Categorical and Combinatorial Constructions of A,D,E Root Systems

A Dissertation Presented

by

Jaimal Thind

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2009

Stony Brook University

The Graduate School

Jaimal Thind

We, the dissertation committee for the above candidate

for the Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation.

Alexander Kirillov Jr. - Dissertation Advisor
Associate Professor, Department of Mathematics

Jason Starr - Chairperson of Defense
Assistant Professor, Department of Mathematics

Leon Takhtajan
Professor, Department of Mathematics

Martin Roček
Professor, Yang Institute for Theoretical Physics

This dissertation is accepted by the Graduate School.

Lawrence Martin
Dean of the Graduate School

Abstract of the Dissertation

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The main results of this dissertation, based on joint work with A. Kirillov Jr., give categorical and combinatorial constructions of the root system R and the Lie algebra $\mathfrak{g}$ from the corresponding Dynkin diagram Γ. In particular, given a Dynkin diagram Γ we produce a canonical quiver $\widehat{\Gamma}$ and show that a choice of a Coxeter element in the Weyl group gives an identification $R \to \widehat{\Gamma}$. Moreover, the bilinear form and root lattice admit explicit descriptions in terms of $\widehat{\Gamma}$. Using this identification, we construct a root basis in $\mathfrak{g}$ so that the structure constants of the Lie bracket are given by paths in $\widehat{\Gamma}$. Hence $\mathfrak{g}$ can be defined purely in terms of $\widehat{\Gamma}$. We also approach this categorically: Given the graph Γ, we construct a triangulated category $\mathcal{C}$ so that the Grothendieck group $\mathcal{K}(\mathcal{C})$ is the corresponding root lattice, so that the indecomposable classes are the roots, and so that the bilinear form admits an explicit description in terms of $\mathcal{C}$. This can be seen as giving a construction analogous to Gabriel [G] and Ringel [R1] which does not require orienting the diagram Γ.

Contents

Acknowledgements

I owe thanks to many people. First, to my advisor Alexander Kirillov Jr., for all his help and particularly for his patience. I would like to thank my family, particularly my parents for their support. Thanks to my sister Kiran, too. Many, many thanks to Kate - enough said! Thanks to the 202 crew: Bickle, Ben, Carrie, Norb, Voghell, Patty and our affiliates. Thanks to members of the various Math crews: Elizabeth, Lindo, Travis, Gabe, Aussie Pete, Mike, Nate, Stas, Samir, Dror, Andrew, K.I. and Al. Much thanks to the many wonderful teachers I've had over the years. In particular, thanks to Prof. Peter Rosenthal for supervising my undergrad research and encouraging me to come to Stony Brook. Thanks also to the staff of the Mathematics Department: Donna, Lucille and Nancy.

Introduction

At the end of the 19th Century simple Lie algebras were completely classified by their root systems, which in turn are classified by Dynkin diagrams. The construction of a root system and a Lie algebra from a Dynkin diagram has several approaches. The standard construction of a root system is to choose a system of simple roots corresponding to vertices of the Dynkin diagram and then use the Weyl group to obtain all roots. The standard construction of the corresponding Lie algebra involves choosing generators corresponding to vertices of the Dynkin diagram and then using the edges to determine the relations (the "Serre relations").

Another approach to constructing the root system from the Dynkin diagram is based on the theory of quiver representations. Briefly, a quiver is an oriented graph $\overrightarrow{\Gamma} = (\Gamma, \Omega)$, where Γ is a graph and Ω is an orientation of Γ. A representation of a quiver is the assignment of a vector space $V(i)$ to each vertex i and a linear map $\Phi_e : V(i) \to V(j)$ for every arrow $e : i \to j$. Denote the Abelian category of representations of a quiver $\overrightarrow{\Gamma}$ by $\mathrm{Rep}(\overrightarrow{\Gamma})$.

In the 1970's Gabriel showed that when the underlying graph of a quiver $\overrightarrow{\Gamma}$ is a Dynkin diagram of type A, D, E the set of isomorphism classes of indecomposable representations are in bijection with the set of positive roots of the corresponding root system (see [G]). Moreover, one can obtain an explicit description of the inner product and the root lattice. Ringel then showed that using this category one could construct the positive part of the corresponding Lie algebra (see [R1]). To obtain all roots and the whole Lie algebra one must consider isomorphism classes of objects in a related category; the "2-periodic derived category" $\mathcal{D}^b(\mathrm{Rep}(\overrightarrow{\Gamma}))/T^2$, where T is the translation functor (see [PX1]). In this approach two different choices of orientation of the same graph give rise to different Abelian categories, which are not equivalent, but instead derived equivalent. The relation between different categories is given by the "BGP reflection functors".

The drawback of the quiver approach is that it requires a choice of orientation of the Dynkin diagram, making the constructions non-canonical. Similarly, the standard construction of a root system and its Lie algebra requires a choice of simple roots and generators respectively. One would like to find a construction which does not require these choices.

Another approach, which is independent of any choice of orientation, was suggested by Ocneanu [Oc], in the setting of quantum subgroups of $SU(2)$. His idea was to give a purely combinatorial construction by studying "essential paths" in the quiver $\widehat{\Gamma} = \Gamma \times \mathbb{Z}_h$, which requires no choice of orientation.

In the case of affine Dynkin diagrams the McKay correspondence provides a tool for avoiding choosing orientations. The classical McKay correspondence identifies affine Dynkin diagrams of type A, D, E and finite subgroups $G \subset SU(2)$. In 2006, Kirillov Jr. studied a geometric approach to McKay correspondence using $\bar{G}$-equivariant coherent sheaves on $\mathbb{P}^1$, where $\bar{G} = G/\pm I$ (see [K]). In particular, it was shown that indecomposable objects in the category $\mathcal{D}^b_{\bar{G}}(\mathbb{P}^1)/T^2$ are in bijection with the roots of the corresponding affine root system, that the inner product and root lattice admit an explicit description in terms of this category, and that although there is no natural choice of simple roots, there is a canonical Coxeter element in the Weyl group. This gives a "categorical construction" of the corresponding root system. Here "categorical construction" means that roots are realized as classes of indecomposable objects in a certain category, and the inner product and root lattice admit explicit description in terms of this category.

Motivated by the above constructions, the main results of this thesis give both combinatorial and categorical constructions of R from Γ which do not require a choice of orientation of Γ. This is based on joint work with A. Kirillov Jr.

The first chapter deals with the combinatorial construction of the root system R. In particular, it is shown that a choice of Coxeter element in the Weyl group gives an identification of the root system R with a certain canonical quiver $\widehat{\Gamma}_{cyc}$ associated to Γ (to be defined in the next section) where vertices correspond to roots, and the root lattice and bilinear form admit explicit descriptions in terms of this quiver.

The second chapter gives a categorical construction of R from Γ. Instead of choosing an orientation of Γ and studying representations of the associated quiver, we study representations of a canonical quiver $\widehat{\Gamma}$ associated to Γ. This construction is very closely related to the preprojective algebra of Γ. In particular, the construction gives a certain periodicity result about the preprojective algebra. It is likely that this periodicity result is known to experts, however in the form given here the result is not easily available in the literature. It is worth noting that this construction immediately implies the combinatorial results given in the first chapter, and in fact motivated the combinatorial construction.

In the third and final chapter the combinatorial framework of Chapter 1 is used to construct a root basis in the Lie algebra $\mathfrak{g}$ so that all structure constants can be obtained from $\widehat{\Gamma}_{cyc}$. This gives a combinatorial construction of $\mathfrak{g}$ analogous to the categorical constructions of Ringel and Peng and Xiao. After giving the construction this is related to the Ringel and Peng and Xiao constructions.

Chapter 1

Preliminaries

1.1 Lie algebras and root systems

Throughout this dissertation let $\mathfrak{g}$ be a simple Lie algebra of type A, D, E and let Γ denote the corresponding Dynkin diagram. Let $\mathfrak{h} \subset \mathfrak{g}$ be a fixed Cartan subalgebra and let R denote the corresponding root system. Let r denote the rank. Let n_{ij} denote the number of edges joining i, j in Γ. Let $(\cdot, \cdot)$ be the invariant bilinear form normalized so that $(\alpha, \alpha) = 2$ for $\alpha \in R$ (note that since Γ has no multiple edges all roots have the same length).

Let $\Pi = \{\alpha_1^\Pi, \dots, \alpha_r^\Pi\}$ be a system of simple roots and let $R_\pm^\Pi$ denote the corresponding splitting into positive and negative roots. Let W denote the corresponding Weyl group. Then the simple reflections $s_{\alpha_i^\Pi}$ generate the Weyl group. To simplify notation, the subscripts and superscripts will be amended, writing $s_{\alpha_i^\Pi} = s_i^\Pi$.

For any $w \in W$ denote by $l^\Pi(w)$ the length of a reduced expression for w in terms of the simple refelctions s_i^Π. Then this does not depend on the choice of reduced expression. It is well known that since Γ has no double edges, any two reduced expressions are related by

$$s_i^\Pi s_j^\Pi = s_j^\Pi s_i^\Pi \text{ for } n_{i,j} = 0 \tag{1.1.1}$$

$$s_i^\Pi s_j^\Pi s_i^\Pi = s_j^\Pi s_i^\Pi s_j^\Pi \text{ for } n_{i,j} = 1 \tag{1.1.2}$$

(see [Bour] for details).

Since the Weyl group acts simply-transitively on the set of simple systems, there is a unique element which takes the system Π to the opposite system $-\Pi$. Denote this element by w_0 and define an automorphism of Γ by $i \mapsto \check{\imath}$, where $\check{\imath}$ is defined by:

$$w_0(\alpha_i^{\Pi}) = \alpha_{\check{\imath}}^{-\Pi}. \tag{1.1.3}$$

The element w_0 is called the longest element, and satisfies $l^{\Pi}(w) \leq l^{\Pi}(w_0)$ for any $w \in W$.

Definition 1.1.1. An element $C \in W$ is called a *Coxeter element* if there exists a simple system $\Pi = \{\alpha_1, \ldots, \alpha_r\}$ and a reduced expression for C of the form $C = s_{i_1}^{\Pi} \cdots s_{i_r}^{\Pi}$ such that each simple reflection appears exactly once.
In such a case the simple system Π is said to be *compatible* with C and $l^{\Pi}(C) = r$.
The order of a Coxeter element is called the *Coxeter number* and is denoted by h.

Remark 1.1.2. The set of Coxeter elements in W form a conjugacy class, and hence the Coxeter number is independent of the choice of Coxeter element (see [Bour]). However, not all simple systems are compatible with a given Coxeter element.

1.2 Quivers and Reflection Functors

A quiver $\overrightarrow{\Gamma}$ is an oriented graph. The vertex set is denoted by Γ_0 and the arrow set is denoted by Γ_1. In what follows a quiver is obtained by orienting a graph Γ. In such a case the quiver is denoted by $\overrightarrow{\Gamma} = (\Gamma, \Omega)$ where Ω is an orientation of the graph Γ. There are two functions $s, t : \Gamma_1 \to \Gamma_0$ called "source" and "target" respectively, defined on an oriented edge $e : i \to j$ by $s(e) = i$ and $t(e) = j$.

For any quiver $\overrightarrow{\Gamma}$ let $P(\overrightarrow{\Gamma})$ be the following algebra. As an algebra it is generated by elements $\{e\}_{e \in \Gamma_1} \cup \{e_i\}_{i \in \Gamma_0}$. Here the elements e_i are thought of as "paths of length 0 from i to i". Viewing a path as a sequence of edges, the multiplication of basis elements is given by concatenation of paths. More formally, the relations are:

1. For edges e, f there is the relation
$$e \cdot f = \begin{cases} ef \text{ if } t(f) = s(e) \\ 0 \text{ otherwise.} \end{cases}$$

2. For an edge e and vertex i there are the relations
$$e_i \cdot e = \begin{cases} e \text{ if } t(e) = i \\ 0 \text{ otherwise} \end{cases}$$
and
$$e \cdot e_i = \begin{cases} e \text{ if } s(e) = i \\ 0 \text{ otherwise.} \end{cases}$$

3. The e_i are "orthogonal idempotents":
$$e_i e_j = \begin{cases} e_i \text{ if } i = j \\ 0 \text{ otherwise.} \end{cases}$$

Definition 1.2.1. The algebra $P(\overrightarrow{\Gamma})$ defined above is called the *path algebra* of $\overrightarrow{\Gamma}$. It is an assiociative algebra with unit given by $1 = \sum_{i \in \Gamma_0} e_i$.

The algebra $P(\overrightarrow{\Gamma})$ is graded by path length and by the source and target of the path. This gives a decomposition
$$P(\overrightarrow{\Gamma}) = \bigoplus_{i,j \in \Gamma; k \in \mathbb{N}} P_{i,j;k} \tag{1.2.1}$$
where $P_{i,j;k}$ is the space spanned by paths of length k from i to j. (Here an edge has length 1, and the idempotent corresponding to a vertex has length 0.)

The preprojective algebra of a quiver $\overrightarrow{\Gamma}$ is defined as follows: Consider the double quiver $\overline{\Gamma}$ which has the same vertex set as $\overrightarrow{\Gamma}$ but for every arrow $e : i \to j$ add an arrow $\overline{e} : j \to i$. Choose a function $\epsilon : \overline{\Gamma}_1 \to \{\pm 1\}$ so that $\epsilon(e) + \epsilon(\overline{e}) = 0$. For each vertex $i \in \overrightarrow{\Gamma}$ define $\theta_i \in P_{i,i;2}$ by
$$\theta_i = \sum_{s(e)=i} \epsilon(e) \overline{e} e \in P_{i,i;2} \tag{1.2.2}$$

Definition 1.2.2. The *preprojective algebra* $\Pi(\Gamma)$ of Γ is defined as $P(\overline{\Gamma})/J$ where J is the ideal generated by the θ_i's. The ideal J is called the "mesh" ideal.

Note that this algebra is independent of the choice of ϵ and depends only on the underlying graph and not on the orientation Ω. (See [L2] for details.)

Fix a field $\mathbb{K}$. A representation of a quiver $\overrightarrow{\Gamma}$ is a choice of vector space $X(i)$ for every vertex in Γ_0 and linear map $x_e : X(i) \to X(j)$ for every edge $e : i \to j$. A morphism $\Phi : X \to Y$ of representations is a collection of linear maps $\Phi(i) : X(i) \to Y(i)$ such that the following diagram is commutative for every edge $e : i \to j$.

$$\begin{tikzcd} X(i) \arrow[r, "x_e"] \arrow[d, "\Phi(i)"] & X(j) \arrow[d, "\Phi(j)"'] \\ Y(i) \arrow[r, "y_e"] & Y(j) \end{tikzcd}$$

Denote the Abelian category of representations of $\overrightarrow{\Gamma}$ by $\mathrm{Rep}(\overrightarrow{\Gamma})$.

Remark 1.2.3. Note that a representation of $\overrightarrow{\Gamma}$ is the same as a module over the path algebra $P(\overrightarrow{\Gamma})$ and that the notion of morphism for each coincide as well. (See [C-B] for details.)

For each vertex i define a representation P_i by setting $P_i(j) = P_{i,j}$ the space spanned by paths from i to j in $\overrightarrow{\Gamma}$. This representation is projective and indecomposable, and any indecomposable projective is isomorphic to P_i for some vertex i (see [G]). For any vertex i define a simple object S_i by setting $S_i(j) = \delta_{i,j}\mathbb{K}$.

Definition 1.2.4. The *Auslander-Reiten* quiver of the category $\mathrm{Rep}(\overrightarrow{\Gamma})$ is defined as follows:

1. The vertices are the set $\mathrm{Ind}(\overrightarrow{\Gamma})$ of non-zero isomorphism classes of indecomposable objects.
2. For vertices $[X], [Y]$ there is one edge $e : [X] \to [Y]$ for each indecomposable morphism $\phi : X \to Y$.

The Auslander-Reiten quiver will be denoted by $AR(\overrightarrow{\Gamma})$. (For more details, such as the definition of indecomposable morphism, see [ARS] Chapter VII.)

Definition 1.2.5. For any representation V the dimension vector is given by $\underline{\dim} V = (\dim V(i))_{i \in \Gamma_0} \in \mathbb{Z}^{\Gamma_0}$. This gives a map $\underline{\dim} : \mathcal{K}(\overrightarrow{\Gamma}) \to \mathbb{Z}^{\Gamma_0}$, where $\mathcal{K}(\overrightarrow{\Gamma})$ is the Grothendieck group of $\mathrm{Rep}(\overrightarrow{\Gamma})$.

Definition 1.2.6. Define the *Euler form* $\langle \cdot, \cdot \rangle_{\overrightarrow{\Gamma}}$ by setting

$$\langle X, Y \rangle_{\overrightarrow{\Gamma}} = \dim \mathrm{Hom}(X, Y) - \dim \mathrm{Ext}^1(X, Y). \tag{1.2.3}$$

Define the *symmetrized Euler form* by

$$(X, Y)_{\overrightarrow{\Gamma}} = \langle X, Y \rangle_{\overrightarrow{\Gamma}} + \langle Y, X \rangle_{\overrightarrow{\Gamma}}.$$

1.3 Definition of $\widehat{\Gamma}$

Let Γ be a finite graph without cycles. So in particular Γ is bipartite. Let $\Gamma = \Gamma_0 \sqcup \Gamma_1$ be a bipartite splitting. Define the quiver $\Gamma \times \mathbb{Z}$ as follows:

vertices: $\Gamma \times \mathbb{Z}$

edges: for each $n \in \mathbb{Z}$ and edge $i - j$ in Γ, there are oriented edges

$(i, n) \to (j, n+1), (j, n) \to (i, n+1)$ in $\mathbb{Z}\Gamma$

For Γ as above with bipartite with splitting $\Gamma = \Gamma_0 \sqcup \Gamma_1$, then $\mathbb{Z}\Gamma$ is disconnected: $\Gamma \times \mathbb{Z} = (\Gamma \times \mathbb{Z})_0 \sqcup (\Gamma \times \mathbb{Z})_1$, where

$$\Gamma \times \mathbb{Z}_k = \{(i, n) \mid n + p(i) \equiv k \mod 2\}$$

where $p(i) = 0$ for $i \in \Gamma_0$ and $p(i) = 1$ for $i \in \Gamma_1$.

Definition 1.3.1. Define the quiver $\widehat{\Gamma}$ by setting

$$\widehat{\Gamma} = \{(i, n) \subset \Gamma \times \mathbb{Z} \mid n + p(i) \equiv 0 \mod 2\} = (\Gamma \times \mathbb{Z})_0 \tag{1.3.1}$$

Let Γ be an A, D, E Dynkin diagram with Coxeter number h, so in particular Γ is bipartite. Define also a cyclic version of $\widehat{\Gamma}$ by setting

$$\widehat{\Gamma}_{cyc} = \{(i, n) \subset \Gamma \times \mathbb{Z}_{2h} \mid n + p(i) \equiv 0 \mod 2\} \tag{1.3.2}$$

Example 1.3.2. For the graph $\Gamma = D_5$ the quiver $\widehat{\Gamma}$ is shown in Figure 1.1.

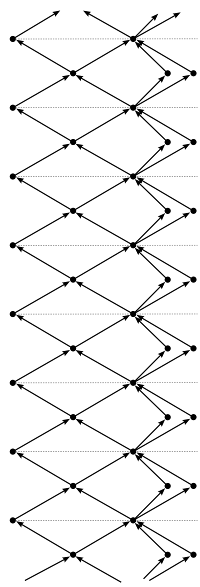

Figure 1.1: The quiver $\widehat{\Gamma}$ for graph $\Gamma = D_5$. For D_5 the Coxeter number is 8, so by identifying the outgoing arrows at the top level and the incoming arrows at the bottom level in this figure, one obtains $\widehat{\Gamma}_{cyc}$.

The following basic properties of $\widehat{\Gamma}$ also hold for $\widehat{\Gamma}_{cyc}$. For brevity only $\widehat{\Gamma}$ is considered.

Define a "twist" map $\tau\colon \widehat{\Gamma} \to \widehat{\Gamma}$ by

$$\tau(i,n) = (i, n+2). \tag{1.3.3}$$

Definition 1.3.3. A function $h\colon \Gamma \to \mathbb{Z}$ satisfying $h(j) = h(i) \pm 1$ if i, j are connected by an edge in Γ and satisfying $h(i) \equiv p(i) \mod 2$, will be called a *height function*. (Here p is the parity function defined in the beginning of this section.)

Definition 1.3.4. Following [G], a connected full subquiver of $\widehat{\Gamma}$ which contains a unique representative of $\{(i,n)\}_{n\in\mathbb{Z}}$ for each $i \in \Gamma$ will be called a *slice*.

Any height function h defines a slice $\Gamma_h = \{(i, h(i)) \mid i \in \Gamma\} \subset \widehat{\Gamma}$; it also defines an orientation Ω_h on Γ where $i \to j$ if i, j are connected by an edge and $h(j) = h(i) + 1$. It is easy to see that two height functions give the same orientation if and only if they differ by an additive constant, or equivalently, if the corresponding slices are obtained one from another by applying a power of τ. Conversely the second coordinate of any slice defines a height function.

Let h be a height function and let $i \in \Gamma$ be a source for the corresponding orientation Ω_h. Define a new height function $s_i^+ h$ by

$$s_i^+ h(j) = \begin{cases} h(j) + 2 & \text{if } \ j = i \\ h(j) & \text{if } \ j \neq i \end{cases}.$$

Similarly, if $i \in \Gamma$ is a sink for the corresponding orientation Ω_h define a new height function $s_i^- h$ by

$$s_i^- h(j) = \begin{cases} h(j) - 2 & \text{if } \ j = i \\ h(j) & \text{if } \ j \neq i \end{cases}.$$

Note that the orientation $\Omega_{s_i^{\pm} h}$ of Γ is obtained by reversing all arrows at i, and that any orientation of Γ can be obtained by a sequence of such operations. It is well-known that for any two height functions h, h' one can be obtained from the other by a sequence of operations $s_i^{\pm}$.

For Γ Dynkin of type A, D, E, with Coxeter number h, define the following permutations on $\widehat{\Gamma}$ (and $\widehat{\Gamma}_{cyc}$).
The "Nakayama" permutation given by:

$$\nu_{\widehat{\Gamma}}(i, n) = (\check{\imath}\ , n + h - 2) \tag{1.3.4}$$

The "Twisted Nakayama" permuation given by:

$$\gamma_{\widehat{\Gamma}}(i, n) = (\check{\imath}\ , n + h) = \tau \circ \nu_{\widehat{\Gamma}}(i, n) \tag{1.3.5}$$

It remains to verify that $\nu_{\widehat{\Gamma}}$ is well-defined. To see this only requires checking that the image does in fact lie in $\widehat{\Gamma}$. Note that if h is even $p(\check{\imath}) = p(i)$ and $k + h = k \mod 2$, so $(\check{\imath}, k + h) \in \widehat{\Gamma}$. If h is odd, then $R = A_{2n}$, $h = 2n + 1$ and $\check{\imath} = 2n - i + 1$, so that $p(\check{\imath}) = p(i) + 1$ and $k + h = k + 1 \mod 2$, so $(\check{\imath}, k + h) \in \widehat{\Gamma}$. Hence the map $\nu_{\widehat{\Gamma}}$ is well-defined.

Example 1.3.5. The maps $\nu_{\widehat{\Gamma}}$ and $\gamma_{\widehat{\Gamma}}$ for the case $\Gamma = A_4$ are shown in Figure 1.2.

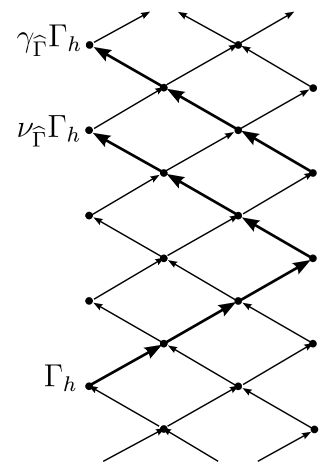

Figure 1.2: The maps $\nu_{\widehat{\Gamma}}$ and $\gamma_{\widehat{\Gamma}}$ in the case $\Gamma = A_4$. A slice Γ_h and its images under $\nu_{\widehat{\Gamma}}$ and $\gamma_{\widehat{\Gamma}}$ are shown in bold.

