The Effective Cone on Symmetric Powers of Curves

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Abstract of the Dissertation

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The dth symmetric power C_d of a smooth complex projective curve (or compact Riemann surface) C is a parameter space for effective divisors of degree d on C, so that the theory of degree-d maps from C to projective space is encoded in the subvarieties of C_d and the relations amongst them. We give a complete description of the cone of codimension-1 subvarieties of C_{g-1} when C is a general curve of genus $g \geq 4$, as well as new bounds for the case C_d in the range $(\frac{g+1}{2}, g-2]$. We also give new information on the movable cone of C_d and the volume function of C_{g-1}.

Dedication

In memory

of

Phyllis Gentle,

Loretta Osipow,

and

Shannon Sandel

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1. Introduction

Given a smooth complex projective algebraic curve C and an integer $d \geq 2$, the *d-th symmetric power* C_d of C is the quotient of the $d-$th Cartesian power C^d by the natural action of the symmetric group $\mathcal{S}_d$. This is a $d-$dimensional smooth projective variety (Proposition 2.2), and it is a fine moduli space parametrizing effective divisors of degree d on C (e.g. Theorem 16.4 in [26]). As such, the information about degree-d linear series on C is encoded in the subvarieties of C_d and the relations amongst them. The primary focus of this work is the codimension-1 subvarieties of C_d.

Cones of divisor classes on C_d have been studied by several authors. Their study began in earnest with the work of Kouvidakis, who in [16] obtained bounds for the effective cone of C_d in the case where C is very general (see Section 2 for the definition). These bounds were obtained by degenerating to smooth curves possessing linear series which are special in the sense of Brill-Noether.

Ciliberto-Kouvidakis [6] and Ross [27] recovered the bounds for the effective cone of C_2 in the case $g \geq 9$ from [16] by degenerating to rational nodal curves. The case $2 \leq g \leq 4$ was completely settled in Theorem 2 of [16], and the best bounds currently known for the remaining genera are due to Debarre for $6 \leq g \leq 8$ (Proposition 8 in [8]) and Ross for $g = 5$ (Section 4 of [27]).

In [23], Pacienza used work of Voisin [28] on curves in K3 surfaces to compute the ample cone of C_k for a very general curve C of genus $2k \geq 6$. In recent unpublished work, Debarre obtained some bounds for the "intermediate" cones on C_d.

The Néron-Severi space $N^1_{\mathbb{R}}(C_d)$ contains two classes x and θ (see Section 2.2 for their definitions) which are linearly independent (Lemma 2.8). In many cases, for instance when C is very general, these classes generate $N^1_{\mathbb{R}}(C_d)$ (Corollary 2.10). Theorem 3 in [16] implies that when $\dim N^1_{\mathbb{R}}(C_d) = 2$, one ray of the effective cone of

C_d is spanned by the fundamental class $2(-\theta+(g+d-1)x)$ of the large diagonal

$$\Delta := \{2p + D' : p \in C, D' \in C_{d-2}\}.$$

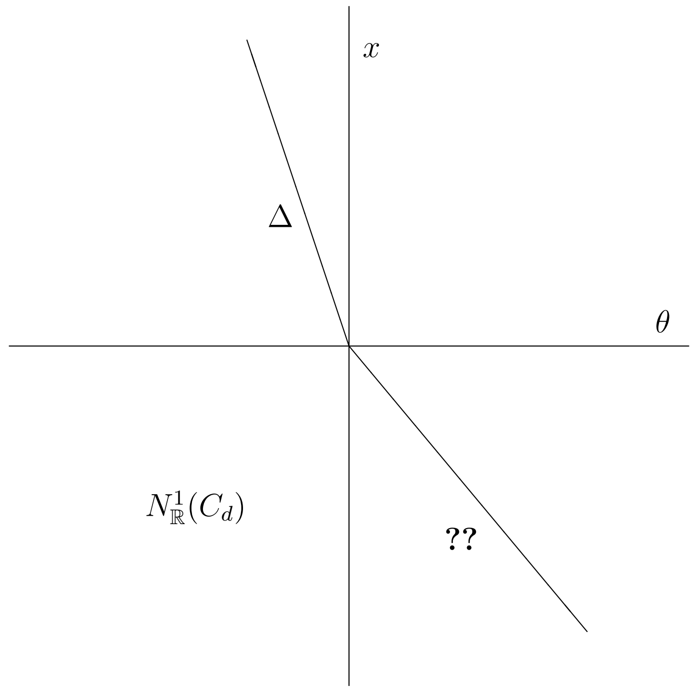

FIGURE 1: The effective cone of C_d.

Part 2 of Theorem 5 in [16] gives the ray spanned by $\theta - x$ as an inner bound for the other part of the effective cone of C_d when the genus g of C is at least 4 and $\frac{g}{2} < d \leq g-1$. We prove the following:

Theorem A. *Let C be a very general curve of genus $g \geq 4$.*

(i) *For $\frac{g+1}{2} < d \leq g-1$, the class $\theta - (1 + \frac{g-d}{g^2-dg+(d-2)})x \in N^1_{\mathbb{R}}(C_d)$ is $\mathbb{Q}-$effective.*

(ii) *The class $\theta - (1+\frac{1}{2g-3})x \in N^1_{\mathbb{R}}(C_{g-1})$ spans a boundary ray of the effective cone of C_{g-1}.*

Combining (ii) with Theorem 3 in [16] yields

Corollary 1.1. *If C is a very general curve of genus $g \geq 4$, the effective cone of C_{g-1} is spanned by the half-diagonal class $-\theta + (2g-2)x$ and the class $\theta - (1+\frac{1}{2g-3})x$.*

We consider Theorem A in light of the bounds given by Theorem 5 of [16] for the effective cone of C_{g-1}. As g tends to infinity, the boundary ray in the fourth quarter of the $(\theta, x)-$plane approaches the ray spanned by $\theta - x$, so that Kouvidakis' inner bound is asymptotically sharp in g relative to the new bound. The ray spanned by $\theta - 2x$ is obtained as an outer bound by degenerating C to a hyperelliptic curve. For any $p \in C$, the class $\theta - x$, which gives the aforementioned inner bound, is represented by the cycle $\Gamma_{g-1}(K_C(-p))$ parametrizing effective divisors of degree $g-1$ subordinate to the complete linear series $|K_C(-p)|$ (see 2.3.2 for the general definition.) Since the dimension of $\Gamma_{g-1}(K_C(-p))$ is the dimension of the linear series $|K_C(-p)|$ (Lemma 2.11) the inner bound in [16] comes from the fact that the canonical series separates 0-jets, i.e. is basepoint-free, and the outer bound is obtained by degenerating to the case in which the canonical linear series fails to separate 1-jets. The divisor we obtain which spans the non-diagonal boundary of the effective cone of C_{g-1} is supported on the set

$$\bigcup_{p \in C} \Gamma_{g-1}(K_C(-2p))$$

and so it reflects very precisely the fact that K_C separates 1-jets. More generally, the divisor we obtain in (i) for C_d in the range $\frac{g}{2} < d \leq g-1$ is supported on

$$\bigcup_{p \in C} \Gamma_d(K_C(-(g-d+1)p)).$$

We obtain more refined information in two distinct, but related, directions. The *stable base locus* $\mathbf{B}(D)$ of a $\mathbb{Q}-$Cartier divisor D on a projective variety X is the set-theoretic intersection of the base loci of the linear systems $|mD|$ (where m varies over all positive integers for which mD is Cartier). The codimension of $\mathbf{B}(D)$ is a rough measure of the size of D; for instance, if D is ample, $\mathbf{B}(D) = \emptyset$, and if D is not $\mathbb{Q}-$effective, $\mathbf{B}(D) = X$;

While the stable base locus is not a numerical invariant in general, we may define a class $\xi \in N^1_{\mathbb{Q}}(X) := H^{1,1}(X) \cap H^2(X, \mathbb{Q})$ to be *stable* if $\mathbf{B}(D) = \mathbf{B}(D')$ for any two divisors D, D' having numerical class ξ (stability for classes in $N^1_{\mathbb{R}}(X)$ will be defined later). Note that by the Nakai-Moishezon criterion (Theorem 3.1) the class of any ample divisor on X is stable.

Theorem B. *Let C be a very general curve of genus $g \geq 4$, and let d be an integer satisfying $\frac{g+1}{2} < d \leq g-1$. Then the class $\theta - x$ on C_d is stable with stable base locus*

$$\{D \in C_d : \dim |D| \geq 1\}$$

of codimension $g - d + 1$.

We now turn to another way of measuring the size of a Cartier divisor. The *volume* of a Cartier divisor D on a projective variety X of dimension n is defined as

$$\mathrm{vol}_X(D) := \limsup_{m \to \infty} \frac{n! h^0(\mathcal{O}_X(mD))}{m^n}$$

Recall that D is *big* if $h^0(\mathcal{O}_X(mD)) = O(m^n)$; it follows from the definitions that $\mathrm{vol}_X(D) > 0$ precisely when D is big. Note that when D is ample, asymptotic Riemann-Roch (e.g. Theorem 1.1.24 in [17]) and Serre vanishing imply that

$$h^0(\mathcal{O}_X(mD)) = \frac{D^n}{n!} m^n + \mathcal{O}(m^{n-1}).$$

Consequently, $\mathrm{vol}_X(D) = D^n$ for any ample divisor D.

