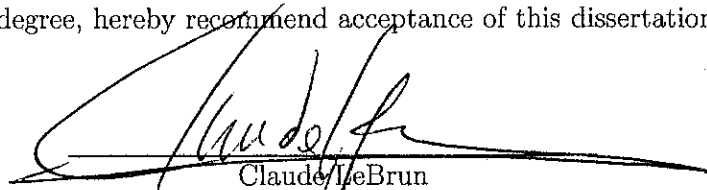


Stony Brook University

The Graduate School

Yujen Shu

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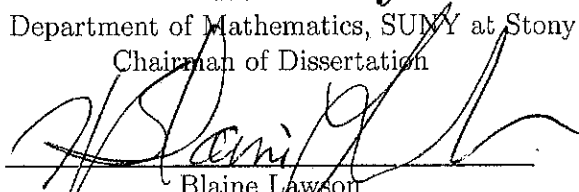
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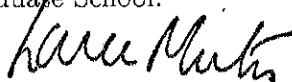
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**Abstract of the Dissertation**  
**Compact Complex Surfaces and Constant**  
**Scalar Curvature Kähler Metrics**

by

Yujen Shu

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in

Mathematics

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In the thesis, I prove the following statement: Every compact complex surface with even first Betti number is deformation equivalent to one which admits an extremal Kähler metric. In fact, this extremal Kähler metric can even be taken to have constant scalar curvature in all but two cases: the deformation equivalence classes of the blow-up of  $\mathbb{P}_2$  at one or two points. The explicit construction of compact complex surfaces with constant scalar curvature Kähler metrics in different deformation equivalence classes is given. The main tool repeatedly applied here is the gluing theorem of C. Arezzo and F. Pacard which states that the blow-up/resolution of

a compact manifold/orbifold of discrete type, which admits cscK metrics, still admits cscK metrics.

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# Chapter 1

## Introduction

Let  $(M, J)$  be a compact complex Kähler manifold, and  $\gamma$  be a Kähler class of  $(M, J)$ . Calabi [8] considers the functional  $\Phi(g) := \int s_g^2 dv_g$  defined on the set of all Kähler metrics  $g$  in the class  $\gamma$ , where  $s_g$  denotes the scalar curvature associated to the metric  $g$ . A Kähler metric is called extremal if it is a critical point of  $\Phi$ . It has been shown that  $g$  is extremal if and only if the gradient of the scalar curvature  $s_g$  is a real holomorphic vector field. In particular,  $g$  is extremal if  $s_g$  is constant. The famous Aubin-Yau theorem [4, 41] asserts that every compact complex manifold  $X$  with negative first Chern class admits a Kähler-Einstein metric in the canonical class  $-c_1(X)$ , which has a negative constant scalar curvature. Followed by applying a deformation argument due to LeBrun and Simanca [24], we know there exists a constant scalar curvature Kähler (cscK) metric in every class near  $-c_1(X)$ .

There are classical obstructions to the existence of cscK metrics related to the automorphism group  $Aut(M, J)$  of  $(M, J)$ . The Matsushima-Lichnerowicz theorem [25, 29] states that the identity component  $Aut_0(M, J)$  of the auto-

morphism group must be reductive if a cscK metric exists. Later, Futaki [13] shows that the scalar curvature of an extremal Kähler metric is constant if and only if its Futaki invariant vanishes identically. Recently Donaldson [10], Chen-Tian [9] and Mabuchi [26, 27] have made substantial progress on relating the existence and uniqueness of extremal Kähler metrics in Hodge Kähler classes to the K-stability of polarized projective varieties. In particular, it has been shown that K-stability is a necessary condition for the existence of cscK metrics for a polarized projective variety. The main result of this thesis is the following:

**Main Theorem:** Every compact complex surface with even first Betti number is deformation equivalent to one which admits an extremal Kähler metric. In fact, this extremal Kähler metric can even be taken to have constant scalar curvature in all but two cases: the deformation equivalence classes of the blow-up of  $\mathbb{P}_2$  at one or two points.

The hypothesis that  $b_1$  is even is equivalent to requiring that  $(M, J)$  admits a Kähler metric [35, 37]. One of the main tools in our proof is the major breakthrough by Arezzo and Pacard [1]: Let  $M$  be a complex manifold/orbifold of discrete type, which admits cscK metrics. Then the blow-up/resolution of  $M$  admits cscK metrics.

To prove the main theorem, we proceed along the Kodaira dimension  $\kappa$  of  $(M, J)$ . The main difficulty lies in the case of  $\kappa = 1$ . First of all, we deal with the case of principal  $E$ -bundles. We could show every principal  $E$ -bundle with even Betti number is covered by  $\mathbb{H} \times \mathbb{C}$ , and it inherits a cscK metric from the product metric. Analyzing the geometric structure of the surface and the local



structure near a multiple fibre, we can generalize this result to the case of all elliptic surfaces whose each fibre has smooth reduction. To construct an elliptic surface with cscK metrics and positive Euler number in each deformation class, we start with the elliptic surface  $S$ , which is obtained by applying logarithmic transform along some fibres on the trivial elliptic bundle. Under certain choices of the base curve and the fibres on which logarithmic transform is applied, there exists an holomorphic involution  $\iota : S \rightarrow S$ . Although the quotient of  $S$  by the action of involution  $\iota$  is singular, we can show its resolution is smooth and carries cscK metrics by an application of Arezzo-Pacard's result [1]. Finally, to show elliptic surfaces with positive Euler number of each deformation class can be constructed in this way, we use the deformation theory of elliptic surfaces [11]: in the case of positive Euler number, the deformation class of an elliptic surface is determined by the diffeomorphism type of the base orbifold and the Euler number  $\chi$ .

In chapter 4, we explain why the complex surfaces of other Kodaira dimensions are deformation equivalent to complex surfaces with cscK metrics. The case of  $\kappa = 2$  is done as an application in Arezzo-Pacard's paper [1] by using the Aubin-Yau theorem [4, 41] and the fact that negative first Chern class implies the automorphism group is discrete. Similarly, the case of  $\kappa = 0$  is a result of Yau's theorem [41] and the fact that all holomorphic vector fields on a minimal complex surface of  $\kappa = 0$  are parallel. The case of ruled surfaces,  $\kappa = -\infty$ , is dealt with by using the result that if a rank 2 vector bundle  $E$  is poly-stable, then the associated ruled surface  $\mathbb{P}(E)$  admits cscK metrics. In the end, we also show that  $\mathbb{P}_2 \# k\overline{\mathbb{P}}_2, k = 1, 2$ , is not deformation equivalent to any complex surface with cscK metrics by showing that the Lie algebra of

holomorphic vector fields of every compact complex surface in the deformation class is not reductive.

Although there exists no cscK metrics on any Kähler class of the blow-up of  $\mathbb{P}_2$  at one or two points, Calabi [8], Arezzo, Parcard, and Singer [2] have shown that there do exist extremal Kähler metrics on them, and the main theorem follows.

## Chapter 2

### Elliptic surfaces

In this chapter, we give a general description of elliptic surfaces, and discuss some special examples: principal  $E$ -bundles and the ones without singular fibres. Then we introduce one important operation: logarithmic transform which we will use later to investigate the geometric structures of elliptic surfaces.

#### 2.1 Properties of elliptic surfaces

**Definition 2.1.1.** *A compact complex surface  $S$  is called an elliptic surface if there is a surjective holomorphic map  $\pi : S \rightarrow B$  over a smooth algebraic curve  $B$ , whose general fibres are smooth elliptic curves. The map  $\pi$  is called an elliptic fibration.*

Throughout this section, we only consider the case that the elliptic surface  $S$  is minimal. That is, it contains no rational curves of self-intersection number

-1. An elliptic surface may have singular, reducible, multiple fibres. Kodaira [19] has classified all possible fibres of an elliptic surface, which are denoted by  ${}_mI_k$ ,  $II$ ,  $III$ ,  $IV$ ,  $I_k^*$ ,  $II^*$ ,  $III^*$ , and  $IV^*$ , where  $k \in \mathbb{N} \cup \{0\}$ . The set of multiple fibres plays a central role in our discussion of elliptic surfaces. And we will recall the definition here.

**Definition 2.1.2.** *Let  $\pi : S \rightarrow B$  be an elliptic fibration and  $p \in B$ . We call  $\pi^{-1}(p)$  a multiple fibre if there exists an integer  $m \geq 1$  and a reduced divisor  $D$  in  $S$  such that as a divisor  $\pi^{-1}(p) = mD$ . The largest  $m$  is called the multiplicity of the fibre.*

In fact, there are only two types of multiple fibres: with  $m$  the multiplicity and  $D$  as above, either  $D$  is a smooth elliptic curve (type  ${}_mI_0$ ) or  $D$  is a reduced cycle of  $n$  rational curves (type  ${}_mI_n$ ).

**Definition 2.1.3.** *Let  $\pi : S \rightarrow B$  be an elliptic fibration. We say that the fibre  $F_p = \pi^{-1}(p)$ ,  $p \in B$ , is singular if it has positive Euler number. In particular, a multiple fibre with smooth reduction is not singular in this sense.*

Except the type  ${}_mI_0$ , all other types have positive Euler numbers [40], and are singular by our definition.

**Lemma 2.1.4.** *A minimal elliptic surface  $S$  has nonnegative Euler number  $\chi$ , and the case  $\chi = 0$  occurs if and only if  $S$  has no singular fibres.*

*Proof.* Let  $\pi : S \rightarrow B$  be an elliptic fibration,  $\Xi \subset B$  be the set of critical values of  $\pi$ , and  $F_b$  be the fibre over  $b$ ,  $b \in B$ .

Set  $F = \bigcup_{b \in \Xi} F_b$ , which is a closed set in  $S$ , and we have the exact sequence

$$\dots \rightarrow H_c^i(S \setminus F, \mathbb{R}) \rightarrow H^i(S, \mathbb{R}) \rightarrow H^i(F, \mathbb{R}) \rightarrow H_c^{i+1}(S \setminus F, \mathbb{R}) \rightarrow \dots$$

where the subscript  $c$  means the cohomology with compact support. Therefore, we deduce the relation among the Euler numbers  $\chi(S) = \chi_c(S \setminus F) + \chi(F)$ . Now  $\pi : S \setminus F \rightarrow B \setminus \Sigma$  is a topological elliptic fibre bundle with fibre  $E$ , and hence we have  $\chi_c(S \setminus F) = \chi_c(B \setminus \Sigma)\chi(E) = 0$ . As a result,

$$\chi(S) = \chi(F) = \sum_{b \in \Sigma} F_b,$$

and the lemma follows.  $\square$

To each elliptic fibration, we can associate two fundamental invariants [5, 19]:

**Definition 2.1.5.** *Let  $\pi : S \rightarrow B$  be an elliptic fibration. We define the following invariants associated to  $S$ :*

1. *The  $j$ -invariant (functional invariant)  $j_S$  of  $S$ . Let  $U$  be the open subset of  $B$  consisting of regular values of  $\pi$ . Let  $f_S$  be the holomorphic map which associate to each point  $b \in B$  the isomorphism class of the elliptic curve  $\pi^{-1}(b) \in \mathbb{C}^+/\mathrm{PSL}_2(\mathbb{Z})$ , where  $\mathbb{C}^+$  is the upper half plane, and  $j$  be the biholomorphic function  $j : \mathbb{C}^+/\mathrm{PSL}_2(\mathbb{Z}) \rightarrow \mathbb{C}$  induced by the elliptic modular function  $\tilde{j} : \mathbb{C}^+ \rightarrow \mathbb{C}$ . Let  $j_S := j \circ f_S/1728 : U \rightarrow \mathbb{C}$ . By the stable reduction theorem [5],  $j_S$  has an extension to a holomorphic function from  $B$  to  $\mathbb{P}_1$ .*
2. *The homological invariant (global monodromy) of  $S$ , which is a sheaf  $G_S$  on the base  $B$ . Let  $U$  be the open subset of  $B$  consisting of regular values of  $\pi$  and  $i : U \rightarrow B$  be the inclusion map. Then we define  $G_S := i_*(R^1\pi_*\mathbb{Z}|U)$ . The sheaf  $R^1\pi_*\mathbb{Z}|U$  is locally constant, and*

it can be constructed from a representation of the fundamental group  
 $L_S : \pi_1(U, b) \rightarrow \text{Aut}(H^1(\pi^{-1}(b), \mathbb{Z})) \cong SL_2(\mathbb{Z})$  .

