On a General Chain Model of the Free Loop Space and String Topology

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Abstract

Let M be a smooth oriented manifold. The homology of M has the structure of a Frobenius algebra. This paper shows that on chain level there is a Frobenius-like algebra structure, whose homology gives the Frobenius algebra of M. Moreover, associated to any Frobenius-like algebra, there is a chain complex whose homology has the structure of a Gerstenhaber algebra and a Batalin-Vilkovisky algebra. And if the Frobenius-like algebra comes from M, it gives the free loop space LM and String Topology of Chas-Sullivan.

Contents

1 Introduction and summary

In this paper we investigate some properties of a chain complex model of the free loop space of a smooth manifold. The purpose of our study is twofold. One is to give a down-to-earth algebraic model of the algebraic structures (the Gerstenhaber and Batalin-Vilkovisky algebras) of String Topology discovered by Chas and Sullivan in [5], and the other is to relate these algebraic structures with some known ones, especially those from the Hochschild complexes of the cochain algebra. Our theory includes non simply connected manifolds.

The paper consists of six sections. In Section 2 we discuss the open Frobenius structure of a manifold and construct a chain complex whose homology gives such a structure. Such a chain complex is called an open DG Frobenius-like algebra. In Section 3 we use the open DG Frobenius-like algebra to construct a chain complex model of the free loop space of a simply connected manifold. In Sections 4 and 5 we give a model of the Gerstenhaber and Batalin-Vilkovisky algebras on the homology of the free loop space of a simply connected manifold obtained in [5]. In the last section, Section 6, we give a chain complex model of the free loop space of a general manifold and construct the associated Gerstenhaber and Batalin-Vilkovisky algebras.

1.1 The open DG Frobenius-like algebra of a manifold

Let M be a smooth, not necessarily closed, oriented n-manifold. Denote by $H_*(M;\mathbb{Q})$ and $H^*_c(M;\mathbb{Q})$ the rational homology and compact cohomology of M respectively. We have that $H^*_c(M;\mathbb{Q})$ with cup product $\cup$ is a graded commutative algebra, and $H_*(M;\mathbb{Q})$ with diagonal map Δ is a graded cocommutative coalgebra. The Poincaré duality says that there is an isomorphism of $\mathbb{Q}$-spaces (grade the cohomology negatively):

$$PD : H^*_c(M;\mathbb{Q}) \xrightarrow{\cong} H_{n+*}(M;\mathbb{Q}).$$

We may also consider $H^*(M;\mathbb{Q})$ and $H^\infty_*(M;\mathbb{Q})$ the rational cohomology and infinite homology (see Definition 2.6) of M. Again $(H^*(M;\mathbb{Q}),\cup)$ and $(H^\infty_*(M;\mathbb{Q}),\Delta)$ are a graded commutative algebra and a (possibly complete) graded cocommutative coalgebra, and we have the Poincaé duality:

$$PD : H^*(M;\mathbb{Q}) \xrightarrow{\cong} H^\infty_{n+*}(M;\mathbb{Q}).$$

By pulling back via PD the coproduct Δ on these two homology groups to the corresponding cohomology groups, the following result is known to algebraic topologists:

Theorem 2.8 (two open Frobenius algebras of a manifold). *Let M be a smooth manifold. Then*

$$(H^*_{\mathrm{c}}(M;\mathbb{Q}),\cup,\Delta) \quad \textit{and} \quad (H^*(M;\mathbb{Q}),\cup,\Delta)$$

form two graded commutative open Frobenius algebras, namely,

(1) $(H^*_{\mathrm{c}}(M;\mathbb{Q}),\cup)$ *is a graded commutative algebra and* $(H^*_{\mathrm{c}}(M;\mathbb{Q}),\Delta)$ *is a graded cocommutative coalgebra with a counit; respectively,* $(H^*(M;\mathbb{Q}),\cup)$ *is a graded commutative algebra with a unit and* $(H^*(M;\mathbb{Q}),\Delta)$ *is a possibly complete graded cocommutative coalgebra;*

(2) *The coproduct Δ is a map of bimodules:*

$$\Delta(\alpha\cup\beta) = \Delta\alpha\cup\beta = \alpha\cup\Delta\beta, \quad \textit{for} \quad \alpha,\beta\in H^*_{\mathrm{c}}(M;\mathbb{Q}) \textit{ or } H^*(M;\mathbb{Q}).$$

Moreover these two Frobenius algebras are dual to each other in the sense that

$$H^*(M;\mathbb{Q}) \cong \mathrm{Hom}(H^{-n-*}_{\mathrm{c}}(M;\mathbb{Q}),\mathbb{Q}),$$

which maps $\cup$ and Δ of the latter to Δ and $\cup$ of the former respectively. And if M is closed, these two Frobenius algebras are identical, and hence have both unit and counit.

In the language of homology groups, the above theorem says that the homology or infinite homology of a manifold with intersection product and diagonal coproduct form two dual open Frobenius algebras. However, we cannot lift these two open Frobenius algebras to the chain level, since the intersection of two chains is partially defined only if they are transversal to each other. In this paper we show that a weaker form of the above open Frobenius algebras exists on the chain level of the manifold. Such a chain model uses the cubical Whitney polynomial differential forms and their appropriate duals (the currents).

Give a smooth manifold M with a smooth cubilation (such a cubilation always exists by the dual decomposition of a smooth triangulation). Recall that *a Whitney polynomial form* (see Definition 2.9) on M is a differential form such that the restriction to each cube is of $\mathbb{Q}$-polynomial coefficients. Denote the set of Whitney forms by $A(M)$, then $A(M)$ forms a commutative DG algebra under the wedge product and the exterior differential. There is a DG subalgebra of $A(M)$, which is the Whitney forms with compact support, and is denoted by $A_{\mathrm{c}}(M)$. Of course if M is closed, $A(M) = A_{\mathrm{c}}(M)$.

By dualizing $A(M)$ properly (for the precise definition, see Definition 2.28) we also get two complete DG coalgebras which model the chain complex and the infinite chain complex of M, and are denoted by $C(M)$ and $C^\infty(M)$ respectively. If M is closed, then

$$C(M) = C^\infty(M) = \mathrm{Hom}(A(M),\mathbb{Q}).$$

Furthermore, we have two embeddings

$$\iota : A_{\mathrm{c}}(M) \longrightarrow C(M) \quad \text{and} \quad \iota : A(M) \longrightarrow C^{\infty}(M)$$

which are given by

$$\alpha \longmapsto \Big\{\beta \mapsto \int_M \alpha \wedge \beta\Big\}.$$

And they are in fact quasi-isomorphisms of $A_{\mathrm{c}}(M)$- and $A(M)$-modules respectively. These properties lead us to define:

Definition 2.17 (open DG Frobenius-like algebra of degree n). *Let k be a field. An open DG Frobenius-like algebra of degree n is a triple (A, E, ι) such that:*

(1) *A is a DG commutative associative algebra over k;*

(2) *E is a (possibly complete) DG cocommutative coassociative coalgebra over A;*

(3) *$\iota : A \longrightarrow E$ is a degree n DG A-module quasi-isomorphism.*

From the definition one deduces that the homology of a DG Frobenius-like algebra is a Frobenius algebra. The theorem is:

Theorem 2.23 (open DG Frobenius-like algebras of a manifold). *Let M be a smooth n-manifold and let $A(M)$ and $A_{\mathrm{c}}(M)$ be the set of Whitney forms and Whitney forms with compact support on M. Let $C(M)$ and $C^{\infty}(M)$ be the two complete DG coalgebras of M. Then the triples*

$$(A_{\mathrm{c}}(M), C(M), \iota) \quad \text{and} \quad (A(M), C^{\infty}(M), \iota)$$

are open DG Frobenius-like algebras of degree n, whose homology groups give the Frobenius algebras of Theorem 2.8.

In this paper we will mostly discuss the open DG Frobenius algebra $(A_{\mathrm{c}}(M), C(M), \iota)$. There is also a concept of open DG Frobenius-like algebras with a group action. The background is this: Let M be smooth manifold, and let $\tilde{M}$ be its universal covering. Denote by G the fundamental group $\pi_1(M)$ of M, then G acts on $\tilde{M}$ by deck transformations. The open DG Frobenius-like algebra on $\tilde{M}$ admits a G-action, whose G-equivariant homology group, as one would expect, is the open Frobenius-like algebra of M. For more details, see Section 2.

1.2 The chain complex model of the free loop space

Let M be a smooth manifold, and assume it is simply connected for a moment. Also in order not to be confused by the notations, we assume M is closed. Denote the open DG Frobenius-like algebra of M by (A, C) for short. Let LM be the free loop space of M. We have a fibration

$$\begin{tikzcd} \Omega M \arrow[r] & LM \arrow[d] \\ & M. \end{tikzcd}$$

The result of Adams ([1]) says that the cobar construction (see Definition 3.8) of the chain complex of M, is quasi-isomorphic to the chain complex of ΩM. From this we may obtain that if C is the complete DG coalgebra of M, then the complete cobar construction (see Definition 3.32) $\underline{\hat{\Omega}}(C)$ gives a complete DG coalgebra model of ΩM. Applying a theorem of Brown (Theorem 3.5 or Brown [3], Theorem 4.2) of the twisted tensor product for fibrations we have:

Theorem 3.36 (a complete DG coalgebra of the free loop space). *Let M be a simply connected, smooth closed manifold and C be the dual space of the Whitney forms (the currents). There is a chain equivalence*

$$(C\hat{\otimes}\underline{\hat{\Omega}}(C), b) \xrightarrow{\simeq} (C_*(LM), \partial),$$

where $\hat{\otimes}$ is the complete tensor product, and b is Brown's twisted differential:

$$\begin{aligned} & b(x \otimes [a_1|\cdots|a_n]) \\ := \;& dx \otimes [a_1|\cdots|a_n] + (-1)^{|x|} x \otimes d_{\mathrm{A}}[a_1|\cdots|a_n] \\ & + \sum_i (-1)^{|x'|} x' \otimes \Big([x''|a_1|\cdots|a_n] - (-1)^{(|x''|-1)|[a_1|\cdots|a_n]|}[a_1|\cdots|a_n|x''] \Big) \end{aligned}$$

where d_{A} is the Adams differential on the complete cobar construction, and x', x'' comes from the complete coproduct of C: $\Delta x = \sum x' \otimes x''$.

For more details of the complex $C\hat{\otimes}\underline{\hat{\Omega}}(C)$, see Definition 3.35, and we call it *the complete cocyclic cobar complex of* C. Recall the embedding $A \xrightarrow{\iota} C$, if we view A as currents we have:

Theorem 3.37 (chain complex of the free loop space from the open Frobenius-like algebra). *Let M be a simply connected, smooth closed manifold. Let A be the Whitney forms and C be the currents on M. Define a chain complex $(A\hat{\otimes}\underline{\hat{\Omega}}(C), b)$ with b given by*

$$\begin{aligned} & b(x \otimes [a_1|\cdots|a_n]) \\ := \;& dx \otimes [a_1|\cdots|a_n] + (-1)^{|x|} x \otimes d_{\mathrm{A}}[a_1|\cdots|a_n] \\ & + \sum_i (-1)^{|x|+|\beta_i|} x \wedge \beta_i \otimes \Big([\beta_i^*|a_1|\cdots|a_n] - (-1)^{(|\beta_i|-1)|[a_1|\cdots|a_n]|}[a_1|\cdots|a_n|\beta_i^*] \Big). \end{aligned}$$

Then there is a chain equivalence

$$\iota \otimes id : (A\hat{\otimes}\underline{\hat{\Omega}}(C), b) \xrightarrow{\simeq} (C\hat{\otimes}\underline{\hat{\Omega}}(C), b).$$

In the case that M is not closed, we may apply the open DG Frobenius-like algebra (A_{c}, C) instead in the above to model the chain complex of LM.

As we shall see, the cocyclic cobar complex $C\hat{\otimes}\underline{\hat{\Omega}}(C)$ is in fact the dual complex of *the cyclic bar complex* (see Definition 3.14) of A. As is observed by A. Connes ([8]), there is a cyclic structure on the cyclic bar complex, and one can define on the complex so-called Connes' cyclic B-operator, which characterizes such cyclic structure. Such a cyclic B-operator is later used by Jones to model the S^1-action on the cochain complex model of the free loop space of a manifold (see Jones [15]). We can define a dual version of Connes' cyclic operator (see Definition 3.39) on the cocyclic cobar complex, which then models the S^1-action on the chain complex of LM:

Theorem 3.41 (cyclic B-operator and the S^1-action). *Let M be a simply connected, smooth closed manifold. Let A be the Whitney forms and C be the currents on M. Define $B : C\hat{\otimes}\underline{\hat{\Omega}}(C) \longrightarrow C\hat{\otimes}\underline{\hat{\Omega}}(C)$ by*

$$\begin{aligned} & B(x \otimes [a_1|\cdots|a_n]) \\ := \; & \sum_{i=1}^{n-1}(-1)^{|[a_i|\cdots|a_n]||[a_1|\cdots|a_{i-1}]|}\varepsilon(x)a_i \otimes [a_{i+1}|\cdots|a_n|a_1|\cdots|a_{i-1}], \end{aligned}$$

where ε is the counit. Then $B^2 = 0$ and we have a chain equivalence

$$(C\hat{\otimes}\underline{\hat{\Omega}}(C), b, B) \xrightarrow{\simeq} (C_*(LM), \partial, J),$$

where J is the S^1-action on $C_*(LM)$.

1.3 The Gerstenhaber and Batalin-Vilkovisky algebras

We next apply the open DG Frobenius-like algebra of a manifold to construct a model for the Gerstenhaber and Batalin-Vilkovisky algebras on the homology of the free loop space discovered by Chas and Sullivan in [5]. The algebraic model has also been obtained by Cohen-Jones [7], Félix et al [9], Merkulov [20] and Tradler [24].

Theorem 4.2 (model of the loop product). *Let M be a simply connected, smooth closed manifold, and let (A, C) be the open DG Frobenius-like algebra of M. Define*

$$\bullet : A\hat{\otimes}\underline{\hat{\Omega}}(C) \bigotimes A\hat{\otimes}\underline{\hat{\Omega}}(C) \longrightarrow A\hat{\otimes}\underline{\hat{\Omega}}(C)$$

by

$$(x \otimes [a_1|\cdots|a_n]) \bullet (y \otimes [b_1|\cdots|b_n]) := (-1)^{|y||[a_1|\cdots|a_n]|}x \wedge y \otimes [a_1|\cdots|a_n|b_1|\cdots|b_n],$$

then

$$(A\hat{\otimes}\underline{\hat{\Omega}}(C), \bullet, b)$$

is a DG algebra, which models the Chas-Sullivan loop product on $C_*(LM)$ in [5].

For the definition of the loop product see [5] §2 or Section 4 of this paper. The operator $\bullet$ thus defined in not commutative, but commutative up to homotopy. We have:

Theorem 4.10 (Gerstenhaber algebra of the free loop space). *Let (A, C) be the DG Frobenius-like algebra of a simply connected, smooth closed manifold M. Define an operator*

$$* : A\hat{\otimes}\underline{\hat{\Omega}}(C) \bigotimes A\hat{\otimes}\underline{\hat{\Omega}}(C) \longrightarrow A\hat{\otimes}\underline{\hat{\Omega}}(C)$$

as follows: for $\alpha = x \otimes [a_1|\cdots|a_n], \beta = y \otimes [b_1|\cdots|b_m] \in A\hat{\otimes}\underline{\hat{\Omega}}(C)$,

$$\alpha * \beta = \sum_{i=1}^{n}(-1)^{|y|+|\beta||[a_{i+1}|\cdots|a_n]|}\varepsilon(a_i y)x \otimes [a_1|\cdots|a_{i-1}|b_1|\cdots|b_m|a_{i+1}|\cdots|a_n],$$

where ε is the counit of C, and

$$\{\alpha, \beta\} := \alpha * \beta - (-1)^{(|\alpha|+1)(|\beta|+1)}\beta * \alpha, \quad \text{for} \quad \alpha, \beta \in A\hat{\otimes}\underline{\hat{\Omega}}(C).$$

Then $(H_*(A\hat{\otimes}\underline{\Omega}(C), \bullet, \{,\})$ is a Gerstenhaber algebra, which models the one of Chas-Sullivan in [5].

The structure of a Gerstenhaber algebra (see Definition 4.9) is discovered by Gerstenhaber in his study of the deformation of associative algebras (see Gerstenhaber [10]). He shows that for an associative algebra A, the *Hochschild cohomology* of A (see Definition 3.18) is a Gerstenhaber algebra. The following theorem has been obtained by the authors cited above:

Theorem 4.17 (isomorphism of two Gerstenhaber algebras). *Let M be a simply connected smooth closed manifold, and A be the Whitney forms on M. Then the Gerstenhaber algebra of Chas-Sullivan in [5], which is modeled in the above theorem, is isomorphic to the Hochschild cohomology of A.*

Recall the dual of Connes' cyclic operator $B : C\hat{\otimes}\underline{\hat{\Omega}}(C) \longrightarrow C\hat{\otimes}\underline{\hat{\Omega}}(C)$. We may restrict it to the subspace $B : A\hat{\otimes}\underline{\hat{\Omega}}(C) \longrightarrow C\hat{\otimes}\underline{\hat{\Omega}}(C)$, which is a chain map. Via the isomorphism

$$H_*(A\hat{\otimes}\underline{\hat{\Omega}}(C)) \cong H_*(C\hat{\otimes}\underline{\hat{\Omega}}(C))$$

we obtain a degree one operator on $H_*(A\hat{\otimes}\underline{\hat{\Omega}}(C))$, still denoted by B.

Theorem 5.4 (Batalin-Vilkovisky algebra of the free loop space). *Let M be a simply connected, smooth closed manifold. Then $(H_*(A\hat{\otimes}\underline{\hat{\Omega}}(C)), \bullet, B)$ is a Batalin-Vilkovisky algebra, namely,*

(1) $(H_*(A\hat{\otimes}\underline{\hat{\Omega}}(C)), \bullet)$ *is a graded commutative algebra;*

(2) B *is a second order operator of square zero.*

Such a Batalin-Vilkoviksy algebra models the one of Chas-Sullivan in [5].

Up to now, we have only discussed the case when M is simply connected. For a non simply connected manifold, the above discussion may not hold, since the cobar construction of the chain complex of M is not equivalent to the chain complex of ΩM. However, this will be overcome in Section 6 by lifting the fibration $\Omega M \to LM \to M$ to the universal covering $\tilde{M}$ of M (idea due to M. Mandell). The loops in LM lift to paths on $\tilde{M}$, which admits a $\pi_1(M)$-action, and can be characterized explicitly. By taking the quotient over $\pi_1(M)$ we get back to LM. We may deal with the lifted fibration on $\tilde{M}$ similarly as in the simply connected case, and the $\pi_1(M)$-equivariant homology forms a Batalin-Vilkovisky algebra, as one would expect.

The rest of the paper is devoted to the proof of above theorems. Section 2 discusses Theorem 2.8 and proves Theorem 2.23. Theorems 3.36, 3.37 and 3.41 are proved in Section 3. Theorems 4.2, 4.10 and 4.17 are proved in Section 4. Theorem 5.4 is proved in Section 5. And in Section 6 we prove the above theorems when the manifold is not necessarily simply connected.

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2 DG Frobenius-like algebra of a manifold

In this section we first discuss the open Frobenius structures of a manifold, and then discuss some properties of the Whitney differential forms on the manifold, by which we construct two open DG Frobenius-like algebras, which characterize the Frobenius structures on chain level.

2.1 The open Frobenius structures of a manifold

Definition 2.1 (open Frobenius algebra of degree n)**.** *Let C be a graded vector space over a field k. An open Frobenius algebra on C of degree n is a triple $(C, \cdot, \Delta)$ such that*

(1) *$(C, \cdot)$ is a graded associative algebra, and (C, Δ) is a graded coassociative coalgebra of degree n;*

(2) *$\Delta : C \to C \otimes C$ is a map of bimodules: $\Delta(a \cdot b) = a \cdot \Delta(b) = \Delta(a) \cdot b$, for all $a, b \in C$, or more explicitly, if we write $\Delta a = \sum a' \otimes a''$, then*

$$\sum (a \cdot b)' \otimes (a \cdot b)'' = \sum (a \cdot b') \otimes b'' = \sum a' \otimes (a'' \cdot b). \tag{1}$$

We say C is commutative if it is both graded commutative and cocommutative.

Remark 2.2. In above definition we have assumed the product $\cdot$ has degree zero, and the coproduct Δ has degree n. In some cases the product may have degree n while the coproduct has degree 0. If this situation happens, we may shift the degree of C up by n (denoted by $C[-n]$), then $(C[-n], \cdot, \Delta)$ forms a Frobenius algebra of degree $-n$, by our definition.

Definition 2.3 (unit, augmentation and counit, coaugmentation)**.** *Suppose $(A, \cdot)$ is a graded associative algebra over a field k. A unit of A is a linear map*

$$\eta : k \longrightarrow A$$

such that the following diagram commutes:

$$\begin{tikzcd}
A \otimes k \arrow[rr, "id \otimes \eta"] \arrow[dr, "\cong"'] & & A \otimes A \arrow[dl, "\cdot"] \\
& A & \\
k \otimes A \arrow[rr, "\eta \otimes id"] \arrow[ur, "\cong"] & & A \otimes A. \arrow[ul, "\cdot"']
\end{tikzcd}$$

Also we say A is augmented if there is a nonzero algebra map

$$\varepsilon : A \longrightarrow k,$$

and ε is called an augmentation.

Similarly, suppose (C, Δ) is a graded coassociative coalgebra over k, then a counit of C is a linear map

$$\varepsilon : C \longrightarrow k$$

such that the following diagram commutes:

$$\begin{tikzcd}
C \otimes C \arrow[rr, "\varepsilon \otimes id"] & & k \otimes C \\
& C \arrow[ul, "\Delta"'] \arrow[ur, "\cong"] \arrow[dl, "\Delta"] \arrow[dr, "\cong"'] & \\
C \otimes C \arrow[rr, "id \otimes \varepsilon"] & & C \otimes k.
\end{tikzcd}$$

And we say C is coaugmented if there is a nonzero coalgebra map

$$\eta : k \longrightarrow C,$$

and η is called a coaugmentation.

From now on when mentioning an algebra (respectively, a coalgebra), we always assume it has a unit and an augmentation (respectively, a counit and a coaugmentation) unless specifying otherwise.

Example 2.4 (Frobenius algebra on compact cohomology and homology)**.** Let M be a connected (not necessarily closed) n-manifold, and denote by $H_*(M;\mathbb{Q})$ and $H^*_{\mathrm{c}}(M;\mathbb{Q})$ its homology and compact cohomology respectively. Then $H^*_{\mathrm{c}}(M;\mathbb{Q})$ and $H_*(M;\mathbb{Q})$ form two isomorphic commutative open Frobenius algebras of degree n with a counit but no unit. In fact, by Poincaré duality, there is an isomorphism

$$PD : H^*_{\mathrm{c}}(M;\mathbb{Q}) \xrightarrow{\cong} H_{n+*}(M;\mathbb{Q}),$$

which is given by

$$\langle PD(\alpha), u\rangle := \int_M \alpha \cup u, \quad \text{for} \quad \alpha \in H^*_{\mathrm{c}}(M;\mathbb{Q}), \quad u \in H^*(M;\mathbb{Q}). \tag{2}$$

Define

$$\Delta : H^*_{\mathrm{c}}(M;\mathbb{Q}) \longrightarrow H^*_{\mathrm{c}}(M;\mathbb{Q}) \otimes H^*_{\mathrm{c}}(M;\mathbb{Q})$$

by

$$\Delta\alpha := \left(PD^{-1} \otimes PD^{-1}\right) \circ \Delta(PD(\alpha)).$$

By (2) we have

$$\begin{aligned}
\Delta(\alpha \cup \beta)(x \otimes y) &= \left(PD^{-1} \otimes PD^{-1}\right) \circ \Delta(PD(\alpha \cup \beta))(x \otimes y) \\
&= \Delta(PD(\alpha \cup \beta))(PD^{-1}(x) \otimes PD^{-1}(y)) \\
&= PD(\alpha \cup \beta)(PD^{-1}(x) \cup PD^{-1}(y)) \\
&= \int_M \alpha \cup \beta \cup PD^{-1}(x) \cup PD^{-1}(y),
\end{aligned}$$

while

$$\begin{aligned}
\big(\Delta(\alpha) \cup \beta\big)(x \otimes y) &= \sum \big(\alpha' \otimes (\alpha'' \cup \beta)\big)(x \otimes y) \\
&= \sum \langle \alpha', x\rangle \langle \alpha'' \cup \beta, y\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum \langle \alpha', x\rangle \langle \alpha'' \cup \beta, PD \circ PD^{-1}(y)\rangle \\
&= \sum \langle \alpha', x\rangle \int_M \alpha'' \cup \beta \cup PD^{-1}(y) \\
&= \sum \langle \alpha', x\rangle \langle \alpha'', PD\big(\beta \cup PD^{-1}(y)\big)\big) \\
&= \int_M \alpha \cup \beta \cup PD^{-1}(x) \cup PD^{-1}(y),
\end{aligned}
$$

for all $\alpha, \beta \in H^*_c(M;\mathbb{Q})$ and $x, y \in H_*(M;\mathbb{Q})$. This shows

$$\Delta(\alpha \cup \beta) = \Delta(\alpha) \cup \beta.$$

Similarly, we can show

$$\Delta(\alpha \cup \beta) = \alpha \cup \Delta(\beta).$$

This shows that

$$(H^*_c(M;\mathbb{Q}), \cup, \Delta)$$

forms a (graded) Frobenius algebra. It is commutative since the cup product is commutative, and the counit comes from the fact $H^n(M;\mathbb{Q}) \cong H_0(M;\mathbb{Q}) \cong \mathbb{Q}$. The Poincaré duality map PD also gives an isomorphic Frobenius algebra structure on the homology $H_*(M;\mathbb{Q})$. □

Remark 2.5. In this paper, we grade the cochain complex negatively, and grade the chain complex positively. The coproduct on the cohomology has a grading $-n$, so if we write

$$\Delta : H^p_c(M;\mathbb{Q}) \longrightarrow \bigoplus_{r+s=p} H^r_c(M;\mathbb{Q}) \otimes H_s(M;\mathbb{Q})$$

by Poincaré duality in stead of

$$\Delta : H^p_c(M;\mathbb{Q}) \longrightarrow \bigoplus_{r+s=p-n} H^r_c(M;\mathbb{Q}) \otimes H^s_c(M;\mathbb{Q}),$$

the degree adds formally.

