Self-Dual Metrics on 4-Manifolds

A Dissertation Presented

by

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to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

August 2007

Stony Brook University

The Graduate School

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Abstract of the Dissertation

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Under a vanishing hypothesis, Donaldson and Friedman proved that the connected sum of two self-dual Riemannian 4-Manifolds is again self-dual. Here we prove that the same result can be extended over to the positive scalar curvature case.

Secondly we give an example of a 4-manifold with $b^+ = 0$ admitting a scalar-flat anti-self-dual metric.

Finally we apply the Geometric Invariant Theory(GIT) for Toric Varieties to the Einstein-Weyl Geometry and obtain a partial result.

Contents

Acknowledgements

I would like to thank my Ph.D. advisor Claude LeBrun for his excellent directions, for letting me use his results and ideas, Justin Sawon for his generous knowledge, Alastair Craw for his lectures on the GIT. and many thanks to Ioana Suvaina, Jeff Viaclovsky.

It was a good opportunity to meet Siddhartha Gadgil, Yair Minsky, Dennis Sullivan, Tony Phillips, Bernard Maskit from whom I have learned a great deal of 3-manifold topology and Kleinian Groups in my early years at Stony Brook.

Thanks go to Blaine Lawson and Dusa McDuff for listening my presentations and spending time to write a recommendation letter, Frédéric Rochon for carefully reading my thesis and giving me a nice feedback.

I am also grateful to my friends İbrahim Ünal, Kerim Gülyüz, Selçuk Eren, Cem Kuşçu, Caner Koca in my years at Stony Brook.

Introduction

Let (M, g) be an oriented Riemannian n-manifold. Then by raising an index, the Riemann curvature tensor at any point can be viewed as an operator $\mathcal{R}: \Lambda^2 M \to \Lambda^2 M$ hence an element of $S^2\Lambda^2 M$. It satisfies the algebraic Bianchi identity hence lies in the vector space of *algebraic curvature tensors.* This space is an $O(n)$-module and has an orthogonal decomposition into irreducible subspaces for $n \geq 4$. Accordingly the Riemann curvature operator decomposes as:

$$\mathcal{R} = U \oplus Z \oplus W$$

where

$$U = \frac{s}{2n(n-1)} g \bullet g \quad \text{and} \quad Z = \frac{1}{n-2} \overset{\circ}{Ric} \bullet g$$

s is the scalar curvature, $\overset{\circ}{Ric} = Ric - \frac{s}{n} g$ is the trace-free Ricci tensor, "$\bullet$" is the Kulkarni-Nomizu product, and W is the *Weyl Tensor* which is defined to be what is left over from the first two pieces.

When we restrict ourselves to dimension $n = 4$, the Hodge Star operator $* : \Lambda^2 \to \Lambda^2$ is an involution and has ± 1-eigenspaces decomposing the space of two forms as $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, yielding a decomposition of any operator acting on this space. In particular $W_\pm : \Lambda^2_\pm \to \Lambda^2_\pm$ is called self-dual and anti-self-dual pieces of the Weyl curvature operator. And we call g to be *self-dual*(resp. *anti-self-dual*) *metric* if W_-(resp. W_+) vanishes. In this case [AHS] construct a complex 3-manifold Z called the *Twistor Space* of (M^4, g), which comes with a fibration by holomorphically embedded rational curves :

$$\begin{array}{ccccl} \mathbb{CP}_1 & \rightarrow & Z & & \text{Complex 3-manifold} \\ & & \downarrow & & \\ & & M^4 & & \text{Riemannian 4-manifold} \end{array}$$

This construction drew the attention of geometers, and many examples of Self-Dual metrics and related Twistor spaces were given afterwards. One result proved to be a quite effective way to produce infinitely many examples and became a cornerstone in the field :

Theorem 2.4.1 (Donaldson-Friedman,1989,[DF]). *If* (M_1, g_1) *and* (M_2, g_2) *are compact self-dual Riemannian 4-manifolds with* $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$*,*

Then $M_1 \# M_2$ *also admits a self-dual metric.*

The idea of the proof is to work upstairs in the complex category rather than downstairs. One glues the blown up twistor spaces from their exceptional divisors to obtain a singular complex space $Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2$. Then using the Kodaira-Spencer deformation theory extended by R.Friedman to singular spaces, one obtains a smooth complex manifold, which turns out to be the twistor space of the connected sum.

When working in differential geometry, one often deals with the moduli space of certain kind of metrics. The situation is also the same for the self-dual theory. Many people obtained results on the space of positive scalar curvature self-dual(PSC-SD) metrics on various kinds of manifolds. Since the positivity of the scalar curvature imposes some topological restrictions on the moduli space, people often find it convenient to work under this assumption

However one realizes that there is no connected sum theorem for self-dual positive scalar curvature metrics. Donaldson-Friedman Theorem(2.4.1) does not make any statement about the scalar curvature of the metrics produced. Therefore we attacked the problem of determining the sign of the scalar curvature for the metrics produced over the connected sum, beginning by proving the following, using the techniques similar to that of [LeOM]:

Theorem 4.2.3 (Vanishing Theorem). *Let $\omega : \mathcal{Z} \to \mathcal{U}$ be a 1-parameter standard deformation of Z_0, where Z_0 is as in Theorem (2.4.1), and $\mathcal{U} \subset \mathbb{C}$ is a neighborhood of the origin. Let $L \to \mathcal{Z}$ be the holomorphic line bundle defined by*

$$\mathcal{O}(L^*) = \mathscr{I}_{\tilde{Z}_1}(K_{\mathcal{Z}}^{1/2}).$$

If $(M_i, [g_i])$ has positive scalar curvature, then by possibly replacing $\mathcal{U}$ with a smaller neighborhood of $0 \in \mathbb{C}$ and simultaneously replacing $\mathcal{Z}$ with its inverse image, we can arrange for our complex 4-fold $\mathcal{Z}$ to satisfy

$$H^1(\mathcal{Z}, \mathcal{O}(L^*)) = H^2(\mathcal{Z}, \mathcal{O}(L^*)) = 0.$$

The proof makes use of the Leray Spectral Sequence, homological algebra and Kodaira-Spencer deformation theory, involving many steps. Using this technical theorem next we prove that the Donaldson-Friedman Theorem can be generalized to the positive scalar curvature(PSC) case :

Theorem 5.4.1. *Let (M_1, g_1) and (M_2, g_2) be compact self-dual Riemannian 4-*

manifolds with $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$ for their twistor spaces. Moreover suppose that they have positive scalar curvature.

Then, for all sufficiently small $\mathfrak{t} > 0$, the conformal class $[g_{\mathfrak{t}}]$ obtained on $M_1 \# M_2$ by the Donaldson-Friedman Theorem (2.4.1) contains a metric of positive scalar curvature.

We work on the self-dual conformal classes constructed by the Donaldson-Friedman Theorem (2.4.1). Conformal Green's Functions [LeOM] are used to detect the sign of the scalar curvature of these metrics. Positivity for the scalar curvature is characterized by non-triviality of the Green's Functions. Then the Vanishing Theorem (4.2.3) will provide the Serre-Horrocks[Ser, Hor] vector bundle construction, which gives the Serre Class, a substitute for the Green's Function by Atiyah [AtGr]. And non-triviality of the Serre Class will provide the non-triviality of the extension described by it.

In chapters §1-§3 we review the background material. In §4 the vanishing theorem is proven, and finally in §5 the sign of the scalar curvature is detected.

Secondly, in chapters §6 we prove that a quotient of a $K3$ surface by a free $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action does not admit any metric of positive scalar curvature. This shows that the scalar flat anti self-dual metrics (SF-ASD) on this manifold can not be obtained from a family of metrics for which the scalar curvature changes sign, contrary to the previously known constructions of this kind of metrics on manifolds of $b^+ = 0$.

Finally, in the last chapter, we apply the Geometric Invariant Theory(GIT) for Toric Varieties to the Einstein-Weyl Geometry and obtain a partial result. We compute the image of the quotient of a C^* action on the twistor space of

Honda metrics on the connected sum of three projective planes according to some linearization.

Chapter 1

Four-Dimensional Riemannian Geometry

Let (M, g) be an oriented Riemannian manifold of dimension n. The aim of this chapteris twofold. First of all, we are going to show that the curvature tensor of M splits into three components in all dimensions. Secondly, there is a further splitting happening only in dimension 4. These decompositions are orthogonal and also irreducible according to appropriate inner product and representation.

1.1 Decomposition of Bilinear Forms

To understand the first decomposition, we begin with the corresponding decomposition of the matrix spaces. Whenever we have a square matrix Q, it has an orthogonal decomposition into a skew-symmetric(alternating) and a symmetric part: $Q = (Q - Q^T)/2 + (Q + Q^T)/2$. We verify that the skew-symmetric and symmetric matrices are orthogonal to each other: Let A, S be two square matrices of the same rank which are skew-symmetric(alternating) and symmetric respectively. We have

$$\langle A, S\rangle := trA^TS = tr(-AS) = -trAS = -trSA = -tr(SA)^T = -trA^TS = -\langle A, S\rangle$$

so $\langle A, S\rangle = 0$. We used the property $trAB = trBA$ here, actually holding for any appropriate matrices [Br]. Symmetric matrices have a further decomposition. It comes out of the fact that there is a canonical way to build a trace-free matrix from an arbitrary one by just subtracting the appropriate factor $(trace/n)$ of the identity matrix to kill the trace, namely $\mathring{S} := S - (trS/n)I_n$. Any trace-free matrix is orthogonal to the multiples of the identity matrix. We apply this decomposition to the symmetric part only, since the skew-symmetric matrices are already trace-free. So the square matrix spaces orthogonally decompose as : $M(n, \mathbb{R}) = \Lambda_n \oplus \mathring{S}_n \oplus \mathbb{R}I_n$ where Λ_n and $\mathring{S}_n$ denote the space of alternating and trace-free symmetric n-matrices.

Now, let E be an n-dimensional real vector space. As the general linear group acts on E, it acts on the tensor spaces $T^{(k,l)}E = \otimes^k E^* \otimes \otimes^l E$. For $f^1 \cdots f^k$ in E^* and $v_1 \cdots v_l$ in E, any $\gamma \in GL(E)$ acts as :

$$\gamma(f^1 \otimes \cdots \otimes f^k \otimes v_1 \otimes \cdots \otimes v_l) = \gamma^{-1T} f^1 \otimes \cdots \otimes \gamma^{-1T} f^k \otimes \gamma v_1 \otimes \cdots \otimes \gamma v_l$$

Let q be a non-degenerate quadratic form on E. Then q induces a natural isomorphism between E and E^*. If $\gamma \in O(q)$, we have $\gamma^{-1T} = \gamma$ so E and E^* are isomorphic as $O(q)$-modules, and we may consider the tensor products of E only.

E is an irreducible $O(q)$-module : think it as $(\mathbb{R}^n, O_n)$, if any of the proper subspaces of $\mathbb{R}^n$ is invariant under the action, choose an orthonormal basis $e_1 \cdots e_k$ for it and complete to an orthonormal basis for $\mathbb{R}^n$, $e_1 \cdots e_n$. Then certainly there is an element in O_n moving e_1 to e_n, so an invariant subspace is impossible. But the same is not true for the tensor products e.g. $\otimes^2 E$.

Proposition 1.1.1. *The module* $(\otimes^2 E, O(q))$ *is reducible and has the irreducible, orthogonal decomposition*

$$\otimes^2 E = \Lambda^2 E \oplus S_0^2 E \oplus \mathbb{R}q.$$

Here, q identifies S^2E with S^2E^* and imports the trace invariant of bilinear forms in S^2E^* to S^2E. The induced trace is denoted by $tr_q : S^2E \to \mathbb{R}$. So the notation S_0^2E stands for the space of trace-free symmetric 2-tensors. Then any k in $\otimes^2 E (\approx E^* \otimes E$ via $q)$ can be decomposed as :

$$k = \Lambda^2 k + S_0^2 k + \frac{tr_q k}{n} q$$

where

$$S_0^2 k = S^2 k - \frac{tr_q k}{n} q$$

and

$$S^2 k(x,y) = [k(x,y) + k(y,x)]/2 \quad , \quad \Lambda^2 k(x,y) = [k(x,y) - k(y,x)]/2$$

as usual, by analogy with the matrix decomposition. The link between is provided by q. One can check that $\Lambda^2 E$ and $S_0^2 E$ are irreducible by checking that the dimension of $O(q)$-invariant quadratic forms on $\otimes^2 E$ is 3.

1.2 Refined Decomposition of the Curvature Tensor in Dimension 4

Let (M, g) be an oriented Riemannian manifold of dimension n. We have a linear transformation between the bundles of exterior forms called the Hodge star operator $* : \Lambda^p \to \Lambda^{n-p}$. It is the unique vector bundle isomorphism between $\binom{n}{p}$-dimensional vector bundles defined by

$$\alpha \wedge (*\beta) = g(\alpha, \beta) dV_g$$

for all $\alpha, \beta \in \Lambda^p$, where dV_g is the canonical n-form of g satisfying $dV_g(e_1, , e_n) = 1$ for any oriented orthonormal basis $e_1, \cdots, e_n$. $*$ is defined pointwise but it takes smooth forms to smooth forms, so induces a linear operator $* : \Gamma(\Lambda^p) \to \Gamma(\Lambda^{n-p})$ between infinite dimensional spaces. Notice that $*1 = dV_g$, $*dV_g = 1$ and $*^2 = (-1)^{p(n-p)} Id_{\wedge^p}$. [Besse, AHS, War]

If n is even, the star operates on the middle dimension with $*^2 = (-1)^{n/2} Id_{\wedge^{n/2}}$. Moreover it is conformally invariant in dimension $n/2$: If we rescale the metric by a scalar λ, then $\widetilde{g} = \lambda g$ and $dV_{\widetilde{g}} = \lambda^{n/2} dV_g$ so that their product remains unchanged on $n/2$-forms since the inner product on the cotangent vectors multiplied by λ^{-1}.

If $n = 2$, $*$ acts on Λ^1 or TM^* as well as TM by duality with $*^2 = -Id_{TM}$. So it defines a complex structure on a surface.

The case we are interested is $n = 4$, i.e. we have a Riemannian 4-Manifold and $* : \Lambda^2 \to \Lambda^2$ with $*^2 = Id_{\wedge^2}$ and we have eigenspaces $E_x(\pm 1)$ over each point x denoted $(\Lambda^2_\pm)_x$ and the bundle Λ^2 splits as $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$. We call these bundles, bundle of *self-dual* and *anti-self-dual two forms* respectively.

Remark 1.2.1. *The splitting of two forms turns out to have a great influence on the geometry of 4-manifolds because the Riemann curvature tensor can be considered as an operator on two forms and so also has a corresponding splitting [AHS]*

$$R = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

where A and C are self-adjoint. This representation of R gives the complete decomposition of the curvature tensor into irreducible components by [SinTho69]

$$R \to (trA \ , \ B \ , \ A - \frac{1}{3}trA \ , \ C - \frac{1}{3}trC)$$

where $trA = trC = \frac{s}{4}$, $B = \overset{\circ}{Ric}$ is the trace-free Ricci tensor, and the last two components are the W_+ and W_- the self-dual and anti-self-dual pieces of the conformally invariant Weyl Tensor. So that the matrix becomes

$$R = \begin{pmatrix} W_+ + \frac{s}{12} & \overset{\circ}{Ric} \\ \overset{\circ}{Ric}^* & W_- + \frac{s}{12} \end{pmatrix}.$$

Here $W_\pm$ are the traceless pieces of the appropriate blocks. The scalar curvature is understood to act by scalar multiplication, and the $\overset{\circ}{Ric}$ acts on the anti-self-dual 2-forms by [CLRic]

$$\psi_{ab} \mapsto \overset{\circ}{Ric}_{ac}\psi^c{}_b - \overset{\circ}{Ric}_{bc}\psi^c{}_a.$$

Example 1.2.2 ([Shen]). *Let $M = S^4$ be the 4-sphere with the induced round metric from $\mathbb{R}^5$, so that its scalar curvature $s = 12$, and $Vol(S^4) = 8\pi^2/3$.*

$$R = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 1 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & 1 & 0 & 0 \\ & 0 & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right)$$

so that the Weyl tensor $W = \Theta_{6\times 6}$. Hence this manifold is (locally) conformally flat, both self-dual and anti-self-dual.

Example 1.2.3 ([Shen]). *Let $M = \mathbb{CP}_2$ be the complex projective plane with the Fubini-Study metric, so that its scalar curvature $s = 24$, and $Vol(\mathbb{CP}_2) = \pi^2/2$.*

$$R = \left(\begin{array}{ccc|ccc} 6 & 0 & 0 & & & \\ 0 & 0 & 0 & & 0 & \\ 0 & 0 & 0 & & & \\ \hline & & & 2 & 0 & 0 \\ & 0 & & 0 & 2 & 0 \\ & & & 0 & 0 & 2 \end{array}\right)$$

so that the Weyl tensor

$$W = R - \frac{s}{12} I_{6\times 6} - \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & & & \\ 0 & -2 & 0 & & 0 & \\ 0 & 0 & -2 & & & \\ \hline & & & 0 & 0 & 0 \\ & 0 & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{array}\right) = W_+.$$

Hence this is a self-dual manifold.

Chapter 2

Self-Dual Manifolds and the Donaldson-Friedman Construction

2.1 The Optimal Metric problem

In this section and in the following, we give some motivation for studying SD/ASD metrics. Scalar-flat-anti-self-dual (SF-ASD) metrics are solutions to the *optimal metric* problem. Optimal metric problem is a struggle to find a "best" metric for a smooth manifold. Historically, geometers were interested in constant sectional curvature spaces. As soon as these spaces were classified, there is a question of what to do with manifolds that do not admit any constant sectional curvature. Some of them are metrized by Einstein metrics, which have constant Ricci curvature. However there are still manifolds which do not admit any Einstein metric. At this point SF-ASD metrics come into the picture. More precisely :

Definition 2.1.1 ([LeOM]). *A Riemannian metric on a smooth* 4*-manifold* M *is called an* optimal metric *if it is the absolute minimum of the* L^2 *norm of* *the Riemann Curvature tensor on the space of metrics*

$$\mathcal{K}(g) = \int_M |\mathcal{R}_g|^2 dV_g.$$

Using the orthogonal decomposition it is equal to

$$\mathcal{K}(g) = \int_M \frac{s^2}{24} + \frac{|\overset{\circ}{Ric}|^2}{2} + |W|^2 \ \ dV_g \ .$$

On the other hand, the generalized Gauss-Bonnet Theorem and the Hirzebruch Signature Theorem express the Euler characteristic χ and the signature τ respectively as

$$\chi(M) = \frac{1}{8\pi^2} \int_M \frac{s^2}{24} + |W|^2 - \frac{|\overset{\circ}{Ric}|^2}{2} \ \ dV_g$$

$$\tau(M) = \frac{1}{12\pi^2} \int_M |W_+|^2 - |W_-|^2 \ \ dV_g \ .$$

Combining the two gives the following expression for $\mathcal{K}$,

$$\mathcal{K}(g) = -8\pi^2(\chi + 3\tau)(M) + 2\int_M \frac{s^2}{24} + 2|W_+|^2 \ \ dV_g \ .$$

This yields

Proposition 2.1.2 ([LeOM])**.** *Let M be a smooth compact oriented 4-manifold. If M admits a SF-ASD metric then this metric is optimal. In this case all other optimal metrics are SF-ASD, too.*

For further information on the optimal metric problem, we suggest the excellent survey article [LeOM] by C. LeBrun.