Bigness may be thought of as a birational version of amplitude, in the sense that for any birational map $f : X \dashrightarrow Y$ of projective varieties and any ample divisor D on Y, the pullback $f^*(D)$ is big on X. Similarly, the volume of a big divisor may be thought of as the birational version of the top self-intersection of an ample divisor. For example, if $f : X \to Y$ is a birational *morphism* of projective varieties, and D is a Cartier divisor on Y, then $\mathrm{vol}_Y(D) = \mathrm{vol}_X(f^*(D))$ (Proposition 3.17). Furthermore, $\mathrm{vol}_X(D)$ depends only on the numerical class of D (Proposition 3.15) and extends to a continuous function $\mathrm{vol}_X : N^1_{\mathbb{R}}(X) \to [0, \infty)$ which is homogeneous of degree

n (Proposition 3.16). We obtain the following partial computation of the volume function:

Theorem C. *Let C be a general curve of genus $g \geq 4$. Then for $t \in [0, 1 + \frac{1}{g^2-g-1}]$,*

$$vol_{C_{g-1}}(\theta - tx) = \sum_{k=0}^{g-1} \binom{g-1}{k} \frac{g!}{(k+1)!} t^k (1-t)^{g-1-k}.$$

This result, together with (ii) of Theorem 1, implies that for a general curve C of genus $g \geq 4$,

$$\mathrm{vol}_{C_{g-1}}(\theta - x) = 1$$

$$\mathrm{vol}_{C_{g-1}}(\theta - (1 + \frac{1}{2g-3})x) = 0$$

For any two positive integers d and r, there exist a determinantal subvariety $W_d^r(C)$ of $\mathrm{Pic}^d(C)$ supported on the set

$$\{\mathcal{L} \in \mathrm{Pic}^d(C) : \dim |\mathcal{L}| \geq r\}$$

(Section 3 of Chapter IV in [1]) and a fine moduli variety $G_d^r(C)$ parametrizing linear series of degree d and dimension r (Theorem 3.6 in [1]). It follows from Theorem 16.4 in [26] that $G_d^0(C)$ and C_d are canonically isomorphic; accordingly, we will denote $W_d^0(C)$ by $W_d(C)$.

The residuation map $\tilde{\tau} : \mathrm{Pic}^{2g-2-d}(C) \to \mathrm{Pic}^d(C)$ defined by $\tilde{\tau}(\mathcal{L}) = K_C \otimes \mathcal{L}^{-1}$ restricts to an isomorphism $W_{2g-2-d}^{g-d-1}(C) \simeq W_d(C)$. The rational map $\tau : G_{2g-2-d}^{g-d-1}(C) \dashrightarrow G_d^0(C) = C_d$ to which this isomorphism lifts via the natural forgetful maps $C_d \to W_d(C)$ and $G_{2g-2-d}^{g-d-1}(C) \to W_{2g-2-d}^{g-d-1}(C)$ is the crux of the proofs of our main results.

If C is general, then by Gieseker's theorem (1.6 on p.214 of [1]) $G_{2g-2-d}^{g-d-1}(C)$ is smooth of dimension d, and by the Brill-Noether theorem the loci of indeterminacy of τ and τ^{-1} are both of codimension 2 or greater. Since τ and τ^{-1} fail to be defined at points parametrizing incomplete linear series, τ is an isomorphism precisely when C

does not possess a degree-d linear series of positive dimension. We may then conclude the following from Hartogs' Theorem:

- τ induces an isomorphism

$$\mathrm{Pic}(C_d) \simeq \mathrm{Pic}(G^{g-d-1}_{2g-2-d}(C)).$$

- For any line bundle $\mathcal{L}$ on C_d, τ induces an isomorphism

$$H^0(C_d, \mathcal{L}) \simeq H^0(G^{g-d-1}_{2g-2-d}(C), \tau^*\mathcal{L}).$$

It follows from the first item that τ induces an isomorphism

$$N^1_{\mathbb{R}}(C_d) \simeq N^1_{\mathbb{R}}(G^{g-d-1}_{2g-2-d}(C))$$

of Néron-Severi spaces, and it follows from the second item that for any line bundle $\mathcal{L}$ on C_d,

$$\mathrm{vol}_{C_d}(\mathcal{L}) = \mathrm{vol}_{G^{g-d-1}_{2g-2-d}(C)}(\tau^*\mathcal{L}).$$

For any projective variety X, the closure of the big cone in $N^1_{\mathbb{R}}(X)$ is equal to the closure of the effective cone in $N^1_{\mathbb{R}}(X)$ This, together with the continuity of the volume function implies that $\tau^* : N^1_{\mathbb{R}}(C_d) \to N^1_{\mathbb{R}}(G^{g-d-1}_{2g-2-d}(C))$ maps the big (resp. effective) cone of C_d to the big (resp. effective) cone of $G^{g-d-1}_{2g-2-d}(C)$.

In the special case $d = g-1$, the map τ is a birational involution on C_{g-1}, and τ^* interchanges the two boundary rays of the effective cone of C_{g-1}. As mentioned earlier, Theorem 3 of [16] implies that one boundary ray is spanned by the class of the large diagonal; thus the other ray is determined once we compute τ^* explicitly (Proposition 4.4).

When $\lceil \frac{g}{2} + 1 \rceil \leq d \leq g-1$, the pullback via τ of the divisor on C_d obtained in (i) of Theorem 1 is supported on the set

$$\bigcup_{p \in C} \{(\mathcal{L}, V) \in G^{g-d-1}_{2g-2-d}(C) : V \cap H^0(\mathcal{L}(-(g-d+1)p)) \neq 0\}$$

of all linear series possessing a ramification point of degree $g-d+1$, i.e. the mildest possible ramification. This is a natural generalization of the large diagonal on C_d.

While the proof of Theorem 1 is a bit more involved, its main features can be outlined easily enough. One important fact is that τ^* and $(\tau^{-1})^*$ map certain stable classes in one Néron-Severi space to certain stable classes in the other (Proposition 4.7). For any effective divisor D of degree $g-d$ on C, the Plücker divisor

$$\widehat{X}_D := \{(\mathcal{L}, V) \in G^{g-d-1}_{2g-2-d}(C) : V \cap H^0(\mathcal{L}(-D)) \neq 0\}$$

is ample (Proposition 4.2), and therefore stable. Our explicit calculation of $(\tau^{-1})^*$ (Proposition 4.4) shows that it takes the class of $\widehat{X}_D$ to $\theta - x$.

The study of cones of divisor classes on C_d for an arbitrary curve C is quite difficult. This is partly because special linear series on arbitrary curves are poorly understood. However, special linear series on hyperelliptic curves are understood quite well, and as such they are a natural first step in treating the case of an arbitrary curve. Debarre [8] and Kong [15] have shown that the class $\theta - (g-1)x$ on C_2 spans a boundary ray of the effective cone of C_2 *precisely* when C is hyperelliptic.

By Corollary 2.10, for a general hyperelliptic curve C the Néron-Severi rank of C_d is 2 for all $d \geq 2$. The ray spanned by $\theta - (g-d+1)x$ was given as an inner bound for the effective cone of C_d for $2 \leq d \leq g$ in the unpublished notes of Debarre, and it was shown in Theorem 5 in [16] that this bound is sharp when $d = g-1$ or $d = g$. We prove the following:

Theorem D. *Let C be a general hyperelliptic curve of genus g, and let $2 \leq d \leq g$.*

(i) *The effective cone of C_d is spanned by the half-diagonal class $-\theta + (g+d-1)x$ and the class $\theta - (g-d+1)x$.*

(ii) *For all $t \in [0, g-d+1]$,*

$$vol_{C_d}(\theta - tx) = \frac{g!}{(g-d)!}(1 - \frac{t}{g-d+1})^d$$

Combining Theorems C and D yields

Corollary 1.2. *Let C be a curve of genus $g \geq 4$. Then*

$$vol_{C_{g-1}}(\theta - x) = \begin{cases} 1 & \text{if } C \text{ is general and nonhyperelliptic;} \\ \frac{g!}{2^{g-1}} & \text{if } C \text{ is general hyperelliptic.} \end{cases}$$

We conclude this introduction with some remarks about other classes of curves. By the Mumford-Martens classification (Theorems 5.1 and 5.2 in Chapter IV of [1]) the non-hyperelliptic curves having Brill-Noether loci of the largest possible dimension are the trigonal, smooth plane quintic, and bielliptic curves. Since these are in some sense the "least general" non-hyperelliptic curves, it would be interesting to extend our study to their symmetric powers.

If C is a general trigonal curve or a general plane quintic, it follows from Theorem 2.10 that the Néron-Severi rank of C_d is 2. The following are consequences of the proof of Theorem 5 in [16]:

- If C is a general trigonal curve of genus $g \geq 5$, the class $\theta - 2x$ spans a boundary ray of the effective cone of C_{g-2}.
- If C is a general plane quintic, the class $\theta - 3x$ spans a boundary ray of the effective cone of C_3.

The study of the case where C is bielliptic is complicated by the fact that the Néron-Severi rank of C_d is at least 3. Debarre has shown in [8] that for any non-hyperelliptic curve C of genus $g \geq 5$, the class $\theta - 2x$ on C_2 is nef, and that it fails to be ample *precisely* when C is bielliptic. As far as we know, there are currently no results on the portion of the ample or effective cone lying outside the $(\theta, x)-$plane.

2. Preliminaries on C_d

Conventions: We work over the field of complex numbers. C will always denote a smooth, connected projective curve. The linear series on C given by a space V of global sections of a line bundle $\mathcal{L}$ on C will be denoted by $(\mathcal{L}, V)$. The term $\mathfrak{g}^r_d$ will sometimes be used to refer to a linear series of dimension r and degree d. The empty set has dimension -1.

Definition: If $\mathcal{Z}$ is a subvariety of the moduli space $\mathcal{M}_g$ of smooth projective curves of genus g, we say that a property holds for **a general curve of** $\mathcal{Z}$ if it holds on the complement of the union of countably many subvarieties of $\mathcal{Z}$. If $\mathcal{Z} = \mathcal{M}_g$, we say that the property holds *for a general curve of genus* g.

2.1. **Definition of C_d and basic properties.** Let X be a projective variety. For any integer $d \geq 2$, the symmetric group $\mathfrak{S}_d$ acts on the dth Cartesian power X^d by permuting the factors.

Proposition 2.1. *For any $d \geq 1$, the quotient variety $X_d = X^d/\mathfrak{S}_d$ is projective.*

Proof. By the Theorem on p.66 of [21], X_d has the structure of a variety. Let $\mathcal{O}_X(1)$ be an ample line bundle on X. Then the line bundle

$$\widetilde{\mathcal{O}}(1) := \bigotimes_{\rho \in \mathfrak{S}_d} \rho^* \mathcal{O}_X(1)$$

is $\mathfrak{S}_d$−invariant and ample on X^d. Consequently $\widetilde{\mathcal{O}}(1)$ descends via the quotient map $\pi : X^d \to X_d$ to a line bundle $\mathcal{O}(1)$ on X_d, which is ample by Chevalley's theorem. □

We now specialize to the case where C is a smooth projective curve.