On an open manifold M there is another Poincaré duality which is between the cohomology and the infinite homology of M.

Definition 2.6 (infinite homology, Munkres [22] p. 33)**.** *Let M be a locally finite cubilated (or simplicial, or cell) space. An infinite q-chain on M is a function c from the oriented q-cubes of M to the integers such that $c(\sigma) = -c(\sigma')$ if σ and σ' are opposite orientations of the same cube (or simplex, or cell). (We do not require $c(\sigma) = 0$ for all but finitely many oriented cubes.) Let $C^\infty_q(M)$ denote the group of infinite q-chains. Since M is locally finite the boundary operator*

$$\partial^\infty_q : C^\infty_q(M) \longrightarrow C^\infty_{q-1}(M)$$

as in the ordinary case is well defined and $(\partial^\infty)^2 = 0$. The homology

$$H^\infty_*(M) := \frac{\ker \partial^\infty}{\mathrm{im}\ \partial^\infty}$$

is called the infinite homology of M.

Example 2.7 (Frobenius algebra on cohomology and infinite homology)**.** Let M be a connected (not necessarily closed) n-manifold. Let $H^*(M;\mathbb{Q})$ and $H_*^\infty(M;\mathbb{Q})$ be the cohomology and infinite homology of M. Then $H^*(M;\mathbb{Q})$ with cup product and (possibly complete) coproduct induced from Poincaré duality (see Munkres [22], p. 388)

$$PD : H^*(M;\mathbb{Q}) \xrightarrow[\cap[M]]{\cong} H_{n+*}^\infty(M;\mathbb{Q})$$

gives $(H^*(M;\mathbb{Q}),\cup,\Delta)$ as well as $H_*^\infty(M;\mathbb{Q})$ a commutative open Frobenius algebra of degree n with a unit but no counit. □

Theorem 2.8 (two Frobenius algebras of a manifold)**.** *Let M be a connected (not necessarily closed) n-manifold. Then the two open Frobenius algebras*

$$(H_{\mathrm{c}}^*(M;\mathbb{Q}),\cup,\Delta) \quad \textit{and} \quad (H^*(M;\mathbb{Q}),\cup,\Delta)$$

are dual to each other in the sense that

$$H^*(M;\mathbb{Q}) \cong \mathrm{Hom}(H_{\mathrm{c}}^{-n-*}(M;\mathbb{Q}),\mathbb{Q}).$$

And if M is closed, then they are identical, and hence have both unit and counit.

Proof. This follows from Examples 2.4 and 2.7. □

We usually call the open Frobenius algebra on the (co)homology of a closed manifold, namely, an open Frobenius algebra with a counit which is isomorphic to its dual space (the isomorphism is given by $\langle x,y\rangle = \varepsilon(x\cdot y)$, where ε is the counit), *a closed Frobenius algebra*. In the language of homology, Theorem 2.8 says that on a manifold M, the rational homology or infinite homology of M, together with the intersection product and the diagonal coproduct, form two dual Frobenius algebras.

The Frobenius algebra structure on the homology of a manifold can not be lifted to chain level. For example, a k-chain, where $k < \dim M$, can never intersect with itself in an expected manner. However, we next show that a weaker form, which we would call a Frobenius-like algebra, exists on the manifold. Such an algebraic structure uses the so-called cubical Whitney polynomial differential forms on M, which is discussed in next subsection.

2.2 Cubical Whitney polynomial differential forms

Definition 2.9 (Whitney polynomial differential forms)**.** *Let M be a cubilated topological space. A cubical Whitney polynomial differential form ω on M is a collection of differential forms, one on each cube, such that:*

1. *the coefficients of these forms on each cube are $\mathbb{Q}$-polynomials with respect to the affine coordinates of the cubes;*
2. *they are compatible under restriction to faces, i.e. if τ is face of σ, then $\omega_\sigma|\tau = \omega_\tau$.*

The set of Whintney polynomial forms on M is denoted by $A(M)$.

Proposition 2.10. *Let $A(M)$ be the Whitney polynomial differential forms of a cubilated space M. We have:*

(1) *$A(M)$, under wedge $\wedge$ and exterior differential d, forms a commutative DG algebra;*

(2) *The Whitney forms may be mapped to the cochains of the space as follows:*

$$\begin{aligned} \rho : A(M) &\longrightarrow C^*(M;\mathbb{Q}) \\ \omega &\longmapsto \left\{ I^n \mapsto \int_{I^n} \omega \right\}, \text{ for any } I^n, \end{aligned}$$

which is a chain map.

Proof. (1) holds because $\wedge$ and d are both natural for restriction to faces. (2) follows from Stokes' theorem. □

The following theorem says ρ in fact induces an algebra isomorphism on cohomology, which is the de Rham theorem for Whitney forms.

Theorem 2.11 (de Rham's theorem for Whitney forms)**.** *Let M be a cubilated topological space. Then ρ is a chain equivalence of DG algebras, i.e.*

$$\rho_* : H^*(A(M), d) \xrightarrow[\text{alg}]{\cong} H^*(M;\mathbb{Q}).$$

Before proving the theorem we first show the following:

Lemma 2.12 (extension lemma)**.** *Let ω^r be a form in $A(\partial I^n)$. Then there is a form $\tilde{\omega}^r \in A(I^n)$ such that $\tilde{\omega}|_{\partial I^n} = \omega$.*

Proof. Suppose σ_0, σ_1 are a pair of front and back faces of I^n, say,

$$\sigma_0 = \{(t_1, \cdots, t_n) | t_1 = 0\} \quad \text{and} \quad \sigma_1 = \{(t_1, \cdots, t_n) | t_1 = 1\}.$$

Consider

$$\tilde{\omega}_1 := (1 - t_1) \cdot \omega|_{\sigma_0} + t_1 \cdot \omega|_{\sigma_1},$$

then $\omega_2 := \omega - \tilde{\omega}_1$ vanishes on σ_0 and σ_1. Take another pair of faces of I^n, say σ'_0, σ'_1,

$$\sigma'_0 = \{(t_1, \cdots, t_n) | t_2 = 0\} \quad \text{and} \quad \sigma'_1 = \{(t_1, \cdots, t_n) | t_2 = 1\}.$$

Consider

$$\tilde{\omega}_2 := (1 - t_2) \cdot \omega_2|_{\sigma'_0} + t_2 \cdot \omega_2|_{\sigma'_1}.$$

Since $\tilde{\omega}_2$ vanishes on σ_0, σ_1,

$$\omega_3 := \omega - (\tilde{\omega}_1 + \tilde{\omega}_2)$$

vanishes on σ_0, σ_1 and σ'_0, σ'_1. Continuing this procedure we obtain a sequence of forms on I^n:

$$\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n.$$

Let

$$\tilde{\omega} := \sum_i \tilde{\omega}_i,$$

then $\omega = \tilde{\omega}|_{\partial I^n}$. This proves the lemma. □

Lemma 2.13 (Poincaré lemma). *Let K be a star shaped complex in $\mathbb{R}^n$, and let $\omega^r \in A(K)$, $r < 0$, be closed, then ω is exact, i.e. there is a form ξ^{r+1} such that $\omega = d\xi$.*

In the above lemma we say K is *star shaped* if there is a point $p_0 \in K$ such that if $p \in K$ then the segment $p_0 p$ is in K.

Proof. Suppose K is star shaped from p_0. Define

$$h : I \times K \longrightarrow K$$

by

$$h(t, p) := (1 - t) \cdot p_0 + t \cdot p.$$

Define

$$\xi(p) := \int_{t=0}^{t=1} h^*\omega, \quad p \in K.$$

We claim that $\omega = d\xi$. In fact, since ω is closed and h^* is a chain map,

$$d\xi = d\int_{t=0}^{t=1} h^*\omega = \int_{t=0}^{t=1} dh^*\omega + h^*\omega\Big|_{t=0}^{t=1} = h^*\omega\Big|_{t=0}^{t=1}.$$

However, since $h(0, \cdot) \equiv p_0$ and $h(1, \cdot) = id$, $h^*\omega\big|_{t=0}^{t=1} = \omega - 0 = \omega$, and therefore $d\xi = \omega$. Q.E.D. □

Remark 2.14. The above lemma can be generalized to the complex which is star shaped from a star shaped complex.

Lemma 2.15 (de Rham's theorem for cubles). *For any cube I^n, if $\omega^r \in A(I^n)$, $r < 0$, is closed, then ω is exact, i.e. there is a form $\xi^{r+1} \in A(I^n)$ such that $\omega = d\xi$. If $r = 0$, then ω is a constant function.*

Proof. (1) First the lemma holds for interval: any closed 0-form is a constant function on the interval and any 1-form is closed and also exact.

(2) In general, the Whitney forms on a cube are just the tensor product of the forms on the intervals, and by (1), the lemma holds. □

Proof of Theorem 2.11. Denote by M_k the k-skeleton of M, then by Extension Lemma 2.12 we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A(M_k, M_{k-1}) \arrow[r] \arrow[d, "\rho"] & A(M_k) \arrow[r] \arrow[d, "\rho"] & A(M_{k-1}) \arrow[r] \arrow[d, "\rho"] & 0 \\ 0 \arrow[r] & C^*(M_k, M_{k-1}) \arrow[r] & C^*(M_k) \arrow[r] & C^*(M_{k-1}) \arrow[r] & 0, \end{tikzcd} \tag{3}$$

where $A(M_k, M_{k-1})$ and $C^*(M_k, M_{k-1})$ are the Whitney forms and cochains on M_k which vanish on M_{k-1}. Since the interior of k-cubes in M_k are disjoint, $A(M_k, M_{k-1})$ and $C^*(M_k, M_{k-1})$ may be written as

$$\bigoplus_{I^k \in M_k} A(I^k, \partial I^k) \quad \text{and} \quad \bigoplus_{I^k \in M_k} C^*(I^k, \partial I^k).$$

Suppose for a moment

$$\sum \rho : \bigoplus_{I^k \in M_k} A(I^k, \partial I^k) \longrightarrow \bigoplus_{I^k \in M_k} C^*(I^k, \partial I^k) \tag{4}$$

is a quasi-isomorphism, we can prove the theorem by induction: For $k = 1$,

$$\rho : A(M_{k-1}) \longrightarrow C^*(M_{k-1})$$

is identity map, and therefore induces an isomorphism on cohomology. Now suppose for $k > 1$, ρ is a quasi-isomorphism, then in the long exact sequence induced by (3):

$$\begin{tikzcd}
H^*(A(M_{k-1})) \arrow[r] \arrow[d, "\rho_*"] & H^*(A(M_k, M_{k-1})) \arrow[r] \arrow[d, "\rho_*"] & H^*(A(M_k)) \arrow[r] \arrow[d, "\rho_*"] & H^{*-1}(A(M_{k-1})) \arrow[r] \arrow[d, "\rho_*"] & H^{*-1}(A(M_k, M_{k-1})) \arrow[d, "\rho_*"] \\
H^*(M_{k-1}) \arrow[r] & H^*(M_k, M_{k-1}) \arrow[r] & H^*(M_k) \arrow[r] & H^{*-1}(M_{k-1}) \arrow[r] & H^{*-1}(M_k, M_{k-1}),
\end{tikzcd}$$

the left two and right two ρ_*'s are isomorphisms, and therefore by 5-lemma, the middle one is an isomorphism.

We now prove (4). For this we only need to show

$$\rho : A(I^k, \partial I^k) \longrightarrow C^*(I^k, \partial I^k) \tag{5}$$

is a quasi-isomorphism. This is proved by induction. First, observe that (5) holds for $k = 0$. Now suppose it holds for $k = n - 1$. For $k = n$, notice that similar to (3) we have a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & A(I^n, \partial I^n) \arrow[r] \arrow[d, "\rho"] & A(I^n) \arrow[r] \arrow[d, "\rho"] & A(\partial I^n) \arrow[r] \arrow[d, "\rho"] & 0 \\
0 \arrow[r] & C^*(I^n, \partial I^n) \arrow[r] & C^*(I^n) \arrow[r] & C^*(\partial I^n) \arrow[r] & 0.
\end{tikzcd} \tag{6}$$

By the acyclicity of $A(I^n)$ (Lemma 2.15) and $C^*(I^n)$, from the induced long exact sequence we see that showing (5) is equivalent to showing

$$\rho : A(\partial I^n) \longrightarrow C^*(\partial I^n) \tag{7}$$

is a quasi-isomorphism. Suppose ω^r is a closed form in $A(\partial I^n)$ and if $r = 1 - n$, then $\int_{\partial I^n} \omega = 0$. Let $\sigma_1 = \{(t_1, \cdots, t_n) | t_1 = 1\}$ be a face of I^n. Then by Poincaré Lemma $\omega|_{\partial I^n - \sigma_1} = d\xi$ for some $\xi \in A(\partial I^n - \sigma_1)$. By Extension Lemma 2.12 we can extend ξ to $\tilde{\xi}$ on ∂I^n. Now $\omega - d\tilde{\xi}$ is closed on ∂I^n and vanishes on $\partial I^n - \sigma_1$. If $r = 1 - n$, then $0 = \int_{\partial I^n} \omega = \int_{\partial I^n} \omega - d\tilde{\xi} = \int_{\sigma_1} \omega - d\tilde{\xi}$. By our assumption that (5) holds for $k = n - 1$, we obtain that $(\omega - d\tilde{\xi})|_{\sigma_1} = d\mu$. Extend μ to $\tilde{\mu}$ by zero on the rest of ∂I^n, then $\omega = d(\tilde{\xi} + \tilde{\mu})$. This shows that

$$H^q(A(\partial I^n)) = \begin{cases} \mathbb{Q}, & \text{if} \quad q = 0, 1 - n, \\ 0, & \text{otherwise,} \end{cases}$$

i.e. (7) holds. By the long exact sequence of (6), (5) holds for $k = n$.

As for the product structure, a general theory of algebraic topology called acyclic models implies that any DG algebra which models the cochain algebra of a manifold induces the same product (the cup product) on cohomology. For a very clear treatment of this issue see Vick [25], Section 4 Products, in particular p. 113. Thus de Rham's theorem is proved. □

Remark 2.16. The proof is also given in Cenkl-Porter [4], Theorem 4.1. Since the idea will be used later, we here give a complete proof.

Note that $A(M)$ is bigraded by the degree of the forms and the order of their polynomial coefficients, and both $\wedge$ and d respect the total grading.

2.3 DG Frobenius-like algebra on smooth manifolds

Suppose M is a smooth manifold, then by a theorem of Whitehead [27], it admits a smooth triangulation, and any two such cubilations are combinatorially equivalent. The dual decomposition of the triangulation in fact gives M a smooth cubilation. An observation about the Whitney forms on M is the following: If M is closed, then for each $r \geq 0$ (r is the total degree), the set of Whitney forms of degree r, denoted by $A^r(M)$, is of finite dimension, and $A(M) = \bigoplus_r A^r(M)$. Since wedge product preserves the degree, the dual space of the Whitney forms, i.e. the set of currents, is a complete DG coalgebra, whose homology is the rational homology of M. Moreover, the differential forms embeds into the currents, whose images in the currents are dense. This leads us to define:

Definition 2.17 (open DG Frobenius-like algebra of degree n)**.** *Let k be a field. An open DG Frobenius-like algebra of degree n is a triple (A, E, ι) such that:*

(1) *A is a DG commutative associative algebra over k;*

(2) *E is a (possibly complete) DG cocommutative coassociative coalgebra over A;*

(3) *$\iota : A \longrightarrow E$ is a degree n DG A-module quasi-isomorphism.*

From the definition, a commutative Frobenius algebra is automatically a Frobenius-like algebra, with E being A itself. An open DG Frobenius-like algebra is a generalization of an open Frobenius algebra in the following sense:

Theorem 2.18. *Let (A, E, ι) be an open DG Frobenius-like algebra over a field k. Then the homology of A is a commutative open Frobenius algebra.*

Proof. This follows from the definition. □

2.3.1 On closed manifolds

We next show that the sets of Whitney forms and currents on a closed manifold form an open DG Frobenius-like algebra. First, let us recall some facts:

Lemma 2.19. *If $\{V_m\}$ and $\{W_n\}$ are both inverse limit systems of k-modules, then*

$$\{\bigoplus_{m+n=j} V_m \otimes W_n\}$$

is also an inverse limit system of k-modules.

Definition 2.20. *If $\{V_m\}$ and $\{W_n\}$ are both inverse limit systems of k-modules, and $V = \varprojlim V_m$ and $W = \varprojlim W_n$. Define the complete tensor product of V and W as*

$$V \hat{\otimes} W := \varprojlim \bigoplus_{m+n=j} V_m \otimes W_n.$$

Lemma 2.21. *Let M be a smooth closed manifold and let $C(M)$ be the dual space of $A(M)$. Then the wedge product*

$$\wedge : A(M) \otimes A(M) \longrightarrow A(M)$$

induces a DG mapping

$$\Delta : C(M) \longrightarrow C(M) \hat{\otimes} C(M)$$

which makes $(C(M), \Delta, d)$ into a complete, cocommutative DG coalgebra, where d is now the dual differential.

Proof. First note that we have a filtration for A

$$A_0 \subset A_1 \subset \cdots \subset A_{i-1} \subset A_i \subset A_{i+1} \subset \cdots ,$$

where A_i is the set of Whitney forms whose total degree $\leq i$. Note $C(M) = \mathrm{Hom}(A, \mathbb{Q})$ (in the following we write it as C for short) is the set of linear functionals on A. If $A_i \subset A_j$, we have the restriction map

$$\psi_i^j : \mathrm{Hom}(A_j, \mathbb{Q}) \longrightarrow \mathrm{Hom}(A_i, \mathbb{Q}),$$

and the sequence $\{\mathrm{Hom}(A_i, \mathbb{Q}), \phi_i^j\}$ forms an inverse limit system, and

$$C = \varprojlim \mathrm{Hom}(A_i, \mathbb{Q}).$$

Now we can see that

$$\begin{aligned} \mathrm{Hom}(A \otimes A, \mathbb{Q}) &= \varprojlim \mathrm{Hom}(\bigoplus_{i+j=k} A_i \otimes A_j, \mathbb{Q}) \\ &= \varprojlim \bigoplus_{i+j=k} \mathrm{Hom}(A_i, \mathbb{Q}) \otimes \mathrm{Hom}(A_j, \mathbb{Q}) \\ &= C(A) \hat{\otimes} C(A). \end{aligned}$$

Therefore $\wedge : A \otimes A \to A$ induces the diagonal

$$\Delta : C \longrightarrow \mathrm{Hom}(A \otimes A, \mathbb{Q}) = C \hat{\otimes} C$$

making C into a complete DG coalgebra. The cocommutativity of C comes from the the commutativity of A. $\square$

Since $A(M)$ computes the rational cohomology of M, by the Universal Coefficient Theorem, $C(M)$ computes the rational homology of M.

Definition 2.22. *Let M be a smooth closed manifold, and let $A(M)$ be the Whitney polynomial differential forms and $C(M)$ be its dual, the currents. Then*

$$(C(M), \Delta, d)$$

is called the complete DG coalgebra model of M.

In the following, we shall write $A(M)$ and $C(M)$ as A and C for short.

Theorem 2.23 (open DG Frobenius-like algebra of a closed manifold)**.** *Let M be a smooth closed n-manifold and let A be the Whitney forms and C be the currents on M. Recall the embedding of p-forms to $(n-p)$-currents by*

$$\iota : A \longrightarrow C : \alpha \longmapsto \Big\{\beta \mapsto \int_M \alpha \wedge \beta\Big\}, \tag{8}$$

then (A, C, ι) forms a complete open DG Frobenius-like algebra of degree n over $\mathbb{Q}$.

Proof. We have to show (A, C) satisfies (1), (2) and (3) in Definition 2.17. (1) follows from the discussion of previous subsection.

We now show (2). First, note that C is a DG A-module, where the action of A is given by

$$\begin{array}{ccc} C \otimes A & \longrightarrow & C \\ x \otimes \alpha & \longmapsto & \big\{\beta \mapsto \langle x, \alpha \wedge \beta\rangle\big\}, \end{array}$$

for $\alpha, \beta \in A$ and $x \in C$. Now for any $\alpha, \beta, u, v \in A$, by definition,

$$\Delta(\alpha\beta)(u \otimes v) = \iota(\alpha\beta)(uv) = \int_M \alpha\beta uv,$$

and

$$\begin{aligned} \big(\Delta(\alpha) \cdot \beta\big)(u \otimes v) &= (-1)^{|u||(\iota\alpha)''\beta|}\langle(\iota\alpha)', u\rangle\langle(\iota\alpha)'' \cdot \beta, v\rangle \\ &= (-1)^{|u||(\iota\alpha)''\beta|}\langle(\iota\alpha)', u\rangle\langle(\iota\alpha)'', \beta v\rangle \\ &= (-1)^{|u||(\iota\alpha)''\beta|}\Delta\alpha(u \otimes \beta u) \\ &= (-1)^{|u||\beta|}\int_M \alpha u\beta v = \int_M \alpha\beta uv. \end{aligned}$$

This means

$$\Delta(\alpha\beta) = \Delta(\alpha) \cdot \beta.$$

Similarly, one can prove

$$\Delta(\alpha\beta) = \alpha \cdot \Delta(\beta).$$

We now show (3). In fact, the embedding of A in C is dense comes from the fact that on homology level,

$$\begin{array}{rcl} \iota_* : H^*(A, d) & \longrightarrow & H_{n+*}(C, d) = \mathrm{Hom}(H^{-n-*}(A), \mathbb{Q}) \\ \alpha & \longmapsto & \Big\{\beta \mapsto \displaystyle\int_M \alpha \cup \beta\Big\}, \quad \beta \in H^*(A, d) \end{array}$$

is an isomorphism by Poincaré duality.

At last, the commutativity comes from the commutativity of the wedge product. The unit of A comes from

$$\eta : k \longrightarrow A$$

viewed as a constant function on M, and counit of C comes from the dual of η. Q.E.D. $\square$

Theorem 2.24 (explicit formula for Δ)**.** *Let A be the Whitney forms of M and C be the currents. Since A is the direct limit of finite dimensional spaces, if we denote by $\{\beta_i\}$ the basis of A, and denote by β_i^* their dual vectors, then*

$$\Delta\alpha = \sum_i \iota(\alpha \wedge \beta_i)\hat{\otimes}\beta_i^*.$$

Therefore, we can formally write (compare Remark 2.5)

$$\begin{array}{rcl} \Delta : A & \longrightarrow & A\hat{\otimes}C \\ \alpha & \longmapsto & \sum_i(\alpha \wedge \beta_i)\hat{\otimes}\beta_i^*. \end{array}$$

Proof. Denote by

$$\alpha_p := \sum_{i \leq p} \iota(\alpha \wedge \beta_i)\hat{\otimes}\beta_i^*,$$

then

$$\left\{\alpha_p = \sum_{i\leq p} \iota(\alpha \wedge \beta_i)\hat{\otimes}\beta_i^*\right\}\Big|_{(A\otimes A)_p}$$

forms an inverse limit system, whose inverse limit is $\sum_i \iota(\alpha \wedge \beta_i)\hat{\otimes}\beta_i^*$. Now for any $u, v \in A$,

$$\Delta\alpha(u \otimes v) = \iota(\alpha)(u \wedge v) = \int_M \alpha \wedge u \wedge v,$$

while

$$\begin{aligned} \sum_i \iota(\alpha \wedge \beta_i)\hat{\otimes}\beta_i^*(u \otimes v) &= \sum_i (-1)^{|u||b_i|}\Big(\int_M \alpha \wedge \beta_i \wedge u\Big)\langle\beta_i^*, v\rangle \\ &= \int_M \alpha \wedge u \wedge v. \end{aligned}$$

Therefore

$$\Delta\alpha = \sum_i \iota(\alpha \wedge \beta_i)\hat{\otimes}\beta_i^*.$$

Q.E.D. □

Remark 2.25. James McClure has constructed a theory of intersection product at chain level by using the PL chains on the manifold (see McClure [18]).