2.2 Self-Dual Gauge Fields

In this section we will describe the relationship between self-dual connections on the bundle of self-dual 2-forms and the metric structure of the 4-manifold, and will see the general distinction between the self-duality of the metric and the self-duality of its Levi-Civita connection. We follow [AHS, MoGa].

Let $\pi : P \to M$ be a principal G-bundle over a 4-manifold. A *connection* for this bundle is a 4-dimensional distribution $\mathcal{H}$ on P which is *horizontal* in the sense that the restriction of the differential

$$\pi_{*p} : \mathcal{H}_p \xrightarrow{\sim} T_{\pi(p)}X$$

is an isomorphism at each point p of P. Furthermore we require that this distribution is *invariant* under the G-action. Using a connection one gets a unique lifting for the paths in M requiring that the tangent vectors of the lifting lie in the distribution. Moreover, for a smooth path between two points in the manifold, a connection determines an isomorphism between the fibers over these points which is equivariant with respect to the G-actions on these fibers. It is achieved by the unique path lifting property of the connection. By this way we are able to connect the points on the distinct fibers, and this is the reason for the name connection.

We have an equivalent way of defining a connection in terms of differential 1-forms. Let $\mathfrak{g}$ be the Lie algebra of G. Then, a connection on the principal bundle is equivalent to a differential 1-form $\omega \in A^1(P;\mathfrak{g}) = \Gamma(\Lambda^1(P;\mathfrak{g}))$ satisfying the properties

- Under the G-action, ω transforms via the adjoint representation of G on its Lie algebra
$$\omega_{pg}(v_p \cdot g) = \mathrm{ad}_g \omega_p(v_p)$$
for any $p \ , \ v_p \ , \ g$ in $P \ , \ T_pP \ , \ G$ respectively.
- For any p in P the pullback $R_p^*\omega = \omega_{MC}$ along the right translation embedding
$$R_p : G \hookrightarrow P.$$

Here, the *adjoint representation* $\mathrm{ad} : G \to Aut(\mathfrak{g})$ is the differential of the inner automorphism

$$Ad_g : G \to G \quad , \quad x \mapsto gxg^{-1}$$

at the identity [KN]p38-40. It is an automorphism since every automorphism of a Lie group induces and automorphism at the tangent space at identity by its differential. See also page 35.

$\omega_{MC} \in A^1(G;\mathfrak{g})$ is called the *Maurer-Cartan form* defined to be the unique 1-form invariant under left multiplication and whose value at the identity of G is the identity linear map $T_eG \to \mathfrak{g}$. So that its value on a vector $v_g \in T_gG$ is equal to $L_{g^{-1}*}v_g$, hence it is often denoted $g^{-1}dg$.

According to the differential 1-form description of a connection, we can define its curvature to be the $\mathfrak{g}$-valued 2-form

$$\Omega = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$$

because of ad-invariance it descents to M as a section of $\mathrm{ad}P \otimes \Lambda^2$, where $\mathrm{ad}P$ is the adjoint bundle, vector bundle associated to P via the adjoint representation, sometimes also denoted by $\mathfrak{g}$.

On a vector bundle E over M, a *connection* is defined by its covariant derivative

$$\nabla : A^0(E) \to A^1(E)$$

a first order linear differential operator where $A^p(E) = \Gamma(\Lambda^p \otimes E)$ is the space of smooth sections of $\Lambda^p \otimes E$. The covariant derivative has a natural extension

$$\mathrm{D}_1 : A^1(E) \to A^2(E)$$

$$\mathrm{D}_1(e \otimes \alpha) = \nabla e \wedge \alpha + e \otimes \mathrm{d}\alpha$$

by forcing the Leibnitz' rule, where $e \in A^0(E)$, $\alpha \in A^1(M)$. The *curvature* is then defined to be the composition of these two operators $\Omega = \mathrm{D}_1\nabla \in A^2(\mathrm{End}E)$.

The relationship between connections on principal bundles and on vector bundles is as follows. By a representation $\rho : G \to \mathrm{Aut}E$, we define the associated vector bundle $P \times_G E$ for a vector space E. A local section of P gives a distinguished local basis $\{e_i\}$ of E. As an example, consider the principal $GL(n, \mathbb{R})$ bundle of n-frames on a manifold. A point p of this principal bundle is a basis for the vector space T_pM, and a local section gives a local basis. We pull back ω via the section and then apply the representation to get a matrix of 1-forms ω_{ij} since the image of the representation is a matrix group. Finally we define the covariant derivative by $\nabla e_i = \sum_j \omega_{ij} \otimes e_j$. Conversely, whenever we have a covariant derivative ∇ preserving a G-structure on a vector bundle E, we gather the G-frames to construct the principal bundle on which ω is naturally defined.

Incidentally, according to a theoretical physicist the curvature form $\Omega \in A^2(M; \mathrm{ad}P)$ is called a *gauge field*, and the connection form $\omega \in A^1(M; \mathrm{ad}P)$ is called a *gauge potential*. On a 4-manifold, a connection is said to be *self-dual* if its curvature $\Omega \in A^2_+(M; \mathrm{ad}P)$, i.e. $*\Omega = \Omega$, and *anti-self-dual* if $\Omega \in A^2_-(M; \mathrm{ad}P)$ i.e. $*\Omega = -\Omega$.

For a Riemannian 4-manifold (M, g), we induce the Riemannian connection on the $SO(3)$ bundle Λ^2_+ by the previous construction or by the covariant differentiation of self-dual 2-forms. The adjoint bundle $\mathrm{ad}P = P \times_{SO(3)} \mathfrak{so}(3)$ is the bundle Λ^2_+ itself, where

$$\mathfrak{so}(3) = \{M \in GL(3, \mathbb{R}) : tr(M) = 0, M^T = -M\} \approx \mathbb{R}^3$$

because of the relations

$$e^{trM} = det\ e^M = 1 \quad \text{and} \quad e^{M^T+M} = I.$$

The curvature of the induced connection is then the part of the Riemann curvature tensor which lies in

$$\mathrm{ad}P \otimes \Lambda^2 = \Lambda^2_+ \otimes \Lambda^2$$

so that

$$\Omega = A \oplus B^* \in A^2(\Lambda^2_+)$$

according to our refined decomposition in (1.2). Since $B^* \in A^2_-(\Lambda^2_+)$, the self-duality of our Riemannian connection amounts to $B \equiv 0$ i.e. iff the metric is Einstein. We thus established

Proposition 2.2.1 ([AHS])**.** *Let (M, g) be a Riemannian 4-manifold. Then the connection induced by the Riemannian connection on the bundle of self-dual 2-forms Λ^2_+ is self-dual if and only if the metric is an Einstein metric.*

Similarly for an Einstein manifold the Riemannian connection on Λ^2_- is anti-self-dual. This shows that the self-duality of a connection is not directly related to the self-duality of the base space, rather it is related to the property of being Einstein. If M is a spin manifold, the spinor bundle $\mathbb{V}_+$ is self-dual and $\mathbb{V}_-$ is anti-self-dual as bundles with $SU(2)$ connections. See [AHS].

Let E be a hermitian vector bundle. Then on the space of connections, we have the *Yang-Mills* functional defined by

$$YM(A) = \frac{1}{8\pi^2}\int_M |\Omega_A|^2 \, dV_g$$

i.e. square of the L^2 norm of the curvature of the connection up to a constant. Theoretical physicists are usually eager to find the global minimum of this functional, which is analogous to the optimal metric problem discussed in Sec (2.1).[1] This is a more general problem since we can have a more general vector bundle. To solve this problem, choose a connection. Then the first Pontrjagin class is represented by the following 4-form

$$p_1(E) = (c_1^2 - 2c_2)(E) = -\frac{1}{4\pi^2} tr(\Omega^2) \;\in\; H^4(M, \mathbb{Z}).$$

On the other hand, the 2-forms satisfy

$$\Phi \wedge \Phi = \Phi_+ \wedge *\Phi_+ + \Phi_- \wedge (- * \Phi_-) = (|\Phi_+|^2 - |\Phi_-|^2) dV_g$$

also we have the relation

$$tr(\xi^2) = -|\xi|^2$$

for the elements in the Lie algebra $\mathfrak{su}(n)$ of skew adjoint matrices, as can be seen easily for the case $n = 3$

$$\begin{bmatrix} 0 & a & b \\ -\bar{a} & 0 & c \\ -\bar{b} & -\bar{c} & 0 \end{bmatrix}^2 = \begin{bmatrix} -|a|^2 - |b|^2 & * & * \\ * & -|a|^2 - |c|^2 & * \\ * & * & -|b|^2 - |c|^2 \end{bmatrix}.$$

[1]Thanks to Levent Akant for this remark.

Since Ω is in $A^2(\mathrm{ad}E)$, combining these two relations and evaluating on the orientation class of the manifold, first Pontrjagin number of E becomes the nonnegative integer

$$p_1(E)[M] = \frac{1}{4\pi^2}\int_M (|\Omega_+|^2 - |\Omega_-|^2)\, dV_g \ \in \ \mathbb{Z}.$$

Then we have the following inequality

$$YM(A) = \frac{1}{8\pi^2}\int_M (|\Omega_+^A|^2 + |\Omega_-^A|^2)\, dV_g \ \geq \ \frac{1}{8\pi^2}\int_M (|\Omega_+^A|^2 - |\Omega_-^A|^2)\, dV_g = 2p_1(E)[M],$$

right hand side of which is a topological constant. So the Yang-Mills functional is bounded below by this topological invariant of the vector bundle E. This inequality is an equality if and only if $\Omega_-^A \equiv 0$, i.e. the connection is self-dual.

2.3 Self-Dual Manifolds and their Twistor Spaces

For any oriented self-dual Riemannian 4-manifold (M, g), Atiyah, Hitchin and Singer [AHS] constructed a complex 3-manifold Z called the *Twistor Space* of M^4, and we have a fibration by holomorphically embedded rational curves :

$$\begin{array}{ccccl} \mathbb{CP}_1 & \rightarrow & Z & & \text{complex 3-manifold} \\ & & \downarrow & & \\ & & M^4 & & \text{Riemannian 4-manifold} \end{array}$$

Example [[AHS]] $\begin{array}{c}\mathbb{CP}_3\\ \downarrow\\ S^4\end{array}$ is the fibration for the 4-sphere with its standard round metric inherited from the Euclidean space $\mathbb{R}^5$.

Example [[AHS]] $\begin{array}{c}F_{1,2}\mathbb{C}^3\\ \downarrow\\ \mathbb{CP}_2\end{array}$ is the fibration for the complex projective plane with the Fubini-Study metric. The twistor space is the flag variety

$$F_{1,2}\mathbb{C}^3 := \{(\mathbb{C}^1, \mathbb{C}^2) : \mathbb{C}^1 \subset \mathbb{C}^2 \subset \mathbb{C}^3\}.$$

Example [[Poon86]] $\begin{array}{c}\widetilde{Q}_{2,2}\\ \downarrow\\ \mathbb{CP}_2\#\mathbb{CP}_2\end{array}$ is the fibration for any self-dual metric on $2\mathbb{CP}_2$ with positive scalar curvature, e.g. the ones constructed by Poon and/or LeBrun. In this case, the twistor space $\widetilde{Q}_{2,2} = Res(Q_{2,2})$ is the small resolution of the intersection locus $Q_{2,2} = Q_2 \cap Q_2 \subset \mathbb{CP}_5$ of two quadrics in $\mathbb{CP}_5$.

Poon considers the natural square root of the anti-canonical line bundle on the twistor space as in Section (4.1), computes the number of holomorphic sections as $h^0(Z, K^{-1/2}) = 6$ using the Riemann-Roch Theorem. Then, using the map

$$Z \dashrightarrow \mathbb{CP}_5$$

$$p \mapsto [s_0(p) : \cdots : s_6(p)]$$

he tries to embed Z into the space of holomorphic sections of its line bundle, shows that this maps is generically one-to-one, the image lies in a 2-dimensional space of quadrics. See [Poon86] for details of this beautiful construction.

Example [[Poon92]] $\begin{array}{ccc} Z & \xrightarrow{2:1} & \mathbb{CP}_3 \\ \downarrow & & \\ 3\mathbb{CP}_2 & & \end{array}$ is the fibration for generic self-dual metric of positive scalar curvature on the connected sum $3\mathbb{CP}_2$. The twistor space is the small resolution of the twofold cover of $\mathbb{CP}_3$ branched along a quartic which is the zero locus of the polynomial

$$B(Z) = Z_0 Z_1 Z_2 Z_3 - Q^2(Z)$$

where Q is a real positive definite quadric. This quartic has 13 singular points which are ordinary double points or nodal points, 1 of which is real under an anti-holomorphic involution.

Poon again uses Riemann-Roch Theorem to compute $h^0(Z, K^{-1/2}) = 4$ this time. Shows that the map to the space of holomorphic sections $Z \dashrightarrow \mathbb{CP}_3$ is generically two to one, replaces the points where the map is undefined by rational curves etc.Consult [Poon92] for details.

2.4 The Donaldson-Friedman Theorem

One of the main improvements in the field of self-dual Riemannian 4-manifolds is the connected sum theorem of Donaldson and Friedman [DF] published in 1989. If M_1 and M_2 admit self-dual metrics, then under certain circumstances their connected sum admits, too . This helped us to create many examples of self-dual manifolds. If we state it more precisely :

Theorem 2.4.1 (Donaldson-Friedman[DF]). *Let (M_1, g_1) and (M_2, g_2)be compact self-dual Riemannian 4-manifolds and Z_i denote the corresponding twistor spaces. Suppose that $H^2(Z_i, \mathcal{O}(TZ_i)) = 0 \quad for \quad i = 1, 2.$*

Then, there are self-dual conformal classes on $M_1 \# M_2$ whose twistor spaces arise as fibers in a 1-parameter standard deformation of $Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2$.

We devote the rest of this section to understand the statement and the ideas in the proof of this theorem since our main result (5.4.1) is going to be a generalization of this celebrated theorem.

The idea is to work upstairs in the complex category rather than downstairs. So let $p_i \in M_i$ be arbitrary points in the manifolds. Consider their inverse images $C_i \approx \mathbb{CP}_1$ under the twistor fibration, which are twistor lines, i.e. rational curves invariant under the involution. Blow up the twistor spaces Z_i along these rational curves. Denote the exceptional divisors by $Q_i \approx \mathbb{CP}_1 \times \mathbb{CP}_1$ and the blown up twistor spaces by $\widetilde{Z}_i = Bl(Z_i, C_i)$. The normal bundles for the exceptional divisors is computed by :

Lemma 2.4.2 (Normal Bundle)**.** *The normal bundle of Q_2 in $\widetilde{Z}_2$ is computed to be*

$$NQ_2 = N_{Q_2/\widetilde{Z}_2} \approx \mathcal{O}(1,-1) := \pi_1^*\mathcal{O}_{\mathbb{P}_1}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}_1}(-1)$$

where the second component is the fiber direction in the blowing up process.

Proof. We split the computation into the following steps

1. We know that $N_{C_2/Z_2} \approx \mathcal{O}(1) \oplus \mathcal{O}(1)$ and we compute its second wedge power as

   $$c_1(\wedge^2\mathcal{O}(1)\oplus\mathcal{O}(1))[\mathbb{P}_1] = c_1(\mathcal{O}(1)\oplus\mathcal{O}(1))[\mathbb{P}_1] = (c_1\mathcal{O}(1)+c_1\mathcal{O}(1))[\mathbb{P}_1] = 2$$

   by the Whitney product identity of the characteristic classes. so we have

   $$\wedge^2 N_{C_2/Z_2} \approx \mathcal{O}_{\mathbb{P}_1}(2)$$

2. $K_Q = \pi_1^*K_{\mathbb{P}_1} \otimes \pi_2^*K_{\mathbb{P}_1} = \pi_1^*\mathcal{O}(-2) \otimes \pi_2^*\mathcal{O}(-2) = \mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_1}(-2,-2)$

3. $K_Q = K_{\widetilde{Z}_2} + Q|_Q = \pi^*K_{Z_2} + 2Q|_Q = \pi^*(K_{Z_2}|_{\mathbb{P}_1}) + 2Q|_Q = \pi^*(K_{\mathbb{P}_1} \otimes \wedge^2 N^*_{\mathbb{P}_1/Z_2}) + 2Q|_Q = \pi^*(\mathcal{O}(-2) \otimes \mathcal{O}(-2)) + 2Q|_Q = \pi^*\mathcal{O}(-4) + 2Q|_Q$
   since the second component is the fiber direction, the pullback bundle

will be trivial on that so $\pi^*\mathcal{O}(-4) = \mathcal{O}(-4,0)$ solving for $Q|_Q$ now gives us

$$N_{Q/\widetilde{Z}_2} = Q|_Q = (K_Q \otimes \pi^*\mathcal{O}(-4)^*)^{1/2} = (\mathcal{O}(-2,-2)\otimes\mathcal{O}(4,0))^{1/2} = \mathcal{O}(1,-1)$$

$\square$

We then construct the complex analytic space Z_0 by identifying Q_1 and Q_2 so that it has a normal crossing singularity

$$Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2.$$

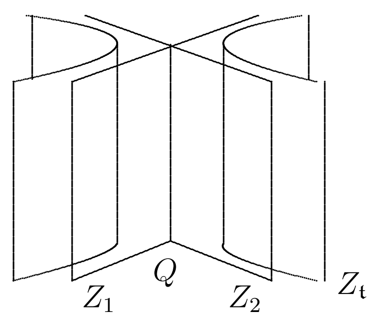
Carrying out this identification needs a little bit of care. We interchange the components of $\mathbb{CP}_1 \times \mathbb{CP}_1$ in the gluing process so that the normal bundles $N_{Q_1/\widetilde{Z}_1}$ and $N_{Q_2/\widetilde{Z}_2}$ are dual to each other. Moreover we should respect to the real structures. The real structures σ_1 and σ_2 must agree on Q obtained by identifying Q_1 with Q_2, so that the real structures extend over Z_0 and form the anti-holomorphic involution $\sigma_0 : Z_0 \to Z_0$.

