Complete Set of Eigenfunctions of the Quantum Periodic Toda Chain

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Abstract of the Dissertation

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The quantum periodic Toda chain is a system of particles whose quantum behavior is governed by the Hamiltonian operator

$$\mathbf{H} = \left(-\frac{1}{2} \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} + \sum_{k=1}^{N-1} e^{x_k - x_{k+1}} + e^{x_N - x_1} \right).$$

Building on the previous works of Gutzwiller [11] and Sklyanin [31] , Pasquier and Gaudin [7] was able to find quantization conditions for this system by introducing an integral transform which turned the Schrodinger equation into the Baxter equation. They gave the solution for the Baxter equation, but were not able to state how to obtain the actual eigenfunctions due to the

lack of any inverse transform. Kharchev and Lebedev [19] succeeded in constructing a more explicit integral transform and its inverse, which they used to prove that Pasquier-Gaudin solutions can be inverted to give an eigenfunction for the quantum periodic Toda chain Hamiltonian. However, they did not know whether these solutions formed a complete set.

We answer this question affirmatively, that all eigenfunctions of the quantum periodic Toda chain arise from the Pasquier-Gaudin solutions, in the form of integral representation obtained explicitly by Kharchev and Lebedev. This will, in addition, show that the joint spectrum of commuting Hamiltonians of the Periodic Toda chain is simple.

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God has been faithful throughout my graduate studies, and I have learned to trust Him more through these years. So in the beginning of a new intellectual journey, I dedicate my thesis to Him.

Chapter 1

Introduction

We consider the eigenfunction expansion problem associated with the differential equation

$$\left(-\frac{1}{2}\sum_{k=1}^{N}\frac{\partial^2}{\partial x_k^2}+\sum_{k=1}^{N-1}e^{x_k-x_{k+1}}\ +e^{x_N-x_1}\right)\tilde{\Psi}=E\tilde{\Psi}. \tag{1.0.1}$$

This is called the Schrödinger equation for the quantum periodic Toda chain, which is a quantum mechanical system obtained from quantization of the classical periodic Toda chain.

1.1 The Classical Toda Chain

Consider a Hamiltonian system of N particles of unit mass in one dimension with interaction between the neighboring particles and no external forces acting on them. Let q_k and p_k $(k=1,\ldots,N)$ be the position and the momentum of the k-th particle. The Hamiltonian of the system is written as

$$\mathbf{H}=\frac{1}{2}\sum_{k=1}^{N}p_k^2+\sum_{k=1}^{N-1}V(q_k-q_{k+1}),$$

where the $\sum_{k=1}^{N-1}V(q_k-q_{k+1})$ term is called the potential term, and it describes the interaction between the neighboring particles. Then the evolution of the system is described by the the canonical Hamilton's equations

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} \ , \ \ \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}. \tag{1.1.1}$$

Unless the potential term is a polynomial of order 2 or less, this equation is nonlinear. In the nonlinear case, it is usually impossible to obtain the solutions explicitly. In a series of papers [33, 34], M. Toda introduced system of infinitely many particles with exponential interaction between them. He studied soliton solutions and periodic solutions of the system. Periodic solutions are solutions that satisfy periodic condition $q_{k+N} = q_k$ and $p_{k+N} = p_k$ for some integer N. The periodic solutions satisfy the finite particle system described by the Hamiltonian

$$\mathbf{H} = \frac{1}{2}\sum_{k=1}^{N} p_k^2 + \sum_{k=1}^{N-1} e^{q_k - q_{k+1}} + e^{q_N - q_1}.$$

This system is called the periodic Toda chain, and when $e^{q_N - q_1}$ is missing it is called the open Toda chain. Flaschka [5] and Manakov [25] independently found N functionally independent quantities that are conserved throughout the evolution, for both open and periodic N particle Toda chains. Such a system is said to be a completely integrable Hamiltonian system. Using the conserved quantities, Moser [26] and Kac and van Moerbeke [17],[18] respectively gave complete explicit solutions for the open and periodic Toda chains. Classical periodic Toda chain have regained the attention of physicists because of its appearance in N=2 Supersymmetric Yang-Mills theory [29, 27].

1.2 Quantum Open Toda chain

The Toda chain is a system in classical mechanics. After it has been completely solved, the question of whether the corresponding quantum mechanical system can also be solved explicitly gained interest. The Schrödinger equation for quantum Toda chain is obtained from the classical Toda chain by the standard procedure called the canonical quantization. For the quantum open Toda chain, the Schrödinger equation is

$$\left(-\frac{1}{2}\sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} + \sum_{k=1}^{N-1} e^{x_k - x_{k+1}}\right)\tilde{\Psi} = E\tilde{\Psi}, \tag{1.2.1}$$

where we have set $\hbar = 1$, which appears in the standard form of Schrödinger equation.

Surprisingly, the solution of this equation naturally appeared in the representation theory of Lie groups. Consider the Iwasawa decomposition of $\mathrm{SL}(N, \mathbb{R}) = \mathrm{ODU}$. Where O is the subgroup of orthogonal matrices, D is the subgroup of diagonal matrices of positive real numbers with determinant 1 and U is the subgroup of upper triangular matrices with 1's on the diagonal. Parametrizing each subgroup introduces a local coordinate system near the identity of $\mathrm{SL}(N, \mathbb{R})$. One can write down left invariant vector fields with respect to this coordinate system. Then consider the universal enveloping algebra of the Lie algebra of $\mathrm{SL}(N, \mathbb{R})$. Since set of left invariant vector fields with the Lie bracket is isomorphic to the Lie algebra, one can write any element of this universal enveloping algebra as a differential operator on $C^\infty(\mathrm{SL}(N, \mathbb{R}))$. Most notably, there is a natural second order central element of the universal algebra called the Casimir element, which under this realization will become a second order differential operator. Then we consider a function in $C^\infty(D)$, and find an appropriate extension of that function to an element in $C^\infty(\mathrm{SL}(N, \mathbb{R}))$, so that when Casimir element acts on it, some of the derivatives disappear and we get the differential equation of the quantum open Toda chain. The eigenfunction then coincides with Whittaker functions which is well known in representation theory. This was the observation of B. Kostant [22]. Semenov-Tian-Shansky [30] later proved Plancherel theorem for the eigenfunction expansion via Whittaker functions.

1.3 Quantum Periodic Toda chain

First attempt to find solutions of the equation (1.0.1) was done in Gutzwiller's papers [10],[11]. He constructed eigenfunctions for the case $N = 2, 3, 4$ by using a method he discovered, which nowadays is called Quantum Separation of Variables. For the eigenfunctions of the N particle quantum periodic Toda chain, Gutzwiller used an ansatz of a formal series of the eigenfunctions of the $N-1$ quantum open Toda chain. He found that the coefficients a_k of the series satisfy Baxter equation of the following form:

$$-a_{k-1} + a_{k+1} = t(k) a_k, \tag{1.3.1}$$

where $t(k)$ is some polynomial of k. This led to the solutions of (1.0.1) along with quantization condition involving a Hill determinant. The solutions for general N case was not dealt due to the lack of explicit solutions for the open quantum Toda chain and the algebraic complexity of deducing the Baxter's equation. Later, Sklyanin [31] used R-matrix formalism of Fadeev and Takhtajan [32] to drastically simplify the process of deriving the Baxter equations for general N particle case. Benefiting from Sklyanin's work and inspired by ideas from statistical mechanics, Pasquier and Gaudin [7] reinterpreted the Baxter's equation as an equation of operators, $\Lambda(u)Q(u) = i^N Q(u+i) - i^{-N} Q(u-i)$. They used clever arguments using the asymptotic information of the spectrum of $Q(v)$ to construct appropriate solutions of the Baxter equation along with the condition for their existence, which they concluded as the quantization condition for the quantum periodic Toda chain. But the paper did not address the problem of how to recover the solution of the quantum periodic Toda chain from the solution of the Baxter's equation they constructed. Nor did they rule out the possibility that other solutions of Baxter's equation fitting their description might exist. Kharchev and Lebedev [19], [20] worked with an integral transform using the eigenfunctions of the open Toda chain in the spirit of the original Gutzwiller's approach. When applied to the eigenfunctions of the periodic Toda chain, one obtains the Baxter's equation equivalent to Pasquier-Gaudin's. And because their integral transform had explicit inverse transform, they could use the solution constructed by Pasquier and Gaudin and invert it back to get the actual eigenfunction of the periodic Toda chain. They proved that inverse integral transform of the Pasquier-Gaudin solutions does converge and it does satisfy the quantum periodic Toda equation. So we now have only one question remaining - whether the inverse integral transform of the Pasquier-Gaudin solutions give complete set of solutions of the periodic Toda chain. By complete set, we mean that the solutions produces an eigenfunction expansion associated to the equation (1.0.1). The main goal of this thesis is to answer this question affirmatively.

For the readers who are already familiar with Pasquier-Gaudin [7] and Kharchev-Lebedev [19]'s work, we note here that the main goal of thesis was achieved by using uniform bounds rather than asymptotics. The estimates then are used to give some strong restrictions to the possible solutions of the Baxter's equation. The assumption that the solutions of the Baxter's equation takes the form of separated variables is not needed at all, but rather it will follow from the restrictions acquired from the uniform bounds.

Chapter 2

Preliminary Materials and Statement of the Main Theorem

Except for the statement of the main theorem and its corollaries, all materials in this chapter are from Kharchev and Lebedev [19] and [20], with slight changes in notation. Throughout this thesis, we will denote the Hamiltonian operator for the periodic Toda chain $\left(-\frac{1}{2}\sum_{k=1}^{N}\frac{\partial^2}{\partial x_k^2}+\sum_{k=1}^{N-1}e^{x_k-x_{k+1}}+e^{x_N-x_1}\right)$ by $\mathbf{H}$, so that the Schrödinger equation (1.0.1) can be written as $\mathbf{H}\tilde{\Psi}=E\tilde{\Psi}$. For the open Toda chain, we write $\mathbf{h}$ to denote $\left(-\frac{1}{2}\sum_{k=1}^{N}\frac{\partial^2}{\partial x_k^2}+\sum_{k=1}^{N-1}e^{x_k-x_{k+1}}\right)$.

2.1 Change of Variables and the Structure of the Eigenfunctions

It is convenient to make use of the center of mass

$$X_C=\frac{1}{N}(x_1+\cdots+x_N).$$

So we introduce a change of variables

$$u_j = \sqrt{\frac{j}{j+1}}\left(x_{j+1} - \frac{x_1 + \cdots + x_j}{j}\right) \qquad\qquad j = 1, \ldots, N-1\ ,$$

$$u_N = \sqrt{N}X_C = \frac{1}{\sqrt{N}}(x_1 + \cdots + x_N), \tag{2.1.1}$$

which is a slight modification of the usual Jacobi coordinates[a]. The difference is the presence of the coefficients $\sqrt{\frac{j}{j+1}}$ and $\sqrt{N}$, which is introduced so that the form of the Laplacian is invariant. We also introduce vectors

$$\mathbf{u} = (u_1, \ldots, u_{N-1}) \quad , \quad \mathbf{x} = (x_1, \ldots, x_{N-1}) \tag{2.1.2}$$

and use $(\mathbf{u}, u_N)$ (resp. $(\mathbf{x}, x_N)$) to mean $(u_1, \ldots, u_N)$ (resp. $(x_1, \ldots, x_N)$). In this change of variables, the equation for quantum periodic Toda chain (1.0.1) becomes

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial u_N^2} - \frac{1}{2}\triangle_{\mathbf{u}} + V(\mathbf{u})\right)\tilde{\Psi}(\mathbf{u}, u_N) = E\tilde{\Psi}(\mathbf{u}, u_N), \tag{2.1.3}$$

where $\triangle_{\mathbf{u}} = \left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \cdots + \frac{\partial^2}{\partial u_{N-1}^2}\right)$, and

$$V(\mathbf{u}) = \exp(-\sqrt{2}u_1) + \exp\left(\frac{N}{\sqrt{N^2-N}}u_{N-1} + \sum_{k=1}^{N-2}\frac{1}{\sqrt{k^2+k}}u_k\right) + \\ + \sum_{k=2}^{N-2}\exp\left(\frac{1}{\sqrt{k^2-k}}u_{k-1} - \frac{k+1}{\sqrt{k^2+k}}u_k\right). \tag{2.1.4}$$

The fact that the potential term $V(\mathbf{u})$ in the above equation is independent of u_N allows us to say that the generalized eigenfunctions[b] of the quantum periodic Toda chain have the following structure

$$\tilde{\Psi}(\mathbf{u}, u_N) = \Psi(\mathbf{u})e^{i\frac{E_1}{\sqrt{N}}u_N} \tag{2.1.5}$$

[a]This is a convenient coordinate system used in many body problems. (c.f. [21])

[b]Generalized eigenfunction means that is a solution of the eigenvalue problem in the sense of distributions.

for some constant [c] E_1. Plugging in (2.1.5) into (2.1.3) we get a reduced equation

$$\left(-\frac{1}{2}\triangle_{\mathbf{u}} + V(\mathbf{u})\right)\Psi(\mathbf{u}) = \tilde{E}\Psi(\mathbf{u}), \tag{2.1.6}$$

where $\tilde{E} = E - \frac{E_1^2}{N}$. Observe that $V(\mathbf{u})$ increases without bound in all directions. In such case, $-\frac{1}{2}\triangle_{\mathbf{u}} + V(\mathbf{u})$ has pure point spectrum and the $L^2(\mathbb{R}^{N-1})$ eigenfunctions form a complete set of basis for $L^2(\mathbb{R}^{N-1})$. Note that once we have all the $L^2(\mathbb{R}^{N-1})$ eigenfunctions, then functions of the form (2.1.5) can be used to produce an eigenfunction expansion since the $e^{i\frac{E_1}{\sqrt{N}}u_N}$ factor can be thought as the kernel of the Fourier transform. Hence to solve the eigenfunction expansion problem, it is enough to find all the eigenfunctions of this reduced equation. The eigenfunction $\Psi(\mathbf{u})$ itself has nice properties. It is an analytic function on $\mathbb{R}^{N-1}$ because $-\frac{1}{2}\triangle_{\mathbf{u}} + V(\mathbf{u})$ is an analytic hypoelliptic operator. It can also be analytically continued to an entire function on $\mathbb{C}^{N-1}$ (cf. [3]) since $V(\mathbf{u})$ is entire. In the next chapter, we will show that it is a Schwartz class[d] function.

