Rigidity of Rank-One Factors of Compact Symmetric Spaces

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Abstract of the Dissertation

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In this dissertation we study the minimal submanifolds of product spaces and prove global rigidity theorems for such submanifolds. Specifically, we consider closed minimal submanifolds $M \subseteq M_1 \times M_2$ where M_1 and M_2 are compact symmetric spaces. If both factors have rank one and M is minimal and satisfies two uniform bounds on its induced data, then it must be a totally geodesic subspace of the first factor. If only the first factor is of rank one, then that factor itself is rigid in this way.

In particular this implies that these minimal submanifolds are isolated from minimal submanifolds that are not of this type.

This analysis does not apply to the exceptional case of the Cayley Plane since this space does not admit a submersion from a Euclidean sphere.

Contents

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Introduction

Symmetric spaces can in many ways be considered to be the model spaces in Riemannian geometry. A wide variety of interesting geometric ideas can be studied on them. Questions of holonomy, comparison geometry, pinching of curvature, Einstein metrics and the geometry of isometries and Killing fields all have very precise resolutions for Riemannian symmetric spaces.

One example of this is the study of totally geodesic submanifolds. A submanifold is totally geodesic if the ambient connection and the induced Levi-Civita connection on the submanifold coincide. In general these submanifolds are rare. In dimension greater than one, their existence indicates a certain degree of local symmetry. Such a submanifold is preserved by the ambient geodesic spray of any of its tangent spaces.

For symmetric spaces there is a very precise characterisation of totally geodesic submanifolds. Using the algebraic structure of the isometry group, one can show that the totally geodesic submanifolds that contain a point $p \in M$ are in an exact one-to-one correspomndence with subspaces $V \in T_pM$ that are preserved by the Riemannian curvature tensor at p.

In this dissertation, we study the totally geodesic submanifolds of a Riemannian symmetric space of compact type. This will be done in the context of minimal submanifolds.

The minimality of a submanifold is a variational condition. An immersed submanifold $i : M \to X$ is minimal if it is a critical point of the volume function

$$\mathrm{vol}_p : \mathrm{Imm}_p(X) \to \mathbb{R}$$

defined on the set of p-dimensional immersed submanifolds with given boundary conditions. Minimality is a classical and natural condition and the knowledge of the set of minimal submanifolds of a space, and closed ones in particular, contributes greatly to an understanding of the space. One case of this is the important role of geodesics in Riemannian geometry.

The most naive question to ask may be of how many minimal submanifolds there are near a given one. A common approach to this is to consider the Jacobi operator on the submanifold. The zero eiegnspace of this operator corresponds to infinitesimal deformations through minimal submanifolds. The approach that we take in this dissertation is different though. One result that we prove is that there is a C^3-open neighbourhood in the set of immersions into a compact-type symmetric space so that any closed minimal submanifold in it must be totally geodesic and of a particular type. These totally geodesic subspaces are isolated from other minimal submanifolds.

Specifically, we show that if a Riemannian symmetric space X of compact type splits as a product

$$X = X_1 \times X_2$$

where X_1 is symmetric and of rank one, then any closed minimal submanifold $M \subset X$ that is close (in a precise sense) to a totally geodesic factor $X_1 \times \{p\}$ must be totally geodesic and of the same form. In this sense, the rank one subspaces are rigid and isolated from other minimal submanifolds of X. This discussion will be made more clear in Chapters 4 and 5.

The organisation of this dissertation is quite simple. Chpaters 1 and 2 are primarily background material on symmetric spaces and minimal submanifolds respectively. They include some technical statements from other sources, but no original proofs.

Chapter 3 contains some elementary calculations that are used later. The main results and proofs are contained in Chapters 4 and 5. These chapters are organised so that the principal results are made at the start of Chapter 4 with the proofs in the following sections. Chapter 5 consists of the extension of the main results for $X_1 = S^n$ to include $\mathbb{CP}_n$ and $\mathbb{HP}_n$.

Chapter 1

Elementary Background on Symmetric Spaces

1.1 Riemannian Symmetric Spaces

The content of this dissertation is in large part the proof of a theorem that pertains to the submanifolds of a space with a large degree of symmetry. To be precise, we will take the ambient manifold to be a *symmetric space* and the submanifolds to be *minimal* and of a particular type with respect to the algebraic structure of the ambient space. We will outline all of these concepts in detail in the coming chapters and then ultimately state and prove these statements in chapters 4 and 5. The principal reference for this chapter is Kobayashi and Nomizu [4] which contains an excellent introduction to symmetric spaces. We will assume that the manifold M is connected at all times.

Let (M, g) be a Riemannian manifold. For $x \in M$ and U a sufficiently small neighbourhood of x we can define a map s_x on U by

$$s_x(\exp(X)) = \exp(-X)$$

where exp is the exponential map at x and X is a tangent vector at x. We then have that $s_x^2 = Id$ and so s_x is a self-diffeomorphism of U. We refer to s_x as the *symmetry* at x. The defining characteristic of s_x is that it reverses geodesics through x. The derivative of s_x at x is $-Id$.

The map s_x can be locally defined at any point in a Riemannian manifold. The Riemannian manifold (M, g) is a *locally symmetric space* if for every

$x \in M$, s_x is an isometry of U, for sufficiently small U. (M, g) is a *symmetric space* if for each x, s_x can be extended to a global isometry of M.

Two facts that we will note follow trivially from this definition. The first is that a symmetric space is complete. This follows since geodesics can be extended indefinitely, by a correct use of the symmetries along it. The second fact is that the isometry group of a symmetric space acts transitively. The large collection of symmetries indicates that this should be so. The manifold M is then of the form $M = G/H$.

We also note that if R is the curvature tensor of (M, g) and ∇ is the Levi-Civita connection then we have

$$\nabla R = 0. \tag{1.1}$$

∇R is clearly invariant under the action of all isometries. That is, $g^*(\nabla R) = \nabla R$. However, for any $x \in M$,

$$\begin{aligned} s_x^*(\nabla R)(X,Y,Z,W) &= s_x\Big((\nabla_{s_x X} R)_{s_x Y, s_x Z}(s_x W)\Big) \\ &= (-1)^5 (\nabla R)(X,Y,Z,W). \end{aligned}$$

We conclude that $\nabla R = 0$.

The observation that $M = G/H$ omogeneous will allow us to consider the Lie group G of isometries of (M, g) instead of M itself. We assume that G is connected. The action of G and of the isotropy of a point will be our principal focus. We fix a point $o \in M$. The action of the symmetry s_o on M induces an action on G. We define a homomorphism σ by

$$\sigma(g) = s_o \circ g \circ s_o^{-1}.$$

It is clear that $\sigma^2 = Id$. It is not necessary that $s_o \in G$ but conjugation by an element sends the component of the identity to itself so $\sigma \in G$ for $g \in G$. σ in turn induces an involutive automorphism of the Lie algebra $\mathfrak{g}$. This automorphism will be used extensively in the following section to determine the metric and curvature of the space M.

A converse statement is that if s_o is an involutive diffeomorphism of $M = G/H$ that fixes $o = [H]$ and has derivative -1 at o then for any other $x \in M$ we can define s_x by

$$s_x = g \circ s_o \circ g^{-1} \quad \text{where } x = g(o).$$

s_x is a diffeomorphism with square the identity, that fixes x and acts by -1 tangent to x. We will give criteria for when the maps s_x are all isometries of M.

Let σ be the involutive automorphism of the group G. It then induces an involutive automorphism of the Lie algebra $\mathfrak{g}$. $\sigma^2 = Id$ so $\mathfrak{g}$ splits as a direct sum $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ where $\mathfrak{h}$ is the $+1$-eigenspace of σ and $\mathfrak{m}$ is the -1-eigenspace.

If G is a Lie group, we can identify the tangent space to G at the identity with $\mathfrak{g}$, the set of left-invariant vector fields on G. Similarly, suppose that M is a Riemannian symmetric space and $M = G/H$. Then $\pi : \mathfrak{g} \to T_oM$ is surjective with kernel $\mathfrak{h}$ so $\pi : \mathfrak{m} \to T_oM$ is an isomorphism. We will make use of this identification throughout. The curvature of M will be able to be expressed in terms of Lie brackets of elements of $\mathfrak{m}$.

In the following section we will almost exclusively consider $\mathfrak{g}$, $\mathfrak{m}$ and σ.

1.2 Orthogonal Symmetric Lie Algebras

In this section we describe some properties of Lie algebras of Killing fields to symmetric spaces and how the geometry of the symmetric space can be determined purely algebraically. We emphasise at this stage that we are exclusively interested in real Lie algebras.

A *symmetric Lie algebra* is a triple $(\mathfrak{g}, \mathfrak{h}, \sigma)$ consisting of a Lie algebra $\mathfrak{g}$ with an automorphism σ of $\mathfrak{g}$ that satisfies $\sigma^2 = 1$, together with $\mathfrak{h} = \{X \in \mathfrak{g}\,;\ \sigma(X) = X\}$. A symmetric Lie algebra is the infinitesimal model for a symmetric quotient space $M = G/H$.

Since $\sigma^2 = 1$ on $\mathfrak{g}$ we can split $\mathfrak{g}$ into the $+1$ and -1 eigenspaces of σ. Specifically, let $\mathfrak{m} = \{X \in \mathfrak{g}\,;\ \sigma(X) = -X\}$. We then have the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

It follows that $\mathfrak{h}$ and $\mathfrak{m}$ together satisfy

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h},$$
$$[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m},$$
$$[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}.$$

That is, $\mathfrak{h}$ acts on $\mathfrak{m}$. We assume that this action is *effective* in the sense that no non-zero element of $\mathfrak{h}$ annihilates every element of $\mathfrak{m}$. For brevity we won't continue to refer to this assumption.

The action of $\mathfrak{h}$ on $\mathfrak{m}$ is by the restriction of the adjoint action. Denote the induced algebra of endomorphisms by $ad_{\mathfrak{m}}(\mathfrak{h})$. We consider the connected group $K \subseteq GL(\mathfrak{m})$ of endomorphisms of $\mathfrak{m}$ that has Lie algebra $ad_{\mathfrak{m}}(\mathfrak{h}) \subseteq \mathfrak{gl}(\mathfrak{m})$. If K is compact we say that $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is an *orthogonal symmetric Lie algebra.* In this case, $\mathfrak{m}$ has an inner product that is invariant under the action of K, and hence $\mathfrak{h}$.

If $\mathfrak{g}$ is a semi-simple Lie algebra one can express $(\mathfrak{g}, \mathfrak{h}, \sigma)$ as a direct sum of symmetric Lie algebras of the form

1. $(\mathfrak{g}' + \mathfrak{g}', \Delta(\mathfrak{g}'), \sigma')$ where $\mathfrak{g}'$ is simple,
2. $(\mathfrak{g}', \mathfrak{h}', \sigma')$ where $\mathfrak{g}'$ is simple.

More precisely, the Lie algebra $\mathfrak{g}$ decomposes as a direct sum of simple ideals. The automorphism permutes the simple ideals so the decomposition can be expressed as

$$\mathfrak{g} = (\mathfrak{g}_1 + \mathfrak{g}_1') + \cdots + (\mathfrak{g}_k + \mathfrak{g}_k') + \mathfrak{g}_{k+1} + \cdots + \mathfrak{g}_n. \tag{1.2}$$

σ restricts to be an isomorphism between $\mathfrak{g}_i$ and $\mathfrak{g}_i'$ for $i = 1, \ldots, k$ and restricts to be an automorphism on $\mathfrak{g}_j$ for $j = k+1, \ldots, n$.

Example. In the first of the two distinguished cases σ' is defined by $\sigma'(X, Y) = (Y, X)$. $\mathfrak{h}' = \Delta(\mathfrak{g}')$ is the diagonal $\{(X, X)\,;\, X \in \mathfrak{g}'\}$ and the transverse subspace is $\mathfrak{m}' = \{(X, -X)\,;\, X \in \mathfrak{g}'\}$.

The map σ extends to the product $G \times G$ by $(g, h) \mapsto (h, g)$ where G is a corresponding Lie group. The quotient $G \times G/\Delta G$ is diffeomorphic to G via the map $g \mapsto [(g, e)]$ and the symmetry can be understood on G as follows.

$$\sigma(g, e) = (e, g) = (g^{-1}g, g) = (g^{-1}, e) \cdot (g, g) \sim (g^{-1}, e)$$

so the map σ on $G \times G$ descends to a map on the quotient, or on the group G itself by

$$\tilde{\sigma} : g \mapsto g^{-1}.$$

Example. Consider the orthogonal symmetric Lie algebra $(\mathfrak{so}(n+m), \mathfrak{so}(n)+\mathfrak{so}(m), \sigma)$ where σ is conjugation by the matrix

$$\left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & -I_m \end{array}\right)$$

on $\mathfrak{so}(n+m)$. This is the symmetric Lie algebra associated to the Grassmannian of n-planes in $\mathbb{R}^{n+m}$. An element of $\mathfrak{so}(n+m)$ decomposes into blocks

$$\left(\begin{array}{c|c} A & -B^T \\ \hline B & C \end{array}\right)$$

where $A \in \mathfrak{so}(n)$, $C \in \mathfrak{so}(m)$ and B is an $m \times n$-matrix. $\mathfrak{h}$ is the set where $B = 0$ and $\mathfrak{m}$ is where $A = C = 0$.

A symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is said to be *irreducible* if in the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, the subalgebra $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ acts irreducibly on $\mathfrak{m}$ by the adjoint representation. Otherwise it is referred to as *reducible*. This perhaps seems like an unusual definition, but one may consider $[\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$ to be the smallest subalgebra of $\mathfrak{g}$ that contains all information about the space $\mathfrak{m}$ and the symmetry σ.

If $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is an orthogonal symmetric Lie algabra and $\mathfrak{g}$ is semi-simple, the irreduciblity of $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is equivalent to the condition that $(\mathfrak{g}, \mathfrak{h}, \sigma)$ cannot be decomposed into two or more proper factors, as in Equation 1.2. In this case, $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$ and $\mathfrak{h}$ acts irreducibly on $\mathfrak{m}$.

A result that is fundamental to the consideration of the Riemannian geometry of symmetric spaces is the following relation between tensor fields on the manifold G/H and tensors on the vector space $\mathfrak{m}$.

Theorem 1.1. [4] *There is a one-to-one correspondence between G-invariant tensor fields on the manifold $M = G/H$ and $Ad(H)$-invariant tensors on the vector space $\mathfrak{m}$.*

This is particularly of use when we consider inner products. There is a one-to-one correspondence between metrics on M for which G acts by isometries, and inner products on $\mathfrak{m}$ that are invariant under the adjoint action of H.

Observation. Suppose $\langle\,,\,\rangle$ is a G-invariant metric on $M = G/H$. Then the map s_o that descends from the automorphism σ is an isometry of M and hence the maps s_x are as well.

That is, an involutive automorphism of G and a G-invariant inner product on M give M the structure of a Riemannian symmetric space.

For example, let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be an orthogonal symmetric Lie algebra. Then the group of transformations by the adjoint action of H is compact and there is necessarily then an H-invariant inner product on $\mathfrak{m}$.

The canonical symmetric bilinear form on a Lie algebra $\mathfrak{g}$ is the *Killing form.* This is defined as

$$\kappa(X, Y) = \mathrm{Tr}(ad_X \circ ad_Y).$$

κ is invariant under the adjoint action of $\mathfrak{g}$ on itself.

Theorem 1.2. *The Killing form is non-degenerate if and only if the Lie algebra* $\mathfrak{g}$ *is semi-simple.*

We can further refine our definitions to consider when the Killing form is positive definite or negative definite. An orthogonal symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is said to be of *compact type* if the Killing form for $\mathfrak{g}$ is negative definite on the subspace $\mathfrak{m}$. It is of *non-compact type* if it is positive definite. These definitions are made because of the implications they make on the associated simply-connected symmetric M. If κ is negative definite on $\mathfrak{m}$, $-\kappa$ defines an invariant metric on M with Ricci curvature bounded above zero. By Myers theorem M must be compact. We will elaborate below.

We will assume that the Lie algebra $\mathfrak{g}$ is of compact type. The Killing form restricts to $\mathfrak{m}$ and induces an inner product by

$$\langle X, Y\rangle = -\kappa(X, Y).$$

This is the inner product on $\mathfrak{m}$ that we will consider and this will induce the Riemannian metric on M that we will study.

We must also consider another bilinear form on $\mathfrak{m}$ that will be determined as follows. For X and Y in $\mathfrak{m}$, ad_Y maps $\mathfrak{m}$ to $\mathfrak{h}$, and ad_X sends $\mathfrak{h}$ to $\mathfrak{m}$. We consider $ad_X \circ ad_Y$ as an endomorphism of $\mathfrak{m}$, and take its trace. Define B by

$$B(X, Y) = \mathrm{Tr}_{\mathfrak{m}}(ad_X \circ ad_Y).$$

We will give an extended explicit derivation of some properties of B.

Proposition 1.3. *B is a symmetric bilinear endomorphism on $\mathfrak{m}$.*

Proof. Bilinearity is obvious. If $\langle \, , \, \rangle$ is the metric arising from the Killing form, B can be expressed as

$$B(X,Y) = \sum_i \langle [X,[Y,e_i]], e_i \rangle = \sum_i \langle e_i, [Y,[X,e_i]] \rangle = B(Y,X)$$

where $\{e_i\}$ is a Killing-orthonormal basis for $\mathfrak{m}$. This follows since κ is *ad*-invariant. □

Proposition 1.4. *B is invariant under the action of $\mathfrak{h}$ on $\mathfrak{m}$.*

Proof. Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and let H be the connected subgroup of G with corresponding algebra $\mathfrak{h}$. The invariance of B under the action of $\mathfrak{h}$ on $\mathfrak{m}$ is equivalent to its invariance under the action of H. The adjoint action of H on $\mathfrak{g}$ is by automorphisms so for $h \in H$ and $X \in \mathfrak{m}$,

$$ad_{Ad_h(X)} = Ad_h \circ ad_X \circ Ad_h^{-1}.$$

This implies that

$$\begin{aligned}
(Ad_h^* B)(X,Y) &= B(Ad_h(X), Ad_h(Y)) \\
&= \mathrm{Tr}_{\mathfrak{m}}(Ad_h \circ ad_X \circ ad_Y \circ Ad_h^{-1}) \\
&= \mathrm{Tr}_{\mathfrak{m}}(ad_X \circ ad_Y) \\
&= B(X,Y)
\end{aligned}$$

and B is H-invariant. □

That is, κ and B are H-invariant symmetric bilinear forms on $\mathfrak{m}$. If $(\mathfrak{g}, \mathfrak{h}, \sigma)$ is irreducible they can be compared.

Proposition 1.5. *Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be an irreducible orthogonal symmetric Lie algebra of compact type with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Let κ be the restriction of the Killing form to $\mathfrak{m}$ and let B be the H-invariant symmetric form that was defined above. Then there exists $\rho > 0$ such that*

$$B(X,Y) = \rho\, \kappa(X,Y)$$

for all X, $Y \in \mathfrak{m}$.

Proof. We diagonalise the form B in terms of κ. That is, we can define an endomorphism of $\mathfrak{m}$ that we also denote by B by

$$B(X,Y) = \kappa(B(X),Y).$$

B is symmetric so we can find an orthonormal basis of eigenvectors for B. Let $V_\rho \subseteq \mathfrak{m}$ be a non-trivial eigenspace. Then V_ρ is invariant under the action of $\mathfrak{h}$ since B and κ are invariant. Since $\mathfrak{m}$ is an irreducible representation of $\mathfrak{h}$ we must have that $V_\rho = \mathfrak{m}$ and $B = \rho\,\kappa$.