Now we will be trying to deform the singular space Z_0, for which the Kodaira-Spencer's standard deformation theory does not work since it is only for manifolds it does not tell anything about the deformations of the singular spaces. We must use the theory of deformations of a compact reduced complex analytic spaces, which is provided by R.[Friedman]. This generalized theory is quite parallel to the theory of manifolds. The basic modification is that the roles of $H^i(\Theta)$ are now taken up by the groups $T^i = \mathrm{Ext}^i(\Omega^1, \mathcal{O})$.

We have assumed $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$ so that the deformations of Z_i

are unobstructed. Donaldson and Friedman are able to show that $T^2_{Z_0} = \mathrm{Ext}^2_{Z_0}(\Omega^1, \mathcal{O}) = 0$ so the deformations of the singular space is unobstructed. We have a versal family of deformations of Z_0. This family is parameterized by a neighborhood of the the origin in $\mathrm{Ext}^1_{Z_0}(\Omega^1, \mathcal{O})$. The generic fiber is non-singular and the real structure σ_0 extends to the total space of this family.[2]

$$\begin{aligned} \omega : \quad & \mathcal{Z} \to \mathcal{U} \qquad \text{for} \qquad Z_0 = \widetilde{Z}_1 \cup_Q \widetilde{Z}_2 \\ & Z_0 \longmapsto 0 \end{aligned}$$



Instead of working with the entire versal family, it is convenient to work with certain subfamilies, called *standard deformations*:

Definition 2.4.3 ([LeOM]). *A* 1-parameter standard deformation *of Z_0 is a flat proper holomorphic map $\omega : \mathcal{Z} \to \mathcal{U} \subset \mathbb{C}$ of a complex 4-manifold to an open neighborhood of 0, together with an anti-holomorphic involution $\sigma : \mathcal{Z} \to \mathcal{Z}$, such that*

- $\omega^{-1}(0) = Z_0$
- $\sigma|_{Z_0} = \sigma_0$
- *σ descents to the complex conjugation in $\mathcal{U}$*

[2] Thanks to C.LeBrun for the figure.

- *ω is a submersion away from $Q \subset Z_0$*
- *ω is modeled by $(x, y, z, w) \mapsto xy$ near any point of Q.*

We also define

Definition 2.4.4 (Flat Map[H]). *Let K be module over a ring A. We say that K is* flat *over A if the functor $L \mapsto K \otimes_A L$ is an exact functor for all modules L over A.*
Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say $\mathcal{F}$ is flat *over Y if the stalk $\mathcal{F}_x$ is a flat $\mathcal{O}_{y,Y}$-module for any x. Where $y = f(x)$, $\mathcal{F}_x$ is considered to be a $\mathcal{O}_{y,Y}$-module via the natural map $f^\# : \mathcal{O}_{y,Y} \to \mathcal{O}_{x,X}$. We say X is* flat *over Y if $\mathcal{O}_X$ is.*

Then for sufficiently small, nonzero, real $\mathfrak{t} \in \mathcal{U}$ the complex space $Z_\mathfrak{t} = \omega^{-1}(\mathfrak{t})$ is smooth and one can show that it is the twistor space of a self-dual metric on $M_1 \# M_2$.

Chapter 3

The Leray Spectral Sequence

Given a continuous map $f : X \to Y$ between topological spaces, and a sheaf $\mathcal{F}$ over X, the *q-th direct image sheaf* is the sheaf $R^q(f_*\mathcal{F})$ on Y associated to the presheaf $V \to H^q(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$. This is actually the right derived functor of the functor f_*. The Leray Spectral Sequence is a spectral sequence $\{E_r\}$ with

$$E_2^{p,q} = H^p(Y, R^q(f_*\mathcal{F}))$$

$$E_\infty^{p,q} = H^{p+q}(X, \mathcal{F})$$

The first page of this spectral sequence reads :

$$\begin{array}{c|cccc}
 & \vdots & \vdots & \vdots & \\
 & H^0(Y, R^2(f_*\mathcal{F})) & H^1(Y, R^2(f_*\mathcal{F})) & H^2(Y, R^2(f_*\mathcal{F})) & \cdots \\
 & H^0(Y, R^1(f_*\mathcal{F})) & H^1(Y, R^1(f_*\mathcal{F})) & H^2(Y, R^1(f_*\mathcal{F})) & \cdots \\
E_2 & H^0(Y, R^0(f_*\mathcal{F})) & H^1(Y, R^0(f_*\mathcal{F})) & H^2(Y, R^0(f_*\mathcal{F})) & \cdots \\
\hline
\end{array}$$

A degenerate case is when $R^i(f_*\mathcal{F}) = 0$ for all $i > 0$.

Remark 3.0.5. *This is the case if $\mathcal{F}$ is flabby for example. Remember that to* *be* flabby[1] *means that the restriction map* $r : \mathcal{F}(B) \to \mathcal{F}(A)$ *is onto for open sets* $B \subset A$*. In this case* $H^i(X, \mathcal{F}) = 0$ *for* $i > 0$ *as well as* $H^i(U, \mathcal{F}|_U) = 0$ *for* U *open, because the restriction of a flabby sheaf to any open subset is again flabby by definition. That means* $H^q(f^{-1}(.), \mathcal{F}|_.) = 0$ *for all* $q > 0$ *so* $R^i(f_*\mathcal{F}) = 0$*.*

When the spectral sequence degenerates this way, the second and succeeding rows of the first page vanish. And because $V \to H^0(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$ is the presheaf of the direct image sheaf, we have $R^0 f_* = f_*$. So the first row consist of $H^i(Y, f_*\mathcal{F})$'s. Vanishing of the differentials cause immediate convergence to $E_\infty^{i,0} = H^{i+0}(X, \mathcal{F})$. So we got:

Proposition 3.0.6. *If* $R^i(f_*\mathcal{F}) = 0$ *for all* $i > 0$*, then* $H^i(X, \mathcal{F}) = H^i(Y, f_*\mathcal{F})$ *naturally for all* $i \geq 0$*.*

As another example, the following proposition reveals a different sufficient condition for this degeneration. See [Voisin] v2 , p124 for a sketch of the proof:

Proposition 3.0.7 (Small Fiber Theorem)**.** *Let* $f : X \to Y$ *be a holomorphic, proper and submersive map between complex manifolds,* $\mathcal{F}$ *a coherent analytic sheaf or a holomorphic vector bundle on* X*. Then* $H^i(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)}) = 0$ *for all* $y \in Y$ *implies that* $R^i(f_*\mathcal{F}) = 0$*.*

As an application of these two propositions, we obtain the main result of this chapter :

Proposition 3.0.8. *Let* Z *be a complex* n*-manifold with a complex* k*-dimensional submanifold* V*. Let* $\widetilde{Z}$ *denote the blow up of* Z *along* V*, with blow up map*

[1] Flasque in French.

$\pi : \widetilde{Z} \to Z$. Let $\mathcal{G}$ denote a coherent analytic sheaf(or a vector bundle) over Z. Then we can compute the cohomology of $\mathcal{G}$ on either side i.e.

$$H^i(\widetilde{Z}, \pi^*\mathcal{G}) = H^i(Z, \mathcal{G}).$$

Proof. The inverse image of a generic point on Z is a point, else a $\mathbb{P}^{n-k-1}$. We have

$$H^i(f^{-1}(y), \pi^*\mathcal{G}|f^{-1}(y)) = H^i(\mathbb{P}^{n-k-1}, \mathcal{O}) = H^{0,i}_{\bar{\partial}}(\mathbb{P}^{n-k-1}) = 0$$

at most, since the cohomology of $\mathbb{P}^{n-k-1}$ is accumulated in the middle for $i > 0$. So that we can apply Proposition (3.0.7) to get $R^i(\pi_*\pi^*\mathcal{G}) = 0$ for all $i > 0$. Which is the hypothesis of Proposition (3.0.6), so we get $H^i(\widetilde{Z}, \pi^*\mathcal{G}) = H^i(Z, \pi_*\pi^*\mathcal{G})$ naturally for all $i \geq 0$, and the latter equals $H^i(Z, \mathcal{G})$ since $\pi_*\pi^*\mathcal{G} = \mathcal{G}$ by the combination of the following two lemmas. □

Lemma 3.0.9 (Projection Formula[H]p124)**.** *If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathcal{F}$ is an $\mathcal{O}$-module, and if $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank. Then there is a natural isomorphism*

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) = f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E},$$

in particular for $\mathcal{F} = \mathcal{O}_X$

$$f_*f^*\mathcal{E} = f_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{E}.$$

Lemma 3.0.10 (Zariski's Main Theorem,Weak Version[H]p280)**.** *Let $f : X \to$*

Y be a birational projective morphism of noetherian integral schemes, and assume that Y is normal. Then $f_*\mathcal{O}_X = \mathcal{O}_Y$.

Proof. The question is local on Y. So we may assume that Y is affine and equal to $SpecA$. Then $f_*\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$-algebras, so $B = \Gamma(Y, f_*\mathcal{O}_X)$ is a finitely generated A-module. But A and B are integral domains with the same quotient field, and A is integrally closed, we must have $A = B$. Thus $f_*\mathcal{O}_X = \mathcal{O}_Y$. $\square$

Chapter 4

Vanishing Theorem

In this chapter, we are going to prove that a certain cohomology group of a line bundle vanishes. For that we need some definitions and lemmas.

First of all, the canonical bundle of a twistor space Z has a natural square root, equivalently Z is a spin manifold as follows:

4.1 Natural square root of the canonical bundle of a twistor space

Remember that, using the Riemannian connection of M, we can split the tangent bundle $T_xZ = T_xF \oplus (p^*TM)_x$. The complex structure on $(p^*TM)_x$ is obtained from the identification $\cdot\varphi : T_xM \longleftrightarrow (\mathbb{V}_+)_x$ provided by the Clifford multiplication of a non-zero spinor $\varphi \in (\mathbb{V}_+)_x$. This identification is linear in φ as φ varies over $(\mathbb{V}_+)_x$. So we have a nonvanishing section of $\mathcal{O}_Z(1) = \mathcal{O}_{\mathbb{P}\mathbb{V}_-}(1)$ with values in $Hom(TM, \mathbb{V}_+)$ or $Hom(p^*TM, p^*\mathbb{V}_+)$ trivializing the bundle

$$\mathcal{O}_Z(1) \otimes Hom(p^*TM, p^*\mathbb{V}_+) = \mathcal{O}_Z(1) \otimes p^*TM^* \otimes p^*\mathbb{V}_+ \approx p^*TM^* \otimes \mathcal{O}_Z(1) \otimes p^*\mathbb{V}_+$$

hence yielding a natural isomorphism

$$p^*TM \approx \mathcal{O}_Z(1) \otimes p^*\mathbb{V}_+, \tag{4.1}$$

where $\mathcal{O}_Z(1) = \mathcal{O}_{\mathbb{P}\mathbb{V}_-}(1)$ is the positive Hopf bundle on the fiber.

The Hopf bundle exist locally in general, so as the isomorphism. If M is a spin manifold, $\mathbb{V}_\pm$ exist globally on M and $\mathcal{O}_Z(1)$ exist on Z, so our isomorphism holds globally. Furthermore, we have a second isomorphism holding for any projective bundle, obtained as follows (see [Fulton] p434-5 , [Zheng] p108) :

Let E be a complex vector bundle of rank-$(n+1)$ over M, $p : \mathbb{P}E \to M$ its projectivization. We have the imbedding of the tautological line bundle $\mathcal{O}_{\mathbb{P}E}(-1) \hookrightarrow p^*E$. Giving the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}E}(-1) \to p^*E \to p^*E/\mathcal{O}_{\mathbb{P}E}(-1) \to 0,$$

tensoring by $\mathcal{O}_{\mathbb{P}E}(1)$ gives

$$0 \to \mathcal{O}_{\mathbb{P}E} \to \mathcal{O}_{\mathbb{P}E}(1) \otimes p^*E \to T_{\mathbb{P}E/M} \to 0$$

where $T_{\mathbb{P}E/M} \approx Hom(\mathcal{O}(-1), \mathcal{O}(-1)^\perp) = Hom(\mathcal{O}(-1), p^*E/\mathcal{O}(-1)) = \mathcal{O}(1) \otimes p^*E/\mathcal{O}(-1)$ is the relative tangent bundle of $\mathbb{P}E$ over M, originally defined to be ${\Omega^1_{\mathbb{P}E/M}}^*$. Taking $E = \mathbb{V}_-$, TF denoting the tangent bundle over the fibers:

$$0 \to \mathcal{O}_Z \to \mathcal{O}_Z(1) \otimes p^*\mathbb{V}_- \to TF \to 0$$

so, we got our second isomorphism :

$$TF \oplus \mathcal{O}_Z \approx \mathcal{O}_Z(1) \otimes p^*\mathbb{V}_- \tag{4.2}$$

Now we are going to compute the first chern class of the spin bundles $\mathbb{V}_\pm$, and see that $c_1(\mathbb{V}_\pm) = 0$. Choose a connection ∇ on $\mathbb{V}_\pm$. Following [KN]v2,p307 it defines a connection on the associated principal $\mathfrak{su}(2)$ bundle P, with connection one form $\omega \in A^1(P, \mathfrak{su}(2))$ defined by the projection [Morita]p50 $T_uP \to V_u \approx \mathfrak{su}(2)$ having curvature two form $\Omega \in A^2(P, \mathfrak{su}(2))$ defined by [KN]v1,p77 :

$$\Omega(X, Y) = d\omega(X, Y) + \frac{1}{2}[\omega(X), \omega(Y)] \text{ for } X, Y \in T_uP.$$

We define the first polynomial functions f_0, f_1, f_2 on the lie algebra $\mathfrak{su}(2)$ by

$$det(\lambda I_2 + \frac{i}{2\pi}M) = \sum_{k=0}^{2} f_{2-k}(M)\lambda^k = f_0(M)\lambda^2 + f_1(M)\lambda + f_2(M) \text{ for } M \in \mathfrak{su}(2).$$

Then these polynomials $f_i : \mathfrak{su}(2) \to \mathbb{R}$ are invariant under the adjoint action of $SU(2)$, denoted $f_i \in I^1(SU(2))$, namely

$$f_i(\mathrm{ad}_g(M)) = f_i(M) \text{ for } g \in SU(2)\ , M \in \mathfrak{su}(2)$$

where $\mathrm{ad}_g : \mathfrak{su}(2) \to \mathfrak{su}(2)$ is defined by $\mathrm{ad}_g(M) = R_{g^{-1}*}L_{g*}(M)$.
If we apply any $f \in I^1(SU(2))$ after Ω we obtain:

$$f \circ \Omega\ :\ T_uP \times T_uP \longrightarrow \mathfrak{su}(2) \longrightarrow \mathbb{R}.$$

It turns out that $f \circ \Omega$ is a closed form and projects to a unique 2-form say $\overline{f \circ \Omega}$ on M i.e. $f \circ \Omega = \pi^*(\overline{f \circ \Omega})$ where $\pi : P \to M$. By the way, a q-form φ on P projects to a unique q-form, say $\overline{\varphi}$ on M if $\varphi(X_1 \cdot\cdot X_q) = 0$ whenever at least one of the X_i's is vertical and $\varphi(R_{g*}X_1 \cdot\cdot R_{g*}X_q) = \varphi(X_1 \cdot\cdot X_q)$. $\overline{\varphi}$ on M defined by $\overline{\varphi}(V_1 \cdot\cdot Vq) = \varphi(X_1 \cdot\cdot X_q), \pi(X_i) = V_i$ is independent of the choice of X_i's. See [KN]v2p294 for details.

So, composing with Ω and projecting defines a map w : $I^1(SU(2)) \to H^2(M, \mathbb{R})$ called the Weil homomorphism, it is actually an algebra homomorphism when extended to the other gradings.

Finally, the chern classes are defined by $c_k(\mathbb{V}_\pm) := \left[\overline{f_k \circ \Omega}\right]$ independent of the connection chosen. Notice that $f_2(M) = det(\frac{i}{2\pi}M), f_1(M) = tr(\frac{i}{2\pi}M)$ in our case. And if $M \in \mathfrak{su}(2)$ then $e^M \in SU(2)$ implying $1 = det(e^M) = e^{trM}$ and $trM = 0$. But Ω is of valued $\mathfrak{su}(2)$, so if you apply the $f_1 = tr$ after Ω you get 0. Causing $c_1(\mathbb{V}_\pm) = \left[\overline{tr(\frac{i}{2\pi}\Omega)}\right] = 0$.

One last remark is that $\overline{f_k \circ \Omega} = \gamma_k$ in the notation of [KN], $\gamma_1 = P^1(\frac{i}{2\pi}\Theta) = tr(\frac{i}{2\pi}\Theta)$ in the notation of [GH]p141,p407. And $\Omega = \pi^*\Theta$ in the line bundle case.

Vanishing of the first chern classes mean that the determinant line bundles of $\mathbb{V}_\pm$ are diffeomorphically trivial since $c_1(\wedge^2\mathbb{V}_\pm) = c_1\mathbb{V}_\pm = 0$. Combining this with the isomorphisms (4.1) and (4.2) yields:

$$\wedge^2 p^*TM = \wedge^2(\mathcal{O}_Z(1) \otimes p^*\mathbb{V}_+) = \mathcal{O}_Z(2) \otimes \wedge^2 p^*\mathbb{V}_+ = \mathcal{O}_Z(2) = \mathcal{O}_Z(2) \otimes \wedge^2 p^*\mathbb{V}_-$$

$$= \wedge^2(\mathcal{O}_Z(1) \otimes p^*\mathbb{V}_-) = \wedge^2(TF \oplus \mathcal{O}_Z) = \oplus_{2=p+q}(\wedge^p TF \otimes \wedge^q \mathcal{O}_Z) =$$

$$TF \otimes \mathcal{O}_Z = TF$$

since TF is a line bundle. Taking the first chern class of both sides:

$$c_1(p^*TM) = c_1(\wedge^2 p^*TM) = c_1TF.$$

Alternatively, this chern class argument could be replaced with the previous taking wedge powers steps if the reader feels more comfortable with it.
Last equality implies the decomposition:

$$c_1Z = c_1(p^*TM \oplus TF) = c_1(p^*TM) + c_1TF = 2c_1TF.$$

So, TF^* is a differentiable square root for the canonical bundle of Z. If M is not spin $\mathbb{V}_\pm, \mathcal{O}_Z(1)$ are not globally defined, but the complex structure on their tensor product is still defined, and we can still use the isomorphisms (4.1),(4.2) for computing chern classes of the almost complex structure on Z using differential forms defined locally by the metric. Consequently our decomposition is valid whether M is spin or not.

One more word about the differentiable square roots is in order here. A differentiable square root implies a holomorphic one on complex manifolds since in the sheaf sequence:

$$\begin{array}{ccccccc} .. & \to & H^1(M,\mathcal{O}^*) & \to & H^2(M,\mathbb{Z}) & \to & .. \\ & & L & \mapsto & c_1(L) & \mapsto & 0 \\ & & & & \frac{1}{2}c_1(L) & \mapsto & 0 \end{array}$$

$c_1(L)$ maps to 0 since it is coming from a line bundle, and if it decomposes, $\frac{1}{2}c_1(L)$ maps onto 0 too, that means it is the first chern class of a line bundle.