Proposition 2.2. *C_d is smooth.*

Proof. This is a consequence of the fundamental theorem of symmetric polynomials. $\square$

Note that this is false if we replace C by a smooth surface.

We now introduce a class of morphisms that will prove useful in the next section. If $d_1, ..., d_k$ is a partition of d with $d_1 \geq ... \geq d_k$, then the Cartesian product $C_{d_1} \times ... \times C_{d_k}$ is a quotient of $C^d \simeq C^{d_1} \times ... \times C^{d_k}$ by the natural action of $\mathcal{S}_{d_1} \times ... \times \mathcal{S}_{d_k}$. Since the latter group embeds into $\mathcal{S}_d$ compatibly with the action of $\mathcal{S}_d$ on C^d, the addition map

$$\sigma_{d_1,...,d_k} : C_{d_1} \times ... \times C_{d_k} \to C_d$$

$$(D_1, ..., D_k) \mapsto D_1 + ... + D_k$$

is a finite morphism of smooth projective varieties whose degree is the multinomial coefficient $\binom{d}{d_1 ... d_k}$. In particular, $\sigma_{1,...,1}$ is the canonical quotient map from C^d to C_d.

2.2. **The Néron-Severi Space of C_d.** The *Néron-Severi group* $NS(X)$ of a smooth projective variety X is the additive group of divisors on X modulo algebraic equivalence. (See Exercise 1.7 in Chapter V of [13] for the definition of algebraic equivalence and p.364 in *loc. cit.* for the definition of numerical equivalence.)

Proposition 2.3. *For any $d \geq 2$ there is a canonical isomorphism*

$$NS(C_d) \simeq \mathbb{Z} \oplus NS(J(C)).$$

Proof. Since $H^2(C_d, \mathbb{Z})$ is torsion-free (e.g. [19]) for all $d \geq 1$, algebraic and numerical equivalence of divisors coincide on C_d. By descent, pullback via the quotient map induces an isomorphism of $\mathrm{Pic}(C_d)$ with the $\mathfrak{S}_d-$invariant part of $\mathrm{Pic}(C^d)$.

By the Lefschetz theorem on (1,1)-classes, the exponential sequence on C^d gives rise to the $\mathfrak{S}_d-$equivariant exact sequence

$$0 \to \mathrm{Pic}^0(C^d) \to \mathrm{Pic}(C^d) \to H^2(C^d, \mathbb{Z}) \cap H^{1,1}(C^d) \to 0$$

so that $NS(C_d) = (H^2(C^d, \mathbb{Z}) \cap H^{1,1}(C^d))^{\mathfrak{S}_d} = H^2(C^d, \mathbb{Z})^{\mathfrak{S}_d} \cap H^{1,1}(C^d)$.

The $\mathfrak{S}_d-$invariant part of the Künneth decomposition of $H^2(C^d, \mathbb{Z})$ is

$$H^2(C^d, \mathbb{Z})^{\mathfrak{S}_d} \simeq H^2(C, \mathbb{Z}) \oplus \mathrm{Sym}^2 H^1(C, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathrm{Sym}^2 H^1(C, \mathbb{Z}).$$

The copy of $H^2(C, \mathbb{Z})$ sitting in $H^2(C^d, \mathbb{Z})^{\mathfrak{S}_d}$ is generated by the sum of the fiber classes associated to the projections of C^d onto its d factors, so it is contained in $H^{1,1}(C^d)$. Because of the principal polarization, we may identify $\mathrm{Sym}^2 H^1(C, \mathbb{Z})$ with the additive group $\mathrm{End}^s(H^1(C, \mathbb{Z}))$ of endomorphisms of $H^1(C, \mathbb{Z})$ symmetric with respect to the polarization. The (1,1)-part of this is the group of Hodge structure endomorphisms of $H^1(C, \mathbb{Z})$ symmetric with respect to the polarization, which is in turn isomorphic to $NS(J(C))$ by Proposition 5.2.1 in [3]. $\square$

Definition: The **Néron-Severi space** $N^1_{\mathbb{R}}(X)$ of a smooth projective variety X is $NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

As an immediate consequence of Proposition 2.3, we have

Corollary 2.4. *For any integer $d \geq 2$ and any C, $N^1_{\mathbb{R}}(C_d) \simeq \mathbb{R} \oplus N^1_{\mathbb{R}}(J(C))$.* $\square$

We now introduce the basic classes of divisors on C_d.

Definition: For a given $p \in C$ and $d \geq 2$, X_p is the unique reduced effective divisor on C_d supported on the set $\sigma_{d-1,1}(C_{d-1} \times \{p\})$.

Lemma 2.5. *For all $p \in C$, the divisor X_p on C_d is ample.*

Proof. Fix a point $p \in C$ and an integer $d \geq 2$. For $i = 1, .., .d$, let $\pi_i : C^d \to C$ be projection onto the ith factor, and let $\pi : C^d \to C_d$ be the canonical quotient map. Then

$$\pi^*(X_p) = {\pi_1}^*(p) + ... + {\pi_d}^*(p)$$

Since this divisor is ample, it follows from Corollary 1.2.28 on p.34 of [17] that X_p is ample. $\square$

Remark: This result is generalized in Lemma 4.2.

The natural map $\mathrm{Div}(C) \to \mathrm{Pic}(C)$ which maps each divisor on C to its associated line bundle restricts to the degree-d part of each group, giving a map $\mathrm{Div}^d(C) \to \mathrm{Pic}^d(C)$. If we compose this with the natural inclusion $C_d \hookrightarrow \mathrm{Div}^d(C)$, we get the morphism

$$a_d : C_d \to \mathrm{Pic}^d(C)$$

$$D \mapsto \mathcal{O}_C(D)$$

which will be referred to as the **Abel map.** We will use this term to refer to both $a_d : C_d \to \mathrm{Pic}^d(C)$ and the map of a_d onto its image $W_d(C) \subseteq \mathrm{Pic}^d(C)$.

For each $\mathcal{L} \in \mathrm{Pic}^d(C)$, the fibre $a_d^{-1}(\mathcal{L})$ is equal the complete linear system $\mathbb{P}(H^0(\mathcal{L}))$. Furthermore, the differential of the Abel map at $D \in C_d$ is given by the coboundary map $H^0(\mathcal{O}_D(D)) \to H^1(\mathcal{O}_C)$ induced by the short exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(D) \to \mathcal{O}_D(D) \to 0.$$

We can summarize this discussion as follows:

Proposition 2.6. *Let C be a curve and let $d \geq 2$ be an integer. Then $a_d : C_d \to W_d(C)$ is*

(i) *an immersion at $D \in C_d$ precisely when $h^0(\mathcal{O}_C(D)) = 1$.*

(ii) *a submersion at $D \in C_d$ precisely when $h^1(\mathcal{O}_C(D)) = 0$.*

(iii) *an isomorphism precisely when C does not possess a degree-d linear series of positive dimension.* $\square$

Note that by Riemann-Roch, if $d \geq 2g - 1$ the map a_d exhibits C_d as a $\mathbb{P}^{d-g}$−bundle over $\mathrm{Pic}^d(C)$.

If the genus of C is at least 1, then the divisors X_p and X_q are not linearly equivalent for $p \neq q$. However, varying p over C gives an algebraic, and hence numerical, family of divisors on C_d.

Definition: The class $x \in N^1_{\mathbb{R}}(C_d)$ is the common numerical class of the divisors X_p on C_d.

Similarly, pulling the translates of the theta-divisor on $J(C)$ back to C_d via the Abel map gives another algebraic family of divisors on C_d.

Definition: The class $\theta \in N^1_{\mathbb{R}}(C_d)$ is the numerical class of the pullback of a theta-divisor on $J(C)$.

Lemma 2.7. *For $2 \leq d \leq g$ and $0 \leq k \leq d$, the intersection product $x^{d-k}\theta^k$ is equal to $\frac{g!}{(g-k)!}$.*

Proof. Let $q_1, ..., q_{d-k}$ be general points of C. Then $\bigcap_{i=1}^{d-k} X_{q_i} \subseteq C_d$ has numerical class x^{d-k} and is isomorphic to C_k. Its image under the Abel map $C_d \to W_d(C)$ is the translate of $W_k(C)$ by $\mathcal{O}_C(q_1 + ... + q_{d-k})$. By the Poincare Formula (p.25 in [1]), the class w_k of any translate of $W_k(C)$ in $\text{Pic}^d(C)$ is $\frac{1}{(g-k)!} \cdot \theta^{g-k}$, so $x^{d-k}\theta^k$ is equal to the intersection $w_k \cdot \Theta^k$ on $\text{Pic}^d(C)$, which in turn is equal to $\frac{g!}{(g-k)!}$. $\square$

Lemma 2.8. *If C is a curve of genus $g \geq 1$, then for all $d \geq 2$ and all $e \in \{1, ..., d-1\}$ the classes $x^i\theta^{e-i}$ $(0 \leq i \leq e)$ are linearly independent.*

Proof. See Exercise B-1 on p.328 of [1]. $\square$

So far we have seen that the Neron-Severi rank of C_d is always at least 2. We will use a result of Pirola (which refines an earlier result of Lefschetz) to show that in the cases of interest to us, it is exactly 2. Recall that by the Torelli theorem (e.g. p.245 in [1]) the assignment of a curve C of genus g to its polarized Jacobian $(J(C), \Theta)$ gives an isomorphism of $\mathcal{M}_g$ with the Jacobian locus $\mathcal{J}_g$ in the moduli space $\mathcal{A}_g$ of $g-$dimensional principally polarized abelian varieties.

Proposition 2.9. *((ii) of Proposition 3.4 in [25]) Let $g \geq 2$ be an integer, and let Y be a subvariety of codimension $\leq g-2$ in $\mathcal{J}_g$. Then the rank of the Neron-Severi group of a general point of Y is 1.*

Remark: Proposition 3.4 in *loc. cit.* says that the statement is true if we replace $\mathcal{J}_g$ by the locus of genus-g hyperelliptic Jacobians, $g-$dimensional abelian varieties with fixed polarization type, or $g-$dimensional Prym varieties.