These two invariants are not unrelated. There is a natural compatibility between them. From the definition, the functional invariant  $j_S$  takes the value of  $\infty$  only at singular fibres. Let  $\pi : S \rightarrow B$  be an elliptic fibration. Assume the functional invariant  $j_S$  is not identically 0 or 1. Let  $U := B \setminus j^{-1}(\{0, 1, \infty\})$ . Composing with the canonical projection  $SL_2(\mathbb{Z}) \rightarrow \mathbb{P}SL_2(\mathbb{Z})$ , the equivalence class of the representation  $G_S : \pi_1(U) \rightarrow SL_2(\mathbb{Z})$  induces a representation  $\tilde{G}_S : \pi_1(U) \rightarrow \mathbb{P}SL_2(\mathbb{Z})$ . On the other hand, the elliptic modular function  $\tilde{j} : \mathbb{C}^+ \rightarrow \mathbb{C}$  is unbranched outside the preimage of 0 and 1. The covering  $\tilde{j} : \mathbb{C}^+ \setminus j^{-1}(\{0, 1\}) \rightarrow \mathbb{C} \setminus \{0, 1\}$  therefore induces an equivalence class of a representation  $\tilde{j}_* : \pi_1(\mathbb{C} \setminus \{0, 1\}) \rightarrow \mathbb{P}SL_2(\mathbb{Z})$ . Then the meromorphic function  $j_S$  gives a map  $j_{S*} : \pi_1(U) \rightarrow \mathbb{P}SL_2(\mathbb{Z})$ . Making the identification of the fibre  $F_b$  with the quotient  $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$  before and after going around the loop, we see that  $j_{S*}$  can be thought of as the monodromy of the period  $\mathbb{Z} \oplus \mathbb{Z}\tau$ . But it is slightly cruder than the monodromy since it does not distinguish  $\pm\tau$ . The discussion gives the following commutative diagram

$$\begin{array}{ccc} \pi_1(U) & \xrightarrow{G_S} & SL_2(\mathbb{Z}) \\ j_{S*} \downarrow & & \downarrow \\ \mathbb{P}SL_2(\mathbb{Z}) & \xlongequal{\quad} & \mathbb{P}SL_2(\mathbb{Z}). \end{array}$$

**Definition 2.1.6.** Let  $L : \pi_1(U) \rightarrow SL_2(\mathbb{Z})$  be an equivalence class of a representation, and  $h : U \rightarrow \mathbb{C}$  be a meromorphic function with  $h(u) \neq 0, 1, \infty$  for  $u \in U$ . We say  $L$  belongs to  $h$  if it induces the representation defined by

$h$ .

Kodaira has the following result about the classification of elliptic fibrations without multiple fibres.

**Theorem 2.1.7.** (Kodaira [19]) *Let  $B$  be a Riemann surface,  $U = B \setminus \{p_1, \dots, p_k\}$ , and  $h$  be a meromorphic function on  $B$  with  $h(u) \neq 0, 1, \infty$  for  $u \in U$ .*

1. *If  $k \geq 1$ , then there are exactly  $2^{g(B)+k-1}$  inequivalent homological invariants  $L$  that belong to  $h$ , where  $g(B)$  is the genus of the curve  $B$ .*
2. *Given  $h$  and a homological invariant  $L$  belonging to  $h$ , there exists a unique minimal elliptic fibration  $f : S \rightarrow B$  with these invariants, admitting a section.*
3. *Let  $\mathcal{F}(h, L)$  denote the set of all elliptic fibrations, without multiple fibres, with given invariants  $h$  and  $L$ . Given  $S'$  an element of  $\mathcal{F}(h, L)$ , there exist a covering  $B = \cup V_i$  with  $f_i : S_i := f^{-1}(V_i) \rightarrow V_i$  being the restriction of  $f$ , and a cocycle  $\xi_{ij} \in H^1(\mathcal{T})$ , where  $\mathcal{T}$  is the sheaf of local holomorphic sections of the unique elliptic surface in (2), such that  $S'$  is obtained by gluing the collections of  $S_i$  together using the  $\xi_{ij}$ . In particular, the set  $\mathcal{F}(h, L)$  is parameterized by the abelian group  $H^1(\mathcal{T})$ .*

**Definition 2.1.8.** *The unique minimal elliptic surface  $S^{\mathfrak{B}}$ , which admits the invariants  $j_S$  and  $G_S$  of  $\pi : S \rightarrow B$  and a section, is called the basic elliptic surface associated with  $S$ .*

From Theorem 2.1.7, we see that an elliptic surface without multiple fibres is locally isomorphic to its basic elliptic surface.

## 2.2 Elliptic fibre bundles

A special case of elliptic surfaces is the total space of a holomorphic elliptic fibre bundle  $\pi : S \rightarrow B$  which is locally trivial. Let  $E$  be an elliptic curve. A holomorphic elliptic fibre bundle  $S \xrightarrow{\pi} B$  is determined by the associated class  $\xi \in H^1(B, \mathcal{A}_B)$  where  $\mathcal{A}_B$  is the sheaf of germs of local holomorphic maps from  $B$  to  $\text{Aut}(E)$ .

**Definition 2.2.1.** *Let  $E$  be an elliptic curve. The  $E$ -fibre bundle  $S \xrightarrow{\pi} B$  is called a principal bundle if the structure group can be reduced to  $E$ .*

Let  $S \xrightarrow{\pi} B$  be a principal  $E$ -bundle with the associated class  $\xi \in H^1(B, \mathcal{E}_B)$ , where  $\mathcal{E}_B$  is the sheaf of germs of local holomorphic maps from  $B$  to  $E$ . The long exact sequence induced by the universal covering sequence is written as

$$H^1(B, \Gamma) \rightarrow H^1(B, \mathcal{O}_B) \rightarrow H^1(B, \mathcal{E}_B) \xrightarrow{c} H^2(B, \Gamma) \rightarrow 0,$$

where  $\Gamma$  is the lattice such that  $\mathbb{C}/\Gamma = E$ .

**Lemma 2.2.2.** ([5]) *Let  $E$  be an elliptic curve and  $E_B$  be the sheaf of the germs of locally constant map  $B \rightarrow E$ . A principal  $E$ -bundle with class  $\xi \in H^1(B, \mathcal{E}_B)$  can be defined by a locally constant cocycle in  $H^1(B, E_B)$  if and only if  $c(\xi) = 0$ .*

*Proof.* Let  $\Gamma$  be the period of  $E$ . That is,  $E = \mathbb{C}/\Gamma$ . The diagram of sheaves

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{C}_B & \longrightarrow & E_B \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \Gamma & \longrightarrow & \mathcal{O}_B & \longrightarrow & \mathcal{E}_B \longrightarrow 0 \end{array}$$



induces the commutative diagram in cohomology

$$\begin{array}{ccccccc}
H^1(B, \mathbb{C}) & \longrightarrow & H^1(B, E) & \xrightarrow{\gamma} & H^2(B, \Gamma) & \longrightarrow & H^2(B, \mathbb{C}) \\
\downarrow & & \downarrow & & \parallel & & \\
H^1(B, \mathcal{O}_B) & \longrightarrow & H^1(B, \mathcal{E}_B) & \xrightarrow{c} & H^2(B, \Gamma) & & 
\end{array}$$

Since  $H^2(B, \Gamma) = H^2(B, \mathbb{Z}) \otimes \Gamma$ , and  $H^2(B, \mathbb{Z})$  is free of torsion, we know the map  $\gamma$  is a zero map, and  $H^1(B, \mathbb{C}) \rightarrow H^1(B, E)$  is surjective. Moreover, the map  $H^1(B, \mathbb{C}) \rightarrow H^1(B, \mathcal{O}_B)$  is given by projecting onto the (0,2)-factor of  $H^1(B, \mathbb{C})$  in the Hodge decomposition, and is therefore surjective. Chasing the diagram in cohomology therefore gives us the lemma.  $\square$

**Proposition 2.2.3.** [5] *Let  $E$  be an elliptic curve. Every  $E$ -principal bundle admits a holomorphic  $\mathbb{C}^*$ -bundle as a unramified covering.*

*Proof.* Let  $\Gamma$  be the period of  $E$ . The diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{2\pi\sqrt{-1}} & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^* \longrightarrow 0 \\
& & \downarrow i & & \parallel & & \downarrow q \\
0 & \longrightarrow & \Gamma & \longrightarrow & \mathbb{C} & \longrightarrow & E \longrightarrow 0,
\end{array}$$

where  $q$  is the quotient with respect to some subgroup  $\langle \tau \rangle$  of  $\mathbb{C}^*$ , induces the diagram in the cohomology

$$\begin{array}{ccccccc}
H^1(B, \mathcal{O}_B) & \longrightarrow & H^1(B, \mathcal{O}_B^*) & \longrightarrow & H^2(B, \mathbb{Z}) & \longrightarrow & 0 \\
\parallel & & \downarrow H^1(q) & & \downarrow & & \\
H^1(B, \mathcal{O}_B) & \longrightarrow & H^1(B, \mathcal{E}_B) & \xrightarrow{c} & H^2(B, \Gamma) & \longrightarrow & 0.
\end{array}$$

Let  $\xi \in H^1(B, \mathcal{E}_B)$  be the cocycle class defining the  $E$ -principal bundle. Since

$H^2(B, \Gamma) = H^2(B, \mathbb{Z}) \otimes \Gamma$ , there exists a primitive embedding  $\mathbb{Z} \rightarrow \Gamma$  such that  $c(\xi)$  is in the image of the map  $H^2(B, \mathbb{Z}) \rightarrow H^2(B, \Gamma)$ . Chasing this cohomology diagram gives us the existence of  $\eta \in H^1(B, \mathcal{O}_B^*)$  such that  $\xi = H^1(q)\eta$ . That is, the  $E$ -principal bundle defined by  $\xi$  is a  $\mathbb{Z}$ -quotient of a holomorphic  $\mathbb{C}^*$ -bundle defined by  $\eta \in H^1(\mathcal{O}_B)$  where the  $\mathbb{Z}$ -action is generated by the multiplication with  $\tau$ .  $\square$

The topological version of  $H^1(B, \mathcal{O}_B) \rightarrow H^1(B, \mathcal{E}_B) \rightarrow H^2(B, \Gamma) \rightarrow 0$  tells us that topologically principal  $E$ -bundles are classified by the associated Chern class  $c(\xi) \in H^2(B, \Gamma)$ . The Chern class  $c(\xi)$  vanishes if and only if the  $E$ -bundle  $S \rightarrow B$  is topologically trivial. In this case, the first Betti number  $b_1(S) = b_1(B) + 2$  is even. On the other hand, if  $c(\xi) \neq 0$ , the splitting  $E = S^1 \times S^1$  would imply the existence of an  $S^1$ -bundle  $X$  over  $B$  such that  $S = X \times S^1$ . Therefore the Gysin sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(B, \mathbb{Z}) & \longrightarrow & H^1(X, \mathbb{Z}) & \longrightarrow & H^0(B, \mathbb{Z}) \\ & & \searrow^{\delta} & & \searrow & & \\ & & H^2(B, \mathbb{Z}) & \longrightarrow & H^2(X, \mathbb{Z}) & \longrightarrow & H^1(B, \mathbb{Z}) \longrightarrow 0 \end{array}$$

tells us that  $b_1(S) = b_1(B) + 1$  is odd, where  $\delta$  is the multiplication with the Chern class  $c(\xi)$ .