4.2 Vanishing Theorem

Let $\omega : \mathcal{Z} \to \mathcal{U}$ be a 1-paramater standard deformation of Z_0, where $\mathcal{U} \subset \mathbb{C}$ is an open disk about the origin. Then the invertible sheaf $K_{\mathcal{Z}}$ has a square root as a holomorphic line bundle as follows:

We are going to show that the Steifel-Whitney class $w_2(K_{\mathcal{Z}})$ is going to vanish. We write $\mathcal{Z} = \mathcal{U}_1 \cup \mathcal{U}_2$ where $\mathcal{U}_i$ is a tubular neighborhood of $\widetilde{Z}_i$, $\mathcal{U}_1 \cap \mathcal{U}_2$ is a tubular neighborhood of $Q = \widetilde{Z}_1 \cap \widetilde{Z}_2$. So that $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{U}_1 \cap \mathcal{U}_2$ deformation retracts on $\widetilde{Z}_1, \widetilde{Z}_2$ and Q. Since $Q \approx \mathbb{P}_1 \times \mathbb{P}_1$ is simply connected, $H^1(\mathcal{U}_1 \cap \mathcal{U}_2, \mathbb{Z}_2) = 0$ and the map r_{12} in the Mayer-Vietoris exact sequence :

$$\begin{array}{ccccccccc} .. & \to & H^1(\mathcal{U}_1 \cap \mathcal{U}_2, \mathbb{Z}_2) & \to & H^2(\mathcal{U}_1 \cup \mathcal{U}_2, \mathbb{Z}_2) & \stackrel{r_{12}}{\to} & H^2(\mathcal{U}_1, \mathbb{Z}_2) \oplus H^2(\mathcal{U}_2, \mathbb{Z}_2) & \to & .. \\ & & \| & & \in & & & & \\ & & 0 & & w_2(K_{\mathcal{Z}}) & & & & \end{array}$$

is injective. Therefore it is enough to see that the restrictions $r_i(w_2(K_{\mathcal{Z}})) \in H^2(\mathcal{U}_i, \mathbb{Z}_2)$ are zero. For that, we need to see that $K_{\mathcal{Z}}|_{\widetilde{Z}_i}$ has a radical :

$$K_{\mathcal{Z}}|_{\widetilde{Z}_1} \stackrel{(1)}{=} (K_{\widetilde{Z}_1} - \widetilde{Z}_1)|_{\widetilde{Z}_1} \stackrel{(2)}{=} (K_{\widetilde{Z}_1} + Q)|_{\widetilde{Z}_1} \stackrel{(3)}{=} ((\pi^* K_{Z_1} + Q) + Q)|_{\widetilde{Z}_1} = \\ 2(\pi^* K_{Z_1}^{1/2} + Q)|_{\widetilde{Z}_1}$$

where (1) is the application of the adjuction formula on $\widetilde{Z}_1$, $K_{\widetilde{Z}_1} = K_{\mathcal{Z}}|_{\widetilde{Z}_1} \otimes [\widetilde{Z}_1]$. (2) comes from the linear equivalence of 0 with Z_t on $\widetilde{Z}_1$, and Z_t with Z_0:

$$0 = \mathcal{O}(Z_t)|_{\widetilde{Z}_1} = \mathcal{O}(Z_0)|_{\widetilde{Z}_1} = \mathcal{O}(\widetilde{Z}_1 + \widetilde{Z}_2)|_{\widetilde{Z}_1} = \mathcal{O}(\widetilde{Z}_1 + Q)|_{\widetilde{Z}_1}$$

(3) is the change of the canonical bundle under the blow up along a submanifold, see [GH]p608. K_{Z_1} has a natural square root as we computed in the

previous section, so $\pi^* K_{Z_1}^{1/2} \otimes [Q]$ is a square root of $K_{\mathcal{Z}}$ on $\widetilde{Z}_1$. Similarly on $\widetilde{Z}_2$, so $K_{\mathcal{Z}}$ has a square root $K_{\mathcal{Z}}^{1/2}$.

Before our vanishing theorem, we are going to mention the Semicontinuity Principle and the Hitchin's Vanishing theorem, which are involved in the proof:

Lemma 4.2.1 (Semicontinuity Principle[Voisin]v1p232). *Let $\phi : \mathcal{X} \to \mathcal{B}$ be a family of complex compact manifolds With fiber $X_b, b \in \mathcal{B}$. Let $\mathcal{F}$ be a holomorphic vector bundle over $\mathcal{X}$, then*

The function $b \mapsto h^q(X_b, \mathcal{F}|_{X_b})$ is upper semicontinuous. In other words, we have $h^q(X_b, \mathcal{F}|_{X_b}) \leq h^q(X_0, \mathcal{F}|_{X_0})$ for b in a neighborhood of $0 \in \mathcal{B}$.

Lemma 4.2.2 (Hitchin Vanishing[HitLin][Poon86]). *Let Z be the twistor space of an oriented self-dual riemannian manifold of positive scalar curvature with canonical bundle K, then*

$$h^0(Z, \mathcal{O}(K^{n/2})) = h^1(Z, \mathcal{O}(K^{n/2})) = 0 \ for \ all \ n \geq 1.$$

Theorem 4.2.3 (Vanishing Theorem). *Let $\omega : \mathcal{Z} \to \mathcal{U}$ be a 1-parameter standard deformation of Z_0, where Z_0 is as in Theorem (2.4.1), and $\mathcal{U} \subset \mathbb{C}$ is a neighborhood of the origin. Let $L \to \mathcal{Z}$ be the holomorphic line bundle defined by*

$$\mathcal{O}(L^*) = \mathscr{I}_{\widetilde{Z}_1}(K_{\mathcal{Z}}^{1/2})$$

If $(M_i, [g_i])$ has positive scalar curvature, then by possibly replacing $\mathcal{U}$ with a smaller neighborhood of $0 \in \mathbb{C}$ and simultaneously replacing $\mathcal{Z}$ with its inverse *image, we can arrange for our complex 4-fold $\mathcal{Z}$ to satisfy*

$$H^1(\mathcal{Z}, \mathcal{O}(L^*)) = H^2(\mathcal{Z}, \mathcal{O}(L^*)) = 0.$$

Proof. The proof proceeds by analogy to the techniques in [LeOM], and consists of several steps :

1. It is enough to **show that $\mathbf{H^j(Z_0, \mathcal{O}(L^*)) = 0}$ for $\mathbf{j = 1, 2}$:**

   Since that would imply $h^j(Z_t, \mathcal{O}(L^*)) \leq 0$ for $j = 1, 2$ in a neighborhood by the semicontinuity principle. Intuitively, this means that the fibers are too small, so we can apply Proposition (3.0.7) to see $R^j\omega_*\mathcal{O}(L^*) = 0$ for $j = 1, 2$. The first page of the Leray Spectral Sequence reads :

$$\begin{array}{c|cccc}
 & \vdots & \vdots & \vdots & \\
 & H^0(\mathcal{U}, R^3\omega_*\mathcal{O}(L^*)) & H^1(\mathcal{U}, R^3\omega_*\mathcal{O}(L^*)) & H^2(\mathcal{U}, R^3\omega_*\mathcal{O}(L^*)) & \cdots \\
 & 0 & 0 & 0 & \cdots \\
 & 0 & 0 & 0 & \cdots \\
E_2 & H^0(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*)) & H^1(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*)) & H^2(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*)) & \cdots \\
\hline
\end{array}$$

   Remember that

$$E_2^{p,q} = H^p(\mathcal{U}, R^q\omega_*\mathcal{O}(L^*))$$

$$E_\infty^{p,q} = H^{p+q}(\mathcal{Z}, \mathcal{O}(L^*))$$

and that the differential

$$d_2(E_2^{p,q}) \subset E_2^{p+2,q-1}.$$

Vanishing of the second row implies the immediate convergence of the first row till the third column because of the differentials, so

$$E_\infty^{p,0} = E_2^{p,0} \text{ i.e. } H^{p+0}(\mathcal{Z}, \mathcal{O}(L^*)) = H^p(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*)) \text{ for } p \leq 3$$

hence $H^p(\mathcal{Z}, \mathcal{O}(L^*)) = H^p(\mathcal{U}, R^0\omega_*\mathcal{O}(L^*))$, for $p \leq 3$.

Since $\mathcal{U}$ is one dimensional, $\omega : \mathcal{Z} \to \mathcal{U}$ has to be a flat morphism, so the sheaf $\omega_*\mathcal{O}(L^*)$ is coherent[Gun, Bon]. $\mathcal{U}$ is an open subset of $\mathbb{C}$ implying that it is Stein. And the so called Theorem B of Stein Manifold theory characterizes them as possesing a vanishing higher dimensional($p > 0$) coherent sheaf cohomology[Lew]p67,[H]p252,[Gun, Bon]. So $H^p(\mathcal{U}, \omega_*\mathcal{O}(L^*)) = 0$ for $p > 0$. Tells us that $H^p(\mathcal{Z}, \mathcal{O}(L^*)) = 0$ for $0 < p \leq 3$.

2. Related to Z_0, we have the **Mayer-Vietoris like** sheaf exact sequence

$$0 \to \mathcal{O}_{Z_0}(L^*) \to \nu_*\mathcal{O}_{\widetilde{Z}_1}(L^*) \oplus \nu_*\mathcal{O}_{\widetilde{Z}_2}(L^*) \to \mathcal{O}_Q(L^*) \to 0$$

where $\nu : \widetilde{Z}_1 \sqcup \widetilde{Z}_2 \to Z_0$ is the inclusion map on each of the two components of the disjoint union $\widetilde{Z}_1 \sqcup \widetilde{Z}_2$. This gives the long exact cohomology sequence piece :

$$0 \to H^1(\mathcal{O}_{Z_0}(L^*)) \to H^1(Z_0, \nu_*\mathcal{O}_{\widetilde{Z}_1}(L^*) \oplus \nu_*\mathcal{O}_{\widetilde{Z}_2}(L^*)) \to H^1(\mathcal{O}_Q(L^*)) \to$$
$$H^2(\mathcal{O}_{Z_0}(L^*)) \to H^2(Z_0, \nu_*\mathcal{O}_{\widetilde{Z}_1}(L^*) \oplus \nu_*\mathcal{O}_{\widetilde{Z}_2}(L^*)) \to 0$$

due to the fact that :

3. $\mathbf{H^0}(\mathcal{O}_\mathbf{Q}(\mathbf{L}^*)) = \mathbf{H^2}(\mathcal{O}_\mathbf{Q}(\mathbf{L}^*)) = \mathbf{0}$ **:** To see this, we have to understand the restriction of $\mathcal{O}(L^*)$ to Q :

$$L^*|_Q = (\tfrac{1}{2}K_{\mathcal{Z}} - \widetilde{Z}_1)|_{\widetilde{Z}_2}|_Q = (\tfrac{1}{2}(K_{\widetilde{Z}_2} - \widetilde{Z}_2) - \widetilde{Z}_1)|_{\widetilde{Z}_2}|_Q = (\tfrac{1}{2}(K_{\widetilde{Z}_2} + Q) - Q)|_{\widetilde{Z}_2}|_Q = \tfrac{1}{2}(K_{\widetilde{Z}_2} - Q)|_{\widetilde{Z}_2}|_Q = \tfrac{1}{2}(K_Q - Q - Q)|_{\widetilde{Z}_2}|_Q = (\tfrac{1}{2}K_Q - Q)|_{\widetilde{Z}_2}|_Q = \tfrac{1}{2}K_Q|_Q \otimes NQ^{-1}_{\widetilde{Z}_2} = \mathcal{O}(-2,-2)^{1/2} \otimes \mathcal{O}(1,-1)^{-1} = \mathcal{O}(-2,0)$$

here, we have computed the normal bundle of Q in $\widetilde{Z}_2$ in Lemma (2.4.2) as $\mathcal{O}(1,-1)$, where the second component is the fiber direction in the blowing up process. So the line bundle L^* is trivial on the fibers. Since $Q = \mathbb{P}_1 \times \mathbb{P}_1$, we have

$$H^0(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}(-2,0)) = H^0(\mathbb{P}_1 \times \mathbb{P}_1, \pi_1^*\mathcal{O}(-2)) = H^0(\mathbb{P}_1, \pi_{1*}\pi_1^*\mathcal{O}(-2)) = H^0(\mathbb{P}_1, \mathcal{O}(-2)) = 0$$

by the Leray spectral sequence and the projection formula since $H^k(\mathbb{P}_1, \mathcal{O}) = 0$ for $k > 0$. Similarly

$$H^2(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}(-2,0)) = H^2(\mathbb{P}_1, \mathcal{O}(-2)) = 0$$

by dimensional reasons. Moreover, for the sake of curiosity

$$H^1(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}(-2,0)) = H^1(\mathbb{P}_1, \mathcal{O}(-2)) \approx H^0(\mathbb{P}_1, \mathcal{O}(-2) \otimes \mathcal{O}(-2)^*)^* = H^0(\mathbb{P}_1, \mathcal{O})^* = \mathbb{C}.$$

4. $\mathbf{H^1}(\widetilde{\mathbf{Z}}_\mathbf{2}, \mathcal{O}_{\widetilde{\mathbf{Z}}_\mathbf{2}}(\mathbf{L}^*)) = \mathbf{H^2}(\widetilde{\mathbf{Z}}_\mathbf{2}, \mathcal{O}_{\widetilde{\mathbf{Z}}_\mathbf{2}}(\mathbf{L}^*)) = \mathbf{0}$ **:** These are applications of Hitchin's second Vanishing Theorem and are going to help us to simplify our exact sequence piece.

$$H^1(\widetilde{Z}_2, \mathcal{O}_{\widetilde{Z}_2}(L^*)) = H^1(\widetilde{Z}_2, \mathcal{O}(K_{\mathcal{Z}}^{1/2} - \widetilde{Z}_1)|_{\widetilde{Z}_2}) = H^1(\widetilde{Z}_2, \mathcal{O}(K_{\mathcal{Z}}^{1/2} - Q)|_{\widetilde{Z}_2}) = H^1(\widetilde{Z}_2, \pi^* K_{Z_2}^{1/2}) = H^1(Z_2, \pi_* \pi^* K_{Z_2}^{1/2}) = H^1(Z_2, K_{Z_2}^{1/2}) = 0$$

by the Leray spectral sequence, projection formula and Hitchin's Vanishing theorem for Z_2, since it is the twistor space of a positive scalar curvature space. This implies $H^2(Z_2, K_{Z_2}^{1/2}) \approx H^1(Z_2, K_{Z_2}^{1/2})^* = 0$ because of the Kodaira-Serre Duality. Hence our cohomological exact sequence piece simplifies to

$$0 \to H^1(\mathcal{O}_{Z_0}(L^*)) \to H^1(\widetilde{Z}_1, \mathcal{O}_{\widetilde{Z}_1}(L^*)) \to H^1(\mathcal{O}_Q(L^*)) \to H^2(\mathcal{O}_{Z_0}(L^*)) \to H^2(\widetilde{Z}_1, \mathcal{O}_{\widetilde{Z}_1}(L^*)) \to 0$$

5. $\mathbf{H^k(\mathcal{O}_{\tilde{Z}_1}(L^* \otimes [Q]_{\tilde{Z}_1}^{-1})) = 0}$ **for** $\mathbf{k = 1, 2, 3}$ **:** This technical result is going to be needed to understand the exact sequence in the next step. First we simplify the sheaf as

$$(L^* - Q)|_{\widetilde{Z}_1} \overset{def}{=} (\tfrac{1}{2}K_{\mathcal{Z}} - \widetilde{Z}_1 - Q)|_{\widetilde{Z}_1} = \tfrac{1}{2}K_{\mathcal{Z}}|_{\widetilde{Z}_1} \overset{adj}{=} \tfrac{1}{2}(K_{\widetilde{Z}_1} - \widetilde{Z}_1)|_{\widetilde{Z}_1} = \tfrac{1}{2}(K_{\widetilde{Z}_1} + Q)|_{\widetilde{Z}_1}.$$

So

$$H^k(\widetilde{Z}_1, L^* - Q) = H^k(\widetilde{Z}_1, (K_{\widetilde{Z}_1} + Q)/2) \overset{sd}{\approx} H^{3-k}(\widetilde{Z}_1, (K_{\widetilde{Z}_1} - Q)/2)^* = H^{3-k}(\widetilde{Z}_1, \tfrac{1}{2}\pi^* K_{Z_1})^* \overset{lss}{=} H^{3-k}(Z_1, \tfrac{1}{2}\pi_*\pi^* K_{Z_1})^* \overset{pf}{=} H^{3-k}(Z_1, K_{Z_1}^{1/2})^* \overset{sd}{\approx} H^k(Z_1, K_{Z_1}^{1/2})$$

and one of the last two terms vanish in any case for $k = 1, 2, 3$. So we apply the Hitchin Vanishing theorem for dimensions 0 and 1.

6. **Restriction maps to Q :** Consider the exact sequence of sheaves on $\widetilde{Z}_1$:

$$0 \to \mathcal{O}_{\widetilde{Z}_1}(L^* \otimes [Q]^{-1}_{\widetilde{Z}_1}) \to \mathcal{O}_{\widetilde{Z}_1}(L^*) \to \mathcal{O}_Q(L^*) \to 0.$$

   The previous step implies that the restriction maps :

$$H^1(\mathcal{O}_{\widetilde{Z}_1}(L^*)) \xrightarrow{restr_1} H^1(\mathcal{O}_Q(L^*)) \quad \text{and} \quad H^2(\mathcal{O}_{\widetilde{Z}_1}(L^*)) \xrightarrow{restr_2} H^2(\mathcal{O}_Q(L^*))$$

   are isomorphism. In particular $H^2(\mathcal{O}_{\widetilde{Z}_1}(L^*)) = 0$ due to (3). Incidentally, this exact sheaf sequence is a substitute for the role played by the Hitchin Vanishing Theorem, for the $\widetilde{Z}_2$ components in the cohomology sequence. It also assumes Hitchin's theorems for the $\widetilde{Z}_1$ component.

7. **Conclusion :** Our cohomology exact sequence piece reduces to

$$0 \to H^1(\mathcal{O}_{Z_0}(L^*)) \to H^1(\widetilde{Z}_1, \mathcal{O}_{\widetilde{Z}_1}(L^*)) \xrightarrow{restr_1} H^1(\mathcal{O}_Q(L^*)) \to$$
$$H^2(\mathcal{O}_{Z_0}(L^*)) \to 0$$

   the isomorphism in the middle forces the rest of the maps to be 0 and hence we get $H^1(\mathcal{O}_{Z_0}(L^*)) = H^2(\mathcal{O}_{Z_0}(L^*)) = 0$.