1.4 The Dynkin Case

In the theory of quiver representations, quivers whose underlying graph is Dynkin of type A, D, E play a special role. In this dissertation they also play an important role, so this section gives a brief summary of the relevant results needed for what follows.

Recall the maps $\nu_{\widehat{\Gamma}}$ and $\gamma_{\widehat{\Gamma}}$ defined on $\widehat{\Gamma}$ by Equation 1.3.4 and Equation 1.3.5 respectively. These maps give a nice combinatorial description of the Auslander-Reiten quiver of $\mathrm{Rep}(\overrightarrow{\Gamma})$, using $\widehat{\Gamma}$ and the Nakayama permutation ν defined by Equation 1.3.4. This is done as follows.

Fix a vertex $i_0 \in \Gamma$. Then given an orientation Ω of Γ there is a unique slice, denoted Γ_Ω in $\widehat{\Gamma}$ identifying the vertex $i_0 \in \Gamma$ with the vertex $(i_0, p(i_0)) \in \widehat{\Gamma}$ and $\overrightarrow{\Gamma}$ with Γ_Ω. This determines a unique height function $h_\Omega : \Gamma \to \mathbb{Z}$. The following Theorem describes the Auslander-Reiten quiver of $\overrightarrow{\Gamma}$ with a full subquiver of $\widehat{\Gamma}$.

Theorem 1.4.1. *[G] The Auslander-Reiten quiver $AR(\Gamma, \Omega)$ of* $\mathrm{Rep}(\Gamma, \Omega)$ *can be identified with the full subquiver of $\widehat{\Gamma}$ lying between the slice $\Gamma_{\Omega^{opp}}$ and the slice $\nu_{\widehat{\Gamma}}(\Gamma_{\Omega^{opp}})$. Explicitly*

$$AR(\overrightarrow{\Gamma}) = \{(i,k) \in \widehat{\Gamma} | h_{\Omega^{op}}(i) \le k \le h_{\Omega^{op}}(i) + h - 2\}.$$

Moreover, the projective representations P_i, correspond to the slice $\Gamma_{\Omega^{opp}}$.

For a proof see [G, Proposition, p.50].

Example 1.4.2. For the case $\Gamma = A_4$ and the orientation Ω given by $1 \leftarrow 2 \leftarrow 3 \leftarrow 4$ the Auslander-Reiten quiver is shown in Figure 1.3 as the shaded region.

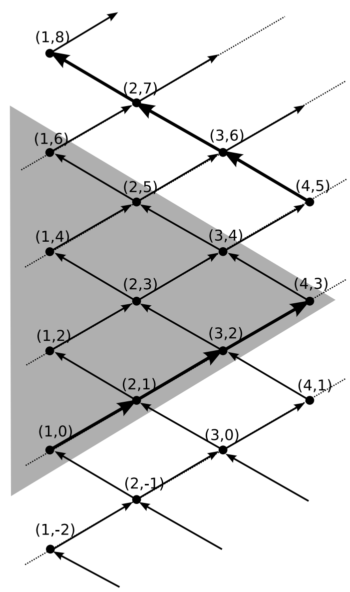

Figure 1.3: The Aulander-Reiten quiver is shown as the shaded region. The slices $\Gamma_{\Omega^{opp}}$ and $\nu_{\widehat{\Gamma}}(\Gamma_{\Omega^{opp}})$ are shown in bold.

On $\mathrm{Rep}(\overrightarrow{\Gamma})$ there are functors ν and τ defined by:

$$\mathrm{Hom}(X, Y) = (\mathrm{Ext}^1(Y, \tau X))^* \qquad (1.4.1)$$

$$\nu(X) = (\mathrm{Hom}_P(X, P))^* \qquad (1.4.2)$$

where in the second line a representation X is identified with a module over the path algebra $P = P(\overrightarrow{\Gamma})$.
In terms of the Auslander-Reiten quiver, these correspond exactly to the maps $\nu_{\widehat{\Gamma}}$ and $\tau_{\widehat{\Gamma}}$.

Remark 1.4.3. In the setting of equivariant sheaves on $\mathbb{P}^1$ considered in [K], the functor τ is given by tensoring with the dualizing sheaf $\mathcal{O}(-2)$.

Now consider the corresponding derived category, denoted by $\mathcal{D}^b(\overrightarrow{\Gamma})$. Recall that an object in $\mathcal{D}(\overrightarrow{\Gamma})$ can be thought of as a choice of complex $X^\bullet(i)$ for each vertex $i \in \Gamma$ together with maps of complexes $x_e : X^\bullet(i) \to X^\bullet(j)$ for each edge $e : i \to j$. In $\mathcal{D}^b(\overrightarrow{\Gamma})$ the indecomposable objects, up to isomophism, are of the form $X[k]$, where X is an indecomposable object in $\mathrm{Rep}(\overrightarrow{\Gamma})$ considered as a complex concentrated in degree 0. The Auslander-Reiten quiver of the derived category is the quiver $\widehat{\Gamma}$ defined in Section 1.3. For more details about the structure of the derived category see [Hap].

The fundamental result for this theory is Gabriel's Theorem, which relates representations of Γ with the corresponding root system.

Theorem 1.4.4. *Let Γ be a Dynkin graph of type A, D, E, let Ω be any orientation of Γ, and let $\overrightarrow{\Gamma} = (\Gamma, \Omega)$ be the corresponding quiver. Then the map $\underline{\dim} : \mathcal{K}(\overrightarrow{\Gamma}) \to \mathbb{Z}^{\Gamma_0}$ gives an isomorphism of lattices. Hence the Grothendieck group of $\mathrm{Rep}(\overrightarrow{\Gamma})$ is identified with the root lattice of the corresponding root system. Moreover, under this bijection the set $\mathrm{Ind} \subset \mathcal{K}(\overrightarrow{\Gamma})$ is identified with the set of positive roots R_+ of the corresponding root system, the symmetrized Euler form $(X,Y)_{\overrightarrow{\Gamma}} = \langle X,Y\rangle_{\overrightarrow{\Gamma}} + \langle Y,X\rangle_{\overrightarrow{\Gamma}}$ gives an inner product on $\mathcal{K}(\overrightarrow{\Gamma})$, and the Auslander-Reiten translation τ gives a Coxeter element in the corresponding Weyl group.*

Gabriel's Theorem provides a bijection between indecomposable representations and positive roots. A natural question is how this bijection depends on the choice of orientation of the Dynkin diagram. To fully understand this it is necessary to pass to the derived category $D^b(\overrightarrow{\Gamma})$. In fact, one must consider the "root" category $\mathcal{D}^b(\overrightarrow{\Gamma})/T^2$ where T is the translation functor of the derived category. For more details on this see [PX1].

Let $\overrightarrow{\Gamma} = (\Gamma, \Omega)$ and let i be a source for Ω. Define a functor $RS_i^+ : \mathcal{D}^b(\Gamma, \Omega) \to \mathcal{D}^b(\Gamma, s_i\Omega)$ by setting

$$RS_i^+ X(j) = \begin{cases} Cone(X(i) \to \bigoplus_{i\to k} X(k)) & \text{if } \ i = j \\ X(j) & \text{otherwise.} \end{cases}$$

For an edge $e : j \to k$ in Ω let $\overline{e}$ denote the corresponding edge in $S_i\Omega$, the map $RS_i^+(x_{\overline{e}})$ is given by

$$RS_i^+(x_{\overline{e}}) = \begin{cases} x_e & \text{if } s(e) \neq i \\ (0, \iota_j) : X(j) \to X^{\bullet+1}(i) \bigoplus \oplus_{i \to j} X(j) & \text{if } s(e) = i \end{cases}$$

where ι_j is the embedding of $X(j)$ into $\oplus X(j)$.

Similarly for i a sink define $LS_i^- : \mathcal{D}^b(\Gamma, \Omega) \to \mathcal{D}^b(\Gamma, s_i\Omega)$ by

$$LS_i^- X(j) = \begin{cases} Cone(\bigoplus_{i \to k} X(k) \to X(i)) & \text{if } \ i = j \\ X(j) & \text{otherwise.} \end{cases}$$

For an edge $e : j \to k$ in Ω let $\overline{e}$ denote the corresponding edge in $S_i\Omega$, the map $LS_i^-(x_{\overline{e}})$ is given by

$$LS_i^-(x_{\overline{e}}) = \begin{cases} x_e & \text{if } t(e) \neq i \\ (\iota_j^{+1}, 0) : X(j) \to (\oplus_{j \to i} X^{\bullet+1}(j)) \bigoplus X^{\bullet}(i)) & \text{if } t(e) = i \end{cases}$$

where ι_j is the embedding of $X(j)$ into $\oplus X(j)$.

These are the derived functors of the well-known "BGP reflection functors" (see [GM]). These functors provide a derived equivalence between the categories $\mathcal{D}^b(\Gamma, \Omega)$ and $\mathcal{D}^b(\Gamma, s_i\Omega)$. Note that the functors RS_i^+ and LS_i^- are inverse to each other. The name reflection functor comes from the action of these functors on the Grothendieck group. In the setting of Gabriel's Theorem, the Grothendieck group of $\mathcal{D}^b(\overrightarrow{\Gamma})/T^2$ is isomorphic to the root lattice, indecomposable objects correspond to all roots and the reflection functors act on the Grothendieck group as the corresponding simple reflections in the Weyl group of the associated root system.

Chapter 2

Combinatorial Construction

As discussed in the introduction, there are several approaches to constructing a root system from the corresponding Dynkin diagram. This chapter gives a combinatorial construction of the root system R corresponding to a Dynkin graph Γ of type A, D, E. Rather than choosing a set of simple roots, as in the standard construction, a choice of Coxeter element is made. This choice then gives a canonical identification of R with the quiver $\widehat{\Gamma}_{cyc}$, in which roots correspond to vertices and the bilinear form and root lattice admit explicit descriptions purely in terms of $\widehat{\Gamma}_{cyc}$. This is then related to the construction of R from Γ using quiver theory.

This construction is motivated by the categorical construction given in the next chapter. However, all the proofs are independent and purely combinatorial. The main results of this chapter are summarized in the following Theorem.

Theorem 2.0.5. *Let Γ be an A, D, E Dynkin diagram, let R denote the corresponding root system and W its Weyl group. Fix $C \in W$ a Coxeter element. Then there is a canonical bijection $\Phi : R \to \widehat{\Gamma}_{cyc}$ with the following properties:*

1. *It identifies the Coxeter element C with the "twist" $\tau : \widehat{\Gamma}_{cyc} \to \widehat{\Gamma}_{cyc}$. Hence the natural projection map $\pi : \widehat{\Gamma}_{cyc} \to \Gamma$ given by $\pi(i, n) = i$ gives a bijection between orbits of the Coxeter element and vertices of the Dynkin diagram.*

2. *It gives a bijection between simple systems Π compatible with C and*

height functions $h : \Gamma \to \widehat{\Gamma}_{cyc}$.

3. *For each height function* h *there is an explicit description of the corresponding positive roots and negative roots as disjoint connected subquivers of* $\widehat{\Gamma}_{cyc}$*, as well as a reduced expression for the longest element* w_0 *in the Weyl group.*

4. *There is a de-symmetrization* $\langle \cdot, \cdot \rangle$ *of the inner product on* R*, which is analogous to the Euler form* $\langle \cdot, \cdot \rangle$ *in the category* $\mathrm{Rep}(\overrightarrow{\Gamma})$ *defined in Section 1.2. Moreover, under the bijection* Φ *this form admits an explicit description in terms of paths in* $\widehat{\Gamma}_{cyc}$.

2.1 Canonical Indexing Set

Let $R \subset E$ be a root system of type A, D, E. Instead of fixing a choice of simple roots, recall that for different simple root systems Π, Π', there is a unique element $w \in W$ such that $w(\Pi) = \Pi'$, which therefore gives a canonical bijection between simple roots $\alpha \in \Pi$ and $\alpha' \in \Pi'$. Therefore, it is possible to use a single index set Γ for indexing simple roots in each of the simple roots systems. More formally, this can be stated as follows.

Proposition 2.1.1. *There is a canonical indexing set* Γ*, which depends only on the root system* R*, such that for any simple root system* Π *there is a bijection*

$$\begin{aligned} \Gamma &\to \Pi \\ i &\mapsto \alpha_i^\Pi \end{aligned}$$

which is compatible with the action of W*: if* $\Pi' = w(\Pi)$*, then* $w(\alpha_i^\Pi) = \alpha_i^{\Pi'}$.

For any $i, j \in \Gamma$, $i \neq j$, define

$$n_{ij} = -(\alpha_i^\Pi, \alpha_j^\Pi) \in \{0, 1\}$$

(this obviously does not depend on the choice of simple root system Π); taking vertices indexed by Γ with i, j connected by n_{ij} unoriented edges gives the Dynkin diagram of R; abusing the notation, this diagram will also be denoted by Γ.

2.2 Coxeter element and compatible simple root systems

From now on, fix a Coxeter element $C \in W$ and denote by h the Coxeter number, i.e. the order of C.

Recall that a simple root system $\Pi = \{\alpha_1, \dots, \alpha_r\}$ is *compatible* with C if there is a reduced expression $C = s_{i_1} \cdots s_{i_r}$. By definition, for any Coxeter element there exists at least one compatible simple root system. However, not every simple root system is compatible with given C. More precisely, there is the following result.

Lemma 2.2.1. *For a given simple root system Π and Coxeter element C let $l^\Pi(C)$ be the length of a reduced expression for C given in terms of the simple reflections s_i^Π. Then $l^\Pi(C) \geq r$ and C is compatible with Π if and only if $l^\Pi(C) = r$.*

Proof. Let $\omega_i \in E$ be the fundamental weights. Then it is immediate from the definition that $s_i(\omega_j) = \omega_j$ for $j \neq i$ and $s_i(\omega_i) = -\omega_i + \sum_j n_{ij}\omega_j$. If $l^\Pi(C) < r$ then $C = s_{i_1} \cdots s_{i_l}$, and there exists $i \in \Gamma$ such that $i \neq i_k$ for any k. Hence $C(\omega_i) = \omega_i$. However the Coxeter element has no fixed vectors in E (see [Kos2, Lemma 8.1]). Thus, $l^\Pi(C) \geq r$.

Now suppose that $l^\Pi(C) = r$, so $C = s_{i_1} \cdots s_{i_r}$. Then the argument above shows that every $i \in \Gamma$ must appear in $\{i_1, \dots, i_r\}$. Since $|\Gamma| = r$, it must appear exactly once, so C is compatible with Π. □

The next proposition describes the set of Coxeter elements compatible with a fixed set of simple roots.

Proposition 2.2.2. *[Shi]*

1. *Let C be a Coxeter element, Π a simple root system compatible with C. Choose a reduced expression for C and define an orientation on Γ as follows: $i \to j$ if $n_{ij} = 1$ and i precedes j in a reduced expression for C: $C = \dots s_i^\Pi \dots s_j^\Pi \dots$. Then this orientation does not depend on the choice of a reduced expression for C.*

2. For fixed Π*, the correspondence*

$$\{\textit{Coxeter elements compatible with } \Pi\} \to \{\textit{ orientations of } \Gamma\}$$

defined in Part 1, is a bijection.

Proof. For a fixed set of simple roots Π, if there are two expressions $C = s_{i_1} \cdots s_{i_r} = s_{i'_1} \cdots s_{i'_r}$ for a Coxeter element, we can obtain one from another by the operations $s_i s_j \to s_j s_i$ for $i, j \in \Gamma$ satisfying $n_{ij} = 0$. As mentioned in Section 1.1 any two reduced expressions for an element $w \in W$ can be obtained from each other by using operations $s_i s_j \to s_j s_i$ if $n_{ij} = 0$ and $s_j s_i s_j \to s_i s_j s_i$ if $n_{ij} = 1$; since for a Coxeter element every simple reflection appears only once, the second operation does not apply. Thus, the orientation on Γ does not depend on the choice of reduced expression.

Conversely, given an orientation on Γ, define a complete order on Γ as follows: $\Gamma = \{i_1, \dots, i_r\}$ so that all arrows are of the form $i_k \to i_l$ with $k < l$; thus, this orientation is obtained from the Coxeter element $s^{\Pi}_{i_1} \dots s^{\Pi}_{i_r}$. One easily sees that the order is defined uniquely up to interchanging i, j with $n_{ij} = 0$ and thus the Coxeter element is independent of this choice of order. □

Example 2.2.3. For the root system $R = A_n$ and $\Pi = \{\alpha_1 = e_1 - e_2, \dots, \alpha_n = e_n - e_{n+1}\}$ the Coxeter element $C = s_1 \cdots s_n$ corresponds to the orientation $1 \to 2 \to \cdots \to n$.

What is of interest in the present situation, however, is the opposite direction: given a fixed Coxeter element C, how to describe all simple root systems Π which are compatible with C.

For fixed compatible pair (C, Π) the following result shows how to construct another set of simple roots compatible with C, and describes how the corresponding orientations of Γ relate.

Proposition 2.2.4. *Let* C*,* Π *be compatible and* $i \in \Gamma$ *be a sink (or source) for the corresponding orientation of* Γ *as defined in Proposition 2.2.2.*

1. $C = s_{i_1} \cdots s_{i_{r-1}} s_i$ *(if* $i \in \Gamma$ *is a sink) or* $C = s_i s_{i_1} \cdots s_{i_{r-1}}$ *(if* $i \in \Gamma$ *is a source).*

2. $\Pi' = s_i^{\Pi}(\Pi)$ *is also a set of simple roots compatible with* C. *In this case,* Π *is said to be obtained from* Π' *by elementary reflection. Note that elementary reflection* s_i *can only be applied to* Π *when* i *is a sink or source for the orientation defined by* Π.

3. *The orientation of* Γ *corresponding to* Π' *is obtained from the orientation corresponding to* Π *by reversing the arrows at* i. *Thus a sink becomes a source, and vice versa.*

Proof. The proof is given for $i \in \Gamma$ a sink. The proof for a source is almost identical.

1. If i is a sink, all s_j which do not commute with s_i must precede s_i in the reduced expression for C. Thus, s_i can be moved to the end of the reduced expression.

2. Denoting temporarily $s_j = s_j^{\Pi}$, $s'_j = s_j^{\Pi'}$, then $s'_j = s_i s_j s_i$. Then, using Part 1, write

$$\begin{aligned} C &= s_{i_1} \cdots s_{i_{r-1}} s_i \\ &= (s_i s_i) s_{i_1} (s_i s_i) s_{i_2} (s_i s_i) \cdots (s_i s_i) s_{i_{r-1}} s_i \\ &= s_i (s_i s_{i_1} s_i)(s_i s_{i_2} s_i) \cdots (s_i s_{r-1} s_i) \\ &= s_i s'_{i_1} \cdots s'_{i_{r-1}} = s'_i s'_{i_1} \cdots s'_{i_{r-1}} \end{aligned}$$

   Hence C is compatible with Π'.

3. From $C = s'_i s'_{i_1} \cdots s'_{i_{r-1}}$ and Proposition 2.2.2 we see that i is a source for the orientation obtained from Π' and that the orientation is obtained by reversing all the arrows to i.

$\square$

Theorem 2.2.5. *Fix a Coxeter element* $C \in W$. *Then:*

1. *The map*

$$\{\textit{Simple root systems } \Pi \textit{ compatible with } C\} \rightarrow \{\textit{orientations of } \Gamma\}$$

   defined in Proposition 2.2.2 is surjective. Two different simple root systems Π, Π', *both compatible with* C, *give the same orientation of* Γ *if and only if* $\Pi' = C^k \Pi$ *for some* $k \in \mathbb{Z}$.

2. *If Π, Π' are two simple root systems, both compatible with C, then Π' can be obtained from Π by a sequence of elementary reflections s_i as in Proposition 2.2.4.*

Proof. The fact that the map is surjective easily follows from Proposition 2.2.4 and the fact that any two orientations of a graph without cycles can be obtained one from the other by a sequence of operations s_i: sink $\leftrightarrow$ source.

Now, assume that two simple root systems Π, Π' give the same orientation. Denoting as before $s_i = s_i^{\Pi}$, $s'_i = s_i^{\Pi'}$, then for some complete order of Γ

$$C = s_{i_1} \dots s_{i_r} = s'_{i_1} \dots s'_{i_r}$$

(note that the order is the same for s_i and s'_i!). Let $w \in W$ be such that $w(\Pi) = \Pi'$; then $s'_i = w s_i w^{-1}$ and therefore $C = (w s_{i_1} w^{-1}) \dots (w s_{i_r} w^{-1}) = w C w^{-1}$, so w commutes with C. However, it is known (see [Sp]) that the centralizer of the Coxeter element is the cyclic group generated by C. Thus, $w = C^k$.

Finally, to prove the last part, note that it is well known that any two orientations can be obtained one from another by a sequence of elementary reflections (reversing all arrows at a sink or a source). Thus, if Π, Π' are compatible with C, then applying a sequence of elementary reflections s_i as in Proposition 2.2.4, one can obtain from Π' a simple root system Π'' which gives the same orientation as Π. By Part 1, it means that $\Pi'' = C^k \Pi$. But notice that the simple root system $C(\Pi)$ can be obtained from Π by a sequence of reflections s_i: namely, if $C = s_{i_1}^{\Pi} \dots s_{i_r}^{\Pi}$, then consider the sequence of simple root systems

$$\begin{aligned}
&\Pi_0 = \Pi, \quad \Pi_1 = s_{i_1}^{\Pi}(\Pi_0)\\
&\Pi_2 = s_{i_2}^{\Pi_1}(\Pi_1) = s_{i_1} s_{i_2} s_{i_1}(\Pi_1) = s_{i_1} s_{i_2}(\Pi)\\
&\dots\\
&\Pi_r = s_{i_r}^{\Pi_{r-1}}(\Pi_{r-1}) = s_{i_1} \dots s_{i_r}(\Pi) = C(\Pi).
\end{aligned}$$

One easily sees that i_k is a source for Π_{k-1}, so the above sequence of elementary reflections is well-defined. $\square$

Corollary 2.2.6. *For a given Coxeter element C, there is a canonical bijection*

$$\{\textit{Simple root systems } \Pi \textit{ compatible with } C\} / C \to \{\textit{orientations of } \Gamma\}$$

In particular, this shows that the number of simple root systems compatible with C is equal to $h2^{r-1}$, where h is the Coxeter number and r is the rank. For example, for the root system of type A_{n-1}, where the Coxeter number is n and the rank is $n-1$, this gives $n2^{n-2}$ (compared with the number of all simple root systems, equal to $n!$).

A graphical description of the set of all compatible simple root systems in terms of "height functions" will be given later, in Theorem 2.4.5.

2.3 Representatives of C-Orbits

As before, fix a Coxeter element C and choose a simple system Π compatible with C. Define an order $\leqslant$ on Γ by $i \leqslant j$ if there exists an oriented path $i \to \cdots \to j$, with the orientation defined by Π as in Proposition 2.2.2. In this case, one easily sees that s_i must precede s_j in the reduced expression for C.

Using this relation define $\beta_i^\Pi \in R$ by

$$\beta_i^\Pi = \sum_{j \leqslant i} \alpha_j^\Pi. \tag{2.3.1}$$

Proposition 2.3.1.

1. *The β_i^Π are a basis of the root lattice, and $\alpha_i = \beta_i^\Pi - \sum_{j \to i} \beta_j^\Pi$.*
2. *Let $C = s_{i_1}^\Pi \cdots s_{i_r}^\Pi$ be a reduced expression for C. Then*
$$\beta_{i_k}^\Pi = s_{i_1}^\Pi \cdots s_{i_{k-1}}^\Pi (\alpha_{i_k}^\Pi).$$
3. *$\{\beta_1^\Pi, \dots, \beta_r^\Pi\} = \{\alpha \in R_+^\Pi \mid C^{-1}\alpha \in R_-^\Pi\}$ where $R_\pm^\Pi$ are the sets of positive and negative roots defined by the simple root system Π.*
4. *β_i^Π are representatives of the C-orbits in R.*

Remark 2.3.2. In the theory of quiver representations, the simple representations X_i correspond to the simple roots α_i^Π in R, while the projective representations P_i correspond to the C-orbit representatives β_i^Π. This correspondence will be discussed in detail in Section 2.8.