**Proposition 2.2.4.** *Let  $E$  be an elliptic curve,  $\pi : S \rightarrow B$  be a principal  $E$ -bundle. Then  $S$  admits cscK metrics if and only if the first Betti number  $b_1$  of  $S$  is even.*

*Proof.* The only if part is an immediate consequence of the Hodge decomposition for a compact Kähler manifold. To prove the if part, first recall that Lemma 2.2.2 tells us that the surface  $S$  is defined over some open covering

$\{U_i\}$  of  $B$  by patching the pieces  $U_i \times E$  together according to some locally constant cocycle in  $H^1(B, E)$ . Endow the flat metric on  $E$ . Fix a cscK metric on the curve  $B$  (Depending on the genus of  $B$ , it is either the Fubini-Study, flat, or hyperbolic metric.), and restrict it to each open set  $U_i$ . Then we can see that the product metrics on all pieces  $U_i \times E$  are respected by the patching procedure since the cocycle is locally constant. Therefore we obtain a globally defined cscK metric on the surface  $S$ .  $\square$

In fact, we have  $H^1(B, E) = \text{Hom}(H_1(B, \mathbb{Z}), E) = \text{Hom}(\pi_1(B), E)$  from the universal coefficient theorem. Let  $\xi \in H^1(B, E)$  be the locally constant cocycle which determines the principal  $E$ -bundle  $S \rightarrow B$ . Regard  $\xi$  as a homomorphism from  $\pi_1(B)$  to  $E$ , which acts on  $\mathbb{H} \times E$ . Then the principal  $E$ -bundle  $S$  is obtained from the quotient of  $\mathbb{H} \times E$  by the action of  $\xi$ . In other words, the universal cover of every principal  $E$ -bundle with even first Betti number is  $\mathbb{H} \times \mathbb{C}$  and each deck transformation can be expressed by  $(z, w) \mapsto (\alpha(z), w + t)$  where  $\alpha(z) \in \text{PSL}_2(\mathbb{C})$ , and  $t$  is a constant.

**Corollary 2.2.5.** *Let  $\pi : S \rightarrow B$  be an elliptic fibre bundle with even first Betti number. If the functional invariant  $j_S$  is constant and the homological invariant  $G_S$  is trivial, then there exists an elliptic curve  $E$  such that  $\pi : S \rightarrow B$  is a principal  $E$ -bundle. In particular,  $S$  admits a cscK metric.*

*Proof.* Since  $S$  has no singular fibre and  $j_S$  is constant, all fibres are isomorphic to some elliptic curve  $E$ . Since  $J_S$  is constant and  $G_S$  is trivial, the basic elliptic surface of  $S$  is the product  $B \times E$ . Using Theorem 2.1.7, the surface  $S$  is obtained from  $B \times E$  by twisting according to some cocycle  $\xi \in H^1(B, \mathcal{E}_B)$ . Therefore  $S$  is a principal  $E$ -bundle with even Betti number. The corollary

follows then from Proposition 2.2.4. □

**Theorem 2.2.6.** *Let  $\pi : S \rightarrow B$  be an elliptic fibration whose each fibre is smooth and whose first Betti number  $b_1$  is even. Assume that the genus of  $B$  is at least 2. Then there exists a principal  $E$ -bundle  $S' \rightarrow B$  which is a fibre-preserving étale covering of  $S \rightarrow B$ .*

*Proof.* Since  $S$  has no singular fibres, the  $j$ -invariant  $j_S$  can not take the value of  $\infty$ , therefore it is constant, and every fibre  $E$  is isomorphic. Since  $G_S$  belongs to  $J_S$ , the homological invariant is equivalent to the monodromy representation  $G_S : \pi_1(B) \rightarrow \mathbb{Z}_m$ . (Here the value of  $m$  depends on the value of  $j_S$ : if  $j_S$  is 0, then  $m=2, 3$ , or  $6$ ; if  $j_S$  is 1, then  $m=4$ , or  $6$ ; if  $j_S$  is  $\neq 0, 1$ , then  $m=2$  or  $4$ ; otherwise,  $m=2$ .) Therefore there exists a connected unramified cover  $B' \rightarrow B$  of degree  $m$  such that the pullback  $S' = B' \times_B S$  is an elliptic surface over  $B'$  with fibre  $E$ , constant  $j_{S'}$ , trivial  $G_{S'}$ , and even Betti number. From Corollary 2.2.5,  $S'$  is obtained by twisting  $B' \times E$  according to some constant cocycle  $\xi' \in H^1(B', E)$ . Therefore  $S' \rightarrow B'$  is a principal  $E$ -bundle and by construction, it is an étale  $m$ -cover of  $S$  and it is locally isomorphic to every fibre of  $S \rightarrow B$ . The theorem follows. □

## 2.3 Elliptic surfaces whose each fibre has a smooth reduction

One crucial tool in studying the elliptic surfaces is Kodaira's formula (see

[5] p.161) of the canonical divisor of an elliptic surface  $\pi : S \rightarrow B$ :

$$K_S = \pi^*(K_B + D) + \sum (m_i - 1)F_i$$

where the  $F_i$ 's are the multiple fibres of multiplicity  $m_i$  and  $D$  is some divisor of  $B$  with  $\deg D = \chi(\mathcal{O}_S)$ . The formula implies that  $K_S^2 = 0$ . Usually we assume  $B$  is an orbifold with orbifold points  $P_i$  of order  $m_i$  corresponding to a fibre  $F_i$  of multiplicity  $m_i$ . Let  $\tau_S = \chi(\mathcal{O}_S) - \chi^{orb}(B)$ , where  $\chi^{orb}(B)$  is the orbifold Euler number defined by

$$\chi^{orb}(B) := \chi^{top}(B) - \sum_{j=1}^k \left(1 - \frac{1}{m_j}\right).$$

Using that the plurigenera  $P_m(S) = h^0(S, K^{\otimes m}) = m\tau_S$  when  $m$  is divisible by  $m_i$  for all  $i$  and some extra thoughts, Wall [40] shows that the sign of  $\tau_S$  determines the Kodaira dimension of an elliptic surface.

**Lemma 2.3.1.** (*Wall [40]*) *Let  $\pi : S \rightarrow B$  be an elliptic fibration and  $\tau_S$  be defined as above. Then the Kodaira dimension  $\kappa$  of  $S$  is  $-\infty, 0$ , or  $1$  corresponding to  $\tau_S < 0$ ,  $\tau_S = 0$ , or  $\tau_S > 0$ , respectively.*

*Proof.* See [40] Lemma 7.1. □

In the following, we are interested in elliptic surfaces of Kodaira dimension  $\kappa = 1$ , which are sometimes called properly elliptic surfaces. We will exploit some properties of properly elliptic surfaces without singular fibres, and use them later to construct elliptic surfaces with cscK metrics. Let us start with a feature of the base of an elliptic fibration.

**Definition 2.3.2.** *An orbifold Riemann surface is called good if its orbifold universal cover admits a cscK metric.*

**Lemma 2.3.3.** *Let  $S$  be a minimal elliptic surface with  $\kappa(S) = 1$  and without singular fibres. Then the base  $B$  is a good orbifold.*

*Proof.* Using Noether's formula,  $\chi(\mathcal{O}_S) = 1/12(K_S^2 + \chi) = 1/12\chi = 0$ . By Lemma 2.3.1, we have  $\chi^{orb}(B) = -\tau_S < 0$ . The lemma follows from Troyanov's argument [39] that an orbifold Riemann surface  $B$  is always good if  $\chi^{orb}(B) \leq 0$ .  $\square$

Smooth elliptic fibration without multiple fibres are completely classified by Kodaira [19]. If there are multiple fibres, we need some more work to reduce it to a smooth fibration:

**Lemma 2.3.4. (local version)** *Let  $\pi : S \rightarrow \Delta$  be an elliptic fibration over the unit disk  $\Delta$ , and the fibre  $F_0 = \pi^{-1}(0)$  is a multiple fibre with multiplicity  $n$ . Let  $\delta_n : \Delta \rightarrow \Delta$  be the map  $z \rightarrow z^n$ ,  $\bar{S} = \Delta \times_{\Delta} S$  be the fibre product with respect to  $\delta_n$ , and  $S'$  be the normalisation of  $\bar{S}$ . Then  $S'$  is a nonsingular surface with no multiple fibre and  $S' \rightarrow S$  is an unramified cover.*

*Proof.* Let  $x, y$  be the local coordinate on  $S$  such that  $\pi(x, y) = x^n$ . Then the fibre product  $\bar{S}$  can be expressed as

$$\begin{aligned} U \times_{\Delta} \Delta &= \{(x, y, z) | x^n = z^n\} \\ &= \bigcup_{\zeta \in \mu_n} U_{\zeta}, \end{aligned}$$

where  $U_{\zeta} = \{(x, y, \zeta x) | (x, y) \in U\} \cong U$ , and  $\mu_n$  is the group of the  $n$ -th root of unity. The normalisation  $S'$  locally is the disjoint union of the  $n$

components  $U_\zeta$ , and therefore is smooth. Identifying  $U_\zeta$  with  $U$ , we can define  $\pi' : S' \rightarrow \Delta$  by  $(x, y) \mapsto \zeta x$  on  $U_\zeta$ , which has no multiple fibres. Moreover, the group  $\mu_n$  acts on  $S'$  freely by interchanging these components  $U_\zeta$ , and we have  $S'/u_n = S$ .  $\square$

**Definition 2.3.5.** *The fibration  $\pi' : S' \rightarrow \Delta$  constructed in the above proof is called the  $n$ -th root fibration of  $\pi$ .*

**Lemma 2.3.6.** *([6] Lemma 6.7) Let  $\pi : S \rightarrow B$  be an elliptic surface whose base  $B$  is a good orbifold and whose fibres are either smooth or multiples of smooth elliptic curves. Then there exists a branched Galois cover  $p_1 : B' \rightarrow B$  with Galois group  $G$ , a surface  $S'$  and a commutative diagram*

$$\begin{array}{ccc} S' & \longrightarrow & S \\ \downarrow \pi' & & \downarrow \pi \\ B' & \xrightarrow{p_1} & B \end{array}$$

*such that the action of  $G$  on  $B'$  lifts to  $S'$ ,  $p_1$  induces an isomorphism  $S'/G \rightarrow S$ , and every fibre of  $\pi'$  is smooth.*

*Proof.* Since  $B$  is a good 2-orbifold Riemann surface, there exists a smooth Riemann surface  $B'$ , and a branched covering map  $p_1 : B' \rightarrow B$  with the group of deck transformation  $G$ . We can assume  $p_1$  is a Galois cover. (If it is not, then  $\pi_1(B')$  is not a normal subgroup of  $\pi_1(B)$ . We can then take the intersection  $H$  of all conjugate groups of  $\pi_1(B')$  in  $\pi_1(B)$ , which is a subgroup of  $\pi_1(B')$  of finite index and is normal in  $\pi_1(B)$ . Therefore there exists a further finite cover  $B'' \rightarrow B' \rightarrow B$  such that the group of deck transformations of  $B''$  over  $B$  is  $H$ , which acts transitively on the preimage of any point on  $B$ .) Now let

$p_1 : B' \rightarrow B$  be a Golois cover with the Golois group  $G$ . We have  $B'/G = B$ . Consider the pull-back  $\bar{S} := B' \times_B S$  which has singularities at the pull-back of the multiple fibres. Let  $\Sigma \subseteq B$  be the set of multiple points. Outside the fibres over  $\Sigma$ , there is an étale cover

$$\bar{S}_1 := \bar{S} \setminus \bigcup_{b \in \Sigma} \pi'^{-1}(p_1^{-1}(b)) \longrightarrow S_1 := S \setminus \bigcup_{b \in \Sigma} \pi^{-1}(b),$$

and the action of  $G$  lifted to  $\bar{S}_1$  acts freely such that  $\bar{S}_1/G = S_1$ . Let  $S'$  be the normalisation of  $\bar{S}$ . From Lemma 2.3.4,  $S'$  is smooth, and the action of  $G$  lifted to  $\bar{S}'$  acts on  $S'$  freely such that  $S'/G = S$ .  $\square$