$\square$

Chapter 5

The Sign of the Scalar Curvature

In this chapter, we are going to detect the sign of the scalar curvature of the metric we consider on the connected sum. We use Green's Functions for that purpose. Positivity for the scalar curvature is going to be characterized by nontriviality of the Green's Functions. Then our Vanishing Theorem will provide the Serre-Horrocks vector bundle construction, which gives the Serre Class, a substitute for the Green's Function by Atiyah[AtGr]. And nonzeroness of the Serre Class will provide the non-triviality of the extension described by it.

5.1 Distributions

In this section, we will give a digression on distributions on $\mathbb{R}^n$ following [GH]p366.

Definition 5.1.1. *A* distribution *on $\mathbb{R}^n$ is a linear map $T : C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ that is continuous in the C^∞ topology.*

We say that a distribution is of *order p* if it is continuous in the C^p-

topology. The C^p-topology is defined on $C_c^\infty(\mathbb{R}^n)$ by saying that a sequence $\varphi_n \to 0$ in case there is a compact set K with all $supp\varphi \subset K$ and with

$$\frac{\partial^{\alpha_1+..+\alpha_n}\varphi_n}{\partial x_1^{\alpha_1}..\partial x_n^{\alpha_n}}(x) \to 0$$

uniformly for $x \in K$ and all $(\alpha_1, .., \alpha_n)$ satisfying $\alpha_1 + .. + \alpha_n \leq p$. The C^∞-topology is defined by saying that $\varphi_n \to 0$ in case all $supp\varphi \subset K$ and $\varphi_n \to 0$ in the C^p-topology for each p.

Example The Dirac delta function or distribution $\delta_0 : C_c^\infty(\mathbb{R}^n) \to \mathbb{C}$ is the distribution defined by

$$\delta_0(\varphi) = \varphi(0)$$

Example Let f be a locally L^1 function on $\mathbb{R}^n$, then we define the distribution T_f of order zero by

$$T_f(\varphi) = \int_{\mathbb{R}^n} \varphi f dx$$

for $dx = dx_1 \wedge \ldots \wedge dx_n$ orienting $\mathbb{R}^n$. This is the standard way to produce the distribution corresponding to a function.

Now we are going to define the derivatives of distributions. We are going to make the definition so that it would agree with the usual differentiation for the distributions coming from the T_* construction, i.e. rather than differentiating the distribution, one passes to the function it corresponds, differentiate there, and return to the space of distributions by the T_* construction.

So in the space of distributions, we define the differentiation by $D_i = \partial/\partial x^i$ to be

$$(D_i L)(\varphi) = -L(D_i \varphi)$$

for a distribution L. Notice that if L has a corresponding function, i.e. $L = T_f$ for some f of class C^1, then one easily shows that[GH]

$$D_i T_f = T_{D_i f}$$

via the integration by parts and the Stokes theorem.

Example Here we are going to describe an antiderivative for the one variable δ-distribution. Consider the locally L^1 function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \begin{cases} 0, & for\ x < 0 \\ 1, & for\ 0 \leq x \end{cases}$$

F is not differentiable as you notice.So consider the distribution T_F, the distributional derivative of T_F is computed to be

$$DT_F(\varphi) = T_{F'}(\varphi) = -T_F(\varphi') = -\int_{-\infty}^{\infty} F\varphi' dx = -\int_0^{\infty} \varphi' dx = \varphi(0)$$

as φ is compactly supported, $\lim_{t\to\infty}\varphi(t) = 0$. Hence $T_F' = \delta_0$.

Definition 5.1.2. *A distribution T is said to be* smooth *if $T = T_\psi$ for some $\psi \in C^\infty(\mathbb{R}^n)$.*

Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be a nonnegative function supported in a neighborhood of the origin with total integral

$$\int_{\mathbb{R}^n} \chi dx = 1$$

Supposed moreover that χ is radially symmetric, i.e. in polar coordinates $x = r\omega$, $\chi(x) = \chi(r)$. We distort χ without changing its integral via

$$\chi_\epsilon(x) = \frac{1}{\epsilon^n}\chi(\frac{x}{\epsilon})$$

If $supp\chi = K$ then $supp\chi_\epsilon = \epsilon K$ and

$$\int_{\mathbb{R}^n} \chi_\epsilon dx = \int_{\epsilon K} \chi_\epsilon dx = \int_K \chi dx = 1$$

as we expect. Now for any function $\varphi \in C_c^\infty(\mathbb{R}^n)$ we have

$$\min_{x\in\epsilon K} \varphi(x) \leq \int_{\mathbb{R}^n} \chi_\epsilon \varphi dx \leq \max_{x\in\epsilon K} \varphi(x)$$

letting $\epsilon \to 0$ right and left end side approaches $\varphi(0)$ hence

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \chi_\epsilon \varphi dx = \varphi(0)$$

i.e. $T_{\chi_\epsilon} \to \delta_0$ as $\epsilon \to 0$. Thus we have smoothed the δ_0-function.

5.2 Green's Function Characterization

In this section, we define the Green's Functions. To get a unique Green's Function, we need an operator which has a trivial kernel. So we begin with a compact Riemannian 4-manifold (M, g), and assume that its *Yamabe Laplacian* $\Delta + s/6$ has trivial kernel.This is automatic if g is conformally equivalent to a metric of positive scalar curvature, impossible if it is conformally equivalent to a metric of zero scalar curvature because of the Hodge Laplacian, and may or may not happen for a metric of negative scalar curvature. Since the Hodge Laplacian Δ is self-adjoint, $\Delta + s/6$ is also self-adjoint implying that it has a trivial cokernel, if once have a trivial kernel. Therefore it is a bijection and we have a unique smooth solution u for the equation $(\Delta + s/6)u = f$ for any smooth function f. It also follows that it has a unique distributional solution u for any distribution f. Let $y \in M$ be any point. Consider the Dirac delta distribution δ_y at y defined by

$$\delta_y : C^\infty(M) \to \mathbb{R} \quad , \quad \delta_y(f) = f(y)$$

intuitively, this behaves like a function identically zero on $M - \{y\}$, and infinity at y with integral 1. Then there is a unique distributional solution G_y to the equation

$$(\Delta + s/6)G_y = \delta_y$$

called the *Green's Function* for y. Since δ_y is identically zero on $M - \{y\}$, elliptic regularity implies that G_y is smooth on $M - \{y\}$.

About y, one has an expansion

$$G_y = \frac{1}{4\pi^2}\frac{1}{\varrho_y^2} + O(\log \varrho_y)$$

near ϱ_y denotes the distance from y. In the case (M, g) is self-dual this expansion reduces to [AtGr]

$$G_y = \frac{1}{4\pi^2}\frac{1}{\varrho_y^2} + \textit{bounded terms}$$

We also call G_y to be the *conformal Green's function* of (M, g, y).

This terminology comes from the fact that the Yamabe Laplacian is a *conformally invariant* differential operator as a map between sections of some real line bundles. For any nonvanishing smooth function u, the conformally equivalent metric $\tilde{g} = u^2 g$ has scalar curvature

$$\tilde{s} = 6u^{-3}(\Delta + s/6)u$$

A consequence of this is that $u^{-1}G_y$ is the conformal Green's function for $(M, u^2 g, y)$ if G_y is the one for (M, g, y).

Any metric on a compact manifold is conformally equivalent to a metric of constant scalar curvature sign. Since if $u \not\equiv 0$ is the eigenfunction of the lowest eigenvalue λ of the Yamabe Laplacian,

$$\tilde{s} = 6u^{-3}\lambda u = 6\lambda u^{-2}$$

for the metric $\tilde{g} = u^2 g$. Actually a more stronger statement is true thanks to the proof[LP] of the Yamabe Conjecture, any metric on a compact manifold is conformally equivalent to a metric of constant scalar curvature(CSC). Also if two metrics with scalar curvatures of fixed signs are conformally equivalent, then their scalar curvatures have the same sign.

The sign of Yamabe constant of a conformal class, meaning the sign of the constant scalar curvature of the metric produced by the proof of the Yamabe conjecture is the same as the sign of the smallest Yamabe eigenvalue λ for any metric in the conformal class.

Before giving our characterization for positivity, we need some lemmas, beginning by reminding the :

Lemma 5.2.1 (Normal Coordinates[CFKS]p242,[Pet]p54,[dC]p86)**.** *Let m be a point on a Riemannian manifold M. There exist a coordinate system x^i in a neighborhood of m so that*

(a) $x^i(m) = 0$

(b) $g_{ij}(m) = \delta_{ij}$

(c) $\Gamma^k_{ij}(m) = 0$

(d) $\frac{\partial}{\partial x^i} g_{jk}(m) = 0$

Proof. One gets such a coordinate system by picking an orthonormal basis e_i in T_mM, and letting the point p have coordinates x^i if it is in the image of the vector $\sum x^ie_i$ under the exponential map exp_p. Incidentally, the geodesic from m to p has length $|x|$ and it is tangent to $\sum x^ie_i$. The neighborhood we are working on is a normal neighborhood which is by definition obtained by exponentiating a neighborhood of 0 in T_mM on which the exponential map is a diffeomorphism [Bo]p331. So the geodesics passing through m are given by linear equations (are lines) in this coordinate system which immediately implies $\Gamma^k_{ij}(m) = 0$ because of the reduction of the geodesic equation $\ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0$ for $k = 1..n$ [dC]p62. Notice that (d) follows from the previous conditions since

$$\begin{aligned}
g_{kl}\Gamma^l_{ij} + g_{jl}\Gamma^l_{ik} &= g_{kl}g^{lt}\{\partial_i g_{jt} + \partial_j g_{it} - \partial_t g_{ij}\}/2 + g_{jl}g^{lt}\{\partial_i g_{kt} + \partial_k g_{it} - \partial_t g_{ik}\}/2 \\
&= \delta^t_k\{\partial_i g_{jt} + \partial_j g_{it} - \partial_t g_{ij}\}/2 + \delta^t_j\{\partial_i g_{kt} + \partial_k g_{it} - \partial_t g_{ik}\}/2 \\
&= \{\partial_i g_{jk} + \partial_j g_{ik} - \partial_t g_{ik}\}/2 + \{\partial_i g_{kj} + \partial_k g_{ij} - \partial_j g_{ik}\}/2 \\
&= \partial_i g_{jk}
\end{aligned}$$

□

Remark 5.2.2. *The classical Laplacian of a function in local coordinates is given by [Pet]p55 :*

$$\Delta_c f = \sqrt{detg_{ij}}^{-1}\partial_k(\sqrt{detg_{ij}}g^{kl}\partial_l f)$$

reduces to

$$\sqrt{detg_{ij}}^{-1}\sqrt{detg_{ij}}g^{kl}\partial_k\partial_l f = \delta^{kl}\partial_k\partial_l f = \sum_{k=1}^{n}\partial^2_k f$$

at m in normal coordinates (a)-(d). Another reduction is to $g^{ij}\partial_i\partial_j f$ using harmonic coordinates [Pet]p286 obtained by a much serious process depending on solving the Dirichlet problem. We will be using the negative of this operator and call it the Modern Laplacian or Hodge Laplacian. It is positive definite and extendable over the forms. Defined by $\Delta = d^*d + dd^*$, reducing to d^*d on functions. Sources after 1970's tend to use the modern one[AtGr, War], though a few geometers [dC, Pet] and most analists[PrWe, GilTru] still use the classical one. So we have conventions

$$\Delta f = -\Delta_c f = d^*df = d^*(\nabla f^\flat) = div_B \nabla f$$

where div_B is defined in [Besse] negative of [dC, Pet].

Now we are going to state the maximum principles we are interested in. Before, we have some definitions.

Definition 5.2.3 ([PrWe]p56)**.** *Consider the differential operator $L_c = \sum_{i,j=1}^n a^{ij}(x_1..x_n)\frac{\partial^2}{\partial x_i \partial x_j}$ arranged so that $a^{ij} = a^{ji}$. It is called* elliptic *at a point $x = (x_1..x_n)$ if there is a positive quantity $\mu(x)$ such that*

$$\sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \geq \mu(x)\sum_{i=1}^n {\xi_i}^2$$

for all n-tuples of real numbers $(\xi_1..\xi_n)$. The operator is said to be uniformly elliptic in a domain Ω if the inequality holds for each point of Ω and if there is a positive constant μ_0 such that $\mu(x) \geq \mu_0$ for all x in Ω. Ellipticity of a more general second order operator is defined via its second order term.

In the matrix language, the ellipticity condition asserts that the symmetric matrix $[a^{ij}]$ is positive definite at each point x.

Now, for convenience, we state the weak max/minimum principle for sub/super harmonic functions though we will not be using it [GilTru]p15,32 :

Theorem 5.2.4 (Weak max/min principle)**.** *Let L_c be elliptic in the bounded domain Ω with no zero order term. Suppose that $L_c u \geq 0 (\leq 0)$ in Ω with $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$. Then the maximum(minimum) of u in $\bar{\Omega}$ is achieved on $\partial\Omega$*

Consequently, a function satisfying the above hypothesis e.g. a classically sub(super) harmonic function can not assume an interior maximum(minimum) value unless it is constant. This is similar to the one dimensional case, where if the second derivative of a function is nonnegative, it can not assume an interior maximum, unless it is constant. The proof involves a classical argument by the mean value inequality. Result carries over any compact Riemannian manifold. Around any point take the harmonic coordinate neighborhood. This will be an open cover of the manifold. Take its finite subcover. Look at the open set inside which the maximum is taken. The function is going to be constant there and the argument will spread over to the touching neighbor open set, finally cover the whole manifold. So the function is going to be constant everywhere. That's why the harmonic functions on a compact Riemannian manifold are only constants as a different proof is sketched for the oriented case by [dC]p85 and [Pet]p55.

Next comes the maximum principle that we are going to use [PrWe]p64 :

Lemma 5.2.5 (Hopf's strong maximum principle)**.** *Let u satisfy the differen-*

tial inequality

$$(L_c + h)u \geq 0 \quad with \quad h \leq 0$$

where L_c is uniformly elliptic in Ω and coefficients of L_c and h bounded. If u attains a nonnegative maximum at an interior point of Ω, then u is constant.

So if for example the maximum of u is attained in the interior and is 0, then u has to vanish.

An application of this principle provides us with a criterion of determining the sign of the Yamabe Constant using Green's Functions:

Lemma 5.2.6 (Green's Function Characterization for the Sign[LeOM])**.** *Let (M, g) be a compact Riemannian 4-manifold with $Ker(\Delta + s/6) = 0$, i.e. the Yamabe Laplacian has trivial kernel, taking $\Delta = d^*d$[AtGr]. Fix a point $y \in M$. Then for the conformal class $[g]$ we have the following assertions :*

1. It does not contain a metric of zero scalar curvature

2. It contains a metric of positive scalar curvature iff $G_y(x) \neq 0$ for all $x \in M - \{y\}$

3. It contains a metric of negative scalar curvature iff $G_y(x) < 0$ for some $x \in M - \{y\}$

Proof. Proceeding as in [LeOM], $[g]$ has three possibilities for its Yamabe Type, one of 0,+,−. Since the Yamabe Laplacian is conformally invariant as acting on functions with conformal weight, we assume that either $s = 0$ or $s > 0$ or else $s < 0$ everywhere.

$s = 0$: Then $(\Delta + 0/6)f = \Delta f = 0$ is solved by any nonzero constant function f. Therefore $Ker(\Delta + s/6) \neq 0$, which is not our situation.

$s > 0$: For the smooth function $G_y : M - \{y\} \to \mathbb{R}$, $G_y^{-1}((-\infty, a])$ is closed hence compact for any $a \in \mathbb{R}$. Hence it has a minimum say at m on $M - \{y\}$. We also have $(\Delta + s/6)G_y = 0$ on $M - \{y\}$. At the minimum, choose normal coordinates so that $\Delta G_y(m) = -\sum_{k=1}^{4} \partial_k^2 G_y(m)$ by the Remark (5.2.2). Second partial derivatives are greater than or equal to zero, $\Delta G_y(m) \leq 0$ so $G_y(m) = -\frac{6}{s}\Delta G_y(m) \geq 0$. We got nonnegativity, but need positivity, so assume $G_y(m) = 0$.

Then the maximum of $-G_y$ is attained and it is nonnegative with $(\Delta_c - s/6)(-G_y) = 0 \geq 0$. So the strong maximum principle(5.2.5) is applicable and $-G_y \equiv 0$. This is impossible since $G_y(x) \to \infty$ as $x \to y$, hence $m \neq 0$ and $G_y > 0$. Note that the weak maximum principle was not applicable since we had $G_y \geq 0$, implied $\Delta_c G_y = \frac{s}{6} G_y \geq 0$ though we got a minimum rather than a maximum. Also note that $\nabla G_y(m) = 0$ at a minimum though this does not imply $div \nabla G_y(m) = 0$.

$s < 0$: In this situation we have

$$\frac{1}{6}\int_M s G_y dV = \int_M (\Delta + s/6) G_y dV = \int_M \delta_y dV = 1 > 0$$

implying $G_y < 0$ at some point. Besides, at some other point it should be zero since $G_y(x) \to +\infty$ as $x \to y$.

$\square$

5.3 Cohomological Characterization

Now let (M^4, g) be a compact self-dual Riemannian manifold with the twistor space Z. One of the basic facts of the twistor theory[HitLin] is that for any open set $U \subset M$ and the correponding inverse image $\widetilde{U} \subset Z$ in the twistor space, there is a natural isomorphism

$$pen : H^1(\widetilde{U}, \mathcal{O}(K^{1/2})) \xrightarrow{\sim} \{\text{smooth complex-valued solutions of } (\Delta + s/6)u = 0 \text{ in } U\}$$

which is called *the Penrose transform*[BaSi, HitKä, AtGr], where $K = K_Z$. Since locally $\mathcal{O}(K^{1/2}) \approx \mathcal{O}(-2)$ e.g. $Z = \mathbb{CP}_3$, for a cohomology class $\psi \in H^1(\widetilde{U}, \mathcal{O}(K^{1/2}))$, the value of the corresponding function pen_ψ at $x \in U$ is obtained by restricting ψ to the twistor line $P_x \subset Z$ to obtain an element

$$pen_\psi(x) = \psi|_{P_x} \in H^1(P_x, \mathcal{O}(K^{1/2})) \approx H^1(\mathbb{CP}_1, \mathcal{O}(-2)) \approx \mathbb{C}.$$

Note that pen_ψ is a section of a line bundle, but the choice of a metric g in the conformal class determines a canonical trivialization of this line bundle [HitKä], and pen_ψ then becomes an ordinary function. Taking $U = M - \{y\}$ we have $(\Delta + s/6)G_y = 0$ on U in the uniquely presence of the conformal Green's functions(5.2) and $G_y(x)$ is regarded as a function of x corresponds to a canonical element

$$pen^{-1}(G_y) \in H^1(Z - P_y, \mathcal{O}(K^{1/2}))$$

where P_y is the twistor line over the point y.