Proof. The first two parts are easily obtained by explicit computation (see, e.g. [Kos2, Theorem 8.1]). The other two parts are known; a proof can be found in [Bour, Chapter VI, §1, Proposition 33]. □

Example 2.3.3. For the root system $R = A_n$ with simple roots $\Pi = \{\alpha_1 = e_1 - e_2, \dots, \alpha_n = e_n - e_{n+1}\}$ and Coxeter element $C = s_1 s_2 \cdots s_n$ the corresponding C-orbit representatives are $\beta_i^\Pi = \sum_{j \leqslant i} \alpha_j^\Pi = e_1 - e_i$.

Example 2.3.4. Let $\Gamma = \Gamma_0 \sqcup \Gamma_1$ be a bipartite splitting and

$$C = (\Pi_{i \in \Gamma_0} s_i)(\Pi_{i \in \Gamma_1} s_i).$$

Note that in the corresponding orientation of Γ all the arrows go from Γ_0 to Γ_1. So Γ_0 are sources and Γ_1 are sinks. Then $\beta_i^\Pi = \alpha_i^\Pi$ for $i \in \Gamma_0$ and $\beta_i^\Pi = \alpha_i^\Pi + \sum_j n_{ij} \alpha_j^\Pi = -C(\alpha_i^\Pi)$ for $i \in \Gamma_1$. In this case the β_i^Π obtained are the same C-orbit representatives as in [Kos] except that our case the β_i^Π for $i \in \Gamma_1$ are shifted by C.

Example 2.3.5. As a special case of the previous example, consider the Dynkin diagram of type D_{2n+1}, with simple root system

$$\Pi = \{\alpha_1 = e_1 - e_2, \dots, \alpha_{2n-1} = e_{2n-1} - e_{2n}, \alpha_{2n} = e_{2n} - e_{2n+1}, e_{2n+1} = e_{2n} + e_{2n+1}\}$$

and

$$\Gamma_0 = \{2, 4, \dots, 2n, 2n+1\}, \qquad \Gamma_1 = \{1, 3, \dots, 2n-1\}$$

The corresponding Coxeter element is given by $C = (\Pi_{i \in \Gamma_0} s_i)(\Pi_{i \in \Gamma_1} s_i)$, and the C-orbit representatives are

$\beta_i^\Pi = \alpha_i$ for $i \in \Gamma_0 = \{2, 4, \dots, 2n, 2n+1\}$
$\beta_1^\Pi = e_1 - e_3, \beta_3^\Pi = e_2 - e_5, \dots e_{2i+1} = e_{2i} - e_{2i+3}, \dots, \beta_{2n-3} = e_{2n-2} - e_{2n-1}$
$\beta_{2n-1}^\Pi = e_{2n-2} + e_{2n}$

Figure 2.1 shows the corresponding orientation, as well as the roots α_i and β_i for D_5.

A natural question is how the set of β_i change when the simple root system Π is changed (keeping C fixed). By Theorem 2.2.5, it suffices to describe how β_i change under elementary reflections.

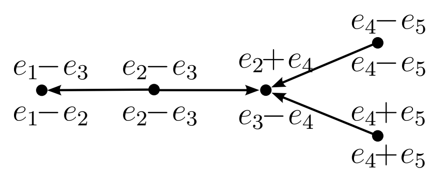

Figure 2.1: The Dynkin Diagram D_5. The C-orbit representative β_i^Π is above the node $i \in \Gamma$, and the simple root α_i^Π is below the node $i \in \Gamma$.

Proposition 2.3.6. *Let Π be a simple root system compatible with C, and let $i \in \Gamma$ be a sink for the corresponding orientation. Then*

$$\beta_j^{s_i\Pi} = \begin{cases} C^{-1}\beta_i^\Pi & \text{for } j = i \\ \beta_j^\Pi & \text{for } j \neq i \end{cases}$$

Similarly, if i is a source, then

$$\beta_j^{s_i\Pi} = \begin{cases} C\beta_i^\Pi & \text{for } j = i \\ \beta_j^\Pi & \text{for } j \neq i \end{cases}$$

Proof. For brevity deonte $s_j = s_j^\Pi$, $s'_j = s_j^{s_i\Pi}$. For i a sink, it is possible to write $C = s_{i_1} \cdots s_{i_{r-1}} s_i$ and $\beta_i = s_{i_1} \cdots s_{i_{r-1}} \alpha_i$. Hence

$$\begin{aligned} C^{-1}\beta_i &= (s_i s_{i_{r-1}} \cdots s_{i_1})(s_{i_1} \cdots s_{i_{r-1}})\alpha_i \\ &= s_i \alpha_i \\ &= \alpha_i^{s_i\Pi} \\ &= \beta_i^{s_i\Pi} \end{aligned}$$

since i is a source for $s_i\Pi$.
Now take $i_j \neq i$. Then

$$\begin{aligned} \beta_{i_j}^\Pi &= s_{i_1} \cdots s_{i_{j-1}} \alpha_{i_j} \\ &= (s_i s_i) s_{i_1} (s_i s_i) \cdots (s_i s_i) s_{i_{j-1}} (s_i s_i) \alpha_{i_j} \\ &= s_i s'_{i_1} \cdots s'_{i_{j-1}} (s_i \alpha_{i_j}) \\ &= \beta_{i_j}^{s_i\Pi} \end{aligned}$$

Similarly, for i a source, write $C = s_i s_{i_1} \cdots s_{i_{r-1}}$ and $\beta_i = \alpha_i$. Hence

$$\begin{aligned} C\alpha_i &= s_i s_{i_1} \cdots s_{i_{r-1}} \alpha_i \\ &= s'_{i_1} \cdots s'_{i_{r-1}} s_i \alpha_i \\ &= \beta_i^{s_i \Pi} \end{aligned}$$

since i is a sink for $s_i \Pi$.
For $i_j \neq i$ a similar calculation to the case i a sink gives $\beta_{i_j}^{\Pi} = \beta_{i_j}^{s_i \Pi}$. □

Theorem 2.3.7. *Let R/C be the set of C-orbits in R. Then there is a bijection*

$$\begin{aligned} \Gamma &\to R/C \\ i &\mapsto C\text{-orbit of } \beta_i^{\Pi} \end{aligned}$$

which does not depend on the choice of a simple root system Π compatible with C.

Proof. The fact that it is a bijection follows from Proposition 2.3.1. To show independence of the choice of Π, notice that by Proposition 2.3.6, if Π, Π' are obtained one from another by an elementary reflection, then the C-orbit of β_i^{Π} and $\beta_i^{\Pi'}$ coincide. On the other hand, by Theorem 2.2.5, any two simple root systems compatible with C can be obtained one from another by elementary reflections. □

For future use, the following proposition is given here, which describes the action of C on β_i. Its geometric meaning will become clear in Section 2.5.

Proposition 2.3.8. $C\beta_i^{\Pi} = -\beta_i^{\Pi} + \sum_{j \to i} C\beta_j^{\Pi} + \sum_{j \leftarrow i} \beta_j^{\Pi}$.

Proof. Let $i \in \Gamma$ be a sink. Then there are no $j \leftarrow i$ and from the proof of Proposition 2.3.6, $-C^{-1}\beta_i^{\Pi} = \alpha_i = \beta_i^{\Pi} - \sum_{j \to i} \beta_j^{\Pi}$. Applying C to this equation and rearranging gives the statement of the proposition.

Now if $i \in \Gamma$ is not a sink apply a sequence of reflections s_j with $j \geq i$ to make it one. This process replaces Π by another compatible simple root system Π'. By Proposition 2.3.6,

1. $\beta_i^{\Pi} = \beta_i^{\Pi'}$

2. if $j \to i$ for Π then $\beta_j^{\Pi} = \beta_j^{\Pi'}$

3. if $j \leftarrow i$ for Π then $\beta_j^{\Pi'} = C^{-1}\beta_j^{\Pi}$.

Then the argument of the first paragraph of the proof gives

$$\begin{aligned}
\beta_i^{\Pi'} + C\beta_i^{\Pi'} &= \sum_{j \to i \text{ in } \Pi'} C\beta_j^{\Pi'} \\
&= \sum_{j \to i \text{ in } \Pi} C\beta_j^{\Pi'} + \sum_{j \leftarrow i \text{ in } \Pi} C\beta_j^{\Pi'} \\
&= \sum_{j \to i \text{ in } \Pi} C\beta_j^{\Pi} + \sum_{j \leftarrow i \text{ in } \Pi} C(C^{-1}\beta_j^{\Pi}) \\
&= \sum_{j \to i \text{ in } \Pi} C\beta_j^{\Pi} + \sum_{j \leftarrow i \text{ in } \Pi} \beta_j^{\Pi}.
\end{aligned}$$

□

2.4 Identification of R and $\widehat{\Gamma}_{cyc}$

The goal of this section is to show that the set of vertices of $\widehat{\Gamma}_{cyc}$ can be (almost) canonically identified with the root system R. To do so, fix one of the vertices $i_0 \in \Gamma$ and choose an identification

$$\Phi_0 \colon (C\text{– orbit of } \beta_{i_0}) \to \{(i_0, n) \mid n \in \mathbb{Z}_{2h}, n + p(i_0) \equiv 0 \mod 2\} \subset \widehat{\Gamma}_{cyc}$$

which identifies the Coxeter element with the twist: $\Phi_0(C\beta) = \tau\Phi_0(\beta)$.

Remark 2.4.1. The seeming arbitrariness in the choice of Φ_0 could have been avoided if $\mathbb{Z}_{2h}$ were replaced with a suitable $\mathbb{Z}_{2h}$-torsor in the definition of $\widehat{\Gamma}_{cyc}$.

Theorem 2.4.2. *Let C be a fixed Coxeter element and Π a simple root system compatible with C. Then there exists a unique bijection $\Phi^{\Pi} \colon R \to \widehat{\Gamma}_{cyc}$ with the following properties*

1. *It identifies the Coxeter element with the twist: $\Phi^{\Pi}(C\beta) = \tau\Phi^{\Pi}(\beta)$.*

2. *It agrees with the identification $R/C \to \Gamma$ given in Theorem 2.3.7: $\Phi^{\Pi}(\beta_i^{\Pi}) = (i, h(i))$ for some $h \colon \Gamma \to \mathbb{Z}_{2h}$.*

3. *If, in the orientation defined by* Π*, we have* $i \to j$*, and* $\Phi^\Pi(\beta_i^\Pi) = (i, n)$*, then* $\Phi^\Pi(\beta_j^\Pi) = (j, n+1)$ *(see Figure 2.2).*

4. *On the* C*–orbit of* β_{i_0}*,* Φ^Π *coincides with* Φ_0*.*

Proof. Since β_i^Π are representatives of C-orbits in R (Proposition 2.3.1), and each C-orbit has period h, it suffices to define $\Phi^\Pi(\beta_i^\Pi)$. On the other hand, condition (4) uniquely defines $\Phi^\Pi(\beta_{i_0}^\Pi)$, and it is easy to see that given $\Phi^\Pi(\beta_{i_0}^\Pi)$, conditions (2) and (3) uniquely determine $\Phi^\Pi(\beta_i^\Pi)$ for all $i \in \Gamma$ (since Γ is connected and simply-connected). □

Theorem 2.4.3. *The bijection* $\Phi \colon R \to \widehat{\Gamma}_{cyc}$*, defined in Theorem 2.4.2, does not depend on the choice of* Π*.*

Proof. By Theorem 2.2.5 it suffices to check that if Π' is obtained from Π by elementary reflection, then $\Phi^\Pi = \Phi^{\Pi'}$. To do so it suffices to check that Φ^Π satisfies all the defining properties of $\Phi^{\Pi'}$. The only property which is not obvious is (3): if in the orientation defined by Π', $i \to j$, and $\Phi^\Pi(\beta'_i) = (i, n)$, then $\Phi^\Pi(\beta'_j) = (j, n+1)$, where for brevity we denoted $\beta_i = \beta_i^\Pi$ and $\beta'_i = \beta_i^{\Pi'}$.

Let $\Pi' = s_k\Pi$. If i, j are both distinct from k, then by Proposition 2.3.6, $\beta_i = \beta'_i$, $\beta_j = \beta'_j$, so property (3) for $\Phi^{\Pi'}$ coincides with the one for Φ^Π. So the only case to consider is when $i = k$ or $j = k$; in these cases, s_k reverses the orientation of the edge between i and j.

If $i = k$, then for Π we have $j \to i$, so i is a sink for Π. Then by Proposition 2.3.6 $\beta'_j = \beta_j$, $\beta'_i = C^{-1}\beta_i$. By definition, if $\Phi^\Pi(\beta_i) = (i, n)$, then $\Phi^\Pi(\beta_j) = (j, n-1)$, and $\Phi^\Pi(\beta'_i) = \tau^{-1}(i, n) = (i, n-2)$ (see Figure 2.2). Thus, condition (3) for pair β'_i, β'_j is satisfied.

Case $j = k$ is done similarly. □

Example 2.4.4. For a Dynkin digram of type A_4, with $\Pi = \{e_1 - e_2, \dots, e_4 - e_5\}$ and $C = s_1s_2s_3s_4$, the map $\Phi \colon R \to \widehat{\Gamma}_{cyc}$ is shown in Figure 2.3. For A_n the figure is similar.

As an immediate application of this construction, it is possible to give an explicit description of the set of all simple root systems compatible with C purely in terms of $\widehat{\Gamma}_{cyc}$.

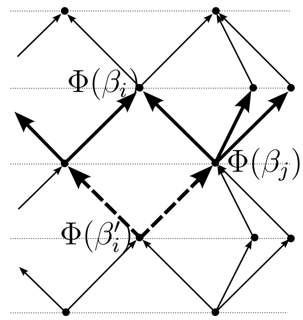

Figure 2.2: Roots under the identification Φ.

Theorem 2.4.5. *Let Π be a simple root system compatible with C. Then the function $h^{\Pi}\colon \Gamma \to \mathbb{Z}_{2h}$ defined by $\Phi(\beta_i^{\Pi}) = (i, h^{\Pi}(i))$ is a height function. Conversely, every height function can be obtained in this way from a unique simple root system compatible with C.*

Proof. The fact that h^{Π} is a height function is obvious from the definition of Φ. To check that any height function can be obtained from some Π compatible with C, note that by Theorem 2.2.5, for given height function h there is a simple root system Π which would give the same orientation of Γ as h. Therefore, we would have $h^{\Pi} = h + a$ for some constant $a \in \mathbb{Z}_{2h}$, which must necessarily be even. Take $\Pi' = C^{a/2}(\Pi)$; then $h = h^{\Pi'}$. $\square$

Corollary 2.4.6. *For a given Coxeter element C, there is a canonical bijection*

$$\{\textit{Simple root systems } \Pi \textit{ compatible with } C\} \to \{\textit{Height functions on } \Gamma\}.$$

The elementary reflections s_i can also be easily described in terms of height functions.

Proposition 2.4.7. *If Π is a simple root system compatible with C, i is a sink for the orientation defined by Π, and h^{Π}, $h^{s_i\Pi}$ is the corresponding height functions as defined in Theorem 2.4.5, then*

$$h^{s_i\Pi}(j) = \begin{cases} h(j) - 2 & j = i \\ h(j) & j \neq i \end{cases}$$

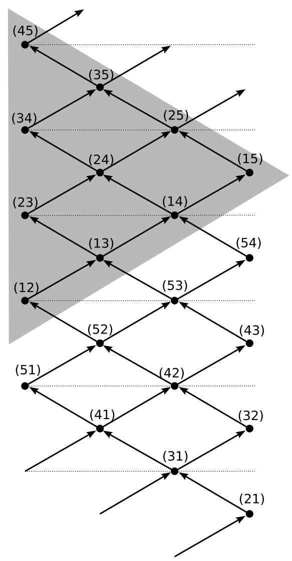

Figure 2.3: The map Φ for the root system of type A_4. The figure shows, for each vertex in $\widehat{\Gamma}_{cyc}$, the corresponding root $\alpha \in R$; the notation (ij) stands for $e_i - e_j$. The set of positive roots (with respect to usual polarization of R) is shaded. Recall that the quiver $\widehat{\Gamma}_{cyc}$ is periodic.

Simiarly, if i is a source, then

$$h^{s_i\Pi}(j) = \begin{cases} h(j)+2 & j=i \\ h(j) & j\neq i \end{cases}$$

(*see Figure 2.4*).

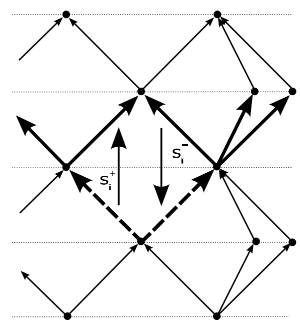

Figure 2.4: The action of simple reflections on height functions.

The proof is immediate from Proposition 2.3.6.

Remark 2.4.8. Note that in this case the height function $h^{s_i\Pi}$ is the same as $s_i^{\pm}h^{\Pi}$ as defined in Section 1.3, and abusing notation, from now on both will be denoted by s_ih^{Π}.

2.5 Root lattice

In the previous section a canonical bijection $\Phi\colon R \to \widehat{\Gamma}_{cyc}$ was constructed, which is independent of the choice of a simple root system Π. Among other things, it identified the set of all compatible Π with the set of "height functions", by using $\Phi(\beta_i^{\Pi})$. In this section this isomorphism is studied further. In particular, a description of the root lattice in terms of $\widehat{\Gamma}_{cyc}$ is given.

Theorem 2.5.1. *Let the lattice Q be defined by*

$$Q = \mathbb{Z}^{\widehat{\Gamma}_{cyc}}/J, \tag{2.5.1}$$

where J is the ideal generated by the following relations, for each $(i,n)\in\widehat{\Gamma}_{cyc}$,

$$(i,n) - \sum_j (j,n+1) + (i,n+2) = 0 \tag{2.5.2}$$

(*where the sum is over all vertices* $j \in \Gamma$ *connected to* i). *Then*

1. *The identification* $\Phi\colon R \to \widehat{\Gamma}_{cyc}$ *defined in Theorem 2.4.3 descends to an isomorphism of lattices* $Q(R) \to Q$, *where* $Q(R)$ *is the root lattice of* R.

2. *For any height function* h, *the elements* $(i, h(i)) \in \widehat{\Gamma}_{cyc}$ *form a basis of* Q.

Proof. Let h be a height function. Then the classes $(i, h(i))$ for $i \in \Gamma$, generate the lattice Q. Indeed, let Q_h be the subgroup generated by $(i, h(i))$. It follows from relations (2.5.2) that if i is a source for h, then $(i, h(i) + 2)$ is in Q_h; thus, $Q_h = Q_{s_i h}$. Since any height function can be obtained from h by succesive applications of elementary reflections, it follows that Q_h contains all $(i, n) \in \widehat{\Gamma}_{cyc}$, so $Q_h = Q$. In particular, this implies that $\mathrm{rank}(Q) \leq r$.

Next, we show that Φ^{-1} descends to a well-defined map $Q \to Q(R)$. To do so requires verifying that

$$\Phi^{-1}(i, n) - \sum_j \Phi^{-1}(j, n+1) + \Phi^{-1}(i, n+2) = 0$$

in $Q(R)$. Choosing a simple root system Π such that $\Phi^{-1}(i, n) = \beta_i^{\Pi}$, this relation is equivalent to the relation

$$\beta_i^{\Pi} - \Big(\sum_{j \to i} C\beta_j^{\Pi} + \sum_{j \leftarrow i} \beta_j^{\Pi}\Big) + C\beta_i^{\Pi} = 0$$

proved in Proposition 2.3.8.

Since $\Phi^{-1}\colon \widehat{\Gamma}_{cyc} \to R$ is a bijection, the map $\Phi^{-1}\colon Q \to Q(R)$ is surjective. Since it has already been shown that $\mathrm{rank}(Q) \leq r$, this implies that $\mathrm{rank}(Q) = r$, so that Φ^{-1} is an isomorphism, and that for fixed height function h, the classes $(i, h(i))$ form a basis of Q. $\square$

Remark 2.5.2. Relations (2.5.2) are motivated by almost split exact sequences in the theory of quiver representations, or by the short exact sequence of coherent sheaves on $\mathbb{P}^1 = \mathbb{P}(V)$: $0 \to \mathcal{F} \to \mathcal{F}(1) \otimes V \to \mathcal{F}(2) \to 0$.

2.6 Euler form

Recall the Euler form

$$\langle X, Y\rangle = \dim \mathrm{RHom}(X, Y) = \dim \mathrm{Hom}(X, Y) - \dim \mathrm{Ext}^1(X, Y)$$

on $\mathrm{Rep}(\overrightarrow{\Gamma})$ defined in Section 1.2. In this section the analog of such a form is defined on both R and $\widehat{\Gamma}_{cyc}$ and used to give a definition of the inner product on R in terms of $\widehat{\Gamma}_{cyc}$.

Define a bilinear form $\langle\cdot,\cdot\rangle^{\Pi}$ on the root lattice $Q(R)$ by

$$\langle \beta_i^{\Pi}, \alpha_j^{\Pi}\rangle^{\Pi} = \delta_{ij}. \tag{2.6.1}$$

By Proposition 2.3.1, this completely determines $\langle\cdot,\cdot\rangle^{\Pi}$. Note that this form is non-degenerate but not symmetric.

Theorem 2.6.1.

1. $\langle \beta_i^{\Pi}, \beta_j^{\Pi}\rangle^{\Pi} = \begin{cases} 1 & i \leqslant j \\ 0 & \text{otherwise} \end{cases}$ *= the number of paths* $(i \to \cdots \to j)$.

2. $\langle\cdot,\cdot\rangle^{\Pi}$ *is integer valued on* R *and satisfies*

$$\langle x, y\rangle^{\Pi} + \langle y, x\rangle^{\Pi} = (x, y) \tag{2.6.2}$$

$$\langle x, y\rangle^{\Pi} = -\langle y, C^{-1}x\rangle^{\Pi}. \tag{2.6.3}$$

where $(\cdot,\cdot)$ *is the* W*-invariant inner product in* E *normalized so that* $(\alpha, \alpha) = 2$ *for* $\alpha \in R$.

Note that the equation $\langle x, y\rangle = -\langle y, C^{-1}x\rangle$ in R corresponds to the statement of Serre Duality $\mathrm{Hom}(X, Y) = \mathrm{Ext}^1(Y, X(-2))^*$ in the theory of equivariant sheaves described in [K], or equivalently, the identity $\mathrm{Ext}^1(X, Y) = D\,\mathrm{Hom}(X, \tau Y)$ in Auslander–Reiten theory (see [ARS]).

Proof. Since throughout the proof the simple root system Π will be fixed, the superscripts will be dropped, writing β_i for β_i^{Π}, etc.

1. Obvious from the definition of β_j.

2. To see that $\langle\cdot,\cdot\rangle^\Pi$ symmetrizes to $(\cdot,\cdot)$, use the identity $\alpha_i = \beta_i - \sum_{k\to i}\beta_k$ (see Proposition 2.3.1), which gives

$$\langle\alpha_i,\alpha_j\rangle^\Pi = \langle\beta_i - \sum_{k\to i}\beta_k, \alpha_j\rangle^\Pi = \begin{cases} 1 & \text{if } i=j \\ -1 & \text{if } j\to i \\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}\langle\alpha_i,\alpha_j\rangle^\Pi + \langle\alpha_j,\alpha_i\rangle^\Pi &= \begin{cases} 2 & \text{if } i=j \\ -1 & \text{if } i\to j \text{ or } j\to i \\ 0 & \text{otherwise}\end{cases} \\ &= (\alpha_i,\alpha_j).\end{aligned}$$

To prove relation (2.6.3), it suffices to prove that for all k, i one has

$$\langle\beta_k, C^{-1}\beta_i\rangle^\Pi = -\langle\beta_i,\beta_k\rangle^\Pi. \tag{2.6.4}$$

This will be proved by fixing k and using induction in i, using the partial order defined by the orientation. So assume that (2.6.4) is true for all $j \geqslant i$.