B can be expressed as

$$B(X,Y) = \sum_i \langle [X,[Y,e_i]], e_i\rangle = -\sum_i \langle [X,e_i],[Y,e_i]\rangle$$

so it is negative semi-definite and $\rho \geq 0$. If $\rho = 0$, we have $[X,e_i] = 0$ for all $X \in \mathfrak{m}$ and for all e_i in the basis. This in turn implies that $[\mathfrak{m},\mathfrak{m}] = 0$. However, as was noted earlier, if $(\mathfrak{g},\mathfrak{h},\sigma)$ is an irreducible and orthogonal symmetric Lie algebra we have $[\mathfrak{m},\mathfrak{m}] = \mathfrak{h}$. We then conclude that $\rho > 0$. □

Example. We return to the example $(\mathfrak{g}+\mathfrak{g}, \Delta(\mathfrak{g}), \sigma)$ that we gave earlier that is associated to a compact Lie group. In this case we can see that $\mathfrak{h} = \Delta(\mathfrak{g}) = \{(X,X)\}$ and $\mathfrak{m} = \{(X,-X)\}$. One can easily verify that for $\tilde{X}$, $\tilde{Y}$ in $\mathfrak{m}$

$$\begin{aligned} \kappa(\tilde{X},\tilde{Y}) &= \mathrm{Tr}_{\mathfrak{g}+\mathfrak{g}}(ad_{\tilde{X}} \circ ad_{\tilde{Y}}) \\ &= 2\mathrm{Tr}_{\mathfrak{m}}(ad_{\tilde{X}} \circ ad_{\tilde{Y}}) \\ &= 2B(ad_{\tilde{X}} \circ ad_{\tilde{Y}}) \end{aligned}$$

so $\rho = 1/2$.

We note that ρ is entirely dependent on the triple $(\mathfrak{g},\mathfrak{h},\sigma)$ and is not subject to any arbitrary choices. It is specified in the same way that $\mathfrak{m}$ is specified by $\mathfrak{g}$ and σ.

If $(\mathfrak{g},\mathfrak{h},\sigma)$ is not irreducible we can also compare B and κ. Suppose that

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_1 + \cdots + \mathfrak{g}_k, \\ \mathfrak{h} &= \mathfrak{h}_1 + \cdots + \mathfrak{h}_k, \\ \mathfrak{m} &= \mathfrak{m}_1 + \cdots + \mathfrak{m}_k, \end{aligned}$$

where $(\mathfrak{g}_i, \mathfrak{h}_i, \sigma_i)$ is an irreducible orthogonal symmetric Lie algebra of compact type. and $\sigma_i = \sigma_{\mathfrak{g}_i}$. Then the Killing form of $\mathfrak{g}_i$ is equal to the restriction of the Killing form of $\mathfrak{g}$. Also, since $[\mathfrak{m}_i, \mathfrak{m}_j] = 0$ for $i \neq j$ we have that $B(\mathfrak{m}_i, \mathfrak{m}_j) = 0$. $B|_{\mathfrak{m}_i}$ is an $\mathfrak{h}_i$ invariant symmetric form on $\mathfrak{m}_i$ so by the same argument to above we have

$$B|_{\mathfrak{m}_i} = \rho_i \, \kappa|_{\mathfrak{m}_i} \quad \text{for } \rho_i > 0 \tag{1.3}$$

$$\text{and,} \quad B = \sum_i B_{\mathfrak{m}_i} = \sum_i \rho_i \, \kappa|_{\mathfrak{m}_i}. \tag{1.4}$$

If we define $\rho = \min\{\rho_i\}$ then for any $X \in \mathfrak{m}$

$$-B(X, X) \geq \rho \|X\|^2.$$

This inequality is used in the chapters to follow.

We can make this calculation more explicit and more in perspective by noting that the tensor is related to the intrinsic geometry of M. This uses an identity that we prove in the following section. We have previously defined a metric on a symmetric space by restricting the negative of the Killing form to $\mathfrak{m}$. This defines a G-invariant metric on M with respect to which all the geodesic symmetries are isometries. Equation 1.5 states that at o and with respect to the identification $T_o M = \mathfrak{m}$ we have

$$R_{X,Y} Z = -[[X, Y], Z]$$

for X, Y, Z in $\mathfrak{m}$. The (unnormalised) Ricci curvature is then given by

$$\begin{aligned} Ric(X, Y) &= \operatorname{Tr}\{Z \mapsto R_{Z,X} Y\} = -\sum_i \langle [[e_i, X], Y], e_i \rangle \\ &= -\operatorname{Tr}_{\mathfrak{m}} (\operatorname{ad}_X \circ \operatorname{ad}_Y) \\ &= -B(X, Y). \end{aligned}$$

That is, $-B$ is exactly the Ricci curvature of M. This in particular shows that if M is irreducible, $Ric = \rho\langle \, , \, \rangle$ so is Einstein. In the reducible case we can say that

$$-\operatorname{Tr}_{\mathfrak{m}} (\operatorname{ad}_X \circ \operatorname{ad}_X) \geq \rho \|X\|^2$$

where ρ is the smallest Ricci curvature of any direction tangent to M.

The next quantity that we will define is motivated by the relationship of symmetric spaces to semi-simple Lie groups. If $\mathfrak{g}$ is a complex semi-simple Lie algebra, the *rank* of $\mathfrak{g}$ is the dimension of a maximal abelian subalgebra of $\mathfrak{g}$. The algebraic structure of $\mathfrak{g}$ can be very precisely obtained by combinatorial data from such a subalgebra. For example, in the algebra $\mathfrak{so}(2n)$, the span of elements of the form

$$\begin{pmatrix} 0 & -x_1 & & & \\ x_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -x_n \\ & & & x_n & 0 \end{pmatrix}$$

forms a maximal abelian subalgebra. The rank of $\mathfrak{so}(2n)$ is then n.

For an orthogonal symmetric Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, we define the *rank* of $(\mathfrak{g}, \mathfrak{h}, \sigma)$ to be the dimension of a maximal subspace V of $\mathfrak{m}$ for which $[X, Y] = 0$ for X, Y in V. We again use the Grassmannian as an example.

Example. Let $M = G(n, n+m)$ be the Grassmannian of n-planes in $\mathbb{R}^{n+m}$. This is a Riemannian symmetric space with orthogonal symmetric Lie algebra $(\mathfrak{so}(n+m), \mathfrak{so}(n) + \mathfrak{so}(m), \sigma)$. The subspace $\mathfrak{m}$ is spanned by elements of the form

$$E_{ij} = \begin{pmatrix} 0 & -e_{ij}^T \\ e_{ij} & 0 \end{pmatrix}$$

where e_{ij} is an $m \times n$ matrix with a 1 in the (i, j) slot and zeroes elsewhere. E_{ij} and E_{kl} commute if and only if $i \neq k$ and $j \neq l$. The dimension of a maximal commuting subspace is thus the smaller of n and m. We then have that $\mathrm{rk}(\mathfrak{so}(n+m), \mathfrak{so}(n) + \mathfrak{so}(m), \sigma) = \min(n, m)$

In the following section, in Equation 1.6, we see that these commuting subspaces in $\mathfrak{m}$ correspond exactly to subspaces of T_oM on which the sectional curvature vanishes. The theorems for rigidity of minimal submanifolds are for the sphere and other rank-1 symmetric spaces, in part because of the strict positivity of the curvature.

1.3 Geometry of Riemannian Symmetric Spaces

In this section we will relate the calculations of the previous section that explicitly gave the metric that we are interested in to the global geometry of the symmetric space $M = G/H$. As previously, this primarily comes from [4].

Our first calculation is to determine the curvature of the Levi-Civita connection for a G-invariant metric on M. This will be done using the principal-bundle formalism.

We consider the homogeneous manifold $M = G/H$ where $H = G_p$ is the stabiliser of some point $p \in M$. G is a principal H-bundle over M. In particular, G can be considered to be a sub-bundle of the the tangent frame bundle of M. This can be shown by considering the map

$$\begin{aligned} G &\to P_{GL}(M), \\ g &\mapsto g_*\{e_i\} \end{aligned}$$

where $\{e_i\}$ is a fixed frame of vectors at p. G is transitive on M so this determines an H-structure on M. In particular, this means that the associated vector bundle for the isotropy representation of H, which is identified with the adjoint representation on $\mathfrak{m}$, is the tangent bundle to M.

If the frame $\{e_i\}$ is orthonormal with respect to some H-invariant inner product at p, G is mapped to the orthonormal frame bundle of the G-invariant Riemannian metric on M.

We can use this description to state a convenient characterisation of the Levi-Civita connection on a a Riemannian symmetric space. It is determined by the conditions on the connection 1-form on the bundle $G \to G/H$:

1. ω is a left-invariant $\mathfrak{h}$-valued 1-form on G.

2. At $e \in G$, $\omega(X_{\mathfrak{h}} + X_{\mathfrak{m}}) = X_{\mathfrak{h}}$ with respect to the splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

Equivalently, left translation of $\mathfrak{m}$ ($\subseteq T_eG$) defines a distribution on G. From its definition every element of the distribution projects on the corresponding tangent space to M and since $\mathfrak{m}$ is $Ad(H)$-invariant, the distribution is invariant under H. In the context of considering G as a bundle over M, and looking at the naturally induced connection on it we will refer to $\mathfrak{h}$ as the *vertical* subspace of $\mathfrak{g}$ and $\mathfrak{m}$ as the *horizontal* subspace.

This distribution defines the connection form ω. The curvature of the connection is essentially the derivative of ω. Specifically, let Ω be the curvature form on G of the canonical connection that we have defined. This is an $\mathfrak{h}$-valued 2-form on G. For X and Y tangent to G, Ω is defined by the equation

$$\Omega(X, Y) = d\omega(X_{\mathfrak{m}}, Y_{\mathfrak{m}}).$$

Here $X_{\mathfrak{m}}$ and $Y_{\mathfrak{m}}$ are the horizontal components of X and Y respectively. If $X, Y \in \mathfrak{m}$ are horizontal vector fields they are everywhere in the kernel of ω. We then have

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]) = -\omega([X, Y])$$
$$\textit{ie.,} \qquad \Omega(X, Y) = -\omega([X, Y]) = -[X, Y]_{\mathfrak{h}} = -[X, Y].$$

Generally when we relate the two formalisms of vector bundles and principal bundles, we need to know how the structure group (the fibre of the principal bundle) is represented on the vector space (the fibre of the vector bundle).

In the case at hand, the structure group of the manifold $M = G/H$ is H, the fibre of the associated vector bundle is $\mathfrak{m}$ and the representation of H on $\mathfrak{m}$ is the adjoint representation.

We summarise this information.

Theorem 1.6. *Let $M = G/H$ be a Riemannian symmetric space with riemannian metric induced by an H-invariant inner product on $\mathfrak{m}$. For $p = [H] \in M$ let X, Y and Z be tangent to M at p. Then using the identification of T_pM with $\mathfrak{m}$, the curvature tensor of M is given by*

$$R_{X,Y}Z = -[[X, Y], Z]. \tag{1.5}$$

The sectional curvature of the metric is given by

$$\begin{aligned} \kappa(X \wedge Y) &= \langle R_{X,Y}Y, X\rangle \\ &= -\langle [[X, Y], Y], X\rangle \\ &= \langle [X, Y], [X, Y]\rangle \end{aligned} \tag{1.6}$$

for $X, Y \in \mathfrak{m}$ orthogonal and unit length so the sectional curvature is non-negative and the zero curvature planes correspond to commuting elements of $\mathfrak{m}$. If $(mathfrakg, \mathfrak{h}, \sigma)$ has rank one there are no such pairs of commuting planes so the sectional curvature is strictly positive.

We consider these for the symmetric Lie algebra $(\mathfrak{so}(n+1), \mathfrak{so}(n), \sigma)$ with associated Riemannian symemtric space S^n. The metric obtained from the Killing form on $\mathfrak{so}(n+1)$ is a constant multiple of the standard metric on S^n. The sectional curvature is in this case $\frac{1}{2(n-1)}$. The curvature transformation is given by

$$\overline{R}_{X,Y}Z = -\mathrm{ad}([X,Y])(Z) = \frac{1}{2(n-1)}\Big(- \langle X, Z\rangle Y + \langle Y, Z\rangle X\Big).$$

We next consider submanifolds of riemannian symmetric spaces that respect some of the symmetry of the space. Specifically, we consider the totally geodesic submanifolds of $M = G/H$. The fundamental fact that we require is the following.

Theorem 1.7. [4] *Let $M = G/H$ be a Riemannian symmetric space amd $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ the canonical decomposition. Then there is a natural one-to-one correspondence between the set of linear subspaces $\mathfrak{m}'$ of $\mathfrak{m}$ such that $[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}'$ and the set of complete totally geodesic submanifolds through the origin $[H]$ of M.*

This further emphasises the importance of the curvature tensor. These are exactly the subspaces of $\mathfrak{m}$ that are preserved by the curvature tensor. We call a subspace $\mathfrak{m}'$ that satisfies $[[\mathfrak{m}', \mathfrak{m}'], \mathfrak{m}'] \subset \mathfrak{m}'$ a *Lie Triple System.*

1.4 Rank-One Symmetric Spaces

In this section we describe the classification of rank-one symmetric spaces of compact type. Earlier in this chapter we have seen that an orthogonal symmetric Lie algebra $(\mathfrak{g}, \mathfrak{h}, \sigma)$, where $\mathfrak{g}$ is a semi-simple Lie algebra, splits into the direct sum of irreducible subspaces. This can be expressed in the statement that a simply-connected Riemannian symmetric space with semi-simple isometry group has de Rham decomposition consisting of a product of irreducible symmetric spaces.

The theorem that we will prove in the coming chapters can generally be considered that the irreducible components of a compact space that have rank one, other than the Cayley Plane, are rigid when considered as minimal submanifolds.

This proof is done on a case-by-case basis, given that there is a classification of rank-one symmetric spaces of compact type. This is a classical result and we will outline its proof in this section. For brevity though we will omit many of the technicalities and subtleties.

Little in this section is original. It is primarily a summary of the excellent reference [2] by Chavel and the proof of Gluck, Warner and Ziller [3] of a theorem of Wong, Wolf, Escobales and Ranjan. This section will be entirely expository, with many absent or imprecise proofs. In this section we will consider M to be a rank-one Riemannian symmetric space of compact type and of dimension n. Here $M = G/H$ where H is the isotropy group ofo se point $o \in M$.

To start we make the definition that a Riemannian homogeneous space G/H is *two-point homogeneous* if for any points $p, q, r, s \in G/H$ such that $d(p,q) = d(r,s)$ there is a an element of G that sends p to r and q to s. This is equivalent to the linearised isotropy action being transitive on the unit sphere at a point. In terms of the Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ in the symmetric case we have the proposition.

Proposition 1.8. ([2], page 61) *A Riemannian symmetric space $M = G/H$ is two-point homogeneous if and only if the associated orthogonal symmetric Lie algebra $\mathfrak{g}$ satisfies*

$$\mathfrak{m} = \mathbb{R}X + [\mathfrak{h}, X] \quad \textit{for all } X \in \mathfrak{m}$$

For a fixed $X \in \mathfrak{m}$ we consider the map $B_X : Y \mapsto [[X,Y],X]$. This is symmetric with respect to the metric on $\mathfrak{m}$ and so can be orthogonally diagonalised. If $Y_1, \ldots, Y_{n-1}$ is a basis of eigenvectors we then have that

$$[X, Y_i] \in \mathfrak{h},$$

and if all of the eigenvectors are non-zero, $[[X,Y_i],X] = \lambda_i Y_i$ then span the orthogonal complement to $X \in \mathfrak{m}$. This is sufficient to show that M is two-point homogeneous. This assumption on the eigenvectors is equivalent to all sectional curvatures through X being non-zero. From the previous section, if the symmetric space is of rank one all sectional curvatures are positive so the space is two-point homogeneous.

This in particular means that all non-constant geodesics are homotopic and equivalent to a simple closed one. They all have the same length which can be normalised to be 2π.

For $X \in T_oM$ we define the number

$$s(X) = \sup\{t\,;\, d(o, \exp_o(tX)) = t\} > 0.$$

This is the value of t for which the unit speed geodesic in that direction ceases to be length minininising. The tangential cut locus is the set

$$TC(o) = \{s(X)X\,;\,|X| = 1\} \subseteq T_oM$$

and the cut locus is

$$C(o) = \{\exp_o(s(X)X)\,;\,|X| = 1\} \subseteq M.$$

The above choice for geodesic loops then gives that the tangential cut locus for a rank one symmetric space is

$$TC(o) = \{X\,;\,|X| = \pi\} = \pi S^{n-1}.$$

Otherwise there would exist simple geodesic loops of length less than 2π.

We next consider the map

$$Y \mapsto R_{Y,X}X$$

for X a unit tangent vector. Then (see [2]) by considering Jacobi fields along the geodesic $\gamma(t) = \exp_o(tX)$ we can see that the only possible eigenvalues for the map are $\frac{1}{4}$ and 1.

We define λ to be the dimension of the 1-eigenspace. By the two-point homogeneity considered above this number is independent of the the point $o \in M$ and the direction $X \in T_oM$.

We consider the space $C(o) \subseteq M$ and consider the map $\exp_o : \pi S^{n-1} \to C(o)$. From the two-point homogeneity of M, the isotropy group at o acts transitively on $\pi S^{n-1} \subseteq T_oM$ and so also on $C(o)$. An orbit of a smooth proper group action is a smooth submanifold so $C(o)$ is a smooth, compact submanifold.

For reasons of invariance the map $\exp_o$ is a submersion. For $p \in C(o)$ and $\pi X \in \pi S^{n-1}$ (with $|X| = 1$) such that $\exp_o(\pi X) = p$, the kernel of $(d\exp_o)_{\pi X}$ can be identified with the set of Jacobi fields along the geodesic $\gamma(t) = \exp_o(tX)$ that vanish at o and p. The dimension of this space is called the *multiplicity* of p with respect to o along γ.

For $X \in T_oM$ with $|X| = 1$ we consider vectors that satisfy

$$R_{Y,X}X = \kappa Y. \tag{1.7}$$

Recall that the possible values of κ are $1/4$ and 1. If Y satisfies Equation 1.7 at o and is parallel translated along γ then since $\nabla R = 0$ Y continues to satisfy Equation 1.7 along γ. We can then see that the Jacobi fields along γ that vanish at o are given by

$$J_\kappa(t) = \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa} t) Y(t).$$

This vanishes at $p = \exp_o(\pi X)$ if $\kappa = 1$ and does not for $\kappa = 1/4$. The kernel of $(d\exp_o)_{\pi X}$ is then equal to the $\kappa = 1$ eigenspace of the above curvature map. Let $V_1 \subseteq T_o M$ be the vector subspace spanned by the X and the $\kappa = 1$ eigenspace. By an invariance argument one can show that the fibre of the submersion that maps to p is equal to the great sphere

$$\exp_o^{-1}(p) = \pi S^{n-1} \cap V_1.$$

In summary then, we have a submersion

$$\exp_o : \pi S^{n-1} \to C(o)$$

of a sphere by totally geodesic spheres. Since H acts transitively on πS^{n-1} and the way the fibres have been characterised we can easily see that the fibres are of constant distance apart. The proof of the classification of rank-one symmetric spaces is then completed by by referring to results on the restrictions that exist for fibrations of spheres by parallel totally geodesic subspaces. In particular we quote the theorem

Theorem 1.9. (Wong, Wolf, Escobales, Ranjan) *If an open set in a round sphere is filled by pieces of parallel great spheres then that filling is a portion of a Hopf fibration.*

This received an elementary unified proof by Gluck, Warner and Ziller [3] that we descibe. In our description we only consider the case of a sphere fibred by spheres, neglecting the words "piece" and "portion".