2.3.2 On open manifolds

Let M be a smooth, not necessarily closed, cubilated manifold. Denote by Λ the set of cubes in M, then Λ is a partially ordered set where for $\alpha, \beta \in \Lambda$, $\alpha \leq \beta$ iff α is a face of β.

Lemma 2.26. *Let $A(M)$ be the set of Whitney forms with finite degree. Denote by A_α^r the set of Whitney forms of degree r on α, then*

$$A(M) = \varprojlim_\alpha A_\alpha = \varprojlim_\alpha \varinjlim_r A_\alpha^r.$$

Proof. This follows from the definitions of the inverse and direct limit systems. □

Denote by $A_c(M)$ the set of Whitney forms with compact support, then $A_c(M)$ is a DG subalgebra of $A(M)$, whose cohomology is the rational compact cohomology $H_c^*(M;\mathbb{Q})$ of M.

Definition 2.27 (currents on open manifolds)**.** *Let M be a smooth cubilated manifold. Define*

$$C^\alpha := \mathrm{Hom}(A_\alpha, \mathbb{Q}) = \varprojlim_r \mathrm{Hom}(A_\alpha^r, \mathbb{Q}),$$

which is the set of currents on α, and

$$C(M) := \varinjlim_\alpha C^\alpha, \quad \text{and} \quad C^\infty(M) := \widehat{\varinjlim_\alpha C^\alpha},$$

where $\widehat{\varinjlim_\alpha C^\alpha}$ is the completion of $C(M)$ with respect to the filtration given by Λ, namely,

$$C^\infty(M) := \prod_{\alpha\in\Lambda} C^\alpha \Big/ \eta_\alpha(x) \sim \eta_\beta \phi_\beta^\alpha(x), \quad \forall x \in C^\alpha,$$

where $\eta_\alpha : C^\alpha \longrightarrow \prod_\alpha C^\alpha$ is the identity map, and $\phi_\beta^\alpha(x)$ is the push forward of x to β.

One sees that if M is closed, then $A(M) = A_c(M)$ and $C(M) = C^\infty(M) = \mathrm{Hom}(A(M), \mathbb{Q})$.

Lemma 2.28. *Let M be a smooth cubilated manifold, and let $C(M)$ and $C^\infty(M)$ be as in above definition. We have:*

(1) *$C(M)$ is a complete DG coalgebra with a counit and*

$$H_*(M;\mathbb{Q}) \cong H_*(C(M), d);$$

(2) *$C^\infty(M)$ is a complete DG coalgebra with no counit and*

$$H_*^\infty(M;\mathbb{Q}) \cong H_*(C^\infty(M), d),$$

where $H_*^\infty(M;\mathbb{Q})$ is the rational infinite homology of M (see Definition 2.6).

Proof. We sketch the proof of (1). First, notice that if α and β are cubes in M and $\alpha \le \beta$, by Extension Lemma 2.12, we have $A_\alpha|_\beta = A_\beta$. Moreover, the following diagram commutes:

$$\begin{tikzcd} A_\alpha \otimes A_\alpha \arrow[r, "\wedge"] \arrow[d] & A_\alpha \arrow[d] \\ A_\beta \otimes A_\beta \arrow[r, "\wedge"] & A_\beta. \end{tikzcd}$$

This implies dually

$$\{C_\alpha \longrightarrow C_{\alpha\times\alpha}\}$$

is a direct limit system of linear maps and therefore

$$C(M) \longrightarrow C(M \times M)$$

is well defined. However $C(M)\otimes C(M)$ is dense in $C(M\times M)$, and if we denote by $C(M)\hat{\otimes}C(M)$ the completion $C(M)\otimes C(M)$ in $C(M\times M)$, then the above map is in fact

$$C(M) \longrightarrow C(M)\hat{\otimes}C(M).$$

Thus $C(M)$ is a complete DG coalgebra.

Define

$$\begin{array}{rcl} \rho : C_*(M;\mathbb{Q}) & \longrightarrow & C(M) \\ a & \longmapsto & \Big\{\omega \mapsto \displaystyle\int_a \omega\Big\}, \quad \text{for any } \omega \in A(M), \end{array}$$

where $C_*(M;\mathbb{Q})$ is the rational cubical chain complex of M. One can check that ρ is a chain map and the proof of the quasi-isomorpshim is completely analogous (dual) to de Rham's theorem for Whitney forms (Theorem 2.11). And the counit of $C(M)$ comes from the nonzero funtional on the constant function.

(2) follows from the same argument once we define the chain map

$$\begin{array}{rcl} \rho : C^{\infty}_*(M;\mathbb{Q}) & \longrightarrow & C^{\infty}(M) \\ a & \longmapsto & \Big\{\omega \mapsto \displaystyle\int_a \omega\Big\}, \quad \text{for any } \omega \in A_{\mathrm{c}}(M). \end{array}$$

Q.E.D. □

Moreover, $A(M)$ and $A_{\mathrm{c}}(M)$ embed into $C^{\infty}(M)$ and $C(M)$ respectively:

$$\iota : A(M) \longrightarrow C^{\infty}(M) \quad \text{and} \quad A_{\mathrm{c}}(M) \longrightarrow C(M)$$

by

$$\alpha \longmapsto \Big\{\beta \mapsto \int_M \alpha\wedge\beta\Big\},$$

which are isomorphisms on the corresponding (co)homology groups. Therefore we have

Theorem 2.29 (DG Frobenius-like algebras of an open manifold)**.** *Let M be a smooth, cubilated open manifold. Let $A(M), A_{\mathrm{c}}(M), C(M)$ and $C^{\infty}(M)$ be as above. Then*

$$(A(M), C^{\infty}(M), \iota) \quad \textit{and} \quad (A_{\mathrm{c}}(M), C(M), \iota)$$

are two open DG Frobenius-like algebras, whose homology groups give the two open Frobenius algebras of M in Theorem 2.8.

Proof. The proof follows from above lemma (Lemma 2.28). □

2.3.3 On manifolds with a group action

Lemma 2.30. *Let (A, E, ι) be a DG Frobenius-like algebra and G be a group. Suppose both A and E are G-modules. Suppose also that $\iota : A \longrightarrow E$ is G-equivariant, then $(A/G, E/G, \iota/G)$ is also an open DG Frobenius-like algebra, where A/G and E/G means $A\otimes_{k[G]} k$ and $E\otimes_{k[G]} k$ respectively.*

Proof. This follows directly from the definition of a DG Frobenius-like algebra. □

Suppose N is a smooth cubilated manifold and $G \in \mathrm{Diff}(N)$ is a discrete subgroup of the diffeomorphism group of N, which acts on N freely and properly. Without loss of generality we may also assume G preserves the cubilation. Denote such a structure by a pair (N, G). As an example, let M be a smooth cubilated manifold and $G = \pi_1(M)$ be the fundamental group of M. Let $\tilde{M}$ be the universal covering of M, then $(\tilde{M}, G)$ is such a pair, where G acts on $\tilde{M}$ by deck transformations.

Now suppose (N, G) is such a pair. In last subsection we have constructed a DG Frobenius-like algebra for N: $(A_{\mathrm{c}}(N), C(N), \iota)$. Since $G \in \mathrm{Diff}(N)$, both the differetial forms $A_{\mathrm{c}}(N)$ and the currents $C(N)$ can be pushed forward, and therefore are G-modules.

Lemma 2.31. *Let (N, G) be as above. Then*

$$\iota : A_{\mathrm{c}}(N) \longrightarrow C(N)$$

is G-equivariant.

Proof. Note that given $g \in G$ and $\omega \in A_{\mathrm{c}}(N)$, the pushforward of ω under g is given by

$$g_*(\omega) := (g^{-1})^*(\omega).$$

Now for any $\omega \in A_{\mathrm{c}}(N)$ and $\eta \in A(N)$,

$$\iota(g_*(\omega))(\eta) = \int_N g_*(\omega) \wedge \eta,$$

while

$$\begin{aligned} g_*(\iota\omega)(\eta) &= (\iota\omega)(g^*(\omega)) \\ &= \int_N \omega \wedge g^*(\eta) \\ &= \int_N g^*((g^{-1})^*(\omega) \wedge \eta) \\ &= \int_{g_*(N)} (g^{-1})^*(\omega) \wedge \eta \\ &= \int_N (g^{-1})^*(\omega) \wedge \eta \\ &= \int_N g_*(\omega) \wedge \eta. \end{aligned}$$

They are equal, so ι is G-equivariant. □

Theorem 2.32 (DG Frobenius-like algebra of manifold with a group action)**.** *Let N be a smooth cubilated manifold and $G \in \mathrm{Diff}(N)$ be a discrete subgroup of the diffeomorphism group of N, which acts on N freely and properly. Then $(A_{\mathrm{c}}(N)/G, C(N)/G, \iota/G)$ is quasi-isomorphic to the DG Frobenius-like algebra of N/G.*

In particular, if M is a smooth manifold and $\tilde{M}$ is its universal covering. Let $G = \pi_1(M)$ be the fundamental group of M. Then then DG Frobenius-like algebra $(A_{\mathrm{c}}(\tilde{M})/G, C(\tilde{M})/G, \iota/G)$ is quasi-isomorphic to the DG Frobenius-like algebra $(A_{\mathrm{c}}(M), C(M), \iota)$ of M.

Proof. The theorem follows from the fact that $C(N)/G$ is quasi-isomorphic to $C(N/G)$. □

3 DG coalgebra of the free loop space

In this section we first recall Brown's twisted tensor product theory for fiber bundles and then apply it to the case of the free loop space, to obtain a complete DG coalgebra model of LM.

3.1 Twisted tensor product theory

Recall the definition of a twisting cochain:

Definition 3.1 (twisting cochain). *Let (C, d) be a DG coalgebra over a field k and (A, δ) be a DG algebra. A twisting cochain is a degree -1 linear map $\Phi = \bigoplus \Phi_q : C_q \to A_{q-1}$ such that*

(1) $\Phi_0(\varepsilon) = 0$*, where ε is the counit;*

(2) $\delta \circ \Phi_q = -\Phi_{q-1} \circ d - \sum_k (-1)^k \Phi_k \cup \Phi_{q-k}$.

Remark 3.2. In Brown [3], the second equation is

$$\delta \circ \Phi_q = \Phi_{q-1} \circ d - \sum (-1)^k \Phi_k \cup \Phi_{q-k}.$$

There is no negative sign in front of $\Phi_{q-1} \circ d$. However, these two definitions are equivalent. In Brown's proof of the existence of twisting cochain, if we set $\Phi_1(T) = T_0 - T$, then all of Brown's results will hold in our case. The reason for us to modify the definition is to make it compatible with later discussion.

Let (M, p) be a connected pointed topological space, and $S_*(M)$ be the 1-reduced singular chain complex of M. The Alexander-Whitney diagonal approximation gives a DG coassociative coalgebra on $S_*(M)$. Now let $C_*(\Omega M)$ be the chain complex of the based loop space of M at base point p. We have

Theorem 3.3 (Brown [3] Theorem 4.1). *For each pathwise connected space M, there exists a twisting cochain $\Phi_M \in \bigoplus C^q(S_q(M); C_{q-1}(\Omega M))$ satisfying the following properties:*

(1) *If T is the constant 0-simplex then $\Phi_M(T) = 0$;*

(2) *If $T \in S_1(M)$ is a 1-simplex and $T_0 \in S_1(M)$ is the constant 1-simplex, then $\Phi_M(T) = T_0 - T$, where in the right side of the equality T and T_0 are viewed as 0-simplices in $C_*(\Omega M)$;*

(3) *Φ_M is natural, i.e. if there is a map $f : M \to \tilde{M}$ and $\bar{f} : \Omega M \to \Omega\tilde{M}$ is induced by f, then $\bar{f}_\# \circ \Phi_M = \Phi_{\tilde{M}} \circ f_\#$.*

Now let $F \to E \xrightarrow{\pi} (M, p)$ be a fibration with fiber $F = \pi^{-1}(p)$. Suppose the fibration is transitive, which means it is a Hurewicz fibration satisfying the homotopy covering property. Taking any loop $\gamma \in \Omega_p M$, for any point $f \in F$ we may lift γ in E ending at f. Denoting the initial point of the path to be γf, we get a continuous action of $\Omega_p M$ on F, which induces an action on chain level:

$$\circ : \ C_*(\Omega M) \otimes C_*(F) \longrightarrow C_*(F).$$

In fact $C_*(F)$ is a left DGA $C_*(\Omega M)$-module under the action $\circ$.

Definition 3.4. *Suppose Φ is the twisting cochain of Theorem 3.3. Define an operator ∂_Φ on $S_*(M)\otimes C_*(F)$ as follows:*

$$\partial_\Phi(x\otimes f) := \partial x\otimes f + (-1)^{|x|}x\otimes\partial f + \sum(-1)^{|x'|}x'\otimes\Phi(x'')\circ f.$$

Then $\partial_\Phi^2=0$. We call ∂_Φ the twisted differential and $S_*(M)\otimes C_*(F)$ the twisted tensor product.

Theorem 3.5 (Brown [3] Theorem 4.2)**.** *For a transitive fiber bundle $F\to E\to M$, there is a chain equivalence*

$$\phi : (S_*(M)\otimes C_*(F),\partial_\Phi)\longrightarrow (C_*(E),\partial).$$

Now for the free loop space of a manifold, $\Omega M\to LM\to(M,p)$, there is a natural lifting function given as follows: for any $\gamma:[0,1]\to M$, $\gamma(0)=q,\ \gamma(1)=p$, then

$$\begin{array}{rccc}\gamma: & \Omega_p M & \longrightarrow & \Omega_q M,\\ & x & \longmapsto & \gamma x\gamma^{-1}.\end{array} \tag{9}$$

Let us call this lifting function *the natural lifting function*. In fact the natural lifting function makes $LM\to M$ a transitive fiber bundle.

Lemma 3.6 (see also McCleary [16])**.** *The action of $C_*(\Omega M)$ on itself induced by the natural lifting function (9) is given by*

$$\begin{array}{rccc}\circ: & C_*(\Omega M)\otimes C_*(\Omega M) & \longrightarrow & C_*(\Omega M),\\ & \alpha\otimes x & \longmapsto & \alpha\circ x := \sum\alpha' x S(\alpha''),\end{array}$$

where $\Delta\alpha=\sum\alpha'\otimes\alpha''$, and S is induced from the inverse map $\gamma\mapsto\gamma^{-1}$ in $\Omega_p(M)$.

Proof. This can be seen from the action of ΩM:

$$\begin{array}{rccc}\gamma: & \Omega_p M & \longrightarrow & \Omega_q M,\\ & x & \longmapsto & \gamma x\gamma^{-1},\end{array}$$

which induces on chain level

$$a\circ x = \mu\circ(Id\otimes\mu)\circ(Id\otimes Id\otimes S)\circ(Id\otimes T)\circ(\Delta\otimes Id)(a\otimes x),$$

where T is the twisting function, and Δ is the diagonal. The formula of the action is exactly the one given in the lemma. $\square$

Theorem 3.7. *Let M be a connected manifold and ΩM be its based loop space. Let $C_*(\Omega M)$ be the singular chain complex of ΩM and $S_*(M)$ be the 1-reduced chain complex of M. We have a chain equivalence*

$$\phi : (S_*(M)\otimes C_*(\Omega M),\partial_\Phi)\xrightarrow{\simeq}(C_*(LM),\partial),$$

where

$$\partial_\Phi(x\otimes a) := \partial x\otimes a + (-1)^{|x|}x\otimes\partial a + \sum(-1)^{|x'|}x'\otimes\Phi(x'')\circ a,$$

for $x\in S_*(M)$ and $a\in C_*(\Omega M)$ with $\circ$ the action given in Lemma 3.6.

Proof. Apply Brown's theorem (Theorem 3.5) to the fibration $\Omega M\to LM\to M$. $\square$

3.2 The cyclic bar complex and the cocyclic cobar complex

In this subsection we discuss some algebras. These concepts are important in the understanding of the free loop space.

3.2.1 The cocyclic cobar complex of a DG coalgebra

We begin with the definition of the cobar construction of a DG coalgebra:

Definition 3.8 (cobar construction)**.** *Let (C, Δ, d) be a DG coalgebra and let $\bar{C}$ be the kernel of the counit. The cobar construction of C, denoted by $\underline{\Omega}(C)$, is a DG algebra defined as follows: As an algebra $\underline{\Omega}(C)$ is the tensor algebra*

$$\bigoplus_{n\geq 0}(\Sigma\bar{C})^{\otimes n}$$

generated by the $\Sigma\bar{C}$ of $\bar{C}$, where Σ means the degree of $\bar{C}$ is shifted down by one. Elements of $\underline{\Omega}(C)$ are written as $[a_1|\cdots|a_n]$, where $a_i \in \bar{C}$, and the unit is given by $[\ ]$. The differential d_{A} on $[x]$is given by:

$$d_{\mathrm{A}}[x] := -[dx] - \sum(-1)^{|x'|}[x'|x''], \quad \textit{for} \quad x \in \bar{C}$$

where x' and x'' comes from the reduced coproduct $\bar{\Delta}x = \sum x' \otimes x'' - 1 \otimes x - x \otimes 1$, and then extends to $\underline{\Omega}(C)$ by derivation. By coassociativity of C, we have that $d_{\mathrm{A}}^2 = 0$.

Lemma 3.9. *The identity map*

$$\begin{array}{rcl} \tau : C & \longrightarrow & \underline{\Omega}(C) \\ a & \longmapsto & [a] \end{array}$$

is a twisting cochain (see Definition 3.1). Moreover, it is universal in the sense that for any twisting cochain $\Phi : C \to A$, where A is a DG algebra, there is a DG algebra map $\eta : \underline{\Omega}(C) \to A$ such that the following diagram commutes:

$$\begin{tikzcd} & \underline{\Omega}(C) \arrow[d, "\eta"] \\ C \arrow[ur, "\tau"] \arrow[r, "\Phi"] & A. \end{tikzcd}$$

Proof. Define η as follows:

$$\begin{array}{rccl} \eta : & \underline{\Omega}(C) & \longrightarrow & A \\ & [a_1|\cdots|a_n] & \longmapsto & \Phi(a_1)\cdots\Phi(a_n). \end{array}$$

Since $\Phi : C \to A$ is a degree -1 map and $\Phi(1) = 0$, it is in fact a map $\Sigma\bar{C} \to A$. Since $\underline{\Omega}(C)$ is the free algebra generated by $\Sigma\bar{C}$, by the property of freeness, η is a well defined algebra map. To show η is a chain map, note that by the definition of twisting cochain, we have

$$\begin{array}{rcl} \delta(\eta[a]) & = & \delta \circ \Phi(a) \\ & = & -\Phi(da) - \sum(-1)^{|a'|}\Phi(a')\Phi(a'') \end{array}$$

$$\begin{aligned} &= -\eta([da]) - \sum(-1)^{|a'|}\eta([a'|a'']) \\ &= \eta(d_{\mathrm{A}}[a]). \end{aligned}$$

This proves the lemma. □

Theorem 3.10. *Let C be a DG cocommutative coalgebra, then the cobar construction of C, $\underline{\Omega}(C)$ is a DG Hopf algebra, which is the universal enveloping algebra of the free DG Lie algebra generated by $\Sigma\bar{C}$, with the differential*

$$\begin{aligned} d_{\mathscr{L}} : \mathscr{L}(\Sigma\bar{C}) &\longrightarrow \mathscr{L}(\Sigma\bar{C}) \\ \Sigma\alpha &\longmapsto -\Sigma da - \sum(-1)^{|a'|}[\Sigma a', \Sigma a'']. \end{aligned}$$

In particular, any element in $\Sigma\bar{C}$ is primitive in the sense that

$$\Delta\Sigma a = \Sigma a \otimes 1 + 1 \otimes \Sigma a.$$

Proof. See Quillen [23], Proposition 6.2. □

For more details of Hopf algebras, Lie algebras, etc., see Milnor-Moore [21] or Quillen [23].

Remark 3.11. In [1] Adams proves that the cobar construction of the 2-reduced singular chain complex of a simply connected manifold is quasi-isomorphic, as DG algebras, to the singular chain complex of ΩM. His construction is similar to that of Brown (Theorem 3.3). In this sense we may view Brown's twisting cochain Φ as the identity map, too.

Definition 3.12 (the cocyclic cobar complex)**.** *Let (C, d) be a cocommutative DG coalgebra and $\underline{\Omega}(C)$ be its cobar construction. Define an operator*

$$b : C \otimes \underline{\Omega}(C) \longrightarrow C \otimes \underline{\Omega}(C)$$

by

$$\begin{aligned} & b(x \otimes [a_1|\cdots|a_n]) \\ := \; & dx \otimes [a_1 \cdots |a_n] + (-1)^{|x|} x \otimes d_{\mathrm{A}}[x_1|\cdots|a_n] \\ & + \sum(-1)^{|x'|} x' \otimes \tau x'' \circ [a_1|\cdots|a_n] \\ = \; & dx \otimes [a_1|\cdots|a_n] \\ & -(-1)^{|x|} x \otimes \left(\sum(-1)^{|[a_1|\cdots|a_i]|}\left([a_1|\cdots|da_i|\cdots|a_n] + (-1)^{|a_i'|}[a_1|\cdots|a_i'|a_i''|\cdots|a_n]\right)\right) \\ & + \sum(-1)^{|x'|} x' \otimes \left([x''|a_1|\cdots|a_n] - (-1)^{|[a_1|\cdots|a_n]|(|x''|-1)}[a_1|\cdots|a_n|x'']\right), \end{aligned}$$

Where in the first equality, $\circ$ is the left adjoint action of the Hopf algebras $\underline{\Omega}(C)$. By Lemma 3.9, b is a twisted differential, $b^2 = 0$. The complex $(C \otimes \underline{\Omega}(C), b)$ is called the cocyclic cobar complex of C.

In the above definition, the second equality holds because the images of C are all primitive, and for primitive elements, say τx, we have $\Delta\tau x = \tau x \otimes 1 + 1 \otimes \tau x$ and $S\tau x = -\tau x$.