Using Proposition 2.3.8, rewrite the left-hand side of (2.6.4) as

$$\langle\beta_k, C^{-1}\beta_i\rangle^\Pi = -\langle\beta_k,\beta_i\rangle^\Pi + \sum_{j\to i}\langle\beta_k,\beta_j\rangle^\Pi + \sum_{j\leftarrow i}\langle\beta_k, C^{-1}\beta_j\rangle^\Pi.$$

The first two terms can be rewritten as

$$-\langle\beta_k,\beta_i\rangle^\Pi + \sum_{j\to i}\langle\beta_k,\beta_j\rangle^\Pi = -\langle\beta_k,\alpha_i\rangle^\Pi = -\delta_{ik}.$$

The last term, using the induction assumption, can be rewritten as

$$\begin{aligned}\sum_{j\leftarrow i}\langle\beta_k, C^{-1}\beta_j\rangle^\Pi &= -\sum_{j\leftarrow i}\langle\beta_j,\beta_k\rangle^\Pi \\ &= -\sum_{j\leftarrow i}(\text{number of paths } j\to\cdots\to k).\end{aligned}$$

Thus, the left-hand side of (2.6.4) is

$$\begin{aligned}\langle \beta_k, C^{-1}\beta_i\rangle^{\Pi} &= -\delta_{ik} - \sum_{j\leftarrow i}(\text{number of paths } j \to \cdots \to k)\\ &= -(\text{number of paths } i \to \cdots \to k) = -\langle \beta_i, \beta_k\rangle^{\Pi}\end{aligned}$$

which proves (2.6.4).

□

Theorem 2.6.2. *For fixed C, the form $\langle\cdot,\cdot\rangle^{\Pi}$ does not depend on the choice of simple root system Π compatible with C. Thus this form will be denoted by $\langle\cdot,\cdot\rangle$ and called the* Euler form *defined by C.*

Proof. Consider the difference $\ll \cdot,\cdot \gg = \langle\cdot,\cdot\rangle^{\Pi_1} - \langle\cdot,\cdot\rangle^{\Pi_2}$. Since these two forms have the same symmetrization, $\ll \cdot,\cdot \gg$ is skew-symmetric and satisfies (2.6.4). Thus,

$$\ll x, y \gg = - \ll y, C^{-1}x \gg = \ll C^{-1}x, y \gg$$

so $\ll (1 - C^{-1})x, y \gg = 0$. Since 1 is not an eigenvalue for C^{-1} (see [Kos2, Lemma 8.1]), the operator $1 - C^{-1}$ is invertible, so the form $\ll \cdot,\cdot \gg$ must be identically zero. □

Proposition 2.6.3. *Let $\langle\cdot,\cdot\rangle$ be the Euler form defined by C in Theorem 2.6.2.*

1. *The form $\langle\cdot,\cdot\rangle$ is C–invariant: $\langle Cx, Cy\rangle = \langle x, y\rangle$.*
2. *$\langle x, y\rangle = (x, (1 - C^{-1})^{-1}y) = ((1 - C)^{-1}x, y)$*
3. *Let ω_i^{Π} be fundamental weights: $(\omega_i^{\Pi}, \alpha_j^{\Pi}) = \delta_{ij}$. Then $\beta_i^{\Pi} = (1 - C)\omega_i^{\Pi}$*
4. *The operator $(1 - C)$ is an isomorphism $P(R) \to Q(R)$, where $Q(R)$, $P(R)$ are root and weight lattices of R respectively.*

Proof.

1. Choose a compatible simple root system Π. Then by definition $\langle \beta_i^{\Pi}, \alpha_j^{\Pi}\rangle = \delta_{ij}$. On the other hand, $C(\Pi)$ is also a compatible simple root system, and $\alpha_i^{C(\Pi)} = C\alpha_i^{\Pi}$, $\beta_i^{C(\Pi)} = C\beta_i^{\Pi}$, so

$$\langle C\beta_i^{\Pi}, C\alpha_j^{\Pi}\rangle = \delta_{ij} = \langle \beta_i^{\Pi}, \alpha_j^{\Pi}\rangle.$$

2. First write $\langle x,y\rangle = (x,Ay)$ for some A. Then using (2.6.3) one obtains:

$$\begin{aligned}(x,y) = \langle x,y\rangle + \langle y,x\rangle &= \langle x,y\rangle - \langle x, C^{-1}y\rangle \\ &= (x,Ay) - (x, AC^{-1}y) \\ &= (x, A(1-C^{-1})y).\end{aligned}$$

   Hence $A = (1-C^{-1})^{-1}$. (Note that $1-C^{-1}$ is invertible since 1 is not an eigenvalue for C^{-1}.) The second identity is proved in a similar way.

3. From Part (2) use that $\delta_{ij} = ((1-C)^{-1}\beta_i^{\Pi}, \alpha_j^{\Pi})$ and hence $\beta_i^{\Pi} = (1-C)\omega_i^{\Pi}$.

4. Since ω_i form a basis of $P(R)$, and β_i form a basis of $Q(R)$ (Proposition 2.3.1), this follows from part (3).

□

As an immediate corollary, one obtains the following result, describing the Euler form in $\widehat{\Gamma}_{cyc}$.

Proposition 2.6.4. *There exists a unique function $\langle\cdot,\cdot\rangle_{\widehat{\Gamma}_{cyc}} : \widehat{\Gamma}_{cyc}\times\widehat{\Gamma}_{cyc} \to \mathbb{Z}$ satisfying*

1. *$\langle (i,n),(j,n)\rangle = \delta_{ij}$,*
   $\langle (i,n),(j,n+1)\rangle$ =the number of paths $(i,n)\to\cdots\to(j,n+1)$ =number of edges between i,j in Γ (note that this number is either zero or one).

2. *For any $q=(k,m)\in\widehat{\Gamma}_{cyc}$ this function satisfies*

$$\langle q,(i,n)\rangle_{\widehat{\Gamma}_{cyc}} - \sum_{j-i}\langle q,(j,n+1)\rangle_{\widehat{\Gamma}_{cyc}} + \langle q,(i,n+2)\rangle_{\widehat{\Gamma}_{cyc}} = 0.$$

Proof. Uniqueness easily follows by induction: for fixed $q=(i,n)$, condition (1) defines $\langle q,(*,n)\rangle_{\widehat{\Gamma}_{cyc}}$, $\langle q,(*,n+1)\rangle_{\widehat{\Gamma}_{cyc}}$. Then condition (2) can be used to define $\langle q,(*,n+2)\rangle_{\widehat{\Gamma}_{cyc}}$. Continuing in this way, one observes that these two conditions completely determine $\langle\cdot,\cdot\rangle_{\widehat{\Gamma}_{cyc}}$.

To prove existence, note that the form $\langle q_1,q_2\rangle_{\widehat{\Gamma}_{cyc}} = \langle \Phi^{-1}(q_1),\Phi^{-1}(q_2)\rangle$, where $\langle\cdot,\cdot\rangle$ is the Euler form on R defined in Theorem 2.6.2, satisfies all required properties. □

Note that the proof of uniqueness actually gives a very simple and effective algorithm for computing $\langle\cdot,\cdot\rangle_{\widehat{\Gamma}_{cyc}}$.

Theorem 2.6.5. *For a simply-laced Dynkin diagram Γ, define the set $\widehat{\Gamma}_{cyc}$ and lattice Q by Equation 1.3.2, Equation 2.5.1 respectively. Let $\langle\cdot,\cdot\rangle_{\widehat{\Gamma}_{cyc}}$ be the Euler form defined in Proposition 2.6.4, and let $(x,y)_{\widehat{\Gamma}_{cyc}} = \langle x,y\rangle_{\widehat{\Gamma}_{cyc}} + \langle y,x\rangle_{\widehat{\Gamma}_{cyc}}$.*

Then $(\cdot,\cdot)_{\widehat{\Gamma}_{cyc}}$ is a positive definite symmetric form on Q, and $\widehat{\Gamma}_{cyc} \subset Q$ is a root system with Dynkin diagram Γ.

Proof. Let R be a root system with Dynkin diagram Γ, and C a Coxeter element in the corresponding Weyl group. Then the map Φ constructed in Theorem 2.4.3 identifies $R \to \widehat{\Gamma}_{cyc}$, $Q(R) \to Q$ and the inner product $(\cdot,\cdot)$ in Q with $(\cdot,\cdot)_{\widehat{\Gamma}_{cyc}}$. □

2.7 Positive roots and the longest element in the Weyl group

Let Π be a simple root system compatible with C and R_+^Π the corresponding set of positive roots. Then bijection $\Phi\colon R \to \widehat{\Gamma}_{cyc}$ constructed in Theorem 2.4.3 identifies R_+^Π with a certain subset in $\widehat{\Gamma}_{cyc}$. This subset can be identified with the usual Auslander–Reiten quiver of the category of representations of the quiver $\overrightarrow{\Gamma}$ where the orientation of Γ is opposite to that defined by Π; this will be discussed in detail in Section 2.8.

In this section an explicit description of the set $\Phi(R_+^\Pi)$ is given in terms of $\widehat{\Gamma}_{cyc}$.

Theorem 2.7.1. *Let Π be a simple root system compatible with C, and $-\Pi = \{-\alpha \mid \alpha \in \Pi\}$ be the opposite simple root system. Let h^Π, $h^{-\Pi}\colon \Gamma \to \mathbb{Z}_{2h}$ be the corresponding height functions as defined in Theorem 2.4.5.*

Let Δ^Π be the set of all vertices of $\widehat{\Gamma}_{cyc}$ "between" correponding slices:

$$\Delta^\Pi = \{(i,n) \in \widehat{\Gamma}_{cyc} \mid h^\Pi(i) \le n < h^{-\Pi}(i)\}. \qquad (2.7.1)$$

(*see Figure 2.5*)*. The subset Δ^Π will be considerd as a quiver, with the same edges as $\widehat{\Gamma}_{cyc}$.*

1. *The map* Φ *gives an identification* $R_+^\Pi \to \Delta^\Pi$.

2. *Define a partial order on* Δ^Π *by* $q_1 \preceq q_2$ *if there exists a path from* q_1 *to* q_2 *in* Δ^Π, *and extend it to a complete order, writing*

$$\Delta^\Pi = \{\alpha(1) = (i_1, n_1), \alpha(2) = (i_2, n_2), \ldots, \alpha(l) = (i_l, n_l)\}$$

so that $\alpha(a) \preceq \alpha(b) \implies a \le b$. *Then*

$$s_{i_1}^\Pi \ldots s_{i_l}^\Pi$$

is a reduced expression for the longest element w_0^Π *of the Weyl group.*

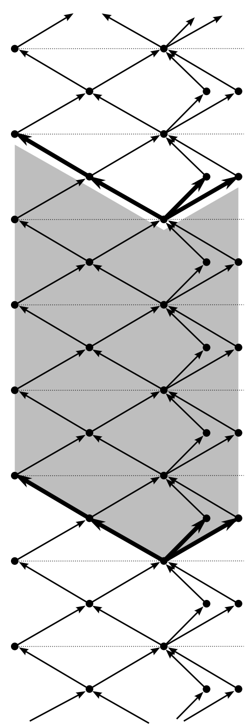

Figure 2.5: Positive roots in $\widehat{\Gamma}_{cyc}$, for diagram of type D_5. Bold lines show $\Phi(\beta_i^\Pi)$ and $\Phi(\beta_i^{-\Pi})$; the shaded area is the set $\Delta = \Phi(R_+^\Pi)$.

Proof. First we prove that $\Delta^\Pi \subset \Phi(R_+^\Pi)$. Indeed, choose $i \in \Gamma$ and let k to be the smallest positive integer such that $\beta = \Phi^{-1}(i, h^\Pi(i) + 2k) \notin R_+^\Pi$.

Then β satisfies $\beta \in R_+^{-\Pi}$, $C^{-1}\beta \in R_+^{\Pi}$. By Proposition 2.3.1 applied to $-\Pi$, then it must be the case that $\beta = \beta_i^{-\Pi}$, so $h^{\Pi}(i) + 2k = h^{-\Pi}(i)$. Therefore, $(i, n) \in \Phi(R_+^{\Pi})$ for all n satisfying $h^{\Pi}(i) \leq n < h^{-\Pi}(i)$.

Reversing the roles of $\Pi, -\Pi$, gives that the set $\Delta^{-\Pi} = \{(i, n) \in \widehat{\Gamma}_{cyc} \mid h^{-\Pi}(i) \leq n < h^{\Pi}(i)\}$ staisfies $\Delta^{-\Pi} \subset \Phi(R_+^{-\Pi}) = \Phi(-R_+^{\Pi})$. Since $\widehat{\Gamma}_{cyc} = \Delta^{\Pi} \sqcup \Delta^{-\Pi}$, this forces $\Delta^{\Pi} = \Phi(R_+^{\Pi})$. In particular, $l = |\Delta^{\Pi}| = |R_+^{\Pi}|$.

To prove the second part, consider a sequence of sets $\Delta_0 = \Delta^{\Pi}$, $\Delta_1 = \Delta_0 \setminus \{\alpha(1)\} = \{\alpha(2), \dots, \alpha(l)\}$, $\Delta_k = \Delta_{k-1} \setminus \{\alpha(k)\} = \{\alpha(k+1), \dots, \alpha(l)\}$, $\Delta_l = \varnothing\}$.

It is immediate from the definitions that one can write

$$\Delta_k = \{(i, n) \in \widehat{\Gamma}_{cyc} \mid h_k(i) \leq n < h^{-\Pi}(i)\} \tag{2.7.2}$$

for the sequence of height functions $h_0 = h^{\Pi}$, $h_1 = s_{i_1} h_0$, $\dots$, $h_k = s_{i_k} h_{i_{k-1}}$, $\dots$, $h_l = h^{-\Pi}$, and i_k is a source for h_{k-1}. In the same way, define a sequence of corresponding simple root systems $\Pi_0 = \Pi$, $\Pi_1 = s_{i_1}\Pi, \dots, \Pi_l = -\Pi$. One easily sees that

$$\Pi_2 = s_{i_2}^{\Pi_1}(\Pi_1) = s_{i_1}^{\Pi} s_{i_2}^{\Pi} s_{i_1}^{\Pi}(\Pi_1) = s_{i_1}^{\Pi} s_{i_2}^{\Pi}(\Pi).$$

Repeating this argument, gives $\Pi_k = s_{i_1}^{\Pi} \dots s_{i_k}^{\Pi}(\Pi)$; in particular,

$$-\Pi = \Pi_l = s_{i_1}^{\Pi} \dots s_{i_l}^{\Pi}(\Pi).$$

Since $l = |R_+| = l(w_0)$, the word $s_{i_1}^{\Pi} \dots s_{i_l}^{\Pi}$ is a reduced expression for the longest element w_0^{Π}. $\square$

Remark 2.7.2. The reduced expression for w_0 given in the previous theorem is given by source to sink operations taking the slice Γ_{Π} to $\Gamma_{-\Pi}$ and thus is adapted to Ω_{Π}. (A reduced expression $s_{i_1} \cdots s_{i_l}$ is *adapted* to Ω if i_k is a source for $s_{i_1} \cdots s_{i_{k-1}}\Omega$.)

Example 2.7.3. Consider the case $R = A_4$ from Example 2.4.4. The construction above gives the expression $w_0 = s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_1 s_2 s_1$ for the longest element.

Note that the second part of the theorem is equivalent to the algorithm for constructing a reduced expression for w_0 in terms of the Auslander–Reiten quiver given in [Béd] (in the form given here it is reformulated in [Z, Theorem 1.1]).

Example 2.7.4. Let Π be such that it defines a bipartite orientation on Γ, as in Example 2.3.4, so that $C = (\Pi_{i\in\Gamma_0} s_i)(\Pi_{i\in\Gamma_1} s_i)$. Assume additionally that the Coxeter number $h = 2g$ is even. Then it is known that $w_0 = C^g$ (see [Kos]), and thus $-\Pi = C^g(\Pi)$, and the corresponding height functions are related by $h^{-\Pi} = h^{\Pi} + 2g$. In this case, the set $\Phi(R_+^{\Pi})$ is a "horizontal strip". However, as the example of type A shows (see Example 2.4.4), in general the set $\Phi(R_+^{\Pi})$ can have a more complicated shape.

Recall the automorphism $i \mapsto \check{\imath}$ of defined by Equation 1.1.3. Equivalently, $\check{\imath}$ can be defined by setting $-\beta_i^{\Pi} = \beta_{\check{\imath}}^{-\Pi}$, where β_i^{Π} are the C-orbit representatives defined above. Thus for the root systems of type A, D_{2n+1}, E_6 this map corresponds to the diagram automorphism, while for D_{2n}, E_7, E_8 this map is just the identity (corresponding to the fact that $-Id = C^{h/2} \in W$).

Recall the map $\gamma_{\widehat{\Gamma}_{cyc}} : \widehat{\Gamma}_{cyc} \to \widehat{\Gamma}_{cyc}$ by the formula

$$\gamma_{\widehat{\Gamma}_{cyc}}(i, k) = (\check{\imath}, k + h).$$

Lemma 2.7.5. *Under the identification $\Phi : R \to \widehat{\Gamma}_{cyc}$ the map $-Id$ corresponds to $\gamma_{\widehat{\Gamma}_{cyc}}$.*

Proof. $\Phi(-\beta_i^{\Pi}) = \Phi(\beta_{\check{\imath}}^{-\Pi}) = (\check{\imath}, h^{-\Pi}(\check{\imath})) = \gamma_{\widehat{\Gamma}_{cyc}}(i, h^{\Pi})$.

For the last equality, since the maps Φ and $\gamma_{\widehat{\Gamma}_{cyc}}$ are compatible with the simple reflection s_j when $j \in \Gamma$ is a sink or source, it is enough to consider the case where Π gives a bipartite splitting $\Gamma = \Gamma_0 \sqcup \Gamma_1$ and $C = (\prod_{i\in\Gamma_0} s_i^{\Pi})(\prod_{i\in\Gamma_1} s_i^{\Pi})$.

If $h = 2g$ is even, then $w_0^{\Pi} = C^g$ (see [Kos]) and hence $h^{\Pi}(i) + h = h^{-\Pi}(\check{\imath})$.

If $h = 2g + 1$ is odd, then $R = A_{2g}$. Then $w_0^{\Pi} = (\prod_{i\in\Gamma_0} s_i^{\Pi})C^g$ and $\check{\imath} = 2g + 1 - i$, so that $\check{\imath} \in \Gamma_1$ if $i \in \Gamma_0$ and $\check{\imath} \in \Gamma_0$ if $i \in \Gamma_1$. If $\check{\imath} \in \Gamma_0$, then

$$h^{-\Pi}(\check{\imath}) = w_0^{\Pi} h^{\Pi}(\check{\imath}) = h^{\Pi}(\check{\imath}) + 2g + 2 = h^{\Pi}(i) - 1 + 2g + 2 = h^{\Pi}(i) + h.$$

For $\check{\imath} \in \Gamma_1$ a similar calculation shows that $h^{-\Pi}(\check{\imath}) = h^{\Pi}(i) + h$.

$\square$

Recall that a choice of compatible simple roots Π gives a height function h^{Π}, and hence a slice given by $\Gamma_{\Pi} = \{(i, h^{\Pi}(i))\}$. The following Proposition gives another characterisation of $\Delta^{\Pi} \subset \widehat{\Gamma}_{cyc}$.

Proposition 2.7.6. *Let* Π *be compatible with* C*. Then* $\gamma_{\widehat{\Gamma}_{cyc}}(\Gamma_\Pi) = \Gamma_{-\Pi}$*, and so the subset* $\Delta^\Pi \subset \widehat{\Gamma}_{cyc}$ *can be identified as the full subquiver lying "between"* Γ_Π *and* $\gamma_{\widehat{\Gamma}_{cyc}}(\Gamma_\Pi) = \Gamma_{-\Pi}$*:*

$$\Delta^\Pi = \{(i,n) \in \widehat{\Gamma} | h^\Pi \leq n < h^{-\Pi}\}$$

where h^Π *and* $h^{-\Pi}$ *are the height functions corresponding to the slices* Γ_Π *and* $\Gamma_{-\Pi}$ *respectively.*

Proof. This follows immediately from Lemma 2.7.5 and the identification of Δ^Π with the set of roots lying between h^Π and $h^{-\Pi}$ given in Theorem 2.7.1.

□

2.8 Relationship between $AR(\Gamma, \Omega)$ and $\widehat{\Gamma}_{cyc}$

The goal of this section is to show that for any choice of Π compatible with C there is an identification of the Auslander-Reiten quiver of $\overrightarrow{\Gamma} = (\Gamma, \Omega^{op}_{h^\Pi})$ with the subset $\Delta^\Pi_+ \subset \widehat{\Gamma}_{cyc}$.

Let Π be a simple root system compatible with C. This gives a height function h^Π, and hence gives an orientation Ω on Γ, as well as a slice Γ_Ω. Consider the natural projection $P_\Pi : \widehat{\Gamma} \to \widehat{\Gamma}_{cyc}$ given by $P_\Pi(i,k) = (i, \overline{k})$.

Proposition 2.8.1. *Let* Π *be compatible with* C*, let* Ω_Π *denote the corresponding orientation of* Γ*, and let* τ *be the Auslander-Reiten translation of the category* $\mathrm{Rep}(\Gamma, \Omega^{op}_\Pi)$*.*

1. $P_\Pi \circ \tau^{-1} = \tau_{\widehat{\Gamma}_{cyc}} \circ P_\Pi$.
2. $P_\Pi \circ \nu = \nu_{\widehat{\Gamma}_{cyc}} \circ P_\Pi$
3. *The map* P_Π *identifies the Auslander-Reiten quiver* $AR(\Gamma, \Omega^{opp})$ *in* $\widehat{\Gamma}$ *with* Δ^Π *in* $\widehat{\Gamma}_{cyc}$.
4. *Under the identification* P_Π*, the projective representation* P_i *corresponds to* β^Π_i.

Proof.

1. This follows easily just by definitions:

$$\begin{aligned} P_\Pi \circ \tau_{\widehat{\Gamma}}(i,k) &= P_\Pi(i, k+2) \\ &= (i, \overline{k+2}) \\ &= \tau_{\widehat{\Gamma}_{cyc}}(i, \overline{k}) \\ &= \tau_{\widehat{\Gamma}_{cyc}} P_\Pi(i,k). \end{aligned}$$

2. Again by definitions:

$$\begin{aligned} P_\Pi(\nu(i,k)) &= P_\Pi(\check{\imath}, k+h-2) \\ &= (\check{\imath}, \overline{k+h-2}) \\ &= (\check{\imath}, \overline{k} + \overline{h-2}) \\ &= \nu_{\widehat{\Gamma}_{cyc}}(i, \overline{k}) \\ &= \nu_{\widehat{\Gamma}_{cyc}} \circ P_\Pi(i,k). \end{aligned}$$

3. As in Section 1.4 fix a vertex $i_0 \in \Gamma$ to identify $AR(\Gamma, \Omega^{op})$ with a full subquiver of $\widehat{\Gamma}$. A choice of compatible simple system Π gives a unique height function h^Π, which also gives an orientation Ω, and a unique slice Γ_Ω in $\widehat{\Gamma}_{cyc}$. As in Section 1.2 fix a vertex $i \in \Gamma$ to identify $AR(\Gamma, \Omega^{op})$ with a full subquiver of $\widehat{\Gamma}$. Then this gives a unique slice Γ_Ω through $(i, h^\Pi(i))$ in $\widehat{\Gamma}$. By the construction of the map P_Π these two slices are identified. Parts 1 and 2 show that P_Π identifies τ with $\tau_{\widehat{\Gamma}_{cyc}}$ and ν with $\nu_{\widehat{\Gamma}_{cyc}}$. Then the description of $AR(\Gamma, \Omega^{opp})$ given in Theorem 1.4.1, and the description of Δ^Π given in Proposition 2.7.6, shows that the map P_Π identifies $AR(\Gamma, \Omega^{opp})$ with Δ^Π. (Compare Example 1.4.2 with Example 2.4.4.)