The Hopf fibrations are given using the normed division algebras $\mathbb{K}$ and are given by sending a nonzero point in $\mathbb{K}^{n+1}$ to the line that it spans. For $\mathbb{K} = \mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ the Hopf maps are given by

$$\begin{aligned} S^1 \to S^{2n+1} &\to \mathbb{CP}_n \\ S^3 \to S^{4n+3} &\to \mathbb{HP}_n \\ S^7 \to S^{15} \to S^8 &= \mathbb{OP}_1. \end{aligned}$$

Following [3], if there is a fibration of S^{n-1} by parallel totally geodesic $(k-1)$-spheres, then one can construct a fibration of S^{n-1-k} by totally geodesic $(k-1)$-spheres. Proceeding inductively, one has that k divides n. Let $F : S^{k-1} \to S^{n-1} \to M$ be such a fibration , If P is a fibre of F then it spans a k-dimensional subspace of $\mathbb{R}^n$ which we also denote by P. This defines a map

$$F : M \to G_k(\mathbb{R}^n)$$

into the appropriate Grassmannian. The derivative of this map at P can be considered as

$$\begin{aligned} F_* : T_P M &\to \mathrm{Hom}(P, P^\perp), \\ \text{or} \quad F_* : T_P M \otimes P &\to P^\perp \end{aligned} \tag{1.8}$$

where P denotes either a point in M or the span of the fibre over this point. $P^\perp$ is the orthogonal complement to P in $\mathbb{R}^n$.

If $n = 2k$, which is the first nontrivial case, $P, T_P M$ and $P^\perp$ are all k-dimensional and the expression 1.8, since the spaces have been mutually identified, can be identified with the multiplication in a normed division algebra. A classical theorem of Hurwitz identifies this with $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, according to dimension. It follows that there is an isometry of $\mathbb{R}^n$ that sends the fibration F to the Hopf fibration. The case $n > 2k$ can be reduced to this case.

We also note the explicit theorem from [3].

Theorem 1.10. *There is no open set in S^{23} that can be filled by pieces of parallel great 7-spheres.*

This demonstrates the non-existence of projective spaces over the octonions past the Cayley Plane.

This is sufficient for the classification of rank-one symmetric spaces. If M is a compact rank-one symmetric space and $o \in M$, the sphere of radius π in $T_o M$ can be fibred by parallel totally geodesic spheres. The tangent space to the fibre at $\pi X \in \pi S^{n-1}$ consists of $+1$-eigenvectors of the curvature map

$$Y \to R_{Y,X} X.$$

The orthogonal complement consists of $1/4$-eigenvectors. This determines the sectional curvatures of the manifold at o and hence the curvature operator at o. This determines the decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$$

and hence the symmetric space G/H. The fibration must be a Hopf fibration so the symmetric space must be one of S^m, $\mathbb{RP}_m$, $\mathbb{CP}_m$, $\mathbb{HP}_m$ or $\mathbb{OP}_2$.

Chapter 2

Minimal Submanifolds

2.1 The Second Fundamental Form

Let $f : M^p \to \bar{M}^n$ be a smooth immersion. We can pull the ambient tangent bundle back to M and see that $f^*(T_{\bar{M}})$ splits as a direct sum

$$f^*(T_{\bar{M}}) = T_M \oplus N_M \tag{2.1}$$

into the tangent and normal compnents. One can also pull the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$ back to $f^*(T_{\bar{M}})$. By projection to the two factors this induces connections on T_M and N_M. That is, for $\in \mathcal{E}(T_M)$ and $\nu \in \mathcal{E}(N_M)$,

$$\begin{aligned} \nabla^M X &= (\overline{\nabla} X)^T \\ \nabla^N \nu &= (\overline{\nabla} \nu)^N. \end{aligned}$$

∇^M is the Levi-Civita connection for T_M for the metric induced by the immersion. With respect to the splitting of equation 2.1 on M the connection $\bar{\nabla}$ decomposes as

$$\overline{\nabla} = \begin{pmatrix} \nabla^M & -A \\ B & \nabla^N \end{pmatrix} \tag{2.2}$$

Here A and B are tensors that relate T_M and N_M. Explicitly they are given by

$$\begin{aligned} A^\nu(X) &= -(\overline{\nabla}_X \nu)^T \\ B(X, Y) &= (\overline{\nabla}_X Y)^N. \end{aligned}$$

for X and Y tangent vectors and ν normal. The following is clear and easy to prove.

Proposition 2.1. *1. A and B are tensorial in X, Y and ν. Ie., they are only dependent on these vectors at the given point.*

2. $B(X, Y) = B(Y, X)$.

3. $\langle A^\nu(X), Y\rangle = \langle X, A^\nu(Y)\rangle$

4. $\langle A^\nu(X), Y\rangle = \langle \nu, B(X, Y)\rangle$

The first part of the proposition means that we can consider A and B to be sections of bundles defined over M. For example, B is at each point a symmetric bilinear form on T_M with values in N_M. The negative sign in 2.3 is given to ensure part (4) of the proposition, which is that A and B are transposes of one another. A and B will together be referred to as the *Second Fundamental Form* of the immersion. The tensor A will be considered as a section of Riemannian vector bundle $H(M) = Hom(N_M, S(M))$. Here $S(M)$ is the space of symmetric linear endomorphisms of T_M. In our consideration of Simons's work we will consider this space further.

An immersed submanifold M of $\overline{M}$ is *totally geodesic* if the second fundamental form of the immersion vanishes identically. In this case the Levi-Civita connection of the ambient manifold is (in directions tangent to the submanifold) the direct sum of the connections on the subspaces. Parallel transport with respect to the ambient connection along tangential paths preserves tangential and normal vectors. The ambient curvature tensor, when acting on tangential vectors, coincides with the curvature of the submanifold.

An immersed submanifold M of $\bar{M}$ is said to be *minimal* if the trace of the second fundamental form vanishes identically. That is, we define the *mean curvature* of a submanifold to the the normal vector field K defined by

$$K = \mathrm{Tr} B = \sum_i B(e_i, e_i)$$

where $\{e_i\}$ is an orthonormal basis for T_M. The submanifold M is minimal if $K \equiv 0$. This is equivalent to

$$\mathrm{Tr} A^\nu = 0$$

for all normal vectors ν. The condition of minimality is natural because the mean curvature can be considered the gradient vector of the area function, considered on the space of immersions of one fixed manifold in another. An immersion is a critical point of the area (possibly area minimising) if the mean curvature vanishes identically. $K = 0$ is the Euler-Lagrange equation for the variational problem.

Example. We give an example of a helicoid-like minimal surface in $S^2 \times S^1$. We recall the definition of the helicoid. This is a minimal submanifold of $\mathbb{R}^3$ obtained by raising a line in $\mathbb{R}^2$ at constant speed and simultaneously rotating it at constant speed. We have a similar example in $S^2 \times \mathbb{R}$.

Let $(x, y, z) = (\cos\theta\ \sin\phi, \sin\theta\ \sin\phi, \cos\phi)$ be a coordinate description for S^2 and let $t \in \mathbb{R}$. We can define an embedding of $S^1 \times \mathbb{R}$ into $S^2 \times \mathbb{R}$ by

$$(\cos\phi, \sin\phi, t) \mapsto (\cos(\alpha t)\ \sin\phi, \sin(\alpha t)\ \sin\phi, \cos\phi, t)$$

where α is an arbitrary fixed real number. The great circle $S^2 \cap \{y = 0\}$ is rotated at rate α as it is lifted at constant unit speed. The image of the embedding has tangent space spanned by the orthogonal unit vectors

$$\begin{aligned} e_1 &= \partial_\phi = (\cos(\alpha t)\ \cos\phi, \sin(\alpha t)\ \cos\phi, -\sin\phi, 0) \\ e_2 &= \frac{1}{|\partial_t|}\partial_t = \frac{1}{\sqrt{1 + \alpha \sin^2\phi}}(-\alpha\sin(\alpha t)\ \sin\phi, \alpha\cos(\alpha t)\ \sin\phi, 0, 1) \end{aligned}$$

and with respect to this orthonormal basis the second fundamental form A^ν is given by the matrix

$$A^\nu = \frac{\alpha\cos\phi}{1 + \alpha\sin^2\phi}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.3}$$

It is thus trace free so for any value of α the helicoid-like submanifold of $S^2 \times \mathbb{R}$ is minimal. We can also note that the submanifold is invariant under the action of the group $\frac{\pi}{\alpha}\mathbb{Z}$ acting isometrically by translation in the real factor. If $\alpha = k/2$ where k is a positive integer the submanifold is invariant by $2\pi\mathbb{Z}$ so descends to the quotient of the ambient space by this group. This gives a minimal submanifold of $S^2 \times S^1$ where S^1 is the circle of unit radius.

2.2 The Fundamental Second Order Equation for A

In this section we give the second order differential equation for A that is fundamental to this dissertation. The equation relates the Laplacian of A to an algebraic expression involving A and the ambient curvature of $\bar{M}$. It was derived by Simons [8] and used extensively to study submanifolds of spheres and of euclidean space. The principal results given in this dissertation are of a similar type to some in that paper.

Let (M, g) be a smooth Riemannian manifold and V a vector bundle over M. Suppose that V is endowed with an inner product defined on its fibres and a connection ∇ that preserves the metric. We say that V with this structure is a a smooth Riemannian vector bundle. Then, for $\psi \in \mathcal{E}(V)$, we have $\nabla\psi \in \mathcal{E}(T^* \otimes V)$. $T^* \otimes V$ is a Riemannian vector bundle so we can differentiate again. We define the second derivative of ψ in the directions X and Y to be

$$\begin{aligned}\nabla_{X,Y}\psi &= \nabla_X(\nabla\psi)(Y) \\ &= \nabla_X\nabla_Y\psi - \nabla_{\nabla_X Y}\psi\end{aligned}$$

where the symbol ∇ should be interpreted carefully. This expression is symmetric in X and Y. Define $\nabla^2\psi$ to be the section of V given by

$$\begin{aligned}\nabla^2\psi &= \operatorname{Tr}(X, Y \mapsto \nabla_{X,Y}\psi) \\ &= \sum_i \nabla_{e_i,e_i}\psi.\end{aligned}$$

The important property of this *rough* Laplacian that we will require is that it is, in good situations, negative semi-definite.

Proposition 2.2. *Let M be a compact oriented riemannian manifold with boundary. Let $\varphi, \psi \in \mathcal{E}(V)$ be sections of V. Then we have*

$$\int_M \langle\nabla^2\psi, \varphi\rangle \mathrm{vol} = -\int_M \langle\nabla\psi, \nabla\varphi\rangle \mathrm{vol} + \int_{\partial M} X \lrcorner \,\mathrm{vol} \tag{2.4}$$

where X is the vector field on M defined by

$$\langle X, Y\rangle = \langle\nabla_Y\psi, \varphi\rangle.$$

This is a standard result that follows by Stokes' Theorem. In particular we can conclude that if $\partial M = \emptyset$, ∇^2 is negative semi-definite.

We will use this on the vector bundle $H(M) = Hom(N_M, S(M))$ for M a submanifold, N_M its normal bundle and $S(M)$ the bundle of symmetric transformations of T_M. The natural connection on $H(M)$ is given as follows. Let $A \in \mathcal{E}(H(M))$, $\nu \in \mathcal{E}(N_M)$, $X, Y \in \mathcal{E}(T_M)$. Then,

$$(\nabla_X A)^\nu(Y) = \nabla_X^M(A^\nu Y) - A^{\nabla_X^N \nu}(Y) - A^\nu(\nabla_X Y).$$

The second derivative can also be derived similarly. In the case that A and B define the second fundamental form of M we have two identities.

Theorem 2.3. [8] *Let $M \subseteq \bar{M}$ be a minimal submanifold and let $\bar{R}$ denote the riemannian curvature of $\bar{M}$. Then*

$$(\nabla_x B)(y,z) - (\nabla_y B)(x,z) = (\bar{R}_{x,y}z)^N \qquad \forall x, y, z \in T_M \tag{2.5}$$

$$\sum_{i=1}^{p} (\nabla_{e_i} B)(e_i, z) = \sum_{i=1}^{p} (\bar{R}_{e_i,z} e_i)^N \quad \forall z \in T_M. \tag{2.6}$$

where $\{e_i\}$ is a local frame on M. The first equation holds for all submanifolds, the second holds if M is minimal.

These two expressions can be re-interpreted by instead considering B as a 1-form with values in a the bundle $Hom(T_M, N_M)$. The first equation here, known as Codazzi's Equation, is the anti-symmetrization of the derivative. The second equation is the trace of the derivative of B. The equations can be re-expressed as

$$\begin{aligned} d^\nabla B &= \pi^N \bar{R} \\ \delta^\nabla B &= \mathcal{R}. \end{aligned}$$

where $\mathcal{R}$ is a Ricci-like transformation in $Hom(T_M, N_M)$. The operator d^∇ sends bundle valued 1-forms to bundle valued 2-forms. The operator δ^∇ is the formal adjoint of the differential operator $\nabla : Hom(T_M, N_M) \to \Lambda^1 \otimes Hom(T_M, N_M)$. In this way the equations can be considered inhomogeneous Yang-Mills-type equations. The inhomogeneity is only in terms of the ambient curvature and the first order behaviour of the submanifold.

An additional similarity with the Yang-Mills equations is the fact that one (Second Bianchi Identity) holds for all connections. The other comes from the variational problem. In this case, equation 2.5 holds for all submanifolds. Equation 2.6 holds only if the submanifold is minimal.

In particular, one can conclude that the second fundamental form satisfies a first order elliptic equation. It also however satisfies a second order equation that is the basis for our investigation. We first must define some terms that appear in the equation.

We recall that the second fundamental form A is a section of the bundle $Hom(N_M, S(M))$. For any normal vector ν, A^ν is a symmetric transformation of T_M. We can then consider A^t to be the transpose of A, considering $S(M)$ and N_M to have their natural metrics. We define $\tilde{A}$ to be the section of $Hom(N_M, N_M)$

$$\tilde{A} = A^t \circ A.$$

$\tilde{A}$ satisfies

$$\begin{aligned} \langle \tilde{A}(\nu), \eta \rangle &= \langle A^\nu, A^\eta \rangle \\ &= \mathrm{Tr}(A^\nu \circ A^\eta) \\ &= \sum_i \langle A^\nu(e_i), A^\eta(e_i) \rangle. \end{aligned} \tag{2.7}$$

We now define a slightly less simple operator. We note that the map

$$(\nu, \eta) \mapsto \mathrm{ad}(A^\nu) \circ \mathrm{ad}(A^\eta)$$

is a bilinear form on N_M with values in $Hom(S(M), S(M))$. We define $\underset{\sim}{A}$ to be the trace of this map. That is,

$$\begin{aligned} \underset{\sim}{A} &= \sum_j \mathrm{ad}A^{\nu_j} \mathrm{ad}A^{\nu_j} \\ \text{and} \quad \langle \underset{\sim}{A}(s_1), s_2 \rangle &= \sum_j \langle [A^{\nu_j}, [A^{\nu_j}, s_1]], s_2 \rangle \\ &= \sum_j \langle [A^{\nu_j}, s_1], [A^{\nu_j}, s_2] \rangle \end{aligned} \tag{2.8}$$

where s_1 and s_2 are symmetric transformations of T_M. From these expressions we can see that $\tilde{A}$ and $\underset{\sim}{A}$ are symmetric and positive semi-definite operators on the respective spaces. We will consider $A \circ \tilde{A}$, $\underset{\sim}{A} \circ A \in \mathcal{E}(\mathrm{Hom}(N_M, S(M)))$.

The two remaining tensors on M that we will define are dependent on A, together with the ambient curvature of $\bar{M}$.

Let $\overline{R}(A)$ be the section of $\mathrm{Hom}(N_M, S(M))$ defined by

$$\langle \overline{R}(A)^W X, Y\rangle = \sum_{i=1}^{p} \left\{ \begin{array}{l} 2\langle \overline{R}_{e_i,Y} B(X,e_i), W\rangle + 2\langle \overline{R}_{e_i,X} B(Y,e_i), W\rangle \\ -\langle A^W(X), \overline{R}_{e_i,y} e_i\rangle - \langle A^W(Y), \overline{R}_{e_i,X} e_i\rangle \\ +\langle \overline{R}_{e_i,B(X,Y)} e_i, W\rangle - 2\langle A^W(e_i), \overline{R}_{e_i,X} Y\rangle \end{array} \right\}$$

where $\{e_i\}$ is an orthonormal basis for T_M. To simplify the notation in the later sections we will denote the six terms in this expression by (1) to (6). For example,

$$\langle (2)^W X, Y\rangle = 2\sum_{i=1}^{p} \langle \overline{R}_{e_i,X} B(Y,e_i), W\rangle.$$

For a fixed normal vector W, $\overline{R}(A)^W$ can be seen to be symmetric in X and Y. Terms (2) and (4) are the transposes of terms (1) and (3) respectively. The fifth term is symmetric since B is.

We will show that (6) is symmetric. Firstly, we use the first Bianchi identity to observe

$$\overline{R}_{e_i,X} Y = \overline{R}_{e_i,Y} X - \overline{R}_{X,Y} e_i.$$

Next, recall that a symmetric transformation (for example, A^W) is orthogonal to a skew-symmetric one (such as $\overline{R}_{X,Y}$). Thus,

$$\begin{aligned} \langle (6)^W X, Y\rangle &= -2\sum_i \langle A^W(e_i), \overline{R}_{e_i,X} Y\rangle \\ &= -2\sum_i \langle A^W(e_i), \overline{R}_{e_i,Y} X\rangle + 2\langle A^W, \overline{R}_{X,Y}\rangle \\ &= \langle (6)^W Y, X\rangle. \end{aligned}$$

as was required.

We now define the final distiguished section of $Hom(N_M, S(M))$ that we need. For X, Y tangent vectors to M and W normal to M and $\{e_i\}$ an orthonormal basis for T_M, define $\overline{R}'$ by

$$\langle \overline{R}'^W X, Y\rangle = \sum_i \big(\langle \overline{\nabla}_X (\overline{R})_{e_i,y} e, W\rangle + \langle \overline{\nabla}_{e_i} \overline{R}_{e_i,X} Y, W\rangle\big).$$

$\overline{R}'$ is defined independently of the basis for T_M and is linear in its arguments. It is a smooth section of $Hom(N_M, S(M))$ that is dependent on A and the ambient derivative $\overline{\nabla}(\overline{R})$ of curvature. This now allows us to give the theorem proved by Simons that shows that A satisfies a second order elliptic equation.

Theorem 2.4. [8] *Let M be a minimal submanifold of a riemannian manifold and let A be its second fundamental form, considered as a section of $Hom(N_M, S(M))$. Then A satisfies*

$$\nabla^2 A = -A \circ \tilde{A} - \underset{\sim}{A} \circ A + \overline{R}(A) + \overline{R}'. \tag{2.9}$$

This equation was given in the paper of Simons [8] and was fundamental to many of his important results there, including an intrinsic rigidity theorem for totally geodesic submanifolds of S^n.

We make some observations that we will use in the following chapters. The second term $(2)^W$ is the transpose of $(1)^W$ and $(4)^W$ is the transpose of $(3)^W$. That is,

$$\begin{aligned} \langle (1)^W X, Y \rangle &= \langle X, (2)^W Y \rangle \\ \langle (3)^W X, Y \rangle &= \langle X, (4)^W Y \rangle. \end{aligned}$$

Since A is symmetric, this in particular means that

$$\begin{aligned} \langle (2), A \rangle &= \sum_j \mathrm{Tr}\big(A^{\eta_j *} \circ (2)^{\eta_j}\big) \\ &= \sum_j \mathrm{Tr}\big((2)^{\eta_j *} \circ A^{\eta_j}\big) \\ &= \sum_j \mathrm{Tr}\big((1)^{\eta_j} \circ A^{\eta_j}\big) \\ &= \langle (1), A \rangle. \end{aligned}$$

Similarly, $\langle (3), A \rangle = \langle (4), A \rangle$. Thus, in the estimation of $\langle \overline{R}(A), A \rangle$ we only need to consider (1), (3), (5) and (6).

Chapter 3

Geometry of Projections

This aim of this dissertation is to give some results on minimal submanifolds. We consider the ambient space to be the Riemannian product

$$\overline{M} = M_1 \times M_2$$

and suppose $M \subseteq \overline{M}$ is a minimal submanifold. For the results in the coming chapters, it is necessary to understand the ambient curvature and how it relates and restricts to the tangent and normal spaces to M. The curvature of $\overline{M}$ is the pull-backs of the curvature tensors of M_1 and M_2. We must then consider the projection maps π_1 and π_2 from M to M_1 and M_2. The required terms are tensorial so we can just consider these terms on vector spaces.