Corollary 2.10. *If C is general among curves of genus $g \geq 2$ possessing a $\mathfrak{g}^1_e$ (where $e \geq 2$), then for all $d \geq 2$, the Néron-Severi rank of C_d is 2. In particular, this is true of both the general curve of genus g and the general hyperelliptic curve of genus g.*

Proof. Since the dimension of the $d-$gonal locus in $\mathcal{M}_g$ is $\min\{3g-3, 2g+2d-5\}$, its image under the Torelli embedding is of codimension $\max\{0, g-2d+2\}$ in the Jacobian locus J_g. So the result follows immediately from Propositions 2.3 and 2.9. $\square$

2.3. **Some important algebraic cycles on C_d.** From this point on, all references to the "second quarter" and "the fourth quarter" refer to the second and fourth quarters of the $(\theta, x)-$plane, respectively.

2.3.1. *Diagonal Loci.* Here we introduce the loci in C_d which parametrize the various types of nonreduced divisors of degree d on C.

Definition: Let $d \geq 3$ be an integer, and let $(n_1, ..., n_k)$ and $a := (a_1, ..., a_k)$ be $k-$tuples of positive integers satisfying $n_1 \geq n_2 \geq ... \geq n_k$ and $\sum_{j=1}^{k} a_j n_j = d$. Define the map

$$\phi_a : C_{n_1} \times ... \times C_{n_k} \to C_d$$

$$\phi_a((D_1, ..., D_k)) = \sum_{j=1}^{k} a_j D_j.$$

Then $\Delta_{a_1,...,a_k} := (\phi_a)_*(C_{n_1} \times ... \times C_{n_k})$ is **the diagonal locus associated to a.**

Note that $\Delta_{2,1,..,1}$ is the large diagonal Δ we have defined earlier. We will refer to the curve Δ_d as the "small diagonal" of C_d.

Remark: In the sequel, the phrase *small diagonal* will be used to refer to both the locus Δ_d in C_d ands it pullback to the Cartesian power C^d via the natural quotient map.

2.3.2. *The loci of divisors subordinate to a given linear series.* The cycles we define in this subsection and the next play an essential role in our main results.

Let $d \geq 2$ be an integer, and let $(\mathcal{L}, V)$ be a linear series of degree n and dimension r on C. Recall from [1] that there is a natural rank-d vector bundle $E_{\mathcal{L}}$ on C_d whose fibre over $D \in C_d$ is $H^0(D, \mathcal{L}|_D)$. As such, there is a morphism

$$\alpha_V : V \otimes \mathcal{O}_{C_d} \to E_{\mathcal{L}}$$

whose fibre over each $D \in C_d$ is the restriction map $V \to H^0(D, \mathcal{L}|_D)$. The latter fails to be injective precisely when D is subordinate to $(V, \mathcal{L})$, i.e. when $V \cap H^0(\mathcal{L}(-D)) \neq 0$.

Definition: The cycle $\Gamma_d(V, \mathcal{L})$ is the degeneracy locus of α_V. If $(V, \mathcal{L})$ is complete, we will write $\Gamma_d(\mathcal{L})$ instead.

Note that $\Gamma_d(V, \mathcal{L})$ is supported on the set

$$\{D \in C_d : V \cap H^0(\mathcal{L}(-D)) \neq 0\}.$$

The following result computes the fundamental class of $\Gamma_d(V, \mathcal{L})$; we refer to p.342 of [1] for the proof.

Lemma 2.11. *(3.2 on p. 342 of* [1]*) Let C be a curve of genus g, and let $n, d,$ and r be integers satisfying $n \geq d \geq r$. Then $\Gamma_d(V, \mathcal{L})$ is $r-$dimensional, and its fundamental class is*

$$\sum_{k=0}^{d-r} \binom{n-g-r}{k} \frac{x^k \theta^{d-r-k}}{(d-r-k)!}.$$

2.3.3. *The loci of divisors moving in a linear system of given dimension.* The final class of cycles we will consider are the natural cycles contracted by the Abel map a_d.

Definition: For any curve C and any two positive integers r and d,

$$C_d^r := a_d^{-1}(W_d^r(C)).$$

Note that the support of C_d^r is the set

$$\{D \in C_d : \dim |D| \geq r\}.$$

Part (iii) of Proposition 2.6 can be restated as saying that a_d is an isomorphism precisely when $C_d^1 = \emptyset$.

2.4. **Diagonal Calculations.** We collect some special cases of Proposition 5.1 on p.358 of [1] that will be used in the sequel.

Proposition 2.12. *Let C be a curve of genus g. The fundamental class of the small diagonal Δ_d in C_d is*

$$dx^{d-2} \cdot \Big(((d-1)g+1)x - (d-1)\theta \Big)$$

Proof. By Proposition 5.1 on p.358 of [1], this class is

$$\sum\nolimits_{0 \leq \beta \leq \alpha \leq d-1} \frac{(-1)^{\alpha+\beta}}{\beta!(\alpha-\beta)!} \Big(d(\beta+1-g) + d^2(g-\beta) \Big) x^{d-1-\alpha}\theta^{\alpha}.$$

The result is thus immediate when $d = 2$; when $d \geq 3$, it follows from the fact that $\sum_{1 \leq \beta \leq \alpha} (-1)^{\beta} \beta \binom{\alpha}{\beta} = 0$ for all $\alpha \geq 2$. $\square$

Proposition 2.13. *The numerical class of $\Delta_{g-d+1,d}$ is*

$$d(g-d+1)x^{g-3} \cdot \Big\{ \Big(d(g-d+1)(g^2-g) - (g^2+1)(g-2) \Big) x^2$$

$$+ \Big((2-2d)g^2 + (2d^2-3)g - (2d^2-d-2) \Big) x\theta + (d-1)(g-d)\theta^2 \Big\}.$$

Proof. By Proposition 5.1 on p.358 of ACGH, the class of $\Delta_{g-d+1,d}$ is the coefficient of t_1t_2 in the expression

$$\sum_{0\le\beta\le\alpha\le g-1}\frac{(-1)^{\alpha+\beta}}{\beta!(\alpha-\beta)!}\Big(1+(g-d+1)t_1+dt_2\Big)^{2-g+\beta}\Big(1+(g-d+1)^2t_1+d^2t_2\Big)^{g-\beta}x^{g-1-\alpha}\theta^\alpha.$$

This coefficient is equal to

$$d(g-d+1)\sum_{\alpha=0}^{g-1}\frac{(-1)^\alpha}{\alpha!}\Bigg\{\Big((d-1)g-d(d-1)\Big)\sum_{\beta=0}^{\alpha}(-1)^\beta\binom{\alpha}{\beta}\beta^2$$

$$+\Big((2-2d)g^2+(2d^2-d-2)g-(d^2-d-1)\Big)\sum_{\beta=0}^{\alpha}(-1)^\beta\binom{\alpha}{\beta}\beta$$

$$+\Big((d-1)g^3-(d^2-2)g^2+(d^2-d-1)g+2\Big)\sum_{\beta=0}^{\alpha}(-1)^\beta\binom{\alpha}{\beta}\Bigg\}x^{g-1-\alpha}\theta^\alpha$$

Since the three sums over β are equal to 0 for all $\alpha\ge 3$, we finally obtain

$$d(g-d+1)x^{g-3}\Bigg\{\Big((d-1)g^3-(d^2-2)g^2+(d^2-d-1)g+2\Big)x^2$$

$$+\Big((2-2d)g^2+(2d^2-3)g-(2d^2-2d-1)\Big)x\theta+\Big((d-1)g-d(d-1)\Big)\theta^2\Bigg\}.$$

$\square$

3. Stable Base Loci, Cones of Divisor Classes, and The Volume Function

In this section we introduce the parts of the asymptotic theory of linear series that will be used in our work.

3.1. **From Ample to Nef and Big.**

We state the following definition for completeness.

Definition: A Cartier divisor D on a projective variety X is **ample** if for any coherent sheaf $\mathcal{F}$ on X, there exists a positive integer n_0 such that for all $n \geq n_0$ the sheaf $\mathcal{F}(mD)$ is globally generated.

The following important result (e.g. Theorem 5.1 in [13]) implies that ampleness, unlike very ampleness, is a *numerical* property of a divisor.

Theorem 3.1. *(Nakai-Moishezon) Let D be a Cartier divisor on a projective variety X. Then L is ample if and only if for all integral subschemes Y of X*

$$D^{\dim(Y)} \cdot Y > 0.$$

The following definitions are important generalizations of ampleness. The first of these is directly inspired by the Nakai-Moishezon theorem:

Definition: A Cartier divisor D on a projective variety X is **nef** if for any integral subscheme Y,

$$D^{\dim(Y)} \cdot Y \geq 0.$$

Definition: A Cartier divisor D on a projective variety X is **semiample** if for some $m \in \mathbb{N}$ the linear system $|mD|$ is basepoint-free.

Every semiample divisor is nef, but the converse is not true; even though nefness is a numerical property by definition, semiampleness is not. To see why, consider a smooth projective variety X with $h^1(\mathcal{O}_X) > 0$ and compare the trivial line bundle on X to a non-torsion element of $\mathrm{Pic}^0(X)$.

Proposition 3.2. *Let $f : X \to Y$ be a morphism of projective varieties, and let D be an ample divisor on Y. Then f^*D is semiample on X. In particular, f^*D is nef.*

Proof. If D is an ample divisor on Y, then by definition there exists a positive integer n for which $\mathcal{O}_Y(nD)$ is globally generated, and our result follows from the fact that global generation is preserved under pullback via morphisms. □

The following is Theorem 1.1.24 in [17]; we refer to *loc. cit.* for its proof.

Theorem 3.3. ***(Asymptotic Riemann-Roch)*** *Let D be a Cartier divisor on an irreducible projective variety of dimension n. Then for $m >> 0$,*

$$\chi(X, \mathcal{O}_X(mD)) = \frac{D^n}{n!} \cdot m^n + O(m^{n-1}).$$

If D is nef, we have from Theorem 1.4.40 in [17] that for $m >> 0$, $h^i(\mathcal{O}_X(mD)) = O(m^{n-i})$. (This is a generalization of the Serre vanishing property of ample divisors.) Combining this with Theorem 3.3 yields

Corollary 3.4. *Let D be a nef Cartier divisor on an irreducible projective variety X of dimension n. Then for $m >> 0$,*

$$h^0(X, \mathcal{O}_X(mD)) = \frac{D^n}{n!} \cdot m^n + O(m^{n-1}).$$

Definition: A Cartier divisor D on an irreducible projective variety X is *big* if $h^0(X, \mathcal{O}_X(mD)) = O(m^n)$ as $m \to \infty$.