**Theorem 2.3.7.** *Let  $S$  be a minimal properly elliptic surface of Kähler type whose fibres are either smooth or multiples of smooth elliptic curves. Then the universal cover is biholomorphic to  $\mathbb{H} \times \mathbb{C}$ , and  $S$  inherits a cscK metric from  $\mathbb{H} \times \mathbb{C}$ .*

*Proof.* (cf. [40]) Let  $\pi : S \rightarrow B$  be the elliptic fibration. By Lemma 2.3.6, there exists a smooth finite cover  $B'$ , and a branched Golois covering map  $p_1 : B' \rightarrow B$  with Golois group  $G$ . Lift the action of  $G$  to the normalization  $S'$  of the pullback  $B' \times_B S$ , which is a smooth elliptic fibration over  $B'$ . Then we can see that  $G$  acts freely on  $S'$  and the quotient  $S'/G = S$ . From Theorem 2.2.6, there is an étale cover  $B''$  of  $B'$  such that the pull-back surface  $S''$  is a principal  $E$ -bundle with even first Betti number. That is,  $S''$  is obtained by twisting the trivial bundle  $B'' \times E$  by a locally constant cocycle in  $H^1(B'', E)$ . We can then see  $\mathbb{H} \times \mathbb{C}$  is the universal cover of  $S'', S'$ , and  $S$ . And the action



of every deck transformation  $\gamma$  of  $\mathbb{H} \times \mathbb{C}$  over  $S''$  can be written explicitly as

$$\gamma(z, w) = (\alpha_\gamma(z), w + t_\gamma),$$

where  $\alpha_\gamma \in \mathrm{PSL}_2(\mathbb{C})$ ,  $t_\gamma \in \mathbb{C}$ . So far, we have the following diagram

$$\begin{array}{ccccc} \mathbb{H} \times \mathbb{C} & \longrightarrow & S'' & \longrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{H} & \longrightarrow & B'' & \longrightarrow & B \end{array}$$

Without loss of generality, we can assume  $S''$  is a normal cover of  $S$ . Let  $H$  be the group of deck transformations of  $\mathbb{H} \times \mathbb{C}$  over  $S$ , and  $\Gamma$  be the group of deck transformations of  $\mathbb{H} \times \mathbb{C}$  over  $S''$ . The action of  $\Gamma$  on  $\mathbb{H} \times \mathbb{C}$  descends to a discrete group  $\bar{\Gamma}$  acting on the base  $\mathbb{H}$  and the quotient  $\mathbb{H}/\bar{\Gamma} = B''$  is the base of  $S''$ . By Theorem 2.2.4, we have that the group  $\Gamma$  is a subgroup of  $\mathrm{PSL}_2 \times \mathbb{C}$  and  $S''$  inherits the product metric from  $\mathbb{H} \times \mathbb{C}$ . To show  $S$  inherits the product metric from  $\mathbb{H} \times \mathbb{C}$ , it suffices to show that every deck transformation  $h \in H$  preserves the product metric. Given  $h \in H$ , since all fibres of  $S''$  are isomorphic, and the group of deck transformations of  $S''$  over  $S$  preserves the elliptic structure, we can write the action of  $h \in H$  on  $\mathbb{H} \times \mathbb{C}$  as

$$h(z, w) = (\alpha_h(z), \epsilon_h w + a_h(z)),$$

where  $\epsilon_h$  is the  $k$ -th root of unity for  $k = 2, 4$ , or  $6$ . For a covering transformation  $h$ ,  $\epsilon_h$  is constant. A direct computation shows that

$$\begin{aligned}
h^{-1}(z, w) &= (\alpha_h^{-1}(z), \epsilon_h^{-1}w - \epsilon_h^{-1}a_h(\alpha_h^{-1}(z))), \\
h\gamma h^{-1}(z, w) &= (\alpha_h\alpha_\gamma\alpha_{h^{-1}}(z), w - a_h(\alpha_{h^{-1}}(z)) + \epsilon_h t_\gamma + a_h(\alpha_\gamma\alpha_{h^{-1}}(z))).
\end{aligned}$$

It follows that  $a_h(\alpha_\gamma(z)) - a_h(z) = t_{h\gamma h^{-1}} - \epsilon_h t_\gamma$  is a constant. For a fixed  $h$ , the cocycle  $\{t_{h\gamma h^{-1}} - \epsilon_h t_\gamma\} \in \text{Hom}(\pi_1(B''), \mathbb{C}) \cong H^1(B'', \mathbb{C})$  defines a principal  $\mathbb{C}$ -bundle  $X$  over  $B''$ , for which  $a_h(z)$  defines a section. Therefore the principle  $\mathbb{C}$ -bundle is trivial, and  $t_{h\gamma h^{-1}} - \epsilon_h t_\gamma = 0$ . It follows that the holomorphic function  $a_h(z)$  is constant on the orbits of  $\bar{\Gamma}$ , and can be regarded as a holomorphic function of the compact surface  $B''$ . Thus  $a_h(z)$  is constant. This concludes the proof.  $\square$

## 2.4 Logarithmic transformation

Logarithmic transformation is an important operation introduced by Kodaira [19]. It enables us to replace a smooth fibre in an elliptic fibration by a multiple fibre. (For more details see [16], Chap. 4.5). Let  $\pi : S \rightarrow B$  be an elliptic fibration over a smooth curve  $B$ . Let  $U \subset B$  be an open neighborhood of  $p \in B$ ,  $E_0$  be the smooth fiber over  $p$ . Choose some local coordinate  $z$  with  $z(0) = p$ . Denote  $\Sigma := \pi^{-1}(U)$ . For every  $m \in \mathbb{N}$ , consider the diagram

$$\begin{array}{ccc}
\tilde{\Sigma} & \longrightarrow & \Sigma \\
\downarrow \tilde{\pi} & & \downarrow \pi \\
\tilde{U} & \xrightarrow{\phi} & U
\end{array}$$

where  $\phi$  is a cyclic cover of degree  $m$  given by  $\tilde{z} \mapsto \tilde{z}^m$ , and  $\tilde{\Sigma} = \phi^*\Sigma =$

$\tilde{U} \times_U \Sigma$ . Let  $\Lambda(z)$  be a family of lattices such that  $\Sigma = U \times \mathbb{C}/\Lambda(z)$ , then  $\tilde{\Sigma} = \tilde{U} \times \mathbb{C}/\Lambda(\tilde{z}^m)$ . Let  $\beta(z)$  be a local  $m$ -torsion section of  $\Sigma \rightarrow U$  and  $G \in \text{Aut}(\tilde{\Sigma})$  be the cyclic group generated by

$$(\tilde{z}, t) \mapsto (e^{\frac{2\pi\sqrt{-1}}{m}} \tilde{z}, t + \beta(\tilde{z}^m)) \mod \Lambda(\tilde{z}^m).$$

Denote  $\Sigma' := \tilde{\Sigma}/G$ . We then have an isomorphism  $\alpha : \Sigma' \setminus (\phi^* E_0/G) \cong \Sigma \setminus E_0$ , where  $E_0 = \pi^{-1}(p)$ . By setting  $S' = (S \setminus E_0) \cup_{\alpha} \Sigma'$ , we get an elliptic fibration  $S' \rightarrow B$ , which is isomorphic to  $S$  away from  $E_0$  and has a multiple fibre with reduction isomorphic to  $E_0/G$  of multiplicity  $m$  over  $p$ .

Since we simply remove  $T^2 \times D^2 (\approx \Sigma)$  and glue it back in (as  $\Sigma'$ ) with a different fibration, we can reformulate the logarithmic transformation in the following way: Consider an elliptic surface  $\pi : S \rightarrow C$  and fix a generic fibre  $\pi^{-1}(t) = F$ . We denote a closed tubular neighborhood of the fibre  $F$  in  $S$  by  $\bar{\Sigma}$  with the interior  $\Sigma$ ;  $\bar{\Sigma}$  is diffeomorphic to  $T^2 \times D^2$ . Deleting the interior  $\Sigma$  from  $S$  and regluing  $T^2 \times D^2$  via a smooth map  $\phi : T^2 \times S^1 \rightarrow \text{Boundary}(S \setminus \Sigma)$  with multiplicity  $m$ , we get a new manifold  $S'$ . The diffeomorphism type of the resulting manifolds depends on the multiplicity of  $m$  of the map  $\phi$ .

A remark here is that the process of logarithmic transform is quite violent. A non-algebraic surface may be obtained from an algebraic one by a logarithmic transform, and vice versa.

## Chapter 3

### A construction of elliptic surfaces of cscK metrics

In this chapter, we construct elliptic surfaces with cscK metrics.

#### 3.1 Arezzo-Pacard Theorem

The main tool we use here is the remarkable theorem 3.1.1, 3.1.2 by Arezzo-Pacard [1]. Kronheimer [21] has shown that there exist asymptotically locally Euclidean resolutions of singularities of the type  $\mathbb{C}^2/\Gamma$ , where  $\Gamma$  is a finite subgroup of  $SU(2)$ . Arezzo and Pacard [1] glue this resolution to an orbifold with isolated singularities of local model  $\mathbb{C}^2/\Gamma$ , which has no nontrivial holomorphic vector fields, study the partial differential equations arising from the perturbation of the Kähler forms suggested in [24], and they succeed constructing cscK metrics on the resulting desingularisation. This is a major breakthrough in the construction of manifolds with cscK metrics, and we state their results

here.

**Theorem 3.1.1.** (Arezzo, Pacard [1]) *Let  $M$  be a compact cscK manifold. Assume there are no nontrivial holomorphic vector fields that vanish somewhere on  $M$ . Then, given finitely many points  $p_1, p_2, \dots, p_n$  on  $M$ , the blow-up of  $M$  at  $p_1, p_2, \dots, p_n$  carries a cscK metric.*

**Theorem 3.1.2.** (Arezzo, Pacard [1]) *Let  $M$  be a compact 2-dimensional cscK orbifold with isolated singularities. Assume there are no nontrivial holomorphic vector fields on  $M$ . Let  $p_1, p_2, \dots, p_n$  be finitely many points on  $M$ , each of which has a neighborhood biholomorphic to  $\mathbb{C}^2/\Gamma_j$ , where  $\Gamma_j$  is a finite subgroup of  $SU(2)$ . Then the minimal resolution of  $M$  at  $p_1, p_2, \dots, p_n$  carries cscK metrics.*

The first result we obtain by applying Arezzo-Pacard Theorem 3.1.1, 3.1.2 is that an elliptic surface obtained by blowing up a minimal properly elliptic surface of Kähler type, which has no singular fibres, at finitely many points admits cscK metrics.