2.2 Complete Set of Commuting Hamiltonians: The R Matrix Approach.

We review Faddeev and Takhtajan [32]'s construction of commuting Hamiltonians for the periodic Toda chain. Let L_k be

$$L_k(\lambda) = \begin{pmatrix} \lambda - \mathbf{P}_k & e^{-x_k} \\ -e^{x_k} & 0 \end{pmatrix}$$

for $k = 1, \ldots, N$. Here, $\mathbf{P}_k = -i\frac{\partial}{\partial x_k}$, which is the momentum operator. Define matrix $T(\lambda)$ as

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \stackrel{def}{=} L_N(\lambda)\cdots L_1(\lambda).$$

Let

$$\hat{t}(\lambda) = \operatorname{tr} T(\lambda) = A(\lambda) + D(\lambda).$$

$\hat{t}(\lambda)$ can be viewed as polynomial in λ with operator coefficients.

[c] E_1 is the eigenvalue of the operator $\mathbf{H}_1$, to be introduced in the next section.

[d] A function $f(\mathbf{x})$ belongs to Schwartz class if $f(\mathbf{x})$ is smooth and if $f(\mathbf{x})$ and all its partial derivatives decay to zero faster than any inverse polynomial at infinity.

Proposition 2.2.1. *The coefficients of $\hat{t}(\lambda)$ gives commuting Hamiltonians, i.e.*

$$\hat{t}(\lambda) = \sum_{k=0}^{N} (-1)^k \lambda^{N-k} \mathbf{H}_k,$$

where $\mathbf{H}_k$'s are self-adjoint operators such that $[\mathbf{H}_i, \mathbf{H}_j] = 0$ for all integers $1 \leq i, j \leq N$. Moreover, the Hamiltonian for quantum periodic toda chain can be written as $\mathbf{H} = \frac{1}{2}(\mathbf{H}_1)^2 - \mathbf{H}_2$.

Proof. Let P be a matrix on $\mathbb{R}^2 \otimes \mathbb{R}^2$ such that $P(\mathbf{a} \otimes \mathbf{b}) = \mathbf{b} \otimes \mathbf{a}$. Define R-matrix as $I_2 \otimes I_2 + \frac{i}{\lambda-\mu} P$. Written explicitly,

$$R(\lambda - \mu) = \begin{pmatrix} 1 + \frac{i}{(\lambda-\mu)} & & & \\ & 1 & \frac{i}{(\lambda-\mu)} & \\ & \frac{i}{(\lambda-\mu)} & 1 & \\ & & & 1 + \frac{i}{(\lambda-\mu)} \end{pmatrix}.$$

Then the commutation relations $[e^{x_k}, \mathbf{P}_k] = ie^{x_k}$ and $[e^{-x_k}, \mathbf{P}_k] = -ie^{-x_k}$ can be used to verify that matrices L_k satisfy the quantum Yang-Baxter equation[e] $R(L_k(\lambda) \otimes I_2)(I_2 \otimes L_k(\mu)) = (L_k(\mu) \otimes I_2)(I_2 \otimes L_k(\lambda))R$ holds. Multiplying these identities for $k = 1, \ldots, N$, we get

$$R(T(\lambda) \otimes I_2)(I_2 \otimes T(\mu)) = (T(\mu) \otimes I_2)(I_2 \otimes T(\lambda))R. \tag{2.2.1}$$

Taking the trace of this equation, we obtain $[\hat{t}(\lambda), \hat{t}(\mu)] = 0$. Since λ and μ are arbitrary, this means that the coefficients of the polynomials commute with each other. The first few Hamiltonians can be written as

$$\mathbf{H}_1 = \sum_{k=1}^{N} \mathbf{P}_k,$$

$$\mathbf{H}_2 = \sum_{j<k} \mathbf{P}_j \mathbf{P}_k - \sum_{k=1}^{N} e^{x_k - x_{k+1}},$$

$$\mathbf{H}_3 = \sum_{j<k<l} \mathbf{P}_j \mathbf{P}_k \mathbf{P}_l + \cdots .$$

Now it is straightforward to see that $\mathbf{H} = \frac{1}{2}(\mathbf{H}_1)^2 - \mathbf{H}_2$. □

[e]The Yang-Baxter equation is fundamental in the theory of classical and quantum integrable systems. The argument described here first appeared in [36].

holds, i.e. some symmetric functions of $\boldsymbol{\gamma}$ are the eigenvalues. Kostant [22] observed that $\phi_{\boldsymbol{\gamma}}$ are the Whittaker functions that was previously studied in representation theory of Lie groups. The analytic properties of Whittaker functions [16, 12, 13] yield that equation (2.3.3) defines $\phi_{\boldsymbol{\gamma}}$ uniquely up to a constant if the following conditions are imposed :

(a) The solution vanishes very rapidly as $(x_k - x_{k+1}) \to \infty$.

$$\phi_{\boldsymbol{\gamma}}(\mathbf{x}) \sim \exp\{-2e^{(x_k - x_{k+1})}\} \quad \text{as} \;\; (x_k - x_{k+1}) \to \infty.$$

(b) As a function of $\boldsymbol{\gamma}$, $\phi_{\boldsymbol{\gamma}}$ is symmetric under any permutation

$$\phi_{\dots\gamma_j\dots\gamma_k\dots} = \phi_{\dots\gamma_k\dots\gamma_j\dots}.$$

(c) $\phi_{\boldsymbol{\gamma}}$ can be analytically continued to an entire function of $\boldsymbol{\gamma} \in \mathbb{C}^{N-1}$ and the following asymptotics hold:

$$\psi_\gamma \sim |\gamma_j|^{(2-N)/2} \exp\left\{-\frac{\pi}{2}(N-2)|\gamma_j|\right\}$$

as $|\operatorname{Re}\gamma_j| \to \infty$ in a finite strip of the complex plane.

Condition (a) alone fixes the eigenfunctions up to a common $\boldsymbol{\gamma}$-related factor. Conditions (b) and (c) together fixes the factor up to a constant. To fix the constant factor and to make the definition explicit, we introduce the definition given by Kharchev and Lebedev.

Definition 2.3.1. We define $\phi_{\boldsymbol{\gamma}}$ inductively as follows.

(1) $\phi_{\gamma_1} = \exp(i\gamma_1 x_1)$.

(2) Let $\phi_{\gamma_1,\dots,\gamma_{n-1}}(x_1,\dots,x_{n-1})$ be the $n-1$ particle generalized eigenfunction. Then $\phi_{\lambda_1,\dots,\lambda_n}(x_1,\dots,x_n)$ is defined by the following $n-1$ fold integral

$$\begin{aligned} \phi_{\lambda_1,\dots,\lambda_n}(x_1,\dots,x_n) &= \int_{\mathcal{C}} Q(\boldsymbol{\gamma};\boldsymbol{\lambda})\phi_{\boldsymbol{\gamma}}(x_1,\dots,x_{n-1}) \times \\ &\quad \times e^{\left(ix_N\left(\sum_{j=1}^{n}\lambda_j - \sum_{j=1}^{n-1}\gamma_j\right)\right)}\mu(\boldsymbol{\gamma})d\boldsymbol{\gamma}, \end{aligned} \tag{2.3.4}$$

where the integration is performed along the horizontal lines with $\operatorname{Im}\gamma_j > \max_k\{\operatorname{Im}\lambda_k\}$, and $\mu(\gamma)$ and $Q(\boldsymbol{\gamma}|\boldsymbol{\lambda})$ is defined by

$$\mu(\boldsymbol{\gamma}) = \frac{(2\pi)^{1-n}}{(n-1)!}\prod_{j<k}\frac{\gamma_k-\gamma_j}{\pi}\sinh\left(\pi(\gamma_k-\gamma_j)\right), \tag{2.3.5}$$

$$Q(\gamma_1,\ldots,\gamma_{n-1}|\lambda_1,\ldots,\lambda_n) = \prod_{j=1}^{n-1}\prod_{k=1}^{n}\Gamma(i\lambda_k - i\gamma_j). \tag{2.3.6}$$

This system of generalized eigenfunctions is complete as can be seen from the Plancherel theorem proved by Semenov-Tian-Shansky [30], which says that the integral operator

$$Uf(\boldsymbol{\gamma}) = \int_{\mathbb{R}^{N-1}} f(\mathbf{y})\overline{\phi_{\boldsymbol{\gamma}}(\mathbf{y})}d\mathbf{y} \tag{2.3.7}$$

is a unitary operator from $L^2(\mathbb{R}^{N-1}, d\mathbf{x})$ to $L^2(\mathbb{R}^{N-1}, \mu(\boldsymbol{\gamma})d\boldsymbol{\gamma})$. Being unitary means that the adjoint operator gives the inverse transform, i.e.

$$\int_{\mathbb{R}^{N-1}} Uf\,(\boldsymbol{\gamma})\,\phi_{\boldsymbol{\gamma}}(\mathbf{x})\mu(\boldsymbol{\gamma})d\boldsymbol{\gamma} = f(\mathbf{x}) \tag{2.3.8}$$

holds in L^2 sense.

In the $N = 3$ case, the definition of $\phi_{\boldsymbol{\gamma}}$ and the Plancherel theorem of Semenov-Tian-Shansky can be restated using the MacDonald function[f] $\mathrm{K}_\nu(x)$ as

$$\phi_{\gamma_1,\gamma_2}(x_1,x_2) = 2e^{\frac{i}{2}(\gamma_1+\gamma_2)(x_1+x_2)}\mathrm{K}_{i(\gamma_1-\gamma_2)}(2e^{\frac{x_1-x_2}{2}})$$

and

$$f(x_1,x_2) = \frac{1}{8\pi^2}\int_{\mathbb{R}^2\times\mathbb{R}^2}\Big(f(y_1,y_2)e^{\frac{i}{2}(\gamma_1+\gamma_2)(x_1+x_2-y_1-y_2)}\times$$
$$\times\overline{\mathrm{K}_{i(\gamma_1-\gamma_2)}(2e^{\frac{y_1-y_2}{2}})}\mathrm{K}_{i(\gamma_1-\gamma_2)}(2e^{\frac{x_1-x_2}{2}})\sinh(\gamma_2-\gamma_1)\Big)\,d\mathbf{y}d\boldsymbol{\gamma}.$$

This special case is known as Kontorovich-Lebedev Transform (c.f. [4]).

[f] Also known as the modified Bessel function of second kind.