Lemma 3.13. *b is a coderivation with respect to the coproduct induced from two coalgebras.*

Proof. For $x \otimes a \in C \otimes \underline{\Omega}(C)$, suppose

$$\Delta x = \sum x' \otimes x'', \quad \Delta a = \sum a' \otimes a'',$$

ignoring the signs for a moment (they can be dealt with systematically), we have

$$\Delta(x \otimes a) = \sum x' \otimes a' \bigotimes x'' \otimes a''$$

and

$$b(x \otimes a) = dx \otimes a + x \otimes d_{\mathrm{A}} a + \sum x' \otimes \tau x'' \circ a,$$

thus

$$\begin{aligned}
b(\Delta(x \otimes a)) \quad = \quad & \sum \Big(dx' \otimes a' \bigotimes x'' \otimes a'' + x' \otimes d_{\mathrm{A}} a'' \bigotimes x'' \otimes a'' \Big) && (10)\\
& + \sum \Big(x' \otimes a' \bigotimes dx'' \otimes a'' + x \otimes a' \bigotimes x'' \otimes d_{\mathrm{A}} a'' \Big) && (11)\\
& + \sum (x')' \otimes \tau (x')'' \circ a' \bigotimes x'' \otimes a'' && (12)\\
& + \sum x' \otimes a' \bigotimes (x'')' \otimes \tau (x'')'' \circ a'', && (13)
\end{aligned}$$

while

$$\begin{aligned}
\Delta(b(x \otimes a)) \quad = \quad & \Delta \Big(dx \otimes a + x \otimes d_{\mathrm{A}} a + \sum x' \otimes \tau x'' \circ a \Big) \\
= \quad & \Delta \Big(dx \otimes a + x \otimes d_{\mathrm{A}} a \Big) && (14)\\
& + \sum (x')' \otimes (\tau x'' \circ a)' \bigotimes (x')'' \otimes (\tau x'' \circ a)''. && (15)
\end{aligned}$$

Note (10)+(11)=(14), we only need to show (12)+(13)=(15). Let us look at (15): first, note in (15), there is a factor $(\tau x'' \circ a)'$ and $(\tau x'' \circ a)''$ which are the factors of the coproduct $\Delta(\tau x \circ a)$, while

$$\begin{aligned}
& \Delta(\tau x \circ a) \\
= \quad & \Delta \Big(\tau x'' \cdot a - a \cdot \tau x'' \Big) \\
= \quad & \Delta \tau x'' \cdot \Delta a - \Delta a \cdot \Delta \tau x'' \\
= \quad & \Big(\tau x'' \otimes 1 + 1 \otimes \tau x'' \Big) \Big(\sum a' \otimes a'' \Big) - \Big(\sum a' \otimes a'' \Big) \Big(\tau x'' \otimes 1 + 1 \otimes \tau x'' \Big) \\
= \quad & \sum \Big(\tau x'' \circ a' \bigotimes a'' + a' \bigotimes \tau x'' \circ a'' \Big),
\end{aligned}$$

so to show (12)+(13)=(15), it is the same to show

$$\begin{aligned}
& \sum (x')' \otimes \tau (x')'' \circ a' \bigotimes x'' \otimes a'' + x' \otimes a' \bigotimes (x'')' \otimes \tau (x'')'' \circ a'' && (16)\\
= \quad & \sum (x')' \otimes \tau x'' \circ a' \bigotimes (x')'' \otimes a'' + (x')' \otimes a' \bigotimes (x')'' \otimes \tau x'' \circ a''. && (17)
\end{aligned}$$

Comparing the first and the second term in (16) and (17) respectively, by coassociativity and cocommutativity of C we see they are equal. $\square$

3.2.2 The cyclic bar complex of a DG algebra

Definition 3.14 (cyclic bar complex)**.** *Let (A,d) be a (negatively) graded commutative DG algebra over k, with unit $\eta : k \to A$. The reduced cyclic bar complex (also called the reduced Hochschild chain complex) of A, denoted by $A \otimes \underline{\Omega}(A)$ or $(HC_*(A;A), b)$, is a chain complex defined as follows: As a vector space, $HC_*(A;A) = \bigoplus_n A \otimes (S\bar{A})^{\otimes n}$, where $\bar{A}$ is the cokernel of the unit and $S\bar{A}$ is $\bar{A}$ with degree shifted up by one. Denote elements of $HC_*(A;A)$ by $v \otimes [a_1|\cdots|a_n]$, then the differential b is given by*

$$\begin{aligned}
& b(v\otimes[a^1|\cdots|a^n]) \\
:= \;& dv\otimes[a^1|\cdots|a^n] - \sum_i (-1)^{|v|+|[a^1|\cdots|a^{i-1}]|} v\otimes[a^1|\cdots|da^i|\cdots|a^n] \\
& + \sum_i (-1)^{|v|+|[a^1|\cdots|a^i]|} v\otimes[a^1|\cdots|a^i a^{i+1}|\cdots|a^n] \\
& + (-1)^{|v|} x a^1 \otimes [a^2|\cdots|a^n] - (-1)^{|v||a^n|+(|a^n|-1)|[a^1|\cdots|a^{n-1}]|} a^n v \otimes [a^1|\cdots|a^{n-1}].
\end{aligned}$$

The homology of the cyclic bar complex is called the Hochschild homology of A.

Definition 3.15 (bar construction)**.** *In the above definition, we call the free DG coalgebra*

$$\bigoplus_{n\geq 0} (S\bar{A})^{\otimes n}$$

with differential defined by

$$\begin{aligned}
d_{\mathrm{A}}([a^1|\cdots|a^n]) \;:=\; & -\sum_i (-1)^{|[a^1|\cdots|a^{i-1}]|}[a^1|\cdots|da^i|\cdots|a^n] \\
& + \sum_i (-1)^{|[a^1|\cdots|a^{i-1}]|}[a^1|\cdots|a^{i-1}|a^i a^{i+1}|\cdots|a^n]
\end{aligned}$$

the bar construction of A, and is also denoted by $\underline{\Omega}(A)$ (compare Definition 3.8).

Given a DG algebra A, there is a natural product on its cyclic bar complex, called *the shuffle product*. Let us now describe. If $(a_1, \cdots, a_p)$ and $(b_1, \cdots, b_q)$ are two ordered sets, then a shuffle σ of $(a_1, \cdots, a_p)$ and $(b_1, \cdots, b_q)$ is a permutation of $(a_1, \cdots, a_p, b_1, \cdots, b_q)$ such that $\sigma(a_i)$ occurs before $\sigma(a_j)$ and $\sigma(b_i)$ occurs before $\sigma(b_j)$ whenever $i < j$. The shuffle of two sets appears in our discussion in the following way: Let V be a vector space and $T(V)$ be the tensor algebra generated by V, it is a Hopf algebra viewed as the universal enveloping algebra of the free Lie algebra generated by V, i.e. $T(V) = U\mathscr{L}(V)$. Denote elements of $T(V)$ by $a_1 \otimes \cdots \otimes a_p$, then

$$\Delta(a_1\otimes\cdots\otimes a_p) = \sum_\sigma a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(i)} \bigotimes a_{\sigma(i+1)}\otimes\cdots\otimes a_{\sigma(p)}, \tag{18}$$

where the sum runs over all σ such that $(1, \cdots, p)$ is a shuffle of $(1, \cdots, i)$ and $(i+1, \cdots, p)$.

Now in the cyclic bar complex of a DG algebra A, given $x \otimes [a^1|\cdots|a^p]$ and $y \otimes [b^1|\cdots|b^q]$, define their *shuffle product* as

$$(v\otimes[a^1|\cdots|a^p]) \times (w\otimes[b^1|\cdots|b^q])$$

$$:= \quad (-1)^{|w||[a^1|\cdots|a^p]|}\sum_\sigma(-1)^\epsilon vw\otimes\sigma([a^1]\otimes\cdots\otimes[a^p]\otimes[b^1]\otimes\cdots\otimes[b^q]),$$

where σ runs over all shuffles of $([a^1],\cdots,[a^p])$ and $([b^1],\cdots,[b^q])$, and ϵ is the sign of the permutation of the elements.

Lemma 3.16. *Let A be a commutative DG algebra over k. Then*

$$(A\otimes\underline{\Omega}(A),\times,b)$$

is a commutative DG algebra.

Proof. See, for example, Getzler et al [13], Proposition 4.1. □

Proposition 3.17. *Let C be a cocommutative DG coalgebra and A be its dual DG algebra. Then there is a non-degenerate DG pairing*

$$\langle\ ,\ \rangle : C\otimes\underline{\Omega}(C)\bigotimes A\otimes\underline{\Omega}(A)\quad\longrightarrow\quad k$$

given by

$$\langle a_0\otimes[a_1|\cdots|a_n], b^0\otimes[b^1|\cdots|b^n]\rangle=\prod_{i=0}^n\langle a_i,b^i\rangle.$$

Moreover, the pairing respects the coproduct of $C\otimes\underline{\Omega}(C)$ and the shuffle product of $A\otimes\underline{\Omega}(A)$, namely, for any $\alpha\in C\otimes\underline{\Omega}(C)$ and $\mu,\nu\in A\otimes\underline{\Omega}(A)$,

$$\langle\alpha,\mu\times\nu\rangle=\sum\langle\alpha',\mu\rangle\langle\alpha'',\nu\rangle,$$

where $\Delta\alpha=\sum\alpha'\otimes\alpha''$.

Proof. This follows from a direct computation. □

Finally, we introduce the definition of the Hochschild cochain complex and the Hochschild cohomology of a DG algebra, which will be used later:

Definition 3.18 (Hochschild cohomology)**.** *Let (A,d) be a DG algebra over k. The Hochschild cochain complex $(HC^*(A;A),b)$ is the vector space $\bigoplus_n \mathrm{Hom}(A^{\otimes n};A)$ with differential b defined as follows: for $f\in\mathrm{Hom}(A^{\otimes n};A)$, bf is the sum of two terms $b_{\mathrm{I}}f\in\mathrm{Hom}(A^{\otimes n};A)$ and $b_{\mathrm{II}}f\in\mathrm{Hom}(A^{\otimes(n+1)};A)$, where*

$$\begin{aligned}&(b_{\mathrm{I}}f)(a_0\otimes\cdots\otimes a_{n-1})\\ =\;&(-1)^{|a_0|+\cdots+|a_{n-1}|+n-1}d(f(a_0\otimes\cdots\otimes a_{n-1}))\\ &+\sum_i(-1)^{|a_0|+\cdots+|a_{i-1}|-i+1}f(a_0\otimes\cdots\otimes da_i\otimes\cdots\otimes a_{n-1})\end{aligned}$$

and

$$\begin{aligned}&(b_{\mathrm{II}}f)(a_0\otimes\cdots\otimes a_n)\\ =\;&(-1)^{|a_0|}a_0f(a_1\otimes\cdots\otimes a_n)-(-1)^{|a_0|+\cdots+|a_n|-n}f(a_0\otimes\cdots\otimes a_{n-1})a_n\\ &+\sum_i(-1)^{|a_0|+\cdots+|a_i|-i+1}f(a_0\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_n).\end{aligned}$$

We have $b^2=0$. The cohomology of $(HC^*(A;A),b)$ is called the Hochschild cohomology of A, and denoted by $HH^*(A;A)$.

3.2.3 The cyclic structure and A_∞-structures

The cyclic bar complex of an algebra has a cyclic structure, which was first observed by Connes in his study of non commutative geometry. The cyclic operator thus obtained, always denoted by B, was later studied by Jones to model the S^1-action on the free loop space (see Jones [15], Getzler-Jones [12], and Getzler et al [13]). Let us recall their results.

Definition 3.19 (Connes' cyclic B-operator)**.** *Let A be a DG algebra and $HC_*(A;A)$ be the reduced cyclic bar complex of A. Define the Connes cyclic operator B on $HC_*(A;A)$ as*

$$\begin{aligned} & B(a^0\otimes[a^1|\cdots|a^n]) \\ := & \sum_{i=0}^{n}(-1)^{|[a_0|\cdots|a_{i-1}]||[a_i|\cdots|a_n]|}1\otimes[a^i|\cdots|a^n|a^0|\cdots|a^{i-1}], \end{aligned}$$

where 1 is the unit.

Lemma 3.20. *Let B as above, then*

$$B^2=0, \quad \textit{and} \quad bB+Bb=0.$$

Proof. See, for example, Connes [8], Lemma 30. ☐

Definition 3.21 (the dual of Connes' cyclic B-operator)**.** *Let C be a DG coalgebra. Define on the cocyclic cobar complex $C\otimes\underline{\Omega}(C)$ the following operator*

$$\begin{aligned} & B(x\otimes[a_1|\cdots|a_n]) \\ := & \sum_{i=1}^{n-1}(-1)^{|[a_i|\cdots|a_n]||[a_1|\cdots|a_{i-1}]|}\varepsilon(x)a_i\otimes[a_{i+1}|\cdots|a_n|a_1|\cdots|a_{i-1}], \end{aligned}$$

where ε is the counit of C.

Lemma 3.22. *$B^2=0$ and $bB+Bb=0$.*

Proof. Since the cocyclic cobar complex is dual to the cyclic bar complex (see Proposition 3.17), it follows from Lemma 3.20. ☐

Definition 3.23 (cyclic homology)**.** *Let A be a (negatively graded) DG algebra over k, and u be a parameter of degree -2. Let $A\otimes\underline{\Omega}(A)[u]$ be $A\otimes\underline{\Omega}(A)$ tensor with $k[u]$, where u commutes with the shuffle product (Lemma 3.16) and denote $b_B:=b+uB$. Then $b_B^2=0$, and we call*

$$(A\otimes\underline{\Omega}(A)[u],b_B)$$

the cyclic complex of A, and its homology is called the cyclic homology of A.

Analogously, let C be a DG coalgebra over k, and v be a parameter of degree 2. Define on $C\otimes\underline{\Omega}(C)[v]$ a differential operator b_B by

$$b_B(v^n\otimes x):=\begin{cases} v^n\otimes bx+v^{n-1}\otimes Bx, & \textit{for} \quad n\geq 1,\\ bx, & \textit{for} \quad n=0,\end{cases}$$

Then $b_B^2=0$ and the complex

$$(C\otimes\underline{\Omega}(C)[v],b_B)$$

is called the cyclic chain complex of C, and its homology is called the cyclic homology of C.

The cyclic bar complex and the cyclic complex of a DG algebra are related to the free loop space as follows: If A is the de Rham algebra of a simply connected, smooth manifold, then $(A\otimes\underline{\Omega}(A), b)$ and $(A\otimes\underline{\Omega}(A), b_B)$ give the cochain and the S^1-equivariant cochain complex model of LM respectively (see below). We first recall Borel's definition of equivariant homology.

Definition 3.24. *Let G be compact Lie group and M be a manifold with a G-action. Let BG be the classifying space of G and EG be its total space. The G-equivariant homology and cohomology of M, denoted by $H_*^G(M)$ and $H_G^*(M)$ respectively, are the homology and cohomology of $EG\times_G M$, i.e.*

$$H_*^G(M) := H_*(EG\times_G M) \quad \text{and} \quad H_G^*(M) := H^*(EG\times_G M).$$

Lemma 3.25. *Let X be a topological space with an S^1-action given by $f : S^1\times X\to X$. Then at chain level we have two operations:*

$$\begin{array}{rccc} J: & C_*(X) & \longrightarrow & C_{*+1}(X) \\ & \alpha & \longmapsto & (-1)^{|\alpha|} f_\#(z\times\alpha) \end{array} \tag{19}$$

and

$$\begin{array}{rccc} I: & C^*(X) & \longrightarrow & C^{*+1}(X) \\ & w & \longmapsto & (-1)^{|w|} f^\#(w)/z, \end{array} \tag{20}$$

where z is the fundamental cycle of S^1, and $\times$ and $/$ are the cross product and the slant product respectively (for the definition of these two products, see [14], p. 278-280). Modulo degenerate chains, $I^2 = J^2 = 0$ and $\partial J + J\partial = 0$, $\delta I + I\delta = 0$.

Proof. See, for example, Jones [15], §4. □

Definition 3.26. *Let X be an S^1-space and let $C_*(X)$ and $C^*(X)$ be the singular chain and cochain complexes of X respectively. Let u be a parameter of degree -2 and v be a parameter of degree 2. Define*

$$\partial_J : C_*(X)[v] \longrightarrow C_*(X)[v]$$

and

$$\delta_I : C^*(X)[u] \longrightarrow C^*(X)[u]$$

as follows:

$$\partial_J(v^n\otimes\alpha) := \begin{cases} \partial\alpha\otimes v^n + J(\alpha)\otimes v^{n-1}, & \textit{for} \quad n\geq 1, \\ \partial\alpha, & \textit{for} \quad n=0. \end{cases}$$

and

$$\delta_I(w\otimes u^n) := \delta w\otimes u^n + I(w)\otimes u^{n+1}, \ \textit{for } n\geq 0.$$

Proposition 3.27. *Let X be an S^1-space. With ∂_J, δ_I defined above, there are chain equivalences*

$$\begin{array}{rcl} \phi : (C_*(X)[v], \partial_J) & \xrightarrow{\simeq} & (C_*^{S^1}(X), \partial), \\ \psi : (C^*(X)[u], \delta_I) & \xrightarrow{\simeq} & (C^*_{S^1}(X), \delta). \end{array}$$

Proof. See Getzler et al [13], Proposition 1.5. □

Theorem 3.28 (Jones)**.** *Let M be a simply connected manifold and $A(M)$ be the (de Rham) cochain complex of M. Then there is a chain map*

$$\phi : (A \otimes \underline{\Omega}(A), b, B) \longrightarrow (C^*(LM), \delta, I),$$

which leads to chain equivalence

$$\tilde{\phi} : (A \otimes \underline{\Omega}(A)[u], b_B) \xrightarrow{\simeq} (C^*(LM)[u], \delta_I).$$

Proof. See Jones [15], §4 or Getzler et al [13], Theorem 2.1. □

However, the cocyclic chain complex $(C \otimes \underline{\Omega}(C)[u], \Delta, b_B)$ of C, is no more a DG coalgebra, i.e. b_B is no more a coderivation. In the cyclic bar complex case, Getzler, Jones and Patrick have observed that there is a perturbation of the product, such that the complex is associative up to homotopy. And they have found an A_∞-algebra structure on the cyclic bar complex ([13]).

Definition 3.29. *Let V be a graded vector space over a field k. An A_∞-coalgebra on V consists of V and a sequence of linear operators*

$$m_n : V \to V^{\otimes n}, \quad \deg(m_n) = n - 2,$$

satisfying the following conditions:

(1) *$m_1 = d : V \to V$ is a differential;*

(2) *$m_2 = \Delta : V \to V^{\otimes 2}$ is a chain map (called coproduct), i.e. d is a coderivation with respect to Δ:*

$$\Delta \circ d = (d \otimes Id + Id \otimes d) \circ \Delta; \tag{21}$$

(3) *$m_3 : V \to V^{\otimes 3}$ is a chain homotopy for the coassociativity of Δ, i.e.*

$$m_3 \circ d + (d \otimes Id^{\otimes 2} + Id \otimes d \otimes Id + Id^{\otimes 2} \otimes d) \circ m_3 = (\Delta \otimes Id) \circ \Delta - (Id \otimes \Delta) \circ \Delta; \tag{22}$$

(4) *$m_n : V \to V^{\otimes n}$, $n \geq 4$ are the higher homotopy operators, and all m_n satisfy the general formula:*

$$\sum_{k=1}^{n} (-1)^k \Big(\sum_{j=0}^{k-1} Id^{\otimes j} \otimes m_{n-k+1} \otimes Id^{\otimes k-j-1} \Big) \circ m_k = 0, \tag{23}$$

where

$$\begin{aligned} & Id^{\otimes j} \otimes m_{n-k+1} \otimes Id^{\otimes k-j-1}(a_1 \otimes a_2 \otimes \cdots \otimes a_k) \\ := \; & (-1)^{(n-k+1)(|a_1|+\cdots+|a_j|)} a_1 \otimes \cdots \otimes a_j \otimes m_{n-k+1}(a_{j+1}) \otimes a_{j+2} \cdots \otimes a_k. \end{aligned}$$

Note that (21) and (22) satisfy the general formula (23), and we shall write $\Big(\sum_{j=0}^{k-1} Id^{\otimes j} \otimes m_{n-k+1} \otimes Id^{\otimes k-j-1} \Big) \circ m_k$ as $m_{n-k+1} \circ m_k$ for short.

In other words, an A_∞-coalgebra on A is the free associative algebra (the tensor algebra) generated by A with degrees shifted down by one, and a derivation d on it with $d^2 = 0$.

Proposition 3.30. *If V is an A_∞-coalgebra, then its homology $H_*(V, m_1)$ is a graded coalgebra.*

Theorem 3.31. *Let C be a cocommutative DG coalgebra over a field k of characteristic 0, and let u be a parameter of degree 2. There is a sequence of linear operators*

$$m_n : C \otimes \underline{\Omega}(C)[u] \longrightarrow (C \otimes \underline{\Omega}(C)[u])^{\otimes n}, \quad n = 1, 2, \cdots$$

which extends to the free coalgebra generated by $C \otimes \underline{\Omega}(C)[u]$ by derivation, satisfying $m_1 = b_B$ and $m_1 + m_2 + \cdots$ is a coderivation of square 0, i.e. $(C \otimes \underline{\Omega}(C)[u], \{m_n\})$ forms an A_∞-coalgebra.

Proof. The theorem is a corollary of Getzler et al [13], Proposition 4.3. □

3.3 The complete DG coalgebra of the free loop space

In last section we have constructed a complete DG coalgebra which models the chain complex of a manifold. The goal of this subsection is to apply the complete DG coalgebra to Brown's twisted tensor product model of the free loop space.

Definition 3.32 (complete cobar construction)**.** *Let C be a complete DG coassociative coalgebra with counit $\varepsilon : C \to k$. The complete cobar construction of C, denoted by $\hat{\underline{\Omega}}(C)$, is the direct sum of the complete tensor products:*

$$\bigoplus_{n \geq 0} \underbrace{\Sigma\bar{C}\hat{\otimes} \cdots \hat{\otimes}\Sigma\bar{C}}_{n},$$

where $\bar{C}$ is the kernel of the counit $\varepsilon : C \to k$, and Σ means shifting the degrees of the elements down by one.

Lemma 3.33. *Define on $\Sigma\bar{C}$ an operator*

$$\begin{array}{rcl} d_{\mathrm{A}} : \Sigma\bar{C} & \longrightarrow & \Sigma\bar{C}\hat{\otimes}\Sigma\bar{C} \\ \Sigma x & \longmapsto & -\Sigma dx - \sum(-1)^{|x'|}\Sigma x'\hat{\otimes}\Sigma x'', \end{array}$$

where $\Sigma x'$ and $\Sigma x''$ comes from the reduced coproduct of C (with degree shifted down by one):

$$\sum x' \otimes x'' := \Delta x - x \otimes 1 - 1 \otimes x.$$

Then d_{A} extends to $\hat{\underline{\Omega}}(C)$ under completion, with $d_{\mathrm{A}}^2 = 0$. Therefore, $(\hat{\underline{\Omega}}(C), d_{\mathrm{A}})$ is a DG algebra under completion. If moreover, C is cocommutative, then $(\hat{\underline{\Omega}}(C), d_{\mathrm{A}})$ is a complete DG coalgebra, where

$$\Delta\Sigma x := \Sigma x \otimes 1 + 1 \otimes \Sigma x, \Sigma x \in \Sigma\bar{C}$$

and extends to $\hat{\underline{\Omega}}(C)$ diagonally under completion.

Proof. We can extend d_{A} to $\Sigma\bar{C} \otimes \Sigma\bar{C}$ by derivation, however, in order to show $d_{\mathrm{A}}^2 = 0$, we have to extend it to $\Sigma\bar{C}\hat{\otimes}\Sigma\bar{C}$. The point here is that, the coproduct of C

$$\Delta : C \longrightarrow C\hat{\otimes}C$$

is a continuous map, by which we mean it maps a Cauchy sequence to a Cauchy sequence (in the projective topology), and therefore for an element in $C\hat{\otimes}C$, suppose it is approximated by a

Cauchy sequence $\{\sum_{i+j=n} f_i \otimes g_j\}$, where $\Delta \otimes id + id \otimes \Delta$ can be defined and whose images are also a Cauchy sequence. One may define the image of the limit under $id \otimes \Delta + \Delta \otimes id$ to be the limit of the images. And since the difficult part of extending d_{A} to $\Sigma\bar{C}\hat{\otimes}\Sigma\bar{C}$ is the same as that of extending $\Delta \otimes id + id \otimes \Delta$ to $C\hat{\otimes}C$, such a difficulty can be overcome by the above argument. $\square$

Theorem 3.34 (complete DG coalgebra of the based loop space)**.** *Let M be a simply connected, cubilated space. Let A be the Whitney polynomial differential forms of M and C be the currents. Then the complete cobar construction $\underline{\Omega}(C)$ gives a complete DG coalgebra model of the chain complex of ΩM.*

The proof is deferred to Section 6, where a more general case is proved. However, we can see this from K.-T. Chen's iterated integral theory: In [6], Chen proves that if M is a simply connected, smooth closed manifold, then the bar construction (Definition 3.15) of the differential forms of M gives a DG algebra model of the cochain complex of LM. Chen's proof still holds when we restrict our study on the Whitney forms. By definition, the complete cobar construction is exactly the dual complex of the bar complex, and therefore the theorem holds.

Definition 3.35 (complete cocyclic cobar complex)**.** *Let C be a complete DG coalgebra. Consider the complete tensor product $C\hat{\otimes}\hat{\underline{\Omega}}(C)$ of C and $\hat{\underline{\Omega}}(C)$, and define on it an operator:*

$$\begin{aligned} & b(x \otimes [a_1|\cdots|a_n]) \\ := \quad & dx \otimes [a_1|\cdots|a_n] + (-1)^{|x|} x \otimes d_{\mathrm{A}}[a_1|\cdots|a_n] \\ & -\sum (-1)^{|x'|} x' \otimes \Big([x''|a_1|\cdots|a_n] - (-1)^{|[x'']||[a_1|\cdots|a_n]|}[a_1|\cdots|a_n|x'']\Big), \end{aligned}$$

which extends to $C\hat{\otimes}\hat{\Omega}(C)$ under completion. Then $b^2 = 0$, and we call the complex $(C\hat{\otimes}\underline{\Omega}(C), b)$ the complete cocyclic cobar complex of C.