**Proposition 3.1.3.** *Let  $S$  be a minimal properly elliptic surface of Kähler type, which has no singular fibres. Then every nontrivial holomorphic vector field on  $S$  has no zeros.*

*Proof.* By Lemma 2.3.3, the assumption implies the base  $B$  is a good orbifold. Let  $\mathfrak{h}(S)$  be the algebra of holomorphic vector fields. Let  $\zeta \in \mathfrak{h}(S)$  be nontrivial. Since the base  $B$  is a 2-orbifold with  $\chi^{\text{orb}}(B) < 0$ ,  $\zeta$  is vertical. By Theorem 2.3.7,  $\mathbb{H} \times \mathbb{C}$  is the universal cover of  $S$ . Thus  $\zeta$  can be lifted to a holomorphic vector field  $\tilde{\zeta} = f(z) \frac{\partial}{\partial w}$  on  $\mathbb{H} \times \mathbb{C}$ , where  $f$  is a holomorphic

function on  $\mathbb{H}$ ,  $z$  is the coordinate of  $\mathbb{H}$ , and  $w$  is the coordinate on  $\mathbb{C}$ . Since  $f$  is invariant under the action of  $\pi_1(B)$ , it defines a holomorphic function on  $B$ , and is constant. Therefore we conclude that  $\zeta$  is induced by  $c \frac{\partial}{\partial w}$ , which is everywhere nonzero.  $\square$

**Corollary 3.1.4.** *Let  $S$  be a minimal properly elliptic surface of Kähler type, which has no singular fibres. Then every surface obtained by blowing up  $S$  at finitely many points admits cscK metrics.*

*Proof.* The corollary follows from Arezzo-Pacard Theorem 3.1.1 and Proposition 3.1.3.  $\square$

## 3.2 The construction of elliptic surfaces with cscK metrics

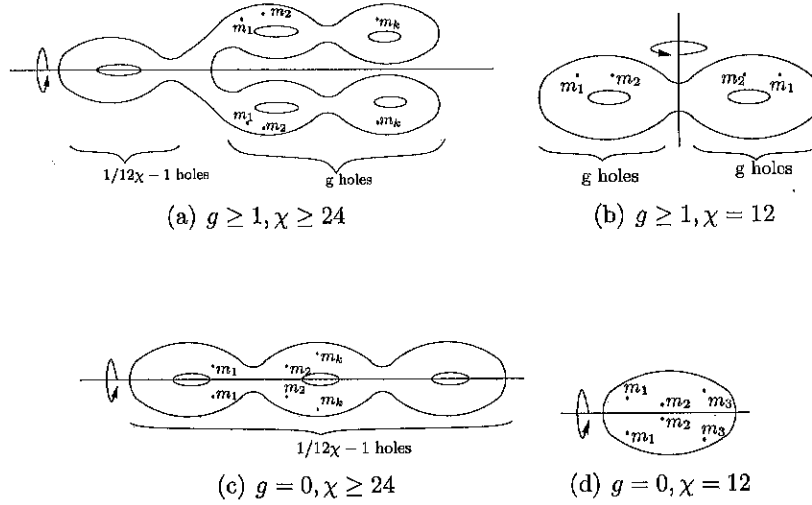
**Lemma 3.2.1.** *Let  $\chi$  be a positive multiple of 12,  $g$  be a nonnegative integer. Then there exists a smooth compact Riemann surface  $(\Sigma_{g'}, J)$  of genus  $g' := \frac{1}{12}\chi + 2g - 1$ , which admits an isometric holomorphic involution  $\psi$  with  $\frac{1}{6}\chi$  fixed points.*

*Proof.* When  $g'$  is at least 2, such a surface  $(\Sigma_{g'}, J, \psi)$  exists due to Thurston's pants decomposition [32] of a Riemann surface  $\Sigma_{g'}$ , which states that a hyperbolic Riemann surface  $R$  of genus  $l$  always contains  $3l - 3$  simple closed geodesics such that cutting  $R$  along these geodesics decomposes  $R$  into  $2g - 2$  pairs of pants, and the fact that for any pair of pants, the length of boundaries

provide the coordinates for Teichmüller space, and can be chosen arbitrarily. When  $g' = 0, 1$ , this involution  $\psi$  amounts to a  $180^\circ$  rotation of  $S^2$  about an axis, or the Weierstrass involution of a torus.  $\square$

This lemma tells us that given two numbers  $\chi = 12d > 0$ , and  $g \geq 0$ , there exists a Riemann surface of genus  $g' = \frac{1}{12}\chi + g - 1$ , and it admits a holomorphic surjective holomorphic map  $\Sigma_{g'} \rightarrow B$  of degree 2, where  $B$  is a Riemann surface of genus  $g$ .

Our goal is to construct an elliptic surface with Euler number  $\chi$ , base curve  $B$  of genus  $g$  and  $k$  multiple fibres of multiplicities  $m_1, m_2, \dots, m_k$ , respectively, where the numbers  $\chi, g, k$  and  $m_1, m_2, \dots, m_k$  are given.



Now let  $E = \mathbb{C}/\Gamma$  be a fixed elliptic curve, and  $\phi : E \rightarrow E$  be the Weierstrass involution, which is an isometry with respect to the flat metric on  $E$  with 4 fixed points. Let  $\hat{S}$  be the product  $\Sigma_{g'} \times E$ . Choose arbitrarily  $k$  points  $P_1, P_2, \dots, P_k$  on  $\Sigma_{g'}$  outside the fixed points of  $\Psi$ . Let  $Q_i := \psi(P_i) \in \Sigma_{g'}$ .

Apply the logarithmic transformation of order  $m_i = n_i$  at these  $2k$  points  $P_i, Q_i$ , where  $i = 1, \dots, k$ . Let  $S$  be the resulting surface. The involution  $\psi \times \phi$  on the product  $\Sigma_{g'} \times E$  can be extended to an involution  $\iota$  on  $S$ , which has  $\frac{2}{3}\chi$  fixed points.

From the construction,  $S \rightarrow \Sigma_{g'}$  is an elliptic fibration with  $2k$  multiple fibres. Now take the quotient of  $S$  by the action of  $\iota$ . The resulting surface  $S' = S/\mathbb{Z}_2$  is singular and has  $\frac{2}{3}\chi$  ordinary double points. Let  $S''$  be a minimal resolution, which is obtained by replacing each singular point by a  $(-2)$ -curve. we can see that  $S'' \rightarrow B$  is an elliptic fibration with  $k$  multiple fibres of order  $m_1, \dots, m_k$  respectively.

**Proposition 3.2.2.** *The elliptic surface  $S''$  constructed above has Euler characteristic number  $\chi$ .*

*Proof.* Recall from the construction that the  $\frac{2}{3}\chi$  double points come from the fixed points of  $\iota$ , and they are locally modeled by  $\mathbb{C}^2/\mathbb{Z}_2$ . To resolve the singularities, we take the blow-up  $\tilde{S}$  of  $S$  at the  $\frac{2}{3}\chi$  double points, and extend the map  $\iota$  to an automorphism  $\tilde{\iota}$  of  $\tilde{S}$ , then  $S'' = \tilde{S}/\mathbb{Z}_2$  is the nonsingular complex surface obtained by replacing each double point with a  $(-2)$ -curve. Let  $p_1 : \tilde{S} \rightarrow S''$ , and  $p_2 : \tilde{S} \rightarrow S$  be the quotient map and the blow-down, respectively.

$$\begin{array}{ccccc} S & \xleftarrow{p_2} & \tilde{S} & \xrightarrow{p_1} & S'' \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma_{g'} & \xlongequal{\quad} & \Sigma_{g'} & \longrightarrow & B \end{array}$$

We claim that the irregularity  $q := h^0(S'', \Omega_{S''}^1) = g$  : If  $\eta$  is a nonzero holomorphic 1-form on  $S''$ , then  $p_1^*\eta$  is a holomorphic 1-form on  $\tilde{S}$  invariant under



$\tilde{\iota}$ . Since  $H^0(S, \Omega_S^1) \rightarrow H^0(\tilde{S}, \Omega_{\tilde{S}}^1)$  is an isomorphism by Hartog's extension theorem, there exists a 1-form  $\xi$  on  $S$  invariant under  $\iota$ , such that  $p_2^*(\xi) = p_1^*(\eta)$ . Since  $S$  is an elliptic surface without singular fibres, its universal cover is  $\mathbb{H} \times \mathbb{C}$ , and we know that every holomorphic 1-form on  $S$  comes from either the base  $\Sigma_{g'}$  or the fibre. However, since  $\phi^*dz = -dz$  on the elliptic curve  $E$ , the only 1-forms on  $S$  invariant under  $\iota$  are induced from holomorphic 1-forms of  $\Sigma_{g'}$  invariant under  $\psi$ . That is,  $q = \dim_{\mathbb{C}}\{\omega \in H^0(\Sigma_{g'}, \Omega^1) | \psi^*\omega = \omega\}$ . Since every holomorphic 1-form on  $\Sigma_{g'}$  invariant under the action of  $\psi$  corresponds to a holomorphic 1-form on  $\Sigma_{g'}/\psi = B$ , we conclude that  $q = g$ . Using similar argument as above, we can also show  $p_g = \frac{1}{12}\chi + 1 + g$ : the only holomorphic 2-forms on  $S$  invariant under  $\iota$  are induced from the exterior product of holomorphic 1-forms of  $E$  and those of  $\Sigma_{g'}$ . Because  $\phi^*dz = -dz$ , we can take "anti-symmetric" 1-forms on  $\Sigma_{g'}$ , i.e.  $\psi^*\omega = -\omega$ . Since every 1-form on  $S$  can be decomposed as the sum of a symmetric 1-form and an anti-symmetric one, we have  $p_g = h^0(\Sigma_{g'}, \Omega^1) - g = (\frac{1}{12}\chi + 2g - 1) - g = \frac{1}{12}\chi + g - 1$ . From Noether's formula, we can therefore compute the Euler number of  $S''$  as

$$\begin{aligned} 12\chi(\mathcal{O}_{S''}) - K_{S''}^2 &= 12(h^0(\mathcal{O}_{S''}) - h^1(\mathcal{O}_{S''}) + h^2(\mathcal{O}_{S''})) - 0 \\ &= 12(1 - g + (\frac{1}{12}\chi + g - 1)) \\ &= \chi, \end{aligned}$$

where the first equality holds since  $S''$  is minimal.  $\square$

**Proposition 3.2.3.** *Let  $S''$  be an elliptic surface constructed above. If  $S''$  has even first Betti number  $b_1$ , then  $S''$  admits cscK metrics.*

*Proof.* Since  $S''$  has even first Betti number, it admits a Kähler form  $\omega$ . Note

that the pullback form  $p_1^*\omega$  is positive semi-definite on  $\tilde{S}$ , which is degenerate on the exceptional divisors. Consider the Eguchi-Hanson metric on the total space of  $\mathcal{O}(-1)$ . By gluing the Kähler potentials carefully, we can get a Kähler form  $\tilde{\omega}$  on  $\tilde{S}$ , therefore the first Betti number  $b_1(\tilde{S})$  is even. Because blowing up at points does not change the first Betti number,  $b_1(S)$  is even too. Therefore  $S$  is a minimal properly elliptic surface of Kähler type, which has no singular fibres. By Theorem 2.3.7,  $S$  admits cscK metrics. Furthermore, the involution  $\iota : S \rightarrow S$  is an isometric automorphism. This gives  $S' = S/\iota$  a cscK orbifold metric. In view of Arezzo-Pacard Theorem 3.1.2, to show that the minimal resolution  $S''$  admits a cscK metric, it suffices to show that there are no nontrivial holomorphic vector fields on the orbifold  $S'$ : If  $\zeta'$  is a holomorphic vector field on  $S'$ , it can be lifted to a holomorphic vector field  $\zeta$  on  $S$ , which is invariant under the action of  $\iota$ , and  $\zeta$  has zeros at the fixed points of  $\iota$ . By Proposition 3.1.3,  $\mathfrak{h}(S)$  consists of only parallel holomorphic vector fields. Therefore  $\zeta$  is trivial, and so is  $\zeta'$ .  $\square$

**Proposition 3.2.4.** *Let  $S''$  be an elliptic surface constructed as above. If  $S''$  has even first Betti number  $b_1$ , then  $S''$  admits no nontrivial holomorphic vector fields. In particular, every blow-up of  $S''$  at finitely many points admits a cscK metric.*

*Proof.* Let  $\zeta \in \mathfrak{h}(S'')$  be a holomorphic vector field. The vector field  $\zeta$  restricted to the  $(-2)$ -curves is tangent to these curves, therefore  $\zeta$  can be lifted to a holomorphic vector field  $\tilde{\zeta}$  on  $\tilde{S}$ , which is invariant under  $\tilde{\iota}$ . Let  $p_2 : \tilde{S} \rightarrow S$  be the natural blow-down map. By Hartog's extension theorem, the push-forward  $p_{2*}\tilde{\zeta}$  is well-defined holomorphic vector field on  $S$  which vanishes at

all fixed points of  $\iota$ . As we have shown in the proof of Theorem 3.2.3,  $b_1(S)$  is even. Therefore,  $S$  is a Kähler type minimal elliptic surface with  $\kappa(S) = 1$  and has no singular fibres. By Proposition 3.1.3,  $p_{2*}\tilde{\zeta}$  can only be parallel, and therefore it vanishes everywhere. The last statement follows from a direct application of Arezzo-Pacard Theorem 3.1.1.  $\square$

## Chapter 4

### Compact complex surfaces and cscK metrics

In this chapter, we will show that every complex surface, which is not in the deformation class of  $\mathbb{P}_2 \# k\bar{\mathbb{P}}_2$ ,  $k = 1$  or  $2$ , with  $b_1$  even is deformation equivalent to a surface which admits cscK metrics. We will proceed with the help of classification theorem of compact complex surfaces: Let  $S$  be a complex surface, and  $K_S$  be the canonical line bundle. We can define the pluri-canonical map  $\iota_{K_S^{\otimes k}} : S \rightarrow \mathbb{P}(H^0(S, K_S^{\otimes k}))^*$ , which is a rational map, not defined at the base locus of the linear system  $|K_S^{\otimes k}|$ . The Kodaira dimension  $\kappa$  of  $S$  is defined to be the maximal dimension of the image  $\iota_{K_S^{\otimes k}}(S)$  for  $k \geq 1$ .