2.4 Integral Transform Using Eigenfunctions of the N-1 Open Toda Lattice.

We introduce the auxiliary function

$$\tilde{\Phi}_{\boldsymbol{\gamma}}(x_1,\ldots,x_N) = e^{-ix_N(\gamma_1+\cdots+\gamma_{N-1})}\phi_{\boldsymbol{\gamma}}(x_1,\ldots,x_{N-1}). \tag{2.4.1}$$

First observation is that

$$\frac{\partial}{\partial u_N}\tilde{\Phi}_{\boldsymbol{\gamma}}(\mathbf{x},x_N) = \frac{1}{\sqrt{N}}(\frac{\partial}{\partial x_1}+\cdots+\frac{\partial}{\partial x_N})\tilde{\Phi}_{\boldsymbol{\gamma}}(\mathbf{x},x_N) = 0.$$

So it is independent of u_N and we may define, in the changed coordinates

$$\Phi_{\boldsymbol{\gamma}}(\mathbf{u}) = \tilde{\Phi}_{\boldsymbol{\gamma}}(\mathbf{u},u_N). \tag{2.4.2}$$

The integral transform introduced by Kharchev and Lebedev is[g]

$$\xi_{\mathbf{E}}(\boldsymbol{\gamma}) = \int_{\mathbb{R}^{N-1}} \Psi_{\mathbf{E}}(\mathbf{u})\overline{\Phi_{\overline{\boldsymbol{\gamma}}}(\mathbf{u})}d\mathbf{u}. \tag{2.4.3}$$

It will be shown in the appendix that if $\xi_{\mathbf{E}}(\boldsymbol{\gamma})$ satisfies some nice conditions, then using the Plancherel theorem we can recover the original function by

$$\frac{1}{2\pi}\int_{\mathbb{R}^{N-1}} \xi_{\mathbf{E}}(\boldsymbol{\gamma})\,\Phi_{\boldsymbol{\gamma}}(\mathbf{u})\mu(\boldsymbol{\gamma})d\boldsymbol{\gamma} = \Psi_{\mathbf{E}}(\mathbf{u}). \tag{2.4.4}$$

2.5 The Baxter Equation

Let

$$t_{\mathbf{E}}(\lambda) = \sum_{k=0}^{N}(-1)^k\lambda^{N-k}E_k,$$

so that

$$\hat{t}(\lambda)\tilde{\Psi}_{\mathbf{E}}(\mathbf{x}) = t_{\mathbf{E}}(\lambda)\tilde{\Psi}_{\mathbf{E}}(\mathbf{x}).$$

Let $\mathbf{e}_j$, $j=1,\ldots,N$ be the standard unit vectors.

[g] The $\boldsymbol{\gamma}$ is conjugated so that the integral is holomorphic rather than anti-holomorphic.

Proposition 2.5.1. *Suppose $E_1 = 0$. Then*

$$t_{\mathbf{E}}(\gamma_j)\xi_{\mathbf{E}}(\boldsymbol{\gamma}) = i^N \xi_{\mathbf{E}}(\boldsymbol{\gamma} + i\mathbf{e}_j) + i^{-N}\xi_{\boldsymbol{E}}(\boldsymbol{\gamma} - i\mathbf{e}_j) \tag{2.5.1}$$

holds, which we call the Baxter equation.

Lemma 2.5.2.

$$\hat{t}(\gamma_j)\Phi_{\boldsymbol{\gamma}}(\mathbf{u}) = i^{-N}\Phi_{\boldsymbol{\gamma}+i\mathbf{e}_j}(\mathbf{u}) + i^N \Phi_{\boldsymbol{\gamma}-i\mathbf{e}_j}(\mathbf{u}).$$

Proof. From equation (2.2.1) we obtain the following relation.

$$(\lambda - \mu + i)D_N(\mu)C_N(\lambda) = (\lambda - \mu)C_N(\lambda)D_N(\mu) + iD_N(\lambda)C_N(\mu).$$

Now set $\mu = \gamma_j$ and apply both sides to $\phi_{\boldsymbol{\gamma}}(\mathbf{x})$. Then using (2.3.3) and the fact that $D_N(\gamma_j)$ does not contain x_N and therefore commutes with e^{x_N}, we have

$$C_N(\lambda)\ (D_N(\gamma_j)\phi_{\boldsymbol{\gamma}}(\mathbf{x})) = -e^{x_N}(\lambda - \gamma_j + i)\prod_{k\neq j}(\lambda - \gamma_k + i\delta_j^k)\ (D_N(\gamma_j)\phi_{\boldsymbol{\gamma}}(\mathbf{x}))\,,$$

where δ_j^k is the Kronecker delta function. Along with this, we can show that $D_N(\gamma_j)\phi_{\boldsymbol{\gamma}}(\mathbf{x})$ satisfies the conditions (a)-(c) in section 2.3. Then the uniqueness of the Whittaker function tells us that $D_N(\gamma_j)\phi_{\boldsymbol{\gamma}}(\mathbf{x})$ is a constant multiple of $\phi_{\boldsymbol{\gamma}+i\mathbf{e}_j}(\mathbf{x})$. To determine the coefficients, we look at the asymptotics of both functions at large values of $x_{k+1} - x_k$ for $k = 1, \ldots, N-2$. We omit this process as it is sketched in [19], and simply state that we get[h]

$$D_N(\gamma_j)\phi_{\boldsymbol{\gamma}}(\mathbf{x}) = i^N e^{x_N}\phi_{\boldsymbol{\gamma}+i\mathbf{e}_k}(\mathbf{x}).$$

Now, by considering the quantum determinant (2.2.3), we immediately get also

$$A_N(\gamma_j)\phi_{\boldsymbol{\gamma}}(\mathbf{x}) = i^{-N}e^{-x_N}\phi_{\boldsymbol{\gamma}-i\mathbf{e}_k}(\mathbf{x}).$$

These two equations yield the desired result. ◻

Now we prove proposition 2.5.1.

[h]For the case $N = 2$, this becomes a well known relation in Bessel functions.

Proof. Let $\hat{t}'(\lambda)$ be gotten from $\hat{t}(\lambda)$ by changing variables from $(\mathbf{x}, x_N)$ to $(\mathbf{u}, u_N)$, and then deleting all terms containing $\frac{\partial}{\partial u_N}$. Since $\mathbf{H}_1\Psi_{\mathbf{E}}(\mathbf{u})$ does not depend on u_N,

$$\hat{t}(\lambda)\Psi_{\mathbf{E}}(\mathbf{u}) = \hat{t}'(\lambda)\Psi_{\mathbf{E}}(\mathbf{u}).$$

Using this we have

$$\begin{aligned}
t_{\mathbf{E}}(\gamma_j)\xi_{\mathbf{E}}(\boldsymbol{\gamma}) &= t_{\mathbf{E}}(\gamma_j)\int_{\mathbb{R}^{N-1}} \Psi_{\mathbf{E}}(\mathbf{u})\overline{\Phi_{\overline{\boldsymbol{\gamma}}}(\mathbf{u})}d\mathbf{u} \\
&= \int_{\mathbb{R}^{N-1}} \left(\hat{t}'(\gamma_j)\Psi_{\mathbf{E}}(\mathbf{u})\ \right)\overline{\Phi_{\overline{\boldsymbol{\gamma}}}(\mathbf{u})}d\mathbf{u} \\
&= \int_{\mathbb{R}^{N-1}} \Psi_{\mathbf{E}}(\mathbf{u})\left(\overline{\hat{t}'(\overline{\gamma_j})\Phi_{\overline{\boldsymbol{\gamma}}}(\mathbf{u})}\right)d\mathbf{u} \\
&= \int_{\mathbb{R}^{N-1}} \Psi_{\mathbf{E}}(\mathbf{u})\left(\overline{i^{-N}\Phi_{\overline{\boldsymbol{\gamma}}-i\mathbf{e}_j}(\mathbf{u}) + i^{N}\Phi_{\overline{\boldsymbol{\gamma}}+i\mathbf{e}_j}(\mathbf{u})}\right)d\mathbf{u},
\end{aligned}$$

where integration by parts was used in going from the second line to third line[i], and the last equality is essentially the equation in the lemma in terms of variable $\mathbf{u}$. □

Proposition 2.5.3. $\xi_{\mathbf{E}}(\boldsymbol{\gamma})$ *is entire in each* γ_j*'s.*

Proof. $\overline{\Phi_{\overline{\boldsymbol{\gamma}}}(\mathbf{u})}$ is entire. So the proof is complete if can justify the interchange of differentiation and integration, i.e.

$$\frac{\partial}{\partial\gamma_j}\int_{\mathbb{R}^{N-1}} \Psi_{\mathbf{E}}(\mathbf{u})\overline{\Phi_{\overline{\boldsymbol{\gamma}}}(\mathbf{u})}d\mathbf{u} = \int_{\mathbb{R}^{N-1}} \frac{\partial}{\partial\gamma_j}\Psi_{\mathbf{E}}(\mathbf{u})\overline{\Phi_{\overline{\boldsymbol{\gamma}}}(\mathbf{u})}d\mathbf{u}.$$

This is justified as long as the integrand $\Psi_{\mathbf{E}}(\mathbf{u})\overline{\Phi_{\overline{\boldsymbol{\gamma}}}(\mathbf{u})}$ and all its first partial derivatives are bounded by $\mathrm{L}^1(\mathbb{R}^{N-1})$ functions of $\mathbf{u}$ uniformly with respect to $\boldsymbol{\gamma}$ (See [6]). The existence of such $\mathrm{L}^1(\mathbb{R}^{N-1})$ functions will be apparent from the bounds we prove in the next chapter. □

[i]The fact that the boundary terms vanish follows from the rapid decay of $\Psi_{\mathbf{E}}(\mathbf{u})$ which will follow from the estimates proved in the next chapter.

2.6 Fundamental Solutions of the Baxter Equation and the Hill Determinant

Sklyanin [31] considered a solution of the Baxter equation (2.5.1) taking the form of products of one variable functions as follows.

$$\xi_{\mathbf{E}}(\boldsymbol{\gamma}) = \prod_{j=1}^{N-1} \sigma_j(\gamma_j). \tag{2.6.1}$$

Substituting this into the Baxter equation (2.5.1), we obtain the following functional-difference equation.

$$t_{\mathbf{E}}(z)\sigma(z) = i^N \sigma(z+i) + i^{-N}\sigma(z-i). \tag{2.6.2}$$

This approach is called the quantum separation of variables or Sklyanin's separation of variables.

Definition 2.6.1. The two fundamental solutions $\sigma_+(z)$ and $\sigma_-(z)$ of the functional difference equation (2.6.2) are

$$\sigma_\pm(z) = e^{-N\pi z} \frac{K_\pm(z)}{\prod_{k=1}^{N} \Gamma\big(1 \mp i(z-\theta_k(\mathbf{E}))\big)}. \tag{2.6.3}$$

where $\theta_k(\mathbf{E})$'s are N zeros of $t_{\mathbf{E}}(z)$ and $K_\pm(z)$ are functions defined by the following (semi-)infinite determinants:

$$K_+(z) = \begin{vmatrix} 1 & \frac{1}{t_{\mathbf{E}}(z+i)} & 0 & \\ \frac{1}{t_{\mathbf{E}}(z+2i)} & 1 & \frac{1}{t_{\mathbf{E}}(z+2i)} & \\ 0 & \frac{1}{t_{\mathbf{E}}(z+3i)} & 1 & \ddots \\ & & \ddots & \ddots \end{vmatrix}, \tag{2.6.4}$$

$$K_-(z) = \begin{vmatrix} \ddots & \ddots & & \\ \ddots & 1 & \frac{1}{t_{\mathbf{E}}(z-3i)} & 0 \\ & \frac{1}{t_{\mathbf{E}}(z-2i)} & 1 & \frac{1}{t_{\mathbf{E}}(z-2i)} \\ & 0 & \frac{1}{t_{\mathbf{E}}(z-i)} & 1 \end{vmatrix}. \tag{2.6.5}$$

The determinants $K_{\pm}(z)$ converge absolutely and uniformly with respect to z on compact sets away from poles [35]. Hence $K_{\pm}(z)$ are meromorphic. By performing row expansion in the top row of $K_+(z-i)$ we get the following identity

$$K_+(z-i) = K_+(z) - \frac{1}{t_{\mathbf{E}}(z)t_{\mathbf{E}}(z+i)} K_+(z+i). \tag{2.6.6}$$

Similarly we have,

$$K_-(z+i) = K_-(z) - \frac{1}{t_{\mathbf{E}}(z)t_{\mathbf{E}}(z-i)} K_-(z-i). \tag{2.6.7}$$

In the definition of $\sigma_{\pm}(z)$ we see that the poles of $K_{\pm}(z)$ are canceled by the poles of Γ functions so that $\sigma_{\pm}(z)$ are entire functions. The fact that $\sigma_{\pm}(z)$ satisfy (2.6.2) follows from equations (2.6.6) and (2.6.7).

We introduce another important object, the Wronskian of two solutions $\sigma_1(z)$ and $\sigma_2(z)$ of (2.6.2),

$$W[\sigma_1;\sigma_2](z) := \sigma_1(z)\sigma_2(z+i) - \sigma_1(z+i)\sigma_2(z). \tag{2.6.8}$$

Using the Baxter equation (2.6.2) we see that

$$W[\sigma_1;\sigma_2](z+i) = (-1)^N W[\sigma_1;\sigma_2](z). \tag{2.6.9}$$

An immediate consequence of this is that Wronskian vanishes at z_0 if and only if there exists some coefficients a and b (not both zero) such that $a\sigma_1(z_0+ni) + b\sigma_2(z_0+ni) = 0$ for all integers n.