Theorem 3.36 (complete DG coalgebra model of the free loop space)**.** *Let M be a simply connected, smooth closed manifold, and let A be the set of Whitney forms and C be the set of currents on M. Then there is a chain equivalence*

$$\psi : (C\hat{\otimes}\hat{\underline{\Omega}}(C), b) \xrightarrow{\simeq} (C_*(LM), \partial).$$

Proof. The proof is the dual version of Chen [6] and Jones [15], see also Getzler et al [13]. We can also see it from twisted tensor product point of view: Since C gives a chain model of M and $\hat{\underline{\Omega}}(C)$ gives a chain model of ΩM, and the identity map

$$\tau : C \longrightarrow \hat{\underline{\Omega}}(C)$$

is the twisting cochain of Brown (Remark 3.11), applying Theorem 3.7 we obtain the chain equivalence. $\square$

Theorem 3.37. *Let M be a simply connected, smooth closed n-manifold and let A be the Whitney forms of M and C the the currents. Consider the tensor product $A\hat{\otimes}\hat{\underline{\Omega}}(C)$, and define an operator*

$$b : A\hat{\otimes}\hat{\underline{\Omega}}(C) \longrightarrow A\hat{\otimes}\hat{\underline{\Omega}}(C)$$

by (compare Theorem 2.24):

$$\begin{aligned} & b(x\otimes[a_1|\cdots|a_n]) \\ := \; & dx\otimes[a_1|\cdots|a_n]+(-1)^{|x|}x\otimes d_{\mathrm{A}}[a_1|\cdots|a_n] \\ & +\sum_i(-1)^{|x|+|b_i|}x\wedge\beta_i\otimes\Big([\beta_i^*|a_1|\cdots|a_n]-(-1)^{(|b_i|-1)|[a_1|\cdots|a_n]|}[a_1|\cdots|a_n|\beta_i^*]\Big), \end{aligned}$$

then $b^2=0$. The embedding of A in C gives a quasi-isomorphism

$$A\hat{\otimes}\underline{\hat{\Omega}}(C)\xhookrightarrow{\iota\otimes id}C\hat{\otimes}\underline{\hat{\Omega}}(C)$$

which induces isomorphism

$$H_*(A\hat{\otimes}\underline{\hat{\Omega}}(C)\xrightarrow{\cong}H_*(LM).$$

Proof. Since the identity map $\tau: C\to\underline{\hat{\Omega}}(C)$ is a twisting cochain (now in complete sense) as before, b is a twisted differential, i.e. $b^2=0$ (compare Definitions 3.12 and 3.35). Now we show

$$\iota\otimes id: A\hat{\otimes}\underline{\hat{\Omega}}(C)\longrightarrow C\hat{\otimes}\underline{\hat{\Omega}}(C)$$

is a chain map: Since $\iota: A\to C$ is a chain map, $\iota\otimes id$ preserves the differential part (i.e. those terms only containing the differential) on both sides, so we only need to check $\iota\otimes id$ preserves the diagonal part (i.e. those terms only involving the coproduct). However, by Theorem 2.24, Δ of A factors through $A\hat{\otimes}C\to C\hat{\otimes}C$, which implies that $\iota\otimes id$ also preserves the diagonal part, therefore $\iota\otimes id$ is a chain map. Before showing $\iota\otimes id$ is a quasi-isomorphism, we claim:

Lemma 3.38. *Let Ω_0 be the degree zero currents of M, then the the ideal generated by Ω_0 is acyclic.*

Proof. This is a dual version of Getzler et al [13], Proposition 2.4. ☐

Denote by $\widetilde{\underline{\hat{\Omega}}(C)}$ the complete cobar construction modulo the ideal, then

$$C\hat{\otimes}\widetilde{\underline{\hat{\Omega}}(C)}\quad\text{and}\quad C\hat{\otimes}\underline{\hat{\Omega}}(C)$$

are quasi-isomorphic, so are $A\hat{\otimes}\widetilde{\underline{\hat{\Omega}}(C)}$ and $A\hat{\otimes}\underline{\hat{\Omega}}(C)$. Give a filtration of $A\hat{\otimes}\widetilde{\underline{\hat{\Omega}}(C)}$ and $C\hat{\otimes}\widetilde{\underline{\hat{\Omega}}(C)}$ by

$$F_p=\bigoplus_{q\le p}A^q\hat{\otimes}\widetilde{\underline{\hat{\Omega}}(C)},\quad\text{and}\quad F_p=\bigoplus_{q\le p}C_q\hat{\otimes}\widetilde{\underline{\hat{\Omega}}(C)},$$

then the filtration is complete and moreover $\iota\otimes id$ preserves the filtration. The E^2-terms of the associated spectral sequences are, since M is simply connected,

$$H^*(M)\otimes H_*(\Omega M),\quad\text{and}\quad H_*(M)\otimes H_*(\Omega M),$$

which are isomorphic, and therefore by the comparison theorem of spectral sequences (see, for example, McCleary [17] Theorem 3.26, p. 82), $\iota\otimes id$ is a quasi-isomorphism. ☐

Definition 3.39 (the dual of Connes' cyclic B-operator in complete cocyclic cobar complex)**.** *Let C be a complete cocommutative DG coalgebra, and $C\hat{\otimes}\underline{\hat{\Omega}}(C)$ be the complete cocyclic cobar complex of C. Define the dual of Connes' cyclic B-operator*

$$B : C\hat{\otimes}\underline{\hat{\Omega}}(B) \longrightarrow C\hat{\otimes}\underline{\hat{\Omega}}(C)$$

as

$$B(x \otimes [a_1|\cdots|a_n]) \;\; := \;\; \sum_{i=1}^{n-1}(-1)^{|[a_i|\cdots|a_n]||[a_1|\cdots|a_{i-1}]|}\varepsilon(x)a_i \otimes [a_{i+1}|\cdots|a_n|a_1|\cdots|a_{i-1}],$$

where ε is the counit of C.

Lemma 3.40. *$B^2 = 0$ and $bB + Bb = 0$.*

Proof. The proof is the same as Lemma 3.22. □

We may also consider the cyclic homology of a complete cocommutative DG coalgebra (Definition 3.23). As a corollary of Jones (Theorem 3.28), we have:

Theorem 3.41. *Let M be a simply connected, smooth closed manifold, A be the Whitney forms and C be the currents on M. We have chain equivalence*

$$(C\hat{\otimes}\underline{\hat{\Omega}}(C), b, B) \longrightarrow (C_*(LM), \partial, J),$$

which induces an isomorphism

$$H_*(C\hat{\otimes}\underline{\hat{\Omega}}(C)[u], b_B) \xrightarrow{\cong} H_*^{S^1}(LM).$$

Proof. This is a direct corollary of Theorem 3.28. □

4 The Chas-Sullivan loop product

In this section we use the DG Frobenius-like algebra of a manifold M to give a model for the Chas-Sullivan loop product defined in [5]. The study of the commutativity of the loop product leads to a Gerstenhaber algebra structure on the free loop space, which is isomorphic to the Hochschild cohomology of the Whitney forms on the manifold.

4.1 Model of the Chas-Sullivan loop product

Lemma 4.1. *Let M be a simply connected, cubilated smooth closed manifold, and let A be the Whitney polynomial forms and C be the currents. Define a product*

$$\bullet : A\hat{\otimes}\underline{\hat{\Omega}}(C) \bigotimes A\hat{\otimes}\underline{\hat{\Omega}}(C) \longrightarrow A\hat{\otimes}\underline{\hat{\Omega}}(C)$$

by

$$\big(\alpha \otimes [a_1|\cdots|a_n]\big) \bullet \big(\beta \otimes [b_1|\cdots|b_m]\big) := (-1)^{|\beta||[a_1|\cdots|a_n]|}\alpha \cdot \beta \otimes [a_1|\cdots|a_n|b_1|\cdots|b_m]. \tag{24}$$

Then $(A \otimes \underline{\Omega}(C), \bullet)$ forms a DG algebra.

Proof. From the definition we see that $\bullet$ is associative, also in Theorem 3.37 we have shown b is a differential. Therefore we only need to show b is a derivation. Taking $\alpha \otimes x$ and $\beta \otimes y$ in $A\hat{\otimes}\underline{\Omega}(C)$, ignoring the signs, we have

$$\begin{aligned}
& b((\alpha \otimes x) \bullet (\beta \otimes y)) && (25)\\
= \ & b(\alpha \cdot \beta \otimes xy) \\
= \ & d(\alpha \cdot \beta) \otimes x \cdot y + \alpha \cdot \beta \otimes d_{\mathrm{A}}(x \cdot y) && (26)\\
& + \sum (\alpha \cdot \beta)' \otimes \tau(\alpha \cdot \beta)'' \circ (x \cdot y), && (27)
\end{aligned}$$

while

$$\begin{aligned}
& b(\alpha \otimes y) \bullet (\beta \otimes y) + (\alpha \otimes x) \bullet b(\beta \otimes y) && (28)\\
= \ & (d\alpha) \cdot \beta \otimes x \cdot y + \alpha \cdot \beta \otimes d_{\mathrm{A}}(x) \cdot y && (29)\\
& + \sum \alpha' \cdot \beta \otimes (\tau\alpha'' \circ x) \cdot y && (30)\\
& + \alpha \cdot (d\beta) \otimes x \cdot y + \alpha \cdot \beta \otimes x \cdot d_{\mathrm{A}}(y) && (31)\\
& + \sum \alpha \cdot \beta' \otimes x \cdot (\tau\beta'' \circ y). && (32)
\end{aligned}$$

To show (25)=(28), noting that (26)=(29)+(31), we only need to show (27)=(30)+(32), i.e.

$$\sum (\alpha \cdot \beta)' \otimes \tau(\alpha \cdot \beta)'' \circ (x \cdot y) = \sum \alpha' \cdot \beta \otimes (\tau\alpha'' \circ x) \cdot y + \sum \alpha \cdot \beta' \otimes x \cdot (\tau\beta'' \circ y).$$

By the Frobenius-like condition (Definition 2.17) it is the same for us to show

$$\tau z \circ (x \cdot y) = (\tau z \circ x) \cdot y + x \cdot (\tau z \circ y),$$

where $z = (\alpha\beta)''$. However, by Theorem 3.10, all τz are primitive and the primitive elements act as derivation, we are done. $\square$

Let us briefly recall the loop product define in [5]. For the free loop space LM of a manifold M, denote by $C_*(LM)$ the chain complex of the total space. For $x, y \in C_*(LM)$ two chains in general position (transversal), consider their projections in M, denoted by $\tilde{x}$ and $\tilde{y}$ respectively (we would like to call them the "shadow" of x and y, since such a "projection" does not preserve dimension and is usually not a chain map). The Chas-Sullivan loop product is defined as follows: first intersect $\tilde{x}$ and $\tilde{y}$ in M, then over the intersection set, do the Pontryajin product pointwisely. From this we get a chain in $C_*(LM)$, denoted by $x \bullet y$, which is usually called *the Chas-Sullivan loop product* of x and y:

$$\begin{aligned}
\bullet: \quad C_*(LM) \otimes C_*(LM) &\longrightarrow C_*(LM),\\
x \otimes y &\longmapsto x \bullet y.
\end{aligned}$$

Chas and Sullivan showed that ∂ is derivation with respect to $\bullet$. A theorem of Wilson ([28]) says that although the above product is defined on transversal chains, it already catches all the homology information of $C_*(LM)$, thus the Chas-Sullivan loop product is well-defined on the homology space $H_*(LM)$. Denote $\mathbb{H}_*(LM) = H_*(LM)[n]$, then $\mathbb{H}_*(LM)$ is a graded algebra with the product having degree 0.

Theorem 4.2 (model for the loop product)**.** *Let M be a simply connected, smooth closed manifold. Then the product $\bullet$ in Lemma 4.1 gives a model of the loop product in [5].*

Proof. Let us denote by

$$\phi : A\hat{\otimes}\underline{\hat{\Omega}}(C) \longrightarrow C_*(LM)[n]$$

the chain model of the free loop space. In last section we have shown that ϕ is a chain map, so here we only need to show ϕ is an algebra map. First let us consider $\phi(\alpha\otimes x)$ and $\phi(\beta\otimes y)$. They are two chains in LM, whose geometric pictures are the traces obtained by moving x (resp. y) along α (resp. β). Their shadows in M are α and β respectively. Now $\phi(\alpha\otimes x)\bullet\phi(\beta\otimes y)$ is a chain in LM described as follows: The shadow is $\alpha\cdot\beta$, and for any point $q\in\alpha\cdot\beta$, suppose there is a path γ connecting p and q, i.e.

$$\gamma : [0,1]\to\alpha\cdot\beta\subset M,\quad \gamma(0)=q,\quad \gamma(1)=p,$$

then by naturality of the twisting cochain, the fiber over q is the Pontryajin product

$$\gamma_{\#}(x)\cdot\gamma_{\#}(y), \tag{33}$$

where $\gamma_{\#}$ is the chain map induced from

$$\begin{array}{rccc} \gamma: & \Omega_p M & \longrightarrow & \Omega_q M, \\ & x & \longmapsto & \gamma\cdot x\cdot\gamma^{-1}. \end{array} \tag{34}$$

On the other hand, $\phi((-1)^{|x||\beta|}\alpha\cdot\beta\otimes x\cdot y)$ is a chain in LM described as follows: its shadow is also $\alpha\cdot\beta$, and the fiber over q is

$$\gamma_{\#}(x\cdot y). \tag{35}$$

In order to show

$$\phi(\alpha\otimes x)\bullet\phi(\beta\otimes y)=\phi((-1)^{|x||\beta|}\alpha\cdot\beta\otimes x\cdot y),$$

we only need to show (33)=(35):

$$\gamma_{\#}(x)\cdot\gamma_{\#}(y)=\gamma_{\#}(x\cdot y). \tag{36}$$

However, look at the path action (34), we have

$$\gamma(x\cdot y)=\gamma(x)\cdot\gamma(y),$$

for any $x,y\in\Omega_p M$, and on chain level, it exactly gives equality (36). □

4.2 Gerstenhaber algebra on the loop homology

In [5], Chas and Sullivan show that the loop product on the loop homology of a manifold is commutative. They show that, at chain level, the loop product is homotopy commutative, and such a homotopy comes from a new binary operator, which is inspired from the pre-Lie operator of Gerstenhaber on the Hochschild cochain complex of an associative algebra (Gerstenhaber [10]). The authors then discover that with the loop product and the pre-Lie operator, the homology of the free loop space forms a Gerstenhaber algebra.

4.2.1 Commutativity of the loop product

We first give a description of the pre-Lie operator $*$ defined in [5]: for two chains $\alpha, \beta \in C_*(LM)$ in general position, we have $\tilde{\alpha}$ is transversal to loops in β. Form a chain $\alpha * \beta$ given by the following loops: for any loop γ in β, first go around γ from the base point till the intersection point with $\tilde{\alpha}$, then go around the loops in α, and finally go around the rest of γ.

Definition 4.3. *Let A be a DG Frobenius-like algebra of a simply connected, smooth close manifold, and let $A\hat{\otimes}\underline{\hat{\Omega}}(C)$ be the twisted tensor product. Define an operator*

$$*: A\hat{\otimes}\underline{\hat{\Omega}}(C) \bigotimes A\hat{\otimes}\underline{\hat{\Omega}}(C) \longrightarrow A\hat{\otimes}\underline{\hat{\Omega}}(C)$$

as follows: for $\alpha = x\otimes[a_1|\cdots|a_n], \beta = y\otimes[b_1|\cdots|b_m] \in A\hat{\otimes}\underline{\hat{\Omega}}(C)$,

$$\alpha * \beta = \sum_{i=1}^{n}(-1)^{|y|+|\beta||[a_{i+1}|\cdots|a_n]|}\varepsilon(a_i y)x\otimes[a_1|\cdots|a_{i-1}|b_1|\cdots|b_m|a_{i+1}|\cdots|a_n], \tag{37}$$

where ε is the counit of C.

Lemma 4.4. *Let A be as above. Then for any $\alpha, \beta \in A\hat{\otimes}\underline{\hat{\Omega}}(C)$,*

$$b(\alpha * \beta) = b\alpha * \beta + (-1)^{|\alpha|+1}\alpha * b\beta + (-1)^{|\alpha|}(\alpha \bullet \beta - (-1)^{|\alpha||\beta|}\beta \bullet \alpha). \tag{38}$$

In particular, $(H_*(A\hat{\otimes}\underline{\hat{\Omega}}(C), \bullet)$ is a graded commutative algebra.

Proof. Take arbitrary $\alpha = x\otimes[a_1|\cdots|a_n], \beta = y\otimes[b_1|\cdots|b_m] \in A\hat{\otimes}\underline{\hat{\Omega}}(C)$. First observe that the expressions of $b(\alpha*\beta)$, $b\alpha * \beta$ and $\alpha * b\beta$ have two parts, one contains those terms involving the differentials of the entries in α and β (we call *the differential part*), the other contains those terms involving the coproducts of the entries in α and β (we call *the diagonal part*).

From the construction of $*$, we observe that the differential parts of both sides of (38) are equal. So we only need to check the diagonal parts. In fact, the diagonal part of $b(\alpha*\beta)$ equals

$$\sum_i \varepsilon(a_i y)x'\otimes[x''|a_1|\cdots|a_{i-1}|b_1|\cdots|b_m|a_{i+1}|\cdots|a_n] \tag{39}$$

$$+ \sum_i \varepsilon(a_i y)x'\otimes[a_1|\cdots|a_{i-1}|b_1|\cdots|b_m|a_{i+1}|\cdots|a_n|x''] \tag{40}$$

$$+ \sum_{i\neq j} \varepsilon(a_i y)x\otimes[a_1|\cdots|a_j'|a_j''|\cdots|a_{i-1}|b_1|\cdots|b_m|a_{i+1}|\cdots|a_n] \tag{41}$$

$$+ \sum_{i,j} \varepsilon(a_i y)x\otimes[a_1|\cdots|a_{i-1}|b_1|\cdots|b_j'|b_j''|\cdots|b_m|a_{i+1}|\cdots|a_n], \tag{42}$$

and the diagonal part of $b\alpha * \beta$ equals

$$\sum \varepsilon(x'' y)x'\otimes[b_1|\cdots|b_m|a_1|\cdots|a_n] \tag{43}$$

$$+ \sum_i \varepsilon(a_i y)x'\otimes[x''|a_1|\cdots|a_{i-1}|b_1|\cdots|b_m|a_{i+1}|\cdots|a_n] \tag{44}$$

$$+ \sum_i \varepsilon(a_i y)x'\otimes[a_1|\cdots|a_{i-1}|b_1|\cdots|b_m|a_{i+1}|\cdots|a_n|x''] \tag{45}$$

$$+ \quad \sum \varepsilon(x''y)x' \otimes [a_1|\cdots|a_n|b_1|\cdots|b_n] \tag{46}$$

$$+ \quad \sum_{i\neq j} \varepsilon(a_i y)x \otimes [a_1|\cdots|a_{i-1}|b_1|\cdots|b_m|a_{i+1}|\cdots|a'_j|a''_j|\cdots|a_m] \tag{47}$$

$$+ \quad \sum_{i} \varepsilon(a'_i y)x \otimes [a_1|\cdots|a_{i-1}|b_1|\cdots|b_m|a''_i|a_{i+1}|\cdots|a_m] \tag{48}$$

$$+ \quad \sum_{i} \varepsilon(a''_i y)x \otimes [a_1|\cdots|a_{i-1}|a'_i|b_1|\cdots|b_m|a_{i+1}|\cdots|a_m], \tag{49}$$

while the diagonal part of $\alpha * b\beta$ equals

$$\sum_{i} \varepsilon(a_i y')x \otimes [a_1|\cdots|a_{i-1}|y''|b_1|\cdots|b_m|a_{i+1}|\cdots|a_m] \tag{50}$$

$$+ \quad \sum_{i} \varepsilon(a_i y')x \otimes [a_1|\cdots|a_{i-1}|b_1|\cdots|b_m|y''|a_{i+1}|\cdots|a_m] \tag{51}$$

$$+ \quad \sum_{i,j} \varepsilon(a_i y)x \otimes [a_1|\cdots|a_{i-1}|b_1|\cdots|b'_j|b''_j|\cdots|b_n|a_{i+1}|\cdots|a_n]. \tag{52}$$

Note that A is a Frobenius-like algebra, by (1), we see that (39) and (44) cancel, so do (40) and (45), (41) and (47), (42) and (52), (48) and (51), (49) and (50). Two terms left are (43) and (46). However, from the connectedness and (1), in (43) we have

$$\sum \varepsilon(x''y)x' = \sum \varepsilon(x'')x'y = xy$$

so up to sign,

$$(43) = xy \otimes [b_1|\cdots|b_m|a_1|\cdots|a_n] = \beta \bullet a.$$

Similarly, we have $(46) = \alpha \bullet \beta$. Thus the lemma is proved. □

Definition 4.5 (pre-Lie algebra)**.** *Let V be a graded vector space over k. A pre-Lie structure on V is a degree one binary operator*

$$* : V \otimes V \longrightarrow V$$

such that

$$(\gamma * \alpha) * \beta - (-1)^{(|\alpha|+1)(|\beta|+1)}(\gamma * \beta) * \alpha = \gamma * (\alpha * \beta - (-1)^{(|\alpha|+1)(|\beta|+1)}\beta * \alpha). \tag{53}$$

We call $(V, *)$ a pre-Lie algebra (or pre-Lie system).

Lemma 4.6. *Let $(V, *)$ be a pre-Lie algebra. Define*

$$\begin{array}{rccl} \{,\}: & V \otimes V & \longrightarrow & V \\ & a \otimes b & \longmapsto & a * b - (-1)^{(|a|+1)(|b|+1)} b * a, \end{array}$$

then $(V, \{,\})$ is a degree one Lie algebra.

Proof. See Gerstenhaber [10], Theorem 1. □

Lemma 4.7. *Let A be as above. Then $(A \hat{\otimes} \underline{\hat{\Omega}}(C), *)$ is a pre-Lie algebra.*

Proof. Take arbitrary three elements $\alpha = x\otimes[a_1|\cdots|a_n], \beta = y\otimes[b_1|\cdots|b_m], \gamma = z\otimes[c_1|\cdots|c_l] \in A\hat{\otimes}\underline{\hat{\Omega}}(C)$. Up to sign, the four items in (53) are:

$$\begin{aligned}
& (\gamma*\alpha)*\beta \\
= & \sum_{i\neq j}\varepsilon(c_iy)\varepsilon(c_jx)z\otimes[c_1|\cdots|c_{i-1}|b_1|\cdots|b_m|c_{i+1}|\cdots|c_{j-1}|a_1|\cdots|a_n|c_{j+1}|\cdots|c_l] \quad (54)\\
+ & \sum_{i,j}\varepsilon(c_ix)\varepsilon(a_jy)z\otimes[c_1|\cdots|c_{i-1}|a_1|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|c_{i+1}|\cdots|c_l], \quad (55)
\end{aligned}$$

$$\begin{aligned}
& (\gamma*\beta)*\alpha \\
= & \sum_{i\neq j}\varepsilon(c_ix)\varepsilon(c_jy)z\otimes[c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|c_{i+1}|\cdots|c_{j-1}|b_1|\cdots|b_m|c_{j+1}|\cdots|c_l] \quad (56)\\
+ & \sum_{i,j}\varepsilon(c_iy)\varepsilon(b_jx)z\otimes[c_1|\cdots|c_{i-1}|b_1|\cdots|b_{j-1}|a_1|\cdots|a_m|b_{j+1}|\cdots|b_m|c_{i+1}|\cdots|c_l], \quad (57)
\end{aligned}$$

$$\begin{aligned}
& \gamma*(\alpha*\beta) \\
= & \sum_{i,j}\varepsilon(c_ix)\varepsilon(a_jy)z\otimes[c_1|\cdots|c_{i-1}|a_1|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|c_{i+1}|\cdots|c_l], \quad (58)
\end{aligned}$$

and

$$\begin{aligned}
& \gamma*(\beta*\alpha) \\
= & \sum_{i,j}\varepsilon(c_iy)\varepsilon(b_jx)z\otimes[c_1|\cdots|c_{i-1}|b_1|\cdots|b_{j-1}|a_1|\cdots|a_n|b_{j+1}|\cdots|b_m|c_{i+1}|\cdots|c_l]. \quad (59)
\end{aligned}$$

Note that (54) and (56) cancel, so do (55) and (58), (57) and (59). Thus (53) holds. □

Corollary 4.8. *Let A be as above. Then*

$$(A\hat{\otimes}\underline{\hat{\Omega}}(C), \{,\}, b)$$

is a degree one DG Lie algebra. In particular, $(H_*(A\hat{\otimes}\underline{\hat{\Omega}}(C), \{,\})$ is a degree one graded Lie algebra.

Proof. The degree one Lie algebra follows from the above lemma and the theorem of Gerstenhaber (Lemma 4.6). Lemma 4.4 shows b respects $\{,\}$: in fact, for any $\alpha,\beta\in A\hat{\otimes}\underline{\hat{\Omega}}(C)$,

$$\begin{aligned}
b\{\alpha,\beta\} &= b(\alpha*\beta-(-1)^{(|\alpha|+1)(|\beta|+1)}\beta*\alpha)\\
&= \big(b\alpha*\beta+(-1)^{|\alpha|+1}\alpha*b\beta\big)-(-1)^{(|\alpha|+1)(|\beta|+1)}\big(b\beta*\alpha+(-1)^{|\beta|+1}\beta*b\alpha\big)\\
&= \{b\alpha,\beta\}+(-1)^{|\alpha|+1}\{\alpha,b\beta\}.
\end{aligned}$$

This proves the corollary. □

Definition 4.9 (Gerstenhaber algebra)**.** *Let V be a graded vector space over a field k. A Gerstenhaber algebra on V is a triple $(V,\cdot,\{,\})$ such that*

(1) *$(V,\cdot)$ is a graded commutative algebra;*

(2) $(V, \{,\})$ *is a graded degree one Lie algebra;*

(3) *the bracket is a derivation for both variables.*

Now we are ready to show the famous theorem of Chas and Sullivan, where the Lie bracket $\{,\}$ is called *the loop bracket:*

Theorem 4.10 (Gerstenhaber algebra of the free loop sapce)**.** *Let M be a simply connected, smooth closed manifold and LM its free loop space. Let A be Whitney forms and C be the currents on M. Then*

$$(H_*(A\hat{\otimes}\underline{\hat{\Omega}}(C), \bullet, \{,\})$$

is a Gerstenhaber algebra, which models the Gerstenhaber algebra on $\mathbb{H}_*(LM)$ obtained in [5].

