hat{F} - h$ has the desired properties. $\square$

3.2 Proof of Theorem A

We now turn to the proof of our main theorem. Our goal is to take any $\{a_\gamma\} \in \mathfrak{H}^2_\varphi(\Gamma)$ and to construct an $F \in \mathscr{H}^2_\varphi(\mathbb{C})$ such that $F(\gamma) = a_\gamma$. In order to simplify the notation a little we define $B_\gamma := B_{\rho_\gamma}(\gamma)$. For each $\gamma \in \Gamma$, let H_γ be a function satisfying the conclusions of Lemma 2.1.1 where $r = \rho_\gamma$. Define functions $F_\gamma : B_\gamma \to \mathbb{C}$ by

$$F_\gamma(z) := a_\gamma e^{H_\gamma(z)}.$$

It then follows that F_γ is holomorphic in B_γ and $F_\gamma(\gamma) = a_\gamma$. Furthermore, we have that

$$\begin{aligned}
\int_{B_\gamma} |F_\gamma(z)|^2 e^{-\varphi(z)} dA_z &= \int_{B_\gamma} |a_\gamma|^2 \exp(2\operatorname{Re} H_\gamma(z) - \varphi(z)) dA_z \\
&\lesssim |a_\gamma|^2 e^{-\varphi(\gamma)} \int_{B_\gamma} dA_z \\
&\lesssim |a_\gamma|^2 e^{-\varphi(\gamma)} \rho_\gamma^2.
\end{aligned}$$

Let $\eta : [0,\infty) \to [0,1]$ be a smooth function which is identically 1 on $[0,\frac{1}{4}]$ and identically 0 on $[\frac{3}{4},\infty)$. Define the function $\hat{F} : \mathbb{C} \to \mathbb{C}$ by

$$\hat{F}(z) := \sum_{\gamma\in\Gamma} F_\gamma(z)\eta_\gamma(z) \tag{3.2.1}$$

where $\eta_\gamma := \eta\left(\frac{|z-\gamma|^2}{\rho_\gamma^2}\right)$. Then $\hat{F}$ is well defined, $\hat{F}(\gamma) = a_\gamma$, and we have the estimates

$$\int_{\mathbb{C}} |\hat{F}(z)|^2 e^{-\varphi(z)} dA_z \leq \sum_{\gamma\in\Gamma} \int_{B_\gamma} |F_\gamma(z)|^2 e^{-\varphi(z)} dA_z \lesssim \sum_{\gamma\in\Gamma} |a_\gamma|^2 e^{-\varphi(\gamma)} \rho_\gamma^2 \quad (3.2.2)$$

and therefore

$$\int_{\mathbb{C}} |\hat{F}(z)|^2 e^{-\varphi(z)} dA_z \lesssim \sum_{\gamma\in\Gamma} |a_\gamma|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^{2n_\gamma}} < +\infty.$$

Our assumption on $D^+_\varphi(\Gamma)$ implies that for r sufficiently large

(1) $n_\Gamma(z,r) \lesssim r^2$.

(2) there exists $\delta > 0$ such that

$$\begin{aligned} \Delta\psi_r(z) &\geq \Delta\varphi_r(z) - \frac{n_\Gamma(z,r)}{\pi r^2} \\ &\geq \Delta\varphi_r(z) - (1-\delta) ⨍_{B_r(z)} \Delta\varphi(\xi) dA_\xi \\ &\geq m\delta > 0. \end{aligned}$$

We fix such an r and δ for the remainder of the proof. From $|\varphi - \varphi_r| \leq C_r$ (Lemma 2.4.1) it follows that the estimates for $\hat{F}$ given above hold for φ replaced with φ_r. Observe that $\bar{\partial}\hat{F}$ is supported on $\cup_{\gamma\in\Gamma}T_\gamma$ and so we have that