What is this interesting cohomology class? The answer was discovered by Atiyah [AtGr] involving the *Serre Class* of a complex submanifold. Which is a construction due to Serre [Ser] and Horrocks [Hor]. We now give the definition of the Serre class via the following lemma:

Lemma 5.3.1 (Serre-Horrocks Vector Bundle,Serre Class)**.** *Let W be a (possibly non-compact) complex manifold, and let $V \subset W$ be a closed complex submanifold of complex codimension 2, and $N = N_{V/W}$ be the normal bundle of V. For any holomorphic line bundle $L \to W$ satisfying*

$$L|_V \approx \wedge^2 N \quad \text{and} \quad H^1(W, \mathcal{O}(L^*)) = H^2(W, \mathcal{O}(L^*)) = 0$$

There is a rank-2 holomorphic vector bundle $E \to W$ called the Serre-Horrocks bundle *of (W, V, L) , together with a holomorphic section ζ satisfying*

$$\wedge^2 E \approx L \quad , \quad d\zeta|_V : N \xrightarrow{\sim} E \quad \text{and} \quad \zeta = 0 \text{ exactly on V.}$$

The pair (E, ζ) is unique up to isomorphism if we also impose that the isomorphism $\det d\zeta : \wedge^2 N \to \wedge^2 E|_V$ should agree with a given isomorphism $\wedge^2 N \to L|_V$. They also give rise to an extension

$$0 \to \mathcal{O}(L^*) \to \mathcal{O}(E^*) \xrightarrow{\cdot\zeta} \mathscr{I}_V \to 0,$$

the class of which is defined to be the Serre Class $\lambda(V) \in \mathrm{Ext}^1_W(\mathscr{I}_V, \mathcal{O}(L^*))$, *where $\mathscr{I}_V$ is the ideal sheaf of V, and this extension determines an element of $H^1(W - V, \mathcal{O}(L^*))$ by restricting to $W - V$.*

Proof. Consult [LeOM] for a proof. □

For an alternative treatment of Serre's class via the Grothendieck class consult [AtGr]. We are now ready to state the answer of Atiyah:

Theorem 5.3.2 (Atiyah[AtGr]). *Let (M^4, g) be a compact self-dual Riemannian manifold with twistor space Z, and assume that the conformally invariant Laplace operator $\Delta = d^*d + s/6$ on M has no global nontrivial solution so that the Green's functions are well defined. Let $y \in M$ be any point, and $P_y \subset Z$ be the corresponding twistor line.*
Then the image of the Serre class $\lambda(P_y) \in \mathrm{Ext}^1_Z(\mathscr{I}_{P_y}, \mathcal{O}(K^{1/2}))$ in $H^1(Z - P_y, \mathcal{O}(K^{1/2}))$ is the Penrose transform of the Green's function G_y times a non-zero constant. More precisely

$$pen^{-1}(G_y) = \frac{1}{4\pi^2}\lambda(P_y)$$

Now thanks to this remarkable result of Atiyah, we can substitute the Serre class for the Green's functions in our previous characterization 5.2.6 and get rid of them to obtain a better criterion for positivity as follows :

Proposition 5.3.3 (Cohomological Characterization , [LeOM]). *Let (M^4, g) be a compact self-dual Riemannian manifold with twistor space Z. Let P_y be a twistor line in Z.*
Then the conformal class $[g]$ contains a metric of positive scalar curvature if and only if $H^1(Z, \mathcal{O}(K^{1/2})) = 0$, and the Serre-Horrocks vector bundle(5.3.1) on Z taking $L = K^{-1/2}$ associated to P_y satisfies $E|_{P_x} \approx \mathcal{O}(1) \oplus \mathcal{O}(1)$ for every twistor line P_x

Proof. $\Rightarrow$: If a conformal class contains a metric of positive scalar curvature g, then we can show that $Ker(\Delta + \frac{s}{6})$ is trivial as follows: Let $(\Delta + \frac{s}{6})u = 0$ for some smooth function $u : M \to \mathbb{R}$ and $s > 0$. Since M is compact, u has a minimum say at some point m. At the minimum one has

$$\Delta u(m) = -\sum u_{kk}(m) \leq 0$$

because of the normal coordinates (5.2.1) about m, modern Laplacian and second derivative test. So that

$$\Delta u = -\frac{su}{6} \leq 0 \quad \text{implying} \quad u \geq 0 \quad \text{everywhere.}$$

If we integrate over M on gets 0 for the Laplacian of a function so

$$0 = \int_M \Delta u \, dV = \int_M -\frac{su}{6} dV \;\; \text{hence} \;\; \int_M su \, dV = 0 \;\; \text{implying} \;\; u \equiv 0 \;\; \text{since} \;\; s > 0$$

that is to say that the kernel is zero.

Remember the Penrose Transform map

$$pen : H^1(M, \mathcal{O}(K^{1/2})) \xrightarrow{\sim} Ker(\Delta + \frac{s}{6})$$

implies that $H^1(M, \mathcal{O}(K^{1/2})) = 0$, also by Serre Duality

$$H^2(M, K^{1/2}) \approx H^{0,2}_{\bar\partial}(M, K^{1/2}) \overset{SD}{\approx} H^{3,1}_{\bar\partial}(M, K^{1/2*})^* \approx H^1(M, K \otimes K^{-1/2})^* = H^1(M, K^{1/2})^* = 0$$

also

$$\wedge^2 NP_y = \wedge^2 \mathcal{O}_{P_y}(1) \oplus \mathcal{O}_{P_y}(1) = \bigoplus_{2=p+q} \wedge^p \mathcal{O}(1) \otimes \wedge^q \mathcal{O}(1) = \wedge^1 \mathcal{O}(1) \otimes \wedge^1 \mathcal{O}(1) = \mathcal{O}_{\mathbb{P}_1}(2) = K^{-1/2}|_{P_y}$$

since $K^{-1/2}|_{P_y} = \mathcal{O}_{\mathbb{P}_3}(4)^{1/2}|_{P_y} = \mathcal{O}_{P_y}(2)$. So that the hypothesis for the Serre-Horrocks vector bundle construction (5.3.1) for $L = K^{-1/2}$ is satisfied. Then we have the image of the Serre class

$$4\pi^2 pen^{-1}(G_y) = \lambda(P_y) \in H^1(Z - P_y, K^{1/2})$$

So

$$4\pi^2 G_y(x) = pen_{\lambda(P_y)}(x) = \lambda(P_y)|_{P_x} \in H^1(P_x, \mathcal{O}(K^{1/2})) \approx \mathbb{C}$$

where

$$H^1(P_x, \mathcal{O}(K^{1/2})) \approx H^1(\mathbb{CP}_1, \mathcal{O}(-2)) \approx H^0(\mathbb{CP}_1, \Omega^1(\mathcal{O}(-2)^*)) = H^0(\mathbb{CP}_1, \mathcal{O}) \approx \mathbb{C}$$

By the Green's Function Characterization (5.2.6) we know that $4\pi^2 G_y(x) \neq 0$. So $\lambda(P_y)|_{P_x} \in H^1(P_x, \mathcal{O}(K^{1/2}))$ is also nonzero.

Since $\lambda(P_y)$ corresponds to the extension

$$0 \to \mathcal{O}(K^{1/2}) \to \mathcal{O}(E^*) \to \mathcal{I}_{P_y} \to 0$$

If we restrict to $Z - P_y$

$$0 \to \mathcal{O}(K^{1/2}) \to \mathcal{O}(E^*) \to \mathcal{O} \to 0$$

dualizing we obtain

$$0 \to \mathcal{O} \to \mathcal{O}(E) \to \mathcal{O}(K^{-1/2}) \to 0$$

now restricting this extension to P_x

$$0 \to \mathcal{O}_{\mathbb{P}_1} \to \mathcal{O}(E)|_{P_x} \to \mathcal{O}(2) \to 0$$

So since $G_y(x) \neq 0$, we expect that this extension is nontrivial. Let's figure out the possibilities. First of all, by the theorem of Grothendieck[VB]p22 every holomorphic vector bundle over $\mathbb{P}_1$ splits. In our case $E|_{P_x} = \mathcal{O}(k) \oplus \mathcal{O}(l)$ for some $k, l \in \mathbb{Z}$. Moreover if we impose $k \geq l$, this splitting is uniquely determined[VB].

Secondly, any short exact sequence of vector bundles splits topologically by [VB]p16. In our case, topologically we have $E|_{P_x} \overset{t}{=} \mathcal{O} \oplus \mathcal{O}(2)$. So, setting the chern classes to each other we have

$$c_1(E|_{P_x})[P_x] = c_1(\mathcal{O}(k) \oplus \mathcal{O}(l))[\mathbb{P}_1] = c_1\mathcal{O}(k) + c_1\mathcal{O}(l)[\mathbb{P}_1] = k + l$$

equal to

$$c_1(E|_{P_x})[P_x] = c_1(\mathcal{O} \oplus \mathcal{O}(2))[\mathbb{P}_1] = c_1\mathcal{O} + c_1\mathcal{O}(2)[\mathbb{P}_1] = 0 + 2 = 2.$$

Hence $l = 2-k$. We now have $E|_{P_x} = \mathcal{O}(k) \oplus \mathcal{O}(2-k)$. Our extension becomes

$$0 \to \mathcal{O}_{\mathbb{P}_1} \to \mathcal{O}(k) \oplus \mathcal{O}(2-k) \to \mathcal{O}(2) \to 0$$

The inclusion $\mathcal{O} \hookrightarrow \mathcal{O}(k) \oplus \mathcal{O}(2-k)$ gives a trivial holomorphic subbundle. It has one complex dimensional space of sections. So these sections are automatically sections of $\mathcal{O}(k) \oplus \mathcal{O}(2-k)$, too. This implies

$$0 \neq H^0(\mathcal{O}(k) \oplus \mathcal{O}(2-k)) = H^0(\mathcal{O}(k)) \oplus H^0(\mathcal{O}(2-k))$$

Imposing $k, 2-k \geq 0$ by the Kodaira Vanishing Theorem[GH] since the direct sum elements $\mathcal{O}(k)$ and $\mathcal{O}(2-k)$ should possess sections. Also, from uniqueness $k \geq l = 2-k$. Altogether we have $2 \geq k \geq 1$. From the two choices, $k = 2$ gives the trivial extension $\mathcal{O}(2) \oplus \mathcal{O}$, $k = 1$ gives the nontrivial extension $E|_{P_x} = \mathcal{O}(1) \oplus \mathcal{O}(1)$ as we expected. See the following remark for existence.

$\Leftarrow$: For the converse, if $E|_{P_x} = \mathcal{O}(1) \oplus \mathcal{O}(1)$ then we already showed that this is the nontrivial extension hence $G_y(x) \neq 0$, so that the scalar curvature is positive by the Green's Function Characterization (5.2.6) $\square$

Remark 5.3.4. *The nontrivial extension of $\mathcal{O}$ by $\mathcal{O}(2)$ exists by the Euler exact sequence*

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus n+1} \xrightarrow{\mathcal{E}} T'\mathbb{P}^n \to 0$$

[GH]p409 for $n = 1$.

5.4 The Sign of the Scalar Curvature

We are now ready to approach the problem of determining the sign of the Yamabe constant for the self-dual conformal classes constructed in Theorem (2.4.1). The techniques used here are analogous to the ones used by LeBrun in [LeOM].

Theorem 5.4.1. *Let (M_1, g_1) and (M_2, g_2) be compact self-dual Riemannian 4-manifolds with $H^2(Z_i, \mathcal{O}(TZ_i)) = 0$ for their twistor spaces. Moreover suppose that they have positive scalar curvature.*

Then, for all sufficiently small $\mathfrak{t} > 0$, the self-dual conformal class $[g_\mathfrak{t}]$ obtained on $M_1 \# M_2$ by the Donaldson-Friedman Theorem (2.4.1) contains a metric of positive scalar curvature.

Proof. Pick a point $y \in (M_1 \# M_2) \backslash M_1$. Consider the real twistor line $P_y \subset \widetilde{Z}_2$, and extend this as a 1-parameter family of twistor lines in $P_{y_\mathfrak{t}} \subset Z_\mathfrak{t}$ for $\mathfrak{t}$ near $0 \in \mathbb{C}$ and such that $P_{y_\mathfrak{t}}$ is a real twistor line for $\mathfrak{t}$ real. By shrinking $\mathcal{U}$ if needed, we may arrange that $\mathcal{P} = \cup_\mathfrak{t} P_{y_\mathfrak{t}}$ is a closed codimension-2 submanifold of $\mathcal{Z}$ and $H^1(\mathcal{Z}, \mathcal{O}(L^*)) = H^2(\mathcal{Z}, \mathcal{O}(L^*)) = 0$ by the Vanishing Theorem (4.2.3). Next we check that $L|_\mathcal{P} \approx \wedge^2 N_\mathcal{P}$. Over a twistor line $P_{y_\mathfrak{t}}$ we have

$$\wedge^2 N_\mathcal{P}|_{P_{y_\mathfrak{t}}} = \wedge^2(\mathcal{O}(1) \oplus \mathcal{O}(1)) = \mathcal{O}_{P_{y_\mathfrak{t}}}(2)$$

by considering the first Chern classes. On the other hand, notice that the restriction of L^* to any smooth fiber $Z_\mathfrak{t}$, $\mathfrak{t} \neq 0$ is simply $K^{1/2}$:

$$L^*|_{Z_\mathfrak{t}} = (\frac{1}{2}K_\mathcal{Z} - \widetilde{Z}_1)|_{Z_\mathfrak{t}} = \frac{1}{2}K_\mathcal{Z}|_{Z_\mathfrak{t}} = \frac{1}{2}(K_{Z_\mathfrak{t}} - Z_\mathfrak{t})|_{Z_\mathfrak{t}} = \frac{1}{2}K_{Z_\mathfrak{t}}|_{Z_\mathfrak{t}}.$$

Here, $\widetilde{Z}_1|_{Z_\mathfrak{t}} = 0$ because of the fact that $\widetilde{Z}_1$ and $Z_\mathfrak{t}$ does not intersect for $\mathfrak{t} \neq 0$. The normal bundle of $Z_\mathfrak{t}$ is trivial, because of the fact that we have a standard deformation. Then

$$L|_{P_{y_\mathfrak{t}}} = K_{Z_\mathfrak{t}}^{-1/2}|_{P_{y_\mathfrak{t}}} = TF|_{P_{y_\mathfrak{t}}} = \mathcal{O}_{P_{y_\mathfrak{t}}}(2) \quad \text{for} \quad \mathfrak{t} \neq 0$$

since TF of Sec (4.1) is the square-root of the anti-canonical bundle. For the case $\mathfrak{t} = 0$, we need the fact that $L^*|_{\widetilde{Z}_2} = \pi^* K_{Z_2}^{1/2}$ which we have computed in the step 4 of the proof of the vanishing theorem (4.2.3). This yields

$$L|_{P_{y_0}} = \pi^* K_{Z_2}^{-1/2}|_{\widetilde{Z}_2}|_{P_{y_0}} = \mathcal{O}_{P_{y_0}}(2).$$

Then the Serre-Horrocks construction (5.3.1) is available to obtain the holomorphic vector bundle $E \to \mathcal{Z}$ and a holomorphic section ζ vanishing exactly along $\mathcal{P}$, also, the corresponding extension

$$0 \to \mathcal{O}(L^*) \to \mathcal{O}(E^*) \xrightarrow{\cdot\zeta} \mathscr{I}_{\mathcal{P}} \to 0$$

gives us the Serre class $\lambda(\mathcal{P}) \in H^1(\mathcal{Z} - \mathcal{P}, \mathcal{O}(L^*))$.

Since $L^*|_{Z_\mathfrak{t}} = K_{Z_\mathfrak{t}}^{1/2}$ for $\mathfrak{t} \neq 0$ by the above computation, Proposition (5.3.2) of Atiyah tells us that the restriction of $\lambda(\mathcal{P})$ to $Z_\mathfrak{t}$, $\mathfrak{t} > 0$, has Penrose transform equal to a positive constant times the conformal Green's function of $(M_1 \# M_2, g_\mathfrak{t}, y_\mathfrak{t})$ for any $\mathfrak{t} > 0$.

Now, we will restrict (E, ζ) to the two components of the divisor Z_0. We begin by restricting to $\widetilde{Z}_2$. We have $L|_{P_{y_0}} = \mathcal{O}_{P_0}(2) = \wedge^2 NP_{y_0}$ and

$$H^k(\widetilde{Z}_2, L^*) = H^k(\widetilde{Z}_2, \pi^* K_{Z_2}^{1/2}) = H^k(Z_2, \pi_* \pi^* K_{Z_2}^{1/2}) = H^k(Z_2, K_{Z_2}^{1/2}) = 0$$

for $k = 1, 2$ because of the projection lemma, Leray spectral sequence and the Hitchin's Vanishing theorem for positive scalar curvature on M_2. So that we have the Serre-Horrocks bundle for the triple $(\widetilde{Z}_2, P_{y_0}, L|_{\widetilde{Z}_2} = \pi^* K_{Z_2}^{-1/2})$. On the other hand it is possible to construct the Serre-Horrocks bundle E_2 for the triple $(Z_2, P_{y_0}, K_{Z_2}^{-1/2})$ for which all conditions are already checked to be satisfied. In the construction of these Serre-Horrocks bundles, if we stick to a chosen isomorphism $\wedge^2 N \to L|_{P_{y_0}}$, these bundles are going to be isomorphic by (5.3.1). The splitting type of E on the twistor lines corresponding to the points in $M_2 - \{y_0, p_2\}$ supposed to be the same as the splitting type of E_2, which is $\mathcal{O}(1) \oplus \mathcal{O}(1)$ since Z_2 already admits a self-dual metric of positive scalar curvature.