Proof. We have shown that $H_*(A\hat{\otimes}\underline{\hat{\Omega}}(C)$ is a graded commutative algebra (Lemma 4.4) and a degree one graded Lie algebra (Corollary 4.8). Next we show that the bracket is a derivation with respect to the loop product for both variables. By symmetry we only need to show, for $\alpha, \beta, \gamma \in H_*(A\hat{\otimes}\underline{\hat{\Omega}}(C))$,

$$\{\alpha \bullet \beta, \gamma\} = \alpha \bullet \{\beta, \gamma\} + (-1)^{|\beta|(|\gamma|+1)}\{\alpha, \gamma\} \bullet \beta.$$

This immediately follows from the following Lemma 4.11. □

Lemma 4.11. *Let A be as above. Then for $\alpha = x \otimes [a_1|\cdots|a_n], \beta = y \otimes [b_1|\cdots|b_m], \gamma = z \otimes [c_1|\cdots|c_l] \in A\hat{\otimes}\underline{\hat{\Omega}}(C)$,*

(1) $(\alpha \bullet \beta) * \gamma = \alpha \bullet (\beta * \gamma) + (-1)^{|\beta|(|\gamma|+1)}(\alpha * \gamma) \bullet \beta$;

(2) *setting*

$$\begin{aligned} & h(\alpha \otimes \beta \otimes \gamma) \\ = & \sum_{i<j} (-1)^{\epsilon} \varepsilon(c_i x)\varepsilon(c_j y) z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|c_{i+1}|\cdots|c_{j-1}|b_1|\cdots|b_m|c_{j+1}|\cdots|c_l], \end{aligned}$$

where $\epsilon = |\gamma|(|\alpha| + |\beta|) + |x| + |y| + |\alpha||[c_{i+1}|\cdots|c_n]| + |\beta||[c_{j+1}|\cdots|c_n]|$, we have

$$(b \circ h - h \circ b)(\alpha \otimes \beta \otimes \gamma) = \gamma * (\alpha \bullet \beta) - (\gamma * \alpha) \bullet \beta - (-1)^{(|\alpha|+1)|\gamma|}\alpha \bullet (\gamma * \beta).$$

Proof. (1) comes immediately from the definitions of $\bullet$ and $*$. We prove (2). In fact, up to sign,

$$\begin{aligned} & \gamma * (\alpha \bullet \beta) - (\gamma * \alpha) \bullet \beta - \alpha \bullet (\gamma * \beta) \\ = & \sum_i \varepsilon(c_i xy) z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|b_1|\cdots|b_m|c_{i+1}|\cdots|c_l] && (60) \\ + & \sum_i \varepsilon(c_i x) yz \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|c_{i+1}|\cdots|c_l|b_1|\cdots|b_m] && (61) \\ + & \sum_i \varepsilon(c_i y) xz \otimes [a_1|\cdots|a_n|c_1|\cdots|c_{i-1}|b_1|\cdots|b_m|c_{i+1}|\cdots|c_l], && (62) \end{aligned}$$

while

$$b \circ h(\alpha, \beta, \gamma)$$

$$
\begin{aligned}
&= \sum_{i<j} \varepsilon(c_ix)\varepsilon(c_jy)dz \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|c_{i+1}|\cdots|c_{j-1}|b_1|\cdots|b_m|c_{j+1}|\cdots|c_l] && (63)\\
&+ \sum_{i<j,r} \varepsilon(c_ix)\varepsilon(c_jy)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|dc_r|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l] && (64)\\
&+ \sum_{i<j,r} \varepsilon(c_ix)\varepsilon(c_jy)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c'_r|c''_r|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l] && (65)\\
&+ \sum_{i<j,p} \varepsilon(c_ix)\varepsilon(c_jy)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|da_p|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l] && (66)\\
&+ \sum_{i<j,p} \varepsilon(c_ix)\varepsilon(c_jy)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a'_p|a''_p|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l] && (67)\\
&+ \sum_{i<j,q} \varepsilon(c_ix)\varepsilon(c_jy)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|db_q|\cdots|b_m|\cdots|c_l] && (68)\\
&+ \sum_{i<j,q} \varepsilon(c_ix)\varepsilon(c_jy)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b'_q|b''_q|\cdots|b_m|\cdots|c_l] && (69)\\
&+ \sum_{i<j} \varepsilon(c_ix)\varepsilon(c_jy)z' \otimes [z''|c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l] && (70)\\
&+ \sum_{i<j} \varepsilon(c_ix)\varepsilon(c_jy)z' \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l|z''], && (71)
\end{aligned}
$$

and

$$
\begin{aligned}
&h(b\alpha,\beta,\gamma)\\
&= \sum_{i<j} \varepsilon(c_idx)\varepsilon(c_jy)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|c_{i+1}|\cdots|c_{j-1}|b_1|\cdots|b_m|c_{j+1}|\cdots|c_l] && (72)\\
&+ \sum_{i<j,p} \varepsilon(c_ix)\varepsilon(c_jy)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|da_p|\cdots|a_n|\cdots|c_{i-1}|b_1|\cdots|b_m|\cdots|c_l] && (73)\\
&+ \sum_{i<j,p} \varepsilon(c_ix)\varepsilon(c_jy)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a'_p|a''_p|\cdots|a_n|\cdots|c_{i-1}|b_1|\cdots|b_m|\cdots|c_l] && (74)\\
&+ \sum_{i<j} \varepsilon(c_ix')\varepsilon(c_jy)z \otimes [c_1|\cdots|c_{i-1}|x''|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l] && (75)\\
&+ \sum_{i<j} \varepsilon(c_ix')\varepsilon(c_jy)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|x''|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l], && (76)
\end{aligned}
$$

and

$$
\begin{aligned}
&h(\alpha,b\beta,\gamma)\\
&= \sum_{i<j} \varepsilon(c_ix)\varepsilon(c_jdy)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|c_{j+1}|\cdots|c_{j-1}|b_1|\cdots|b_m|c_{j+1}|\cdots|c_l] && (77)\\
&+ \sum_{i<j,q} \varepsilon(c_ix)\varepsilon(c_jy)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|db_q|\cdots|b_m|\cdots|c_l] && (78)\\
&+ \sum_{i<j,q} \varepsilon(c_ix)\varepsilon(c_jy)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b'_q|b''_q|\cdots|b_m|\cdots|c_l] && (79)\\
&+ \sum_{i<j} \varepsilon(c_ix)\varepsilon(c_jy')z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|y''|b_1|\cdots|b_m|\cdots|c_l] && (80)
\end{aligned}
$$

$$+ \sum_{i<j} \varepsilon(c_i x)\varepsilon(c_j y')z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b_m|y''|\cdots|c_l], \tag{81}$$

and $h(\alpha, \beta, b\gamma)$ is the sum of these four parts: Part I equals

$$\sum_{i<j} \varepsilon(c_i x)\varepsilon(c_j y)dz \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|c_{j+1}|\cdots|c_{j-1}|b_1|\cdots|b_m|c_{j+1}|\cdots|c_l] \tag{82}$$

$$+ \sum_{i<j,r} \varepsilon(c_i x)\varepsilon(c_j y)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|dc_r|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l] \tag{83}$$

$$+ \sum_{i<j,r} \varepsilon(c_i x)\varepsilon(c_j y)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c'_r|c''_r|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l], \tag{84}$$

Part II equals

$$\sum_{i<j} \varepsilon(dc_i x)\varepsilon(c_j y)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l] \tag{85}$$

$$+ \sum_{i<j} \varepsilon(c'_i x)\varepsilon(c_j y)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|c''_i|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l] \tag{86}$$

$$+ \sum_{i<j} \varepsilon(c''_i x)\varepsilon(c_j y)z \otimes [c_1|\cdots|c_{i-1}|c'_i|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l], \tag{87}$$

Part III equals

$$\sum_{i<j} \varepsilon(c_i x)\varepsilon(dc_j y)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l] \tag{88}$$

$$+ \sum_{i<j} \varepsilon(c_i x)\varepsilon(c'_j y)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b_m|c''_j|\cdots|c_l] \tag{89}$$

$$+ \sum_{i<j} \varepsilon(c_i x)\varepsilon(c''_j y)z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|c'_j|b_1|\cdots|b_m|\cdots|c_l] \tag{90}$$

and Part IV equals

$$\sum_{i} \varepsilon(c'_i x)\varepsilon(c''_i y)z \otimes z \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots a_n|b_1|\cdots|b_m|c_{i+1}|\cdots|c_l] \tag{91}$$

$$+ \sum_{i<j} \varepsilon(c_i x)\varepsilon(c_j y)z' \otimes [z''|c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l] \tag{92}$$

$$+ \sum_{i<j} \varepsilon(c_i x)\varepsilon(c_j y)z' \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l|z''] \tag{93}$$

$$+ \sum_{j} \varepsilon(z'' x)\varepsilon(c_j y)z' \otimes [a_1|\cdots|a_n|c_1|\cdots|c_{j-1}|b_1|\cdots|b_m|\cdots|c_l] \tag{94}$$

$$+ \sum_{i} \varepsilon(c_i x)\varepsilon(z'' y)z' \otimes [c_1|\cdots|c_{i-1}|a_1|\cdots|a_n|\cdots|c_l|b_1|\cdots|b_m|\cdots|c_l]. \tag{95}$$

Note that (63) and (82) are equal, so are (64) and (83), (65) and (84), (66) and (73), (67) and (74), (68) and (78), (69) and (79), (70) and (92), (71) and (93), (75) and (87), (76) and (86), (80) and (90), and (81) and (89). Also (72)+(85)=0, (77)+(88)=0, so the remaining terms in $b\circ h(\alpha,\beta,\gamma) - h\circ b(\alpha,\beta,\gamma)$ are (91)+(94)+(95), and after simplifying, it is exactly (60)+(61)+(62). Thus (2) is proved. $\square$

The above lemma is much similar to [5], Lemma 4.6, with a minor modification.

Remark 4.12. We have shown in Theorem 4.2 that $(H_*(A\hat{\otimes}\hat{\underline{\Omega}}(C)), \bullet)$ models the Chas-Sullivan product. Strictly speaking, since the bracket $\{,\}$ presented above comes from the commutator of $*$, while $*$ is not a chain map, one may be skeptical that $\{,\}$ really models the Chas-Sullivan loop product, even though the above constructions follow [5] step by step. However, in [5] and in Section 5 of this paper, the bracket is uniquely determined by the S^1-action, as the deviation of S^1-action from being a derivation. The S^1-operator does come from a chain map, therefore $\{,\}$ does model the Chas-Sullivan loop bracket.

4.2.2 Brace algebra with a product

The operators $\bullet$ and $*$ defined in last subsection are in fact a part of a more general structure, called *a homotopy Gerstenhaber algebra* by Gerstenhaber and Voronov ([26]), or *a brace algebra with a product* by McClure and Smith ([19]).

Definition 4.13 (brace algebra with a product, [26])**.** *Let $V = (\bigoplus V^n, d)$ be a chain complex with b of degree 1. V is called a brace algebra with a product if it is equipped with a product $\bullet$ making it into an associative DG algebra, and a collection of braces*

$$\begin{array}{ccc} V\otimes V^{\otimes n} & \longrightarrow & V \\ (x, x_1, \cdots, x_n) & \longmapsto & x\{x_1, \cdots, x_n\}, \end{array}$$

for all $n \geq 0$, satisfying the following identities:

(1) *for $x, x_1, \cdots, x_n, y_1, \cdots, y_m \in V$,*

$$\begin{aligned} & x\{x_1, \cdots, x_n\}\{y_1, \cdots, y_m\} \\ = & \sum_{0\leq i_1\leq\cdots\leq i_n\leq m} (-1)^{\epsilon} x\{y_1, \cdots, y_{i_1}, x_1\{y_{i_1+1}, \cdots\}, \cdots, x_n\{y_{i_n+1}, \cdots\}, \cdots, y_m\}, \end{aligned}$$

where $\epsilon = \sum_{p=1}^{n} |x_p| \sum_{j=1}^{i_p} |y_j|$;

(2) *for $x_1, x_2, y_1, \cdots, y_n \in V$,*

$$(x_1 \bullet x_2)\{y_1, \cdots, y_n\} = \sum_{k=0}^{n} (-1)^{\epsilon} x_1\{y_1, \cdots, y_k\} \bullet x_2\{y_{k+1}, \cdots, y_n\},$$

where $\epsilon = (|x_2| + 1)\sum_{p=1}^{k} |y_p|$;

(3) *for $x, x_1, \cdots, x_{n+1} \in V$,*

$$\begin{aligned} & d(x\{x_1, \cdots, x_{n+1}\}) - (dx)\{x_1, \cdots, x_{n+1}\} \\ & -(-1)^{|x|}\sum_{i=1}^{n+1}(-1)^{|x_1|+\cdots+|x_{i-1}|} x\{x_1, \cdots, dx_i, \cdots, x_{n+1}\} \\ = & (-1)^{(|x|+1)|x_1|} x_1 \bullet x\{x_2, \cdots, x_{n+1}\} + (-1)^{|x|+|x_1|+\cdots+|x_n|} x\{x_1, \cdots, x_n\} \bullet x_{n+1} \\ & -(-1)^{|x|}\sum_{i=1}^{n}(-1)^{|x_1|+\cdots+|x_i|} x\{x_1, \cdots, x_i \bullet x_{i+1}, \cdots, x_{n+1}\}. \end{aligned}$$

Theorem 4.14 (brace algebra with a product of the free loop space)**.** *If M is a simply connected, smooth manifold, then the chain model of LM, $A\hat{\otimes}\underline{\hat{\Omega}}(C)$, has the structure of a brace algebra with a product.*

Proof. Take $x = x \otimes [a_1|\cdots|a_p], x_1 = x_1 \otimes [b^1_1|\cdots|b^1_{q_1}], \cdots, x_n = x_n \otimes [b^n_1|\cdots|b^n_{q_n}] \in A\hat{\otimes}\underline{\hat{\Omega}}(C)$, then define

$$\begin{aligned} & x\{x_1, \cdots, x_n\} \\ := & \sum_{1\leq i_1<\cdots<i_n\leq p} (-1)^{\epsilon}\varepsilon(a_{i_1}x_1)\cdots\varepsilon(a_{i_n}x_n) \\ & \qquad\qquad x \otimes [a_1|\cdots|a_{i_1-1}|b^1_1|\cdots|b^1_{q_1}|\cdots|a_{i_n-1}|b^n_1|\cdots|b^n_{q_n}|\cdots|a_p], \end{aligned}$$

where ϵ is the sign as in (37). Similar to the computations in last subsection one checks that it satisfies all the conditions listed in above definition, though the computation is tedious. □

4.2.3 Two Gerstenhaber algebras are isomorphic

Recall the results of Gerstenhaber in his study of the deformation of associative algebras ([10]):

Definition 4.15 (the product and bracket of the Hochschild cochain complex)**.** *Let A be a (DG) algebra over a field k and let $HC^*(A;A)$ be its Hochschild cochain complex (Definition 3.18). Define the product $\cdot$, the pre-Lie operator $*$, and the bracket $\{,\}$ on $HC^*(A;A)$ as follows: for $f \in \mathrm{Hom}(A^{\otimes n}; A)$, $g \in \mathrm{Hom}(A^{\otimes m}; A)$, up to sign,*

(1) *for any $a_1, \cdots, a_{m+n} \in A$,*

$$(f \cdot g)(a_1, \cdots, a_{m+n}) := f(a_1, \cdots, a_n)g(a_{n+1}, \cdots, a_{m+n}); \tag{96}$$

(2) *for any $a_1, \cdots, a_{n+m-1} \in A$,*

$$(f * g)(a_1, \cdots, a_{n+m-1}) := \sum_{i=1}^{n} f(a_1, \cdots, a_{i-1}, g(a_i, \cdots, a_{i+m-1}), \cdots, a_{n+m-1}); \tag{97}$$

(3) *the bracket is the commutator of $*$:*

$$\{f, g\} := f * g - (-1)^{(|f|+1)(|g|+1)} g * f. \tag{98}$$

Theorem 4.16 (Gerstenhaber [10])**.** *Let A be a DG associative algebra over a field k and let the operators $\cdot$, $*$ and $\{,\}$ be given by the above definition, then Lemmas (4.4) and (4.11) hold. Therefore the Hochschild cohomology $(HH^*(A;A), \cdot, \{,\})$ is a Gerstenhaber algebra.*

Theorem 4.17 (isomorphism of two Gerstenhaber algebras, see also Cohen-Jones [7], Merkulov [20], Tradler [24] and Félix et al [9])**.** *Let M be a simply connected smooth closed manifold and A be the Whitney forms on M. Then*

$$HH^*(A;A) \xrightarrow{\cong} \mathbb{H}_*(LM)$$

are isomorphic as Gerstenhaber algebras.

Proof. In fact, let C be set of the currents on M, then the Hochschild cochain complex is chain equivalent to $A\hat{\otimes}\underline{\hat{\Omega}}(C)$ (see Lemma 3.32 and Theorem 3.37):

$$HC^*(A;A) \simeq A\hat{\otimes}\underline{\hat{\Omega}}(C).$$

For $f, g \in HC^*(A;A)$, we may write them as $f = u \otimes [a_1|\cdots|a_n], g = v \otimes [b_1|\cdots|b_m] \in A\hat{\otimes}\underline{\hat{\Omega}}(C)$, the operators $\cdot$, $*$ and $\{,\}$ defined above by (96), (97) and (98) can be rewritten as

$$f \cdot g = uv \otimes [a_1|\cdots|a_n|b_1|\cdots|b_m]$$

and

$$f * g = \sum_{i=1}^{n} \langle a_i, v\rangle u \otimes [a_1|\cdots|a_{i-1}|b_1|\cdots|b_m|a_{i+1}|\cdots|a_n],$$

and

$$\{f, g\} := f * g - (-1)^{(|f|+1)(|g|+1)} g * f.$$

Comparing them with the loop product (24) and pre-Lie operator (37), we see that $\mathbb{H}_*(LM)$ and $HH^*(A;A)$ are isomorphic as Gerstenhaber algebras. $\square$

5 Batalin-Vilkovisky algebra on the loop homology

Let J be the S^1-action on the loop homology. In [5], Chas and Sullivan prove that $(\mathbb{H}_*(LM), \bullet, J)$ forms a Batalin-Vilkovisky algebra. Namely, J on homology is not a derivation with respect to $\bullet$, but the deviation from being a derivation of J is a derivation. One deduces that, for $\alpha, \beta \in \mathbb{H}_*(LM)$,

$$\{a, b\} := (-1)^{|\alpha|} J(\alpha \bullet \beta) - (-1)^{|\alpha|} J(\alpha) \bullet b - \alpha \bullet J(\beta)$$

defines a degree one graded Lie algebra on $\mathbb{H}_*(LM)$. Chas and Sullivan show that this Lie bracket is in fact the loop bracket on homology.

Definition 5.1 (Batalin-Vilkovisky algebra)**.** *Let V be a graded vector space over a field k. A Batalin-Vilkovisky algebra on V is a triple $(V, \bullet, \Delta)$ such that:*

(1) *$(V, \bullet)$ is a graded commutative algebra;*

(2) *$\Delta : V \to V$ is degree one operator with $\Delta^2 = 0$;*

(3) *The deviation from being a derivation of Δ with respect to $\bullet$ is a derivation for both variables, namely,*

$$(-1)^{|\alpha|}\Delta(\alpha \bullet \beta) - (-1)^{|\alpha|}\Delta(\alpha) \bullet b - \alpha \bullet \Delta(\beta)$$

is a derivation for both $\alpha, \beta \in V$.

Proposition 5.2. *Let $(V, \bullet, \Delta)$ be a Batalin-Vilkovisky algebra. Define $[\ ,] : V \otimes V \longrightarrow V$ by*

$$[\alpha, \beta] := (-1)^{|\alpha|}\Delta(\alpha \bullet \beta) - (-1)^{|\alpha|}\Delta(\alpha) \bullet b - \alpha \bullet \Delta(\beta), \text{ for } \alpha, \beta \in V,$$

then $(V, \bullet, [\ ,])$ forms a Gerstenhaber algebra.

Proof. See Getzler [11], Proposition 1.2. $\square$

In other words, a Batalin-Vilkovisky algebra is a special kind of Gerstenhaber algebra.

Lemma 5.3. *Let M be a simply connected, smooth closed manifold and LM be its free loop space. Then*

$$\{\alpha,\beta\} = (-1)^{|\alpha|}B(\alpha\bullet\beta) - (-1)^{|\alpha|}B(\alpha)\bullet b - \alpha\bullet B(\beta), \text{ for } \alpha,\beta\in\mathbb{H}_*(LM), \tag{99}$$

where $\{,\}$ and $\bullet$ are the loop bracket and the loop product respectively, and B is the induced S^1-action on $\mathbb{H}_*(LM)$.

More precisely, let A be the DG Frobenius-like algebra of M and $A\hat{\otimes}\underline{\hat{\Omega}}(C)$ be be the twisted tensor product, and let B be the dual Connes cyclic operator (Definition 3.39) on $C\hat{\otimes}\underline{\hat{\Omega}}(C)$, then there is a linear map

$$h : A\hat{\otimes}\underline{\hat{\Omega}}(C)\bigotimes A\hat{\otimes}\underline{\hat{\Omega}}(C)\longrightarrow C\hat{\otimes}\underline{\hat{\Omega}}(C)$$

such that for any $\alpha,\beta\in A\hat{\otimes}\underline{\hat{\Omega}}(C)$,

$$(b\circ h - h\circ b)(\alpha\otimes\beta) = \{\alpha,\beta\} - (-1)^{|\alpha|}B(\alpha\bullet\beta) - (-1)^{(|\beta|+1)(|a|+1)}\beta\bullet B(\alpha) + \alpha\bullet B(\beta). \tag{100}$$

Proof. First note that $A\hat{\otimes}\underline{\hat{\Omega}}(C)$ embeds in $C\hat{\otimes}\underline{\hat{\Omega}}(C)$, the operator B is well defined. For $\alpha = x\otimes[a_1|\cdots|a_n], \beta = y\otimes[b_1|\cdots|b_m]\in A\hat{\otimes}\underline{\hat{\Omega}}(C)$, define

$$\phi(\alpha,\beta) := \sum_{i<j}\varepsilon(x)\varepsilon(a_jy)a_i\otimes[a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}]$$

and

$$\psi(\alpha,\beta) := \sum_{k<l}\varepsilon(y)\varepsilon(b_lx)b_k\otimes[b_{k+1}|\cdots|b_{l-1}|a_1|\cdots|a_n|b_{l+1}|\cdots|b_m|b_1|\cdots|b_{k-1}],$$

and let $h=\phi+\psi$. We show h thus defined satisfies (100).