Proposition 2.6.2. *The zeros of $W[\sigma_+;\sigma_-](z)$ coincides with the zeros of the infinite determinant*

$$\begin{vmatrix} \ddots & \ddots & & & & & \\ \ddots & 1 & \frac{1}{t_{\mathbf{E}}(z-2i)} & & & & \\ & \frac{1}{t_{\mathbf{E}}(z-i)} & 1 & \frac{1}{t_{\mathbf{E}}(z-i)} & & 0 & \\ & & \frac{1}{t_{\mathbf{E}}(z)} & 1 & \frac{1}{t_{\mathbf{E}}(z)} & & \\ & & & \frac{1}{t_{\mathbf{E}}(z+i)} & 1 & \frac{1}{t_{\mathbf{E}}(z+i)} & \\ & 0 & & & \frac{1}{t_{\mathbf{E}}(z+2i)} & 1 & \ddots \\ & & & & & \ddots & \ddots \end{vmatrix}.$$

Proof. The infinite determinant converges absolutely and uniformly on compact sets (see [35]). So it defines a meromorphic function. Let $\mathcal{H}(z)$ be the function defined by the above determinant. By performing a row expansion in the row containing $\frac{1}{t_{\mathbf{E}}(z)}$ and performing column expansions to the subsequent matrices in the column that contains $\frac{1}{t_{\mathbf{E}}(z-i)}$ and $\frac{1}{t_{\mathbf{E}}(z+i)}$, one obtains

$$\begin{aligned}\mathcal{H}(z) &= -\frac{1}{t_{\mathbf{E}}(z)}\frac{1}{t_{\mathbf{E}}(z-i)}K_-(z-i)K_+(z)+\\ &\quad +K_+(z)K_-(z)-\frac{1}{t_{\mathbf{E}}(z)}\frac{1}{t_{\mathbf{E}}(z+i)}K_-(z)K_+(z+i).\end{aligned}$$

Then applying (2.6.7) gives us the following identity.

$$\mathcal{H}(z) = K_+(z)K_-(z+i) - \frac{K_+(z+i)K_-(z)}{t_{\mathbf{E}}(z)t_{\mathbf{E}}(z+i)}.$$

Using the definition of $\sigma_\pm(z)$ in (2.6.1), we have

$$W[\sigma_+,\sigma_-](z) = i^{-N}\mathcal{H}(z)\prod_{k=1}^{N}\pi^{-1}\sinh\pi(z-\theta_k) \tag{2.6.10}$$

and the poles of $\mathcal{H}(z)$ cancel all zeros of $\prod_{k=1}^{N}\pi^{-1}\sinh\pi(z-\theta_k)$. ☐

The determinant $\mathcal{H}(z)$ is called the Hill determinant. It is known that [35]

Proposition 2.6.3. *The Hill determinant can be represented as*

$$\mathcal{H}(z) = 1 + \sum_{k=1}^{N}\alpha_k(\mathbf{E})\coth(z-\theta_k(\mathbf{E})), \tag{2.6.11}$$

where $\alpha_k(\mathbf{E})$ are some constants depending on $\mathbf{E}$. It has exactly N zeros $\delta_1(\mathbf{E}),\ldots,\delta_N(\mathbf{E})$ in the strip $0 \le \operatorname{Im} z < 1$.

2.7 Statement of the Main Theorem

We introduce the Pasquier and Gaudin [7] solutions of the Baxter equation (2.5.1).

Definition 2.7.1. Let

$$\begin{aligned}\Xi = \{ \quad \mathbf{E} \quad &\in \mathbb{R}^N \Big| \exists (a,b) \in \mathbb{R}^2 \diagdown \{(0,0)\} \text{ such that} \\ &a\sigma_+(\delta_j(\mathbf{E})) + b\sigma_-(\delta_j(\mathbf{E})) = 0 \text{ for } j = 1, \dots, N\}\end{aligned}$$

and also define

$$\Xi_0 = \{\mathbf{E} \in \Xi \mid E_1 = 0\}.$$

The equations in the definition of the set Ξ is the quantization condition of the quantum periodic Toda chain first proposed by Gutzwiller [11] after establishing it for $N = 2, 3, 4$.

Definition 2.7.2. For any $\mathbf{E} \in \Xi$, the Pasquier-Gaudin solutions of the Baxter Equation (2.5.1) is

$$\xi_{\mathbf{E}}(\boldsymbol{\gamma}) = \prod_{j=1}^{N-1} e^{N\gamma_j} \frac{a\sigma_+(\gamma_j) + b\sigma_-(\gamma_j)}{\prod_{k=1}^{N} \sinh \pi(\gamma_j - \delta_k(\mathbf{E}))},$$

where a and b are the pair of constants satisfying $a\sigma_+(\delta_j(\mathbf{E})) + b\sigma_-(\delta_j(\mathbf{E})) = 0$ as in the definition of Ξ.

Kharchev and Lebedev [19] proved the following theorem.

Theorem 2.7.3. *Suppose $\mathbf{E} \in \Xi$ and $\xi_{\mathbf{E}}(\boldsymbol{\gamma})$ is a Pasquier-Gaudin solution. Then*

$$\Psi_{\mathbf{E}}(\mathbf{u}) = \frac{1}{2\pi} \int_{\mathbb{R}^{N-1}} \xi_{\mathbf{E}}(\boldsymbol{\gamma})\, \Phi_{\boldsymbol{\gamma}}(\mathbf{u}) \mu(\boldsymbol{\gamma}) d\boldsymbol{\gamma} \tag{2.7.1}$$

is an $L^2(\mathbb{R}^{N-1})$ function and $\tilde{\Psi}_{\mathbf{E}}(\mathbf{u}, u_N) = \Psi_{\mathbf{E}}(\mathbf{u}) e^{i\frac{E_1}{\sqrt{N}} u_N}$ satisfies equation (2.3.1).

Now we state the main theorem to be proved in this thesis.

Theorem 2.7.4. *(Main Theorem) Suppose there exist $\mathbf{E} = (E_1, E_2, \dots, E_N)$ and an $L^2(\mathbb{R}^{N-1})$ function $\Psi_{\mathbf{E}}(\mathbf{u})$ such that $\tilde{\Psi}_{\mathbf{E}}(\mathbf{u}, u_N) = \Psi_{\mathbf{E}}(\mathbf{u}) e^{i\frac{E_1}{\sqrt{N}} u_N}$ satisfies equation (2.3.1). Then $\mathbf{E} \in \Xi$ and*

$$\int_{\mathbb{R}^{N-1}} \Psi_{\mathbf{E}}(\mathbf{u}) \overline{\Phi_{\overline{\boldsymbol{\gamma}}}(\mathbf{u})} d\mathbf{u} = \prod_{j=1}^{N-1} e^{N\pi\gamma_j} \frac{a\sigma_+(\gamma_j) + b\sigma_-(\gamma_j)}{\prod_{k=1}^{N} \sinh \pi(\gamma_j - \delta_k(\mathbf{E}))} \tag{2.7.2}$$

for some constants a and b satisfying $a\sigma_+(\delta_1(\mathbf{E})) + b\sigma_-(\delta_1(\mathbf{E})) = 0$.

Since it is apparent from definition 2.7.2 that Pasquier-Gaudin solutions are unique up to a constant multiple for a given $\mathbf{E} \in \Xi$, we have

Corollary 2.7.5. *The joint spectrum of of commuting Hamiltonians $\mathbf{H}_k$'s of the Periodic Toda chain is simple.*

For any $\mathbf{E} \in \Xi_0$, the normalized joint eigenfunction is

$$\chi_{\mathbf{E}}(\mathbf{u}) = \frac{1}{\mathcal{N}_{\mathbf{E}}} \int_{\mathbb{R}^{N-1}} \prod_{j=1}^{N-1} e^{N\pi\gamma_j} \frac{a\sigma_+(\gamma_j) + b\sigma_-(\gamma_j)}{\prod_{k=1}^{N} \sinh \pi(\gamma_j - \delta_k(\mathbf{E}))} \, \Phi_{\boldsymbol{\gamma}}(\mathbf{u})\mu(\boldsymbol{\gamma})d\boldsymbol{\gamma}, \tag{2.7.3}$$

where $\mathcal{N}_{\mathbf{E}}$ is a normalization constant.

The two theorems 2.7.3 and 2.7.4 establish a bijection between the eigenfunctions and the Pasquier-Gaudin solutions. Considering the structure of the generalized eigenfunctions described in (2.1.5) and the discussion thereafter, we have the following eigenfunction expansion.

Corollary 2.7.6. *Let $f(\mathbf{u}, u_N) \in L^2(\mathbb{R}^N) \cap C(\mathbb{R}^N)$. For each $\mathbf{E} \in \Xi_0$, define*

$$\hat{f}_{\mathbf{E}}(\gamma) = \int_{\mathbb{R}^N} f(\mathbf{u}, u_N)\chi_{\mathbf{E}}(\mathbf{u})e^{i\gamma u_N} d\mathbf{u}du_N.$$

Then

$$f(\mathbf{u}, u_N) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\sum_{\mathbf{E}\in\Xi_0} \hat{f}_{\mathbf{E}}(\gamma)\chi_{\mathbf{E}}(\mathbf{u})e^{-iu_N\gamma} \right) d\gamma$$

and this expansion can be extended to hold on $L^2(\mathbb{R}^N)$ functions.

Chapter 3

Estimates for the Eigenfunctions

3.1 Uniform Bounds for Eigenfuctions of Periodic Toda Chain and Its Derivatives.

The Agmon metric is a metric introduced by S. Agmon [1] to study the decay of L^2-eigenfunctions of second order elliptic PDEs.

Definition 3.1.1. The Agmon metric is

$$\rho_{\lambda,V}(\mathbf{x},\mathbf{y}) := \inf_{\substack{\mathbf{r}\,:\,[0,1]\to\mathbb{R}^n \\ \text{piecewise } C^1 \text{ and} \\ \mathbf{r}(0)=\mathbf{x},\mathbf{r}(1)=\mathbf{y}}} \left\{ \int_0^1 (V(\mathbf{r}(t)) - \lambda)_+^{\frac{1}{2}} \, \|\mathbf{r}'(t)\| dt \right\},$$

where $(f)_+$ means $\max\{f, 0\}$.

There are several decay results involving the Agmon metric. The version we need in this paper is a pointwise bound proved by B. Simon and Carmona [2].

Theorem 3.1.2. *Let V be a real and continuous function on $\mathbb{R}^n$ that is bounded below. Suppose λ is a constant that makes $\{\mathbf{x}|\lambda - V(\mathbf{x}) \geq 0\}$ a* *compact set. Then any $L^2(\mathbb{R}^n)$ eigenfunction f of the equation $(-\triangle+V)f = \lambda f$ satisfies*

$$|f| \leq C_\epsilon e^{-(1-\epsilon)\rho_{\lambda,\breve{V}}(\mathbf{0},\mathbf{x})}, \tag{3.1.1}$$

where ϵ is any positive number, C_ϵ is some constant that depends only on ϵ and $\breve{V}$ is a function obtained from V by

$$\breve{V}(\mathbf{x}) = \inf_{|\mathbf{x}-\mathbf{y}|\leq 1} V(\mathbf{y}).$$

In the theorem, where $\breve{V}$ is defined, we chose a ball of radius 1 centered at $\mathbf{x}$ to take the infimum. But actually, the radius can be any small size as long as it is fixed. We note one observation here, that if V_1 and V_2 are two functions such that $V_1 \geq V_2$ everywhere on $\mathbb{R}^n$, then $\breve{V_1} \geq \breve{V_2}$.

Corollary 3.1.3. *The eigenfunction $\Psi_{E_2,\ldots,E_N}(\mathbf{u})$ satisfies the bound*

$$|\Psi_{\mathbf{E}}(\mathbf{u})| \ll \exp\left(-Ce^{C'\|\mathbf{u}\|}\right) \tag{3.1.2}$$

for some positive constants C,C'. Here $\ll$ is the Hardy symbol which means that $\leq$ holds for some positive constant multiple of the right side.