Secondly, we restrict (E, ζ) to $\widetilde{Z}_1$. Alternatively we restrict the Serre class $\lambda(\mathcal{P})$ to $H^1(\widetilde{Z}_1, \mathcal{O}(L^*))$ where

$$L^*|_{\widetilde{Z}_1} = \tfrac{1}{2}K_{\mathcal{Z}} - \widetilde{Z}_1|_{\widetilde{Z}_1} = \tfrac{1}{2}K_{\mathcal{Z}} + Q|_{\widetilde{Z}_1} \overset{adj}{=} \tfrac{1}{2}(K_{\widetilde{Z}_1} - \widetilde{Z}_1) + Q|_{\widetilde{Z}_1} =$$
$$\tfrac{1}{2}(K_{\widetilde{Z}_1} + Q) + Q|_{\widetilde{Z}_1} = \tfrac{1}{2}(\pi^* K_{Z_1} + 2Q) + Q|_{\widetilde{Z}_1} = \pi^* \tfrac{1}{2} K_{Z_1} + 2Q|_{\widetilde{Z}_1},$$

and show that it is non-zero on every real twistor line away from Q here. Remember that we have the the restriction isomorphism obtained in the step 6 of the proof of the vanishing theorem (4.2.3)

$$H^1(\mathcal{O}_{\widetilde{Z}_1}(L^*)) \xrightarrow{\sim} H^1(\mathcal{O}_Q(L^*)) \approx \mathbb{C}$$

as a consequence of Hitchin's Vanishing theorems for positive scalar curvature on M_1, as mentioned in the step 5, and $H^1(\mathcal{O}_Q(L^*)) = H^1(\mathbb{P}_1 \times \mathbb{P}_1, \mathcal{O}(-2, 0)) = \mathbb{C}$, as computed in the step 3. This shows that if there is a rational curve of Q on which the Serre class is non-zero, then this class is non-zero and a generator of $H^1(\mathcal{O}_{\widetilde{Z}_1}(L^*))$. The Serre-Horrocks bundle construction on Z_2 shows us that $E|_{C_2} = \mathcal{O}(1) \oplus \mathcal{O}(1)$ where C_2 is the twistor line on which the blow up is done. We know that $Q = \mathbb{P}_1 \times \mathbb{P}_1 \approx \mathbb{P}(NC_2)$. So that the exceptional divisor has one set of rational curves which are the fibers, and another set of rational curves, coming from the sections of the projective bundle $\mathbb{P}(NC_2)$. Take the zero section of $\mathbb{P}(NC_2)$, on which E has a splitting type $\mathcal{O}(1) \oplus \mathcal{O}(1)$. So over the zero section in Q, E is going to be the same, hence non-trivial splitting type. This shows that over this rational curve on Q, the Serre-class is nonzero. Hence by the isomorphism above, the Serre-class is the (up to constant) nontrivial class in $H^1(\widetilde{Z}_1, \mathcal{O}(L^*)) \approx \mathbb{C}$.

Next we have to show that this non-trivial class is non-zero on every real twistor line in $\widetilde{Z}_1 - Q$ or $Z_1 - C_1$[1]. For this purpose consider the Serre-Horrocks vector bundle E_1 and its section ζ_1 for the triple $(Z_1, C_1, K_{Z_1}^{-1/2})$, so that $\pi^*\zeta_1$ is a section of π^*E_1 vanishing exactly along Q. Remember the construction of the line bundle associated to the divisor Q in $\widetilde{Z}_1$ [GH]. Consider the local defining functions $s_\alpha \in \mathfrak{M}^*(U_\alpha)$[2] of Q over some open cover $\{U_\alpha\}$ of $\widetilde{Z}_1$. These functions are holomorphic and vanish to first order along Q. Then the corresponding line bundle is constructed via the transition functions $g_{\alpha\beta} = s_\alpha \,/ s_\beta$. Since s_α's transform according to the transition functions, they constitute a holomorphic section s of this line bundle $[Q]$, which vanish up to first order along Q. Local holomorphic sections of this bundle is denoted by $\mathcal{O}([Q])$ and they are local functions with simple poles along Q. If we multiply $\pi^*\zeta_1$ with these functions, we will get a holomorphic section of π^*E_1 on the corresponding local open set, since ζ_1 has a non-degenerate zero on Q, so that it vanishes up to degree 1, there. This guarantees that the map is one to one,

[1] Thanks to C.LeBrun for this trick.

[2] Here, $\mathfrak{M}^*$ stands for the multiplicative sheaf of meromorphic functions which are not identically 0, in the convention of [GH]. Actually the local defining functions here are holomorphic because Q is effective.

and the multiplication embeds $\mathcal{O}([Q])$ into $\pi^* E_1$. The quotient has rank 1, and the transition functions of $\pi^* E_1$ relative to a suitable trivialization will then look like

$$\begin{pmatrix} g_{\alpha\beta} & k_{\alpha\beta} \\ 0 & d_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} \end{pmatrix}$$

where $d_{\alpha\beta}$ stands for the determinant of the transition matrix of the bundle $\pi^* E_1$ in this coordinate chart. Since the bundle $\det \pi^* E_1 \otimes [Q]^{-1}$ has the right transition functions, it is isomorphic to the quotient bundle, hence we have the following exact sequence

$$0 \to [Q] \to \pi^* E_1 \to \pi^* K^{-1/2} \otimes [Q]^{-1} \to 0$$

since $\det E_1 = K_{Z_1}^{-1/2}$ as an essential feature of the Serre-Horrocks construction. This extension of line bundles is classified by an element in

$$\mathrm{Ext}^1_{\widetilde{Z}_1}(\pi^* K^{-1/2} \otimes [Q]^{-1}, [Q]) \approx H^1(\widetilde{Z}_1, \pi^* K^{1/2} \otimes [Q]^2)$$

by [AtGr]. If we restrict our exact sequence to $\widetilde{Z}_1 - Q = Z_1 - C_1$, since the bundle $[Q]$ is trivial on the complement of Q, this extension class will be the Serre class of the triple $(Z_1, C_1, K_{Z_1}^{-1/2})$. Finally, since M_1 has positive scalar curvature, this class is nonzero on every real twistor line in $Z_1 - C_1$. So that non-triviality of the class forced non-triviality over the real twistor lines. In other words E has a non-trivial splitting type over the real twistor lines of $\widetilde{Z}_1$.

So we showed that the Serre-Horrocks vector bundle E determined by $\lambda(\mathcal{P})$ splits as $\mathcal{O}(1) \oplus \mathcal{O}(1)$ on all the σ_0-invariant rational curves in Z_0 which are limits of real twistor lines in $\mathcal{Z}_\mathfrak{t}$ as $\mathfrak{t} \to 0$. It therefore has the same splitting type on all the real twistor lines of $\mathcal{Z}_\mathfrak{t}$ for $\mathfrak{t}$ small. Besides,

$$h^j(Z_t, \mathcal{O}(L^*)) \leq h^j(Z_0, \mathcal{O}(L^*)) = 0 \quad \text{for} \quad j = 1, 2$$

by the semi-continuity principle and the proof of the vanishing theorem (4.2.3). So that via $L^*|_{Z_\mathfrak{t}} \approx K^{1/2}$,

$$H^1(Z_\mathfrak{t}, \mathcal{O}(K^{1/2})) \approx Ker(\Delta + \frac{s}{6}) = 0.$$

Since the two conditions are satisfied, Cohomological characterization (5.3.3) guarantees the positivity of the conformal class. $\square$

Chapter 6

Deformations of Scalar-Flat Anti-Self-Dual metrics and Quotients of Enriques Surfaces

One of the most interesting features of the space of anti-self-dual or self-dual(ASD/SD) metrics on a manifold is that the scalar curvature can change sign on a connected component. That means, one can possibly join two ASD metrics of scalar curvatures of opposite signs by a 1-parameter family of ASD metrics. However,this is not the case, for example for the space of Einstein metrics. There, each connected component has a fixed sign for the scalar curvature.

As a consequence, contrary to the Einstein case, most of the examples of SF-ASD metrics are constructed by first constructing a family of ASD metrics. Then showing that there are metrics of positive and negative scalar curvature in the family, and guaranteeing that there is a scalar-flat member in this family. In the $b^+ = 0$ case actually this is the only way known to construct such metrics on a 4-manifolds. This paper presents an example of a SF-ASD Riemannian 4-manifold which is impossible to obtain by this kind of techniques since it does not have a positive scalar curvature deformation.

§7.4 reviews the known examples of ASD metrics constructed by a deformation changing the sign of the scalar curvature, §6.2 introduces an action on the $K3$ surface and furnish the quotient manifold with a SF-ASD metric, §6.3 shows that the smooth manifold defined in §6.2 does not admit any positive scalar curvature(PSC) or PSC-ASD metric, §6.4 includes some examples and remarks, finally the Appendix(6.5) gives an alternative way to show that the b^+ of the $K3$-surface is nonzero, which is something needed in the preceding section.

6.1 Constructions of SF-ASD metrics

Here we review some of the constructions for SF-ASD metrics on 4-manifolds. We begin with

Theorem 6.1.1 (LeBrun[LeOM]). *For all integers $k \geq 6$, the manifold*

$$k\overline{\mathbb{CP}}_2 = \underbrace{\overline{\mathbb{CP}}_2 \# \cdots \# \overline{\mathbb{CP}}_2}_{k-many}$$

admits a 1-parameter family of real analytic ASD conformal metrics $[g_t]$ for $t \in [0,1]$ such that $[g_0]$ contains a metric of $s > 0$ on the other hand $[g_1]$ contains a metric of $s < 0$.

Corollary 6.1.2 (LeBrun[LeOM]). *For all integers $k \geq 6$, the connected sum $k\overline{\mathbb{CP}}_2$ admits scalar-flat anti-self-dual(SF-ASD) metrics.*[1]

[1] Quite recently, LeBrun and Maskit announced that they have extended this result to the case $k = 5$ with similar techniques, which is the minimal number for these type of connected sums according to [LeSD].

Proof. Let $h_t \in [g_t]$ be a smooth family of metrics representing the smooth family of conformal classes $[g_t]$ constructed in [LeOM]. We know that the smallest eigenvalue λ_t of the Yamabe Laplacian $(\Delta + s/6)$ of the metric h_t exists, and is a continuous function of t. It measures the sign of the conformally equivalent constant scalar curvature metric [LP].

But the theorem(6.1.1) tells us that λ_0 and λ_1 has opposite signs. Then there is some $c \in [0, 1]$ for which $\lambda_c = 0$. Let u be the eigenfunction corresponding to the eigenvalue 0, for the Yamabe Laplacian of h_c , i.e. $(\Delta + s/6)u = 0$. Rescale it by a constant so that it has unit integral.

Rescale the metric h_c so that it has constant scalar curvature [LP]. We have three cases for the scalar curvature, positive, zero or negative. If it is zero then we are done. Suppose $s_c = s > 0$. Since u is a continuous function on the compact manifold, it has a minimum say at m. Choose the normal coordinates around there, so that $\Delta u(m) = -\sum_{k=1}^{4} \partial^2 u(m)$. Second partial derivatives are greater than or equal to zero, $\Delta u(m) \leq 0$ so $u(m) = -\frac{6}{s}\Delta u(m) \geq 0$. Assume $u(m) = 0$. Then the maximum of $-u$ is attained and it is nonnegative with $(-\Delta - s/6)(-u) = 0 \geq 0$. So the strong maximum principle is applicable and $-u \equiv 0$, which is not an eigenfunction. So u is a positive function. For a conformally equivalent metric $\tilde{g}$, the new scalar curvature $\tilde{s}$ is computed to be [Besse]

$$\tilde{s} = 6u^{-3}(\Delta + s/6)u$$

in terms of s. Thus $\widetilde{g} = u^2 h_c$ is a scalar-flat anti-self-dual metric on $k\overline{\mathbb{CP}}_2$ for any $k \geq 6$. The negative scalar curvature case is treated similarly. $\square$

Another construction tells us

Theorem 6.1.3 ([Kim]). *There exist a continuous family of self-dual metrics on a connected component of the moduli space of self-dual metrics on*

$$l(S^3 \times S^1)\# m\mathbb{CP}_2 \text{ for any } m \geq 1 \text{ and for some } \ l \geq 2$$

which changes the sign of the scalar curvature.

6.2 SF-ASD metric on the Quotient of Enriques Surface

In this section we are going to describe what we mean by $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and the scalar-flat anti-self-dual(SF-ASD) metric on it.

Let A and B be real 3×3 matrices. For $x, y \in \mathbb{C}^3$, consider the algebraic variety $V_{2,2,2} \subset \mathbb{CP}_5$ given by the equations

$$\sum_j A_i^j x_j^2 + B_i^j y_j^2 = 0 \quad , \quad i = 1, 2, 3$$

or more precisely,

$$A_1^1 x_1^2 + A_1^2 x_2^2 + A_1^3 x_3^2 + B_1^1 y_1^2 + B_1^2 y_2^2 + B_1^3 y_3^2 = 0$$

$$A_2^1 x_1^2 + A_2^2 x_2^2 + A_2^3 x_3^2 + B_2^1 y_1^2 + B_2^2 y_2^2 + B_2^3 y_3^2 = 0$$

$$A_3^1 x_1^2 + A_3^2 x_2^2 + A_3^3 x_3^2 + B_3^1 y_1^2 + B_3^2 y_2^2 + B_3^3 y_3^2 = 0$$

For generic A and B, this is a complete intersection of three nonsingular quadric hypersurfaces. By the Lefschetz hyperplane theorem, it is simply connected, and

$$K_{V_2} = K_{\mathbb{P}^5} \otimes [V_2^{\mathbb{P}^5}] = \mathcal{O}(-6) \otimes \mathcal{O}(1)^{\otimes 2} = \mathcal{O}(-4)$$

since $[V_2]_h = 2[H]_h$ and taking Poincare duals, similarly

$$K_{V_{2,2}} = K_{V_2} \otimes [V_{2,2}^{\mathbb{P}^5}] = \mathcal{O}(-4) \otimes \mathcal{O}(2) = \mathcal{O}(-2)$$

$$K_{V_{2,2,2}} = K_{V_{2,2}} \otimes [V_{2,2,2}^{\mathbb{P}^5}] = \mathcal{O}(-2) \otimes \mathcal{O}(2) = \mathcal{O}$$

finally. So the canonical bundle is trivial. V is a $K3$ Surface. We define the commuting involutions $\sigma^{\pm}$ by

$$\sigma^+(x, y) = (x, -y) \quad \text{and} \quad \sigma^-(x, y) = (\bar{x}, \bar{y})$$

and since we arranged A and B to be real, $\sigma^{\pm}$ both act on V.

At a fixed point of σ^+ on V, we have $y_j = -y_j = 0$, so $\sum_j A_i^j x_j^2 = 0$. So if we take an invertible matrix A, these conditions are only satisfied for $x_j = y_j = 0$ which does not correspond to a point, so σ^+ is free and holomorphic. At a fixed point of σ^- on V, x_j's and y_j's are all real. If $A_1^j, B_1^j > 0$ for all j then $\sum_j A_i^j x_j^2 + B_i^j y_j^2 = 0$ forces $x_j = y_j = 0$ making σ^- free. At a fixed point $\sigma^-\sigma^+$ on V, $x_j = \bar{x}_j$ and $y_j = -\bar{y}_j$, so x_j's are real and y_j's are purely imaginary. Then y_j^2 is a negative real number. So if we choose $A_2^j > 0$ and $B_2^j < 0$, this forces $x_j = y_j = 0$, again we obtain a free action for $\sigma^-\sigma^+$. Thus choosing A and B within these circumstances $\sigma^{\pm}$ generate a free $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action and we define $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ to be the quotient of $K3$ by this free action. We have

$$\chi = \sum_{k=0}^{4} (-1)^k b_k = 2 - 2b_1 + b_2 = 2 + (2b^+ - \tau) \quad \text{hence} \quad b^+ = (\chi + \tau - 2)/2$$

so, $b^+(K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2) = (24/4 - 16/4 - 2)/2 = 0$, a special feature of this manifold.

Next we are going to furnish this quotient manifold with a Riemannian metric. For that purpose, there is a crucial observation [HitEin] that, for any free involution on $K3$, there exists a complex structure on $K3$ making this involution holomorphic, so the quotient is a complex manifold. We begin by stating the

Theorem 6.2.1 (Calabi-Yau[Ca, Yau, GHJ, Joyce])**.** *Let (M, ω) be a compact Kähler n-manifold. Let ρ be a $(1,1)$-form belonging to the class $2\pi c_1(M)$ so that it is closed.*
Then, there exists a unique Kähler metric with form ω' which is in the same class as in ω, whose Ricci form is ρ.

Intuitively, one can slide the Kähler form ω in its cohomology class and obtain any desired reasonable Ricci form ρ.

Remark 6.2.2. *Since $c_1(K3) = 0$ in our case, taking $\rho \equiv 0$ gives us a Ricci-Flat(RF) metric on the $(K3, \omega)$ surface, the* Calabi-Yau metric. *This metric is hyperkählerian because of the following reason: The holonomy group of Kähler manifolds are a subgroup of U_2. However, Ricci-Flatness reduces the holonomy since harmonic forms are parallel because of the Weitzenböck Formula for the Hodge/modern Laplacian on 2-forms (6.3.6). Scalar flatness and non-triviality* *of b^+ is to be checked. $b_1(K3) = 0$ implies $b^+(K3) = (24 - 16 - 2)/2 = 3$, which is nonzero. Actually b^+ is nontrivial for any Kähler surface since the Kähler form is harmonic and self-dual. Harmonic parallel forms are kept fixed by the holonomy group, a fact that imposes a reduction from U_2 to SU_2 which is the next possible option and isomorphic to Sp_1 in this dimension, hence the Calabi-Yau metric is hyperkähler. Alternatively one can see that the holomorphic forms are also parallel by a similar argument, another reason to reduce the holonomy. So we have at least three almost complex structures I, J, K, parallel with respect to the Riemannian connection. By duality we regard these as three linearly independent self-dual 2-forms, parallelizing Λ_2^+. So any parallel Λ_2^+ form on $K3$ defines a complex structure after normalizing. In other words $aI + bJ + cK$ defines a complex structure for the constants satisfying $a^2 + b^2 + c^2 = 1$, i.e the normalized linear combination. On the other hand*

$$b_1(K3/\mathbb{Z}_2) = b_1(K3) = 0 \quad , \quad b^+(K3/\mathbb{Z}_2) = (12 - 8 - 2)/2 = 1.$$

Since the pullback of harmonic forms stay harmonic, the generating harmonic 2-form on $K3/\mathbb{Z}_2$ comes from the universal cover, so is fixed by the $\mathbb{Z}_2$ action. It is also a parallel self-dual form so its normalization is then a complex structure left fixed by $\mathbb{Z}_2$. So the quotient is a complex surface with $b_1 = 0$ and $2c_1 = 0$ implying that it is an Enriques Surface.

So we saw that any involution or $\mathbb{Z}_2$-action can be made holomorphic by choosing the appropriate complex structure on $K3$. In particular by changing the complex structure, σ^- becomes holomorphic, too and then both $K3/\mathbb{Z}_2^{\pm}$ are *complex manifolds, i.e. Enriques Surfaces, for* $\mathbb{Z}_2^{\pm} = \langle \sigma^{\pm} \rangle$.

Remark 6.2.3. *Even though we managed to make* σ^+ *and* σ^- *into holomorphic actions by modifying the complex structure, it is impossible to provide a complex structure according to which they are holomorphic at the same time. The reason is that, in such a situation the quotient* $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *would be a complex manifold. On the other hand the* Noether's Formula *[Beauville]*

$$\chi(\mathcal{O}_S) = \frac{1}{12}(K_S^2 + \chi(S)) = \frac{1}{12}(c_1^2 + c_2)[S]$$

holds for any compact complex surface [BPV] as a consequence of the Hirzebruch-Riemann-Roch Theorem. It produces a non-integer holomorphic Euler characteristic $\frac{1}{12}\frac{24}{4} = \frac{1}{2}$.