Proposition 3.5. *(Corollary 2.2.7 on p.141 of* [17]*) A Cartier divisor D on an irreducible projective variety X is big if and only if there exists an ample divisor A*

on X, an effective divisor E on X, and a positive integer m such that mD is linearly equivalent to $A + E$. $\square$

3.2. **Stable Base Loci and Cones of Divisor Classes.** Before proceeding further, we must know what it means for a class in $N^1_{\mathbb{R}}(X)$ to be ample, nef, or big.

Definition: Let η be a class in $N^1_{\mathbb{R}}(X)$.

(i) η is **ample** if $\eta = \sum_{j=1}^{m} r_j D_j$ for positive real numbers $r_1, ..., r_m$ and classes $D_1, ..., D_m$ of ample Cartier divisors on X.

(ii) η is **nef** if for all integral subschemes Y of X, $\eta^{\dim Y} \cdot Y \geq 0$.

(iii) η is **big** if $\eta = \sum_{j=1}^{m} r_j D_j$ for positive real numbers $r_1, ..., r_m$ and classes $D_1, ..., D_m$ of big Cartier divisors on X.

Proposition 3.6. *Let X be a projective variety.*

(i) *The closure of the ample cone of X in $N^1_{\mathbb{R}}(X)$ is equal to the nef cone of X.*

(ii) *The closure of the big cone of X is equal to the closure of the effective cone of X.*

Definition: Let X be a smooth projective variety and let D be a divisor on X. Then the **stable base locus** $\mathbf{B}(D)$ of D is $\bigcap_{m=1}^{\infty} \mathrm{Bs}(|mD|)$ where $\mathrm{Bs}(|mD|)$ is the set-theoretic base locus of the linear system $|mD|$.

Remark: Note that if $\dim |mD| = 0$ for all m, then $\mathbf{B}(D) = X$.

Lemma 3.7. *$\boldsymbol{B}(D)$ is a Zariski-closed subset of X.* $\square$

The remark immediately following the definition of semiampleness shows that the stable base locus is not a numerical invariant. Note that $\mathbf{B}(D) = \emptyset$ precisely when D is semiample. However, we have "inner and outer approximations" of the stable base locus which *are* numerical invariants.

Definition: Let D be an $\mathbb{R}-$divisor on X.

(i) The **restricted base locus** of D is

$$\mathbf{B}_-(D) := \bigcup_A \mathbf{B}(D+A)$$

where the intersection is taken over all ample $\mathbb{R}-$divisors A such that $D+A$ is a $\mathbb{Q}-$divisor.

(ii) The **augmented base locus** of D is

$$\mathbf{B}_+(D) := \bigcap_A \mathbf{B}(D-A)$$

where the intersection is taken over all ample $\mathbb{R}-$divisors A such that $D-A$ is a $\mathbb{Q}-$divisor.

Lemma 3.8. *If D is a $\mathbb{Q}-$divisor, then $\boldsymbol{B}_-(D) \subseteq \boldsymbol{B}(D) \subseteq \boldsymbol{B}_+(D)$.* $\square$

Lemma 3.9. *Both $\boldsymbol{B}_-(D)$ and $\boldsymbol{B}_+(D)$ are numerical invariants of D.* $\square$

It is an immediate consequence of the definition that $\mathbf{B}_+(D)$ is Zariski-closed. While it is currently unknown whether $\mathbf{B}_-(D)$ is always Zariski-closed, we do know that it is at worst a countable union of subvarieties (Proposition 1.19 in [10]).

The following theorem of Nakamaye gives a useful characterization of the augmented base locus of a nef and big divisor. We refer to [22] or p.249-251 in [18] for the proof.

Theorem 3.10. *(Theorem 0.3 in [22]) If D is a nef and big divisor on a smooth projective variety X, then $\boldsymbol{B}_+(D)$ is the union of all positive-dimensional subvarieties V of X for which $D^{\dim V} \cdot V = 0$.*

Remark: This result has been generalized to Theorem C of [11], which characterizes augmented base loci of big divisors that need not be nef. However, Theorem 3.10 is entirely sufficient for our purposes.

Ampleness, nefness, and bigness can all be characterized in terms of augmented and restricted base loci:

Lemma 3.11.

(i) *D is ample if and only if $\boldsymbol{B}_+(D) = \emptyset$.*

(ii) *D is nef if and only if $\boldsymbol{B}_-(D) = \emptyset$.*

(iii) *D is big if and only if $\boldsymbol{B}_+(D) \neq X$.*

(iv) *D is pseudoeffective if and only if $\boldsymbol{B}_-(D) \neq X$.* □

We refer to [10] for the proofs. In *loc. cit.*,(i) and (iii) are Example 1.7, while (ii) and (iv) are Example 1.18.

Note that by (i) of Lemma 3.11, D is ample if and only if $\mathbf{B}_-(D) = \mathbf{B}_+(D) = \emptyset$. This helps motivate the following

Definition: A class $\eta \in N^1_{\mathbb{R}}(X)$ is **stable** if $\mathbf{B}_-(\eta) = \mathbf{B}_+(\eta)$. If η is stable, its **stable base locus** $\mathbf{B}(\eta)$ is its augmented (or restricted) base locus.

Note that if D is a stable $\mathbb{Q}-$divisor, then $\mathbf{B}(D)$ does not depend on the numerical class of D. We will need to know for the proof of Theorem 1 that in some sense, most classes in $N^1_{\mathbb{R}}(X)$ are stable.

Proposition 3.12. *(1.26 in* [10]*) The set of stable classes is open and dense in $N^1_{\mathbb{R}}(X)$. In fact, for every $\eta \in N^1_{\mathbb{R}}(X)$ there exists $\epsilon > 0$ such that for any ample class α satisfying $\|\alpha\| < \epsilon$, $\eta - \alpha$ is stable.* □

If $\eta \in N^1_{\mathbb{Q}}(X)$ is a stable class which is big, as stated in the introduction we may measure its size by the codimension of its stable base locus.

Definition: Let X be a smooth projective variety of dimension n, and let $l \in \{0, ..., n\}$ be given. Then

$$\mathfrak{K}_l(X) := \overline{\{\eta \in N^1_{\mathbb{Q}}(X) : \eta \text{ stable}, \dim \mathbf{B}(\eta) \leq l\}}$$

A few remarks are in order. First, by Lemma 3.11, the nef cone of X is equal to $\mathfrak{K}_0(X)$ and the pseudoeffective cone is equal to $\mathfrak{K}_{n-1}(X)$. Secondly, we have the inclusions

$$\mathfrak{K}_0(X) \subseteq \mathfrak{K}_1(X) \subseteq ... \subseteq \mathfrak{K}_{n-1}(X) \subseteq \mathfrak{K}_n(X) = X$$

Finally, by Proposition 3.12, we have the alternate characterizations

$$\mathfrak{K}_l(X) = \{\eta \in N^1_{\mathbb{R}}(X) : \dim \mathbf{B}_-(\eta) \leq l\} = \overline{\{\eta \in N^1_{\mathbb{R}}(X) : \dim \mathbf{B}_+(\eta) \leq l\}}.$$

The cone $\mathfrak{K}_{n-2}(X)$ has been referred to as the **movable** or **modified nef** cone, and has been studied by Boucksom in [4], who proved in *loc. cit.* that it is the closed convex cone generated by pushforwards of ample classes on modifications of X. The *duals* of the $\mathfrak{K}_l(X)$ were computed by Payne in the case where X is a complete $\mathbb{Q}-$factorial toric variety [24], and there is unpublished work of Debarre on the case where X is the projectivization of a rank-2 vector bundle on $\mathbb{P}^2$.

3.3. **The Volume Function.** We will assume for the remainder of this section that X is an irreducible projective variety of dimension n.

Definition: Let D be a Cartier divisor on X. Then the **volume** of D is

$$\mathrm{vol}_X(D) := \limsup_{m\to\infty} \frac{n!h^0(X, mD)}{m^n}.$$

It is an immediate consequence of the definition of bigness that $\mathrm{vol}_X(D) > 0$ precisely when D is big. Also, by Corollary 3.4, we have

Lemma 3.13. *If D is a nef divisor on X, then $vol_X(D) = D^n$.* □

In addition, the following properties of vol_X (as well as other properties that will not be used in the sequel) bear out a strong analogy with the top self-intersection.

Lemma 3.14. *((i) of Proposition 2.2.35 in* [17]*) If k is a positive integer and D is a Cartier divisor on X, then*

$$vol_X(kD) = k^n \cdot vol_X(D).$$

□

As a result, for any $\mathbb{Q}-$Cartier divisor D on X, we may define

$$\mathrm{vol}_X(D) := \frac{1}{k^n} \cdot \mathrm{vol}_X(kD),$$

where k is a positive integer for which mD is an integral Cartier divisor.

Proposition 3.15. *(Proposition 2.2.41 in* [17]*) Let D_1 and D_2 be Cartier divisors on X which are numerically equivalent. Then*

$$vol_X(D_1) = vol_X(D_2).$$

It follows that $\mathrm{vol}_X : N^1_{\mathbb{Q}}(X) \to [0, \infty)$ is a well-defined function. The next result ensures that vol_X can be uniquely extended to a continuous real-valued function on $N^1_{\mathbb{R}}(X)$.

Proposition 3.16. *(Theorem 2.2.44 in* [17]*) Let $\|\cdot\|$ be any norm on $N^1_{\mathbb{R}}(X)$. Then there exists a constant $C > 0$ such that*

$$|vol_X(\eta) - vol_X(\eta')| \leq C \cdot (\max(\|\eta\|, \|\eta'\|))^{n-1} \cdot \|\eta - \eta'\|$$

for any two classes $\eta, \eta' \in N^1_{\mathbb{Q}}(X)$. □

Remark: It has been proved in [5] that vol_X has continuous first derivative.