Proof. Recall that $\Psi_{\mathbf{E}}(\mathbf{u})$ satisfies equation (2.1.6) and hence we can apply theorem 3.1.2. From equation (2.1.4), it is easy to see that there exists a positive real number κ such that

$$V(\mathbf{u}) \geq e^{\kappa\|\mathbf{u}\|}.$$

After defining $V_2(\mathbf{u}) = e^{\kappa\|\mathbf{u}\|}$, we have the following relation between the Agmon metric of the two

$$\rho_{E,\breve{V}}(\mathbf{0},\mathbf{u}) \geq \rho_{E,\breve{V_2}}(\mathbf{0},\mathbf{u}).$$

Let us give a lower bound for $\rho_{\lambda,\breve{V_2}}(\mathbf{0},\mathbf{u})$. First, it is easy to see that $\breve{V_2}(\mathbf{u}) = e^{\kappa(\|\mathbf{u}\|-1)_+}$. Let $\mathbf{r}(t)$ be a piecewise C^1 path such that $\mathbf{r}(0) = 0$ and $\mathbf{r}(1) = \mathbf{u}$. Let $r(t) = \|\mathbf{r}(t)\|$ and let $\mathbf{w}(t) = \frac{\mathbf{r}(t)}{r(t)}$, so that $\mathbf{r}(t) = r(t)\mathbf{w}(t)$. Then from the fact that $\langle \mathbf{w}(t), \mathbf{w}'(t)\rangle = 0$, we have

$$\|\mathbf{r}'(t)\| = \|r'(t)\mathbf{w}(t) + r(t)\mathbf{w}'(t)\| = \sqrt{(r'(t))^2 + (r(t)\ \|\mathbf{w}'(t)\|)^2} \geq |r'(t)|.$$

So that

$$\int_0^1 (e^{\kappa(\|\mathbf{r}(t)\|-1)_+} - E)_+^{\frac{1}{2}} \|\mathbf{r}'(t)\| dt \geq \int_0^1 (e^{\kappa(r(t)-1)_+} - E)_+^{\frac{1}{2}} |r'(t)| dt$$

$$\geq \int_0^1 (e^{\kappa(r(t)-1)_+} - E)_+^{\frac{1}{2}} r'(t) dt \geq \int_0^{\|\mathbf{u}\|} \left(e^{\frac{1}{2}\kappa(r-1)_+} - \sqrt{E} \right) dr$$

$$\geq \frac{2}{\kappa} e^{-\frac{1}{2}\kappa} \left(e^{\frac{1}{2}\kappa\|\mathbf{u}\|} - 1 \right) - \sqrt{E}\|u\|.$$

So in a ball of large radius where

$$\frac{2}{\kappa} e^{-\frac{1}{2}\kappa} \left(e^{\frac{1}{2}\kappa\|\mathbf{u}\|} - 1 \right) - \sqrt{E}\|u\| \geq \frac{1}{\kappa} e^{-\frac{1}{2}\kappa} e^{\frac{1}{2}\kappa\|\mathbf{u}\|}$$

holds, we have

$$|\Psi_{\mathbf{E}}(\mathbf{u})| \leq C_\epsilon e^{-(1-\epsilon)\rho_{E,\check{V}}(\mathbf{0},\mathbf{u})} \leq C_\epsilon e^{-(1-\epsilon)\rho_{E,\check{V}_2}(\mathbf{0},\mathbf{u})} \ll \exp\left(-C e^{C'\|\mathbf{u}\|} \right)$$

for some positive constants C_ϵ, ϵ, C and C'. Inside the ball, $|\Psi_{\mathbf{E}}(\mathbf{u})|$ is bounded by a some constant so we have the desired result. $\square$

We proceed to give upper bounds for the derivatives of Ψ. We start by quoting the following theorem on gradient estimates for Poisson equations from [9]

Theorem 3.1.4. *Suppose ω is a solution to the Poisson equation*

$$\triangle \omega = f$$

on an n-dimensional cube Q of side d, and $f \in C^1(Q) \cap C^2(\bar{Q})$. Then the following inequality holds.

$$\left|\frac{\partial}{\partial x_k} \omega(p^*)\right| \leq \frac{n}{d} \sup_{\partial Q} |\omega| + \frac{d}{2} \sup_{Q} |f|,\ k = 1, \ldots, n\ ,$$

where p^* is the center of the cube.

Using this theorem along with the bound (3.1.2), we obtain

Corollary 3.1.5. *Let Ψ be as in the corollary (3.1.3). Then*

$$\|D^\alpha \Psi(\mathbf{u})\| \leq C_\alpha \exp\left(-C'_\alpha e^{C''_\alpha \|\mathbf{u}\|}\right), \tag{3.1.3}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{N-1})$ is a multi-index and $D^\alpha = \frac{\partial^{|\alpha|}}{\partial u_1^{\alpha_1} \cdots \partial u_{N-1}^{\alpha_{N-1}}}$ for some positive constants C_α, C'_α and C''_α.

Proof. For the case $|\alpha| = 1$, we take a look at the equation (2.1.3) and set $\omega = \Psi$ and $f = (V(\mathbf{u}) - E)\Psi$. The bound (3.1.2) implies that both of the functions are bounded by some function of the form of $c \exp(-c' e^{c''|\mathbf{x}|})$. Now apply the gradient estimate given in the previous theorem by choosing a (multi-)cube of side 1 centered at $\mathbf{u}$ then the inequality is immediate. The bounds for higher derivatives can be obtained inductively by differentiating the equation (2.1.3) and observing that all higher derivatives of V grows at most exponentially while it is always paired with some derivatives of Ψ (which is again bounded by some function of the form of $c \exp(-c' e^{c''|\mathbf{x}|})$ by the previous step.) $\square$

In particular, corollary 3.1.5 implies that Ψ belongs to the Schwartz class.

3.2 Uniform Bounds for Eigenfunctions of Open Toda Chain.

The following inequality is needed in order to bound the Whittaker function, and the proof appears in [14]. We reproduce the proof here.

Lemma 3.2.1. *For real numbers γ_j, $j = 1, \ldots n$ and λ_k, $k = 1, \ldots n+1$, the following inequality holds.*

$$\sum_{j=1}^{n} \sum_{k=1}^{n+1} \left|\gamma_j - \lambda_k\right| - \sum_{1 \leq j < k \leq n+1} \left|\lambda_j - \lambda_k\right| - \sum_{1 \leq j < k \leq n} \left|\gamma_j - \gamma_k\right| \geq 0. \tag{3.2.1}$$

Proof. Since the inequality is invariant under permutations between γ_k's and also between λ_k's, without loss of generality we may assume that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1}$ and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n$. Then one can check that

$$\sum_{1 \leq j < k \leq n+1} \left|\lambda_j - \lambda_k\right| + \sum_{1 \leq j < k \leq n} \left|\gamma_j - \gamma_k\right| = \sum_{k=1}^{n+1} \left(\sum_{j=1}^{k-1} (\gamma_j - \lambda_k) + \sum_{j=k}^{n} (\lambda_k - \gamma_j) \right).$$

Using the above identity, the inequality can be rewritten as

$$\sum_{k=1}^{n+1}\left(\sum_{j=1}^{k-1}(\left|\gamma_j-\lambda_k\right|-(\gamma_j-\lambda_k))+\sum_{j=k}^{n}(\left|\lambda_k-\gamma_j\right|-(\lambda_k-\gamma_j))\right)\geq 0.$$

But this is obvious since $|a|-a\geq 0$. $\square$

Corollary 3.2.2. *Let γ_j, $j=1,\dots n$ and λ_k, $k=1,\dots n{+}1$ be real numbers. Suppose also that $\max_j\{\gamma_j\}\geq 2\max_k\{\lambda_k\}$. Then the following inequality holds.*

$$\sum_{j=1}^{n}\sum_{k=1}^{n+1}\left|\gamma_j-\lambda_k\right|-\sum_{1\leq j<k\leq n+1}\left|\lambda_j-\lambda_k\right|-\sum_{1\leq j<k\leq n}\left|\gamma_j-\gamma_k\right|\geq\max_j\{\gamma_j\}. \quad (3.2.2)$$

If instead, we have $\min_j\{\gamma_j\}\leq 2\min_k\{\lambda_k\}$,

$$\sum_{j=1}^{n}\sum_{k=1}^{n+1}\left|\gamma_j-\lambda_k\right|-\sum_{1\leq j<k\leq n+1}\left|\lambda_j-\lambda_k\right|-\sum_{1\leq j<k\leq n}\left|\gamma_j-\gamma_k\right|\geq-\min_j\{\gamma_j\}. \quad (3.2.3)$$

Proof. Without loss of generality we may assume that $\gamma_1\leq\cdots\leq\gamma_n$ and $\lambda_1\leq\cdots\leq\lambda_{n+1}$. Then $\max_j\{\gamma_j\}=\gamma_n$ and $\lambda_{n+1}\leq\frac{1}{2}\gamma_n$. Let F_n be the left hand side of the first equation, with n to keep track of number of real variables. If we separate the terms in F_n that involve γ_n or λ_{n+1}, we have

$$F_n=F_{n-1}+\sum_{k=1}^{n}\left(\left|\gamma_n-\lambda_k\right|+\left|\gamma_k-\lambda_{n+1}\right|-\left|\lambda_k-\lambda_{n+1}\right|-\left|\gamma_n-\gamma_k\right|\right)$$

$$\geq\sum_{k=1}^{n}\left(\gamma_n-\lambda_k+\left|\gamma_k-\lambda_{n+1}\right|+\lambda_k-\lambda_{n+1}-\gamma_n+\gamma_k\right)=\sum_{k=1}^{n}\left(\left|\gamma_k-\lambda_{n+1}\right|+\gamma_k-\lambda_{n+1}\right)$$

$$\geq\left|\gamma_n-\lambda_{n+1}\right|+\gamma_n-\lambda_{n+1}=2(\gamma_n-\lambda_{n+1})\geq\gamma_n.$$

The inequalities used in the above are $F_{n-1}\geq 0$ (which follows from (3.2.1)) and $|a|+a\geq 0$.
The inequality for $\min_j\{\gamma_j\}\leq 2\min_k\{\lambda_k\}$ can be handled by negating all γ and λ variables. $\square$

Proposition 3.2.3. *The N particle generalized eigenfunction $\phi_{\boldsymbol{\gamma}}(\mathbf{x})$ of quantum open Toda chain satisfies the inequality*

$$\left|\phi_{\boldsymbol{\lambda}}(\mathbf{x})\right| \leq Ce^{C'\|\mathbf{x}\|}\left(\|\boldsymbol{\lambda}\|+1\right)^M \exp\left(-\frac{\pi}{2}\sum_{1\leq j<k\leq N}\left|\mathrm{Re}(\lambda_j-\lambda_k)\right|\right), \quad (3.2.4)$$

for some constants C,C' and some integer M, provided that for each γ_j, $\mathrm{Im}\,\gamma_j$ is bounded in some compact interval of $\mathbb{R}$.

Proof. Let $\phi^{[n]}_{\lambda_1,\ldots,\lambda_n}(\mathbf{x})$ denote the n quantum open Toda chain eigenfunction. We will use induction on n, with induction hypothesis being

$$\left|\phi^{[n]}_{\boldsymbol{\lambda}}(\mathbf{x})\right| \leq C_n e^{C'_n\|\mathbf{x}\|}(\|\boldsymbol{\lambda}\|+1)^{M_n} \exp\left(-\frac{\pi}{2}\sum_{1\leq j<k\leq n}\left|\mathrm{Re}(\lambda_j-\lambda_k)\right|\right) \quad (3.2.5)$$

for some constants C_n and C'_n, and some integer M_n. For the initial step of induction $n=1$, the above bound holds trivially. So assuming the induction hypothesis, we will bound the $n+1$ quantum open Toda chain eigenfunction $\phi^{[n+1]}_{\boldsymbol{\lambda}}(\mathbf{x})$, which is given by the integral

$$\begin{aligned}\phi^{[n+1]}_{\lambda_1,\ldots,\lambda_{n+1}}(x_1,\ldots,x_{n+1}) \;&=\; \int_{\mathcal{C}}\mu(\boldsymbol{\gamma})Q(\boldsymbol{\gamma};\boldsymbol{\lambda})\phi^{[n]}_{\gamma_1,\ldots,\gamma_n}(x_1,\ldots,x_n)\times \\ &\times\exp\left(ix_{n+1}\left(\sum_{j=1}^{n+1}\lambda_j-\sum_{j=1}^{n}\gamma_j\right)\right)d\boldsymbol{\gamma}\end{aligned} \quad (3.2.6)$$