#### 4.1 Kodaira dimension $\kappa = 2$

A complex surface  $S$  in this case is said to be of general type. Since we have  $c_1^2(K) > 0$ , every minimal surface of general type is projective. For a minimal surface of general type,  $\iota_{K_S^{\otimes k}}$  is a globally defined map for  $k \geq 5$ , and

it is an embedding away from some smooth  $(-2)$ -rational curves. The image of these curves are isolated singular points with local structure group  $\Gamma$ , where  $\Gamma$  is a finite subgroup of  $SU_2$  (see [5]). One can get the pluri-canonical model  $X = \iota_{K^{\otimes k}}(S)$  by collapsing these  $(-2)$ -curves. If  $M$  has no  $(-2)$ -curves, it has negative first Chern class  $c_1(M)$ , and Aubin-Yau Theorem [4, 41] asserts that every manifold with negative first Chern class admits a Kähler-Einstein metric. Otherwise, Kobayashi [18] has shown that the pluri-canonical model  $X$  admits a Kähler-Einstein orbifold metric of negative scalar curvature by extending Aubin's proof of the Calabi conjecture. Along with the fact that a complex manifold of general type has no nontrivial holomorphic vector fields, a direct application of Arezzo-Pacard Theorem 3.1.1 gives the following result.

**Theorem 4.1.1.** *(Arezzo-Pacard [1]) Every compact complex surface of general type admits cscK metrics.*

*proof.* See [1] Corollary 8.3. □

## 4.2 Kodaira dimension $\kappa = 1$

Every complex surface  $S$  in this case is a properly elliptic surface. Since an elliptic curve has Euler number  $\chi = 0$ , the Euler characteristic number of an elliptic surface  $S$  is given by the sum of those of singular fibres. In particular,  $\chi(S) \geq 0$  and the equality holds if and only if there are no singular fibres.

**Definition 4.2.1.** *Two smooth compact complex manifolds  $M, N$  are said to be deformation equivalent or of the same deformation type if there exist connected*

reduced complex spaces  $\mathcal{X}$  and  $\mathcal{T}$ , and a proper holomorphic submersion  $\Phi : \mathcal{X} \rightarrow \mathcal{T}$ , together with points  $t_1, t_2 \in \mathcal{T}$  such that for each  $t \in \mathcal{T}$ ,  $M_t := \Phi^{-1}(t)$  is a compact complex submanifold, and  $\Phi^{-1}(t_1) = M$ ,  $\Phi^{-1}(t_2) = N$ .

Since we consider only the reduced spaces, an equivalent definition would be to assume that  $\mathcal{T}$  consists of finitely many irreducible components, each of which is smooth (and which can be taken to be a disk in  $\mathbb{C}$ ). Let  $S_1$  and  $S_2$  be two deformation equivalent surfaces, and let  $\tilde{S}_1$  and  $\tilde{S}_2$  be the blow-ups of  $S_1$  and  $S_2$  at  $r$  points. A straightforward argument shows that  $\tilde{S}_1$  and  $\tilde{S}_2$  are again deformation equivalent.

Now we introduce the following classification of the deformation types of elliptic surfaces with singular fibres.

**Theorem 4.2.2.** *Two elliptic surfaces with positive Euler numbers are deformation equivalent (through elliptic surfaces) if and only if they have the same Euler numbers and their base orbifolds are diffeomorphic.*

*Proof.* See [11] Chap.1 Theorem 7.6 . □

**Theorem 4.2.3.** *A properly elliptic surface of Kähler type is deformation equivalent to a compact complex surface with cscK metrics.*

*Proof.* Let  $\check{S}$  be the minimal model of  $S$ . If  $\chi(\check{S}) = 0$ , then  $\check{S}$  is a minimal properly elliptic surface of Kähler type which has no singular fibres. This is done in Theorem 3.1.3. If  $\chi(\check{S}) > 0$ , a direct application of Theorem 4.2.2 shows that  $\check{S}$  is deformation equivalent to one of the elliptic surfaces we constructed in Section 3. Therefore by Proposition 3.2.4,  $S$  admits cscK metrics. □

### 4.3 Kodaira dimension $\kappa = 0$

Minimal compact complex surfaces of Kähler type with Kodaira dimension  $\kappa = 0$  consist of Enriques surfaces, K3 surfaces, bielliptic surfaces, and Abelian surfaces. Although not all of them are projective, they all admit Kähler metrics. Moreover, each of them has a vanishing real first Chern class. A direct application of the following celebrated theorem by Yau [41] implies that every compact Kähler manifold with vanishing real first Chern class admits a Ricci-flat Kähler-Einstein metric.

**Theorem 4.3.1.** (*Yau [41]*) *Let  $M$  be a compact Kähler manifold with Kähler form  $\omega$ , and  $c_1(M)$  be its real first Chern class. Then every closed real 2-form of  $(1,1)$ -type belonging to the class  $2\pi c_1(M)$  is the Ricci form of one and only one Kähler metric in the Kähler class  $[\omega]$ .*

Since every holomorphic vector field on a Ricci-flat Kähler manifold is parallel, there is no obstruction on the application of Arezzo-Pacard Theorem 3.1.2, and we can reach the following result.

**Theorem 4.3.2.** *Let  $S$  be a compact complex surface of Kähler type and of Kodaira dimension  $\kappa = 0$ . Then  $S$  admits cscK metrics.*

### 4.4 Kodaira dimension $\kappa = -\infty$

In this case,  $S$  is called a ruled surface. A minimal ruled surface is either a geometrically ruled surface or the complex projective plane  $\mathbb{P}^2$ . If  $S$  is a geometrically ruled surface, then there exists a holomorphic rank two vector bundle  $V$  over a curve  $C$  such that  $S$  is isomorphic to the associated  $\mathbb{P}^1$ -bundle  $\mathbb{P}(V)$ . Two vector bundles  $V, V'$  over  $C$  give isomorphic ruled surfaces if and only if  $V' = V \otimes L$  for some holomorphic line bundle  $L$  over  $C$ . It follows that  $c_1(V) \bmod 2$  is a holomorphic invariant of  $S$ . In addition, given rank two vector bundles  $V$  and  $V'$  over the same curve  $C$ , if  $c_1(V) \equiv c_1(V') \bmod 2$ , then the resulting surfaces  $\mathbb{P}(V)$  and  $\mathbb{P}(V')$  are diffeomorphic. Let  $\pi : S = \mathbb{P}(V) \rightarrow C$  be the ruling,  $\sigma_0$  be the class of a holomorphic section of  $\pi$  (it always exists!), and  $f$  be the class of a fibre of  $\pi$ . We can see  $\{\sigma_0, f\}$  is a basis of  $H^2(S, \mathbb{Z})$  and  $\sigma_0^2 \equiv c_1(V) \bmod 2$ . Moreover, the intersection pairing on  $H^2(S, \mathbb{Z})$  is even if  $c_1(V) \equiv 0 \bmod 2$  and is odd if  $c_1(V) \equiv 1 \bmod 2$ . In particular, the homotopy type of  $S$ , indeed the diffeomorphism and the deformation equivalent classes are determined by  $b_1(S)$  and the pairing on  $H^2(S, \mathbb{Z})$ .

For every Riemann surface  $C$  of genus  $g \geq 2$ , we shall see that in each deformation class of geometrically ruled surfaces  $\pi : S \rightarrow C$ , there exists one which admits cscK metrics.

#### 4.4.1 $c_1(V) = 0 \bmod 2$

Let  $C$  be a Riemann surface with the hyperbolic metric,  $\pi_1(C) := \langle a_1, b_1, \dots, a_g, b_g : [a_1, b_1][a_2, b_2] \dots [a_g, b_g] = 1 \rangle$  be the fundamental group of  $C$ , and  $g^{FS}$  be the (multiple of) Fubini-Study metric with curvature 1 on the complex projective line  $\mathbb{P}_1$ . Define the representation  $\rho : \pi_1(C) \rightarrow SU(2)/Z_2$  by



$$\rho(a_1) = \rho(b_1) := \left[ \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right],$$

$$\rho(a_2) = \rho(b_2) := \left[ \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right],$$

and  $\rho(a_j) = \rho(b_j) := [id]$  for  $j \neq 1, 2$ . Then  $\pi_1(C)$  acts isometrically on  $\mathbb{H} \times \mathbb{P}^1$ , and the quotient

$$S := \mathbb{H} \times_{\rho} \mathbb{P}^1 = \mathbb{H} \times \mathbb{P}^1 / \pi_1(C)$$

inherits a Kähler metric  $g$  of constant scalar curvature  $s = -1 + 1 = 0$ .

**Proposition 4.4.1.** *The surface  $S$  defined above is deformation equivalent to the trivial product  $C \times \mathbb{P}^1$ , and it has no nontrivial holomorphic vector fields.*

*Proof.* To prove the first statement, we construct a family of surfaces defined by the following representations  $\rho_t : \pi_1(C) \rightarrow SU(2)/Z_2$ :

$$\rho_t(a_1) = \rho_t(b_1) = \left[ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \right],$$

$$\rho_t(a_2) = \rho_t(b_2) = \left[ \begin{pmatrix} \cos t & e^{it} \sin t \\ -e^{-it} \sin t & \cos t \end{pmatrix} \right],$$

and  $\rho_t(a_j) = \rho_t(b_j) = [id]$  for  $j \neq 1, 2$ . Let  $S_t = \mathbb{H} \times_{\rho_t} \mathbb{P}^1$ . This defines a family of geometrically ruled surfaces in the same deformation class with  $S_0 = C \times \mathbb{P}^1$  and  $S_{\pi/2} = S$ . To show the algebra  $\mathfrak{h}(S)$  of holomorphic vector fields consists of only the zero vector field, we first find by direct computation that the action of  $\pi_1(C)$  by  $\rho$  fixes no points of  $\mathbb{P}^1$ . Since a Riemann surface with genus  $g \geq 2$

admits no nontrivial holomorphic vector fields, every holomorphic vector field  $\xi$  on  $S$  is vertical. In particular,  $\xi$  has zeros and is nonparallel. Since  $S$  carries a scalar flat Kähler metric, the algebra  $\mathfrak{h}_0$  of nonparallel holomorphic vector fields is the complexification of the Lie algebra of nonparallel killing fields. Lift  $\xi$  to a vertical holomorphic vector field  $\tilde{\xi}$  on  $\mathbb{H} \times \mathbb{CP}^1$ , so  $\tilde{\xi} \in \mathfrak{su}_2$ , which is invariant under the action of  $\pi_1(C)$ . This action of  $\pi_1(C)$  is given by composing the representation  $\pi_1(C) \rightarrow SU_2/Z_2$  with the adjoint action of  $SU_2/Z_2$  on its Lie algebra. Since the adjoint action of  $SU_2/Z_2$  on its Lie algebra coincides with the action of  $SO_3$  on  $\mathbb{R}^3$ , every nonzero  $\pi_1(C)$ -invariant vector field  $\tilde{\xi}$  defines an invariant point on  $S^2 \cong \mathbb{P}^1$ . Therefore  $\tilde{\xi} = 0$ , and  $\xi$  is trivial.  $\square$

#### 4.4.2 $c_1(V) = 1 \bmod 2$

**Definition 4.4.2.** Let  $(M, J, \omega)$  be a Kähler manifold of complex dimension  $n$ . The slope of a holomorphic vector bundle  $E$  over  $M$  of rank  $r$  is the number

$$\mu(E) := \frac{1}{r} \int_M c_1(E) \wedge \omega^{n-1}.$$

**Definition 4.4.3.** Let  $(M, J, \omega)$  be a Kähler manifold. A holomorphic vector bundle is said to be stable if  $\mu(F) < \mu(E)$  for any proper sub-bundle  $F \subset E$ .