4. The β_i^Π map to the slice $\Gamma_\Omega \subset \widehat{\Gamma}_{cyc}$. Similarly, the projective representations P_i map to the slice $\Gamma_\Omega \subset \widehat{\Gamma}$. By Part 3, these are identified by P_Π.

$\square$

Given a simple root system Π, compatible with C, we have obtained a bijection of the Auslander-Reiten quiver $AR(\Gamma, \Omega^{opp})$ of the category $\mathrm{Rep}(\Gamma, \Omega^{opp})$ with the subquiver $\Delta^{\Pi} \subset \widehat{\Gamma}_{cyc}$. Both $AR(\Gamma, \Omega^{opp})$ and Δ^{Π} correspond to the set of positive roots R_{+}^{Π}. For $AR(\Gamma, \Omega^{opp})$ the correspondence is the usual identification between $\mathrm{Ind}(\Gamma, \Omega^{opp})$ and positive roots $R_{+}^{\Pi} \subset \oplus_{i \in I} \mathbb{Z}\alpha_i$, given by the dimension vector $\dim X$. For Δ^{Π} it is given by the map $\Phi : R \to \widehat{\Gamma}_{cyc}$ from Section 2.4. The following Theorem shows that the bijection $AR(\Gamma, \Omega^{opp}) \to \Delta^{\Pi}$ agrees with these identifications to R_{+}^{Π}.

Theorem 2.8.2. *Let Π be a compatible simple root system, h^{Π} the corresponding height function and Ω the corresponding orientation. The following diagram is commutative:*

$$
\begin{tikzcd}
 & AR(\Gamma, \Omega^{opp}) \arrow[dl, "\dim"'] \arrow[dd, "P_{\Pi}"] \\
R_{+}^{\Pi} \arrow[dr, "\Phi"'] & \\
 & \Delta^{\Pi}
\end{tikzcd}
$$

Proof. Under the map $\dim : AR(\Gamma, \Omega^{opp}) \to R_{+}^{\Pi}$ the projective representation P_i given by

$$
P_i(j) = \begin{cases} \mathbb{C} & \text{j} \leqslant \text{ i} \\ 0 & \text{otherwise} \end{cases}
$$

is identified with the C-orbit representative β_i^{Π}. By Theorem 1.4.1 these representations form the slice $\Gamma_{\Omega} \subset \widehat{\Gamma}$. The C-orbit representatives map to the slice Γ_{Ω} in $\widehat{\Gamma}_{cyc}$ under the map Φ. By Proposition 2.8.1 these two slices are identified by the map P_{Π}. Hence the diagram commutes on the projectives P_i. The map dim identifies τ^{-1} (where τ is the Auslander-Reiten translation) with C, the map Φ identifies C with $\tau_{\widehat{\Gamma}_{cyc}}$, and P_{Π} identifies τ^{-1} with $\tau_{\widehat{\Gamma}_{cyc}}$. Since any $X \in AR(\Gamma, \Omega^{opp})$ is of the form $\tau^{-k}(P_i)$, and any $\alpha \in R_{+}^{\Pi}$ is of the form $C^k \beta_i^{\Pi}$ this shows that the diagram commutes. $\square$

2.9 Identification of Euler Forms

In this section, it is shown for a choice of compatible simple system Π, the bilinear form $\langle\cdot,\cdot\rangle_R$ on R, constructed in Section 2.6, corresponds to the Euler form $\langle X, Y\rangle = \dim \mathrm{RHom}(X,Y) = \dim \mathrm{Hom}(X,Y) - \dim \mathrm{Ext}^1(X,Y)$ on $\mathrm{Rep}(\Gamma, \Omega^{opp})$, where Ω is the orientation determined by Π.

Note that the Euler form satisfies

$$\langle X, Y\rangle = -\langle Y, \tau X\rangle = \langle \tau X, \tau Y\rangle \tag{2.9.1}$$

(see [C-B], for a proof).

Theorem 2.9.1. *The map $P_\Pi : \mathbb{Z}\Gamma \to \widehat{\Gamma}_{cyc}$ identifies the Euler form $\langle\cdot,\cdot\rangle$ on $AR(\Gamma, \Omega^{opp})$ with the form $\langle\cdot,\cdot\rangle_R$ on R_+^Π.*

Proof. Recall that the C-orbit representative β_i^Π corresponds to the projective representation P_i, while the simple root α_i^Π corresponds to the simple representation X_i. It is well known (see [C-B] p.24) that $\langle P_i, X_j\rangle = \delta_{ij}$. Define a form $\ll \cdot,\cdot \gg$ on $AR(\Gamma, \Omega)$ by $\ll X, Y \gg = \langle P_\Pi(X), P_\Pi(Y)\rangle_R$ where $\langle\cdot,\cdot\rangle_R$ is the Euler Form on R defined in Section 2.6. Then $\ll P_i, X_j \gg = \langle \beta_i^\Pi, \alpha_j^\Pi\rangle_R = \delta_{ij}$, and $\ll \cdot,\cdot \gg$ satisfies Equation 2.9.1. However, since the value of $\ll P_i, X_j \gg$ and Equation 2.9.1 completely determine the form, the two forms $\langle\cdot,\cdot\rangle$ and $\ll \cdot,\cdot \gg$ are equal. $\square$

Chapter 3

Categorical Construction

In the previous chapter, a combinatorial construction of the root system R associated to a Dynkin graph Γ was given. This construction used a Coxeter element in the Weyl group and to identify the root system with the quiver $\widehat{\Gamma}_{cyc}$. In this chapter, a categorical construction of R from Γ is given, which served as motivation for the construction of Chapter 1. As described in the previous chapter, by choosing an orientation of Γ and studying representations of the corresponding quiver one obtains a categorical construction of R. However, in this setup the abelian categories obtained depend on the choice of orientation, although by passing to the 2-periodic derived category the categories become derived equivalent. The equivalences are given by the well-known BGP reflection functors.

In this chapter, given a Dynkin diagram Γ the translation quiver $\widehat{\Gamma} \subset \Gamma \times \mathbb{Z}$ is constructed and a triangulated subcategory $\mathcal{D} \subset \mathcal{D}(\widehat{\Gamma})$ of the corresponding derived category is studied. Given any choice of orientation Ω equivalences $\mathcal{D} \to \mathcal{D}^b(\mathrm{Rep}(\Gamma, \Omega))$ are constructed and shown to be compatible with the reflection functors.

Moreover, this construction is closely related to the preprojective algebra of Γ. We begin by giving a "graphical" description of the Koszul complex of the preprojective algebra. In this setup elements of degree k in the Koszul complex are visualized as paths in $\widehat{\Gamma}$ with k "jumps". This description is then used to construct the indecomposable objects in the category $\mathcal{D}$. We show that our category $\mathcal{D}$ has $\widehat{\Gamma}^{op}$ as its Auslander-Reiten quiver, and use this to relate $\mathcal{D}$ with the mesh category as described in [BBK] and [Hap]. This proves a periodicity result about the preprojective algebra (see Theorem 3.8.1).

The main result of this Chapter are summarized in the following Theorem.

Theorem 3.0.2. *Given a simply-laced Dynkin diagram* Γ *with Coxeter number* h *there exists a triangulated category* $\mathcal{C}$ *with an exact functor* $\mathcal{C} \to \mathcal{C}\colon \mathcal{F} \mapsto \mathcal{F}(2)$ *("twist") with the following properties:*

1. *The category* $\mathcal{C}$ *is 2-periodic:* $T^2 = \mathrm{id}$

2. *For any* $\mathcal{F} \in \mathcal{C}$*, there is a canonical functorial isomorphism* $\mathcal{F}(2h) = \mathcal{F}$*, where* h *is the Coxeter number of* Γ*.*

3. *Let* $\mathcal{K}$ *be the Grothendieck group of* $\mathcal{C}$*. The corresponding root system* R *is identified with the set* $\mathrm{Ind} \subset \mathcal{K}$ *of all indecomposable classes.*

4. *The map* $C\colon [\mathcal{F}] \mapsto [\mathcal{F}(-2)]$ *is a Coxeter element for this root system.*

5. *Set* $\langle X, Y\rangle_{\mathcal{C}} = \dim \mathrm{RHom}(X, Y) = \dim \mathrm{Hom}(X, Y) - \dim \mathrm{Ext}^1(X, Y)$ *(the "Euler form") then the inner product is given by* $(X, Y) = \langle X, Y\rangle_{\mathcal{C}} + \langle Y, X\rangle_{\mathcal{C}}$*.*

6. *There is a natural bijection* $\Phi : \mathrm{Ind} \to \widehat{\Gamma}$*, between indecomposable objects and vertices in* $\widehat{\Gamma}$*. Under this bijection, the Coxeter element* C *defined above is identified with the map* $\tau : (i, n) \mapsto (i, n + 2)$*. Denote the indecomposable object corresponding to* $q = (i, n)$ *by* X_q*.*

7. *The category* $\mathcal{C}$ *has "Serre Duality":*

$$\mathrm{Hom}(X, Y) = (\mathrm{Ext}^1(Y, X(-2)))^*$$

There is also an identification

$$\mathrm{Hom}(X_q, X_{q'}) = \mathrm{Path}(q', q)/J$$

where Path *is the vector space generated by paths in* $\widehat{\Gamma}$ *and* J *is some explicitly described subspace. Thus the form* $\langle \cdot, \cdot\rangle$ *is determined by paths in* $\widehat{\Gamma}$*.*

8. *For every height function* h*, there is a derived functor* $R\rho_h : \mathcal{C} \to \mathcal{D}^b(\Gamma, \Omega_h)/T^2$ *which is an equivalence of triangulated categories.*

3.1 The Category $\mathcal{D}$

In this section a precursor to the conjectured category is introduced and several basic results are established. To begin a few preliminary results are required.

Definition 3.1.1. Let Q be any quiver and $e : i \to j$ in Q an edge in Q. For any object $X \in \mathcal{D}(Q)$ define the complex of vector spaces $Cone_e$ as the cone of the map $x_e : X(i) \to X(j)$. This will be called the "Cone of the edge e".

Remark 3.1.2. Note that $x_e : X(i) \to X(j)$ is an honest morphism between complexes of vector spaces, not a morphism in a derived category.

Lemma 3.1.3. *For any quiver $\overrightarrow{\Gamma}$ and edge $e : i \to j$, $Cone_e$ is functorial.*

Proof. Suppose that $X, Y \in \mathcal{D}(\overrightarrow{\Gamma})$ and that $F : X \to Y$ is a map of complexes of representations. Let $x_e : X(i) \to X(j)$ and $y_e : Y(i) \to Y(j)$ be the maps of complexes corresponding to the edge e. Then $F(j) \circ x_e = y_e \circ F(i)$ and $d_Y \circ F = F \circ d_X$. By definition of $Cone_e$, this gives a map

$$\begin{array}{lcl} Cone_e(x_e) & & = (X^{\dot{+}1}(i) \oplus X(j), d_X(i) + x_e, d_X(j)) \\ \Big\downarrow{\scriptstyle \tilde{F}} & & \\ Cone_e(y_e) & & = (Y^{\dot{+}1}(i) \oplus Y(j), d_Y(i) + y_e, d_Y(j)) \end{array}$$

Also note that this shows that if F is a quasi-isomorphism, then the induced map $\tilde{F}$ is also a quasi-isomorphism.
Let $\Phi \in \mathrm{Hom}_{\mathcal{D}(\overrightarrow{\Gamma})}(X, Y)$ and let $X \leftarrow Z \to Y$ be a roof diagram for Φ. Then the above shows that there is an induced roof diagram $Cone_e(x_e) \leftarrow Cone_e(z_e) \to Cone_e(y_e)$. Hence $Cone_e$ is functorial. ☐

Definition 3.1.4. Let Γ be a finite graph without cycles. Take the subcategory $\mathcal{D} \subset \mathcal{D}(\widehat{\Gamma})$ defined as follows: $Obj_{\mathcal{D}} = \{(X, \{\phi\}_{q \in \widehat{\Gamma}})\}$ where $X \in \mathcal{D}(\widehat{\Gamma})$ is such that for any $q \in \widehat{\Gamma}$ the complex $X(q)$ is bounded and Φ is a collection of isomorphims $\phi(q) : X(\tau q) \to Cone(x_q)$. Here $x_q : X(q) \to \bigoplus_{q \to q'} X(q')$ is the map given by edges $q \to q'$, hence $Cone(x_q)$ is the direct sum of the Cone over the edges $e : q \to q'$ as defined above.

A morphism $\Phi : X \to Y$ in $\mathcal{D}$ is given by a morphism in $\mathcal{D}(\widehat{\Gamma})$ such that the following diagram is commutative:

$$\begin{tikzcd}
X(\tau q) \arrow[r, "F(\tau q)"] \arrow[d, "\Phi_q"] & Y(\tau q) \arrow[d, "\Psi_q"] \\
Cone(X(q) \to \oplus X(q')) \arrow[r, "\tilde{F}(\tau q)"] & Cone(Y(q) \to \oplus X(q'))
\end{tikzcd}$$

The translation functor in $\mathcal{D}$ is the same as that in $\mathcal{D}(\widehat{\Gamma})$. The distinguished triangles in $\mathcal{D}$ will be the distinguished triangles (X, Y, Z) in $\mathcal{D}(\widehat{\Gamma})$ where each of the $X, Y, Z \in \mathcal{D}$.

Definition 3.1.5. An object $X \in \mathcal{D}(\widehat{\Gamma})$ is said to satisfy the *fundamental relation* if for any $q \in \widehat{\Gamma}$ there is a choice of map $z : X(\tau q) \to X(q)[1]$ so that

$$X(q) \xrightarrow{x_q} \bigoplus_{q \to q'} X(q') \xrightarrow{\sum x_{q'}} X(\tau q) \xrightarrow{z} X(q)[1] \tag{3.1.1}$$

is an exact triangle. Here $x_q, x_{q'}$ are the maps corresponding to edges $q \to q'$ and $q' \to \tau q$.

Proposition 3.1.6. *Any object $X \in \mathcal{D}$ satisfies the fundamental relation.*

Proof. By definition of an object $X \in \mathcal{D}$. □

Lemma 3.1.7. *The category $\mathcal{D}$ is full.*

Proof. Let $F \in \mathrm{Hom}_{\mathcal{D}(\widehat{\Gamma})}(X, Y)$ and let q be any vertex. Since for any edge $e : q \to q'$, $Cone(e)$ is functorial, there is an induced map $\tilde{F}$ making the diagram below commute.

$$\begin{tikzcd}
X(\tau q) \arrow[r, "F(\tau q)"] \arrow[d, "\Phi_q"] & Y(\tau q) \arrow[d, "\Psi_q"] \\
Cone(X(q) \to \oplus X(q')) \arrow[r, "\tilde{F}(\tau q)"] & Cone(Y(q) \to \oplus X(q'))
\end{tikzcd}$$

Hence $F \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$. □

It remains to show that the category $\mathcal{D}$ inherits the structure of a triangulated category.

Theorem 3.1.8. *$\mathcal{D}$ is a triangulated subcategory of $\mathcal{D}(\widehat{\Gamma})$.*

Following [GM], the notation T1, T2, T3, T4 for the axioms of a triangulated category will be used for simplicity. The reader can refer to [GM] for details.

Proof. Since triangles and morphisms have been defined above, only the axioms T1 $\to$ T4 remain to be verified.

For T1 the only thing that needs to be checked is that a morphism $X \to Y$ can be completed to a triangle. First note that by construction the objects and morphisms in $\mathcal{D}$ are objects and morphisms in the derived category $\mathcal{D}(\widehat{\Gamma})$ with extra structure. So to show that a morphism can be completed in $\mathcal{D}$ it is enough to show that the completion in $\mathcal{D}(\widehat{\Gamma})$ carries the required extra structure. By construction of the derived category any morphism can be completed to a distinguished triangle which is isomorphic to a triangle of the form

$$X \xrightarrow{F} Y \to Cone(F) \to X[1]$$

where $F : X \to Y$ is a morphism of complexes (see [GM] Chapter 4 §2). Hence it is enough to verify that if $F : X \to Y$ is a morphism of complexes between objects in $\mathcal{D}$ then $Z = Cone(F)$ comes with an identification $Z(\tau q) \to Cone(Z(q) \to \oplus Z_{q'})$. To see this note that:

$$\begin{aligned}
&Cone(ConeF(q) \to \oplus ConeF(q')) = \\
&= Cone(X^{k+1}(q) \oplus Y^k(q) \to \bigoplus (X^{k+1}(q') \oplus Y^k(q'))) \\
&= (X^{k+1}(q) \bigoplus Y^k(q))^{+1} \bigoplus \oplus (X^{k+1}(q') \oplus Y^k(q')) \\
&= X^{k+2}(q) \bigoplus Y^{k+1}(q) \bigoplus (\oplus X^{k+1}(q')) \bigoplus (\oplus Y^k(q')) \\
&= (X^{k+2}(q) \oplus X^{k+1}) \bigoplus (Y^{k+1}(q) \oplus Y^k(q')) \\
&= (X^{k+1}(q) \oplus X^k(q'))^{+1} \bigoplus (Y^{k+1}(q) \oplus Y^k(q')) \\
&= X^{k+1}(\tau q) \oplus Y^k(\tau q) \\
&= Cone(X(\tau q) \to Y(\tau q) \\
&= Cone(F(\tau q)).
\end{aligned}$$

To check that this is an isomorphism of complexes, not just of graded vector spaces, it remains to check that the differentials match. Let δ be the differential of $Cone(F)$ and let d_X, d_Y be the differentials of X, Y. Denote by D the differential of $Cone(X^{k+1}(q) \oplus Y^k(q) \overset{x_q+y_q}{\to} \bigoplus(X^{k+1}(q') \oplus Y^k(q')))$.

$$\begin{aligned}
\delta(\tau q) &= (d_X^{+1}(\tau q) + F^{+1}(\tau q), d_Y(\tau q)) \\
&= (d_X^{+2}(q) + x_q^{+1} + F^{+1}(q), \sum_{q \to q'} d_X^{+1} + F^{+1}(q'), d_Y^{+1}(q) + y_q, \sum_{q \to q'} d_Y(q')) \\
&= (d_X^{+2}(q) + x_q^{+1} + F^{+1}(q), d_Y^{+1}(q) + y_q, \sum_{q \to q'} d_X^{+1} + F^{+1}(q'), \sum_{q \to q'} d_Y(q')) \\
&= (\delta^{+1}(q) + x_q^{+1} + y_q^{+1}, \delta(q')) \\
&= D
\end{aligned}$$

T2 follows by definition of triangles in $\mathcal{D}$.

For T3, Lemma 3.1.7 shows that if the diagram is completed with a morphism in $\mathcal{D}(\widehat{\Gamma})$, then this also a morphism in $\mathcal{D}$.

For T4, again Lemma 3.1.7 shows that by completing the diagram in $\mathcal{D}(\widehat{\Gamma})$, the morphisms are in $\mathcal{D}$ so the diagram can be completed in $\mathcal{D}$ as well. □

3.2 DG Preprojective Algebra

In this section a graphical description of the "derived preprojective algebra" is given. This is then related to the Koszul complex of the preprojective algebra. Later this will be used to construct projectives in the category $\mathrm{Rep}(\widehat{\Gamma})$ and to define indecomposable objects in the category $\mathcal{D}$. This algebra is known to experts, however the presentation given here is not readily available in the literature.

To begin consider the following algebra A.

Definition 3.2.1. Let P be the path algebra of $\overline{\Gamma}$. Let A be the algebra obtained by adjoining to P elements $t_i, i \in \Gamma$, with relations

$$t_i e_j = e_j t_i = \delta_{ij} t_i$$

where $e_i \in P_{i,i;0}$ is the idempotent in P corresponding to the path of length 0 from i to i.

Thus, A is generated by the expressions of the form

$$p_1 t_{i_1} p_2 \dots p_k t_{i_k} p_{k+1} \tag{3.2.1}$$

where each p_a is a path from i_a to i_{a-1}.

The elements of A are pictured as paths in $\widehat{\Gamma}$ with jumps (i,n) to $(i,n+2)$ for each t_i that appears.

Extend the grading of P (See Equation 1.2.1.) to A by letting the t_i be elements of degree 2, so that $t_i \in A_{i,i;2}$. This gives a decomposition of A:

$$A = \bigoplus_{i,j\in\Gamma\ ,\ l\in\mathbb{Z}_+} A_{i,j;l} \tag{3.2.2}$$

The interesting fact is that A has another grading, which is given by the number of jumps (ie the number of t_i which appear in an expression). This gives a further decomposition of A:

$$A = \bigoplus_{n\leq 0} A^n = \bigoplus_{i,j\in\Gamma\ ,\ l\in\mathbb{Z}_+, n\leq 0} A^n_{i,j;l} \tag{3.2.3}$$

where A^{-k} is the subspace in A generated by expressions of the form (3.2.1) with exactly k t_i's. In particular, $A^{-k}_{i,j;l}$ can be thought of as paths from i to j of length l with k jumps.

The graded vector space $A = \bigoplus A^n$ is made into a complex by setting

$$\begin{aligned} d_{-k}\colon A^{-k} &\to A^{-k+1} \\ p_1 t_{i_1} p_2 \dots p_k t_{i_k} p_{k+1} &\mapsto \sum_{a=1}^{k} (-1)^{a+1} p_1 t_{i_1} p_2 \dots p_a \theta_{i_a} p_{a+1} \dots p_k t_{i_k} p_{k+1} \end{aligned} \tag{3.2.4}$$

where $\theta_i \in A_{i,i;2}$ is given by

$$\theta_i = \sum_{s(e)=i} \epsilon(e) e\overline{e} \in P_{i,i;2}. \tag{3.2.5}$$

A routine calculation shows that this definition of d and multiplication in A make it a DG algebra.

3.2.1 The Koszul Complex

Let R be the algebra of functions $\Gamma \mapsto \mathbb{C}$ with pointwise multiplication. Let $\overline{\Gamma}$ be the double quiver, and for each edge joining i and j in Γ let $e^{ij} : i \to j$ and $e^{ji} : j \to i$ denote the corresponding pair of arrows in $\overline{\Gamma}$. Let V be the vector space spanned by the edges of the double $\overline{\Gamma}$ and let L be the R-submodule of $V \otimes V$ generated by the element $\theta = \sum_{e \in \Gamma} [e^{ij}, e^{ji}]$. Consider the embedding $j : L \hookrightarrow V \otimes V$. Denote by J the quadratic ideal generated by L. Then the preprojective algebra of Γ is $\Pi = T_R(V)/J$.

To see this coincides with the definition of Π given in Section 1.2 suppose that if instead of considering the ideal J generated by the element θ above, we consider the ideal J' generated by the elements θ_i. Then $\theta \in J'$ so $J \subset J'$ and $\theta_i = \pm\theta \cdot e_i$, where e^i is the idempotent corresponding to vertex i, so that $\theta_i \in J$. Hence $J = J'$. So the two definitions coincide.

As in [EG], consider the associated Koszul complex

$$K_\bullet = T_R(V \oplus L) = \bigoplus V^{n_1} \otimes L \otimes V^{n_2} \otimes L \otimes \cdots L \otimes V^{n_j}$$

where $V^n = V^{\otimes n}$. The differential d is given by

$$d(v_1 \otimes l_1 \otimes v_2 \otimes \cdots \otimes l_{j-1} \otimes v_j) = \sum_i (-1)^i v_1 \otimes l_1 \otimes v_2 \otimes \cdots v_i \otimes j(l_i) \otimes v_{i+1} \otimes \cdots \otimes l_{j-1} \otimes v_j \tag{3.2.6}$$

where $v_1 \otimes l_1 \otimes v_2 \otimes \cdots \otimes l_{j-1} \otimes v_j \in V^{n_1} \otimes L \otimes V^{n_2} \otimes L \otimes \cdots L \otimes V^{n_j}$.