Let U be a vector space with inner product and suppose that U has two decompositions. Suppose that

$$U = V \oplus V^{\perp} = W_1 \oplus W_2.$$

We suppose that $dimV \leq dimW_1$.

Let π^V and $\pi^{\perp}$ be the orthogonal projection maps onto V and its orthogonal complement $V^{\perp}$ respectively and let π_1 and π_2 be the projections onto the W_1 and W_2 factors. These maps satisfy $\pi^2 = \pi = \pi^*$ where π^* is the adjoint or transpose of π.

We study $\pi^V\pi_1$ and $\pi^{\perp}\pi_1$ when considered to act either as endomorphisms of V and $V^{\perp}$ respectively, or as maps between these spaces.

We will consider a series of maps on and between V and $V^{\perp}$. They are,

$$\pi^V\pi_1 : V \to V \qquad \pi^{\perp}\pi_1 : V^{\perp} \to V^{\perp}$$
$$\pi^{\perp}\pi_1 : V \to V^{\perp} \qquad \pi^V\pi_1 : V^{\perp} \to V$$

$\pi^V\pi_1$ is symmetric on V with respect to the inner product so there is an orthonormal basis $\{e_i\}$ such that $\pi^V\pi_1 e_i = \lambda_i^2 e_i$ for some $\lambda_i \in [0,1]$. If none of the eigenvalues λ_i^2 are 0 or 1, we can then also obtain an orthonormal set $\{\eta_j\}$ in $V^\perp$ that satisfies

$$\begin{aligned}
\pi^\perp\pi_1 e_j &= \sqrt{\lambda_j^2(1-\lambda_j^2)}\ \eta_j, \\
\pi^V\pi_1\eta_j &= \sqrt{\lambda_j^2(1-\lambda_j^2)}\ e_j \\
\pi^\perp\pi_1\eta_j &= (1-\lambda_j^2)\eta_j
\end{aligned}$$

That is, $\pi^\perp\pi_1$ sends an eigenvector for $\pi^V\pi_1$ on V to an eigenvector for $\pi^\perp\pi_1$ on $V^\perp$. The eigenvalues are λ_j^2 and $1-\lambda_j^2$ respectively.

The set $\{\eta_j\}$ does not however span $V^\perp$. The orthogonal complement to the span of the set $\{\eta_j;\ j = 1,\dots,p\}$ is equal to the kernel of $\pi^V\pi_1$ when considered on $V^\perp$. $\pi^\perp\pi_1$ preserves this space and can be orthogonally diagonalised. We can then take as a basis $\{\eta_j;\ j = p+1,\dots,N-p\}$ so that $\pi^\perp\pi_1\eta_j = (1-\lambda_j^2)\eta_j$.

To summarise, we have orthonormal bases for V and $V^\perp$ that diagonalise $\pi^V\pi_1$ and $\pi^\perp\pi_1$ respectively.

All of the above calculations have been only considering the projection onto the first factor. It is also necessary to consider the other factor. We can then see that the basis $\{e_i\}$ defined above satisfies

$$\begin{aligned}
\pi^V\pi_2 e_i &= (1-\lambda_i^2)e_i, \\
\pi^V\pi_2\eta_j &= \lambda_j^2\eta_j \\
\pi^\perp\pi_2 e_i &= -\pi^\perp\pi_1 e_i = -\sqrt{\lambda_i^2(1-\lambda_i^2)}\ \eta_i.
\end{aligned}$$

In summary, an orthonormal basis for $V^\perp$ that we may construct using π_2 differs from the basis $\{\eta_j\}$ by a factor of -1. Any quadratic expression for η_j takes the same values on the new basis.

Finally, we consider the norm of the map $\pi_2 : V \to W_2$. If in a heuristic sense, V is *almost* contained in the subspace W_1, the map π_2 restricted to V is almost the zero map. The norm of π_2 is given by

$$\|\pi_2\|^2 = \mathrm{Tr}(\pi_2^*\pi_2).$$

The transpose of π_2 in this case is obtained by observing

$$\langle \pi_2^* X, Y \rangle = \langle X, \pi_2 Y \rangle = \langle X, Y \rangle = \langle \pi^V X, Y \rangle$$

for $X \in W_2$ and $Y \in V$ so $\pi_2^* = \pi^V$. The norm is then calculated by

$$\|\pi_2\|^2 = \sum_i \langle \pi^V \pi_2 e_i, e_i \rangle = \sum_i (1 - \lambda_i^2)$$

where $\{e_i\}$ is the above distinguished basis of eigenvectors.

The assumption that we will make later is that this norm is small. Suppose that

$$\|\pi_2\| \leq \Lambda.$$

Then,

$$\sum (1 - \lambda_i^2) \leq \Lambda^2.$$

This is the inequality that we wish to use in the calculations of the following chapter.

Chapter 4

Rigidity of Sphere Factors of Symmetric Spaces

In this chapter we give all the main calculations leading to the main rigidity thoerem.

This thesis is a description of some theorems that generalise a result of Simons for rigidity and isolation phenomena for minimal submanifolds of the euclidean sphere. He showed that if a closed minimal submanifold of S^n had second fundamental form uniformly bounded by an explicitly stated constant, it had to be a totally geodesic great sphere. We have a result of a similar sort where the ambient space is a compact symmetric space with semi-simple isometry group. We show that minimal (actually totally geodesic) submanifolds that correspond to the non-exceptional rank-one components are similarly rigid.

We recall theorem 2.4. The second fundamental form A satisfies the second order elliptic equation

$$\nabla^2 A = -A \circ \tilde{A} - \underset{\sim}{A} \circ A + \overline{R}(A) + \overline{R}'.$$

Let M be a closed minimal submanifold of $\overline{M}$. That is, $\partial M = \emptyset$ so from equation 2.4 we have

$$\begin{aligned} 0 \le \int_M \|\nabla A\|^2 &= -\int_M \langle \nabla^2 A, A\rangle \\ &= \int_M \langle A \circ \tilde{A} + \underset{\sim}{A} \circ A, A\rangle - \langle \overline{R}(A) + \overline{R}', A\rangle. \end{aligned} \quad (4.1)$$

The first simplification of this inequality is the following universal statement of Simons.

Theorem 4.1. [8] *Let M^p be a minimal submanifold of $\overline{M}^n$. Then the second fundamental form satisfies*

$$\langle A \circ \tilde{A} + \underset{\sim}{A} \circ A, A \rangle \leq q \|A\|^4$$

where

$$q = 2 - \frac{1}{n-p}.$$

For the next simplification of Equation 4.1 we place an assumption on the geometry of $\overline{M}$. That is, let $\overline{M}$ be locally symmetric. By Equation 1.1, $\nabla R \equiv 0$ and so from its definition.

$$\overline{R}' \equiv 0.$$

The only remaining term is $\langle \overline{R}(A), A \rangle$. The inequality that we wish to have is of the form $\langle \overline{R}(A), A \rangle \geq C\|A\|^2$ where C is a constant that is essentially independent of the submanifold. If we have this inequality we observe that

$$\begin{aligned} 0 &\leq \int_M \langle A \circ \tilde{A} + \underset{\sim}{A} \circ A - \overline{R}(A), A \rangle \\ &\leq \int_M q\|A\|^4 - C\|A\|^2 \\ &= \int_M \|A\|^2 \big(q\|A\|^2 - C \big) \end{aligned} \tag{4.2}$$

Proposition 4.2. *Let M be a closed minimal submanifold of a compact Riemannian symetric space. Suppose that the second fundamental form of M uniformly satifies*

$$\begin{aligned} \langle \overline{R}(A), A \rangle &\geq C\|A\|^2 \\ \|A\|^2 &< \frac{C}{q}. \end{aligned}$$

Then M is totally geodesic.

This follows clearly from the inequality 4.2. We will give a number of cases where results of this type can be proven.

Let $\overline{M} = M_1 \times M_2$ be a Riemannian product and let M be a submanifold of $\overline{M}$. We will consider submanifolds that are close to being contained in the first factor. We consider the projection $\pi_2 : M \to M_2$ and wish to estimate the size of this map. The method that is useful in the present case is to consider the uniform norm of the derivative map $\pi_2 : T_M \to T_{M_2}$.

The norm of $\alpha = \pi_2$ at a point is given by

$$\|\alpha\|^2 = \mathrm{Tr}(\alpha^* \alpha).$$

Theorem 4.3. *Let M be a minimal submanifold of $S^n \times S^m$ for $p < n$. Suppose that the map π_2 uniformly satisfies $\|\pi_2\| \leq \Lambda$. Then the second fundamental form of M satisfies*

$$\langle \overline{R}(A), A\rangle \geq \big((p-1) - \Lambda^2\big(18p^2 + 2p + 3\big)\big)\,\|A\|^2.$$

Proof. This is proven in an extended calculation in the following section. This equation is given in equation 4.7. □

In particular, for small values of Λ, $(p-1) - \Lambda^2\big(18p^2 + 2p + 3\big) > 0$ and we have the estimate that we discussed in Theorem 4.2.

Theorem 4.4. *Let $\Lambda = \sqrt{\frac{p-1}{2(18p^2+2p+3)}}$. Let M be a closed p-dimensional minimal submanifold of $S^n \times S^m$. Suppose that $\|\pi_2\| < \Lambda$ on M and the second fundamental form of M uniformly satisfies $\|A\|^2 < \frac{p-1}{2q}$ for $q = 2 - \frac{1}{n+m-p}$. Then M is a totally geodesic submanifold.*

Proof. This follows from the integration by parts argument given above. □

We can strengthen this result by giving Theorem 4.30. This result is that if M is as we have above, it must actually be a totally geodesic subspace of the first factor. That is,

Theorem 4.5. *There exists $\Lambda > 0$ such that if T is a p-dimensional totally geodesic submanifold of $S^n \times S^m$ and $\|\pi_2\| \leq \Lambda$ then $T \subset S^n \times \{pt\}$.*

We note that the uniform bound on the second fundamental form A is a condition on the second order derivatives of the immersion of M. The bound on π_2 is a condition on the first derivatives of the immersion. Together with our knowledge of the totally geodesic submanifolds of S^n we thus have

Theorem 4.6. *Let f be the embedding of S^p into $S^n \times S^m$ as a totally geodesic sphere in the first factor. Then there is a C^2 neighourhood U of f in the set of immersions such that any minimal immersion contained in U is conjugate to f.*

By the term *conjugate to f* we mean equivalent by the action of isometries of $S^n \times S^m$ and diffeomorphisms of S^p, where these groups act as one might expect.

We now turn our attention to minimal submanifolds of slightly more general symmetric spaces. The spaces that we consider are of the form

$$\overline{M} = S^p \times M_2.$$

M_2 is a compact symmetric space with semi-simple isometry group. The metric that we take on $\overline{M}$ is the one obtained by restricting the Killing form on $\mathfrak{so}(p+1) + \mathfrak{g}$ to $\mathfrak{m}_1 + \mathfrak{m}_2$. We consider the minimal submanifolds of $\overline{M}$.

Firstly we recall the Ricci tensor of $\overline{M}$. We saw previously that if the orthogonal symmetric Lie algebra decomposes into $\mathfrak{g} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_k$ there are positive real numbers ρ_i such that the Ricci tensor is given by

$$\mathrm{Ric} = \rho_1 g_1 + \cdots + \rho_k g_k$$

where the g_i are the metrics induced by the Killing forms on the irreducible subspaces $\mathfrak{m}_i$ of $\mathfrak{g}$. In particular we have that an irreducible compact symmetric space is Einstein. We define ρ to be the smallest of these constants. This is the smalles Ricci curvature of any direction tangent to M.

$$\rho = \min\ \rho_i.$$

We can now state the main theorem of this dissertation. It is also stated as Theorem 4.28.

Theorem 4.7. *There exists $C = C(p, M_2) > 0$ such that for any $0 < \Lambda \leq 1$ and for any p-dimensional closed minimal submanifold M of the symmetric space $\overline{M} = S^p \times M_2$ for which $\|\pi_2\| \leq \Lambda$ the term $\overline{R}(A)$ satisfies*

$$\langle \overline{R}(A), A \rangle \geq \Big((2\rho + \frac{1}{p-1}) - C\Lambda^2 \Big) \|A\|^2.$$

The constant C here is given explicitly in Equation 4.11. The proof of this theorem is lengthy and completed in Section 4.2. In particular, if M is closed and minimal this implies the integral inequality

$$0 \leq \int_M \|A\|^2 \left(\|A\|^2 - \left(\frac{(2\rho + \frac{1}{p-1}) - C\Lambda^2}{q} \right) \right) \tag{4.3}$$

In the following theorem we take $\Lambda^2 = \frac{1}{C(p-1)}$ so that $(2\rho + \frac{1}{p-1}) - C\Lambda^2 = 2\rho$.

Theorem 4.8. *Suppose that M is a p-dimensional closed minimal submanifold of $S^p \times M_2$. If the projection of M to M_2 uniformly satisfies $\|\pi_2\| < \Lambda$ and the second fundamental form of M uniformly satisfies*

$$\|A\|^2 < 2\rho/q$$

where $q = 2 - \frac{1}{\dim M_2}$ then M must be a totally geodesic submanifold.

Combining this with Theorem 4.30 we obtain the theorem

Theorem 4.9. *There exists $\Lambda > 0$ such that if M is a p-dimensional closed minimal submanifold of $S^p \times M_2$ that satisfies*

$$\begin{aligned} \|\pi_2\| &< \Lambda \\ \|A\|^2 &< \frac{1}{q}\left(2\rho + \frac{1}{p-1} - C\Lambda^2\right) \end{aligned}$$

then $M = S^p \times \{q\}$ for some $q \in M_2$.

The second fundamental form A is given by projecting the ambient Levi-Civita connection to the tangent and normal spaces to M. It is thus dependent on the second order derivatives of the map immersing M to $S^p \times M_2$. The projection π_2 is considered on tangent vectors so is dependent on the first derivatives of the map. The conditions $\|\pi_2\| < \Lambda$ and $\|A\|^2 < Const$ then defines a C^2-open set in the set of immersions. We can conclude the following.

Theorem 4.10. *Let $i : S^p \to S^p \times M_2$ be the standard embedding of S^p as a totally geodesic factor. Then there is a C^2-neighbourhood $\mathcal{U}$ of i in the set of immersions such that any minimal immersion in $\mathcal{U}$ is conjugate to i.*

As we have noted earlier, by the term conjugate we mean equivalent with respect to the action of isometries of $\overline{M}$ and diffeomorphisms of M.

However, one can also note that the norm squared of the second fundamental form of a minimal submanifold is closely related to the intrisic scalar curvature. Denote by K the scalar curvature of a submanifold $M \subseteq S^p \times M_2$ in the induced metric. We obtain a theorem where the constraint is on K.

Theorem 4.11. *Let ρ, q, and K_2 be as stated above and let $\Lambda > 0$ satisfy the equation*

$$\frac{1}{q}(\rho + \frac{1}{p-1}) = \Lambda^2 \left(\frac{C}{q} + p + p(p-1)K_2^2 \right).$$

Then if $M \subseteq S^p \times M_2$ is a p-dimensional closed minimal submanifold so that the projection π_2 and the initrinsic scalar curvature uniformly satisfy

$$\begin{aligned} \|\pi_2\| &\leq \Lambda \\ \frac{p}{2} - K &< \frac{\rho}{q} \end{aligned}$$

then $A \equiv 0$ and M is totally geodesic.

Furthermore, there is a (possibly smaller) $\Lambda > 0$ such that $\|\pi_2\| \leq \Lambda$ and $\frac{p}{2} - K < \frac{\rho}{q}$ together imply that $\pi_2 \equiv 0$ and $M = S^p \times pt$ and $K \equiv \frac{p}{2}$.

For the proof we first consider two objects for the submanifold. Let

$$\begin{aligned} \overline{S}(X,Y) &= \langle \overline{R}_{Y,X} X, Y \rangle \\ S(X,Y) &= \langle R_{Y,X} X, Y \rangle \end{aligned}$$

where R is the intrinsic curvature of M and $\overline{R}$ is the curvature of $\overline{M}$ and X and Y are tangent to M. If X and Y are orthogonal and unit length, $\overline{S}$ and S give the sectional curvatures in the plane spanned by X and Y.

Proof. The Gauss curvature equation [4, page 23] states that the curvature operators and the second fundamental form satisfy

$$\langle B(X,Y), B(X,Y) \rangle - \langle B(X,X), B(Y,Y) \rangle = \overline{S}(X,Y) - S(X,Y).$$

Since M is minimal we have that

$$
\begin{aligned}
0 \le \|A\|^2 = \sum_{i,k} \|B(e_i, e_k)\|^2 &= \sum_{i,k} \overline{S}(e_i, e_k) - \sum_{i,k} S(e_i, e_k) \\
&= \sum_{e_i, e_k} \overline{S}(e_i, e_k) - K.
\end{aligned}
$$

We must now estimate the first of these terms. The ambient sectional curvature is given by $\overline{S}(e_i, e_k) = \|[e_i, e_k]\|^2$ so

$$
\begin{aligned}
\|A\|^2 &= \sum_{i,k} \|[\pi_1 e_i, \pi_1 e_k]\|^2 + \sum_{i,k} \|[\pi_2 e_i, \pi_2 e_k]\|^2 - K \\
&\le \frac{1}{2(p-1)} \sum_{ik} \lambda_i^2 \lambda_k^2 + p(p-1)\Lambda^2 K_2^2 - K
\end{aligned}
$$

Here K_2 is given by

$$
K_2 = \max\{\|[X_1, X_2]\|\,;\ X_i \in \mathfrak{m}_i,\ \|X_i\| = 1\}.
$$

K_2^2 is the maximum sectional curvature of a plane tangent to M_2. The assumption of π_2 implies that $\lambda_i^2 \lambda_k^2 = 1 + \lambda_i^2(\lambda_k^2 - 1) + (\lambda_i^2 - 1) \le 1 + 2\Lambda^2$ so we can estimate

$$
\begin{aligned}
\|A\|^2 &\le \frac{1}{2(p-1)} \sum_{ik} 1 + \frac{2\Lambda^2}{2(p-1)} \sum_{ik} 1 + p(p-1)\Lambda^2 K_2^2 - K \\
&= \frac{p}{2} + \Lambda^2 \left(p + p(p-1)K_2^2\right) - K.
\end{aligned}
$$

From the Equation 4.3 we see that we have to estimate $\|A\|^2 - \left(\frac{(2\rho + \frac{1}{p-1}) - C\Lambda^2}{q}\right)$.

$$
\begin{aligned}
\|A\|^2 - \left(\frac{(2\rho + \frac{1}{p-1}) - C\Lambda^2}{q}\right) &= \|A\|^2 - \frac{2\rho + \frac{1}{p-1}}{q} + \frac{C}{q}\Lambda^2 \\
&\le \left(\frac{p}{2} - \frac{\rho}{q}\right) - \left(\frac{\rho + \frac{1}{p-1}}{q}\right) + \frac{C}{q}\Lambda^2 \\
&\quad + \left(p + p(p-1)K_2^2\right)\Lambda^2 - K
\end{aligned}
$$

Now, as stated in the theorem, we take Λ that satisfies

$$
\frac{1}{q}(\rho + \frac{1}{p-1}) = \Lambda^2 \left(\frac{C}{q} + p + p(p-1)K_2^2\right)
$$

Then,

$$\|A\|^2 - \Big(\frac{(2\rho + \frac{1}{p-1}) - C\Lambda^2}{q}\Big) \leq \frac{p}{2} - \frac{\rho}{q} - K.$$

If this term is uniformly negative then necessarily the other term in Equation 4.3 must be zero.