The term $\exp\left(ix_{n+1}\left(\sum_{j=1}^{n+1}\lambda_j-\sum_{j=1}^{n}\gamma_j\right)\right)$ can be bounded by $Ce^{C'|x_{n+1}|}$ for some constants C and C' since the imaginary parts of λ_j and γ_j's are bounded. We use the following bounds for $\mu(\boldsymbol{\gamma})$ and $Q(\boldsymbol{\gamma};\boldsymbol{\lambda})$

$$\left|\mu(\boldsymbol{\gamma})\right| \ll \left(\|\boldsymbol{\gamma}\|+1\right)^{\frac{n(n-1)}{2}} \exp\left(\pi\sum_{1\leq j<k\leq n}\left|\mathrm{Re}(\gamma_j-\gamma_k)\right|\right), \quad (3.2.7)$$

$$\left|Q(\boldsymbol{\gamma};\boldsymbol{\lambda})\right| \ll (\|\boldsymbol{\lambda}\|+1)^a(\|\boldsymbol{\gamma}\|+1)^a \exp\left(-\frac{1}{2}\pi\sum_{j=1}^{n}\sum_{k=1}^{n+1}\left|\mathrm{Re}(\gamma_j-\lambda_k)\right|\right) \quad (3.2.8)$$

for some constant a. The first inequality is apparent from the definition of μ and the second inequality follows from a well known bound (c.f. [28])

$$\left|y\right|^{\underline{x}-\frac{1}{2}}e^{-\frac{\pi}{2}|y|} \ll \left|\Gamma(x+iy)\right| \ll \left|y\right|^{\overline{x}-\frac{1}{2}}e^{-\frac{\pi}{2}|y|}, \quad (3.2.9)$$

where $x+yi$ lies in some vertical strip and $\underline{x}$ and $\overline{x}$ are the minimum and maximum values of x on that strip. Overall, we see that

$$\left|\phi_{\boldsymbol{\lambda}}^{[n+1]}\right| \ll e^{C_n'\|\mathbf{x}\|+C'|x_{n+1}|}\int_{\mathcal{C}}(\|\boldsymbol{\lambda}\|+1)^b(\|\boldsymbol{\gamma}\|+1)^c\times$$
$$\times\exp\left(\frac{\pi}{2}\sum_{1\le j<k\le n}\left|\mathrm{Re}(\gamma_j-\gamma_k)\right|-\frac{\pi}{2}\sum_{j=1}^{n}\sum_{k=1}^{n+1}\left|\mathrm{Re}(\gamma_j-\lambda_k)\right|\right)d\boldsymbol{\gamma}.$$

Now we may use inequalities from the lemma and its corollaries to show that the integral on the right hand side is bounded by some function of the form

$$C_{n+1}e^{C_{n+1}'\|(\mathbf{x},x_{n+1})\|}(\|\boldsymbol{\lambda}\|+1)^{M_{n+1}}\exp\left(-\frac{\pi}{2}\sum_{1\le j<k\le n+1}\left|\mathrm{Re}(\lambda_j-\lambda_k)\right|\right). \tag{3.2.10}$$

To do this, we separate $\mathcal{C}$ into two parts and bound the integral on each region. Put $\mathcal{C}=\mathcal{C}_1\cup\mathcal{C}_2$ where

$$\begin{aligned}
\mathcal{C}_1 &= \left\{\boldsymbol{\gamma}\in\mathcal{C}\middle|\ 2\min_k\{\mathrm{Re}\,\lambda_k\}\le\mathrm{Re}\,\gamma_j\le 2\max_k\{\mathrm{Re}\,\lambda_k\}\text{ for }j=1,\dots,n\right\},\\
\mathcal{C}_2 &= \left\{\boldsymbol{\gamma}\in\mathcal{C}\middle|\ \max_j\{\mathrm{Re}\,\gamma_j\}\ge 2\max_k\{\mathrm{Re}\,\lambda_k\}\text{ or }\min_j\{\mathrm{Re}\,\gamma_j\}\le 2\min\{\mathrm{Re}\,\lambda_k\}\right\}.
\end{aligned}$$

Note that $\mathcal{C}_1$ is a compact set and it is disjoint from $\mathcal{C}_2$.
On $\mathcal{C}_1$, lemma 3.2.1 implies that

$$\int_{\mathcal{C}_1}(\|\boldsymbol{\lambda}\|+1)^b(\|\boldsymbol{\gamma}\|+1)^c\exp\left(\frac{\pi}{2}\sum_{1\le j<k\le n}\left|\mathrm{Re}(\gamma_j-\gamma_k)\right|-\frac{\pi}{2}\sum_{j=1}^{n}\sum_{k=1}^{n+1}\left|\mathrm{Re}(\gamma_j-\lambda_k)\right|\right)d\boldsymbol{\gamma}$$
$$\ll\int_{\mathcal{C}_1}(\|\boldsymbol{\lambda}\|+1)^b(\|\boldsymbol{\gamma}\|+1)^c\exp\left(-\frac{\pi}{2}\sum_{1\le j<k\le n+1}\left|\mathrm{Re}(\lambda_j-\lambda_k)\right|\right)d\boldsymbol{\gamma}$$
$$\ll(\|\boldsymbol{\lambda}\|+1)^b p(\lambda)\exp\left(-\frac{\pi}{2}\sum_{1\le j<k\le n+1}\left|\mathrm{Re}(\lambda_j-\lambda_k)\right|\right),$$

where $p(\lambda)=\int_{\mathcal{C}_1}(\|\boldsymbol{\gamma}\|+1)^c d\boldsymbol{\gamma}$ can be easily seen to be a function of at most a polynomial in λ.
On $\mathcal{C}_2$, the inequality 3.2.2 and 3.2.3 can be applied to yield

$$\int_{\mathcal{C}_2}(\|\boldsymbol{\lambda}\|+1)^b(\|\boldsymbol{\gamma}\|+1)^c\exp\left(\frac{\pi}{2}\sum_{1\le j<k\le n}\left|\mathrm{Re}(\gamma_j-\gamma_k)\right|-\frac{\pi}{2}\sum_{j=1}^{n}\sum_{k=1}^{n+1}\left|\mathrm{Re}(\gamma_j-\lambda_k)\right|\right)d\boldsymbol{\gamma}$$

$$\ll \int_{\mathcal{C}_2} (\|\boldsymbol{\lambda}\|+1)^b(\|\boldsymbol{\gamma}\|+1)^c \exp\left(-\frac{\pi}{2}\sum_{1\le j<k\le n+1}\left|\operatorname{Re}(\lambda_j-\lambda_k)\right| - \frac{\pi}{2}q(\boldsymbol{\gamma})\}\right)d\boldsymbol{\gamma},$$

where

$$q(\boldsymbol{\gamma}) = \begin{cases} \max_j\{\operatorname{Re}\gamma_j\} & \max_j\{\operatorname{Re}\gamma_j\} \ge 2\max_k\{\operatorname{Re}\lambda_k\} \\ -\min_j\{\operatorname{Re}\gamma_j\} & \max_j\{\operatorname{Re}\gamma_j\} < 2\max_k\{\operatorname{Re}\lambda_k\}. \end{cases}$$

The integrand is easily seen to have exponential decay in $\boldsymbol{\gamma}$ variables and therefore the integral converges absolutely. Hence integral on region $\mathcal{C}_2$ is also bounded by (3.2.10) for some appropriate constants C_{n+1},C'_{n+1} and M_{n+1}. □

Corollary 3.2.4. *Fix $\gamma_1,\ldots,\gamma_{k-1},\gamma_{k+1},\ldots,\gamma_{N-1}$. Let $\left|\operatorname{Im}\gamma_k\right|$ be bounded. The auxiliary function $\Phi_{\boldsymbol{\gamma}}$ satisfies*

$$\|\Phi_{\boldsymbol{\gamma}}(\mathbf{u})\| \le Ce^{C'\|\mathbf{u}\|}(\|\boldsymbol{\gamma}\|+1)^M e^{-\frac{(N-2)\pi}{2}|\operatorname{Re}\gamma_k|} \tag{3.2.11}$$

for some constants C and C' and some integer M.

3.3 Unform Bounds for the Integral Transform of Kharchev-Lebedev.

Proposition 3.3.1. *Fix $\gamma_1,\ldots,\gamma_{k-1},\gamma_{k+1},\ldots,\gamma_{N-1}$. Let $\left|\operatorname{Im}\gamma_k\right|$ be bounded. Then for any integer m, there exists a constant C such that*

$$\left|\xi_{\mathbf{E}}(\boldsymbol{\gamma})\right| \le C(|\gamma_k|+1)^{-m}e^{-\frac{(N-2)\pi}{2}|\operatorname{Re}\gamma_k|} \tag{3.3.1}$$

holds.

Proof. From the estimates (3.1.2) and (3.2.11), we easily get

$$\left|\xi_{\mathbf{E}}(\boldsymbol{\gamma})\right| = \left|\int_{\mathbb{R}^{N-1}} \Psi_{\mathbf{E}}(\mathbf{u})\overline{\Phi_{\overline{\boldsymbol{\gamma}}}(\mathbf{u})}d\mathbf{u}\right| \le C(|\gamma_k|+1)^M e^{-\frac{(N-2)\pi}{2}|\operatorname{Re}\gamma_k|} \tag{3.3.2}$$

for some constant C and some integer M. We will improve this bound by performing integration by parts. To do this, note that the recursive definition (2.3.4) for $\phi_{\boldsymbol{\gamma}}$ reveals that

$$\phi_{\boldsymbol{\gamma}}(x_1, \ldots, x_{N-1}) = e^{ix_{N-1}(\gamma_1+\cdots+\gamma_{N-1})} f_{\boldsymbol{\gamma}}(x_1, \ldots, x_{N-2})$$

for some function $f_{\boldsymbol{\gamma}}(x_1, \ldots, x_{N-2})$. Looking at the definition for $\tilde{\Phi}_{\boldsymbol{\gamma}}(x_1, \ldots, x_N)$, we see that

$$\tilde{\Phi}_{\boldsymbol{\gamma}}(x_1, \ldots, x_N) = e^{i(x_{N-1}-x_N)(\gamma_1+\cdots+\gamma_{N-1})} f_{\boldsymbol{\gamma}}(x_1, \ldots, x_{N-2}).$$

Let $v = x_{N-1} - x_N$. Since v is spanned by the variables $\{u_k\}_{k=1}^{N-1}$, we may find a measure $d\mathrm{m}$ such that $dv d\mathrm{m} = d\mathbf{u}$. Now, we perform integration by parts in v. Each integration by parts brings down $(\gamma_1 + \cdots + \gamma_{N-1})^{-1}$ and it differentiates $\Psi_{\mathbf{E}}(\mathbf{u})$ with respect to v. The boundary term goes away and the remaining part is bounded by $C(|\gamma_k| + 1)^{M-1} e^{-\frac{(N-2)\pi}{2}|\operatorname{Re}\gamma_k|}$ due to the bound (3.1.3). We have to be a little careful when $\gamma_1 + \cdots + \gamma_{N-1} = 0$. In that case, first prove the bound away outside some relatively compact open set containing the set $\{\gamma_k \in \mathbb{C} \big| \gamma_1 + \cdots + \gamma_{N-1} = 0\}$, and then use the fact that $\xi_{\mathbf{E}}(\boldsymbol{\gamma})$ is continuous. We will frequently use this type of argument. We may repeat this procedure of integration by parts as many times as we need in order to get the desired bound. $\square$

Chapter 4

Completeness Proof of Pasquier-Gaudin Solutions.

Suppose there exist $\mathbf{E} = (E_1, E_2, \ldots, E_N)$ and an $L^2(\mathbb{R}^{N-1})$ function $\Psi_{\mathbf{E}}(\mathbf{u})$ such that $\tilde{\Psi}_{\mathbf{E}}(\mathbf{u}, u_N) = \Psi_{\mathbf{E}}(\mathbf{u})e^{i\frac{E_1}{\sqrt{N}}u_N}$ satisfies equation (2.3.1). We have proved so far that

$$\xi_{\mathbf{E}}(\boldsymbol{\gamma}) = \int_{\mathbb{R}^{N-1}} \Psi_{\mathbf{E}}(\mathbf{u})\overline{\Phi_{\overline{\boldsymbol{\gamma}}}(\mathbf{u})}d\mathbf{u}$$

satisfies three properties.

(A1) It satisfies the Baxter equation $t_{\mathbf{E}}(\gamma_j)\xi_{\mathbf{E}}(\boldsymbol{\gamma}) = i^N\xi_{\mathbf{E}}(\boldsymbol{\gamma} + i\mathbf{e}_j) + i^{-N}\xi_{\boldsymbol{E}}(\boldsymbol{\gamma} - i\mathbf{e}_j)$. (Proposition 2.5.1)

(A2) It is an entire function in each of its variables γ_j. (Proposition 2.5.3)

(A3) $\left|\xi_{\mathbf{E}}(\boldsymbol{\gamma})\right| \ll (|\gamma_k|+1)^{-m}e^{-\frac{N-2}{2}\pi|\mathrm{Re}(\gamma_k)|}$, when other variables $\gamma_1, \ldots, \gamma_{k-1}, \gamma_{k+1}, \ldots, \gamma_{N-1}$ are fixed. (Proposition 3.3.1)

We will show that the above properties imply that $\mathbf{E} \in \Xi$ and that $\xi_{\mathbf{E}}(\boldsymbol{\gamma})$ is a Pasquier-Gaudin solution (definition 2.7.2). Then the proof of the main theorem 2.7.4 will be complete.

4.1 Liouville's Theorem for Periodic Functions

The following theorem is a very special case of a known theorem about almost periodic functions in [24]. The proof for this special case is elementary, and we include it here.