In fact, up to sign,

$$\begin{aligned}
&\{\alpha,\beta\} - (-1)^{|\alpha|}B(\alpha\bullet\beta) + (-1)^{|\beta|}\beta\bullet B(\alpha) + \alpha\bullet B(\beta)\\
= &\sum_i\varepsilon(a_iy)x\otimes[a_1|\cdots|a_{i-1}|b_1|\cdots|b_m|a_{i+1}|\cdots|a_n] &(101)\\
+ &\sum_i\varepsilon(xy)a_i\otimes[a_{i+1}|\cdots|a_n|b_1|\cdots|b_m|a_1|\cdots|a_{i-1}] &(102)\\
+ &\sum_k\varepsilon(xy)b_k\otimes[b_{k+1}|\cdots|b_m|a_1|\cdots|a_n|b_1|\cdots|b_{k-1}] &(103)\\
+ &\sum_k\varepsilon(b_kx)y\otimes[b_1|\cdots|b_{k-1}|a_1|\cdots|a_n|b_{k+1}|\cdots|b_m] &(104)\\
+ &\sum_i\varepsilon(x)a_iy\otimes[b_1|\cdots|b_m|a_{i+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] &(105)\\
+ &\sum_k\varepsilon(y)b_kx\otimes[a_1|\cdots|a_n|b_{k+1}|\cdots|b_m|b_1|\cdots|b_{k-1}], &(106)
\end{aligned}$$

while

$$b\phi(\alpha,\beta)$$

$$
\begin{aligned}
= & \sum_{i<j} \varepsilon(x)\varepsilon(a_j y) da_i \otimes [a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (107)\\
+ & \sum_{i<j,p} \varepsilon(x)\varepsilon(a_j y) a_i \otimes [a_{i+1}|\cdots|da_p|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (108)\\
+ & \sum_{i<j,p} \varepsilon(x)\varepsilon(a_j y) a_i \otimes [a_{i+1}|\cdots|a'_p|a''_p|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (109)\\
+ & \sum_{i<j,q} \varepsilon(x)\varepsilon(a_j y) a_i \otimes [a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|db_q|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (110)\\
+ & \sum_{i<j,q} \varepsilon(x)\varepsilon(a_j y) a_i \otimes [a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|b'_q|b''_q|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (111)\\
+ & \sum_{i<j} \varepsilon(x)\varepsilon(a_j y) a'_i \otimes [a''_i|a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (112)\\
+ & \sum_{i<j} \varepsilon(x)\varepsilon(a_j y) a''_i \otimes [a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}|a'_i]. & (113)
\end{aligned}
$$

and

$$
\begin{aligned}
& \phi(b\alpha,\beta) & \\
= & \sum_{i<j,p} \varepsilon(a_j y)\varepsilon(x) a_i \otimes [a_{i+1}|\cdots|da_p|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (114)\\
+ & \sum_{i<j,p} \varepsilon(a_j y)\varepsilon(x) a_i \otimes [a_{i+1}|\cdots|a'_p|a''_p|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (115)\\
+ & \sum_{i<j} \varepsilon(x)\varepsilon(a_j y) da_i \otimes [a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (116)\\
+ & \sum_{i<j} \varepsilon(x)\varepsilon(da_j y) a_i \otimes [a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (117)\\
+ & \sum_{i<j} \varepsilon(x)\varepsilon(a'_j y) a_i \otimes [a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|b_m|a''_j|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (118)\\
+ & \sum_{i<j} \varepsilon(x)\varepsilon(a''_j y) a_i \otimes [a_{i+1}|\cdots|a_{j-1}|a'_j|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (119)\\
+ & \sum_{i<j} \varepsilon(x)\varepsilon(a_j y) a'_i \otimes [a''_i|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (120)\\
+ & \sum_{i<j} \varepsilon(x)\varepsilon(a_j y) a''_i \otimes [a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}|a'_i] & (121)\\
+ & \sum_{i} \varepsilon(x)\varepsilon(a''_i y) a'_i \otimes [b_1|\cdots|b_m|a_{i+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] & (122)\\
+ & \sum_{i} \varepsilon(x')\varepsilon(a_i y) x'' \otimes [a_1|\cdots|a_{i-1}|b_1|\cdots|b_m|a_{i+1}|\cdots|a_n] & (123)\\
+ & \sum_{i} \varepsilon(x')\varepsilon(x'' y) a_i \otimes [a_{i+1}|\cdots|a_n|b_1|\cdots|b_m|a_1|\cdots|a_{i-1}], & (124)
\end{aligned}
$$

and

$$\phi(\alpha, b\beta)$$

$$
\begin{aligned}
&= \sum_{i<j} \varepsilon(x)\varepsilon(a_j dy) a_i \otimes [a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] && (125)\\
&+ \sum_{i<j,q} \varepsilon(x)\varepsilon(a_j y) a_i \otimes [a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|db_q|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] && (126)\\
&+ \sum_{i<j,q} \varepsilon(x)\varepsilon(a_j y) a_i \otimes [a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|b'_q|b''_q|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] && (127)\\
&+ \sum_{i<j} \varepsilon(x)\varepsilon(a_j y') a_i \otimes [a_{i+1}|\cdots|a_{j-1}|y''|b_1|\cdots|b_m|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}] && (128)\\
&+ \sum_{i<j} \varepsilon(x)\varepsilon(a_j y') a_i \otimes [a_{i+1}|\cdots|a_{j-1}|b_1|\cdots|b_m|y''|a_{j+1}|\cdots|a_n|a_1|\cdots|a_{i-1}]. && (129)
\end{aligned}
$$

Note that (114) and (108) are identical, so are (116) and (107), (117) and (125), (115) and (109), (120) and (112), (121) and (113), (118) and (129), (119) and (128), (110) and (126), and (111) and (127), therefore the remaining terms of $b\phi(\alpha,\beta) - \phi(b\alpha,\beta) - \phi(\alpha,b\beta)$ are exactly $(101) + (102) + (105)$.

Similarly, the remaining terms of $b\psi(\alpha,\beta) - \phi(b\alpha,b) - \psi(\alpha,b\alpha)$ are $(103)+(104)+(106)$. Thus formula (100) holds. Since $A\hat{\otimes}\underline{\hat{\Omega}}(C)$ and $C\hat{\otimes}\underline{\hat{\Omega}}(C)$ have the same homology, (99) follows from (100). $\square$

The above lemma is similar to Lemma 5.2 in [5]. By this lemma we obtain:

Theorem 5.4 (Batalin-Vilkovisky algebra of the free loop space). *Let M be a simply connected, smooth closed manifold and let A be the Whitney forms and C be the currents on M. Then*

$$(H_*(A\hat{\otimes}\underline{\hat{\Omega}}(C), \bullet, B)$$

is a Batalin-Vilkovisky algebra, which models the Batalin-Vilkoviksy algebra on $\mathbb{H}_*(LM)$ obtained in [5].

Proof. We have shown (Theorem 4.10) that

$$(H_*(A\hat{\otimes}\underline{\hat{\Omega}}(C), \bullet, \{,\})$$

is a Gerstenhaber algebra, therefore the loop bracket $\{,\}$ is a derivation for both variables with respect to $\bullet$. The above lemma says that the deviation of B from being a derivation is exactly the loop bracket. Thus according to Definition 5.1,

$$(H_*(A\hat{\otimes}\underline{\hat{\Omega}}(C), \bullet, B)$$

is a Batalin-Vilkovisky algebra. Note that in [5], the Batalin-Vilkovisky algebra is obtained exactly the same way, we say the Batalin-Vilkoviksy algebra obtained above models the one of [5]. $\square$

6 Batalin-Vilkovisky algebra on a general manifold

In the previous sections, we have only studied the free loop space of a simply connected manifold. The non simply connected case is quite different in nature, for example, the cobar construction of a non simply connected may not be the chain model of the based loop space of that manifold. The goal of this section is to construct a chain complex model of LM, which encodes the fundamental group $\pi_1(M)$, and includes the simply connected manifolds as a special case. The idea is to lift the loops on M to its universal covering $\tilde{M}$, where the loops now becomes paths, which can be characterized explicitly. This idea is due to Mike Mandell, which is informed to the author by James McClure. Since $\tilde{M}$ is simply connected, our algebraic methods may now be applied.

6.1 The chain complex model of LM

We begin with the following observation about the free loop space LM.

Lemma 6.1 (an equivalent characterization of LM)**.** *Let M be a smooth manifold. Denote by G the fundamental group $\pi_1(M)$ and by $\tilde{M}$ the universal covering of M. For any $g \in G$, let*

$$L_g\tilde{M} := \left\{ f : I = [0,1] \to \tilde{M} \middle| f(1) = g \circ f(0) \right\}.$$

Then $\coprod_{g\in G} L_g\tilde{M}$ admits a G-action induced from that on $\tilde{M}$: for $f \in L_g\tilde{M}$, and $h \in G$,

$$\begin{array}{rccc} h\circ f: & [0,1] & \longrightarrow & \tilde{M} \\ & x & \longmapsto & h\circ f(x). \end{array}$$

Note that since $(h\circ f)(1) = h\circ f(1) = h\circ(g\circ f(0)) = hgh^{-1}\circ((h\circ f)(0))$, $h\circ f \in L_{hgh^{-1}}\tilde{M}$. We have a homeomorphism

$$\coprod_{g\in G} L_g\tilde{M} \Big/ G \cong LM,$$

and the following commutative diagram:

$$\begin{tikzcd} \coprod L_g\tilde{M} \arrow[r, "/G"] \arrow[d, "\pi_0"] & LM \arrow[d, "\pi_0"] \\ \tilde{M} \arrow[r, "/G"] & M, \end{tikzcd}$$

where π_0 is the projection of the paths to their starting points.

Proof. The proof follows from the definition of the universal covering space (see, for example, Hatcher [14], §1.3) and the definitions of $L_g\tilde{M}$ and LM. ☐

In the following we shall use this lemma to construction a chain complex model for LM.

6.1.1 The chain complex model of $L_g\tilde{M}$

Recall the definition of $L_g\tilde{M}$:

$$L_g\tilde{M} = \left\{ f : [0,1] \to \tilde{M} \middle| f(1) = g \circ f(0) \right\}.$$

Let

$$\Delta_n := \left\{ (t_1, \cdots, t_n) \in \mathbb{R}^n \middle| 0 \le t_1 \le \cdots \le t_n \le 1 \right\}, \quad n \ge 0 \tag{130}$$

and consider the maps (see [13] §2)

$$\begin{tikzcd} L_g\tilde{M}\times\Delta_n \arrow[r, "\Psi_n"] \arrow[d, "\mathrm{proj}"] & \tilde{M}\times\cdots\times\tilde{M} \\ L_g\tilde{M}, & \end{tikzcd} \tag{131}$$

where Ψ_n are the evaluation maps

$$\Psi_n(f,(t_1,\cdots,t_n)) := (f(0), f(t_1),\cdots,f(t_n)).$$

Since $L_g\tilde{M}$ is the space of continuous maps from I to $\tilde{M}$, we would view $\{\Psi_n\}$ as an approximation of $L_g\tilde{M}$, which intuitively means, as n becomes larger and larger, the chain complex of $\tilde{M}\times\cdots\times\tilde{M}$, which is $C_*(\tilde{M})^{\otimes n+1}$, will go to $C_*(L_g\tilde{M})$ closer and closer, and the limit is $C_*(L_g\tilde{M})$. This approach is called *the cosimplicial approximation of* $L_g\tilde{M}$.

Definition 6.2 (cosimplicial space). *A cosimplicial object $K^\bullet$ in a category $\mathscr{C}$ consists of*

(1) *a sequence of objects $K^0, K^1, K^2, \cdots$ in $\mathscr{C}$;*

(2) *for each $n\ge 0$, a collection of morphisms, called coface maps:*

$$\delta^i : K^{n-1}\longrightarrow K^n,\quad 0\le i\le n;$$

(3) *for each $n\ge 0$, a collection of morphisms, called codegeneracy maps:*

$$\sigma^i : K^{n+1}\longrightarrow K^n,\quad 0\le i\le n.$$

The coface and codegeneracy maps satisfy the following identities:

$$\begin{aligned} \delta^j\circ\delta^i &= \delta^i\circ\delta^{j-1}, \quad \textit{if}\quad i<j,\\ \sigma^j\circ\delta^i &= \begin{cases} \delta^i\circ\sigma^{j-1}, & \textit{if}\quad i<j\\ id, & \textit{if}\quad i=j \quad \textit{or}\quad i=j+1,\\ \delta^{i-1}\circ\sigma^j, & \textit{if}\quad i>j+1,\end{cases}\\ \sigma^j\circ\sigma^i &= \sigma^{i-1}\circ\sigma^j, \quad \textit{if}\quad i>j. \end{aligned}$$

A cosimplicial object in the category of topological spaces and continuous maps is called a cosimplicial space.

Example 6.3 (the standard simplices). Let Δ_n be the standard simplices given by (130). Then $\{\Delta_n\}$ is a cosimplicial space with coface maps $\delta_i:\Delta_{n-1}\to\Delta_n$ and codegeneracy maps $\sigma_i:\Delta_{n+1}\to\Delta_n$ given by

$$\begin{aligned} \delta_0(t_1,\cdots,t_{n-1}) &= (0,t_1,\cdots,t_{n-1})\\ \delta_i(t_1,\cdots,t_{n-1}) &= (t_0,\cdots,t_{i-1},t_i,t_i,\cdots,t_{n-1}),\quad 1\le i\le n-1\\ \delta_n(t_1,\cdots,t_{n-1}) &= (t_1,\cdots,t_{n-1},1)\\ \sigma_i(t_1,\cdots,t_{n+1}) &= (t_1,\cdots,t_i,t_{i+2},\cdots,t_{n+1}). \end{aligned}$$

Example 6.4. Let X be a topological space with a group G-action. Fix $g \in G$. Then $\{X^{\times n+1}\}$ is a cosimplicial space whose coface and codegeneracy maps are given by

$$\begin{aligned} \delta_i(x_0, \cdots, x_{n-1}) &= (x_0, \cdots, x_{i-1}, x_i, x_i, \cdots, x_{n-1}), \quad 0 \le i \le n-1 \\ \delta_n(x_0, \cdots, x_{n-1}) &= (x_0, \cdots, x_{n-1}, g \circ x_0), \\ \sigma_i(x_0, \cdots, x_{n+1}) &= (x_0, \cdots, x_i, x_{i+2}, \cdots, x_{n+1}). \end{aligned}$$

Proposition 6.5. *Let M be a topological space and $\tilde{M}$ be its universal covering. Denote by G the fundamental group $\pi_1(M)$. Fix $g \in G$. Let $\tilde{M}^{\times n+1}$ be the cosimplicial space in Example 6.4. Then the following diagram is commutative:*

$$\begin{tikzcd} L_g\tilde{M} \times \Delta_m \arrow[r, "\Psi_m"] \arrow[d, "id\times s"] & \tilde{M} \times \cdots \times \tilde{M} \arrow[d, "s"] \\ L_g\tilde{M} \times \Delta_n \arrow[r, "\Psi_n"] & \tilde{M} \times \cdots \times \tilde{M}, \end{tikzcd} \tag{132}$$

where s is any morphism of the two cosimplicial spaces, and Ψ_n and Ψ_m are the evaluation maps (131).

Proof. It follows from the definition of these two cosimplicial spaces and Ψ_n. $\square$

We shall not talk much about the cosimplicial space, since all we need in the following is the commutative diagram (132). For more details, see, for example, Bott-Segal [2], §5 or Jones [15]. The above proposition leads us to consider the following two complexes:

Definition 6.6 (cosimplicial chain complex and simplicial cochain complex with a group action)**.** *Let (C, Δ, d) be a coassociative DG coalgebra over field k. Suppose G is a discrete group and C admits a $k[G]$-action, which commutes with Δ. Let $\underline{\Omega}(C)$ be the cobar construction (Definition 3.8) of C. Fix $g \in G$. Define an operator*

$$b_g : C \otimes \underline{\Omega}(C) \longrightarrow C \otimes \underline{\Omega}(C)$$

by

$$\begin{aligned} & b_g(x \otimes [a_1|\cdots|a_n]) \\ := \; & dx \otimes [a_1|\cdots|a_n] - \sum_i (-1)^{|x|+|[a_1|\cdots|a_{i-1}]|} x \otimes [a_1|\cdots|da_i|\cdots|a_n] \\ & + \sum \Big((-1)^{|x'|} x' \otimes [x''|a_1|\cdots|a_n] - (-1)^{(|x'|-1)(|x''|+|[a_1|\cdots|a_n]|)} x'' \otimes [a_1|\cdots|a_n|g_* x'] \Big) \\ & + \sum_i (-1)^{|x|+|[a_1|\cdots|a_{i-1}|a_i']|} x \otimes [a_1|\cdots|a_i'|a_i''|\cdots|a_n], \end{aligned}$$

then $b_g^2 = 0$ and the chain complex is called the cosimplicial chain complex of C.

Analogously, let $(A, \cdot, d)$ be an associative DG algebra over field k. Suppose G is a discrete group and A admits a $k[G]$-action, which commutes with $\cdot$. Let $\underline{\Omega}(A)$ be the bar construction (Definition 3.15) of A. Fix $g \in G$. Define an operator

$$b_g : A \otimes \underline{\Omega}(A) \longrightarrow A \otimes \underline{\Omega}(A)$$

by

$$\begin{aligned}
&b_g(x\otimes[a^1|\cdots|a^n])\\
:=\ & dx\otimes[a^1|\cdots|a^n]+\sum_i(-1)^{|x|+|[a^1|\cdots|a^{i-1}]|}x\otimes[a^1|\cdots|da^i|\cdots|a^n]\\
&+\Big(xa^1\otimes[a^2|\cdots|a^n]-(-1)^{(|a^n|+1)(|x|+|[a^1|\cdots|a^{n-1}]|)}(g^*a^n)x\otimes[a_1|\cdots|a_{n-1}]\Big)\\
&+\sum_i(-1)^{|x|+|[a^1|\cdots|a^{i-1}]|}x\otimes[a^1|\cdots|a^ia^{i+1}|\cdots|a^n],
\end{aligned}$$

then $b_g^2=0$ *and the chain complex is called the simplicial cochain complex of* A.

One sees that the two chain complexes are dual to each other (see Proposition 3.17). In the above definition, if we write $b_g=b_g^{\mathrm{I}}+b_g^{\mathrm{II}}$, where b_g^{I} only involves the differential part, and b_g^{II} only involves the coproduct or product part, then b_g^{I} and b_g^{II} commute, and both have square zero. Also, if G is trivial, then the above two complexes are the cocyclic cobar complex of C (Definition 3.12) and the cyclic bar complex of A (Definition 3.14).

Definition 6.7. *Let* $\tilde M$ *and* $L_g\tilde M$ *as above, and let* $C_*(L_g\tilde M)$ *and* $C_*(\tilde M)$ *be the chain complexes (singular, simplicial or any other appropriate DG coalgebra model) of* $L_g\tilde M$ *and* $\tilde M$ *respectively. Note that there is a chain equivalence (by the Eilenberg-Zilber theorem, see, for example, [17] Theorem 4.36, p. 122)*

$$\mathrm{EZ}: C_*(\underbrace{\tilde M\times\cdots\times\tilde M}_{n})\xrightarrow{\simeq}\underbrace{C_*(\tilde M)\otimes\cdots\otimes C_*(\tilde M)}_{n}.$$

Define (see (131))

$$\begin{aligned}
\bar\phi_{g\#}:\quad C_*(L_g\tilde M)&\longrightarrow \bigoplus_{n\geq 0}C_*(\tilde M)^{\otimes n+1}\\
\alpha\quad&\longmapsto \mathrm{EZ}\circ\Big(\sum_{n\geq 0}\Psi_{n\#}\Big)\Big(\sum_{n\geq 0}\alpha\times\Delta_n\Big).
\end{aligned}$$

In order to keep the degree, we shift the degrees of the last n-entries in $C_*(\tilde M)^{\otimes n+1}$ *down by one, and also modulo those chains that contains counit. Denote the induced map by*

$$\phi_{g\#}:\quad C_*(L_g\tilde M)\longrightarrow C_*(\tilde M)\otimes\underline{\Omega}(C_*(\tilde M)).$$

Similarly, let $C^*(\tilde M)$ *and* $C^*(L_g\tilde M)$ *be the cochain complexes (singular, simplicial or Whitney forms, etc.) of* $\tilde M$ *and* $L_g\tilde M$ *respectively. Define*

$$\begin{aligned}
\phi_g^{\#}:\quad C^*(\tilde M)\otimes\underline{\Omega}(C^*(\tilde M))&\longrightarrow C^*(L_g\tilde M)\\
x\otimes[a^1|\cdots|a^n]&\longmapsto\int_{\Delta_n}\Psi_n^{\#}(x\wedge a^1\wedge\cdots\wedge a^n).
\end{aligned}$$

Theorem 6.8 (two quasi-isomorphisms)**.** *Let* $\tilde M$ *and* $L_g\tilde M$ *be as above. Then the two maps*

$$\phi_{g\#}:\quad (C_*(L_g\tilde M),\partial)\longrightarrow\Big(C_*(\tilde M)\otimes\underline{\Omega}(C_*(\tilde M)),b_g\Big)$$

and

$$\phi_g^{\#}:\quad\Big(C^*(\tilde M)\otimes\underline{\Omega}(C^*(\tilde M)),b_g\Big)\longrightarrow(C^*(L_g\tilde M),\delta)$$

are quasi-isomorphisms.

Proof. Since in the previous chapters we have used the Whitney forms and their dual to model the loop space, we prove the cochain case first.

First we show that $\phi_g^{\#}$ is a chain map. Notice that the notation

$$\int_{\Delta_n} \Psi_n^{\#}(x \wedge a^1 \wedge \cdots \wedge a^n)$$

means integration along the fiber: denote by $e_t : L_g\tilde{M} \to \tilde{M}$ the evaluation of the paths at time t, then

$$\int_{\Delta_n} \Psi_n^{\#}(x \wedge a^1 \wedge \cdots \wedge a^n) = \int_{\Delta_n} e_0^*(x) \wedge e_{t_1}^*(a^1) \wedge \cdots \wedge e_{t_n}^*(a^n).$$

We have

$$\begin{aligned}
& \phi_g^{\#}\big(b_g(x \otimes [a^1|\cdots|a^n])\big) && (133)\\
= \; & \int_{\Delta_n} e_0^*(dx) \wedge e_{t_1}^*(a^1) \wedge \cdots \wedge e_{t_n}^*(a^n) && (134)\\
& + \sum_i \int_{\Delta_n} e_0^*(x) \wedge e_{t_1}^*(a^1) \wedge \cdots \wedge e_{t_i}^*(da^i) \wedge \cdots \wedge e_{t_n}^*(a^n) && (135)\\
& + \sum_i \int_{\Delta_{n-1}} e_0^*(x) \wedge e_{t_1}^*(a^1) \wedge \cdots \wedge e_{t_i}^*(a^i a^{i+1}) \wedge \cdots \wedge e_{t_{n-1}}^*(a^n) && (136)\\
& + \int_{\Delta_{n-1}} e_0^*(xa^1) \wedge e_{t_1}^*(a^2) \wedge \cdots \wedge e_{t_{n-1}}^*(a^n) && (137)\\
& + \int_{\Delta_{n-1}} e_0^*((g^*a^n)x) \wedge e_{t_1}^*(a^2) \wedge \cdots \wedge e_{t_{n-1}}^*(a^{n-1}). && (138)
\end{aligned}$$

Note that $(134) + (135)$ is

$$\int_{\Delta_n} d\big(e_0^*(x) \wedge e_{t_1}^*(a^1) \wedge \cdots \wedge e_{t_n}^*(a^n)\big),$$

which equals

$$d\int_{\Delta_n} e_0^*(x) \wedge e_{t_1}^*(a^1) \wedge \cdots \wedge e_{t_n}^*(a^n) - \int_{\partial\Delta_n} e_0^*(x) \wedge e_{t_1}^*(a^1) \wedge \cdots \wedge e_{t_n}^*(a^n)\big|_{\partial\Delta_n}.$$

Note that by the commutative diagram (132), the second term in the above expression is exactly $(136) + (137) + (138)$, therefore

$$(133) = d\int_{\Delta_n} e_0^*(x) \wedge e_{t_1}^*(a^1) \wedge \cdots \wedge e_{t_n}^*(a^n) = d(\phi_g^{\#}(x \otimes [a^1|\cdots|a^n])),$$

which proves that $\phi_g^{\#}$ is a chain map.

Now we show that $\phi_g^{\#}$ is a quasi-isomorphism. Consider the fibration

$$\begin{tikzcd}
\Big\{f : I \to \tilde{M} \,\Big|\, f(0) = x, f(1) = g \circ x\Big\} \arrow[r] & L_g\tilde{M} \arrow[d, "\pi_0"] \\
& \tilde{M},
\end{tikzcd} \qquad (139)$$

where e_0 is the evaluation of the paths at starting point, then $C^*(L_g\tilde{M})$ is a DG $C^*(\tilde{M})$-bimodule. Give a filtration of $C^*(L_g\tilde{M})$ by

$$F_p = \bigoplus_{q \le p} C^q(\tilde{M}) C^*(L_g\tilde{M}),$$

then the associated (Serre) spectral sequence converges, and its E_2-term is $H^*(\tilde{M}) \otimes H^*(\Omega\tilde{M})$.

Now for the complex $C^*(\tilde{M}) \otimes \underline{\Omega}(C^*(M))$, in order to remove the positive terms (keep in mind that in this paper the cochains are negatively graded), we have to consider Chen's normalized complex (see Getzler et al [13], §2): Denote by Ω^0 the subcomplex of $\underline{\Omega}(C_*(\tilde{M}))$ which contains zero-cochains, then Ω^0 is acyclic (Lemma 3.38). By modulo Ω^0 in $\underline{\Omega}(C_*(\tilde{M}))$, we obtain Chen's normalized chain complex, denoted by

$$C^*(\tilde{M}) \otimes \underline{\Omega}\widetilde{(C^*(\tilde{M}))}.$$

Given a filtration on it by

$$F_p = \bigoplus_{q \le p} C^q(\tilde{M}) \otimes \underline{\Omega}\widetilde{(C^*(\tilde{M}))},$$

then the associated spectral sequence converges with E_2-term

$$H^*(\tilde{M}) \otimes H^*(\underline{\Omega}\widetilde{(C^*(\tilde{M}))}).$$

The following lemma (Lemma 6.9) says that they are isomorphic, so by the comparison theorem of spectral sequences (McCleary [17], Theorem 3.26, p. 82) we see that $\phi_g^{\#}$ is a quasi-isomorphism.