$$
\begin{aligned}
\int_{\mathbb{C}}|\bar{\partial}\hat{F}(z)|^2e^{-\psi_r(z)}dA_z &= \sum_{\gamma\in\Gamma}\int_{T_\gamma}|\bar{\partial}\eta_\gamma(z)|^2|F_\gamma(z)|^2e^{-\psi_r(z)}dA_z \\
&\lesssim \sum_{\gamma\in\Gamma}\frac{1}{\rho_\gamma^2}\int_{T_\gamma}|F_\gamma(z)|^2e^{-\psi_r(z)}dA_z \\
&\lesssim_r \sum_{\gamma\in\Gamma}\frac{1}{\rho_\gamma^{2+2n_\gamma}}\int_{T_\gamma}|F_\gamma(z)|^2e^{-\varphi(z)}dA_z \\
&\lesssim_r \sum_{\gamma\in\Gamma}\frac{1}{\rho_\gamma^{2+2n_\gamma}}\int_{B_\gamma}|F_\gamma(z)|^2e^{-\varphi(z)}dA_z \\
&\lesssim \sum_{\gamma\in\Gamma}|a_\gamma|^2\frac{e^{-\varphi(\gamma)}}{\rho_\gamma^{2n_\gamma}} \\
&< +\infty.
\end{aligned}
$$

In the second to last inequality we used (3.2.2).

Then by Hörmander's Theorem (Theorem 8) there exists a function U such that $\bar{\partial}U = \bar{\partial}\hat{F}$ and

$$\int_{\mathbb{C}}|U(z)|^2e^{-\psi_r(z)}dA_z \lesssim \int_{\mathbb{C}}|\bar{\partial}\hat{F}(z)|^2e^{-\psi_r(z)}dA_z < +\infty.$$

The fact that $e^{-\psi_r(z)}$ is not locally integrable at γ forces $U(\gamma) = 0$ for all $\gamma \in \Gamma$.

We also have that

$$\begin{aligned}\int_{\mathbb{C}} |U(z)|^2 e^{-\varphi(z)} dA_z &\lesssim_r \int_{\mathbb{C}} |U(z)|^2 e^{-\varphi_r(z)} dA_z \\ &\leq \int_{\mathbb{C}} |U(z)|^2 e^{-\psi_r(z)} dA_z \\ &\lesssim \sum_{\gamma \in \Gamma} |a_\gamma|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^{2n_\gamma}} \\ &< +\infty.\end{aligned}$$

where the first inequality follows from our comment about the equivalence of the φ and φ_r norms and the second from Lemma 2.4.2. We now define the function $F := \hat{F} - U$. We immediately see that $F(\gamma) = a_\gamma$ and that F is holomorphic. Finally we have that

$$\int_{\mathbb{C}} |F|^2 e^{-\varphi} dA < +\infty$$

since $\hat{F}$ and U both have finite L^2-norms. The proof is complete. $\square$

3.3 A different norm

For any $r > 0$ define the function

$$\lambda_r(z) := e^{\frac{1}{2} s_r(z)}.$$

Note that $\Gamma = \{z \in \mathbb{C} : \lambda_r(z) = 0\}$, i.e., λ_r acts as a defining function for Γ. A quick calculation shows that given some $\gamma \in \Gamma$

$$|\partial\lambda_r(\gamma)|^2 = \frac{e}{r^2} \prod_{\substack{\tilde{\gamma}\in\Gamma_r(\gamma)\\ \tilde{\gamma}\neq\gamma}} |\gamma - \tilde{\gamma}|^2 e^{-\fint_{B_r(\gamma)} \log|\xi-\tilde{\gamma}|^2 dA_\xi}$$

and so

$$\frac{1}{|\partial\lambda_r(\gamma)|^2} \simeq_r \prod_{\substack{\tilde{\gamma}\in\Gamma_r(\gamma)\\ \tilde{\gamma}\neq\gamma}} \frac{e^{\fint_{B_r(\gamma)} \log|\xi-\tilde{\gamma}|^2 dA_\xi}}{|\gamma - \tilde{\gamma}|^2}. \tag{3.3.1}$$

Remark. Note that we do not claim that $\partial\lambda_r(\gamma)$ exists. In fact it does not. However $|\partial\lambda_r(\gamma)|^2$ does exist as one can readily check.