Theorem 4.1.1. *Suppose f is an entire function and $f(z+2i) = f(z)$. Suppose also that*

$$|f(z)| \leq Ce^{\pi n|\mathrm{Re}(z)|}$$

for some constant C and an integer n. Then

$$f(z) = a_n e^{\pi nz} + a_{n-1}e^{\pi(n-1)z} + \cdots + a_{-n}e^{-\pi nz}.$$

Proof. Let $g(z) = e^{\pi(n+1)z}f(z)$. Let $z = \frac{1}{\pi}\mathrm{Log}\, w$ and

$$h(w) = g(\tfrac{1}{\pi}\mathrm{Log}\, w) = g(z).$$

$h(w)$ is well defined because $g(z+2i) = g(z)$. Also, $h(w)$ is holomorphic on $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. The singularity 0 of $h(w)$ is a removable singularity, which can be seen as follows: As $w \to 0$, $\mathrm{Re}\, z = \frac{1}{\pi}\mathrm{Log}\,|w| \to -\infty$ and $h(w) = g(z) \to 0$ because of the given bound for $f(z)$. So $h(w)$ is entire. The bound for $f(z)$ gives the bound $|h(w)| \leq C|w|^{2n+1}$. Hence by the usual Liouville's theorem and the definition of $g(z)$, $h(w)$ is a polynomial of w of degree less than $2n+1$. Now the constant term of $h(w)$ is zero because $h(0) = 0$. Undoing the change of variables, one gets $f(z) = a_n e^{\pi nz} + a_{n-1}e^{\pi(n-1)z} + \cdots + a_{-n}e^{-\pi nz}$. □

4.2 Estimates for Fundamental Solutions of the Baxter Equation

The growth of $\sigma_\pm(z)$ defined in are described by the following bounds.

Proposition 4.2.1. *Let $\mathcal{M} = \max_k\{\mathrm{Im}\,\theta_k\}$ and $\mathcal{L} = \min_k\{\mathrm{Im}\,\theta_k\}$. There exists a positive real number P such that when $A \leq \mathrm{Im}\, z \leq B$ and $|\mathrm{Re}\, z| \geq P$, the following holds.*

(a) $(1+|z|)^{-B+\mathcal{L}-\frac{1}{2}}e^{\frac{N}{2}\pi|\mathrm{Re}\, z|} \ll \left|\frac{K_+(z)}{\prod_{k=1}^N \Gamma\left(1-i(z-\theta_k)\right)}\right| \ll (1+|z|)^{-A+\mathcal{M}-\frac{1}{2}}e^{\frac{N}{2}\pi|\mathrm{Re}\, z|}$.

(b) $(1+|z|)^{A-\mathcal{M}-\frac{1}{2}} e^{\frac{N}{2}\pi|\operatorname{Re} z|} \ll \left|\frac{K_-(z)}{\prod_{k=1}^{N}\Gamma\left(1+i(z-\theta_k)\right)}\right| \ll (1+|z|)^{B-\mathcal{L}-\frac{1}{2}} e^{\frac{N}{2}\pi|\operatorname{Re} z|}.$

(c) *Suppose $A \geq \mathcal{M}+1$. Then for any constants a and b there exists a positive real r such that $\left|e^{N\pi z}\left(a\sigma_+(z)+b\sigma_-(z)\right)\right| \gg (|z|+1)^r e^{\frac{N}{2}\pi|\operatorname{Re} z|}$ holds.*

Proof. Recall the definition of $K_+(z)$ from equation (2.6.4). $t_{\mathbf{E}}(z)$ is a polynomial of degree 2 or higher. So we may choose a large constant P so that

$$\sum_{n=1}^{\infty}\left|\frac{1}{t_{\mathbf{E}}(z+ni)t_{\mathbf{E}}(z+ni)}\right| < \infty$$

whenever $|\operatorname{Re} z| \geq P$ in a horizontal strip. In fact, we may take P large enough that

$$\sum_{n=1}^{\infty}\left|\frac{1}{t_{\mathbf{E}}(z+ni)t_{\mathbf{E}}(z+ni)}\right| \leq \frac{1}{3}.$$

Hence from propositions B.0.2 and B.0.3 from the appendix, we have

$$\frac{1}{2} \leq K_+(z) \leq \frac{3}{2} \quad \text{if} \quad |\operatorname{Re} z| \geq P.$$

The products of gamma functions in the denominator can be handled by inequality (3.2.9), which implies

$$(1+|z|)^{-B+\mathcal{L}-\frac{1}{2}} e^{\frac{1}{2}\pi|\operatorname{Re} z|} \ll \left|\Gamma\left(1-i(z-\theta_k)\right)\right| \ll (1+|z|)^{-A+\mathcal{M}-\frac{1}{2}} e^{\frac{1}{2}\pi|\operatorname{Re} z|}.$$

This immediately implies inequality (a). Proof for inequality (b) is essentially same. Now for inequality (c), we use inequalities (a) and (b) to get

$$\begin{aligned}\left|e^{N\pi z}\left(a\sigma_+(z)+b\sigma_-(z)\right)\right| &\geq |b||e^{N\pi z}\sigma_-(z)| - |a||e^{N\pi z}\sigma_+(z)| \\ &\gg \left((1+|z|)^{A-\mathcal{M}-\frac{1}{2}} - C(1+|z|)^{-A+\mathcal{M}-\frac{1}{2}}\right) e^{\frac{N}{2}\pi|\operatorname{Re} z|}\end{aligned}$$

For some positive constant C. Since $A-\mathcal{M} \geq 1$,

$$\left((1+|z|)^{A-\mathcal{M}-\frac{1}{2}} - C(1+|z|)^{-A+\mathcal{M}-\frac{1}{2}}\right) \gg (1+|z|)^{A-\mathcal{M}-\frac{1}{2}}$$

for P large enough and we have inequality (c) after setting $r = A-\mathcal{M}-\frac{1}{2}$. ☐

4.3 Solutions of the Baxter Equations Satisfying Certain Estimates

Let us fix $\gamma_1, \ldots, \gamma_{k-1}, \gamma_{k+1}, \ldots, \gamma_{N-1}$ and let $z = \gamma_k$. To simplify our notation, we introduce

$$\sigma(z) \stackrel{def}{=} \xi_{\mathbf{E}}(\gamma_1, \ldots, \gamma_{k-1}, z, \gamma_{k+1}, \ldots, \gamma_{N-1}).$$

Then from the properties (A1)-(A3) we see that $\sigma(z)$ satisfies the following properties[a].

(B1) It satisfies equation (2.6.2).

(B2) It is an entire function.

(B3) It is bounded by $C(|z|+1)^{-m} e^{-\frac{N-2}{2}\pi|\mathrm{Re} z|}$.

Our first step towards proving the main theorem 2.7.4 is to prove the following proposition.

Proposition 4.3.1. *Let $\sigma(z)$ be a function such that satisfies properties (B1)-(B3). Then $\mathbf{E} \in \Xi$ and*

$$\sigma(z) = C e^{N\pi z} \frac{a\sigma_+(z) + b\sigma_-(z)}{\prod_{k=1}^{N} \sinh \pi(z - \delta_k)} \tag{4.3.1}$$

for some constants a and b satisfying $a\sigma_+(\delta_1(\mathbf{E})) + b\sigma_-(\delta_1(\mathbf{E})) = 0$.

Lemma 4.3.2. $e^{N\pi z} W[\sigma(z); \sigma_\pm(z)] = c_\pm$ *for some constants $c_\pm$*

Proof. Let $\mathrm{S} = \{z | \; A \leq \mathrm{Im}\, z \leq A + 2\}$ where A is chosen so that none of the zeros of the Hill determinant lies on the boundary and $A \geq \mathcal{M} + 1$. Part (a) of proposition 4.2.1 implies that there exists a real number p such that

$$|\sigma_+(z)| \ll (1 + |z|)^p e^{\frac{N\pi}{2}}$$

[a] This is same as the functional-difference equation (2.6.2) obtained by using Sklyanin's separation of variables. A priori, we do not know that $\xi_{\mathbf{E}}(\boldsymbol{\gamma})$ is a product of one variable functions. So we cannot use Sklyanin's separation of variables, and we avoid it by saying we have fixed $\gamma_1, \ldots, \gamma_{k-1}, \gamma_{k+1}, \ldots, \gamma_{N-1}$.

and

$$|\sigma_+(z+i)| \ll (1+|z|)^p e^{\frac{N\pi}{2}}$$

hold in $\mathrm{S} \cap \{z \mid |\operatorname{Re} z| > P\}$. Namely, we may set $p = -A + \mathcal{M} - \frac{1}{2}$. Using the bound from property (B3) with m from (B3) chosen to be $m = p+1$, we have

$$|e^{N\pi z} W[\sigma(z); \sigma_+(z)]| \ll (|z|+1)^p e^{\frac{N\pi}{2}|\operatorname{Re} z|} (|z|+1)^{-m} e^{-\frac{(N-2)\pi}{2}|\operatorname{Re} z|} \ll (|z|+1)^{p-m} e^{\pi |\operatorname{Re} z|}.$$

Although the above inequality initially holds in $\mathrm{S} \cap \{z \mid |\operatorname{Re} z| > P\}$, it holds for the whole horizontal strip S because $e^{N\pi z} W[\sigma(z); \sigma_+(z)]$ is continuous. Also $e^{N\pi z} W[\sigma(z); \sigma_+(z)]$ is periodic of period $2i$ because of equation (2.6.9). So we may apply the Liouville's theorem for periodic functions and get

$$e^{N\pi z} W[\sigma(z); \sigma_+(z)] = C_1 e^{\pi z} + C_2 + C_3 e^{-\pi z}.$$

But C_1 and C_3 has to be zero since in the above bound $p - m < 0$. So $e^{N\pi z} W[\sigma(z); \sigma_+(z)]$ is constant. Similar argument can be used for $e^{N\pi z} W[\sigma(z); \sigma_-(z)]$. □

Corollary 4.3.3. *Let $\tilde{\sigma}(z)$ be the linear combination $a\sigma_+(z) + b\sigma_-(z)$ where a and b are any pair of real numbers, not both zero, satisfying $ac_+ + bc_- = 0$. Then $\nu(z) \overset{def}{=} \frac{\sigma(z)}{\tilde{\sigma}(z)}$ is a periodic meromorphic function of period i. Any pole of $\nu(z)$ is a zero of the Hill determinant.*

Proof. From the previous lemma, $W[\sigma(z); \tilde{\sigma}(z)] = e^{-N\pi z}(ac_+ + bc_-) = 0$. This means that $\sigma(z)\tilde{\sigma}(z+i) - \sigma(z+i)\tilde{\sigma}(z) = 0$. Hence

$$\nu(z) \overset{\text{def}}{=} \frac{\sigma(z)}{\tilde{\sigma}(z)} = \frac{\sigma(z+i)}{\tilde{\sigma}(z+i)} = \nu(z+i) \tag{4.3.2}$$

shows that it is periodic. This means that if there is a pole, then all its shifts of integer multiples of i is again a pole. So the poles are the periodic zeros of $a\sigma_+(z) + b\sigma_-(z)$, and therefore they are zeros of the Wronskian $W[\sigma_+(z); \sigma_-(z)]$. Then the definition of the Hill determinant implies that they are zeros of the Hill determinant. □

So we know that the poles, if they exist, are located at some of the δ_k's and its periodic shifts. Hence we may multiply $\sinh(z - \delta_k)$ for some k's to make it an entire function.