Lemma 3.17. *(Birational invariance of volume) Let $\eta : X' \to X$ be a birational projective mapping of $n-$dimensional irreducible varieties. If D is an integral or $\mathbb{Q}-$divisor on X, then*

$$vol_{X'}(\eta^*(D)) = vol_X(D).$$

Proof. See p.153 in [17]. □

3.4. **The Special Case C_d.** Our investigation of the cones $\mathfrak{K}_l(C_d)$ begins by considering the portions of the effective and nef cones of C_d in the second quarter. It is worth noting that unlike the corresponding results for the fourth quarter, Theorem 3.18 and

Proposition 3.19 yield boundary rays of the effective and nef cones, respectively, for *all* symmetric powers of *all* curves.

Theorem 3.18. *(Theorem 3 in* [16]*) If C is a curve of genus g, $d \geq 2$ is an integer, and the Néron-Severi rank of C_d is 2, then the class $2(-\theta + (g+d-1)x)$ of the "big diagonal" in C_d spans a boundary of the effective cone.*

The result and proof that follow are essentially due to Pacienza [23].

Proposition 3.19. *If C is any curve of genus g and $d \geq 3$, the numerical class $-\theta + dgx$ in $N^1_{\mathbb{R}}(C_d)$ is nef and big, and its augmented base locus is Δ_d.*

Proof. By the diagonal calculation in Proposition 2.12, any effective divisor class whose intersection with the small diagonal is zero is a positive multiple of $-\theta + dgx$. By Theorem 3.10, we will be done after constructing a generically finite morphism from C_d to another variety which contracts exactly Δ_d.

We define the multi-difference map $\xi : C^d \to J(C)^{\binom{d}{2}}$ by sending the ordered $d-$tuple $(p_1, ..., p_d)$ of points on C to $\Pi_{1\leq i<j\leq d}\mathcal{O}(p_i - pj)$. The proof of Proposition 3.19 will conclude after that of the following lemma.

Lemma 3.20. *Let C be a smooth projective curve of genus $g \geq 3$, and let d be an integer ≥ 3.*

(i) *If C is nonhyperelliptic, then the restriction of ξ to the complement of the small diagonal in C^d is injective.*

(ii) *If C is hyperelliptic, then the restriction of ξ to the complement of the small diagonal is finite-to-one.*

Proof. Let $(p_1, ..., p_d)$ and $(q_1, ..., q_d)$ be two distinct elements of C^d such that

$$\nu := \xi((p_1, ..., p_d)) = \xi((q_1, ..., q_d)),$$

let $k \in \{1, ..., d\}$ be such that $p_k \neq q_k$, and let $l \in \{1, ..., d\}$ be different from k. Since $p_k - p_l$ is linearly equivalent to $q_k - q_l$ by assumption, we have that $p_k + q_l$ is linearly equivalent to $p_l + q_k$.

If C is nonhyperelliptic, then since $p_k \neq q_k$ we must have $p_k = p_l$ and $q_k = q_l$. Varying l shows that both $(p_1, ..., p_d)$ and $(q_1, ..., q_d)$ are in the small diagonal, and (i) is proved.

If C is hyperelliptic, we may assume that p_k+q_l and p_l+q_k are in the hyperelliptic pencil; otherwise we are done by the argument in the previous case. Assume furthermore that $(p_1, ..., p_d)$ and $(q_1, ..., q_d)$ are not in the small diagonal. Let $m \in \{1, ..., d\}$ be any element distinct from k and l. Then $p_k + q_m$ is linearly equivalent to $p_m + q_k$. If $q_m \neq q_l$, then $\dim |p_k + q_m| = \dim |p_m + q_k| = 0$, and we must have that $p_m = p_k$ and $q_m = q_k$. If $q_m = q_l$, then we must have $p_m = p_l$. Varying m shows that the fibre of $\xi^{-1}(\nu)$ has finite length, and (ii) is proved. □

Conclusion of the proof of Proposition 3.19: We first compose ξ with the $\binom{d}{2}-$th Cartesian power of the Kummer morphism $J(C) \to \mathrm{Kum}_{J(C)}$ (where $\mathrm{Kum}_{J(C)} = J(C)/\langle -1 \rangle$) and then compose the resulting morphism with the natural quotient map $\mathrm{Kum}_{J(C)}^{\binom{d}{2}} \to (\mathrm{Kum}_{J(C)})_{\binom{d}{2}}$, thereby obtaining an $\mathcal{S}_d-$invariant morphism from C^d to the $\binom{d}{2}-$th symmetric power $(\mathrm{Kum}_{J(C)})_{\binom{d}{2}}$ of $\mathrm{Kum}_{J(C)}$. By Lemma 3.20, this contracts precisely the small diagonal in C^d, and thus the induced morphism $C_d \to (\mathrm{Kum}_{J(C)})_{\binom{d}{2}}$ contracts precisely the small diagonal in C_d. □

Determining the boundary of the nef cone in the fourth quarter is considerably more difficult in general. However, we have

Proposition 3.21. *For all $d \geq 2$, the class θ on C_d is nef, and is ample precisely when C does not possess a $\mathfrak{g}^1_d$.*

Proof. Since θ is the class of the pullback of an ample divisor on $\mathrm{Pic}^d(C)$, the nefness of θ follows from Proposition 3.2. The second statement follows from Proposition 2.6. □

As a result of the previous propositions, we have

Corollary 3.22. *Let C be a curve of genus $g \geq 2$, let $d \geq 2$ be an integer, and suppose that the Néron-Severi rank of C_d is 2.*

(i) *The boundary of the nef cone of C_d in the second quarter is spanned by the class $-\theta + dgx$.*

(ii) *If C possesses a $\mathfrak{g}^1_d$, the boundary of the nef cone in the fourth quarter is spanned by the class θ.*

$\square$

4.1. **Proof of Theorem D.** Combining Theorem 3.18 with (1) and (3) of the following result immediately yields Theorem D.

Proposition 4.1. *Let C be a general hyperelliptic curve of genus $g \geq 3$, and let $d \in \{2, ..., g-1\}$ be given.*

(1) The cycle C_d^1 is a divisor which spans the boundary of the effective cone of C_d, and its numerical class is equal to $\theta - (g-d+1)x$, so that $u_{d-1,d} = d-g-1$.

(2) The common boundary of the nef and movable cones of C_d in the fourth quarter is spanned by θ, i.e. $u_{k,d} = 0$ for $0 \leq k \leq d-2$.

(3) For all $t \in [0, g-d+1]$,

$$vol_{C_d}(\theta - tx) = \frac{g!}{(g-d)!}(1 - \frac{t}{g-d+1})^d$$

Remark: Since C_d^1 is neither empty nor of the expected dimension

$$(g - 2(g-d+1)) + 1 = 2d - (g+1)$$

in the case we are considering, the computation of the class of C_d^1 given on p.326 of [1] does not apply.

Proof. (1): By the theorem of Martens (Theorem 5.1 on p.191 of [1]) the loci $W_d^1(C)$ and $W_d^2(C)$ in $\mathrm{Pic}^d(C)$ have respective dimensions $d-2$ and $\max\{-1, d-4\}$, so that $\dim |\mathcal{L}| = 1$ for a general member $\mathcal{L}$ of $W_d^1(C)$. Consequently C_d^1, the inverse image of $W_d^1(C)$ under the Abel map, is a divisor on C_d.

Suppose C_d^1 is big. Then by Proposition 3.5, its class can be expressed as a sum $A+E$ of $\mathbb{Q}-$divisors, where A is ample and E is $\mathbb{Q}-$effective. Since C_d^1 is contracted to the $(d-2)-$dimensional locus $W_d^1(C)$ by the Abel map, and θ is the pullback of an ample class, we have that

$$\theta^{d-1} \cdot A + \theta^{d-1} \cdot E = \theta^{d-1} \cdot C_d^1 = 0.$$

However, the aforementioned property of θ implies that it is the class of a semiample divisor, so we must have $\theta^{d-1} \cdot A > 0$ and $\theta^{d-1} \cdot E \geq 0$, which is absurd.

By geometric Riemann-Roch, the underlying set of C_d^1 is $\{D+E : D \in C_2^1, E \in C_{d-2}\}$ (cf. p.13 of [1]). Put differently, $C_d^1 = \sigma(\pi^{-1}C_2^1)$, where $\pi : C_2 \times C_{d-2} \to C_2$ is projection onto the first factor and $\sigma : C_2 \times C_{d-2} \to C_d$ is the addition map. Consequently the numerical class of C_d^1 is the image of the numerical class of C_2^1 under the "push" map

$$A_{d-2} : H^*_{an}(C_2, \mathbb{Q}) \to H^*_{an}(C_d, \mathbb{Q})$$

which takes a cycle class z to $\sigma_* \pi^* z$.

Since Lemma 3.2 on p.342 of [1] implies that the numerical class of C_2^1 is equal to $\theta - (g-1)x$ and, by Exercise D-8 on p.369 of [1], we have

$$A_{d-2}(x) = (d-1)x$$

$$A_{d-2}(\theta) = \theta + g(d-2)x$$

it follows that C_d^1 has numerical class $A_{d-2}(\theta - (g-1)x) = \theta - (g-d+1)x$.

(2): Since θ is nef and big for $2 \leq d \leq g-1$, this is a consequence of Nakamaye's theorem on base loci.