We can now define an ℓ^2-norm

$$\|\{a_\gamma\}\|_r^2 := \sum_{\gamma\in\Gamma} |a_\gamma|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^2 |\partial\lambda_r(\gamma)|^2}. \tag{3.3.2}$$

We point out that in the above proof of Theorem A, given a $\gamma \in \Gamma$, we needed to estimate from above

$$e^{-s_r(z)} = \frac{e^{\fint_{B_r(z)} \log|\xi-\gamma|^2 dA_\xi}}{|z-\gamma|^2} \prod_{\substack{\tilde{\gamma}\in\Gamma_r(z)\\ \tilde{\gamma}\neq\gamma}} \frac{e^{\fint_{B_r(z)} \log|\xi-\tilde{\gamma}|^2 dA_\xi}}{|z-\tilde{\gamma}|^2}$$

for $z \in T_\gamma$. In fact it was this estimate that forced us to define the $\mathfrak{H}^2_\varphi$-norm the way we did. The next Lemma will (amongst other things) serve the same purpose for the $\mathfrak{H}^2_{\varphi,\lambda_r}$-norm in the proof of Theorem B.

LEMMA 3.3.1. *If $D^+_\varphi(\Gamma) < +\infty$ then for r sufficiently large we have that*

(1) $\rho_\gamma^2 \lesssim_r \frac{1}{|\partial\lambda_r(\gamma)|^2}$ *for all* $\gamma \in \Gamma$.

(2) $e^{-s_r(z)} \lesssim_r \frac{1}{\rho_\gamma^2|\partial\lambda_r(\gamma)|^2}$ *for* $z \in T_\gamma$.

Proof. From the estimates done in Section 2.1 and Section 2.4 it is easy to see that

$$-\frac{1}{r^2} \leq \fint_{B_r(z)} \log|\xi - \gamma|^2 dA_\xi \leq \log 4r^2 \tag{3.3.3}$$

for all $z \in \mathbb{C}$ and $\gamma \in B_r(z)$. Then (3.3.1) and (3.3.3) coupled with the fact that $\rho_\gamma \leq 1$ show that

$$\frac{1}{\rho_\gamma^2|\partial\lambda_r(\gamma)|^2} \gtrsim_r (C_r)^{n_\Gamma(r,\gamma)-1}$$

where $C_r = \frac{e^{-\frac{1}{r^2}}}{r^2} < 1$ for r sufficiently large. As before, the assumption on density implies that

$n_\Gamma(r,\gamma) \lesssim r^2$ for r large enough and so we have that

$$\frac{1}{\rho_\gamma^2|\partial\lambda_r(\gamma)|^2} \gtrsim_r 1.$$

For the second part first observe that since $|\gamma - \tilde{\gamma}| > \rho_\gamma$ and $|z - \gamma| \leq \frac{3\rho_\gamma}{4}$ for $z \in T_\gamma$ we have the estimates

$$\frac{|z-\tilde{\gamma}|}{|\gamma - \tilde{\gamma}|} \geq \frac{|\gamma - \tilde{\gamma}| - |z-\gamma|}{|\gamma-\tilde{\gamma}|} \geq 1 - \frac{\frac{3\rho_\gamma}{4}}{|\gamma - \tilde{\gamma}|} \geq \frac{1}{4}$$

for all $z \in T_\gamma$. Thus we have that

$$\frac{1}{|z-\tilde{\gamma}|^2} \lesssim \frac{1}{|\gamma - \tilde{\gamma}|^2}$$

for all $z \in T_\gamma$. Note that if $\tilde{\gamma} \in \Gamma_r(z) \setminus \Gamma_r(\gamma)$, then $|\gamma - \tilde{\gamma}| \geq r$ and so

$$\frac{1}{|z-\tilde{\gamma}|^2} \lesssim \frac{1}{r^2} \tag{3.3.4}$$

for all $\tilde{\gamma} \in \Gamma_r(z) \setminus \Gamma_r(\gamma)$. From (3.3.3) we get that