Lemma 4.3.4. *Let* $\mathrm{I} \subset \{1, \ldots, N\}$ *be the smallest set that makes*

$$f(z) \stackrel{def}{=} e^{-N\pi z}\nu(z) \prod_{k \in \mathrm{I}} \sinh \pi(z - \delta_k)$$

an entire function. Then $f(z)$ *is a constant function.*

Proof. $f(z)$ is an entire function of period $2i$ by our construction. From bound (c) of proposition 4.2.1 we know that in $\mathrm{S} \cap \{z| \, |\operatorname{Re} z| > P\}$,

$$\left| e^{-N\pi z} \frac{1}{\tilde{\sigma}(z)} \right| = \frac{1}{|e^{N\pi z}\left(a\sigma_+(z) + b\sigma_-(z)\right)|} \ll (1 + |z|)^{-r} e^{\frac{-N\pi}{2}|\operatorname{Re} z|}$$

for some positive real number r. Also, property (B3) for $\sigma(z)$ states that $|\sigma(z)| \ll (|z| + 1)^{-m} e^{-\frac{N-2}{2}\pi|\mathrm{Re} z|}$. So we see that

$$\left|e^{-N\pi z}\nu(z)\right| = \left| e^{-N\pi z} \frac{1}{\tilde{\sigma}(z)} \right| |\sigma(z)| \ll (|z| + 1)^{-1} e^{-(N-1)\pi|\operatorname{Re} z|}$$

outside some compact set of the strip. Now, $|\prod_{k \in \mathrm{I}} \sinh \pi(z - \delta_k)|$ is bounded by some multiple of $e^{\#(\mathrm{I})\,\pi|\operatorname{Re} z|}$. This implies that

$$|f(z)| \ll (|z| + 1)^{-1} e^{\pi|\operatorname{Re} z|}.$$

Hence the Liouville's theorem for periodic functions can be applied to yield that $f(z)$ is a constant. ◻

Proof of proposition 4.3.1. Since $f(z) = C$,

$$C = e^{-N\pi z}\nu(z) \prod_{k \in \mathrm{I}} \sinh(z - \delta_k).$$

Solving this for $\sigma(z)$ gives

$$\sigma(z) = C e^{N\pi z} \frac{a\sigma_+(z) + b\sigma_-(z)}{\prod_{k \in \mathrm{I}} \sinh(z - \delta_k)}.$$

Now, if $\mathrm{I} \subsetneq \{1, \ldots, N\}$, then from proposition 4.2.1 we have

$$|\sigma(z)| \gg e^{-\#(\mathrm{I})\,\pi|\operatorname{Re} z|}\left(e^{N\pi z}(a\sigma_+(z) + b\sigma_-(z))\right) \gg (1 + |z|)^r e^{(\frac{N}{2} - \#(\mathrm{I}))\pi z}.$$

This contradicts the boundedness property (B3). Hence $\mathrm{I} = \{1, \ldots, N\}$ and we have equation (4.3.1). Finally, in order for $\sigma(z)$ to be entire, the periodic zeros of $a\sigma_+(z) + b\sigma_-(z)$ should contain all the zeros of $\prod_{k=1}^{N} \sinh \pi(z - \delta_k)$. More specifically, $a\sigma_+(\delta_k) + b\sigma_-(\delta_k) = 0$ for $k = 1, \ldots, N$. Comparing this with definition 2.7.1, we see that $\mathbf{E} \in \Xi$. ◻

4.4 Proof of the Main Theorem

Proof of the main theorem 2.7.4. Applying proposition 4.3.1 to $\sigma_1(z)$ yields that $\mathbf{E} \in \Xi$ and

$$\sigma_1(z) = Ce^{N\pi z}\frac{a\sigma_+(z) + b\sigma_-(z)}{\prod_{k=1}^N \sinh \pi(z - \delta_k)}.$$

Since this result is obtained after fixing $\gamma_2, \ldots, \gamma_{N-1}$, when considering the above equation in terms of $\xi_{\mathbf{E}}(\boldsymbol{\gamma})$, we should think of C as a function of $\gamma_2, \ldots, \gamma_{N-1}$. Hence

$$\xi_{\mathbf{E}}(\boldsymbol{\gamma}) = C(\gamma_2, \ldots, \gamma_{N-1})e^{N\pi\gamma_1}\frac{a\sigma_+(\gamma_1) + b\sigma_-(\gamma_1)}{\prod_{k=1}^N \sinh(\gamma_1 - \delta_k)}.$$

Applying proposition 4.3.1 to $\sigma_2(z)$ will yield

$$\xi_{\mathbf{E}}(\boldsymbol{\gamma}) = C'(\gamma_1, \gamma_3, \ldots, \gamma_{N-1})e^{N\pi\gamma_2}\frac{a\sigma_+(\gamma_2) + b\sigma_-(\gamma_2)}{\prod_{k=1}^N \sinh(\gamma_2 - \delta_k)}.$$

So we can define a meromorphic function $C''(\boldsymbol{\gamma})$ in two different ways as

$$C''(\boldsymbol{\gamma}) = \frac{C(\gamma_2, \ldots, \gamma_{N-1})}{e^{N\pi\gamma_2}\frac{a\sigma_+(\gamma_2)+b\sigma_-(\gamma_2)}{\prod_{k=1}^N \sinh(\gamma_2-\delta_k)}} = \frac{C'(\gamma_1, \gamma_3, \ldots, \gamma_{N-1})}{e^{N\pi\gamma_1}\frac{a\sigma_+(\gamma_1)+b\sigma_-(\gamma_1)}{\prod_{k=1}^N \sinh(\gamma_1-\delta_k)}}.$$

It is apparent from above that the function $C''(\boldsymbol{\gamma})$ is independent of both γ_1 and γ_2 and hence

$$\xi_{\mathbf{E}}(\boldsymbol{\gamma}) = C''(\gamma_3, \ldots, \gamma_{N-1})e^{N(\gamma_1+\gamma_2)}\frac{a\sigma_+(\gamma_1) + b\sigma_-(\gamma_1)}{\prod_{k=1}^N \sinh(\gamma_1 - \theta_k)} \cdot \frac{a\sigma_+(\gamma_2) + b\sigma_-(\gamma_2)}{\prod_{k=1}^N \sinh \pi(\gamma_2 - \theta_k)}.$$

Repeating this procedure, we have $\xi_{\mathbf{E}}(\boldsymbol{\gamma})$ in the form

$$\xi_{\mathbf{E}}(\boldsymbol{\gamma}) = \tilde{C}\prod_{j=1}^{N-1} e^{N\pi\gamma_j}\frac{a\sigma_+(\gamma_j) + b\sigma_-(\gamma_j)}{\prod_{k=1}^N \sinh \pi(\gamma_j - \theta_k)},$$

where $\tilde{C}$ is a constant because it does not depend on any $\gamma_1, \ldots, \gamma_{N-1}$. Finally, absorbing $\tilde{C}$ into a and b, we get

$$\xi_{\mathbf{E}}(\boldsymbol{\gamma}) = \prod_{j=1}^{N-1} e^{N\pi\gamma_j}\frac{a\sigma_+(\gamma_j) + b\sigma_-(\gamma_j)}{\prod_{k=1}^N \sinh \pi(\gamma_j - \theta_k)}.$$

$\square$

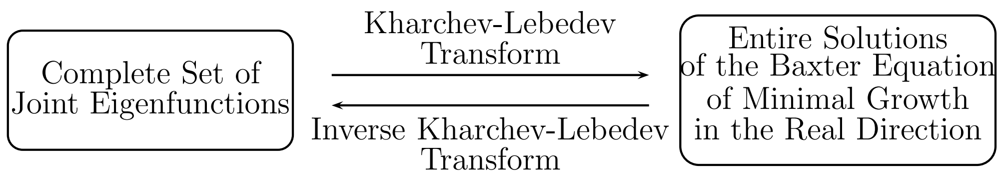

Figure 4.5.1: The Correspondence for the Joint Eigenfunctions

4.5 Discussion

Proposition 4.3.1 reveals an interesting fact about the Baxter equation, considered as a functional difference equation.

Corollary 4.5.1. *The entire solutions of the functional difference equation*

$$(z^N - E_1 z^{N-1} + \cdots + (-1)^N E_N)\sigma(z) = i^N \sigma(z+i) + i^{-N}\sigma(z-i)$$

has the growth of at least $O(e^{-\frac{N\pi}{2}|\operatorname{Re} z|})$ *and solutions satisfying this growth condition exist if and only if* $\mathbf{E} = (E_1, \ldots, E_N) \in \Xi$. *Moreover, when a solution exists, it is unique up to a constant multiple.*

Figure 4.5.1 gives an overall picture of the relation between the quantum periodic Toda chain and this functional difference equation which we called the Baxter equation. The quantum periodic Toda chain is a degenerate case of the integrable periodic Discrete Self-Trapping (DST) chain which in turn is a degenerate case of the Heisenberg XXX spin chain. The Baxter equations appear in these systems also. For example, the Baxter equation for the integrable DST chain [23] is

$$t(z)\sigma(z) = \left(z - \frac{i}{2}\right)^N \sigma(z+i) + i^N \sigma(z+i), \tag{4.5.1}$$

where $t(z)$ is a polynomial with the coefficients being the eigenvalues of the operator $T(z)$ of the DST chain. It will be interesting to find out whether there exists an integral operator that gives the description of figure 4.5.1. My rough conjecture is that even for the DST chain, we can consider the minimally growing solution of the Baxter equation whose existential condition will give us the quantization condition for the DST chain. But the solutions considered may not be entire and we may need to allow some singularities. In light of this conjecture, the meromorphic solutions of the Baxter equations satisfying some boundedness condition can be a meaningful thing to study.

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Appendix A

Inverse Transform of Kharchev-Lebedev Integral

From the inversion formula (2.3.8) and Fourier inversion formula, we have

$$\frac{1}{2\pi}\int_{\mathbb{R}^{2N}} h(\mathbf{y}, y_{_N})\left(\phi_{\boldsymbol{\gamma}}(\mathbf{x})e^{-ix_{_N}\epsilon}\right)\overline{\phi_{\boldsymbol{\gamma}}(\mathbf{y})e^{-iy_N\epsilon}}\mu(\boldsymbol{\gamma})\,dy_{_N}d\epsilon d\mathbf{y}d\boldsymbol{\gamma} = h(\mathbf{x}, x_N) \tag{A.0.1}$$

for any continuous and $L^2(\mathbb{R}^N)$ function h. Let $\mathbf{u}$ and u_N be the change of variables given in section 2.1. The above identity will hold for $h(\mathbf{x}, x_N) = f(\mathbf{u})g(u_N)$ with f and g being Schwartz class functions on $\mathbb{R}^{N-1}$ and $\mathbb{R}$ respectively. In the changed variables, the identity becomes

$$\frac{1}{2\pi}\int_{\mathbb{R}^{2N}} f(\mathbf{u}')g(u'_N)\Phi_{\boldsymbol{\gamma}}(\mathbf{u})e^{-ix_{_N}(\epsilon-\frac{1}{\sqrt{N}}\sum\gamma_k)}\overline{\Phi_{\boldsymbol{\gamma}}(\mathbf{u}')e^{-iy_{_N}(\epsilon-\frac{1}{\sqrt{N}}\sum\gamma_k)}}\mu(\boldsymbol{\gamma})du'_N d\epsilon d\mathbf{u}' d\boldsymbol{\gamma}$$
$$= f(\mathbf{u})g(u_N).$$

We now perform the integration with respect to du'_N and $d\epsilon$, with ϵ shifted by $\frac{1}{\sqrt{N}}\sum\gamma_k$. The integral kernel we have to look at is $e^{i\frac{1}{\sqrt{N}}u'_N\epsilon}e^{-i\left(\frac{1}{\sqrt{N}}u_N+\sum a_k(u_k-u'_k)\right)\epsilon}$. Performing the integrals which is just Fourier transform and its inversion, we have

$$\frac{1}{2\pi}\int_{\mathbb{R}^{2N-2}} g\left(\frac{1}{\sqrt{N}}u_N+\sum a_k(u_k-u'_k)\right)f(\mathbf{u}')\Phi_{\boldsymbol{\gamma}}(\mathbf{u})\overline{\Phi_{\boldsymbol{\gamma}}(\mathbf{u}')}\mu(\boldsymbol{\gamma})d\mathbf{u}'d\boldsymbol{\gamma} = g(u_N)f(\mathbf{u}).$$

Now we use Lebesgue dominated convergence theorem on sequence of $g_n(u_N)$'s which approach to constant function 1. This is possible because $f(\mathbf{u})$ is assumed to be Schwartz class and $\Phi_{\boldsymbol{\gamma}}(\mathbf{u})\overline{\Phi_{\boldsymbol{\gamma}}(\mathbf{u}')}\mu(\boldsymbol{\gamma})$ only has a polynomial growth. Thus we obtain the following identity

$$\frac{1}{2\pi}\int_{\mathbb{R}^{2N-2}} f(\mathbf{u}')\Phi_{\boldsymbol{\gamma}}(\mathbf{u})\overline{\Phi_{\boldsymbol{\gamma}}(\mathbf{u}')}\mu(\boldsymbol{\gamma})d\mathbf{u}'d\boldsymbol{\gamma} = f(\mathbf{u}). \tag{A.0.2}$$

Appendix B

Estimates for Infinite Determinants

Consider the (semi-)infinite determinant

$$P = \begin{vmatrix} 1 & a_1 & 0 & & \\ b_1 & 1 & a_2 & & \\ 0 & b_2 & 1 & \ddots & \\ & & \ddots & \ddots & \end{vmatrix}. \tag{B.0.1}$$

It is known [35] that the above determinant converges absolutely whenever $\sum_j |a_j b_j|$ converges. In such a case we may reorder the terms of the determinant and get

$$P = 1 - \sum_j a_j b_j + \sum_{j \neq k} a_j b_j a_k b_k - \sum_{j,k,l} \cdots .$$

In case $\sum_j |a_j b_j| < 1$ we have

$$|P| \geq 1 - \sum_j |a_j b_j| - \left(\sum_j |a_j b_j| \right)^2 - \left(\sum_j |a_j b_j| \right)^3 \cdots = 2 - \frac{1}{1 - \sum_j |a_j b_j|}.$$

One special case of the above inequality is the following.

Proposition B.0.2. *If* $\sum_j |a_j b_j| \le \frac{1}{3}$ *then*

$$|P| \ge \frac{1}{2}.$$

Similarly, we can prove the following upper bound

Proposition B.0.3. *If* $\sum_j |a_j b_j| \le \frac{1}{3}$ *then*

$$|P| \le \frac{3}{2}.$$