We now relate the Koszul complex $(K_\bullet, d_K)$ to the complex $(A^\bullet, d_A)$ of "paths with jumps". Let $a_1, \ldots, a_r$ be the edges of $\overline{\Gamma}$. Then these form a basis of the space V, and by viewing a path as a sequence of edges, there is an obvious identification between a path $p = a_k a_{k-1} \cdots a_2 a_1$ to the element $a_k \otimes a_{k-1} \otimes \cdots \otimes a_2 \otimes a_1 \in V^k$. We will now identify a path p of length k and its image in V^k, and denote both by p. Denote by $l_i \in L$ the element so that $j(l_i) = \theta_i \in V \otimes V$. Define a map $\Psi : (A^\bullet, d_A) \mapsto (K_\bullet, d_K)$ by

$$\Psi(p_{n+1} t_{i_n} p_n t_{i_{n-1}} \cdots p_2 t_{i_1} p_1) = p_{n+1} l_{i_n} p_n l_{i_{n-1}} \cdots p_2 l_1 p_1 \tag{3.2.7}$$

Proposition 3.2.2. *The map Ψ is an isomorphism of complexes.*

Proof. The identification of paths of length k with V^k described above, and the identification t_i with l_i, shows that the map Ψ gives isomorphisms $A^{-k} \mapsto$

K_k. Checking that Ψ is a chain map is straightforward:

$$\begin{aligned}
\Psi d_A(p_{k+1}t_{i_k}p_kt_{i_{k-1}}\cdots p_2t_{i_1}p_1) &= \Psi(\sum_{n=1}^{k}(-1)^{n+1}p_{k+1}t_{i_k}p_k\cdots p_{n+1}\theta_{i_n}p_n\cdots p_2t_{i_1}p_1 \\
&= \sum_{n=1}^{k}(-1)^{n+1}p_{k+1}l_{i_k}p_k\cdots p_{n+1}\theta_{i_n}p_n\cdots p_2l_{i_1}p_1 \\
&= \sum_{n=1}^{k}(-1)^{n+1}p_{k+1}l_{i_k}p_k\cdots p_{n+1}j(l_{i_n})p_n\cdots p_2l_{i_1}p_1 \\
&= d_K(p_{k+1}l_{i_k}p_kl_{i_{k-1}}\cdots p_2l_{i_1}p_1) \\
&= d_K\Psi(p_{k+1}t_{i_k}p_kt_{i_{k-1}}\cdots p_2t_{i_1}p_1)
\end{aligned}$$

□

3.2.2 The non-Dynkin case

Consider the case where Γ is non-Dynkin. It is known (see [M-V]) that the preprojective algebra of a non-Dynkin graph is Koszul, and that the Koszul complex $K_\bullet$ gives a DG-algebra resolution of Π. These results, together with the isomorphism $\Psi : A^\bullet \to K_\bullet$ gives the following result.

Proposition 3.2.3. *Suppose Γ is non-Dynkin. Then*

$$H^k(A^\bullet) = \begin{cases} \Pi & \textit{if} \ \ k = 0 \\ 0 & \textit{if} \ \ k > 0 \end{cases}$$

so the complex $A^\bullet$ gives a DG-algebra resolution of the preprojective algebra Π.

3.3 Projective Representations of $\widehat{\Gamma}$

Let $q \in \widehat{\Gamma}$ be any vertex. Let $\mathrm{Path}^k(q, v)$ denote the space of "paths with k jumps" from q to v. For $q = (i, n)$ and $v = (j, m)$, this space can be identified with the component $A^{-k}_{i,j;m-n}$, defined by Equation 3.2.3.
Define a representation $X^k_q \in \mathrm{Rep}(\widehat{\Gamma})$ by setting $X^k_q(v) = \mathrm{Path}^k(q, v)$. Composition of paths makes it a module over the path algebra P.

Proposition 3.3.1.

1. *For any k, and any vertex $q \in \widehat{\Gamma}$, the representation X_q^k is projective.*

2. *For any object $X \in \mathrm{Rep}(\widehat{\Gamma})$ we have*

$$\mathrm{Hom}_{\widehat{\Gamma}}(X_q^0, X) = X(q).$$

3. *For any object $X \in \mathrm{Rep}(\widehat{\Gamma})$ we have*

$$\mathrm{Hom}_{\widehat{\Gamma}}(X_q^k, X) = \bigoplus_{v \in \widehat{\Gamma}} \mathrm{Hom}_{\mathbb{C}}(X^{k-1}(q,v), X(\tau v)).$$

Proof.

1. The space X_q^k is freely generated over the path algebra P by elements of the form $p = t_k p_{k-1} \cdots p_2 t_1 p_1$ where the t_i's are jumps and the p_i's are paths.

2. For any $x \in X$ define $\phi_x : X_q^0 \to X$ by $\phi_x(p) = p.x$. This gives the required isomorphism.

3. First the isomorphism $\mathrm{Hom}_{\widehat{\Gamma}}(X_q^k, X) \simeq \bigoplus_{v \in \widehat{\Gamma}} X(2) \otimes (X^{k-1}(q,v))^*$ is established. This isomorphism is given by

$$\phi \mapsto \bigoplus_{p \in X^{k-1}(q,v)} \phi(tp) \otimes p^*$$

with inverse

$$x \otimes \psi_p \mapsto (p_1 t p_2 \overset{\phi_{x,\psi_p}}{\mapsto} p_1 x \psi_p(p_2)).$$

To see this, note that any element $\phi \in \mathrm{Hom}_{\widehat{\Gamma}}(X_q^k, X)$ is determined by where it sends the generators $t_k p_{k-1} \cdots p_2 t_1 p_1$. So for each path $p : q \to v$ with $k-1$ jumps we need to assign an element $x \in \tau v$ which is the value of $\phi(tp)$.

To establish the desired isomorphism, use the standard identification $W \otimes V^* \simeq \mathrm{Hom}(V, W)$ to obtain

$$\mathrm{Hom}_{\widehat{\Gamma}}(X_q^k, X) = \bigoplus_{v \in \widehat{\Gamma}} \mathrm{Hom}_{\mathbb{C}}(X^{k-1}(q,v), X(\tau v)).$$

□

3.4 Indecomposable Objects in $\mathcal{D}$

Using the components of the DG-algebra A defined in Section 3.2 define, for each vertex $q \in \widehat{\Gamma}$, an object $X_q^\bullet \in \mathcal{D}$ as follows: For $q = (i, n)$ and $v = (j, m)$ set

$$X_q^k(v) = A_{i,j;m-n}^{-k}$$

where $A_{i,j;m-n}^{-k}$ is defined by 3.2.3. It remains to check this does, in fact, define an object in $\mathcal{D}$.

Proposition 3.4.1. *For $q \in \widehat{\Gamma}$ there is a canonical isomorphism (up to choice of function ϵ) $X_q(\tau v) \simeq Cone(X_q(v) \to \oplus_{v \to v'} X_q(v'))$ and hence $X_q \in \mathcal{D}$.*

Proof. Let $v = (i, n) \in \widehat{\Gamma}$. For any edge $e : v \to v'$ in $\widehat{\Gamma}$, denote by $\overline{e}$ the corresponding edge $v' \to \tau v$. Define the map $\phi_q : Cone(x_v) \to X(\tau v)$ by

$$\phi_v(x, y) = t_i x + \sum_{e:s(e)=q} \epsilon(e)\overline{e}y \tag{3.4.1}$$

where $x \in X_q^{\bullet+1}(v)$ and $y \in \bigoplus_{e:v \to v'} X_q(v')$. Note that the choice of sign $\epsilon(e)$ is forced by requiring that this map agree with the differentials:

$$\begin{aligned}
\phi_v(d_C(x, y)) &= \phi_v(d_{X_q}x, (-1)^k x_v x + d_{X_q} y) \\
&= t_i d_{X_q}(x) + (-1)^k \sum_{e:s(e)=v} \epsilon(e)\overline{e}x_v(x) + \epsilon(\overline{e})\overline{e}d_{X_q}(y) \\
&= t_i d_{X_q}(x) + (-1)^k \sum_{e:s(e)=v} \epsilon(e)\overline{e}ex + \epsilon(e)\overline{e}d_{X_q}(y) \\
&= t_i d_{X_q}(x) + (-1)^k \theta_i x + \epsilon(e)\overline{e}y \\
&= d_{X_q}(t_i x + \epsilon(e)\overline{e}y) \\
&= d_{X_q}(\phi_v(x, y))
\end{aligned}$$

where $(x, y) \in Cone^k(x_v)$ and d_C denotes the differential on $Cone(x_v)$.

Since paths with jumps form a basis and any path $q \to \tau v$ with k jumps is either a path $p : q \to v'$ with k jumps followed by the edge $\overline{e} : v' \to \tau v$, or is a path $p : q \to v$ with $k - 1$ jumps followed by the jump t_i, the above map gives an isomorphism of complexes. $\square$

Alternatively, for $i \neq j$ let $q = (i, n)$, then for $p = (j, n)$ set $X_q(p) = 0$. Now let $p = (j, n+1)$ and $n_{ij} = 1$ so that $q \to p$ in $\widehat{\Gamma}$. We define $X_q^\bullet(p) := \mathrm{Path}_{\widehat{\Gamma}}(q, p)$ where by this we mean a complex with Path in degree 0, and 0 in all other degrees. Note that by Lemma 3.6.3 this is sufficient to extend to all other vertices using the fundamental relation. Note that it is clear from this definition that X_q is indecomposable.

3.5 Some results about Hom in $\mathcal{D}$

In this section we give some results which will be useful in future sections.

Theorem 3.5.1.

1. *Let $Y \in \mathcal{D}$, and let $q \in \widehat{\Gamma}$. Then there is an isomorphism* $\mathrm{RHom}(X_q, Y) = Y(q)$
2. *Let $q = (j, n)$, $q' = (i, m)$, then* $\mathrm{RHom}(X_q, X_{q'}) = A_{i,j;n-m}$.
3. $\mathrm{Hom}(X_q, X_{q'}) = \mathrm{Path}_{\widehat{\Gamma}}(q', q)/J$ *where J is the mesh ideal, generated by the mesh relations (see Equation 3.2.5).*
4. *Let h be a height function, and Γ_h the corresponding slice. If $q, q' \in \Gamma_h$ then* $\mathrm{Ext}^i(X_q, X_{q'}) = 0$ *for $i > 0$.*

Proof.

1. Let Γ_h be a slice through q. By Lemma 3.6.3 Part 2 $\mathrm{RHom}_{\mathcal{D}}(X_q, Y)$ is determined on the slice Γ_h. On the slice Γ_h the object X_q is concentrated in degree 0 so we can identify $\mathrm{RHom}(X_q, Y)$ with $Y(q)$ by definition of RHom and Proposition 3.3.1 Part 2.
2. By Part 1 we have $\mathrm{RHom}(X_q, X_{q'}) = X_{q'}(q) = A_{i,j;n-m}$.
3. By Part 1 we have
$$\mathrm{Hom}(X_q, X_{q'}) = H^0(X_{q'}(q)) = \mathrm{Path}(q', q)/J.$$
4. By Part 1 we have that $\mathrm{Ext}^k(X_q, X_{q'}) = \mathrm{Path}^k(q', q)/J$. However if $q, q' \in \Gamma_h$ then there are no paths with jumps $q' \to q$, in other words the complex $X_{q'}(q)$ is concentrated in degree 0.

$\square$

3.6 Equivalence of Categories

In this section, for every height function $h : \Gamma \to \mathbb{Z}$ an equivalence of triangulated categories $R\rho_h : \mathcal{D} \to \mathcal{D}^b(\Gamma, \Omega_h)$ is constructed and shown to be compatible with the reflection functors.

Remark 3.6.1. Note that here the equivalence of categories given by a height function, is between the category $\mathcal{D}$ and $\mathcal{D}^b(\Gamma, \Omega_h)$, where as in the case of equivariant sheaves on $\mathbb{P}^1$ considered in [K] the equivalence is between $\mathcal{D}$ and $\mathcal{D}^b(\Gamma, \Omega_h^{op})$ and is given by constructing a tilting object. That can also be done here, however that is not the approach taken.

Recall that any height function h determines an orientation Ω_h, and that the corresponding slice Γ_h is an embedding of the quiver (Γ, Ω_h) in $\widehat{\Gamma}$. So any representation of $\widehat{\Gamma}$ gives a representation of (Γ, Ω_h) by restriction to the slice. So there is a restriction functor $\rho_h : \mathrm{Rep}(\widehat{\Gamma}) \to \mathrm{Rep}(\Gamma, \Omega_h)$ defined by

$$\rho_h(X) = \bigoplus_{q \in \Gamma_h} X(q). \tag{3.6.1}$$

Notice that this functor is exact. Denote by $R\rho_h : \mathcal{D} \to \mathcal{D}^b(\Gamma, \Omega_h)$ the corresponding derived functor.

Theorem 3.6.2. *Let h be a height function, and let Γ_h be the corresponding slice. Then the functor $R\rho_h : \mathcal{D} \to \mathcal{D}^b(\Gamma, \Omega_h)$ is an equivalence of triangulated categories.*

Before proceeding with the Proof of the Theorem, the following preliminary result is required.

Lemma 3.6.3. *Let h be a height function, and let Γ_h be the corresponding slice.*

1. *An object $X \in \mathcal{D}$ is determined up to isomorphism by the collection $\{X^\bullet(q)\}_{q \in \Gamma_h}$ and morphisms corresponding to edges in the slice Γ_h.*

2. *For any $X, Y \in \mathcal{D}$ a morphism $f \in \mathrm{Hom}_{\mathcal{D}}(X, Y)$ is determined by the collection $\{f(q)\}_{q\in\Gamma_h}$.*

Proof.

1. Let $q \in \Gamma_h$ be a source. Since X satisfies the fundamental relations and comes with a fixed isomorphism $X(\tau q) \xrightarrow{\sim} Cone(X(q) \to \bigoplus_{q\to q'} X(q'))$, the complex $X^\bullet(\tau q)$ is determined by $X(q)$ and $X(p)$ for $q \to p$ in $\widehat{\Gamma}$ and morphisms corresponding to the edges joining them. (Noting that the $X(p)$ are in $\{X^\bullet(q)\}_{q\in\Gamma_h}$ since q is a source.) Write $q = (i, n)$ so that i is a source in the quiver (Γ, Ω_h) determined by the height function h. Apply the reflection s_i and consider a source in $q' \in \Gamma_{s_i h}$. Then repeating the argument above and noting that $X(q')$ is in the collection $\{X^\bullet(q)\}_{q\in\Gamma_h}$, and that $X^\bullet(p)$ for $q' \to p$ is in the collection $\{X^\bullet(q)\}_{q\in\Gamma_h} \bigcup X^\bullet(\tau q)$, one sees that $X^\bullet(\tau q')$ is determined. Continuing in this way it follows that for any $p = \tau^k q$ for $q \in \Gamma_h$ the complex $X^\bullet(p)$ is determined by the collection $\{X^\bullet(q)\}_{q\in\Gamma_h}$.

   A similar argument for $q \in \Gamma_h$ a sink can be repeated. This shows that for any $p = \tau^{-k}(q)$ with $q \in \Gamma_h$ the complex $X^\bullet(p)$ is determined by the collection $\{X^\bullet(q)\}_{q\in\Gamma_h}$.

2. Suppose that $F(q) : X(q) \to Y(q)$ is given for all $q \in \Gamma_h$. Take $q \in \Gamma_h$ to be a source, so that for any edge $q \to q'$ in $\widehat{\Gamma}$, q' belongs to the slice Γ_h. Then using the isomorphisms $X(\tau q) \xrightarrow{\sim} Cone(X(q) \to \bigoplus_{q\to q'} X(q'))$ and $Y(\tau q) \xrightarrow{\sim} Cone(Y(q) \to \bigoplus_{q\to q'} Y(q'))$ together with the functoriality of "cone over an edge", the following diagram has a unique completion $F(\tau q)$ making it commutative, which extends F to τq.

$$\begin{tikzcd}
X(\tau q) \arrow[r, "F(\tau q)"] \arrow[d] & Y(\tau q) \arrow[d] \\
Cone(X(q) \to \bigoplus_{q\to q'} X(q')) \arrow[r, "\tilde{F}"] & Cone(Y(q) \to \bigoplus_{q\to q'} Y(q'))
\end{tikzcd}$$

   Continuing in this way (and using a similiar argument for q a sink) it is possible to extend F to all vertices in $\widehat{\Gamma}$.

□

The proof of Theorem 3.6.2 is now given.

Proof. Note that a height function h gives a lifting of the quiver (Γ, Ω_h) to $\widehat{\Gamma}$ and that the image of (Γ, Ω_h) is the slice Γ_h. Hence for any object $Y \in D^b(\Gamma, \Omega_h)$ define an object in $\mathcal{D}$ as follows.

For $i \in \Gamma$ and $q = (i, h(i)) \in \widehat{\Gamma}$ define $X(q) = Y(i)$ and for each edge $e : q \to q' \in \Gamma_h$ define maps $x_e : X(q) \to X(q')$ by $x_e = y_e$ where $y_e : Y(i) \to Y(j)$ and $q' = (j, h(j))$. Then Part 1 of Lemma 3.6.3 shows this determines an object $X \in \mathcal{D}$. Hence for every $Y \in \mathcal{D}^b(\Gamma, \Omega_h)$ there exists $X \in \mathcal{D}$ such that $R\rho_h(X) = Y$. Note that Part 2 of Lemma 3.6.3 implies that for any $X, Y \in \mathcal{D}$ there is an isomorphism $\mathrm{Hom}_{\mathcal{D}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{D}^b(\Gamma,\Omega_h)}(R\rho_h X, R\rho_h Y)$. Together, this shows that $R\rho_h$ is an equivalence, and since $R\rho_h$ is the derived functor of an exact functor it is a triangle functor. $\square$

Corollary 3.6.4. *Let Γ be Dynkin. The Auslander-Reiten quiver of $\mathcal{D}$ is $\widehat{\Gamma}^{op}$, so the objects X_q form a complete list of indecomposable objects in $\mathcal{D}$.*

Proof. By Theorem 3.6.2 the Auslander-Reiten quiver of $\mathcal{D}$ is the isomorphic to that of $\mathcal{D}^b(\Gamma, \Omega_h)$. It is well known (see [Hap] for example) that the Auslander-Reiten quiver of $\mathcal{D}^b(\Gamma, \Omega_h)$ is isomorphic to $\widehat{\Gamma}^{op}$. $\square$

Remark 3.6.5. Usually the Auslander-Reiten quiver of $\mathcal{D}^b(\Gamma, \Omega_h)$ is identified with $\widehat{\Gamma}$ by identifying the projectives with a slice in $\widehat{\Gamma}$ that gives the opposite orientation to Ω_h, and proceeding from there. For reasons that will become clear in Section 3.7, we instead identify it with $\widehat{\Gamma}^{op}$ by identifying the projectives with the slice $\Gamma_h \subset \widehat{\Gamma}^{op}$.

Corollary 3.6.6. *The category $\mathcal{D}$ has Serre Duality:*

$$\mathrm{Hom}_{\mathcal{D}}(X, Y) = (\mathrm{Ext}^1_{\mathcal{D}}(Y, \tau_{\mathcal{D}} X))^*$$

where * denotes the dual space and $\tau_{\mathcal{D}}$ is given by Equation 3.7.3.

Proof. It is well known (see [Hap] Proposition 4.10 p.42) that in the category $\mathcal{D}^b(\Gamma, \Omega_h)$ this relation holds. $\square$

The following theorem shows that the restriction functor is compatible with the reflection functors.

Theorem 3.6.7. *Let i be a source (or sink) for the orientation Ω_h and let $S_i^{\pm}$ denote the corresponding reflection functor. Then the following diagram is commutative.*

$$\begin{tikzcd} & \mathcal{D}^b(\Gamma, \Omega_h) \arrow[dd, "RS_i^{\pm}"] \\ \mathcal{D} \arrow[ur, "R\rho_h"] \arrow[dr, "R\rho_{s_i^{\pm}h}"'] & \\ & \mathcal{D}^b(\Gamma, \Omega_{s_i^{\pm}h}) \end{tikzcd}$$

Proof. Let $q \in \Gamma_h$ be a source. Let $X \in \mathcal{D}$, so that $X(\tau q) \simeq Cone(X(q) \to \oplus X(q'))$. Restriction along the slice $\Gamma_{s_i^+h}$ gives $X(p)$ if $p \neq q$ and $X(\tau q)$ if $p = q$. The reflection functor is defined as $X(p)$ if $p \neq q$ and $Cone(X(q) \to \oplus X(q'))$ if $p = q$, so the diagram commutes. $\square$

3.7 The Mesh Category $\mathcal{B}$

In the remainder of this Chapter, fix Γ to be Dynkin. In this section the definition of the mesh category of a translation quiver (Q, τ) is recalled and then related to the category $\mathcal{D}$.

Let (Q, τ) be a translation quiver (see [ARS] Chapter VII §for details). Define the set of indecomposable objects of the mesh category $\mathcal{B}(Q)$ to be the vertices of Q. Set $\mathrm{Hom}_{\mathcal{B}}(q, q') = \mathrm{Path}(q, q')/J$ where J is the mesh ideal generated by the mesh relations $\sum_{s(e)=i} \epsilon(e)\bar{e}e$ (see Equation 3.2.5).

Consider the mesh category of the translation quiver $(\widehat{\Gamma}^{op}, \tau_{\widehat{\Gamma}})$ where $\tau_{\widehat{\Gamma}}(i, n) = (i, n + 2)$. For simplicity denote this by $\mathcal{B}$ and denote the translation by $\tau_{\mathcal{B}}$.

Remark 3.7.1. Note that we consider the mesh category of $\widehat{\Gamma}^{op}$ instead of $\widehat{\Gamma}$ since the A-R quiver of $\mathcal{D}$ is $\widehat{\Gamma}^{op}$.

It is shown in [BBK] (Section 6) that there are the following automorphisms in $\mathcal{B}$:

1. A Nakayama automorphism $\nu_{\mathcal{B}}$ which commutes with $\tau_{\mathcal{B}}$ and satisfies $\nu_{\mathcal{B}}^2 = \tau_{\mathcal{B}}^{-(h-2)}$. (Here $\nu = \tilde{\beta}^{-1}$ in the notation of [BBK].)

2. An automorphism $\gamma_{\mathcal{B}}$ defined by $\gamma_{\mathcal{B}} := \nu_{\mathcal{B}}\tau_{\mathcal{B}}^{-1}$, which satisfies $\gamma_{\mathcal{B}}^2 = \tau_{\mathcal{B}}^{-h}$.

These automorphisms satisfy:

$$\begin{aligned} \nu_{\mathcal{B}}^2 &= \tau_{\mathcal{B}}^{-(h-2)} \\ \gamma_{\mathcal{B}}^2 &= \tau_{\mathcal{B}}^{-h} \end{aligned} \tag{3.7.1}$$

As before, for any $i \in \Gamma$ define $\check{\imath}$ by $-\alpha_i^{\Pi} = w_0^{\Pi}(\alpha_{\check{\imath}}^{\Pi})$, where $w_0^{\Pi} \in W$ is the longest element and Π is any set of simple roots. Thus for the root systems of type A, D_{2n+1}, E_6 this map corresponds to the diagram automorphism, while for D_{2n}, E_7, E_8 this map is just the identity.

In terms of $\widehat{\Gamma}^{op}$ the maps $\nu_{\mathcal{B}}$ and $\gamma_{\mathcal{B}}$ are given by:

$$\begin{aligned} \nu_{\mathcal{B}}(i,n) &= (\check{\imath}\ , n-h+2) \\ \gamma_{\mathcal{B}}(i,n) &= (\check{\imath}\ , n-h) \end{aligned} \tag{3.7.2}$$

Example 3.7.2. For the graph $\Gamma = A_4$, $\check{\imath}\ = 5-i$ and $h = 5$ so $\nu_{\mathcal{B}}(i,n) = (5-i, n-3)$ and $\gamma_{\mathcal{B}}(i,n) = (5-i, n-5)$. The maps $\nu_{\mathcal{B}}$ and $\gamma_{\mathcal{B}}$ are shown in Figure 3.1.