The final statement of this theorem follows from Theorem 4.30 and by calculating the scalar curvature of those factors in the given metric. □

4.1 Submanifolds of Products of Spheres

In this section we prove Theorem 4.3. That is, we prove that if M is a p-dimensional minimal submanifold of $S^n \times S^m$ for $p < n$ such that
$\pi_2\| \leq \Lambda$ then second fundamental form of M satisfies

$$\langle \overline{R}(A), A\rangle \geq \big((p-1) - \Lambda^2(18p^2 + 2p + 3)\big)\,\|A\|^2.$$

Throughout this section we take S^n and S^m to each have sectional curvatures constantly one. The curvature of $\overline{M}$ is given by

$$\overline{R}_{X_1,X_2}X_3 = -\langle t_1, t_3\rangle t_2 + \langle t_2, t_3\rangle t_1 - \langle s_1, s_3\rangle s_2 + \langle s_2, s_3\rangle s_1.$$

where X_i are tangent vectors and $\pi_1(X_i) = t_i$ and $\pi_2(X_i) = s_i$. We also recall the definition

$$\langle (1)^W X, Y\rangle = \sum_i 2\langle \overline{R}_{e_i,Y} B(X, e_i), W\rangle.$$

Proposition 4.12. *Let M be a p-dimensional submanifold of $\overline{M} = S^n \times S^m$ for $p \leq n$. Then the term (1) satisfies at each point*

$$\langle (1)^W X, Y\rangle = 2\big(-\sum_i \langle e_i, \pi^T \pi_1 B(X, e_i)\rangle\big)\langle \pi^T \pi_1 W, Y\rangle$$
$$+ 2\sum_i \langle \langle \pi^N \pi_1 e_i, W\rangle \pi^T \pi_1 B(X, e_i), Y\rangle \quad \textit{(plus a } \pi_2\textit{-term)}$$

$$\langle (1), A\rangle = -4\sum_{i,j} \lambda_i\lambda_j\sqrt{(1-\lambda_i^2)(1-\lambda_j^2)}\Big(\langle A^{\eta_i}(e_i), A^{\eta_j}(e_j)\rangle - \langle A^{\eta_j}(e_i), A^{\eta_i}(e_j)\rangle\Big) \quad (4.4)$$

where $\{e_i\}$ is an orthonormal basis for T_M that diagonalizes the map $\pi^T\pi_1$, as described in the previous chapter.

Proof. From the expression for the curvature of $S^n \times S^m$ and the definition of the term (1), the π_1-term is given by

$$\begin{aligned}
\langle (1)^W X, Y\rangle &= 2\sum_i \langle -\langle \pi_1 e_i, \pi_1 B(X,e_i)\rangle \pi_1 Y + \langle \pi_1 Y, \pi_1 B(X,e_i)\rangle \pi_1 e_i, W\rangle \\
&= 2\langle (-\sum_i \langle e_i, \pi^T\pi_1 B(X,e_i)\rangle)\pi^T\pi_1 W, Y\rangle \\
&\quad +2\sum_i \langle \langle \pi^N \pi_1 e_i, W\rangle \pi^T\pi_1 B(X,e_i), Y\rangle
\end{aligned}$$

For the second equation, we calculate the inner product by considering the distinguished orthonormal bases that we have described in Chapter 3. We assume $\{e_i\}$ is an orthonormal basis for T_M such that $\pi^T\pi_1 e_i = \lambda_i^2 e_i$ for $i = 1,\ldots,p$. The map $\pi^N\pi_1$ sends the tangent space to the normal space. The image of this map has orthonormal basis $\{\eta_j\}$ that satisfies $\pi^N\pi_2\eta_j = (1-\lambda_j^2)\eta_j$ for $j = 1,\ldots,p$. We extend this to a full basis for N_M by adding η_j that are also eigenvalues of $\pi^N\pi_1$. Denote the eigenvalues also by $1-\lambda_j^2$ for $j = p+1,\ldots,n+m$. The vectors satisfying this second condition are orthogonal to the image of $\pi^N\pi_1$ and so are anihilated by $\pi^T\pi_1$. We have

$$\begin{aligned}
\langle (1), A\rangle_1 &= \sum_{k=1}^{p}\sum_{j=1}^{n+m} \langle (1)^{\eta_j}(e_k), A^{\eta_j}(e_k)\rangle \\
&= -2\sum_{i,j,k} \langle e_i, \pi^T\pi_1 B(e_i,e_k)\rangle \langle \pi^T\pi_1 \eta_j, A^{\eta_j}(e_k)\rangle \\
&\quad +2\sum_{i,j,k} \langle \pi^N\pi_1 e_i, \eta_j\rangle \langle \pi^T\pi_1 B(e_k,e_i), A^{\eta_j}(e_k)\rangle \\
&= -2\sum_{i,j,k} \lambda_i\lambda_j\sqrt{(1-\lambda_i^2)(1-\lambda_j^2)}\langle \eta_j, B(e_i,e_k)\rangle \langle e_j, A^{\eta_j}(e_k)\rangle \\
&\quad\ 2\sum_{i,j,k,l} \lambda_i\lambda_l\sqrt{(1-\lambda_i^2)(1-\lambda_l^2)}\langle \eta_i,\eta_j\rangle \langle B(e_i,e_k),\eta_l\rangle \langle e_l, A^{\eta_j}(e_k)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= -2\sum_{i,j,k}\lambda_i\lambda_j\sqrt{(1-\lambda_i^2)(1-\lambda_j^2)}\langle A^{\eta_i}(e_i),e_k\rangle\langle e_k,A^{\eta_i}(e_j)\rangle \\
&\qquad +2\sum_{i,k,l}\lambda_i\lambda_l\sqrt{(1-\lambda_i^2)(1-\lambda_l^2)}\langle A^{\eta_l}(e_i),e_k\rangle\langle e_k,A^{\eta_i}(e_k)\rangle \\
&= -2\sum_{i,j}\lambda_i\lambda_j\sqrt{(1-\lambda_i^2)(1-\lambda_j^2)}\Big(\langle A^{\eta_i}(e_i),A^{\eta_j}(e_j)\rangle-\langle A^{\eta_j}(e_i),A^{\eta_i}(e_l)\rangle\Big)
\end{aligned}
$$

In the second equation the normal vectors η_j that are considered are only those for $j=1,\dots,p$. The remaining vectors are in the kernel of $\pi^T\pi_1$ so can be omitted.

This is almost sufficient. We now note that this calculation only considers the component of the curvature from the first sphere factor. Only the π_1 term is considered. The component for the second is identical, because the eigenvalues of the map $\pi^T\pi_2$ are $1-\lambda_i^2$ instead of λ_i^2. Making this replacement leaves the final expression the same so the π_2-term equals the π_1 one. Also, $\pi^N\pi_2 e_i = -\pi^N\pi_1 e_i$ so η_i can be replaced by $-\eta_i$. The expressions are all quadratic so this difference can be ignored. □

We also note that

$$\langle(2),A\rangle = \langle(1),A\rangle.$$

The third term of $\overline{R}(A)$ is given by

$$\langle(3)^W X,Y\rangle = -\sum_i\langle A^W(X),\overline{R}_{e_i,Y}e_i\rangle.$$

Proposition 4.13. *The second fundamental form of a minimal submanifold M^p of $\overline{M}$ satisfies*

$$
\begin{aligned}
\langle(3)^W X,Y\rangle &= \big(\mathrm{Tr}\,\pi^T\pi_1\big)\langle\pi^T\pi_1A^WX,Y\rangle-\langle(\pi^T\pi_1)^2A^WX,Y\rangle \\
&\qquad (\textit{plus a } \pi_2\textit{-term,}) \\
\langle(3),A\rangle &= \sum_{i,j}\Big[\lambda_i^2\big(\sum_k\lambda_k^2-\lambda_i^2\big) \\
&\qquad +(1-\lambda_i^2)\big(p-1+\lambda_i^2-\sum_k\lambda_k^2\big)\Big]\|A^{\eta_j}(e_i)\|^2 \quad (4.5)
\end{aligned}
$$

where the bases $\{e_i\}$ and $\{\eta_j\}$ are distinguished in the way considered above.

Proof. We recall,

$$\begin{aligned}\langle (3)^W X, Y\rangle &= -\sum_i \left\langle A^W(X), -\langle \pi_1 e_i, \pi_1 e_i\rangle \pi_1 Y + \langle \pi_1 Y, \pi_1 e_i\rangle \pi_1 e_i\right\rangle \\ &= (\operatorname{Tr} \pi^T\pi_1)\left\langle \pi^T\pi_1 A^W(X), Y\right\rangle - \sum_i \left\langle \pi^T\pi_1 A^W(X), e_i\right\rangle \left\langle e_i, \pi^T\pi_1 Y\right\rangle \\ &= \left(\operatorname{Tr} \pi^T\pi_1\right)\left\langle \pi^T\pi_1 A^W X, Y\right\rangle - \left\langle (\pi^T\pi_1)^2 A^W X, Y\right\rangle\end{aligned}$$

The π_2-terms are obtained in an identical manner. The inner product with A is gotten by taking the π_1 and π_2 terms separately. For example,

$$\begin{aligned}\langle (3)_1, A\rangle &= (\mathrm{Tr}\pi^T\pi_1)\sum_{k,j}\langle \pi^T\pi_1 A^{\eta_j}(e_k), A^{\eta_j}(e_k)\rangle \\ &\qquad -\sum_{k,j}\langle (\pi^T\pi_1)^2 A^{\eta_j}(e_k), A^{\eta_j}(e_k)\rangle \\ &= \big(\sum_l \lambda_l^2\big)\sum_{i,j,k}\langle A^{\eta_j}(e_k), \lambda_i^2 e_i\rangle\langle e_i, A^{\eta_j}(e_k)\rangle \\ &\qquad -\sum_{i,j,k}\langle A^{\eta_j}(e_k), \lambda_i^4 e_i\rangle\langle e_i, A^{\eta_j}(e_k)\rangle \\ &= \sum_{i,j,k}\Big[(\sum_l \lambda_l^2)\lambda_i^2 - \lambda_i^4\Big]\langle A^{\eta_j}(e_i), e_k\rangle\langle e_k, A^{\eta_j}(e_i)\rangle \\ &= \sum_{i,j}\lambda_i^2\big(\sum_k \lambda_k^2 - \lambda_i^2\big)\|A^{\eta_j}(e_i)\|^2\end{aligned}$$

This is the inner product of A with the endomorphism of T_M gotten by only taking the curavature component from the first factor. The inner product with the part from the second factor has eigenvalue λ_i^2 replaced with $1-\lambda_i^2$.

The sum of these expressions is the equation that we require. □

Again we note that

$$\langle (4), A\rangle = \langle (3), A\rangle.$$

The fifth term of $\overline{R}(A)$ is given by

$$\langle (5), A\rangle = \sum_i \left\langle \overline{R}_{e_i, B(X,Y)} e_i, W\right\rangle.$$

This term is similarly estimable.

Proposition 4.14. *As before, if M is a p-dimensional minimal submanifold of $S^n \times S^m$ the terms of $(5)^W X$ and $\langle (5), A\rangle$ that come from the first factor satisfy*

$$\begin{aligned}\langle (5)^W X, Y\rangle &= -\mathrm{Tr}(\pi^T\pi_1)\langle A_X(\pi^N\pi_1 W), Y\rangle + \langle A_X(\pi^N\pi_1\pi^T\pi_1 W), Y\rangle \\ \langle (5), A\rangle &= -\mathrm{Tr}(\pi^T\pi_1)\sum_{j,k}(1-\lambda_j^2)\|A^{\eta_j}(e_k)\|^2 + \sum_{j,k=1}^{p}\lambda_j^2(1-\lambda_j^2)\|A^{\eta_j}(e_k)\|^2.\end{aligned}$$

The π_2-terms, as usual, are obtained by replacing λ_i^2 by $1-\lambda_i^2$.

We note in this proposition that when we assume that $\pi^T\pi_2$ is close to zero, this implies that $1-\lambda_j^2$, as they have been defined, are close to zero only for $j = 1, \ldots, p$. The remaining eigenvalues of $\pi^N\pi_1$ defined on N_M could and will be close to 1 in some cases.

Proof. The component of (5) coming from the first factor is given by

$$\begin{aligned}\langle (5)^W X, Y\rangle &= \sum_i \langle \overline{R}_{e_i, B(X,Y)} e_i, W\rangle \\ &= \sum_i \langle -\langle \pi_1 e_i, \pi_1 e_i\rangle \pi_1 B(X,Y) + \langle \pi_1 B(X,Y), \pi_1 e_i\rangle \pi_1 e_i, W\rangle \\ &= -\mathrm{Tr}(\pi^T\pi_1)\langle A_X(\pi^N\pi_1 W), Y\rangle + \sum_i \langle \pi^T\pi_1 B(X,Y), e_i\rangle\langle e_i, \pi^T\pi_1 W\rangle \\ &= -\mathrm{Tr}(\pi^T\pi_1)\langle A_X(\pi^N\pi_1 W), Y\rangle + \langle A_X(\pi^N\pi_i\pi^T\pi_1 W), Y\rangle,\end{aligned}$$

as required. The inner product is given by

$$\langle (5), A\rangle = -\mathrm{Tr}(\pi^T\pi_1)\sum_{j,k}\langle A_{e_k}(\pi^N\pi_1\eta_j), A_{e_i}(\eta_j)\rangle + \langle A_{e_i}(\pi^N\pi_1\pi^T\pi_1\eta_j), A_{e_i}(\eta_j)\rangle.$$

We can see that this gives us what we require. The bases for T_M and N_M that we take are of eigenvalues for the relevant maps. In particular, there are the coefficients $1-\lambda_j^2$ for the first terms. For the last one, we note that, from equations 3.1 and 3.1, we have that $\pi^N\pi_1\pi^T\pi_1\eta_j = \lambda_j^2(1-\lambda_j^2)\eta_j$ for $j = 1, \ldots, p$ and equals zero for $j = p+1, \ldots, n+m-p$. □

The final term that we will calculate is the sixth. This is slightly more delicate. It is given by

$$\langle (6)^W X, Y\rangle = -2\sum_i \langle A^W(e_i, \overline{R}_{e_i,X} Y\rangle.$$

Proposition 4.15. *The sixth term of $\overline{R}(A)$ in this case satisfies*

$$\begin{aligned}
\langle (6)^W X, Y\rangle &= -2\mathrm{Tr}(\pi^T\pi_1 A^W)\langle \pi^T\pi_1 X, Y\rangle + 2\langle \pi^T\pi_1 A^W(\pi^T\pi_1 X), Y\rangle.\\
\langle (6), A\rangle &= 2\sum_{i,k}\Big(\lambda_i^2\lambda_k^2 + (1-\lambda_i^2)(1-\lambda_k^2)\Big)\Big(\|B(e_i,e_k)\|^2 - \langle B(e_i,e_i), B(e_k,e_k)\rangle\Big).
\end{aligned}$$

In the first case there is also the ubiquitous π_2-term. This is taken into account in the second expression.

Proof.

$$\begin{aligned}
\langle (6), A\rangle &= -2\sum_i \langle A^W(e_i), \overline{R}_{e_i,X}Y\rangle\\
&= -2\sum_i \langle A^W(e_i), -\langle \pi_1 e_i, \pi_1 Y\rangle \pi_1 X + \langle \pi_1 X, \pi_1 Y\rangle \pi_1 e_i\rangle\\
&= -2\sum_i \langle \pi^T\pi_1 A^W(e_i), e_i\rangle\langle \pi^T\pi_1 X, Y\rangle\\
&\qquad +2\sum_i \langle A^W(\pi^T\pi_1 X), e_i\rangle\langle e_i, \pi^T\pi_1 Y\rangle
\end{aligned}$$

as required. We take the inner product of A with the component of $\overline{R}(A)$ coming from the first factor.

$$\begin{aligned}
\langle (6), A\rangle_1 &= -2\sum_{i,j,k}\langle \pi^T\pi_1 A^{\eta_j}(e_i), e_i\rangle\langle \pi^T\pi_1 e_k, A^{\eta_j}(e_k)\rangle\\
&\qquad +2\sum_{jk}\langle \pi^T\pi_1 A^{\eta_j}(\pi^T\pi_1 e_k), A^{\eta_j}(e_k)\rangle\\
&= -2\sum_{i,k}\langle B(\pi^T\pi_1 e_i, e_i), B(\pi^T\pi_1 e_k, e_k)\rangle\\
&\qquad +2\sum_{ik}\langle B(\pi^T\pi_1 e_k, e_i), B(e_k, \pi^T\pi_1 e_i)\rangle \qquad (4.6)\\
&= 2\sum_{ik}\lambda_i^2\lambda_k^2\Big(\|B(e_i,e_k)\|^2 - \langle B(e_i,e_i), B(e_k,e_k)\rangle\Big)
\end{aligned}$$

The π_2 term has $1-\lambda_i^2$ replacing λ_i^2 and the full inner product is the sum of these two terms. □

This completes the first set of calculations. We now use them to explicitly estimate the size of$\langle \overline{R}(A), A\rangle$ under the assumption that the uniform size of $\|\pi_2\|$ is bounded. Specifically we note that the assumption $\|\pi_2\| \leq \Lambda$ implies that $p - \Lambda^2 \leq \sum_i \lambda_i^2 \leq p$.