(3): For $t \in (0, g-d+1]$,

$$\begin{aligned}\mathrm{vol}_{C_d}(\theta - tx) &= \mathrm{vol}_{C_d}\Big((1 - \frac{t}{g-d+1})\theta + (\frac{t}{g-d+1})(\theta - (g-d+1)x)\Big) \\ &= \Big(\frac{t}{g-d+1}\Big)^d \cdot \mathrm{vol}_{C_d}\Big((\frac{g-d+1}{t} - 1)\theta + (\theta - (g-d+1)x)\Big).\end{aligned}$$

Since the Abel map exhibits C_d as the blowup of $W_d(C)$ along the codimension-2 locus $W^1_d(C)$ with exceptional divisor C^1_d–whose class was just determined to be $\theta - (g-d+1)x$–the birational invariance of the volume function implies that

$$\begin{aligned}&(\frac{t}{g-d+1})^d \cdot \mathrm{vol}_{C_d}\Big((\frac{g-d+1}{t}-1)\theta + (\theta-(g-d+1)x)\Big)\\ &= \Big(\frac{t}{g-d+1}\Big)^d \cdot \mathrm{vol}_{W_d(C)}\Big((\frac{g-d+1}{t}-1)\Theta|_{W_d(C)}\Big)\end{aligned}$$

where Θ is the numerical class of the theta-divisor on $\mathrm{Pic}^d(C)$. Since the numerical class of $W_d(C)$ is $\frac{\Theta^{g-d}}{(g-d)!}$ and $\Theta|_{W_d(C)}$ is ample,

$$\mathrm{vol}_{W_d(C)}(\Theta|_{W_d(C)}) = \Theta|^d_{W_d(C)} = \frac{\Theta^{g-d}}{(g-d)!}\cdot\Theta^d = \frac{g!}{(g-d)!}.$$

Therefore we may conclude that

$$\begin{aligned}&\Big(\frac{t}{g-d+1}\Big)^d \cdot \mathrm{vol}_{W_d(C)}\Big((\frac{g-d+1}{t}-1)\Theta|_{W_d(C)}\Big) =\\ &\Big(\frac{t}{g-d+1}\Big)^d \cdot \Big(\frac{g-d+1}{t}-1\Big)^d \cdot \frac{g!}{(g-d)!} = \frac{g!}{(g-d)!}\Big(1-\frac{t}{g-d+1}\Big)^d.\end{aligned}$$

□

In particular, we have

$$\mathrm{vol}_{C_{g-1}}(\theta - x) = \frac{g!}{2^{g-1}}.$$

By a well-known elementary identity, this is an odd integer precisely when g is a power of 2 and fails to be an integer otherwise.

4.2. **Proofs of Theorems A,B, and C.** Throughout this section, C will denote a general non-hyperelliptic curve of genus $g \geq 4$ unless otherwise stated.

4.2.1. *Residuation.* Since we are assuming C is general, its gonality is $\lceil \frac{g}{2}+1 \rceil$. When $\frac{g}{2}+1 \leq d \leq g-1$, Serre duality induces a birational map $\tau : G^{g-d-1}_{2g-2-d}(C) \dashrightarrow C_d$. The loci of indeterminacy of τ and τ^{-1} are $G^{g-d}_{2g-2-d}(C)$ and C^1_d, respectively. Since these loci are both of codimension at least 2 and both varieties are smooth, it follows from Hartogs' Theorem that τ induces an isomorphism of Picard groups, and thus of Neron-Severi groups.

If D is an effective divisor on C of degree $g-d$, where $\frac{g}{2}+1 \leq d \leq g-1$, then D imposes independent conditions on the canonical linear system $|K_C|$, so that the complete linear system $|K_C(-D)|$ is of dimension $d-1$ and degree $g+d-2$. Lemma 2.11 then tells us that the locus $\Gamma_d(K_C(-D))$ is a divisor in C_d with numerical class $\theta - x$.

We may generalize the divisor classes x and θ on C_d in a straightforward fashion to obtain divisor classes on $G^{g-d-1}_{2g-2-d}(C)$.

Proposition 4.2. *For any effective divisor D of degree $r+1$ on C, the set*

$$\widehat{X}_D := \{(V, \mathcal{M}) \in G^r_d(C) : V \cap H^0(\mathcal{M}(-D)) \neq 0\}$$

has the natural structure of an ample divisor on $G^r_d(C)$.

Note that the special case $r = 0$ is Lemma 2.5.

Proof. We first recall some aspects of the construction of $G^r_d(C)$ in Section 3 of Chapter IV of [1]. Fix a Poincare bundle $\mathcal{L}$ on $C \times \mathrm{Pic}^d(C)$, and let D' be an effective divisor on C of degree $2g-d-1$.

If $\eta : C \times \mathrm{Pic}^d(C) \to \mathrm{Pic}^d(C)$ is projection onto the second factor, and Γ is the product divisor $(D + D') \times \mathrm{Pic}^d(C)$, then the direct image sheaf $\eta_*\mathcal{L}(\Gamma)$ is locally free of rank $g+r+1$ and its fibre over a line bundle $\mathcal{M}$ of degree d on C is $H^0(\mathcal{M}(D+D'))$. Indeed, $h^1(\mathcal{M}(D+D')) = 0$ by Serre duality since the degree of $\mathcal{M}(D+D')$ is $2g+r$, so this follows from Riemann-Roch and base change in cohomology (Theorem 2.6 on p.175 of [1]).

If Γ' is the product divisor $D' \times \mathrm{Pic}^d(C)$, then an entirely analogous argument tells us that $\eta_*\mathcal{L}(\Gamma')$ is a rank-g subbundle of $\eta_*\mathcal{L}(\Gamma)$. Since the dual of $\eta_*\mathcal{L}(\Gamma)$ is ample by Proposition 2.2 on p.310 of [1], the Plucker divisor σ' on the Grassman bundle

$G(r+1, \eta_*\mathcal{L}(\Gamma))$ associated to $\eta_*\mathcal{L}(\Gamma')$ is ample by the following lemma. (Recall that a vector bundle E is ample if the hyperplane class on the *subbundle* projectivization $\mathbb{P}_{sub}(E^*)$ of E^* is ample.)

Lemma 4.3. *Let X be a smooth projective variety, and let E be an ample vector bundle of rank s on X. Then for all $s' \leq s$, the Plucker class on the associated Grassman bundle $G(s', E^*)$ of rank-s' subbundles of E^* is ample.*

Proof. If $\nu : G(s', E^*) \to X$ is the structure map, then the determinant of the inclusion $S_{s',E^*} \hookrightarrow \nu^*(E)$ of the tautological subbundle induces the Plucker embedding $G(s', E^*) \hookrightarrow \mathbb{P}_{sub}(\wedge^{s'} E^*)$. The result then follows from the fact that the amplitude of E implies the amplitude of its exterior powers (part (ii) of Corollary 6.1.16 on p.15 of [18]). $\square$

Conclusion of proof of Proposition 4.2: For each line bundle $\mathcal{M}$ of degree d on C, there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathcal{M}) & \xrightarrow{f_1} & H^0(\mathcal{M}(D+D')) & \xrightarrow{g_1} & H^0(\mathcal{M}(D+D')|_{D+D'}) \\ & & {\scriptstyle i}\uparrow & & {\scriptstyle i'}\uparrow & & {\scriptstyle i''}\uparrow \\ 0 & \longrightarrow & H^0(\mathcal{M}(-D)) & \xrightarrow{f_2} & H^0(\mathcal{M}(D')) & \xrightarrow{g_2} & H^0(\mathcal{M}(D')|_{D+D'}) \end{array}$$

in which both rows are exact and all vertical arrows are injective, so that a diagram chase gives the equality

$$(f_1 \circ i)\big(H^0(\mathcal{M}(-D))\big) = f_1\big(H^0(\mathcal{M})\big) \cap i'\big(H^0(\mathcal{M}(D'))\big).$$

Therefore if V is a subspace of $H^0(\mathcal{M})$, we have

$$f_1(V) \cap (f_1 \circ i)\big(H^0(\mathcal{M}(-D))\big) = f_1(V) \cap i'\big(H^0(\mathcal{M}(D'))\big).$$

It follows at once from the definitions that $\widehat{X}_D = G^r_d(C) \cap \sigma'$. Consequently $\widehat{X}_D$, which has a cycle structure induced by its being an intersection of cycles and is the restriction of an ample divisor, is ample. $\square$

As D varies over C_{r+1}, we obtain an algebraic family of divisors whose common numerical class we will denote by $\widehat{X}$. Also, we will denote by $\widehat{\theta}$ the numerical class of the pullback of a theta-divisor in $\mathrm{Pic}^d(C)$.

Remark: By the formula for the fundamental class w^r_d of $W^r_d(C)$ (Theorem 4.4 in Chapter VII of [1]), it follows that $\widehat{\theta}^d = w^{g-d-1}_{2g-2-d} \cdot \theta^d = \frac{g!}{(g-d)!}$.

Proposition 4.4. *Under the isomorphism $\tau^* : N^1_{\mathbb{R}}(C_d) \to N^1_{\mathbb{R}}(G^{g-d-1}_{2g-2-d}(C))$,*

$$\tau^*(\theta) = \widehat{\theta}, \;\; \tau^*(\theta - x) = \widehat{X}.$$

Proof. If $\widetilde{\tau} : \mathrm{Pic}^{2g-2-d}(C) \to \mathrm{Pic}^d(C)$ is the morphism induced by taking Serre duals and $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles on C of respective degrees $2g-2-d$ and d satisfying $\mathcal{L}_1 \otimes \mathcal{L}_2 \simeq K_C$, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}^0(C) \arrow[r, "\cdot\otimes\mathcal{L}_1"] \arrow[d, "(-1)"] & \mathrm{Pic}^{2g-2-d}(C) \arrow[d, "\widetilde{\tau}"] & G^{g-d-1}_{2g-2-d}(C) \arrow[l] \arrow[d, dashed, "\tau"] \\
\mathrm{Pic}^0(C) \arrow[r, "\cdot\otimes\mathcal{L}_2"] & \mathrm{Pic}^d(C) & C_d \arrow[l]
\end{tikzcd}$$

where the leftmost vertical arrow is multiplication by -1 and the left horizontal arrows on the top and bottom are multiplication by $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. Since multiplication by -1 induces the identity on the cohomology of $\mathrm{Pic}^0(C)$, it induces the identity on the Neron-Severi group of $\mathrm{Pic}^0(C)$. Therefore τ^* takes the theta class on C_d to $\widehat{\theta}$ on $G^{g-d-1}_{2g-2-d}(C)$.

If D is an effective divisor of degree $g-d$ on C, then it follows immediately from Riemann-Roch that $D' \in C_d$ satisfying $h^0(D') = 1$ is subordinate to $|K_C(-D)|$ precisely when D is subordinate to $|K_C(-D')|$. This can be rephrased as saying that away from the loci of indeterminacy of τ and τ^{-1}, the divisor $\widehat{X}_D$ is isomorphic to

$\Gamma_d(K_C(-D))$ via τ. Since the fundamental class of $\Gamma_d(K_C(-D))$ is $\theta - x$ by Lemma 2.11, we have that $\tau^*(\theta - x) = \widehat{X}$. $\square$

4.2.2. *Residuation and* vol_{C_d}. Essentially the same Hartogs argument which gives the isomorphism of Picard groups also shows that τ^* and its inverse both preserve the volume function. The next result simultaneously proves Theorem C and (ii) of Theorem A.