**Definition 4.4.4.** A vector bundle  $E$  is said to be polystable if it decomposes as a direct sum of stable vector bundles with the same slope.

In [3], Apostolov and Tonnesen-Friedman have shown that a complex geometrically ruled surface  $M = \mathbb{P}(V)$  over a Riemann surface  $C$ , where  $V$  is a holomorphic rank 2 vector bundle over  $C$ , admits cscK metrics if and only if

the bundle  $V$  is polystable.

**Theorem 4.4.5.** *Let  $C$  be a curve of genus at least 1. Let  $\mathcal{O}(p)$  be the line bundle over  $C$  associated with the divisor of a point  $p \in C$ . Then every nontrivial extension  $V$  of  $\mathcal{O}(p)$  by  $\mathcal{O}$  is stable.*

*Proof.* (See [12].) Let  $V$  be a nontrivial extension of the form

$$0 \rightarrow \mathcal{O} \rightarrow V \rightarrow \mathcal{O}(p) \rightarrow 0.$$

(It exists since  $H^1(C, \mathcal{O}(-p)) = H^0(C, \Omega^1(p)) \neq 0$  if  $C$  has genus at least 1.) The normalized degree  $\mu(V)$  is  $1/2$ . We need to show that for every subline bundle  $L$  of  $V$ ,  $\mu(L) = \deg L$  is less than or equal to zero. If the composite map  $L \rightarrow \mathcal{O}(p)$  is zero, then  $L$  is contained in  $\mathcal{O}$ , and  $\deg L \leq 0 < \mu(V)$ . Otherwise, the map  $L \rightarrow \mathcal{O}(p)$  is nonzero. Thus,  $L^{-1} \otimes \mathcal{O}(p)$  has a nonzero section, which implies  $\deg L \leq 1$ . In particular  $\deg L = 1$  if and only if  $L \cong \mathcal{O}(p)$ , and the exact sequence splits, contrary to the hypothesis. Therefore, we also have  $\deg L \leq 0 < \mu(V)$ . This completes the proof.  $\square$

**Theorem 4.4.6.** *Let  $C$  be a curve of genus at least 1. Let  $\mathcal{O}(p)$  be the line bundle over  $C$  associated with the divisor of a point  $p \in C$ . Then the projectivization  $S := \mathbb{P}(V)$  of every nontrivial extension  $V$  of  $\mathcal{O}(p)$  by  $\mathcal{O}$  admits no nontrivial holomorphic vector fields which vanish somewhere on  $S$ .*

*Proof.* First consider the case that  $C$  has genus at least 2. Let  $S := \mathbb{P}(V)$  be the projectivization of  $V$ , and  $\pi : S \rightarrow C$  be the canonical projection. Denote by  $\text{Aut}(V)$  the automorphism of  $V$  over  $C$ , by  $\text{Aut}(C)$  the automorphism group of  $C$ , and  $\text{Aut}_C(S)$  denotes the automorphism group of  $S$  over  $C$ . That is,

$Aut_C(S) = \{\sigma \in Aut(S) | \pi\sigma = \pi\}$ . Since  $C$  is an irrational curve, there is an exact sequence

$$1 \rightarrow Aut_C(S) \rightarrow Aut(S) \rightarrow Aut(C).$$

It is well known that every Riemann surface  $C$  of genus at least 2 has a discrete automorphism group. Therefore to show that  $S$  admits no nontrivial holomorphic vector fields, it suffices to show the group  $Aut_C(S)$  is discrete. The relation between  $Aut(V)$  and  $Aut_C(S)$  can be found in [17], which states that one has the following exact sequence of groups

$$1 \rightarrow Aut(V)/\Gamma(C, \mathcal{O}^*) \rightarrow Aut_C(S) \rightarrow \Delta \rightarrow 1,$$

where  $\Gamma(C, \mathcal{O}^*)$  is the group of global holomorphic sections of the sheaf  $\mathcal{O}^*$  over  $C$ , and  $\Delta := \{L | L \rightarrow C \text{ is a holomorphic line bundle satisfying } V \otimes L = V\}$  is a subgroup of 2-torsion part of the Jacobian variety of  $C$ , hence  $\Delta$  is discrete. In the case that  $V$  is indecomposable, Maruyama [28] shows that

$$Aut(V) = \left\{ \begin{pmatrix} \alpha & s \\ 0 & \alpha \end{pmatrix} \mid \alpha \in \Gamma(L^* \otimes L) = \mathbb{C}^*, s \in \Gamma(C, (\det V)^{-1} \otimes L^2) \right\},$$

where  $L$  is a maximal sub-bundle of  $V$ . From the proof of Theorem 4.4.5, we know the line bundle  $(\det V)^{-1} \otimes L^2$  has negative degree, and admits no nontrivial holomorphic sections. This implies  $Aut(V) = \mathbb{C}^* = \Gamma(C, \mathcal{O}^*)$ , hence  $Aut_C(S) = \Delta$  is discrete.

If the curve  $C$  has genus 1, that  $Aut_C(S)$  is discrete can be shown by the same argument. It implies that there exists no vertical holomorphic vector

fields on  $S = \mathbb{P}(V)$ . Let  $\xi$  be a holomorphic vector field on  $S$ . Since  $S$  is a fibration with every fibre smooth and compact,  $\xi$  projects to a holomorphic vector field  $\bar{\xi}$  on  $C$ , which can only be parallel. Therefore  $\xi$  vanishes nowhere.

□

**Corollary 4.4.7.** *Let  $S$  be a ruled surface over a curve of genus at least 1, then  $S$  is deformation equivalent to a compact complex surface which admits a cscK metric.*

*Proof.* If  $S$  is minimal, it is deformation equivalent to either the surface we constructed in Section 4.4.1, the trivial bundle  $T \times \mathbb{P}_1$  over a torus, or the projectivization of the nontrivial extension of  $\mathcal{O}(p)$  by  $\mathcal{O}$ . By Theorem 4.4.1 and 4.4.5, each of them admits cscK metrics. If  $S$  is non-minimal,  $S$  is deformation equivalent to the blow-up of the surface we constructed in Section 4.4.1 or the projectivization of the nontrivial extension of  $\mathcal{O}(p)$  by  $\mathcal{O}$ . Each of them admits no nontrivial holomorphic fields by Theorem 4.4.1 and 4.4.6. Therefore a direct application of Arezzo-Pacard Theorem 3.1.1 gives us the conclusion. □

The remaining case is the ruled surfaces over a rational curve  $\mathbb{P}_1$ . First, assume the surface  $S$  over  $\mathbb{P}_1$  is minimal. Then  $S$  is either  $\mathbb{P}_2, \mathbb{P}_1 \times \mathbb{P}_1$ , or  $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k))$ , where  $k \in \mathbb{N}$ . The first two cases admit cscK metrics due to the existence of Fubini-Study metric on projective space. Denote  $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k))$  by  $F_k$ , then  $F_1$  is isomorphic to the one point blow-up of  $\mathbb{P}_2$ , and  $F_k$  is deformation equivalent to  $F_{k'}$  if and only if  $k \equiv k' \pmod{2}$ . Since the automorphism group of  $\mathbb{P}_2$  is  $\mathbb{P}GL_3(\mathbb{C})$ , if  $X$  is the blow-up of  $\mathbb{P}_2$  at more than 3 points in general position, then the automorphism group of  $X$  is trivial.

**Proposition 4.4.8.** *Let  $S$  be a compact complex rational surface. Suppose  $S$  is not deformation equivalent to  $\mathbb{P}_2 \# k\bar{\mathbb{P}}_2$ , where  $k = 1, 2$ . Then  $S$  is deformation equivalent to a complex surface with cscK metrics.*

*Proof.* The assumption implies that  $S$  is either  $\mathbb{P}_2$ , or deformation equivalent to  $\mathbb{P}_1 \times \mathbb{P}_1$ , or  $\mathbb{P}_2 \# k\bar{\mathbb{P}}_1$ , where  $k \geq 3$ . In the first two cases, the Fubini-Study metric and the product of Fubini-Study metrics provide a cscK metric. If  $S$  is the blow-up of  $\mathbb{P}_2$  at  $k$  points, where  $3 \leq k \leq 8$ , Tian and Yau [36] have shown that  $S$  admits a Kähler-Einstein metric. If  $X$  is the blow-up of  $\mathbb{P}_2$  at more than 4 points in the general position, then the automorphism group  $\text{Aut}(X)$  is discrete. Therefore a direct application of Arezzo-Pacard Theorem 3.1.1 shows that  $X = \mathbb{P}_2 \# k\bar{\mathbb{P}}_2$  admits a cscK metric whenever  $k \geq 4$ .  $\square$

## 4.5 Non-existence case

In the section, we will show that  $\mathbb{P}_2 \# k\bar{\mathbb{P}}_2$ ,  $k = 1, 2$ , is not deformation equivalent to any complex surface with cscK metrics. Since the deformation class of  $\mathbb{P}_2 \# \bar{\mathbb{P}}_2$  consists exactly of Hirzebruch surfaces  $F_k = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k))$ , where  $k$  is odd, it suffices to show the Lie algebra of holomorphic vector fields  $\mathfrak{h}$  of  $F_k$  is not reductive whenever  $k > 0$ .

**Definition 4.5.1.** *We say a Lie algebra  $\mathcal{G}$  is reductive if it is the sum of an abelian and a semisimple Lie algebra.*

Recall the theorem of Lichnerowicz and Matsushima [25, 29] tells us that a compact Kähler manifold  $(M, J)$  whose identity component  $\text{Aut}_0(M, J)$  of the automorphism group is not reductive does not admit any cscK metric.

**Theorem 4.5.2.** (*Lichnerowicz-Matsushima [25, 29]*) *Let  $(M, J)$  be a compact manifold with cscK metrics. Then the Lie algebra  $\mathfrak{h}(M)$  of holomorphic vector fields decomposes as a direct sum*

$$\mathfrak{h}(M) = \mathfrak{h}'(M) \oplus \mathfrak{a}(M)$$

*where  $\mathfrak{a}(M)$  is the abelian subalgebra of parallel holomorphic vector fields, and  $\mathfrak{h}'(M)$  is the subalgebra of holomorphic vector fields with zeros. Furthermore,  $\mathfrak{h}'(M)$  is the complexification of the killing fields with zeros. In particular,  $\mathfrak{h}(M)$  is a reductive Lie algebra.*

Let  $E = \mathcal{O} \oplus \mathcal{O}(k)$ ,  $F_k = \mathbb{P}(E)$  be the projectivization, and  $\pi : F_k \rightarrow \mathbb{P}_1$  be the holomorphic ruling. Let  $\mathfrak{h}(M)$  be the Lie algebra of the holomorphic vector fields. Then we have the algebra homomorphism  $\phi : \mathfrak{h} \rightarrow \mathfrak{sl}_2(\mathbb{C})$ , and the following exact sequence:

$$0 \rightarrow \mathfrak{h}^\perp \rightarrow \mathfrak{h} \xrightarrow{\phi} \mathfrak{sl}_2(\mathbb{C}),$$

where  $\mathfrak{h}^\perp$  denotes the algebra of all vertical holomorphic vector fields. Here we use "vertical" to mean they are tangent to the fibres.