Recall that the Auslander-Reiten quiver of $\mathcal{D}$ is $\widehat{\Gamma}^{op}$. This identification is given by $[X_q] \mapsto q$. In terms of arrows, by Theorem 3.5.1 Part 2, for each arrow $q \to q'$ in $\widehat{\Gamma}$ there is an arrow $q' \to q$ in the Auslander-Reiten quiver.

In the category $\mathcal{D}$ define an automorphism $\tau_{\mathcal{D}}$ by

$$\tau_{\mathcal{D}}(X)(q') = X(\tau^{-1}q'). \tag{3.7.3}$$

Notice that

$$X_{\tau q}(q') = \mathrm{Path}^{\bullet}_{\widehat{\Gamma}}(\tau q, q') \simeq \mathrm{Path}^{\bullet}_{\widehat{\Gamma}}(q, \tau^{-1}q') = \tau_{\mathcal{D}}(X_q)(q')$$

so that $\tau_{\mathcal{D}}X_q \simeq X_{\tau q}$ for the indecomposables X_q. In terms of the Auslander-Reiten quiver of $\mathcal{D}$, this identifies $\tau_{\mathcal{D}}$ with the translation τ on $\widehat{\Gamma}^{op}$.

Theorem 3.7.3. *Let h be a height function.*

1. *There are equivalences of additive categories, given by the following commutative diagram:*

$$\begin{tikzcd} \mathcal{B} & \mathcal{D}^b(\Gamma, \Omega_h) \arrow[l, "\Psi_h"'] \\ & \mathcal{D} \arrow[ul, "\psi"] \arrow[u, "R\rho_h"'] \end{tikzcd}$$

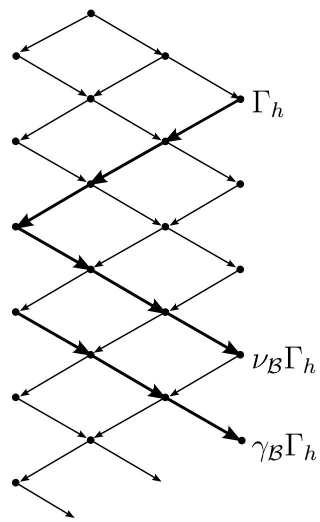

Figure 3.1: The maps $\nu_{\mathcal{B}}$ and $\gamma_{\mathcal{B}}$ in the case $\Gamma = A_4$. A slice Γ_h and its images under $\nu_{\mathcal{B}}$ and $\gamma_{\mathcal{B}}$ are shown in bold. Recall that we are considering the mesh category of $\widehat{\Gamma}^{op}$ as mentioned above.

2. *Under these equivalences the automorphisms $\nu_{\mathcal{B}}$ gives an Nakayama automorphism $\nu_{\mathcal{D}}$ on $\mathcal{D}$ and is identified with the Nakayama automorphism ν in $\mathcal{D}^b(\Gamma, \Omega_h)$*

3. *The map $\tau_{\mathcal{B}}$ can be identified with the Auslander-Reiten translation in $\mathcal{D}^b(\Gamma, \Omega_h)$, and with $\tau_{\mathcal{D}}$ in $\mathcal{D}$.*

4. *The map $\gamma_{\mathcal{B}}$ can be identified with T in $\mathcal{D}^b(\Gamma, \Omega_h)$, and with $T_{\mathcal{D}}$ in $\mathcal{D}$. Hence we can impose a triangulated structure on $\mathcal{B}$ making the equivalences in (1) triangulated equivalences.*

5. *In $\mathcal{D}$ the relation $T^2 = \tau_{\mathcal{D}}^{-h}$ holds and hence the objects X_q satisfy the relation $X_q^{\bullet+2} \simeq X^{\bullet}_{\tau^{-h}q}$.*

Proof.

1. The equivalence ρ_h is from Theorem 3.6.2.
The equivalence Ψ_h is the map which is given $P_i \mapsto (i, h_i)$ on projectives, so that the projectives in $\mathrm{Rep}(\Gamma, \Omega_h)$ map to the slice $\Gamma_h \subset \widehat{\Gamma}^{op}$. (Note that in $\widehat{\Gamma}^{op}$ the arrows are reversed, so this agrees with the usual

identification of projectives with a slice giving the orientation opposite to Ω_h.) This is just the identification of $\mathrm{Ind}(\mathcal{D}^b(\Gamma, \Omega_h))$ with its Auslander-Reiten quiver. That this is an equivalence is well-known, see [Hap] for example.
The equivalence ψ is given by $X_q \mapsto q$. By Corollary 3.6.4 the objects X_q form a complete list of indecomposables, and since

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(X_q, X_{q'}) = H^0(X_{q'}(q)) &= \mathrm{Path}_{\widehat{\Gamma}}(q', q)/J = \mathrm{Path}_{\widehat{\Gamma}^{op}}(q, q')/J \\ &= \mathrm{Hom}_{\mathcal{B}}(q, q')\end{aligned}$$

it follows that this is an equivalence.

2. Follows from (1), since Ψ_h is the identification of $\mathrm{Ind}(\mathcal{D}^b(\Gamma, \Omega_h))$ with its Auslander-Reiten quiver.

3. Again follows from (1).

4. In $\mathcal{D}^b(\Gamma, \Omega_h)$ the Auslander-Reiten translation is defined by $\tau_{\mathcal{D}^b} := T^{-1}\nu$, or equivalently $T = \nu\tau^{-1}$

5. By (4) $\gamma_{\mathcal{B}}$ is identified with $T_{\mathcal{D}}$, by (2) and (3) so there is an identification of Nakayama automorphisms and translations in $\mathcal{B}$ and $\mathcal{D}$. Then using Equation 3.7.1 gives the result.

□

3.8 Periodicity in the Dynkin Case

This section discusses periodicity of the "dg-preprojective algebra" A in the case where Γ is Dynkin. Recall the decomposition given in Section 2:

$$A = \bigoplus_{n \leq 0} A^n = \bigoplus_{i,j \in \Gamma,\ l \in \mathbb{Z}_+, n \leq 0} A^n_{i,j;l} \tag{3.8.1}$$

Note that the differential in A preserves the grading by path length, so this decomposition passes to homology:

$$H^n(A) = \bigoplus_{i,j;l} H^n(A_{i,j;l})$$

Now fix $q = (j, n+l)$ and $q' = (i, n)$. Then by definition of $X_{q'}$, and by Part 2 of Theorem 3.5.1, $\mathrm{RHom}(X_q, X_{q'}) = X_{q'}(q) = A_{i,j;l}$ so the component $A_{i,j;l}$ can be interpreted as the RHom complex of the corresponding indecomposables.

Recall that in the case where Γ was not Dynkin the complex A was a DG resolution of the preprojective algebra Π. In particular, all homology was in degree 0. The decomposition of homology above and the identification $A_{i,j;l} \simeq \mathrm{RHom}_{\mathcal{D}}(X_q, X_{q'})$ makes it clear that this is not the case when Γ is Dynkin.

In the case where Γ is Dynkin there is have the following periodicity result for the Koszul complex of the preprojective algebra and its homology. This is likely known to experts, but does not seem to be easily available in the literature.

Theorem 3.8.1. *There is a quasi-isomorphism of complexes*

$$A^{\bullet+2}_{i,j;l} \simeq A^{\bullet}_{i,j;l+2h}. \tag{3.8.2}$$

In terms of the homology of the complex A this gives:

$$H^{k+2}(A_{i,j;l}) = H^k(A_{i,j;l+2h})$$

Proof. By Theorem 3.7.3 Part 5 there is an identification $X^{\bullet+2}_{q'} \simeq X^{\bullet}_{\tau^{-h}q'}$ in $\mathcal{D}$, so in particular $X^{\bullet+2}_{q'}(q) \simeq X^{\bullet}_{\tau^{-h}q'}(q)$. For $q = (i, n)$ and $q' = (j, n+l)$ there are identifications

$$\begin{aligned} &\mathrm{RHom}(X_q, X_{q'}) = X_{q'}(q) = A_{i,j;l} \\ &\mathrm{RHom}(X_q, X_{\tau^{-h}q'}) = X_{\tau^{-h}q'}(q) = A_{i,j;l+2h}. \end{aligned} \tag{3.8.3}$$

Combining these gives $A^{\bullet+2}_{i,j;l} \simeq A_{i,j;l+2h}$. $\square$

In terms of the category $\mathcal{D}$ this result can be interpreted as follows.

Corollary 3.8.2. *Let $X, Y \in \mathcal{D}$.*

1. $\mathrm{RHom}^{\bullet}(X, \tau^{-h}Y) \simeq \mathrm{RHom}^{\bullet+2}(X, Y)$
2. $\mathrm{Ext}^i(X, \tau^{-h}Y) \simeq \mathrm{Ext}^{i+2}(X, Y)$

Proof. First note that the collection X_q form a complete list of indecomposables in $\mathcal{D}$ (in this section Γ is Dynkin). So it's enough to prove this result for these objects. For these objects, recalling that $\tau X_q \simeq X_{\tau q}$ shows that the first statement follows from Equation 3.8.3 in the proof of Theorem 3.8.1. The second statement follows by taking homology. $\square$

3.9 The quotient category $\mathcal{D}/T^2$

It was shown in [PX1] that the category $\mathcal{D}^b(\Gamma, \Omega)/T^2$ is a triangulated category, and that the set $\mathrm{Ind}(\Gamma, \Omega)$, of classes of indecomposables gives the corresponding root system. More general quotient categories were studied in [Kel], where conditions for a quotient category to inherit a triangulated structure are given.

In this section we consider the quotient category $\mathcal{C} = \mathcal{D}/T^2$ and relate it to Theorem 3.0.2.

Proposition 3.9.1. *The quotient category $\mathcal{D}/T^2$ has the following properties:*

1. *It is triangulated.*
2. $\tau^{-h} = Id = \tau^h$
3. *It has Auslander-Reiten quiver $\widehat{\Gamma}/\tau^h = \widehat{\Gamma}_{cyc}$.*

Proof. Part 1 follows from the main result of [Kel]. The other parts follow from Theorem 3.7.3. $\square$

Let R be the root system corresponding to Γ. Proposition 3.9.1 and Theorem 2.0.5 show that there is a bijection between R and the Auslander-Reiten quiver of the category $\mathcal{C} = \mathcal{D}/T^2$. The following Theorem summarizes this bijection and completes the proof of Theorem 3.0.2.

Theorem 3.9.2. *Let Γ be Dynkin with root system R and let $\mathcal{K}$ be the Grothendieck group of $\mathcal{C}$. Set $\langle X, Y\rangle = \dim \mathrm{RHom}(X, Y) = \dim \mathrm{Hom}(X, Y) - \dim \mathrm{Ext}^1(X, Y)$. The set $\mathrm{Ind} \subset \mathcal{K}$ of isomorphism classes of indecomposable objects in $\mathcal{C}$, with bilinear form given by $(X, Y) = \langle X, Y\rangle + \langle Y, X\rangle$, is isomorphic to R, and $\mathcal{K}$ is isomorphic to the root lattice. Moreover, the translation*

τ gives a Coxeter element for this root system, and $T_{\mathcal{C}}$ gives the longest element.

Chapter 4

Root Bases

Let Γ be a Dynkin graph of type A, D, E. Let $\mathfrak{g}$ and $U_q(\mathfrak{g})$ be the corresponding Lie algebra and quantum group respectively. By choosing an orientation Ω of Γ, one obtains a quiver $\overrightarrow{\Gamma} = (\Gamma, \Omega)$. Ringel used the category $\mathrm{Rep}(\overrightarrow{\Gamma})$ of representations of $\overrightarrow{\Gamma}$ to realise $\mathfrak{n}_+$ and $U_q\mathfrak{n}_+$ (see [R1], [R2]). Peng and Xiao then used a related category, $\mathcal{D}^b \,\mathrm{Rep}(\overrightarrow{\Gamma})/T^2$, to realise the whole Lie algebra $\mathfrak{g}$. As mentioned in the Introduction, the drawback of these constructions is the necessity of choosing an orientation of the Dynkin diagram.

Motivated by these results and the ideas of Ocneanu [Oc], the main goal of this Chapter is to use a Coxeter element, and the results in Chapter 1, to construct a root basis in the Lie algebra $\mathfrak{g}$ and to determine the structure constants of the Lie bracket in purely combinatorial terms.

In Chapter 1 it was shown a choice of Coxeter element gives a bijection between R and a certain quiver $\widehat{\Gamma}_{cyc}$, which identifies roots in R and vertices in $\widehat{\Gamma}$. This bijection then identifies vertices in $\widehat{\Gamma}_{cyc}$ with basis vectors E_α. Using this identification and choice of basis, it is possible to determine the structure constants of the Lie bracket from paths in $\widehat{\Gamma}_{cyc}$. Thus it is possible to realise the Lie algebra $\mathfrak{g}$ completely in terms of the quiver $\widehat{\Gamma}_{cyc}$. This construction is then independent of any choice of orientation of Γ or choice of simple roots.

The main result is summarized in the following Theorem.

Theorem 4.0.3. *Let $\mathfrak{g}$ be a Lie algebra of type A, D, E with fixed Cartan subalgebra $\mathfrak{h}$. This gives a root system R with Weyl group W. Fix a Coxeter* *element $C \in W$.*

1. *The choice of a Coxeter element C gives a root basis $\{E_\alpha\}_{\alpha \in R}$ for $\mathfrak{g}$.*
2. *With this choice of basis the Lie bracket is given by $[E_\alpha, E_\beta] = (-1)^{\langle \alpha, \beta \rangle} E_{\alpha+\beta}$ for $\alpha + \beta \in R$, where $\langle \cdot, \cdot \rangle$ is the de-symmetrization of the bilinear form given in Theorem 2.0.5.*
3. *The identification $R \to \widehat{\Gamma}_{cyc}$ given in Theorem 2.0.5, together with Parts 1 and 2 of Theorem 4.0.3 imply that $\mathfrak{g}$ can be defined combinatorially in terms of $\widehat{\Gamma}_{cyc}$: $\mathfrak{g}$ has root basis given by vertices in $\widehat{\Gamma}_{cyc}$ and the structure constants can be obtained from paths in $\widehat{\Gamma}_{cyc}$.*

4.1 Braid Group Action

Let $U_q(\mathfrak{g})$ be the corresponding quantum group. It is generated by elements $E_i, F_i, K_i^{\pm 1}$, where $i \in \Gamma$. In particular, for $q = 1$ this gives the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

In this section the definition and relevant results of the braid group operators as defined in [J] are reviewed. For more details see [J], or [L].

Fix a system of simple roots Π. Let $E_i, F_i, K_i^{\pm 1}$ denote the corresponding generators of $U_q(\mathfrak{g})$.
For simple roots α_i define operators T_i, T_i' on any finite dimensional module V by setting for $v \in V_\lambda$:

$$T_i(v) = \sum_{a,b,c \geq 0; -a+c-b=m} (-1)^b q^{b-ac} \frac{E_i^{(a)}}{[a]!} \frac{F_i^{(b)}}{[b]!} \frac{E_i^{(c)}}{[c]!} v$$

$$T_i'(v) = \sum_{a,b,c \geq 0; -a+c-b=m} (-1)^b q^{ac-b} \frac{E_i^{(a)}}{[a]!} \frac{F_i^{(b)}}{[b]!} \frac{E_i^{(c)}}{[c]!} v$$

with $m = (\lambda, \alpha_i^\vee)$. (Here $\alpha_i^\vee$ denotes the coroot, not to be confused with the root $\alpha_{\bar{i}}$.)

Then there are unique automorphisms of $U_q(\mathfrak{g})$, also denoted by T_i, T_i' so that for any $u \in U_q(\mathfrak{g})$ and $v \in V$ we have $T_i(uv) = T_i(u)T_i(v)$. The operator T_i acts on weights by the reflection s_i.

The automorphisms T_i satisfy the braid relations:

$$T_iT_j = T_jT_i \quad \text{for} \quad (\alpha_i, \alpha_j) = 0$$

$$T_iT_jT_i = T_jT_iT_j \quad \text{for} \quad (\alpha_i, \alpha_j) = -1$$

For the automorphism T_i there are the following formulae:

$$\begin{aligned}
&T_iE_i = -F_iK_i \\
&T_iF_i = -K_i^{-1}E_i \\
&T_iE_j = E_j \quad \text{for} \quad (\alpha_i, \alpha_j) = 0 \\
&T_iE_j = E_iE_j - q^{-1}E_jE_i \quad \text{for} \quad (\alpha_i, \alpha_j) = -1 \\
&T_iF_j = F_j \quad \text{for} \quad (\alpha_i, \alpha_j) = 0 \\
&T_iF_j = F_iF_j - q^{-1}F_jF_i \quad \text{for} \quad (\alpha_i, \alpha_j) = -1
\end{aligned}$$

In fact, there are automorphisms T_α for any root α. As above, define T_α on a module V by setting for $v \in V_\lambda$:

$$T_\alpha(v) = \sum_{a,b,c \geq 0; -a+c-b=m} (-1)^b q^{b-ac} \frac{E_\alpha^{(a)}}{[a]!} \frac{F_\alpha^{(b)}}{[b]!} \frac{E_\alpha^{(c)}}{[c]!} v$$

$$T'_\alpha(v) = \sum_{a,b,c \geq 0; -a+c-b=m} (-1)^b q^{ac-b} \frac{E_\alpha^{(a)}}{[a]!} \frac{F_\alpha^{(b)}}{[b]!} \frac{E_\alpha^{(c)}}{[c]!} v$$

where $E_\alpha, F_\alpha \in U_q(\mathfrak{g})$ satisfy the $U_q(sl_2)$ relations and $m = (\lambda, \check{\alpha})$.

Lemma 4.1.1. *Let Φ be an automorphism of $U_q(\mathfrak{g})$ such that $E_\alpha = \Phi(E_i)$ and $F_\alpha = \Phi(F_i)$. Then $T_\alpha = \Phi T_i \Phi^{-1}$.*

The automorphisms T_α satisfy relations similar to those of the T_i:

$$
\begin{aligned}
&T_\alpha E_\alpha = -F_\alpha K_\alpha \\
&T_\alpha F_\alpha = -K_\alpha^{-1} E_\alpha \\
&T_\alpha E_\beta = E_\beta \ \text{ for } \ (\alpha, \beta) = 0 \\
&T_\alpha E_\beta = E_\alpha E_\beta - q^{-1} E_\beta E_\alpha \ \text{ for } \ (\alpha, \beta) = -1 \\
&T_\alpha F_\beta = F_\beta \ \text{ for } \ (\alpha, \beta) = 0 \\
&T_\alpha F_\beta = F_\alpha F_\beta - q^{-1} F_\beta F_\alpha \ \text{ for } \ (\alpha, \beta) = -1
\end{aligned}
$$

Since the operators T_i satisfy the braid relations it is possible to define an operator T_w for any $w \in W$. For any reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_k}$ for $w \in W$ define $T_w = T_{i_1} T_{i_2} \cdots T_{i_k}$.

The following Lemma will be useful. It can be found in [J], Proposition 8.20.

Lemma 4.1.2. *If $w \in W$ is such that $w(\alpha_i) \in R_+$, then $T_w(E_i) \in U_q^+$. If, in addition, $w(\alpha_i) = \alpha_j$, then $T_w(E_i) = E_j$.*

For the case to be considered in the following sections, this result gives the following important Corollary.

Corollary 4.1.3. *Let $w_0 \in W$ be the longest element. Then $T_{w_0}(E_{\check{i}}) = T_i E_i = -F_i K_i$.*

Proof. Let $w_0 = s_i w$ be a reduced expression for w_0, so that $T_{w_0} = T_i T_w$. Then since

$$w(\alpha_{\check{i}}) = s_i w_0(\alpha_{\check{i}}) = s_i(-\alpha_i) = \alpha_i$$

the Lemma gives $T_w(E_{\check{i}}) = E_i$, and the result follows by applying T_i. □

4.2 Longest Element and Construction of Root Vectors

Let Π be a simple system, and let $R = R_+ \cup R_-$ be the corresponding polarization. Let w_0 be the longest element. A reduced expression $w_0 =$

$s_{i_1}s_{i_2}\cdots s_{i_l}$ is said to be *adapted* to an orientation Ω of Γ if i_k is a source for $s_{i_1}\cdots s_{i_{k-1}}\Omega$. In particular i_1 is a source of Ω.

Lemma 4.2.1. *Given any orientation Ω, there is a reduced expression adapted to Ω, and moreover, any two expressions adapted to Ω are related by $s_is_j = s_js_i$ with $n_{ij} = 0$.*

Proof. Recall that any height function h determines an orientation Ω_h and that for any orientation Ω there is a choice of h so that $\Omega = \Omega_h$. So take some h corresponding to Ω, then Theorem 2.7.1 gives the existence of such an expression. Note that any reduced expression adapted to Ω gives a sequence of source to sink moves taking the slice Γ_h to the slice Γ_{-h} where Γ_{-h} is the slice corresponding to the simple roots $-\Pi$.

Let $w_0 = s_{i_1}\cdots s_{i_l}$ and $w_0 = s_{i'_1}\cdots s_{i'_l}$ be two different reduced expressions adapted to Ω. Let k be the first index where they differ. Write $i_k = i$ and $i'_k = j$ to simplify notation. Then there are reduced expressions

$$w_0 = ws_iw_1s_jw_2$$

$$w_0 = ws_jw'_1s_jw'_2$$

where s_j does not appear in w_1 and s_i does not appear in w'_1. Thus i, j are both sources for $w\Omega$ and hence $n_{ij} = 0$. Note that since w_1 is obtained as a sequence of source to sink reflections, and since s_j does not appear in w_1, j remains a source during this process. Hence if s_l appears in w_1 then l is not adjacent to j, so that $n_{jl} = 0$. Thus $w_1s_j = s_jw_1$. which gives:

$$\begin{aligned} w_0 &= ws_iw_1s_jw_2 \\ &= ws_is_jw_1w_2 \\ &= ws_js_iw_1w_2 \end{aligned}$$

So it is possible to make the two reduced expressions agree at the index k using only the relation $s_js_l = s_ls_j$ for $n_{jl} = 0$. Continuing in this fashion it is possible to make the expressions agree at every index using only this relation.

$\square$

It is well known that a reduced expression $w_0 = s_{i_1} \cdots s_{i_l}$, adapted to Ω, gives an ordering of the positive roots $R = \{\gamma_1, \ldots, \gamma_l\}$ by setting $\gamma_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_k)$. Such an expression also gives root vectors E_α, F_α for $\alpha \in R_+$ as follows:

$$E_{\gamma_k} = T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}) \tag{4.2.1}$$

$$F_{\gamma_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}) \tag{4.2.2}$$

Note that since the T_i satisfy the braid relation, Lemma 4.2.1 implies that the root vectors defined this way do not depend on the choice of reduced expression adapted to Ω.

Note that if $\gamma_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$ then $\gamma_k = s_{\gamma_{k-1}} \cdots s_{\gamma_1} \alpha_{i_k}$, and the longest element can be expressed as $w_0 = s_{\gamma_l} \cdots s_{\gamma_1}$.

Since $T_{\gamma_k} = (T_{i_1} \cdots T_{i_{k-1}}) T_{i_k} (T_{i_1} \cdots T_{i_{k-1}})^{-1}$ then, as for reflections,

$$T_{i_1} \cdots T_{i_k} = T_{\gamma_k} \cdots T_{\gamma_1}$$

so the root vectors E_{γ_k} given in Equation 4.2.1 can be expressed as

$$E_{\gamma_k} = T_{\gamma_{k-1}} \cdots T_{\gamma_1} E_{i_k}. \tag{4.2.3}$$

Definition 4.2.2. Let Π be a set of simple roots, Ω be an orientation of Γ and let $w_0 = s_{i_1} \cdots s_{i_l}$ be a reduced expression adapted to Ω. A root basis $\{E_\alpha\}_{\alpha \in R}$ is said to be *adapted to the pair* (Π, Ω) if for $\alpha \in R_+$ the vector E_α is given by Equation 4.2.1, or equivalently by Equation 4.2.3.