Proposition 4.16. *Suppose that the endomorphism π_2 of the tangent space uniformly satisfies $\|\pi_2\| \leq \Lambda$ over the submanifold. Then the term* (1) *satisfies*

$$\langle (1), A\rangle \geq -8p^2\Lambda^2\|A\|^2.$$

Proof. We begin with equation 4.4. That is

$$\langle (1), A\rangle = -4\sum_{ij} \lambda_i\lambda_j\sqrt{(1-\lambda_i^2)(1-\lambda_j^2)}\Big(\langle A^{\eta_i}(e_i), A^{\eta_j}(e_j)\rangle - \langle A^{\eta_j}(e_i), A^{\eta_i}(e_j)\rangle\Big).$$

Our hypothesis implies that $\sum_i (1-\lambda_i^2) \leq \Lambda^2$ so we can naively estimate

$$\langle (1), A\rangle \leq 8\Lambda^2 p^2\|A\|^2$$

as required. □

Since $\langle (1), A\rangle = \langle (2), A\rangle$ we also have that

$$\langle (2), A\rangle \geq -8p^2\Lambda^2\|A\|^2.$$

Proposition 4.17. *Suppose again that the map π_2 uniformly satisfies $\|\pi_2\| \leq \Lambda$. Then the second fundamental form of M satisfies*

$$\langle (3), A\rangle \geq \Big((p-1) - \Lambda^2(p-1)\Big)\|A\|^2.$$

Proof. We recall equation 4.5. The hypothesis that $\sum_i (1-\lambda_k^2) \leq \Lambda^2$ implies that $p - \Lambda^2 \leq \sum_i \lambda_i^2 \leq p$ and

$$\begin{aligned} & \lambda_i^2\big(\sum_k \lambda_k^2 - \lambda_i^2\big) + (1-\lambda_i^2)\big(p - 1 + \lambda_i^2 - \sum_k \lambda_k^2\big) \\ \geq\ & (1-\Lambda^2)(p-1-\Lambda^2) - \Lambda^2\cdot\Lambda^2 \\ =\ & (p-1) - \Lambda^2(p-1). \end{aligned}$$

from which the inequality follows. □

Proposition 4.18. *Suppose that the bound $\|\pi_2\| \leq \Lambda$ holds uniformly on M. Then the fifth term of $\overline{R}(A)$ satisfies*

$$\langle (5), A\rangle \geq -(p+3\Lambda^2)\|A\|^2.$$

Proof. The two terms, coming from the projections to the two sphere factors, are

$$\begin{aligned}\langle (5), A\rangle_1 &= -\big(\sum_i \lambda_i^2\big)\sum_{k=1}^{p}\sum_{j=1}^{N-p}(1-\lambda_j^2)\|A^{\eta_j}(e_k)\|^2 + \sum_{j,k=1}^{p}\lambda_j^2(1-\lambda_j^2)\|A^{\eta_j}(e_k)\|^2\\ \langle (5), A\rangle_2 &= -\big(\sum_i (1-\lambda_i^2)\big)\sum_{k=1}^{p}\sum_{j=1}^{N-p}\lambda_j^2\|A^{\eta_j}(e_k)\|^2 + \sum_{j,k=1}^{p}\lambda_j^2(1-\lambda_j^2)\|A^{\eta_j}(e_k)\|^2\end{aligned}$$

In particular, if $0 \leq \sum_{i=1}^p (1-\lambda_i^2) \leq \Lambda^2 \leq 1$, we have that $p-\Lambda^2 \leq \sum_{i=1}^p \lambda_i^2 \leq p$ and

$$\begin{aligned}\langle (5), A\rangle_1 &\geq -p\sum_{j,k}\|A_{e_k}(\eta_j)\|^2 - \Lambda^2\|A\|^2\\ &= -\big(p+\Lambda^2\big)\|A\|^2,\\ \langle (5), A\rangle_2 &\geq -2\Lambda^2\|A\|^2.\end{aligned}$$

The sum of these terms gives the required inequality. □

Proposition 4.19. *In the above situation, if the projection from the tangent space to the second factor uniformly satisfies $\|\pi_2\| \leq \Lambda \leq$ then the sixth term in $\overline{R}(A)$ satisfies*

$$\langle (6), A\rangle \geq \big(1-2\Lambda^2-2p^2\Lambda^2\big)\|A\|^2.$$

Proof. In Proposition 4.15 we consider the expression in the λ's

$$\begin{aligned}\lambda_i^2\lambda_k^2 + (1-\lambda_i^2)(1-\lambda_k^2) &\geq (1-\Lambda^2)^2 - \Lambda^4\\ &= 1-2\Lambda^2.\end{aligned}$$

As a result we have

$$\begin{aligned}&\sum_{i,k}\Big(\lambda_i^2\lambda_k^2 + (1-\lambda_i^2)(1-\lambda_k^2)\Big)\|B(e_i,e_k)\|^2\\ \geq\ &\sum_{i,k}(1-2\Lambda^2)\|B(e_i,e_k)\|^2\\ =\ &(1-2\Lambda^2)\|A\|^2.\end{aligned}$$

We also observe the identity

$$\lambda_i^2\lambda_k^2 + (1-\lambda_i^2)(1-\lambda_k^2) \;=\; 1+ \overbrace{\lambda_i^2(\lambda_k^2-1)+\lambda_k^2(\lambda_i^2-1)}^{\geq 2\Lambda^2}.$$

This allows the second term to be simplified as

$$\begin{aligned}
&\sum_{i,k}\Big(\lambda_i^2\lambda_k^2 + (1-\lambda_i^2)(1-\lambda_k^2)\Big)\langle B(e_i,e_i), B(e_k,e_k)\rangle \\
=\;& \sum_{i,k}\langle B(e_i,e_i), B(e_k,e_k)\rangle + \sum_{i,k}\big(\lambda_i^2(\lambda_i^2-1)+\lambda_k^2(\lambda_i^2-1)\langle B(e_i,e_i), B(e_k,e_k)\rangle \\
\geq\;& -2p^2\Lambda^2\|A\|^2.
\end{aligned}$$

Here we use that the trace of the second fundamental form is zero. Hence,

$$\langle (6), A\rangle \geq \big(1-2\Lambda^2-2p^2\Lambda^2\big)\|A\|^2.$$

□

We combine the conclusions of this collections of lemmas. In summary we have that

$$\begin{aligned}
\langle \overline{R}(A), A\rangle &= \langle (1)+(2)+(3)+(4)+(5)+(6), A\rangle \\
&\geq -8p^2\Lambda^2\|A\|^2 - 8p^2\Lambda^2\|A\|^2 \\
&\qquad\big((p-1)-\Lambda^2(p-1)\big)\|A\|^2 + \big((p-1)-\Lambda^2(p-1)\big)\|A\|^2 \\
&\qquad -\big(p+3\Lambda^2\big)\|A\|^2 + \big(1-2\Lambda^2-2p^2\Lambda^2\big)\|A\|^2 \\
&= \big((p-1)-\Lambda(18p^2+2p+3)\big)\|A\|^2 \qquad (4.7)
\end{aligned}$$

This is sufficient to prove Theorem 4.3 and completes this section.

4.2 Submanifolds of Reducible Symmetric Spaces

In this section we consider minimal submanifolds of products of symmetric spaces. Previously we considered submanifolds of products of spheres. In this case we consider the ambient space to be symmetric and one factor in its decomposition into irreducible comonents is a sphere. Again we consider the tensor $\overline{R}(A)$ and wish to show that it is uniformly bounded below by the second fundamental form.

First we recall some important facts on the Riemannian geometry of symmetric spaces. The symmetric spaces that we will consider are of a certain restricted type, and their metrics will be distinguished as well. We will suppose that $\overline{M}$ is a compact riemannian symmetric space that is diffeomorphic to a homogeneous space G/H where G is a compact group of isometries and H is the kernel of an involutive automorphism of G. We will assume that G is semi-simple. The Lie algebra $\mathfrak{g}$ of G splits into the $+1$ and -1 eigenspaces of the automorphism. That is, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.

The metric geometry that we will assume on $\overline{M}$ can be quite precisely stated as well. In our situation the Killing form is negative definite on $\mathfrak{g}$ and so it induces an inner product on $\mathfrak{g}$ and, by restriction, on the subspace $\mathfrak{m}$. This inner product is invariant by the action of H on $\mathfrak{m}$ and so induces a riemannian metric on $\overline{M}$ that is invariant by G. This is the metric that we will consider.

The curvature operator for this metric is given entirely in terms of the algebraic structure of $\mathfrak{g}$. If we identify T_oM with $\mathfrak{m}$ it is given by

$$\overline{R}_{X,Y}Z = -[[X,Y],Z]$$

for X, Y and Z in $\mathfrak{m}$.

Another assumption that we will make is how the action of $\mathfrak{h}$ on $\mathfrak{m}$ is reducible. The Lie algebra $\mathfrak{g}$ splits as a sum of ideals

$$\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 = \mathfrak{m}_1 + \mathfrak{h}_1 + \mathfrak{m}_2 + \mathfrak{h}_2$$

where $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$. In particular, we will assume that the symmetric Lie algebra $(\mathfrak{g}_1, \mathfrak{h}_1, \sigma)$ is isomorphic to $(\mathfrak{so}(p+1), \mathfrak{so}(p), \sigma)$. This implies that, up to finite coverings, the symmetric space M is

$$\overline{M} = S^p \times M_2$$

where M_2 is another compact symmetric space. Many of the calculations will be done in a little more generality. It should be noted that each of the six terms in the estimate of $\langle \overline{R}(A), A\rangle$ can be controlled where the first factor of the symmetric space is an arbitrary compact-type symmetric space. Only the term (6) is currently preventing an extension of our theorem.

In the following calculations, we suppose that $M \subset \overline{M}$ i minimal submanifold and at a fixed point $x \in M$, $\{e_i\}$ is a basis for T_xM each element of which is an eigenvector for the map $\pi^T\pi_1$. The set $\{\eta_j\}$ is an orthonormal basis for N_M of eigenvectors for $\pi^N\pi_1$. These are as constructed in the previous chapter.

Proposition 4.20. *Let M be a p-dimensional closed minimal submanifold of the compact symmetric space $\overline{M}$. Then the first term in $\overline{R}(A)$ satisfies*

$$\begin{aligned}\langle (1)^W X, Y\rangle &= 2\sum_i \langle \overline{R}_{e_i,Y} B(X,e_i), W\rangle \\ &= 2\sum_i \langle [e_i,[B(X,e_i),W]],Y\rangle \\ \langle (1), A\rangle &= 2\sum_{i,j,k} \langle [A^{\eta_j}(e_k), e_i], [B(e_k,e_i),\eta_j]\rangle\end{aligned}$$

Proof. From the definition of (1) we have that

$$\begin{aligned}\langle (1)^W X, Y\rangle &= -2\sum_i \langle [[e_i,Y],B(X,e_i)],W\rangle \\ &= 2\sum_i \langle [e_i,[B(X,e_i),W]],Y\rangle\end{aligned}$$

$$\begin{aligned}\langle (1), A\rangle &= 2\sum_{ijk} \langle A^{\eta_j}(e_k), [e_i,[B(e_k,e_i),\eta_j]]\rangle \\ &= 2\sum_{i,j,k} \langle [A^{\eta_j}(e_k), e_i], [B(e_k,e_i),\eta_j]\rangle\end{aligned}$$

□

As before, we recall that the term (2) is the transpose of (1). Hence, $\langle (2), A\rangle = \langle (1), A\rangle$.

We now look at the term (3) and take the inner product with A. (3) is given by

$$\langle (3)^W X, Y\rangle = -\sum_i \langle A^W(X), \overline{R}_{e_i,y} e_i\rangle = \sum_i \langle A^W(X), \overline{R}_{y,e_i} e_i\rangle$$

where $\overline{R}$ is the ambient curvature tensor. The right hand side looks very much like the ambient Ricci curvature operator. By this approximate identification, we may say that

$$\langle (3)^W X, Y\rangle = \langle Ric(A^W X), Y\rangle$$

and (3) $= Ric \circ A$. Clearly then, positive Ricci curvature should be a necessary assumption to ensure that $\langle (3), A\rangle$ is comparable to $\|A\|^2$. We have that in this case because of the earlier observtion that the symmetric form $-B$ is exactly the Ricci operator which must then be sufficiently positive. On each irreducible symmetric component of $\overline{M}$, B is a constant positive multiple of the metric.

It should be noted that the identification of the right hand side of (3) with the ambient Ricci curvature is false because we only sum over a basis for the tangent space to the submanifold. The following theorem makes it clear that we must be able to negate the normal directions to use the above argument.

Proposition 4.21. *Let A be the second fundamental form of a minimal submanifold M of a compact symmetric space $\overline{M}$. The third term in the expression $\overline{R}(A)$ satisfies*

$$\begin{aligned} \langle (3)^W X, Y\rangle &= \sum_i \langle [e_i, [A^W(X), e_i]], Y\rangle \\ \langle (3), A\rangle &= -\sum_{jk} \mathrm{Tr}_{\mathfrak{m}}\Big(\mathrm{ad}(A^{\eta_j}(e_k)) \circ \mathrm{ad}(A^{\eta_j}(e_k))\Big) - \sum_{j,k,l} \langle [A^{\eta_j}(e_k), \eta_l], [A^{\eta_j}(e_k), \eta_l]\rangle \\ &\geq \rho \|A\|^2 - \sum_{j,k,l} \langle [A^{\eta_j}(e_k), \eta_l], [A^{\eta_j}(e_k), \eta_l]\rangle \qquad (4.8) \end{aligned}$$

where ρ is the smallest coefficient given in the decomposition of B in Equation 1.3.

This is the most important term in all of our calculations. Our overall aim is to show that the inner product $\langle \overline{R}(A), A\rangle$ is bounded below by a positive multiple of $\|A\|^2$. The term (3) is the one that gives this positivity. The other term can be thought to be small if one thinks that tangent and normal vectors *almost* commute, as happens if the tangent spaces are close to an irreducible factor.

Proof.

$$\begin{aligned} \langle (3)^W X, Y\rangle &= -\sum_i]\langle A^W(X), [[e_i, Y], e_i]\rangle \\ &= \sum_i \langle [e_i, [A^W(X), e_i]], Y\rangle \end{aligned}$$

$$\begin{aligned}
\langle (3), A\rangle &= \sum_{i,j,k} \langle A^{\eta_j}(e_k), [e_i, [A^{\eta_j}(e_k), e_i]]\rangle \\
&= -\sum_{i,j,k} \langle [A^{\eta_j}(e_k), [A^{\eta_j}(e_k), e_i]], e_i\rangle - \sum_{j,k,l} \langle [A^{\eta_j}(e_k), [A^{\eta_j}(e_k), \eta_j]], \eta_j\rangle \\
&\qquad - \sum_{j,k,l} \langle [A^{\eta_j}(e_k), \eta_l], [A^{\eta_j}(e_k), \eta_l]\rangle \\
&= -\sum_{jk} \mathrm{Tr}_{\mathfrak{m}}\Big(\mathrm{ad}(A^{\eta_j}(e_k) \circ \mathrm{ad}(A^{\eta_j}(e_k))\Big) - \sum_{j,k,l} \langle [A^{\eta_j}(e_k), \eta_l], [A^{\eta_j}(e_k), \eta_l]\rangle \\
&= \sum_{j,k} \mathrm{Ric}(A^{\eta_j}(e_k)) - \sum_{j,k,l} \langle [A^{\eta_j}(e_k), \eta_l], [A^{\eta_j}(e_k), \eta_l]\rangle \\
&\geq \rho \|A\|^2 - \sum_{j,k,l} \langle [A^{\eta_j}(e_k), \eta_l], [A^{\eta_j}(e_k), \eta_l]\rangle
\end{aligned}$$

□

Again as before, (4) is the transpose of (3) so $\langle (4), A\rangle = \langle (3), A\rangle$.

In the previous theorem, we considered minimal submanifolds of $S^n \times S^m$ where the projection to the second factor is small. In that case the dimension of the submanifold could be anything less than n. We can't make that assumption in this case.

Proposition 4.22. *Under the same hypotheses as for the previous propositions, we can calculate the fifth term of $\overline{R}(A)$ as*

$$\begin{aligned}
\langle (5)^W X, Y\rangle &= \sum_i \langle A_X\big(\pi^N[e_i, [e_i, W]]\big), Y\rangle \\
\langle (5), A\rangle &= -\sum_{i,j,k,l} \langle A_{e_k}(\eta_l), A_{e_k}(\eta_j)\rangle \langle [e_i, \eta_j], [e_i, \eta_j]\rangle.
\end{aligned}$$

From either of these equations one can see that if tangent and normal vectors almost commute, this is going to be estimably small.

Proof.

$$\begin{aligned}
\langle (5)^W X, Y\rangle &= \sum_i \langle \overline{R}_{e_i, B(X,Y)} e_i, W\rangle \\
&= -\sum_i \langle [[e_i, B(X,Y)], e_i], W\rangle \\
&= \sum_i \langle A_X\big(\pi^N [e_i, [e_i, W]]\big), Y\rangle
\end{aligned}$$

$$\begin{aligned}
\langle (5), A\rangle &= \sum_{ijk} \langle A_{e_k}\big(\sum_l \langle [e_i, [e_i, \eta_j]], \eta_l\rangle \eta_l\big), A_{e_k}(\eta_j)\rangle \\
&= -\sum_{i,j,k,l} \langle A_{e_k}(\eta_l), A_{e_k}(\eta_j)\rangle \langle [e_i, \eta_j], [e_i, \eta_j]\rangle
\end{aligned}$$

□

The final term that we need to consider is the sixth one in $\overline{R}(A)$.

Proposition 4.23. *Let M be a p-dimensional minimal submanifold of $\overline{M} = S^p \times M_2$ where M_2 is a riemannian symmetric space of compact type. The sixth term of $\overline{R}(A)$ satisfies*

$$\langle (6)^W X, Y\rangle = 2\sum_i \langle -[[e_i, X], A^W(e_i)], Y\rangle \tag{4.9}$$

$$\begin{aligned}
\langle (6), A\rangle &= \frac{1}{p-1} \sum_{ik} \lambda_i^2 \lambda_k^2 \Big(\|B(e_i, e_k)\|^2 - \langle B(e_i, e_i), B(e_k, e_k)\rangle \Big) \\
&\quad -2\sum_{ijk} \langle [\pi_2 e_i, \pi_2 e_k], [\pi_2 A^{\eta_j}(e_i), \pi_2 A^{\eta_j}(e_k)]\rangle. \qquad (4.10)
\end{aligned}$$

In this proposition we have assumed that the first factor of the reducible symmetric space is the sphere. This is the first time that we have had to make this assumption. The reason that it is necessary is that in the general case we obtain a term that is negative semi-definite and difficult to estimate in terms of Λ and other terms. In the case of the sphere, this term vanishes identically. We are currently determining whether the rank-one condition is necessary for this term to be controlled.

Proof. We have

$$\begin{aligned}\langle (6)^W X, Y\rangle &= -2\sum_i \langle A^W(e_i), \overline{R}_{e_i,X}Y\rangle \\ &= 2\sum_i \langle A^W(e_i), [[e_i, X], Y\rangle \\ &= 2\sum_i \langle -[[e_i, X], A^W(e_i)], Y\rangle.\end{aligned}$$

To take the inner product of (6) with A we consider the two factors of $S^p \times M_2$ and the projections to them.

$$\begin{aligned}\langle (6), A\rangle &= 2\sum_{ijk} \langle -[[\pi_1 e_i, \pi_1 e_k], \pi_1 A^{\eta_j}(e_i)], \pi^{\eta_j}(e_k)\rangle \\ &\quad -2\sum_{ijk} \langle [\pi_2 e_i, \pi_2 e_k], [A^{\eta_j}(e_i), A^{\eta_j}(e_k)]\rangle\end{aligned}$$

We denote these two terms $\langle (6), A\rangle_1$ and $\langle (6), A\rangle_2$. The first term clearly uses the curvature term for the first factor. This factor is the sphere with Killing metric. The curvature is thus given by

$$R_{XY}Z = -[[X,Y]Z] = \frac{1}{2(p-1)}\Big(-\langle X, Z\rangle Y + \langle Y, Z\rangle X\Big).$$

We then have

$$\langle (6), A\rangle_1 = \frac{2}{2(p-1)}\sum_{ijk} \langle -\langle \pi_1 e_i, \pi_1 A^{\eta_j}(e_i)\rangle \pi_1 e_k + \langle \pi_1 e_k, \pi_1 A^{\eta_j}(e_i)\rangle \pi_1 e_i, A^{\eta_j}(e_k)\rangle$$

This can be seen to be very similar to equation 4.6 from which we obtain the first term that we require. The expression for $\langle (6), A\rangle_2$ is clear. □

This completes our exact calculations of the six terms in $\overline{R}(A)$. To prove the Theorem 4.7 we must estimate them correctly in the case that $\|\pi_2\| \leq \Lambda$. To do this we must first define some constants that will be used. For the first of these we suppose that the symmetric Lie algebra has splitting

$$\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$$

then we define

$$K_1 = \sup\{[X, Y]\,;\, X,\, Y \in \mathfrak{m}_1,\, |X| = |Y| = 1\}.$$

Similarly for $\mathfrak{m}_2$ we define K_2.

Next we recall the summetric bilinear B form on $\mathfrak{m}$ defined by

$$B(X,Y) = \mathrm{Tr}_{\mathfrak{m}}\big(\mathrm{ad}_x \circ \mathrm{ad}_Y\big).$$

Then we proved that if κ_i is the Killing form of $\mathfrak{g}$ restricted to the irreducible component of $\mathfrak{m}$, we have

$$B = \rho_1 \cdot \kappa_1 + \cdots + \rho_k \cdot \kappa_k$$

for some positive constants ρ_i. We define the constant

$$\rho = \min\ \rho_i.$$

Proposition 4.24. *Let M be a p-dimensional minimal submanifold of $S^p \times M'$ where $S^p \times M'$ is symmetric and has the Killing metric. Suppose that π_2 uniformly satisfies $\|pi_2\| \le \Lambda$. Then the first term of $\overline{R}(A)$ satisfies*

$$\langle (1), A\rangle \ = \ \ge -4p^2(N-p)(K_1^2 + K_2^2)\Lambda^2 \|A\|^2.$$

Proof.

$$\begin{aligned}\langle (1), A\rangle &= 2\sum_{ijk} \langle [A^{\eta_j}(e_k), e_k], [B(e_i,e_k), \eta_j]\rangle \\ &= \langle (1), A\rangle_1 + \langle (1), A\rangle_2\end{aligned}$$

where the two terms come from the brackets on the two symmetric components. That is,

$$\langle (1), A\rangle = 2\sum_{ijk} \langle [\pi_1 A^{\eta_j}(e_i), \pi_1 e_k], [\pi_1 B(e_k, e_i 0, \pi_1 \eta_j]\rangle$$

To study this we note that $|\pi_1 X| \le |X|$ for X a tangent vector and the hypothesis $\|\pi_2\| \le \Lambda$ and the fact that $\dim(M) = p = \dim(S^p)$ implies that $|\pi_1 \nu| \le \Lambda |\nu|$ for ν a normal vector to M. Hence,

$$\begin{aligned}\langle (1), A\rangle \qquad & 2\sum_{ijk} \langle [\pi_1 A^{\eta_j}(e_i), \pi_k(e_k)], [\pi_1 B(e_k, e_i), \pi_1 \eta_j]\rangle \\ & \ge -2\sum_{ijk} |A^{\eta_j}(e_k)|\, |B(e_i,e_k)|\, K_1^2\, \Lambda^2 \\ & \ge -2p^2(N-p)\, K_1^2\, \Lambda^2\, \|A\|^2\end{aligned}$$

Similarly, $\langle (1), A\rangle$ is estimable by the corresponding term. □
This also gives the estimate for $\langle (2), A\rangle$.