Proposition 4.5. *Let $\mathcal{L}$ be a line bundle on C_d and $\mathcal{M}$ be a line bundle on $G^{g-d-1}_{2g-2-d}(C)$. Then*

$$H^0(C_d, (\tau^{-1})^*\mathcal{M}) \simeq H^0(G^{g-d-1}_{2g-2-d}(C), \mathcal{M})$$

$$H^0(G^{g-d-1}_{2g-2-d}(C), \tau^*\mathcal{L}) \simeq H^0(C_d, \mathcal{L})$$

In particular, $vol_{C_d}((\tau^{-1})^*\mathcal{M}) = vol_{G^{g-d-1}_{2g-2-d}(C)}(\mathcal{M})$ *and* $vol_{G^{g-d-1}_{2g-2-d}(C)}(\tau^*\mathcal{L}) = vol_{C_d}(\mathcal{L})$.

Proof. Let $U = C_d - C^1_d$ and $V = G^{g-d-1}_{2g-2-d}(C) - G^{g-d}_{2g-2-d}(C)$. Clearly $\tau|_V : V \to U$ is an isomorphism and $\tau^{-1}|_U : U \to V$ is its inverse. These furnish natural isomorphisms

$$H^0(U, \mathcal{L}|_U) \simeq H^0(V, \tau^*\mathcal{L}|_V)$$

$$H^0(V, \mathcal{M}|_V) \simeq H^0(U, (\tau^{-1})^*\mathcal{M}|_U)$$

Since U and V are complements of subvarieties of codimension at least 2 in smooth varieties, these isomorphisms extend to all of C_d and $G^{g-d-1}_{2g-2-d}(C)$. $\square$

As an immediate consequence of Proposition 4.5, we obtain the following:

Corollary 4.6. *τ^* and $(\tau^*)^{-1}$ interchange the pseudoeffective cones of C^d and $G^{g-d-1}_{2g-2-d}(C)$.* $\square$

We now specialize to the case $d = g - 1$. In this scenario τ is a birational involution on C^{g-1}. By Theorem 3.18, the class $-2\theta + (4g - 4)x$ representing the big diagonal in C_{g-1} spans a boundary of the effective cone. Combining this with Corollary 4.6 yields (ii) of Theorem A.

Proof of Theorem C: By Corollary 3.22, the nef cone of C_{g-1} is spanned by the classes θ and $-\theta + (g^2 - g)x$. Since $\tau^*(\theta - tx) = (1-t)\theta + tx$ by Proposition 4.4 and the volume of a nef class is its top self-intersection by Lemma 3.13, we have

$$\mathrm{vol}_{C_{g-1}}(\theta - tx) = \mathrm{vol}_{C_{g-1}}((1-t)\theta + tx)$$

$$= \sum_{k=0}^{g-1} \binom{g-1}{k} t^k (1-t)^{g-1-k} x^k \theta^{g-1-k} = \sum_{k=0}^{g-1} \binom{g-1}{k} \frac{g!}{(k+1)!} t^k (1-t)^{g-1-k}.$$

□

4.2.3. *Residuation and Stable Base Loci.* Here we give the proof of Theorem B, which is an immediate consequence of combining Theorem 3.18 and Proposition 4.4 with the following result.

Proposition 4.7. *Let $\mathcal{L}$ be a stable line bundle on $G^{g-d-1}_{2g-2-d}(C)$ with stable base locus Z and numerical class $a\widehat{X} + b\widehat{\theta}$ satisfying $a > 0$ and $a + b > 0$. Then the line bundle $(\tau^{-1})^*\mathcal{L}$ on C_d is stable with stable base locus $C^1_d \cup \tau^{-1}(Z)$.*

In particular, a line bundle $\mathcal{L}$ on $G^{g-d-1}_{2g-2-d}(C)$ with numerical class in the aforementioned range is stable precisely when the pullback bundle $(\tau^{-1})^*\mathcal{L}$ on C_d is stable.

Proof. Let $\mathcal{L}$ be a stable line bundle on $G^{g-d-1}_{2g-2-d}(C)$ satisfying the hypotheses, and let $\mathcal{M} := (\tau^{-1})^*\mathcal{L}$. Since the stable base locus of a line bundle does not change after taking positive tensor powers, we will assume without loss of generality that $\mathrm{Bs}(|\mathcal{L}|) = Z$ and $\mathrm{Bs}(|\mathcal{M}|) = \mathbf{B}(\mathcal{M})$.

The hypothesis on the coefficients a and b guarantees that the numerical class of $\mathcal{M}$, which is $(a+b)\theta - ax$, lies in the fourth quarter of the $\theta, x-$plane, so that the stable base locus of $\mathcal{M}$ must contain C^1_d. By Proposition 4.5, pullback via τ^{-1} gives a natural isomorphism between $H^0(G^{g-d-1}_{2g-2-d}(C), \mathcal{L})$ and $H^0(C_d, \mathcal{M})$, so

$\mathbf{B}(\mathcal{M}) = \mathrm{Bs}(|\mathcal{M}|) = C_d^1 \cup \tau^{-1}(Z)$. Indeed, if $x \in C_d - C_d^1$ is a basepoint of $|\mathcal{M}|$, then $\tau(x)$ is a basepoint of $|\mathcal{L}|$.

The set of stable classes in in $N^1_{\mathbb{R}}(G^{g-d-1}_{2g-2-d}(C))$ having Z as its stable base locus is open, so its image under $(\tau^{-1})^*$ is open as well. If $t_0 := \frac{a}{a+b}$, then by our previous calculation and Proposition 3.12, we have that for some $\epsilon > 0$, $\theta - tx$ is stable with stable base locus $C_d^1 \cup \tau^{-1}(Z)$ for all t satisfying $0 < |t - t_0| < \epsilon$. We then have by the definitions of the augmented and restricted base loci that
$\mathbf{B}_-(\mathcal{M}) = C_d^1 \cup \tau^{-1}(Z) = \mathbf{B}_+(\mathcal{M})$. □

Consider the case $d = g-1$. It follows from Proposition 4.7 that for t slightly larger than $1 + \frac{1}{g^2-g-1}$, the class $\theta - tx \in N^1_{\mathbb{R}}(C_{g-1})$ is stable with stable base locus $C^1_{g-1} \cup \tau(\Delta_{g-1})$. Since a general curve has only normal Weierstrass points (this follows from, for instance, Theorem 2 of [12]) this is a disjoint union. Furthermore, the dimension of C^1_{g-1} is at least 2 for $g \geq 5$, so that we have examples of non-equidimensional stable base loci.

4.2.4. *Bounds for the case* $\frac{g+1}{2} < d \leq g-2$. We conclude with the following result, which implies (i) of Theorem A.

Proposition 4.8. *Let C be a very general curve of genus $g \geq 4$. For $2 \leq d \leq g$ define the following divisor on C_d :*

$$D_d := \bigcup_{p \in C} \Gamma_d\big(K_C(-(g-d+1)p)\big)$$

Then for $d \neq \frac{g+1}{2}$, the numerical class of D_d is

$$(g-d+1)\big((g^2 - dg + (d-2))\theta - (g^2 - (d-1)g - 2)x\big).$$

In particular, when $\frac{g+1}{2} < d \leq g-1$, the slope of D_d is strictly less than 1 and tends to 1 as $g \to \infty$.

Proof. Fix $d-1$ general points $q_1, ...q_{d-1}$ on C. The two test curves in C_d that we will use to compute the numerical class of D_d are the tiny diagonal Δ_d and the curve

$$\chi_d := \bigcap_{j=1}^{d-1} X_{q_j} = \{p + q_1 + ... + q_{d-1} : p \in C\}$$

The numerical class of χ_d is x^{d-1}, and by Proposition 2.13, the numerical class of Δ_d is $dx^{d-2}(((d-1)g+1)x - (d-1)\theta)$.

The intersection number $\chi_d \cdot D_d$ is the cardinality of the set

$$\left\{ q \in C : \exists p \in C \ni (g-d+1)p + q \leq |K_C(-q_1 - ... - q_{d-1})| \right\}$$

If $(g-d+1)p \leq |K_C(-q_1 - ... - q_{d-1})|$, then there are

$$((2g-2)-(d-1)) - (g-d+1) = g-2$$

points q (counting multiplicity) such that

$$(g-d+1)p + q \leq |K_C(-q_1 - ... - q_{d-1})|,$$

and we may conclude that

$$\chi_d \cdot D_d = (g-2) \cdot \Delta_{g-d+1} \cdot \Gamma_{g-d+1}(K_C(-q_1 - ... - q_{d-1})).$$

Similarly, $\Delta_d \cdot D_d$ is the cardinality of

$$\left\{ q \in C : \exists p \in C \ni (g-d+1)q + dq \leq |K_C| \right\}.$$

Assume the numerical class of D_d is $a\theta - bx$ for $a, b \in \mathbb{Q}$. The fact that the class of $\Gamma_{g+1}(K_C)$ is $x^2 - x\theta + \frac{\theta^2}{2}$ (by Lemma 2.11), the hypothesis that $d \neq \frac{g+1}{2}$, and the computation of $\Delta_{g-d+1,d}$ from Proposition 2.13 all imply that

$$\Delta_d \cdot D_d = d^2g^4 - (2d^3 - d^2 + d)g^3 + (d^4 - 2d)g^2 - (d^4 - 2d^3 - d)g - (2d^2 - 2d)$$

Then our system of equations is

$$ag - b = g^4 - 2dg^3 + (d^2 + 2d - 4)g^2 - (2d^2 - 5d + 1)g - (2d - 2)$$

$$adg - b = dg^4 - (2d^2 - d + 1)g^3 + (d^3 - 2)g^2 - (d^3 - 2d^2 - 1)g - (2d - 2)$$

and it has the solution

$$a = g^3 - (2d - 1)g^2 + (d^2 - 2)g - (d - 1)(d - 2) = (g - d + 1)(g^2 - dg + (d - 2))$$

$$b = g^3 + (2 - 2d)g^2 + (d^2 - 2d - 1)g + (2d - 2) = (g - d + 1)(g^2 - (d - 1)g - 2)$$

$\square$

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