$$\fint_{B_r(z)} \log|\xi - \tilde{\gamma}|^2 dA_\xi - \fint_{B_r(\gamma)} \log|\xi - \tilde{\gamma}|^2 dA_\xi \leq \log 4r^2 + \frac{1}{r^2} \tag{3.3.5}$$

for all $\tilde{\gamma} \in \Gamma_r(z) \cap \Gamma_r(\gamma)$. Putting together (3.3.4) and (3.3.5) and using the assumption on density we get that

$$e^{-s_r(z)} \lesssim \frac{C_r^{n_\Gamma(r,z)}}{\rho_\gamma^2} \prod_{\substack{\tilde{\gamma} \in \Gamma_r(\gamma) \\ \tilde{\gamma} \neq \gamma}} \frac{e^{\fint_{B_r(\gamma)} \log|\xi - \tilde{\gamma}|^2 dA_\xi}}{|\gamma - \tilde{\gamma}|^2} \lesssim_r \frac{1}{\rho_\gamma^2 |\partial \lambda_r(\gamma)|^2}$$

for r sufficiently large. □

3.4 Proof of Theorem B

The proof now follows the same lines as the proof of Theorem A. We take some $\{a_\gamma\} \in \mathfrak{H}^2_{\varphi,\lambda_r}(\Gamma)$ and first construct a smooth extension. Let $\hat{F}$ be as in (3.2.1). Just as before $\hat{F}$ is well defined and $\hat{F}(\gamma) = a_\gamma$. Then by Lemma 3.3.1 we have the estimates

$$\begin{aligned}
\int_{\mathbb{C}} |\hat{F}(z)|^2 e^{-\varphi(z)} dA_z &\lesssim \sum_{\gamma\in\Gamma} |a_\gamma|^2 e^{-\varphi(\gamma)} \rho_\gamma^2 \\
&\lesssim_r \sum_{\gamma\in\Gamma} |a_\gamma|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^2 |\partial\lambda_r(\gamma)|^2} \\
&< +\infty
\end{aligned}$$

for r sufficiently large.

Continuing as before we now estimate

$$\begin{aligned}
\int_{\mathbb{C}} |\bar\partial \hat{F}(z)|^2 e^{-\psi_r(z)} dA_z &= \sum_{\gamma\in\Gamma} \int_{T_\gamma} |\bar\partial \eta_\gamma(z)|^2 |F_\gamma(z)|^2 e^{-\psi_r(z)} dA_z \\
&\lesssim \sum_{\gamma\in\Gamma} \frac{1}{\rho_\gamma^2} \int_{T_\gamma} |F_\gamma(z)|^2 e^{-\psi_r(z)} dA_z \\
&\lesssim_r \sum_{\gamma\in\Gamma} \frac{1}{\rho_\gamma^4 |\partial\lambda_r(\gamma)|^2} \int_{B_\gamma} |F_\gamma(z)|^2 e^{-\varphi(z)} dA_z \\
&\lesssim \sum_{\gamma\in\Gamma} |a_\gamma|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^2 |\partial\lambda_r(\gamma)|^2} \\
&< +\infty
\end{aligned}$$

for r sufficiently large. The rest is identical to the proof of Theorem A. □

3.5 Concluding Remarks

Theorems A and B only provide sufficient conditions for interpolation as opposed to Theorem 4. As mentioned already, we do not believe that the assumptions in Theorem A are necessary. This is because we made a choice in the definition on $n_\gamma = n_\Gamma(\gamma, 1)$ to count points which are within a distance of 1 from γ. This choice might not be optimal for a given sequence though. If, for example, we were dealing with a uniformly separated sequence then there exists some $a > 0$ such that $n_\Gamma(\gamma, a) = 1$. However, if that sequence has a separation constant $\rho \ll 1$ then $n_\Gamma(\gamma, 1)$ is a large and unnecessary overestimate of the number of points close to γ. We conjecture is that the assumptions of Theorem B are in fact necessary for interpolation. Finally, we wish to remark that using much more delicate analytical techniques [21], one can get rid of the $\frac{1}{\rho_\gamma^2}$ factor in the definition of the norm in (3.3.2).

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