**Lemma 4.5.3.** *The algebra homomorphism  $\phi : \mathfrak{h} \rightarrow \mathfrak{sl}_2(\mathbb{C})$  is surjective.*

*Proof.* Given a holomorphic vector field  $\xi \in \mathfrak{h}(\mathbb{P}_1)$ , the generated automorphism group  $h_t = \exp(t\xi) \in \text{Aut}(\mathbb{P}_1) = \text{PGL}_2(\mathbb{C}) = \text{SL}_2(\mathbb{C})/\mathbb{Z}_2$  can be lifted to a family of linear automorphism group  $\tilde{h}_t \in \text{SL}_2(\mathbb{C})$  on  $\mathbb{C}^2$ , which fixes the origin  $O$ , and  $\tilde{h}_0 = \text{identity}$ . Using that  $\mathcal{O}(-1)$  is the one point blow-up of  $\mathbb{C}^2$  at the origin and apply Hartog's extension theorem, the derivative  $\tilde{\xi} = \frac{\partial \tilde{h}_t}{\partial t} \big|_{t=0}$

represents a holomorphic vector field on  $\mathcal{O}(-1)$  such that  $p_*\tilde{\xi} = \xi$ , where  $p : \mathcal{O}(-1) \rightarrow \mathbb{P}_1$  is the natural projection. Since  $\mathcal{O}(1)$  is the dual of  $\mathcal{O}(-1)$ ,  $\mathcal{O}(k)$  is the tensor product of  $k$  copies of  $\mathcal{O}(1)$ , and  $\mathcal{O} \oplus \mathcal{O}(k)$  is the direct sum of bundles  $\mathcal{O}$  and  $\mathcal{O}(k)$ , we deduce that the family of automorphisms  $h_t$  on  $\mathbb{P}_1$  induces a family of automorphisms  $\tilde{h}_t$  on  $E = \mathcal{O} \oplus \mathcal{O}(k)$  such that the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{\tilde{h}_t} & E \\ \downarrow & & \downarrow \\ \mathbb{P}_1 & \xrightarrow{h_t} & \mathbb{P}_1. \end{array}$$

Since  $\tilde{h}_t$  restricted to each fibre is linear, it induces a family of infinitesimal automorphisms on  $F_k = \mathbb{P}(E)$ , and a holomorphic vector field  $\hat{\xi} \in \mathfrak{h}(F_k)$  such that  $\pi_*\hat{\xi} = \xi$ .  $\square$

Recall the exact sequence of groups found by Grothendieck [17]:

$$1 \rightarrow \text{Aut}(V)/\Gamma(C, \mathcal{O}^*) \rightarrow \text{Aut}_C(S) \rightarrow \Delta \rightarrow 1,$$

where  $\Delta$  is a discrete subgroup of Picard group. Assume  $k > 0$ . We have

$$\begin{aligned} \text{Aut}(E) &= \Gamma(\mathbb{P}_1, E \otimes E^*) \\ &= \mathbb{C} \oplus \mathbb{C} \oplus \odot^k V, \end{aligned}$$

where  $V = \mathbb{C}^2$ , and  $\odot^k V = \{b_0 Z_0^k + b_1 Z_0^{k-1} Z_1 + b_2 Z_0^{k-2} Z_1^2 + \dots + b_k Z_1^k | b_0, b_1, \dots, b_k \in \mathbb{C}\}$  is the set of homogeneous polynomials of degree  $k$ . Using the decompo-



sition  $E = \mathcal{O} \oplus \mathcal{O}(k)$ , we can write

$$\Gamma(E \otimes E^*) = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

where  $a, c$  are constants, and  $b \in \odot^k V$ . The group multiplication is given by

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \cdot \begin{pmatrix} a' & 0 \\ b' & c' \end{pmatrix} = \begin{pmatrix} aa' & 0 \\ ab' + bc' & cc' \end{pmatrix}.$$

Observe that  $aI, a \in \mathbb{C}$  induces the identity map on  $\mathbb{P}(E)$ . We can then identify the fibre-preserving automorphism group of  $F_k$  with the group

$$\left\{ \begin{pmatrix} 1 & 0 \\ b & c \end{pmatrix} \mid c \in \mathbb{C}, b \in \odot^k V \right\}.$$

The Lie algebra  $\mathfrak{h}^\perp$  of vertical holomorphic vector fields may be visualized as the set of

$$\left\{ \begin{pmatrix} 0 & 0 \\ * & \times \end{pmatrix} \right\},$$

where  $\times$  stands for an arbitrary complex number, and  $*$  stands for an arbitrary homogeneous polynomial with degree  $k$  in the variables  $z_1, z_2$ . A direct computation shows that the center of  $\mathfrak{h}^\perp$  is trivial, and  $[\mathfrak{h}^\perp, \mathfrak{h}^\perp] \not\subseteq \mathfrak{h}^\perp$ . Therefore,  $\mathfrak{h}^\perp$  is not reductive.

**Theorem 4.5.4.**  *$\mathfrak{h}$  is not reductive. In particular,  $F_k$  admits no cscK metrics whenever  $k > 0$ .*

*Proof.* Suppose that  $\mathfrak{h}$  is reductive. We claim that  $Z(\mathfrak{h}) \subset Z(\mathfrak{h}^\perp)$ , where  $Z(\mathfrak{h}), Z(\mathfrak{h}^\perp)$  denote the centers of  $\mathfrak{h}$  and  $\mathfrak{h}^\perp$ , respectively. In particular,  $Z(\mathfrak{h}) = 0$ . Recall that we have the exact sequence of Lie algebras

$$0 \rightarrow \mathfrak{h}^\perp \rightarrow \mathfrak{h} \xrightarrow{\phi} \mathfrak{sl}_2(\mathbb{C}) \rightarrow 0.$$

Given  $h \in Z(\mathfrak{h})$ ,  $[h, h'] = 0$  for any  $h' \in \mathfrak{h}$  implies  $[\phi(h), \phi(h')] = 0$  in  $\mathfrak{sl}_2(\mathbb{C})$ . Therefore  $\phi(h)$  is in the center of  $\mathfrak{sl}_2(\mathbb{C})$ , which is trivial. It follows that  $h$  is in  $\mathfrak{h}^\perp$ , so  $h \in Z(\mathfrak{h}^\perp)$ . Since  $\mathfrak{h}$  is reductive, and  $\mathfrak{h}^\perp$  is a subideal of  $\mathfrak{h}$ , there exists a subideal  $\tilde{\mathfrak{h}}$  of  $\mathfrak{h}$  such that  $\mathfrak{h} = \mathfrak{h}^\perp \oplus \tilde{\mathfrak{h}}$ . Therefore we have

$$\begin{aligned} \mathfrak{h}^\perp \oplus \tilde{\mathfrak{h}} &= \mathfrak{h} \\ &= [\mathfrak{h}, \mathfrak{h}] \oplus Z(\mathfrak{h}) \\ &= [\mathfrak{h}^\perp \oplus \tilde{\mathfrak{h}}, \mathfrak{h}^\perp \oplus \tilde{\mathfrak{h}}] \\ &= [\mathfrak{h}^\perp, \mathfrak{h}^\perp] \oplus [\tilde{\mathfrak{h}}, \tilde{\mathfrak{h}}]. \end{aligned}$$

This implies  $\mathfrak{h}^\perp = [\mathfrak{h}^\perp, \mathfrak{h}^\perp]$ , which is the contradiction.  $\square$

Next we consider the deformation class of  $\mathbb{P}_2 \# 2\bar{\mathbb{P}}_2$ . Every compact complex ruled surface  $S$  deformation equivalent to  $\mathbb{P}_2 \# 2\bar{\mathbb{P}}_2$  can be realized as a one point blow-up of a Hirzebruch surface  $F_k$  at  $p$  for some point  $p \in F_k$ ,  $k \in \mathbb{N}$ . Denote the surface  $F_k \# \bar{\mathbb{P}}_2$  by  $S$ . Then the Lie algebra  $\mathfrak{h}(S)$  of holomorphic vector fields on  $S$  is isomorphic to the Lie algebra  $\mathfrak{h}'(F_k)$  of holomorphic vector fields vanishing at the point  $p$ , i.e.  $\mathfrak{h}(S) = \mathfrak{h}'(F_k) = \{X \in \mathfrak{h}(F_k) | X(p) = 0\}$ . Consider the algebra homomorphism  $\pi_* : \mathfrak{h}(S) \rightarrow \mathfrak{sl}_2(\mathbb{C})$  composed by  $\mathfrak{h}(S) \rightarrow \mathfrak{h}(F_k)$  and  $\mathfrak{h}(F_k) \xrightarrow{\phi} \mathfrak{sl}_2(\mathbb{C})$ . Let  $\mathfrak{g}$  be the image of  $\pi_*$ . Then  $\mathfrak{g}$  is the Lie algebra of the subgroup of  $SL_2(\mathbb{C})$  fixing the point  $\pi(p)$ . Without loss of generality,

we can assume  $\pi(p) = [1 : 0] \in \mathbb{P}_1$ .

**Theorem 4.5.5.**  $\mathfrak{h}(S)$  is not reductive. In particular,  $F_k \# \bar{\mathbb{P}}_2$  admits no cscK metrics if  $k \geq 0$ .

*Proof.* Suppose  $\mathfrak{h}(S) = \mathfrak{h}'(F_k) = \mathfrak{h}'$  is reductive. Then

$$\begin{aligned} \mathfrak{g} &= \pi_* \mathfrak{h}' \\ &= \pi_*([\mathfrak{h}', \mathfrak{h}'] \oplus Z(\mathfrak{h}')) \\ &= [\pi_* \mathfrak{h}', \pi_* \mathfrak{h}'] + \pi_* Z(\mathfrak{h}') \\ &\leq [\mathfrak{g}, \mathfrak{g}] + Z(\mathfrak{g}). \end{aligned}$$

This is absurd since  $\mathfrak{g} = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \mid a, b \in \mathbb{C} \right\}$  has trivial center, and  $[\mathfrak{g}, \mathfrak{g}] = \left\{ \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \mid * \in \mathbb{C} \right\}$ . □

Recently, Arezzo, Pacard, and Singer [2] have shown that there exists extremal metrics on the one or two points blow-up of  $\mathbb{P}_2 \# k \bar{\mathbb{P}}_2$ . The following corollary follows immediately from their result and Proposition 4.4.8.

**Corollary 4.5.6.** *Let  $S$  be a compact complex surface with even Betti number  $b_1$ . Then  $S$  is deformation equivalent to a complex surface with extremal metrics.*

## Chapter 5

### Remarks

Theorem 4.1.1 and 4.3.2 tell us that every compact complex surface of Kähler type and Kodaira dimension 0 or 2 carries cscK metrics. This is not true for other cases. For example, a pseudo-Hirzebruch surface  $S = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k))$ ,  $k > 0$  does not admit any cscK metrics since its automorphism group is not reductive. Furthermore, although most of elliptic surfaces and ruled surfaces with cscK metrics in this thesis have discrete automorphism group, and we know there is an  $h^{1,1}$ -dimensional family of cscK metrics on the nearby complex surfaces in the deformation class by using LeBrun and Simanca's theorem [24], it is challenging to see, for a fixed complex structure, which Kähler class do these cscK metrics lie in. Even if the first Chern class  $c_1(X)$  of a manifold  $X$  is negative, there are examples by Ross [34] where some Kähler classes do not contain any cscK metric.

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