Proposition 4.25. *In the given situation, if* $\|\pi_2\| \leq \Lambda$ *the third term satisfies*

$$\begin{aligned}\langle (3), A\rangle &\geq \rho\|A\|^2 - \sum_{jkl} \|[A^{\eta_j}(e_k), \eta_l]\|^2 \\ &\geq \rho\|A\|^2 - p(N-p)^2(K_1^2 + K_2^2)\Lambda^2\|A\|^2.\end{aligned}$$

Proof. This follows immmediately from equation 4.8. In particular we note that

$$\begin{aligned}\sum_{jkl} \|[A^{\eta_j}(e_k), \eta_l]\|^2 &= \sum_{jkl} \Big(\|[\pi_1 A^{\eta_j}(e_k), \pi_1\eta_l]\|^2 + \|[\pi_2 A^{\eta_j}(e_k), \pi_2\eta_l]\|^2\Big) \\ &\leq p(N-p)^2(K_1^2 + k_2^2)\Lambda^2\|A\|^2.\end{aligned}$$

The estimate of the first of these terms uses the fact that $|\pi_1\eta_l| \leq \Lambda$ which requires that the dimension of the submanifold is equal to the dimension of the first factor of the symmetric space $S^p \times M_2$. □

Proposition 4.26. *The fifth factor of* $\overline{R}(A)$ *satisfies*

$$\langle (5), A\rangle \geq -p^2(N-p)^2(K_1^2 + K_2^2)\Lambda^2\|A\|^2.$$

Proof. This follows from a similar argument to above, for the equation

$$\langle (5), A\rangle = -\sum_{i,j,k,l} \langle A_{e_k}(\eta_l), A_{e_k}(\eta_j)\rangle \langle [e_i, \eta_j], [e_i, \eta_j]\rangle.$$

□

Proposition 4.27. *Under the assumption that* $\|\pi_2\| \leq \Lambda \leq 1$ *uniformly on* M*, the sixth and final term of* $\overline{R}(A)$ *satisfies*

$$\langle (6), A\rangle \geq \frac{1}{p-1}\|A\|^2 - \frac{4p^2}{p-1}\Lambda^2\|A\|^2 - 2p^2(N-p)K_2^2\Lambda\|A\|^2$$

Proof. This proposition is very similar to the proposition 4.19. As such we recall the identity

$$\lambda_i^2\lambda_k^2 = 1 - \lambda_i^2(1-\lambda_k^2) + (\lambda_i^2 - 1)$$

and calculate the first term of $\langle(6), A\rangle$. Under the hypothesis $\|\pi_2\| \leq \Lambda$ from Theorem 4.23 we have the equation.... (t

$$\begin{aligned}
(p-1)\langle(6), A\rangle_1 &= \sum_{ik} \|B(e_i, e_k)\|^2 \\
&\quad - \sum_{ik} \lambda_i^2(\lambda_k^2 - 1)\big(\|B(e_i, e_k)\|^2 - \langle B(e_i, e_i), B(e_k, e_k)\rangle\big) \\
&\quad + \sum_{ik} (\lambda_i^2 - 1)\big(\|B(e_i, e_k)\|^2 - \langle B(e_i, e_i), B(e_k, e_k)\rangle\big) \\
&\geq \|A\|^2 - 4p^2\Lambda^2\|A\|^2
\end{aligned}$$

The estimate for the term $\langle(6), A\rangle_2$ is similar to the others that we have given and so we omit the calculation. □

This term, $\langle(6), A\rangle$, is the one in which we have only been able to give the result for submanifolds close to rank-one factors. This is because in the calculation of this term for arbitrary compact symmetric spaces, a term arises that is negative semi-definite. Our theorem can be concluded whenever this term vanishes or can be controlled. The first factor being the sphere is one such case.

This completes the explicit calculation of the coefficient that we require. Taking the inequalities in propositions 4.24, 4.25, 4.26 and 4.27 as reference we can define

$$\begin{aligned}
C = C(p, M_2) &= 8p^2(N-p)(K_1^2 + K_2^2) + 2p(N-p)^2(K_1^2 + K_2^2) \\
&\quad + p^2(N-p)^2(K_1^2 + K_2^2) + 4\frac{p^2}{p-1} + 2p^2(N-p)K_2^2.
\end{aligned}$$

This value of C and the above calculations allow us to conclude the following theorem.

Theorem 4.28. *There exists $C = C(p, M_2) > 0$ such that for any $0 < \Lambda \leq 1$ and for any p-dimensional minimal submanifold M of the symmetric space $\overline{M} = S^p \times M_2$ for which $\|\pi_2\| \leq \Lambda$ the term $\overline{R}(A)$ satisfies*

$$\langle\overline{R}(A), A\rangle \geq \Big((2\rho + \frac{1}{p-1}) - C\Lambda^2\Big)\|A\|^2.$$

This coefficient $2\rho + 1/(p-1) - C\Lambda^2$ is independent of the submanifold M. Thus according to the argument at the start of the chapter, we have a criterion for when M must be totally geodesic. This theorem is the analogue of Equation 4.7 in this case.

4.3 Totally Geodesic Submanifolds of Products

In the previous section we gave some sufficient conditions for a minimal submanifold of a riemannian symmetric space to be totally geodesic. The intended result is not so much this as the statement that the minimal submanifold is a particular totally geodesic subspace.

In this section we consider the totally geodesic submanifolds of symmetric spaces. It is a basic result in the theory that complete totally geodesic submanifolds of a symmetric space M with corresponding symmetric Lie algebra $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ are in a one-to-one correspondence with subspaces $\mathfrak{t} \subset \mathfrak{m}$ that satisfy

$$[[\mathfrak{t}, \mathfrak{t}], \mathfrak{t}] \subset \mathfrak{t}.$$

Our study of totally geodesic submanifolds will be via this correspondence.

We outline our hypotheses for this section. Let $(\mathfrak{g}, \mathfrak{h}, \sigma)$ be an orthogonal symmetric Lie algebra with $\mathfrak{g}$ semi-simple and of compact type. Assume that an inner product is given on $\mathfrak{g}$ by the negative of the Killing form. We restrict this to $\mathfrak{m}$ to give an $\mathfrak{h}$-invariant inner product. We will assume that the symmetric Lie algebra is reducible. It splits as

$$\mathfrak{g} = (\mathfrak{m}_1 + \mathfrak{h}_1) + (\mathfrak{m}_2 + \mathfrak{h}_2).$$

Moreover, we assume that the first factor is the symmetric Lie algebra for the round sphere. That is, $(\mathfrak{g}_1, \mathfrak{h}_1, \sigma) = (\mathfrak{so}(n+1), \mathfrak{so}(n), \sigma)$. The important features of this space are that it has rank one and that we know all of its Lie triple systems. The totally geodesic subspaces of the sphere are the great subsphere of the various dimensions so the Lie triple systems are (conjugate to) $(\mathfrak{so}(p+1), \mathfrak{so}(p), \sigma)$. For notational reasons we will continue to refer to the first factor as $\mathfrak{m}_1$.

Proposition 4.29. *Let $\mathfrak{t}$ be a Lie triple system contained in $\mathfrak{m}_1 + \mathfrak{m}_2$. Consider the orthogonal projections $\pi_2 : \mathfrak{t} \to \mathfrak{m}_2$ and $\pi^{\mathfrak{t}} : \mathfrak{m}_1 + \mathfrak{m}_2 \to \mathfrak{t}$. Suppose that $\|\pi^{\mathfrak{t}}\pi_2\|^2 \leq \Lambda^2/p$ for $\Lambda < 1$.*

Then the subalgebra $\mathfrak{t} + [\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{g}$ is simple and isomorphic to $\mathfrak{so}(p+1)$.

Proof.

Firstly, the condition that $\mathfrak{k} = \mathfrak{t} + [\mathfrak{t}, \mathfrak{t}]$ is a subalgebra follows from the observations

$$\begin{aligned} [[\mathfrak{t}, \mathfrak{t}], \mathfrak{t}] &\subset \mathfrak{t} \\ [[\mathfrak{t}, \mathfrak{t}], [\mathfrak{t}, \mathfrak{t}]] &\subset [\mathfrak{t}, [\mathfrak{t}, [\mathfrak{t}, \mathfrak{t}]]] \\ &\subset [\mathfrak{t}, \mathfrak{t}] \end{aligned}$$

This follows by the Jacobi identity and the fact that $\mathfrak{t}$ is a Lie triple system.

Next we consider the subspace $\pi_1(\mathfrak{t}) \subset \mathfrak{m}_1$. This is clearly a Lie triple system of $\mathfrak{m}_1$ and so corresponds to a totally geodesic submanifold of M_1. We know by hypothesis that $M_1 = S^n$ and that the corresponding totally geodesic subspace is a great sphere. This has isometry group $SO(p+1)$.

We consider the map $\pi^{\mathfrak{t}}\pi_2$. The assumption that $\|\pi^{\mathfrak{t}}\pi_2\|^2 \leq \Lambda^2/p$ for $\Lambda < 1$ implies that $\pi_1 : \mathfrak{t} \to \mathfrak{m}_1$ is an injection to its image. We claim that π_1 is injective when considered on $\mathfrak{k} = \mathfrak{t} + [\mathfrak{t}, \mathfrak{t}]$. Let $x = X_1 + X_2$, $y = Y_1 + Y_2$ for X_i, $Y_i \in \mathfrak{m}_i$. We can suppose that X_1 and Y_1 are non-zero. Then,

$$\begin{aligned} [x, y] &= [X_1, Y_1] + [X_2, Y_2] \\ \text{and} \quad \pi_1[x, y] &= [X_1, Y_1] \end{aligned}$$

Suppose that $[X_1, Y_1] = 0$. We have assumed that the symmetric Lie algebra $(\mathfrak{g}_1, \mathfrak{h}_1, \sigma)$ has rank one. This means that the dimension of a maximal subspace of $\mathfrak{m}_1$ on which the brackets vanish is one. In other words,

$$[X_1, Y_1] = 0 \Rightarrow X_1 = \lambda Y_1.$$

We can rescale x and y so that $\lambda = 1$. Then $x - y = X_2 - Y_2 \in \ker \pi_1 \cap \mathfrak{t}$. This implies that $x = y$ and $[x, y] = 0$. That is, π_1 is injective on $[\mathfrak{t}, \mathfrak{t}]$ as well so $\pi_1 : \mathfrak{k} \to \mathfrak{m}_1$ is an isomorphism to its image.

Hence, $\mathfrak{k} = \mathfrak{t} + [\mathfrak{t}, \mathfrak{t}] \cong \mathfrak{so}(p+1)$ and the symmetries correspond. □

In particular, the algebra $\mathfrak{k}$ is simple so the map $\pi_2 : \mathfrak{k} \to \mathfrak{m}_2$ is either identically zero or an isomorphism to its image. In the first case, if $T \subset$

$M_1 \times M_2$ is the corresponding totally geodesic subspace, $\pi_2(T)$ is a point. In the second case, $\pi_2 : T \to \pi_2(T)$ is a covering map. We show that if $\|\pi^T \pi_2\| \leq \Lambda/\sqrt{p}$ uniformly for some sufficiently small Λ dependent only on M_2 this second case cannot occur.

Theorem 4.30. *There exists $\Lambda > 0$ such that if T is a p-dimensional totally geodesic submanifold of $S^n \times M_2$ and $\|\pi_2\| \leq \Lambda$ then $T \subset S^n \times \{pt\}$.*

Proof. As before we consider the maps π_1 and π_2 to the two factors. From the previous proposition if $\Lambda < 1$, $\pi_2(T)$ is either a point or is a totally geodesic subspace of M_2 of the same dimension. We consider the second case and show that for Λ sufficiently small it cannot occur.

Theorem 4.31. [7]*(Area Formula) Let $F : X^n \to Y^{n+m}$ be a C^1 map between Riemannian manifolds. Then*

$$\int_A J_F(x) d\mathcal{H}^n(x) = \int_{F(X)} \mathcal{H}^0(F^{-1}(y) \cap A) d\mathcal{H}^n(y).$$

for any measurable $A \subset X$, where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure on the spaces.

This also holds for locally Lipschitz maps, though that is unnecessary for our considerations. The Jacobian term is given by $(J_F)^2 = \det(dF^* \circ dF)$. In the present case we consider the map π_2 defined on T and estimate the volume of $\pi_2(T)$. We have

$$\begin{aligned} \mathrm{vol}(\pi_2(T)) = \int_{\pi_2(T)} d\mathcal{H}^p(y) &\leq \int_{\pi_2(T)} \int_{\pi_2^{-1}(y)} d\mathcal{H}^0(t) d\mathcal{H}^p(y) \\ &= \int_T J_{\pi_2}(x) d\mathcal{H}^p(x) \end{aligned}$$

The Jacobian term can be estimated. In our context the transpose of $\pi_2 : \mathfrak{t} \to \mathfrak{m}_2$ is $\pi^{\mathfrak{t}} : \mathfrak{m}_2 \to \mathfrak{t}$. The determinant is the product of the eigenvalues of $\pi^{\mathfrak{t}} \pi_2$. Thus,

$$(J_{\pi_2})^2 = \prod_i (1 - \lambda_i^2) \leq \Lambda^p$$

and

$$\mathrm{vol}(\pi_2 T) \leq \Lambda^{\frac{p}{2}} \mathrm{vol}(T).$$

Similarly, by considering the projection to the other factor and observing that the assumption on $\|\pi^{\mathfrak{t}}\pi_2\|$ implies that $\prod_i \lambda_i^2 \geq (1-\Lambda)^p$, we have that

$$\mathrm{vol}(T) \leq \frac{1}{(1-\Lambda)^{\frac{p}{2}}}\mathrm{vol}(S^p)$$

where S^p has the round metric with constant curvature $\frac{1}{2(n-1)}$. This strange curvature value arises because we have the Killing metric on $\mathfrak{so}(n+1)$. Hence we have

$$\mathrm{vol}(\pi_2 T) \leq \left(\frac{\Lambda}{1-\Lambda}\right)^{\frac{p}{2}} \mathrm{vol}(S^p).$$

We show that for sufficiently small Λ this can only occur if $\pi_2 \equiv 0$ and $\pi_2 T$ is a point.

Suppose that $S^p \subset M_2$ is a totally geodesic submanifold. Then for $o \in N$, denote $V = T_o N$. Let U denote a fixed connected open subset of $T_o M_2$ that does not intersect the origin and does not intersect the tangential cut locus of $o \in M_2$. If $V \subseteq T_o M_2$ is a p-dimensional vector subspace we consider $U \cap V$ and consider the function

$$F(V) = \mathrm{vol}(\exp_o(U \cap V)) = \mathcal{H}^p(\exp_o(U \cap V)).$$

This is a positive continuous function on $G(p, T_o M_2)$ (if U is chosen correctly) and so has values bounded away from zero. That is, $F(V) \geq \lambda > 0$ for some small λ and for all V.

If $N \subseteq M_2$ is totally geodesic and $o \in N$ then $N = \exp_o(T_o N)$,

$$\mathrm{vol}(N) \geq \mathrm{vol}(\exp(U \cap T_o N)) \geq \lambda.$$

Thus if we take $\Lambda > 0$ such that $(\Lambda/1-\Lambda)^{p/2}\ \mathrm{vol}(S^p) < \lambda$ then $\|\pi_2\| \leq \Lambda$ uniformly on a totally geodesic T implies that $N = S^p$.

4.4 A Note on Genericity

Throughout the argument giving the proof of Theorem 4.7 we have used distinguished bases $\{e_i\}$ and $\{\eta_j\}$ for T_M and N_M respectively. They were chosen to satisfy

$$\pi^T \pi_1 e_i = \lambda_i^2 e_i$$
$$\pi^N \pi_1 \eta_j = (1-\lambda_j^2)\eta_j$$

and to be connected by the relation

$$\pi^N \pi_1 e_i = \sqrt{\lambda_i^2(1-\lambda_i^2)}\eta_i \quad \text{for } i = 1, \ldots, p.$$

The problem that we note now is that this assignment can be made only if all eigenvalues λ_i^2 are non-zero and not equal to one. This must be taken into account because the problem we are ultimately solving is to give sufficient conditions to have all λ_i's equal to zero. This apparent contradiction can be overcome if we consider precisely what we are doing.

The conclusion of the extensive calculation of Section 4.2 that led to Theorem 4.7 can be stated as:

If $V \subseteq \mathfrak{m}_1 + \mathfrak{m}_2$ is a vector subspace on which all eigenvalues of $\pi^V \pi_1$ are not zero or one and if $\|\pi_2\| \leq \Lambda$ and if A is a tensor on V with the symmetries and type of a second fundamental form then

$$\langle \overline{R}(A), A \rangle \geq \Big((2\rho + \frac{1}{p-1}) - C\Lambda^2 \Big) \|A\|^2.$$

That is, the inequality holds for all A and for all V that lie in a Zariski-open subset of $G(p, \mathfrak{m}_1 + \mathfrak{m}_2)$. By continuity, the inequality must hold for all subspaces $V \subseteq \mathfrak{m}_1 + \mathfrak{m}_2$.

Hence, the conclusion of Theorem 4.7 is valid, even for submanifolds some of whose tangent spaces have 0 as an eigenvector.

Chapter 5

Rigidity of Rank-One Components

We continue our previous calculations regarding minimal submanifolds of symmetric spaces. In the previous chapter we proved a number of theorems on rigidity and isolation phenomena for the totally geodesic factors $S^p \times \{pt\}$ contained in $S^p \times M_2$ where M_2 is a riemannian symmetric space of compact type and where the ambient metric is gotten from the Killing form on the associated Lie algebra of Killing vectors. From these calculations, we can give similar results in the case that the ambient space is $M_1 \times M_2$ where M_1 is an arbitrary compact symmetric space of rank one.

In a similar way to how the previous chapter was directly motivated by the work of Simons for submanifolds of the sphere, this chapter uses the techniques and methods of proof of Lawson [5]. This work also uses the classification of rank-one symmetric spaces so that we can do it one a case-by-case basis.

The fundamental observation that we make is that any rank-one symmetric space admits a submersion from a Euclidean sphere that is compatible with the Riemannian structures of the spaces. This type of structure was studied originally and extensively by O'Neill [6].

5.1 Riemannian Submersions

We consider smooth maps $F : N \to M$ with derivative everywhere surjective. The preimage of any point in M is then a smooth submanifold of N. We will combine this with the Riemannian geometry of the spaces N and M.

Assume that N and M are Riemannian manifolds. At $p \in N$ consider the subspace of T_N orthogonal to the fibre $F^{-1}(F(p))$. F_* is bijection to T_M when restricted to this space. If F_* restricted to this space is an isometry for every $p \in N$ we say that $F : N \to M$ is a *Riemannian submersion*. In the case at hand, in which we are interested in minimal submanifolds, we will also assume that each fibre of the submersion is a totally geodesic submanifold.