4.2.1 Change of Orientation

For a reduced expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_l}$, adapted to Ω, define a new reduced expression $w_0 = s_{i_2} \cdots s_{i_l} s_{\bar{i_1}}$ which is adapted to $s_1 \Omega$. Then this gives a new enumeration of positive roots $\{\gamma'_1, \ldots \gamma'_l\}$, and a new collection of root vectors:

$$\gamma'_1 = s_{i_i}(\gamma_{i_2}), \gamma'_2 = s_{i_1}(\gamma_{i_3}), \ldots, \gamma_l = \alpha_{\bar{i}_1}$$

$$E_{\gamma'} = T_i^{-1}(E_{s_i \gamma}) \quad \text{for} \quad \gamma \neq \alpha_{i_1} \tag{4.2.4}$$

4.2.2 Coxeter Element

Now consider the case where there is a fixed Coxeter element $C \in W$ and hence an identification $R \to \widehat{\Gamma}_{cyc}$ as in Chapter 2. In this case a choice of height function h is identified with a set of simple roots Π compatible with C, and hence determines a polarization $R = R_+^h \cup R_-^h$. A height function also determines a reduced expression for w_0 adapted to the orientation Ω_h. This expression is obtained from $\widehat{\Gamma}_{cyc}$ as a sequence of source to sink reflections which take the slice Γ_{h^Π} to the slice $\Gamma_{h^{-\Pi}}$.

Using this reduced expression, there is an associated ordering of the positive roots which gives a completion of the partial order given by paths in $\widehat{\Gamma}_{cyc}$. Note that the completion depends on the reduced expression.

Now choose a height function h. Then using the reduced expression for w_0 obtained above, it is possible to define a collection of root vectors E_α for $\alpha \in R_+^h$ using Equation 4.2.1.

Under the identification $R \to \widehat{\Gamma}_{cyc}$ suppose that $\alpha = (i, n)$, then $C\alpha = (i, n+2)$. For j connected to i, denote by γ_j the root corresponding to vertex $(j, n+1)$. The collection of roots $\{\alpha, \{\gamma_j\}_{j-i}, C\alpha\}$ is said to satisfy the fundamental relation in $\widehat{\Gamma}_{cyc}$. Such a collection is depicted in Figure 4.1.

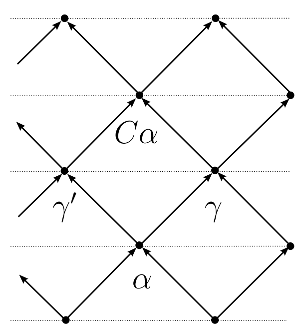

Figure 4.1: A collection of roots $\alpha, \gamma_j, C\alpha \in \widehat{\Gamma}_{cyc}$ satisfying the fundamental relation.

Lemma 4.2.3. *Let $\alpha, \gamma_j, C(\alpha) \in R_+^h$ satisfy the fundamental relation in $\widehat{\Gamma}_{cyc}$. Then the corresponding root vectors satisfy:*

$$E_{C(\alpha)} = (\prod_{j-i} T_{\gamma_j})T_\alpha(E_\alpha) \tag{4.2.5}$$

Proof. Let h be a fixed height function and let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ denote the corresponding set of simple roots and s_i denote the corresponding simple reflections.

Let $\alpha = (i, n), \gamma_j = (j, n+1), C\alpha = (i, n+2)$ and

$$w_0 = ws_i(\prod_{j-i} s_j)s_i w'$$

a reduced expression adapted to Ω_h. Then

$$\begin{aligned}
&E_\alpha = T_w E_{\alpha_i} \text{ and } T_\alpha = T_w T_{\alpha_i} T_w^{-1} \\
&E_{\gamma_j} = T_w T_{\alpha_j} E_{\alpha_j} \text{ and } T_{\gamma_j} = T_w T_{\alpha_i} T_{\alpha_j} T_{\alpha_i}^{-1} T_w^{-1} \\
&E_{C\alpha} = T_w T_{\alpha_i} (\prod_{j-i} T_{\alpha_j}) E_{\alpha_i}
\end{aligned}$$

where the product $\prod_{j-i}$ is taken over all j connected to i in Γ.

On the other hand, using the first two formulae, and comparing with the third one obtains:

$$\begin{aligned}
(\prod_{j-i} T_{\gamma_j}) T_\alpha E_\alpha &= \prod_{j-i} (T_w T_{\alpha_i} T_{\alpha_j} T_{\alpha_i}^{-1} T_w^{-1}) T_w T_{\alpha_i} T_w^{-1} T_w E_{\alpha_i} \\
&= T_w T_{\alpha_i} (\prod_{j-i} T_{\alpha_j}) T_{\alpha_i}^{-1} T_w^{-1} T_w T_{\alpha_i} T_w^{-1} T_w E_{\alpha_i} \\
&= T_w T_{\alpha_i} (\prod_{j-i} T_{\alpha_j}) E_{\alpha_i} \\
&= E_{C\alpha}
\end{aligned}$$

□

Now fix a height function h, and hence a choice of compatible simple roots Π_h, an orientation Ω_h, a reduced expression w_0 adapted to Ω_h and a slice $\Gamma_h \subset \widehat{\Gamma}$. Define a root basis as follows:

1. For $\beta \in \Gamma_h$ choose $E_\beta \in \mathfrak{g}_\beta$.
2. Since any root is of the form $\alpha = C^k \beta$ for some k and some β, define E_α inductively using Equation 4.2.5, beginning with $E_{C\beta}$ for β a source in Γ_h.

Note that for $C^h\beta = \beta$ this procedure produces another root vector $E'_\beta \in \mathfrak{g}_\beta$.

Proposition 4.2.4. *Let E_α, E'_α be the root vectors defined above.*

1. *For $q = 1$ $E'_\alpha = E_\alpha$, so this procedure produces a consistent root basis in $\mathfrak{g}$.*

2. *For $q \neq 1$ $E'_\alpha = K_\alpha^{-1} E_\alpha K_\alpha$.*

Corollary 4.2.5. *Let $U_q\mathfrak{g}$ denote the corresponding quantum group. For $q \neq 1$ there is a $\mathbb{Z}$-torsor of vectors $\{E_\alpha^k\}$ for each root α that are related by $E_\alpha^{k+n} = K_\alpha^n E_\alpha^k K_\alpha^{-n}$.*

Proof. To simplify notation, set $E_{\alpha_i} = E_i, F_{\alpha_i} = F_i, K_{\alpha_i} = K_i$. Then using Corollary 4.1.3 one obtains:

$$\begin{aligned} E'_i &= T_{w_0}(T_{w_0} E_i) \\ &= T_{w_0}(-F_{\check{i}}\ K_{\check{i}}) \\ &= -(T_{w_0} F_{\check{i}})(T_{w_0} K_{\check{i}}) \\ &= -(-K_i^{-1} E_i)(K_i) \\ &= K_i^{-1} E_i K_i \end{aligned}$$

This proves the second part, and to get the first part set $q = 1$ so that $K_i = 1$. ☐

Theorem 4.2.6. *Let h be any height function and denote the associated simple roots and orientation by Π_h and Ω_h respectively.*

1. *The root basis defined above is adapted to the pair (Π_h, Ω_h).*

2. *For this choice of root basis the Lie bracket is given by:*

$$[E_\alpha, E_\beta] = \begin{cases} (-1)^{\langle\alpha,\beta\rangle} E_{\alpha+\beta} & \textit{for } \alpha+\beta \in R \\ 0 & \textit{for } \alpha+\beta \notin R \textit{ and } \alpha \neq -\beta \end{cases}$$

Proof. Let h be the height function used to construct the root basis $\{E_\alpha\}$. By construction this basis is adapted to the pair (Π_h, Ω_h). So it is enough to check that if $\{E_\alpha\}$ is adapted to (Π, Ω), and i is a source for Ω, then $\{E_\alpha\}$ is also adapted to $(s_i\Pi, s_i\Omega)$.

Suppose that $\{E_\alpha\}$ is adapted to (Π, Ω) and that i is a source. Let $\Pi = \{\alpha_1, \ldots, \alpha_r\}$. Then since i is a source and w_0 is adapted to Ω, the corresponding reduced expression for the longest element has the form $w_0 = s_i s_{i_2} \cdots s_{i_l}$. By writing $\gamma_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$, the longest element can be reexpressed as $w_0 = s_{\gamma_l} \cdots s_{\gamma_2} s_{\alpha_i}$. (Note that since i is a source, $\gamma_1 = \alpha_i$.)

Then since $\{E_\alpha\}$ is adapted it is possible to write

$$E_{\gamma_k} = T_{\gamma_{k-1}} \cdots T_{\gamma_2} T_{\alpha_i} E_{\alpha_k}.$$

Now consider the pair $(s_i\Pi, s_i\Omega)$. Denote the simple roots by $\alpha'_j = s_i\alpha_j$ and the corresponding simple reflections $s'_j = s_i s_j s_i$. Then the corresponding reduced expression for the longest element is $w_0 = s'_{i_2} \cdots s'_{i_l} s_i$ and as before if $\gamma'_k = s'_{i_2} \cdots s'_{i_{k-1}}(\alpha'_{i_k})$, then $\gamma'_k = \gamma_{k+1}$ for $k+1 \neq l$ and $\gamma_l = -\alpha_i$.

Now, if $k+1 \neq l$ then

$$\begin{aligned}
E'_{\gamma_k} &= E_{\gamma_{k+1}} \\
&= T_{\gamma_k} \cdots T_{\gamma_2} T_{\alpha_i} E_{\alpha_{i_{k+1}}} \\
&= T_{\gamma_k} \cdots T_{\gamma_2} E_{s_i \alpha_{i_{k+1}}} \quad \text{by Equation 4.2.4} \\
&= T_{\gamma'_{k-1}} \cdots T_{\gamma'_1} E_{\alpha'_k}
\end{aligned}$$

so E'_{γ_k} is given by Equation 4.2.3.

If $k+1 = l$ then,

$$\begin{aligned}
E'_{\gamma_l} &= E_{-\alpha_i} \\
&= T_{w_0}(E_{\alpha_{\bar{i}}}) \\
&= T_{\gamma_l} \cdots T_{\gamma_2} T_{\alpha_i}(E_{\alpha_{\bar{i}}}) \\
&= T'_{\gamma_{l-1}} \cdots T'_{\gamma_1} E_{s_i \alpha_{\bar{i}}} \\
&= T'_{\gamma_{l-1}} \cdots T'_{\gamma_1} E_{\alpha'_{\bar{i}}}
\end{aligned}$$

so again E'_{γ_k} is given by Equation 4.2.3. Hence $\{E_\alpha\}$ is adapted to the pair $(s_i\Pi, s_i\Omega)$.

The proof of the second part will follow from Corollary 4.3.4. $\square$

Note that Proposition 4.2.4 and Proposition 4.2.6 prove the main result, Theorem 4.0.3.

Define $T_C = T_{\alpha_{i_1}} T_{\alpha_{i_2}} \cdots T_{\alpha_{i_r}}$, for some choice of compatible simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\}$, with $C = s_{i_1} s_{i_2} \cdots s_{i_r}$. Since the T_α satisfy the braid relations, the operator does not depend on the choice of compatible simple roots Π.

Proposition 4.2.7. *The root vectors E_α satisfy $T_C E_\alpha = E_{C\alpha}$.*

4.3 Ringel-Hall Algebras

In this section Ringel and Peng and Xiao's approaches to constructing the Lie algebra $\mathfrak{g}$ from quiver theory is reviewed. This is then related to the construction given in the previous section. For more details on Ringel's construction see [R1], [R2], [DX]. For more details on Peng and Xiao's construction see [PX1] and [PX2].

Let Ω be a fixed orientation of Γ and denote by $\overrightarrow{\Gamma} = (\Gamma, \Omega)$ the corresponding quiver. Fix $\mathbb{K}$, a finite field of order p. Let $\mathrm{Rep}(\overrightarrow{\Gamma})$ be the category of representations of this quiver over the field $\mathbb{K}$, and denote by $\mathcal{K}$ its Grothendieck group. Denote by $\mathrm{Ind} \subset \mathcal{K}$ the set of classes of indecomposable objects. Then Gabriel's Theorem gives an identification $\mathrm{Ind} \to R_+$ between indecomposable objects and positive roots of the corresponding root system. Moreover, if $\langle \cdot, \cdot \rangle$ is defined on $\mathcal{K}$ by $\langle X, Y \rangle = \dim \mathrm{Hom}(X, Y) - \dim \mathrm{Ext}^1(X, Y)$, then the form given by $(X, Y) = \langle X, Y \rangle + \langle Y, X \rangle$ is identified with the bilinear form on the root lattice. The form $\langle \cdot, \cdot \rangle$ is called the Euler form.

Ringel then constructed an associative algebra $(\mathcal{H}_p, *)$ as follows:

1. As a vector space $\mathcal{H}_p$ is spanned by $[M] \in \mathcal{K}$.
2. For objects M, N, L define $F^L_{M,N}(p) = |\{X \subset L | X \simeq M \text{ and } L/X \simeq N\}|$. (Since $\mathbb{K}$ is finite, this number is well-defined.)
3. Define an operation $*$ on $\mathcal{H}_p$ by the formula $[M] * [N] = \sum_{[L]} F^L_{M,N}[L]$.

The following Theorem summarizes the main results of Ringel.

Theorem 4.3.1. *Let $(\mathcal{H}_p, *)$ be the algebra defined above.*

1. *For $n = (n_\alpha) \in (\mathbb{Z}_+)^{R_+}$ set $M_n = \bigoplus n_\alpha M_\alpha$ where M_α is the indecomposable corresponding to root α. Then $\{[M_n]\}$ is a PBW-type basis of the algebra $\mathcal{H}_p$, so that all structure constants $F^{[L]}_{[M],[N]}(p)$ are in $\mathbb{Z}[p]$. Hence the Hall algebra can be considered with p as a formal parameter. After making the substitution $q = p^{1/2}$, $\mathcal{H}_q$ can be identified with $U_q\mathfrak{n}_+$.*

2. *For $q = 1$, this gives an isomorphism $\Psi : U\mathfrak{n}_+ \to \mathcal{H}_1$ which is given by $E_\alpha \mapsto [M_\alpha]$, where M_α denotes the indecomposable representation of $\overrightarrow{\Gamma}$ corresponding to root α. In particular the set $\{[M_\alpha]\}$ is a root basis for $\mathfrak{n}_+$.*

3. *In the case $q = 1$, the Lie bracket $[\cdot, \cdot]$ is given by*

$$[M_\alpha, M_\beta] = (-1)^{\langle M_\alpha, M_\beta\rangle} M_{\alpha+\beta}$$

   for $\alpha + \beta \in R$. Here $\langle \cdot, \cdot\rangle$ is the Euler form.

The polynomial $F^L_{M,N}(p)$ appearing in Part 1 of the Theorem is called the "Hall polynomial".

As mentioned before, Peng and Xiao extended the results of Ringel to obtain a description for all of $\mathfrak{g}$. This construction is briefly recalled here. For a more details see [PX1] , [PX2].

Peng and Xiao considered the "root category", $\mathcal{D} = \mathcal{D}^b(\overrightarrow{\Gamma})/T^2$, so that indecomposable objects are in bijection with all roots. If $M \in \mathrm{Rep}(\overrightarrow{\Gamma})$ is indecomposable, then considering this as a complex concentrated in degree 0, M is also indecomposable in $\mathcal{D}$. These objects correspond to positive roots, while their translates, TM, correspond to negative roots. (Up to isomorphism, this is a full description of indecomposable objects in $\mathcal{D}$.) Denote by M_α the class of indecomposable corresponding to root $\alpha \in R_+$. Peng and Xiao then constructed a Lie algebra $\mathcal{H}_\mathcal{D}$ as follows:

1. Set $\mathcal{H}_\mathcal{D} = \mathfrak{N} \oplus \mathfrak{H}$ where $\mathfrak{H} = \mathcal{K}(\mathcal{D})$ and $\mathfrak{N}$ is the free abelian group with basis $\{u_{[M]}\}$ indexed by isomorphism classes of objects $[M]$.

2. Let $h_M = [M] \in \mathcal{K}(\mathcal{D})$.

3. Define a bilinear operation $[\cdot, \cdot]$ on $\mathcal{H}_\mathcal{D}$ by:

(a) $[\mathfrak{H}, \mathfrak{H}] = 0$

(b) $[u_M, u_N] = \sum_{[L]}(F^L_{M,N}(1) - F^L_{N,M}(1))u_L$, for $N \neq TM$, where $F^L_{M,N}$ is the Hall polynomial.

(c) $[u_M, u_{TM}] = \frac{h_M}{d(M)}$ where $d(M) = \dim \mathrm{End}(M)$

(d) $[h_M, u_N] = -(M, N)_{\mathcal{D}} u_N = -[u_N, h_M]$ where $(\cdot, \cdot)_{\mathcal{D}}$ is the symmetrized Euler form on $\mathcal{K}(\mathcal{D})$.

4. For $\alpha \in R_+$, let $h_\alpha = h_M$ where $\dim M_\alpha = \alpha$.

5. For $\alpha \in R_+$, let $M_\alpha = [M_\alpha]$ where $\dim M_\alpha = \alpha$.

6. For $\alpha \in R_+$ let $M_{-\alpha} = -[TM_\alpha]$ where $\dim M_\alpha = \alpha$.

Theorem 4.3.2. *Let $(\mathcal{H}_{\mathcal{D}}, [\cdot, \cdot])$ be defined as above.*

1. *$(\mathcal{H}_{\mathcal{D}}, [\cdot, \cdot])$ is a Lie algebra.*

2. *The collection $\{M_\alpha, M_{-\alpha}\}_{\alpha \in R_+}$ defined above is a root basis for $\mathcal{H}_{\mathcal{D}}$.*

3. *The map given by $E_\alpha \mapsto M_\alpha$, $F_\alpha \mapsto M_{-\alpha}$ and $H_\alpha \mapsto h_\alpha$ for $\alpha \in R_+$ induces an isomorphism of Lie algebras $\mathfrak{g} \to \mathcal{H}_{\mathcal{D}}$. Hence $\mathcal{H}_{\mathcal{D}}$ can be identified with the $\mathbb{Z}$-form of $\mathfrak{g}$.*

For details see [PX1] Section 4.

Recall that given a height function h, there is a corresponding set of simple roots Π_h and a polarization $R = R^h_+ \cup R^h_-$. Let E_α be the root vectors defined in Section 4.2. Define a triangular decomposition $\mathfrak{g} = \mathfrak{n}^h_- \oplus \mathfrak{h} \oplus \mathfrak{n}^h_+$ by setting $\mathfrak{n}^h_\pm = \langle E_\alpha \rangle_{\alpha \in R^h_\pm}$.

A height function h also gives an orientation Ω_h of Γ and hence a quiver $\overrightarrow{\Gamma} = (\Gamma, \Omega_h)$. As above, denote by $\mathcal{K}$ the corresponding Grothendieck group, and by $\mathrm{Ind} \subset \mathcal{K}$ the set of indecomposable classes in $\mathcal{K}$. Then there is a bijection $R^h_+ \to \mathrm{Ind}$, given by $\alpha \mapsto [M_\alpha]$.

Proposition 4.3.3. *Let h be a height function. Then the identification $R^h_+ \to \mathrm{Ind}$ in $\mathrm{Rep}(\overrightarrow{\Gamma})$ induces an isomorphim $U\mathfrak{n}^h_+ \to \mathcal{H}_1$ given by $E_\alpha \mapsto [M_\alpha]$.*
Moreover, the identification $R \to \mathrm{Ind}(\mathcal{D})$ in the root category $\mathcal{D}$ gives an isomorphism $\mathfrak{g} - \mathfrak{h} \to \mathcal{H}_{\mathcal{D}} - \mathfrak{H}$, given by $E_\alpha \mapsto [M_\alpha]$, $E_{-\alpha} \mapsto -[TM_\alpha]$ for $\alpha \in R^h_+$.

Corollary 4.3.4. *The Lie algebra* $\mathfrak{g}$ *can be realised combinatorially in terms of* $\widehat{\Gamma}_{cyc}$*: It has root basis* E_α *for* $\alpha \in \widehat{\Gamma}_{cyc}$ *and Lie bracket given by*

$$[E_\alpha, E_\beta] = \begin{cases} (-1)^{\langle \alpha,\beta \rangle} E_{\alpha+\beta} & \textit{for } \alpha+\beta \in R \\ 0 & \textit{for } \alpha+\beta \notin R \textit{ and } \alpha \neq -\beta \end{cases} \tag{4.3.1}$$

Proof. The only thing to be checked is that in terms of the E_α constructed in Section 4.2, the structure constants of the Lie bracket are given by Equation 4.3.1. For $\alpha, \beta \in R$ with $\alpha \neq -\beta$ there is a choice of compatible simple roots Π so that $\alpha, \beta \in R_+^\Pi$. Let h be the corresponding height function. Then by Proposition 4.3.3 the identification $U\mathfrak{n}_+^h \simeq \mathcal{H}_1$ gives that

$$[E_\alpha, E_\beta] = [M_\alpha, M_\beta] = (-1)^{\langle \alpha,\beta \rangle} M_{\alpha+\beta} = (-1)^{\langle \alpha,\beta \rangle} E_{\alpha+\beta}.$$

□

Remark 4.3.5. Note that the "Euler cocylce" $(-1)^{\langle \cdot,\cdot \rangle}$, defines a cohomologous cocycle, and hence the same extension, as in the construction of $\mathfrak{g}$ given in [FLM].

Example 4.3.6. Consider the case $\Gamma = A_3$, so that $\mathfrak{g} = \mathfrak{sl}_4$. Let $\mathfrak{h}$ be the diagonal matrices. Then the roots are $\alpha = e_i - e_j$ for $i \neq j$, where $e_k(h) = h_{kk}$ for $h \in \mathfrak{h}$. The root space corresponding to root $e_i - e_j$ is $\mathbb{C}E_{ij}$, where E_{ij} is the corresponding matrix unit. For two different choices of Coxeter element C, two different root bases are shown in Figure 4.2. In each case the Lie bracket is then given by the Equation 4.3.1 and the form $\langle \cdot, \cdot \rangle$ can be computed explicitly in terms of $\widehat{\Gamma}_{cyc}$.

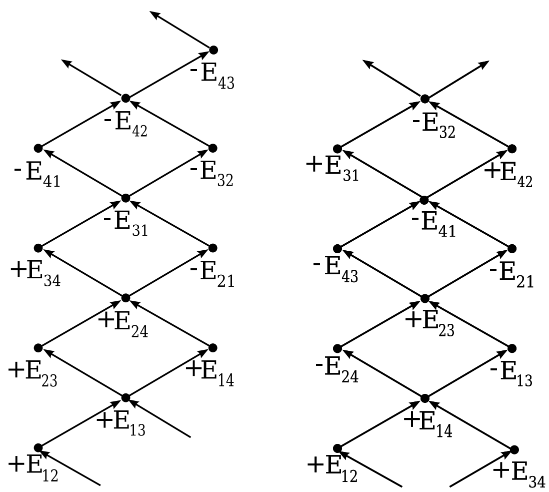

Figure 4.2: Two different root bases for $\mathfrak{sl}_4$ coming from different choices of C. The case $C = (1234)$ is shown in the figure to the left. The case $C = (1243)$ is shown in the figure to the right. For each vertex in $\widehat{\Gamma}_{cyc}$ the corresponding root vector E_α is shown in terms of the matrix units E_{ij}. Recall that $\widehat{\Gamma}_{cyc}$ is periodic, so that arrows leaving the top level are identified with the incoming arrows on the bottom level.

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