Now we go to the $\phi_{g\#}$ case, where all the argument is just the dual of above. First, notice that $\phi_{g\#}$ is a well defined map. In fact, since $\tilde{M}$ is simply connected, $C_*(\tilde{M}) \otimes \Sigma\bar{C}_*(\tilde{M})^{\otimes n}$ is at least $(2n-1)$-connected, therefore if n is big enough, $\Psi_{n\#}(\alpha \times \Delta_n)$ is degenerate, so modulo degenerate chains, $\phi_{g\#}$ is well defined.

We now show $\phi_{g\#}$ is a chain map. In fact,

$$\begin{aligned}
\phi_{g\#}(\partial\alpha) &= \Big(\sum \Psi_{n\#}\Big)\Big(\sum \partial\alpha \times \Delta_n\Big) \\
&= \Big(\sum \Psi_{n\#}\Big)\Big(\sum \partial(\alpha \times \Delta_n) - \sum \alpha \times \partial\Delta_n)\Big) \\
&= \partial \circ \Big(\sum \Psi_{n\#}\Big)\Big(\sum \alpha \times \Delta_n\Big) - \Big(\sum \Psi_{n\#}\Big)\Big(\sum_n \sum_i (-1)^i \alpha \times \delta_i \Delta_{n-1}\Big) \\
&= b_g^{\mathrm{I}} \circ \phi_g(\alpha) + b_g^{\mathrm{II}} \circ \phi_{g\#}(\alpha) = b_g \circ \phi_{g\#}(\alpha).
\end{aligned}$$

That $\phi_{g\#}$ is a quasi-isomorphism is completely analogous to the $\phi_g^{\#}$ case. □

Lemma 6.9 (Adams [1], Chen [6])**.** *Let X be a connected and simply connected manifold and let $\underline{\Omega}(C^*(X))$ be the bar construction of its cochain complex. Then*

$$H^*(\Omega M) \cong H^*(\underline{\Omega}(C^*(X))).$$

Proof. Let PX be the path space of X:

$$PX := \Big\{ f : [0,1] \longrightarrow X \Big| f(1) = x_0 \Big\}.$$

And let Δ_n be the standard n-simplex as (130). Consider the following evaluation:

$$\begin{array}{ccc} \Omega X \times \Delta_n & \xrightarrow{\Psi_n} & \tilde{M} \times \cdots \times \tilde{M} \\ \Big\downarrow{\scriptstyle \mathrm{proj}} & & \\ \Omega X, & & \end{array} \tag{140}$$

where

$$\Psi_n(f,(t_1,\cdots,t_n)) = (f(0), f(t_1),\cdots,f(t_n)).$$

Similar to the above lemma, we obtain a chain complex

$$C^*(X) \otimes \underline{\Omega}(C^*(X))$$

with the boundary operator defined by

$$\begin{aligned} & \tilde{b}(x \otimes [a^1|\cdots|a^n]) \\ := \quad & dx \otimes [a^1|\cdots|a^n] + \sum_i (-1)^{|x|+|[a^1|\cdots|a^{i-1}]|} x \otimes [a^1|\cdots|da^i|\cdots|a^n] \\ & + xa^1 \otimes [a^2|\cdots|a^n]) + \sum_i (-1)^{|x|+|[a^1|\cdots|a^{i-1}]|} x \otimes [a^1|\cdots|a^i a^{i+1}|\cdots|a^n], \end{aligned}$$

and a chain map

$$\phi : (C^*(X) \otimes \underline{\Omega}(C^*(X)), \tilde{b}) \longrightarrow C^*(PX, \delta).$$

We know that PX is contractible thus $C^*(PX)$ is acyclic. Also define an operator

$$\begin{array}{rccc} h: & C^*(X) \otimes \underline{\Omega}(C^*(X)) & \longrightarrow & C^*(X) \otimes \underline{\Omega}(C^*(X)) \\ & x \otimes [a^1|\cdots|a^n] & \longmapsto & 1 \otimes [x|a^1|\cdots|a^n], \end{array}$$

one checks that

$$bh + hb = id,$$

which means that the identity map is homotopic to zero, and therefore the chain complex is also acyclic. By Chen's normalization, we obtain two spectral sequences similar to proof of above lemma and ϕ is an isomorphism on the total space and on the base. The comparison of the Eilenberg-Moore spectral sequences (McCleary [17], Theorem 7.15, p. 252) gives an isomorphism on the fiber

$$H^*(\underline{\Omega}(\widetilde{C^*(X)})) \cong H^*(\Omega X),$$

which proves the lemma. □

6.1.2 The chain complex model of LM

Note that $\coprod_{g\in G} L_g \tilde{M}$ is a disjoint union, by Theorem 6.8 we see that

$$\Big(C_*(\tilde{M}) \otimes \underline{\Omega}(C_*(\tilde{M})) \otimes \mathbb{Q}[G], \sum_{g\in G} \phi_{g\#}\Big)$$

gives a chain model of $\coprod_{g\in G} L_g \tilde{M}$.

Definition 6.10 (the G-action). *Let $C_*(\tilde{M})$ and $C_*(\tilde{M})\otimes\underline{\Omega}(C_*(\tilde{M}))\otimes\mathbb{Q}[G]$ be as above. Define a $\mathbb{Q}[G]$-action on the latter complex by*

$$\begin{array}{ccc}\mathbb{Q}[G]\bigotimes C_*(\tilde{M})\otimes\underline{\Omega}(C_*(\tilde{M}))\otimes\mathbb{Q}[G] & \longrightarrow & C_*(\tilde{M})\otimes\underline{\Omega}(C_*(\tilde{M}))\otimes\mathbb{Q}[G] \\ (h, x\otimes[a_1|\cdots|a_n]\otimes g) & \longmapsto & h_*x\otimes[h_*a_0|\cdots|h_*a_n]\otimes hgh^{-1}.\end{array}$$

Lemma 6.11. *The chain map*

$$\sum_{g\in G}\phi_{g\#}: C_*\Big(\coprod_{g\in G}L_g\tilde{M}\Big)\longrightarrow C_*(\tilde{M})\otimes\underline{\Omega}(C_*(\tilde{M}))\otimes\mathbb{Q}[G]$$

is $\mathbb{Q}[G]$-equivariant.

Proof. This is a chain level version of (131), where the evaluation maps are G-equivariant. ☐

Theorem 6.12. *There are quasi-isomorphisms*

$$C_*(LM)\xrightarrow{\simeq}C_*\Big(\coprod_{g\in G}L_g\tilde{M}\Big)\Big/G\xrightarrow{\simeq}\Big(C_*(\tilde{M})\otimes\underline{\Omega}(C_*(\tilde{M}))\otimes\mathbb{Q}[G]\Big)\Big/G.$$

Proof. Note that the G acts on $\tilde{M}$ freely and properly, so does it on $\coprod_g L_g\tilde{M}$. Therefore at chain level, $\mathbb{Q}[G]$ acts on $C_*\Big(\coprod_g L_g\tilde{M}\Big)$ and $C_*(\tilde{M})\otimes\underline{\Omega}(C_*(\tilde{M}))\otimes\mathbb{Q}[G]$ both freely and properly. By the classical result of algebraic topology (see, for example, McCleary [17] p. 337) the quotient chain complex gives the chain complex of $\coprod_{g\in G}L_g\tilde{M}/G\cong LM$. ☐

Note that taking quotient by G also means tensoring with $\mathbb{Q}$ over $\mathbb{Q}[G]$.

6.1.3 The S^1-action on LM

By lifting the S^1-action on LM to the universal covering, we obtain an $\mathbb{R}$-action on $\coprod_g L_g\tilde{M}$, which is given as follows:

Definition 6.13 (the $\mathbb{R}$-action on $\coprod_{g\in G}L_g\tilde{M}$). *Let $\tilde{M}$ and $L_g\tilde{M}$ be as above. Define an $\mathbb{R}$-action on $L_g\tilde{M}$*

$$\mathbb{R}\times L_g\tilde{M}\longrightarrow L_g\tilde{M}$$

as

$$(q\circ f)(x) := (g^{[q+x]}\circ f)(\{q+x\}),$$

for any $q\in\mathbb{R}$ and $f: I\to\tilde{M}\in L_g\tilde{M}$, where $[q+x]$ is the largest integer no greater than $q+x$ and $\{q+x\}$ is their difference. Consider all the components, we then obtain an $\mathbb{R}$-action

$$\mathbb{R}\times\coprod_{g\in G}L_g\tilde{M}\longrightarrow\coprod_{g\in G}L_g\tilde{M}.$$

Lemma 6.14 (the $\mathbb{Z}$-action). *The embedding of $\mathbb{Z}$ in $\mathbb{R}$ induces a $\mathbb{Z}$-action on $\coprod_{g\in G}L_g\tilde{M}$, which is given by*

$$\begin{array}{ccc}\mathbb{Z}\times L_g\tilde{M} & \longrightarrow & L_g\tilde{M} \\ q\circ f(x) & \longmapsto & g^q\circ f(x), \quad q\in\mathbb{Z}.\end{array}$$

Moreover the $\mathbb{Z}$-action commutes with the action of G. Therefore, if we take the quotient space $\coprod_g L_g\tilde{M}/G$, the $\mathbb{R}$-action can be passed to $\mathbb{R}/\mathbb{Z}$, which is the S^1-action on LM. In other words, the following diagram commutes:

$$\begin{tikzcd} \mathbb{R}\times\coprod L_g\tilde{M} \arrow[r] \arrow[d, "\mathbb{Z}\times G"] & \coprod L_g\tilde{M} \arrow[d, "G"] \\ \mathbb{R}/\mathbb{Z}\times\coprod L_g\tilde{M}/G \arrow[r] & \coprod L_g\tilde{M}/G. \end{tikzcd} \tag{141}$$

Proof. Note that if $f\in L_g\tilde{M}$, then $h\circ f\in L_{hgh^{-1}}\tilde{M}$. For any $q\in\mathbb{Z}$, we have

$$h\circ(q\circ f)=h\circ(g^q\circ f)=(hgh^{-1})^q\circ h\circ f=q\circ(h\circ f).$$

Therefore the $\mathbb{Z}$-action commutes with G. By taking quotient we obtain an action

$$\mathbb{R}/\mathbb{Z}\times\Big(\coprod_{g\in G}L_g\tilde{M}\Big/G\Big)\longrightarrow\Big(\coprod_{g\in G}L_g\tilde{M}\Big/G\Big),$$

which is the S^1-action on LM. $\square$

This lemma leads us to define two operators $\tilde{J}$ and $\tilde{B}$, which is the action of the unit interval on $L_g\tilde{M}$, as follows:

Definition 6.15 (unit interval action on the chain complex)**.** *Denote by ψ the above $\mathbb{R}$-action on $L_g\tilde{M}$:*

$$\psi:\mathbb{R}\times L_g\tilde{M}\longrightarrow L_g\tilde{M}.$$

Define

$$\begin{array}{rccc}\tilde{J}: & C_*(L_g\tilde{M}) & \longrightarrow & C_{*+1}(L_g\tilde{M}) \\ & \alpha & \longmapsto & \psi_*(\Delta_1\times\alpha),\end{array}$$

where Δ_1 is the unit interval in $\mathbb{R}$. And also define an operator $\tilde{B}$ on $C_*(\tilde{M})\otimes\underline{\Omega}(C_*(\tilde{M}))$ as follows

$$\begin{array}{rccc}\tilde{B}: & C_*(\tilde{M})\otimes\underline{\Omega}(C_*(\tilde{M})) & \longrightarrow & C_*(\tilde{M})\otimes\underline{\Omega}(C_*(\tilde{M})) \\ & x\otimes[a_1|\cdots|a_n] & \longmapsto & \sum_i\varepsilon(x)a_i\otimes[a_{i+1}|\cdots|a_n|g_*a_1|\cdots|g_*a_{i-1}].\end{array}$$

Lemma 6.16 (compare Theorem 3.28)**.** *Let $\tilde{J}$ and $\tilde{B}$ be as in above definition. Then*

(1) *$\tilde{J}^2=0$ and $\tilde{B}^2=0$. Moreover, both commute with the $\mathbb{Q}[G]$-action.*

(2) *$\partial\tilde{J}+\tilde{J}\partial=id-g_*$ and $b_g\tilde{B}+\tilde{B}b_g=id-g_*$.*

(3) *both $\tilde{J}$ and $\tilde{B}$ commutes with the $\mathbb{Q}[G]$-action, and the following diagram commutes:*

$$\begin{tikzcd} C_*\Big(\coprod L_g\tilde{M}\Big) \arrow[r, "\tilde{J}"] \arrow[d, "\sum\phi_g"] & C_*\Big(\coprod L_g\tilde{M}\Big) \arrow[d, "\sum\phi_g"] \\ C_*(\tilde{M})\otimes\underline{\Omega}(C_*(\tilde{M}))\otimes\mathbb{Q}[G] \arrow[r, "\tilde{B}"] & C_*(\tilde{M})\otimes\underline{\Omega}(C_*(\tilde{M}))\otimes\mathbb{Q}[G] \end{tikzcd} \tag{142}$$

Proof. (1) $\tilde{J}^2 = 0$ comes from the fact $\tilde{J}^2$ is a degenerate chain, and $\tilde{B}^2 = 0$ comes from the definition. That these two operators commutes with $\mathbb{Q}[G]$ just follows from the definition.

(2) The first equation comes from Lemma 6.14 and the second equation comes from direct computation.

(3) Consider the evaluation maps composed with the unit interval action:

$$\begin{array}{ccccc} [0,1] \times L_g\tilde{M} & \xrightarrow{\psi} & L_g\tilde{M}_n & \xrightarrow{\Psi_n} & \tilde{M} \times \cdots \times \tilde{M} \\ (s, f) & \longmapsto & s \circ f & \longmapsto & (f(s), f(s+t_1), \cdots, f(s+t_n)). \end{array} \tag{143}$$

We see that $\Psi_n\psi$ comes from Ψ_{n+1}: Recall that

$$\Delta_n = \Big\{(t_1, \cdots, t_n) \in \mathbb{R}^n \Big| 0 \le t_1 \le \cdots \le t_n \le 1\Big\},$$

we have a decomposition of $[0,1] \times \Delta_n$ into $n+1$ standard $(n+1)$-simplices:

$$\begin{aligned} \Delta^i_{n+1} &:= \Big\{0 \le s \le \cdots \le s + t_{i-1} \le 1 \le s + t_i \le \cdots \le s + t_n \le 2\Big\} \\ &= \Big\{0 \le s + t_i - 1 \le \cdots \le s + t_n - 1 \le s \le \cdots \le s + t_{i-1} \le 1\Big\}, \end{aligned} \tag{144}$$

and therefore if $\Psi_{n+1\#}(\alpha \times \Delta_{n+1}) = x \otimes [a_1|\cdots|a_{n+1}]$, by changing variables (compare (143) and (144)), we have (up to sign)

$$\begin{aligned} \Psi_{n\#}(\tilde{J}\alpha \times \Delta_n) &= \begin{cases} 0, & \text{if} \quad |x| \ne 0; \\ \sum a_i \otimes [a_{i+1}|\cdots|a_{n+1}|g_*a_1|\cdots|g_*a_{i-1}], & \text{otherwise}, \end{cases} \\ &= \tilde{B} \circ \Psi_{n+1\#}(\alpha \times \Delta_{n+1}), \end{aligned}$$

where in the $|x| \ne 0$ case the value is zero because it is a degenerate chain (the degrees of the two sides are not equal while $\Psi_{n\#}$ is a chain map). This implies (142) if we consider all the $\Psi_{n\#}$'s and all the components of $\coprod_{g\in G} L_g\tilde{M}$. ∎

Theorem 6.17 (S^1-action and the cyclic operator)**.** *Let $\tilde{M}$ and $\coprod_{g\in G} L_g\tilde{M}$ be as above. Then $\tilde{J}$ and $\tilde{B}$ pass to the quotient chain complexes over $\mathbb{Q}[G]$, denoted by J and B, and we have*

(1) $J^2 = B^2 = 0$, $\partial J + J\partial = 0$ *and* $bB + Bb = 0$*;*

(2) *The following diagram commutes:*

$$\begin{tikzcd} C^G_*\Big(\coprod_{g\in G} L_g\tilde{M}\Big) \arrow[r, "J"] \arrow[d] & C^G_{*+1}\Big(\coprod_{g\in G} L_g\tilde{M}\Big) \arrow[d] \\ \Big(C_*(\tilde{M}) \otimes \underline{\Omega}(C_*(\tilde{M})) \otimes \mathbb{Q}[G]\Big)\Big/G \arrow[r, "B"] & \Big(C_*(\tilde{M}) \otimes \underline{\Omega}(C_*(\tilde{M})) \otimes \mathbb{Q}[G]\Big)\Big/G. \end{tikzcd}$$

Proof. The proof follows from the above lemma. ∎

One sees that J is the S^1-action on the chain complex of the free loop space LM, so B models the S^1-action on the chain complex model $\big(C_*(\tilde{M}) \otimes \underline{\Omega}(C_*(\tilde{M})) \otimes \mathbb{Q}[G]\big)/G$.

Lemma 6.18. *Let M, $\tilde{M}$ and G as above. Define*

$$\tilde{B}: \quad \big(C_*(\tilde{M}) \otimes \underline{\Omega}(C_*(\tilde{M})) \otimes \mathbb{Q}[G]\big) \quad \longrightarrow \quad \big(C_*(\tilde{M}) \otimes \underline{\Omega}(C_*(\tilde{M})) \otimes \mathbb{Q}[G]\big)$$

by

$$B(a^0 \otimes [a^1|\cdots|a^n] \otimes g) := \sum_i 1 \otimes [g^* a^i|\cdots|g^* a^n|a^0|\cdots|a^{i-1}] \otimes g,$$

then $\tilde{B}$ can be passed to $\big(C_*(\tilde{M}) \otimes \underline{\Omega}(C_*(\tilde{M})) \otimes \mathbb{Q}[G]\big)/G$, denoted by B, which models the S^1-action on the cochain complex of LM.

Proof. The proof is completely analogous to the above theorem. □

6.2 The Batalin-Vilkovisky algebra

In this subsection we briefly describe the Batalin-Vilkovisky algebra on the homology of the free loop space of a general manifold. The computations are much the same as those in previous chapters, so we will only give the statements and leave the verification to the reader.

As before we consider the DG Frobenius-like algebra on $\tilde{M}$, which is $(A_{\mathrm{c}}(M), C(M), \iota)$ (recall the arguments in §2.3.3). To simplify the notations we write $A_{\mathrm{c}}(\tilde{M}) \otimes \underline{\Omega}(C(\tilde{M})) \otimes g$ as $C_*(L_g\tilde{M})$, and $\big(A_{\mathrm{c}}(\tilde{M}) \otimes \underline{\Omega}(C(\tilde{M})) \otimes \mathbb{Q}[G]\big)/G$ as $C_*^G(\coprod L_g\tilde{M})$ for short.

6.2.1 The loop product on $\mathbb{H}_*(LM)$

The loop product $\bullet$ of Chas and Sullivan is modeled as follows:

Definition 6.19 (loop product)**.** *Let $(A_{\mathrm{c}}(\tilde{M}), C(\tilde{M}), \iota)$ be the DG Frobenius-like algebra of $\tilde{M}$. Define a binary operator $\tilde{\bullet}$ on $C_*(\coprod L_g\tilde{M})$ as follows: for any*

$$\alpha = x \otimes [a_1|\cdots|a_n] \otimes g \in C_*(L_g\tilde{M})$$

and

$$\beta = y \otimes [b_1|\cdots|b_m] \otimes h \in C_*(L_h\tilde{M}),$$

let

$$\alpha \tilde{\bullet} \beta := x \cdot g_*^{-1} y \otimes [a_1|\cdots|a_n|b_1|\cdots|b_m] \otimes gh.$$

On the G-equivariant chain complex $C_*^G(\coprod L_g\tilde{M})$, define a binary operator $\bullet$ as follows: for $[\alpha], [\beta] \in C_*^G(\coprod L_g\tilde{M})$,

$$[\alpha] \bullet [\beta] := \Big[\alpha \tilde{\bullet} \sum_{g\in G} g_*\beta\Big].$$

Lemma 6.20. *The operator $\bullet$ does not depend on the choice of the representatives and is well defined. Moreover, it commutes with the boundary operator b.*

Proof. The fact that $\bullet$ commutes with b follows from a direct computation (compare Definition 4.1 in the simply connected case). To show $\bullet$ does not depend on the choice of representatives, take arbitrary $h, k \in G$,

$$[h_*\alpha] \bullet [k_*\beta] = \Big[h_*\alpha \tilde{\bullet} \sum_{g\in G} g_* k_*\beta\Big] = \Big[h_*\alpha \tilde{\bullet} \sum_{g\in G} g_*\beta\Big] = \Big[h_*\alpha \tilde{\bullet} \sum_{g\in G} g_* h_*\beta\Big] = [\alpha] \bullet [\beta].$$

Also since $\mathbb{Q}[G]$ acts on $C_*(\coprod L_g\tilde{M})$ freely and properly, and the differential forms are compactly supported, $\bullet$ is well defined. $\square$

Therefore we obtain a graded algebra on the homology of LM. As in the simply connected case, such an algebra exactly models the Chas-Sullivan loop product.

6.2.2 The loop bracket and the Batalin-Vilkovisky algebra

Definition 6.21 ($*$ operator and the loop bracket)**.** *Let $(A_c(\tilde{M}), C(\tilde{M}), \iota)$ be the DG Frobenius-like algebra of $\tilde{M}$. Define a binary operator $\tilde{*}$ on $C_*(\coprod L_g\tilde{M})$ as follows: for any*

$$\alpha = x \otimes [a_1|\cdots|a_n] \otimes g \in C_*(L_g\tilde{M})$$

and

$$\beta = y \otimes [b_1|\cdots|b_m] \otimes h \in C_*(L_h\tilde{M}),$$

let

$$\alpha\tilde{*}\beta := \sum_i \varepsilon(a_i y) x \otimes [a_1|\cdots|a_{i-1}|b_1|\cdots|b_m|h_*a_{i+1}|\cdots|h_*a_n] \otimes gh.$$

On the G-equivariant chain complex $C_*^G(\coprod L_g\tilde{M})$, define a binary operator $*$ as follows: for $[\alpha], [\beta] \in C_*^G(\coprod L_g\tilde{M})$,

$$[\alpha] * [\beta] := \Big[\alpha\tilde{*}\sum_{g\in G} g_*\beta\Big].$$

Lemma 6.22 (Gerstenhaber algebra of the free loop space)**.** *Let M and $\tilde{M}$ be as above.*

(1) *On $C_*(\coprod L_g\tilde{M})$,*

$$b(\alpha\tilde{*}\beta) = b\alpha\tilde{*}\beta + \alpha\tilde{*}b\beta + (-1)^{|\alpha|}(\alpha\tilde{\bullet}\beta - (-1)^{|\alpha||\beta|}h_*(h_*^{-1}\beta\tilde{\bullet}\alpha)).$$

(2) *On $C_*^G(\coprod L_g\tilde{M})$, the operator $*$ does not depend on the choice of the representatives and is well defined. Moreover,*

$$b(\alpha * \beta) = b\alpha * \beta + \alpha * b\beta + (-1)^{|\alpha|}(\alpha \bullet \beta - (-1)^{|\alpha||\beta|}\beta \bullet \alpha),$$

which means $\bullet$ is graded commutative on the homology $H_*^G\Big(\coprod L_g\tilde{M}\Big)$.

(3) *The commutator of $*$ forms a degree one Lie algebra, which is compatible with $\bullet$, making*

$$\Big(\mathbb{H}_*^G\Big(\coprod L_g\tilde{M}\Big), \bullet, \{,\}\Big)$$

be a Gerstenhaber algebra.

Proof. These results follow from direct computations (compare §4.2 in the simply connected case). All the computations there can be applied here with a minor modification. $\square$

Theorem 6.23 (Batalin-Vilkovisky algebra of the free loop space)**.** *Let M be a smooth manifold and $\tilde{M}$ be its universal covering. Let B be the cyclic operator defined in Theorem 6.17 on the chain complex model of LM. The homology*

$$\left(\mathbb{H}_*^G\left(\coprod L_g\tilde{M}\right), \bullet, B\right)$$

forms a Batalin-Vilkovisky algebra, which coincides with the one given by [5].

Proof. As in the above Definition 6.19 and Theorem 6.21, the homotopy operator defined in Lemma 5.3 can be applied here, which implies the theorem. □

Remark 6.24. The arguments in Section 6 are independent of Section 3, where the latter is a special case of the former. The reason that we keep Section 3 is that it is more geometric, especially in the identification of the Chas-Sullivan loop product (Theorem 4.2).

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