We use the terminology of fibre bundles in describing the curvature of such spaces. The tangent space to a fibre is referred to as the *vertical* subspace. The orthogonal space to a fibre is called the *horizontal* space. Let X denote a tangent vector at $p \in N$. The vertical and horizontal components of X are denoted $\mathcal{V}X$ and $\mathcal{H}X$ respectively. The fundamental tensor that relates the curvature of N, M and the fibres will be denoted by A and is given by

$$A_X Y = \mathcal{V}\nabla_{\mathcal{H}X}\mathcal{H}Y + \mathcal{H}\nabla_{\mathcal{H}X}\mathcal{V}Y$$

This defines a tensor of type $(1, 2)$ on N. It can be thought of as a measure of the extent to which horizontal parallel translation does not preserve the horizontal and vertical subspaces.

Let R and R' denote the Riemann curvatures of N and M respectively so that $S(X, Y) = \langle R_{Y,X}X, Y\rangle$ and $S'(X, Y) = \langle R'_{Y,X}X, Y\rangle$ denote the sectional curvatures when X and Y are orthogonal and unit length. Then by [6, pg. 465] if X and Y are horizontal vectors to N and V is vertical,

$$\begin{aligned} S'(F_*X, F_*Y) &= S(X, Y) + 3|A_X Y|^2 \\ S(X, V) &= |A_X V|^2. \end{aligned}$$

The second term more generally involves some terms involving the second fundamental form of the fibres. These vanish if we assume the fibres are totally geodesic.

We now give some formulas relating the scalar curvatures of N and M. Following Lawson, we define four terms. At $p \in N$ we let $K(p)$ denote the scalar curvature of N at p, $K'(p)$ denote the scalar curvature of M at $F(p)$, and let $r(p)$ denote the scalar curvature of the fibre at p. Finally we define the *twisting curvature of the submersion* $\tau(p)$ at p by

$$\tau(p) = \sum_j \sum_k S(e_j, \nu_k)$$

where $\{e_j\}$ is an orthonormal basis for the horizontal space at p and $\{\nu_k\}$ is an orthonormal basis for the vertical space at p. We note that $\tau \geq 0$ everywhere. The fundamental relation that we use is

Theorem 5.1. [5, pg. 351]

$$K' = K + \tau - r.$$

We will use this result to prove theorems similar to 4.11, where we will compare the scalar curvature of a minimal submanifold with the scalar curvature of a space that fibres over it.

Finally, in the context of Riemannian submersions, we will recall another construction of Lawson that relates submanifolds of the base of a submersion with submanifolds of the domain. Let $\pi : \overline{M} \to \overline{B}$ be a Riemannian submersion and let M be a submanifold of $\overline{M}$ that respects the map π. That is, suppose there is a submersion $\pi : M \to B$ where B is a submanifold of $\overline{B}$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & \overline{M} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & \xrightarrow{f} & \overline{B} \end{array}$$

commutes. Importantly, we also assume that the submanifold M of $\overline{M}$ contains all of the fibres of the map from $\overline{M}$. We give a number of facts taken from [5] that we will need. Firstly,

Theorem 5.2. [5] *M is a minimal submanifold of $\overline{M}$ if and only if B is a minimal submanifold of $\overline{B}$.*

And secondly, we shall compare the τ functions for the two submersions. We denote the τ functions by τ_M and $\tau_{\overline{M}}$ for the submersions from M and $\overline{M}$ respectively, and we denote the mapping S similarly in the two cases.

Then the Gauss curvature equation for a submanifold $M \subseteq \overline{M}$ is given by

$$\|B(X,Y)\|^2 - \langle B(X,X), B(Y,Y)\rangle = \langle \overline{R}_{Y,X}X, Y\rangle - \langle R_{Y,X}X, Y\rangle$$

for X and Y tangent to M. If M is totally geodesic, the terms on the left vanish and the curvature terms are equal. If, furthermore, Y is tangent to another submanifold $F \subseteq M$ that is totally geodesic in $\overline{M}$ then only the second term on the left vanishes. This occurs in the case at hand where F is the fibre to submersions from M and $\overline{M}$. We assume that this subspace is totally geodesic. Then,

$$\begin{aligned} 0 &\leq \|B(e_i, \nu_k)\|^2 \\ &= S_{\overline{M}}(e_i, \nu_k) - S_M(e_i, \nu_k) \end{aligned} \tag{5.1}$$

since the vector ν_k used to define τ is vertical and tangent to the totally geodesic fibre. This will be used in our calculation of τ_M.

Finally we make an observation for submanifolds $M \subseteq \overline{M} \times M_2$ that fibre over $B \subseteq \overline{B} \times M_2$. This is the observation that the projections π_2 from M to M_2 and from B to M_2 have the same uniform norms. This can be seen from the fact that the fibres to the submersion are wholly contained in the first factor. Also, the (tangential) projection of B to M_2 can be identified with the projection of the horizontal space (those vectors orthogonal to the fibre) to M_2.

5.2 Case 2: $\mathbb{CP}_n \times M_2$

We have named this section *Case 2* to reflect that the first rank-one symmetric space that we considered was the sphere. $\mathbb{CP}_n$ is a natural second one. In this section we show that the $\mathbb{CP}_n \times \{pt\}$ factors in this manifold are rigid as minimal submanifolds, as we have done for spherical factors in previous sections.

First we describe the Riemannian geometry of this space and how this relates to the submersion from S^{2n+1}.

The manifold $\mathbb{CP}_n$ is a Riemannian symmetric space and can be given as the homogeneous space

$$\mathbb{CP}_n = SU(n+1)/S(U(1) \times U(n)).$$

The isotropy group H is isomorphic to $U(n)$. In the decomposition of the orthogonal symmetric Lie algebra $\mathfrak{su}(n+1) = \mathfrak{h} + \mathfrak{m}$ we have $\mathfrak{m}$ the set of elements of the form

$$\left(\begin{array}{c|c} 0 & -\bar{v}^T \\ \hline v & 0 \end{array}\right)$$

for $v \in \mathbb{C}^n$. The metric that we take on $\mathbb{CP}_n$ is induced from the restriction of minus the Killing form on $\mathfrak{su}(n+1)$ to $\mathfrak{m}$. For reasons of invariance and irreducibility this inner product on $\mathfrak{m}$ is equal to a positive multiple of the usual Euclidean inner product, under the above identification of $\mathfrak{m}$ with $\mathbb{C}^n$. One can calculate that the positive constant is equal to $4n$. With this metric the sectional curvatures of $\mathbb{CP}_n$ range between $\frac{1}{4n}$ and $\frac{1}{n}$.

We will consider the fibration $S^1 \to S^{2n+1} \to \mathbb{CP}_n$ where S^1 acts on S^{2n+1} by complex multiplication in $\mathbb{C}^{n+1}$. The sphere is diffeomorphic to the homogeneous $SO(2n+2)/SO(2n+1)$ and the metric on S^{2n+1} that we take is the restriction of minus the Killing form on $\mathfrak{so}(2n+2)$. As we noted in Section 1.3 this is $4n$ times the usual metric on S^{2n+1}.

We can thus see that the fibration $S^1 \to S^{2n+1} \to \mathbb{CP}_n$ is a Riemannian submersion for these metrics. The fibre is one dimensional and the space orthogonal to it is mapped isometrically to the tangent space of $\mathbb{CP}_n$.

We now consider the submanifold geometry in this case. Let M be a $2n$-dimensional submanifold of $\mathbb{CP}_n \times M_2$ and let $N \subseteq \overline{N} = S^{2n+1} \times M_2$ be the submanifold that fibres over it. N is locally the pre-image of M by the projection.

Lemma 5.3. $\tau_N \leq \frac{1}{2}$.

Proof. We recall the definition.

$$\tau_N(p) = \sum_{i,k} S_N(e_i, \nu_k).$$

We can use Equation 5.2. We can assume that the bases $\{e_i\}$ and $\{\nu_k\}$ together diagonalise the map $\pi^T\pi_1$ on N. In particular, $\pi_1\nu_k = \nu_k$ because ν_k is vertical and $|\pi_1 e_i| = \lambda_i$.

$$\begin{aligned} S_N(e_i, \nu_k) &\leq S_{\overline{N}}(e_i, \nu_k) \\ &= \|[\pi_1 e_i, \pi_1 \nu_k]\|^2 + \|[\pi_2 e_i, \pi_2 \nu_k]\|^2 \\ &= \frac{1}{4n}\lambda_i^2 \leq \frac{1}{4n}. \end{aligned}$$

Thus, $\tau_N(p) \leq \frac{1}{4n}2n = \frac{1}{2}$. □

Also, we can note that since the fibre this submersion is S^1, $r = 0$ in this case.

We can now prove the main theorem for this section. Let $\Lambda > 0$ satisfy

$$\left(\frac{C}{q} + 2n + 1 + 2n(2n+1)K_2^2\right)\Lambda^2 = \frac{\rho + \frac{1}{2n}}{q}$$

where $q = 2 - \frac{1}{\dim M_2}$ and ρ is the smallest Ricci curvature of any direction tangent to $S^{2n+1} \times M_2$ and C is defined by equation...... Then we have the theorem.

Theorem 5.4. *Let M be a $2n$-dimensional closed minimal submanifold of $\mathbb{CP}_n \times M_2$ where M_2 is a symmetric space of compact type. Let K' be the intrinsic scalar curvature of the induced metric on M. Suppose that M uniformly satisfies*

$$\begin{aligned} \|\pi_2\| &\leq \Lambda \\ (n+1) - K' &< \frac{\rho}{q}. \end{aligned}$$

Then M is totally geodesic. Furthermore, Λ can be taken sufficiently such that these conditions imply that $\pi_2 \equiv 0$ and $M = \mathbb{CP}_n \times \{pt\}$ and $K' \equiv n+1$.

We use the extensive calculations of the previous chapter. As has been noted above, we lift M to a submanifold of $S^{2n+1} \times M_2$ and show that this space has to satisfy the previously determined conditions to be totally geodesic.

Proof. Let $N \subseteq S^{2n+1} \times M_2$ be the submanifold that fibres over M. Then N is a closed minimal submanifold. Let A_N be its second fundamental form and let K be the scalar curvature of the induced metric on N. Then N uniformly satisfies $\|\pi_2\| \leq \Lambda$ and by Equation 4.3

$$0 \leq \int_M \|A_N\|^2 \Big(\|A_N\|^2 - \big(\frac{(2\rho + \frac{1}{2n}) - C\Lambda^2}{q} \big) \Big).$$

¿From the proof of Theorem 4.11 we obtain the estimate

$$
\begin{aligned}
\|A_N\|^2 \quad - \quad & \Big(\frac{(2\rho+\frac{1}{2n})-C\Lambda^2}{q}\Big) \\
& \le \Big(\frac{2n+1}{2}-\frac{\rho}{q}\Big)-K-\Big(\frac{\rho+\frac{1}{2n}}{q}\Big)+\Big(\frac{C}{q}+2n+1+2n(2n+1)K_2^2\Big)\Lambda^2 \\
& = \Big(\frac{2n+1}{2}-\frac{\rho}{q}\Big)-K \\
& = \frac{2n+1}{2}-\frac{\rho}{q}-K'+\tau-r \\
& \le n+1-K'-\frac{\rho}{q}.
\end{aligned}
$$

If this is negative then necessarily $A_N \equiv 0$ and N is totally geodesic. By considering Lie triple systems for the relevant spaces, $M = \pi(N)$ is then also totally geodesic.

As before, if Λ is sufficiently small, N and M are factors of the respective spaces. □

As we did for the case where the first factor is a sphere, we can express the above inequality in terms of isolation phenomena for embeddings. In this case, we must use the C^3 topology because the condition is on scalar curvature.

Corollary 5.5. *There is a C^3-open neighbourhood of the standard embedding of $\mathbb{CP}_n$ in $\mathbb{CP}_n \times M_2$ in the set of immersions such that any minimal immersion contained in it is conjugate to the standard one.*

5.3 Case 3: $\mathbb{HP}_n \times M_2$

Consider the smooth manifold $\mathbb{HP}_n$ of quaternion lines in $\mathbb{H}^{n+1}$ where scalar multiplication of $\mathbb{H}$ is on the right. This space can be realised as a symmetric homogeneous space G/H where

$$
\begin{aligned}
G &= Sp(n+1) \\
&= \{g \in SO(4(n+1));\ \ [g,I]=[g,J]=[g,K]=0\}.
\end{aligned}
$$

The isotropy group of this action on lines is the group

$$H = Sp(n)\,Sp(1) = Sp(n)\times Sp(1)/\mathbb{Z}_2$$

where $Sp(n)$ preserves the vector $(1, 0, \ldots, 0) \in \mathbb{H}^{n+1}$ and $Sp(1)$ (acting on the right) preserves the line $[1 : 0 : \cdots : 0]$. The diagonal element $(-1, -1)$ acts trivially so is factored out.

The tangent space to HP_n can be identified with $\mathbb{H}^n$, by a similar method to $\mathbb{CP}_n$, and the restriction of the Killing metric to $\mathfrak{m} = \mathbb{H}^n$ is a positive multiple of the standard Euclidean inner product on $\mathbb{H}^n$. One can calculate that this positive multiple is is $8n + 4$.

$\mathbb{HP}_n$ admits a Hopf-type submersion $\pi : S^{4n+3} \to \mathbb{HP}_n$ from a sphere of dimension $4n + 3$. This can be seen from the identification

$$\mathbb{HP}_n = (\mathbb{H}^{n+1})^*/\mathbb{H}^* = S^{4n+3}/S^3$$

by first quotienting out the radial component. The metric that we take on the sphere is $2(4n + 3 - 1) = 8n + 4$ times the usual Euclidean metric. Thus, the derivative of π, restricted to the orthogonal complement to the fibre is an isometry. The fibres of the submersion are the 3-dimensional great spheres contained in the quaternion lines. Thus, π is a Riemannian submersion with totally geodesic fibres so we can use the previous techniques to the Riemannian submersion $\pi : S^{4n+3} \times M_2 \to \mathbb{HP}_n \times M_2$.

If $\mathbb{HP}_n$ is given its standard Riemannian metric for which its curvatures are pinched between 1 and 4 (and for which this submersion if Riemannian if S^{4n+3} has constant unit curvature) then its scalar curvature is $16n(n+2)$. We have homothetically scaled this metric by a factor of $2(4n + 2)$ so its scalar curvature is

$$\frac{16n(n+2)}{2(4n+2)} = \frac{4n(n+2)}{2n+1}.$$

Let M be a closed minimal submanifold of $\mathbb{HP}_n \times M_2$ and let N be the submanifold of $S^{4n+3} \times M_2$ that fibres over M with S^3 fibres. N is then a closed minimal submanifold and the uniform norm of π_2 from N is equal to the uniform norm from M. The map from N to M is a Riemannian submersion with totally geodesic fibres. We consider the previously defined functions K', K, τ_N and r.

Lemma 5.6. *For $p \in N$,*

$$\begin{aligned} r(p) &= \frac{3}{4n+2} \\ \tau_N(p) &\leq \frac{3n}{2n+1} \end{aligned}$$

Proof. This is essentially identical to the case in the previous section, each time taking account of the correct scaling of the metric on S^{4n+3}. □

We can now give the theorem of this section. Suppose that Λ satisfies

$$\left(\frac{C}{q} + 4n + 3 + (4n+2)(4n+3)K_2^2\right)\Lambda^2 = \frac{\rho + \frac{1}{4n+2}}{q}$$

where the constants are defined earlier. C is the term given in Theorem 4.28 for $p = 4n + 3$. K_2 is the maximum value of $\|[X, Y]\|$ for X and Y unit vectors in $\mathfrak{m}_2$. ρ is the smallest Ricci curvature of any direction in $\overline{M}$ and $q = 2 - 1/\dim M_2$.

Theorem 5.7. *Let M be a closed $4n$-dimensional minimal submanifold of $\mathbb{HP}_n \times M_2$ and let K' be its intrinsic scalar curvature. Suppose that M satisfies, for Λ given above,*

$$\|\pi_2\| \leq \Lambda,$$
$$\frac{4n(n+2)}{2n+1} - K' < \frac{\rho}{q}.$$

Then M is totally geodesic. There exists a possibly smaller Λ such that these hypotheses imply that M is a totally geodesic factor $\mathbb{HP}_n \times pt$.

Proof. The proof is essentially identical to the case for the complex projective space in Theorem 5.4. As noted above, the submanifold N of $S^{4n+3} \times M_2$ that factors over M is minimal, closed and $4n+3$-dimensional. We show that N is totally geodesic. Let A be the second fundamental form of N and let K be its intrinsic scalar curvature. Then, following that proof,

$$\begin{aligned}
\|A\|^2 &- \left(\frac{(2\rho + \frac{1}{4n+2}) - C\Lambda^2}{q}\right) \\
&= \frac{4n+3}{2} - \frac{\rho}{q} - K \\
&\qquad + \left[\left(\frac{C}{q} + 4n + 3 + (4n+2)(4n+3)K_2^2\right)\Lambda^2 - \left(\frac{\rho + \frac{1}{4n+2}}{q}\right)\right] \\
&= \frac{4n+3}{2} - K' + \tau + r - \frac{\rho}{q} \\
&\leq \frac{4n+3}{2} + \frac{3n}{2n+1} - \frac{3}{4n+2} - \frac{\rho}{q} - K' \\
&= \frac{4n(n+2)}{2n+1} - K' - \frac{\rho}{q}.
\end{aligned}$$

Hence we assume that this term is negative. By a similar method to the case for $\mathbb{CP}_n$, we can conclude that $A \equiv 0$ and M is totally geodesic. Also as before, by Theorem 4.30 we can take Λ to be possibly even smaller to ensure that $M = \mathbb{HP}_n \times pt$. □

As in the S^p and $\mathbb{CP}_n$ cases, we can show that the $\mathbb{HP}_n$ factors are isolated from other inequivalent minimal embeddings.

Corollary 5.8. *There is a C^3-open neighbourhood of the standard embedding of $\mathbb{CP}_n$ in $\mathbb{CP}_n \times M_2$ in the set of immersions such that any minimal immersion contained in it is conjugate to the standard one.*

5.4 The Case of the Cayley Plane

In the previous two sections of this chapter we have proven that a closed minimal submanifold of $\mathbb{CP}_n \times M_2$ (or $\mathbb{HP}_n \times M_2$) that is sufficiently close to the $\mathbb{CP}_n$ (or $\mathbb{HP}_n$) factor must be totally geodesic and equal to $\mathbb{CP}_n \times \{q\}$ for some $q \in M_2$ (or $\mathbb{HP}_n \times \{q\}$ respectively).

The condition of closeness that we require is that the intrinsic scalar curvature of the submanifold is uniformly close to the constant scalar curvature of the symmetric subspace that it is being compared to, together with the projection to the M_2 factor being uniformly small.

The crucial fact in the proofs of these theorems is that there are Riemannian submersions from spheres onto $\mathbb{CP}_n$ and $\mathbb{HP}_n$. We can lift a minimal submanifold of $\mathbb{CP}_n \times M_2$ to one of $S^{2n+1} \times M_2$ and use the fundamental Theorem 4.9 for submanifolds of this space to show that the submanifold is a $\mathbb{CP}_n$-factor.

According to the classification of rank-one symmetric spaces of compact type, as outlined in Section 1.4 this almost proves a rigidity theorem for all rank-one factors in the decomposition of a symmetric space of compact type. This outstanding case is the Cayley Plane $\mathbb{OP}_2$. This space can be thought of as the set of octonian lines in $\mathbb{O}^3$, although it is a delicate matter to construct it in this way (see [1]).

Perhaps regretably, we have the following theorem.

Theorem 5.9. [3] *No open set in S^{23} can be filled by pieces of parallel great 7-spheres.*

This in particular shows that the set of octonian lines in $\mathbb{O}^4$ cannot be well studied as a symmetric space. However, closer to the issue at hand, it shows that there is not a Riemannian submersion of S^{23} onto $\mathbb{OP}_2$ with fibres totally geodesic 7-spheres. Thus, one cannot use the same techniques as in the other cases to study the minimal submanifolds of $\mathbb{OP}_2$. Without the development of new techniques, this completes this work. With these realisations, we end this dissertation.

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