Now consider another metric on $K3$: the restriction of the *Fubini-Study metric* on $\mathbb{CP}_5$ obtained from the Kähler form

$$\omega_{FS} = \frac{i}{2\pi}\partial\bar{\partial}\log|(x_1, x_2, x_3, y_1, y_2, y_3)|^2$$

We also denote the restriction metric by g_{FS}. It is clear that $\sigma^{\pm}$ leave this form invariant, hence they are isometries of g_{FS}. This is not the metric we are seeking for. This metric has all sectional curvatures lying in the interval $[1, 4]$ and is actually Einstein, i.e. $Ric = 6g$ with constant positive scalar curvature equal to 2 [Pet]p84. Let g_{RF} be the Ricci-Flat Yau metric (6.2.1) taking $\rho \equiv 0$ with Kähler form cohomologous to ω_{FS}. We will show that this metric is invariant under $\sigma^{\pm}$ and projects down to a metric on $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Scalar flatness and being ASD are equivalent notions for Kähler metrics [LeSD], and the local structure does not change under isometric quotients which makes the quotient SF-ASD.

Since σ^+ is holomorphic, the pullback form $\sigma^{+*}\omega_{RF}$ is Kähler and the equalities

$$[\sigma^{+*}\omega_{RF}] = \sigma^{+*}[\omega_{RF}] = \sigma^{+*}[\omega_{FS}] = [\sigma^{+*}\omega_{FS}] = [\omega_{FS}]$$

show that it is cohomologous to the Fubini-Study form. Ricci curvature is preserved and is zero, hence by Calabi uniqueness (6.2.1) we get $\sigma^{+*}g_{RF} = g_{RF}$. Dealing with the anti-holomorphic involution needs a little more care. Think $\sigma^- : K3 \to K3$ as a diffeomorphism. The pullback of a Kähler metric is Kähler with respect to the pullback complex structure. The anti-holomorphicity relation relates the two complex structures by $\sigma^-_* J_1 = -J_2\sigma^-_*$. The pullback Kähler form $\widetilde{\omega}_n = \omega_{\sigma^{-*}g_{FS}} = -\sigma^{-*}\omega_{RF}$ since

$$\widetilde{\omega}_n(u,v) = \sigma^{-*}g_{RF}(J_1u,v) = g_{RF}(\sigma^-_* J_1 u, \sigma^-_* v) = g_{RF}(-J_2\sigma^-_* u, \sigma^-_* v)$$

$$= -\omega_{RF}(\sigma^-_* u, \sigma^-_* v) = -\sigma^{-*}\omega_{RF}(u,v),$$

and hence,

$$[\widetilde{\omega}_n] = [-\sigma^{-*}\omega_{RF}] = -\sigma^{-*}[\omega_{RF}] = -\sigma^{-*}[\omega_{FS}] = -[\sigma^{-*}\omega_{FS}] = -[\omega_{FS}].$$

But this is the form of $\sigma^{-*}g_{FS}$ with respect to the pullback complex structure which is the conjugate(negative) of the original one. Looking from the real point of view, once we have a Kähler metric g, it has a Kähler form corresponding to each supported complex structure on the manifold. Once the complex structure is chosen, the form is obtained by lowering an index

$$\omega_{ab} = \omega(\partial_a, \partial_b) = g(J\partial_a, \partial_b) = g(J_a^{\ c}\partial_c, \partial_b) = J_a^{\ c}\, g(\partial_c, \partial_b) = J_a^{\ c}\, g_{cb} = J_{ab}.$$

So, the form and the complex structure are equivalent from the tensorial point of view. If we conjugate(negate) the complex structure, we should replace the form with its negative. Returning to our case, $\widetilde{\omega}_n$ is the form corresponding to the pullback, hence to the conjugate complex structure. We take its negative to obtain the one corresponding to the original complex structure. So the corresponding form is going to be $\widetilde{\omega} = --\omega_{FS}$ which is ω_{FS} , and again the Calabi uniqueness (6.2.1) implies $\sigma^{-*}g_{RF} = g_{RF}$.

Remark 6.2.4. *There is an alternative argument in [McI]p894 which appears to have a gap: "Fubini-Study metric projects down to the metrics $g^{\pm}_{FS}$ on $K3/\mathbb{Z}_2^{\pm}$. Let $h^{\pm}$ be the Calabi-Yau metric(6.2.1) on $K3/\mathbb{Z}_2^{\pm}$ with Kähler form cohomologous to that of $g^{\pm}_{FS}$. To remedy the ambiguity in the negative side, keep in mind that, σ^{-} fixes the metric and the form on $K3$, though the quotient is not a Kähler manifold initially since it is not a complex manifold, it is locally Kähler. We arrange the complex structure of $K3$ to provide a complex structure to the form, so the quotient manifold is Kähler. Now we have two Kähler metrics on the quotient (for different complex structures) but we do not know much about their curvatures, and want to make it Ricci-Flat, so we use the Calabi-Yau argument. Since $c_1(K3/\mathbb{Z}_2^{\pm}) = 0$ with real coefficients, we pass to the Calabi-Yau metric for $\rho \equiv 0$. $\pi^{\pm}$ denoting the quotient maps, the pullback metrics $\pi^{\pm *}h^{\pm}$ are both Ricci-Flat-Kähler(RFK) metrics on $K3$ with*

Kähler forms cohomologous to that of g_{FS}. Their Ricci forms are both zero. By the uniqueness(6.2.1) of the Yau metric we have $\pi^{+*}h^{+} = \pi^{-*}h^{-}$. Hence this is a Ricci-Flat Kähler metric on $K3$ on which both $\sigma^{\pm}$ act isometrically. This metric therefore projects down to a Ricci-Flat metric on our manifold $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$." The problem is that the pullback metrics $\pi^{\pm *}h^{\pm}$ are Kähler metrics with cohomologous Kähler forms, however they are Kählerian with respect to different complex structures. So the Calabi uniqueness (6.2.1) can not be applied directly.

6.3 Weitzenböck Formulas

Now we are going to show that the smooth manifold $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not admit any positive scalar curvature metric. For that purpose we state the Weitzenböck Formula for the Dirac Operator on spin manifolds. Before that we introduce some notation together with some ingredients of the formula.

The Levi-Civita connection is going to be the linear map we denote by $\nabla : \Gamma(E) \to \Gamma(Hom(TM, E))$ for any vector bundle E over a Riemannian Manifold M. Then we get the adjoint $\nabla^* : \Gamma(Hom(TM, E)) \to \Gamma(E)$ defined implicitly by

$$\int_M \langle \nabla^* S, s\rangle dV = \int_M \langle S, \nabla s\rangle dV$$

and we define the *connection Laplacian* of a section $s \in \Gamma(E)$ by their composition $\nabla^*\nabla s$. Notice that the harmonic sections are parallel for this operator. Using the metric, we can express its action as :

Proposition 6.3.1 ([Pet]p179)**.** *Let (M, g) be an oriented Riemannian mani-*

fold, $E \to M$ a vector bundle with an inner product and compatible connection. Then

$$\nabla^* \nabla s = -tr \nabla^2 s$$

for all compactly supported sections of E.

Proof. First we need to mention the second covariant derivatives and then the integral of the divergence. We set

$$\nabla^2 K(X,Y) = (\nabla \nabla K)(X,Y) = (\nabla_X \nabla K)(Y).$$

Then using the fact that ∇_X is a derivation commuting with every contraction: [KN]v1p124

$$\nabla_X \nabla_Y K = \nabla_X C(Y \otimes \nabla K) = C \nabla_X (Y \otimes \nabla K)$$

$$= C(\nabla_X Y \otimes \nabla K + Y \otimes \nabla_X \nabla K)$$

$$= \nabla_{\nabla_X Y} K + (\nabla_X \nabla K)(Y)$$

$$= \nabla_{\nabla_X Y} K + \nabla^2 K(X,Y)$$

for any tensor K. That is how the second covariant derivative is defined . Higher covariant derivatives are defined inductively.

For the divergence, remember that

$$(divX) dV_g = \mathcal{L}_X dV_g,$$

which is taken as a definition sometimes[KN]v1p281. After combining this with the Cartan's Formula: $\mathcal{L}_X dV = di_X dV + i_X d(dV) = di_X dV$; Stokes' Theorem yields that $\int_M (divX)dV = \int_M \mathcal{L}_X dV = \int_M d(i_X dV) = \int_{\partial M} i_X dV = 0$ for a compact manifold without boundary. This is actually valid even for a noncompact manifold together with a compactly supported vector field.

Now take an open set on M with an orthonormal basis $\{E_i\}_{i=1}^n$. Let s_1 and s_2 be two sections of E compactly supported on the open set. We reduce the left-hand side via multiplying by s_2 as follows:

$$(\nabla^*\nabla s_1, s_2)_{L^2} = \int_M \langle \nabla^*\nabla s_1, s_2\rangle dV = \int_M \langle \nabla s_1, \nabla s_2\rangle dV = \int_M tr((\nabla s_1)^* \nabla s_2) dV$$

$$= \sum_{i=1}^n \int_M \langle (\nabla s_1)^* \nabla s_2(E_i), E_i\rangle dV$$

$$= \sum \int_M \langle (\nabla s_1)^* \nabla_{E_i} s_2, E_i\rangle dV$$

$$= \sum \int_M \langle \nabla_{E_i} s_2, \nabla s_1(E_i)\rangle dV$$

$$= \sum \int_M \langle \nabla_{E_i} s_1, \nabla_{E_i} s_2\rangle dV.$$

Define a vector field X by $g(X,Y) = \langle \nabla_Y s_1, s_2\rangle$. Divergence of this vector field is

$$divX = -d^*(X^\flat) = tr\nabla X = \textstyle\sum_{i=1}^n \langle \nabla_{E_i} X, E_i\rangle = \sum (E_i \langle X, E_i\rangle - \langle X, \nabla_{E_i} E_i\rangle)$$

$$= \textstyle\sum (E_i \langle \nabla_{E_i} s_1, s_2\rangle - \langle \nabla_{\nabla_{E_i} E_i} s_1, s_2\rangle).$$

We know that its integral is zero, so our expression continues to evolve as

$$\sum \int_M \langle \nabla_{E_i} s_1, \nabla_{E_i} s_2\rangle dV - \int_M (divX) dV$$

$$= \sum \int_M \langle \nabla_{E_i} s_1, \nabla_{E_i} s_2 \rangle dV - \sum \int_M (E_i \langle \nabla_{E_i} s_1, s_2 \rangle - \langle \nabla_{\nabla_{E_i} E_i} s_1, s_2 \rangle) dV$$

$$= \sum \int_M (-\langle \nabla_{E_i} \nabla_{E_i} s_1, s_2 \rangle + \langle \nabla_{\nabla_{E_i} E_i} s_1, s_2 \rangle) dV$$

$$= \sum \int_M \langle -\nabla^2 s_1(E_i, E_i), s_2 \rangle dV$$

$$= - \int_M \langle \sum \langle \nabla^2 s_1(E_i), E_i \rangle, s_2 \rangle dV$$

$$= \int_M \langle -tr \nabla^2 s_1, s_2 \rangle dV$$

$$= (-tr \nabla^2 s_1, s_2)_{L^2}$$

So we established that $\nabla^* \nabla s_1 = -tr \nabla^2 s_1$ for compactly supported sections in an open set.

$\square$

Theorem 6.3.2 (Atiyah-Singer Index Theorem[LM]p256,[MoSW]p47)**.** *Let M be a compact spin manifold of dimension $n = 2m$. Then, the index of the Dirac operator is given by*

$$ind(\not{D}^+) = \hat{A}(M) = \hat{\boldsymbol{A}}(M)[M].$$

More generally, if E is any complex vector bundle over M, the index of $\not{D}_E^+ : \Gamma(\mathbb{S}_\pm \otimes E) \to \Gamma(\mathbb{S}_\mp \otimes E)$ is given by

$$ind(\not{D}_E^+) = \{ch(E) \cdot \hat{\boldsymbol{A}}(M)\}[M].$$

For $n=4$, $\hat{\boldsymbol{A}}(M) = 1 - p_1/24$ and the first formula reduces to

$$ind(\not{D}^+) = \hat{A}(M) = \int_M -\frac{p_1(M)}{24} = -\frac{\tau(M)}{8}$$

by the Hirzebruch Signature Theorem.

Let us explain the ingredients beginning with the cohomology class $\hat{\mathbf{A}}(M)$. Consider the power series of the following function[Fr]p108 :

$$\frac{t/2}{\sinh t/2} = \frac{t}{e^{t/2} - e^{-t/2}} = 1 + A_2 t^2 + A_4 t^4 + \dots$$

where we compute the coefficients as

$$A_2 = -\frac{1}{24} \quad , \quad A_4 = \frac{7}{10 \cdot 24 \cdot 24} = \frac{7}{5760}.$$

Consider the Pontrjagin classes $p_1 ... p_k$ of M^{4k}. Represent these as the elementary symmetric functions in the squares of the formal variables $x_1 \cdots x_k$:

$$x_1^2 + \cdots + x_k^2 = p_1 \ , \quad \cdots \quad , \ x_1^2 x_2^2 \cdots x_k^2 = p_k$$

Then $\prod_{i=1}^k \frac{x_i}{e^{x_i/2} - e^{-x_i/2}}$ is a symmetric power series in the variables $x_1^2 \cdots x_k^2$, hence defines a polynomial in the Pontrjagin classes. We call this polynomial $\hat{\mathbf{A}}(M)$

$$\hat{\mathbf{A}}(M) = \prod_{i=1}^k \frac{x_i/2}{\sinh x_i/2}.$$

In lower dimensions we have

$$\hat{\mathbf{A}}(M^4) = 1 - \frac{1}{24}p_1 \quad , \quad \hat{\mathbf{A}}(M^8) = 1 - \frac{1}{24}p_1 + \frac{7}{5760}p_1^2 - \frac{1}{1740}p_2.$$

If the manifold has dimension $n = 4k + 2$, again it has k Pontrjagin classes, and we define the polynomial $\hat{\mathbf{A}}(M^{4k+2})$ by the same formulas.

Secondly, we know that $\not{D}^+ : \Gamma(\mathbb{S}_+) \to \Gamma(\mathbb{S}_-)$ is an elliptic operator, so its kernel is finite dimensional and its image is a closed subspace of finite codimension. The *index* of an elliptic operator is defined to be *dimkernel* $-$ *dimcokernel*. Actually in our case $\not{D}^+$ and $\not{D}^-$ are formal adjoints of each other: $(\not{D}^+\psi, \eta)_{L^2} = (\psi, \not{D}^-\eta)_{L^2}$ for ψ, η compactly supported sections [LM]p114 , [MoSW]p42. Consequently the index becomes $dimker\not{D}^+ - dimker\not{D}^-$. This index is computed from the symbol in the following way. Consider the pullback of $\mathbb{S}_\pm$ to the cotangent bundle T^*M. The symbol induces a bundle isomorphism between these bundles over the complement of the zero section of T^*M. In this way the symbol provides an element in the relative K-theory of $(T^*M, T^*M - M)$. The Atiyah-Singer Index Theorem computes the index from this element in the relative K-theory. In the case of the Dirac operator the index is $\hat{A}(M)$, the so-called *A-hat genus* of M.

Now we are ready to state our main tool

Theorem 6.3.3 (Weitzenböck Formula[Pet]p183,[Besse]p55)**.** *On a spin Riemannian manifold, consider the Dirac operator* $\not{D} : \Gamma(\mathbb{S}_\pm) \to \Gamma(\mathbb{S}_\mp)$*. The Dirac Laplacian can be expressed in terms of the connection/rough Laplacian as*

$$\not{D}^2 = \nabla^*\nabla + \frac{s}{4}$$

where ∇ is the Riemannian connection.

Finally we state and prove our main result :

Theorem 6.3.4. *The smooth manifold $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ does not admit any metric of positive scalar curvature(PSC).*

Proof. If $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ admits a metric of PSC then $K3$ is also going to admit such a metric because one pulls back the metric of the quotient, and obtain a locally identical metric on which the PSC survives.

So we are going to show that the $K3$ surface does not admit any metric of PSC. First of all the canonical bundle of $K3$ is trivial so that $c_1(K3) = 0 = w_2(K3)$ implying that it is a spin manifold.
By the Atiyah-Singer Index Theorem (6.3.2),

$$ind\not{D}^+ = \hat{\mathbf{A}}(M)[M] = -\frac{\tau(M)}{8} = 2$$

for the $K3$ Surface. Since it is equal to $dimker - dimcoker$, this implies that the $dimker\not{D}^+ \geq 2$.

Let $\psi \in ker\not{D}^+ \subset \Gamma(\mathbb{S}_+)$ and consider its image $(\psi, 0)$ in $\Gamma(\mathbb{S}_+ \oplus \mathbb{S}_-)$. Then $\not{D}^2(\psi, 0) = 0$ since $\not{D} = \not{D}^+ \oplus \not{D}^-$. Abusing the notation as $\psi = (\psi, 0)$ the spin Weitzenböck Formula (6.3.3) implies

$$0 = \nabla^*\nabla\psi + \frac{s}{4}\psi.$$

Taking the inner product with ψ and integrating over the manifold yields

$$0 = (\nabla^*\nabla\psi, \psi)_{L^2} + (\frac{s}{4}\psi, \psi)_{L^2} = (\nabla\psi, \nabla\psi)_{L^2} + \frac{s}{4}(\psi, \psi)_{L^2} = \int_M (|\nabla\psi|^2 + \frac{s}{4}|\psi|^2) dV$$

and $s > 0$ implies that $|\nabla\psi| = |\psi| = 0$ everywhere, hence $\psi \equiv 0$. So $ker \not{D}^+ = 0$, which is not the case.

Notice that $s \geq 0$ and $s(p) > 0$ for some point is also enough for the conclusion because then ψ would be parallel and zero at some point implies ψ is zero everywhere. $\square$

Remark 6.3.5. *In the above proof, while taking* $\psi \in ker \not{D}^+$ *some confusion may arise if* $ker \not{D}^+ \subset \Gamma(\mathbb{S}_+)$ *is not specified. A reader might think that* $\not{D}^+$ *acts on* $\Gamma(\mathbb{S}_+ \oplus \mathbb{S}_-)$ *and* ψ *is equal to something like* (ψ, η) *for some nonzero* η*, so that* $\not{D}^2\psi = \not{D}^+\not{D}^-\psi$*.*

Alternatively, we could use the Weitzenböck Formula for the Hodge/modern Laplacian to show that there are no PSC anti-self-dual(ASD) metrics on $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$. This is a weaker conclusion though sufficient for our purposes

Theorem 6.3.6 (Weitzenböck Formula 2[LeOM])**.** *On a Riemannian manifold, the Hodge/modern Laplacian can be expressed in terms of the connection/rough Laplacian as*

$$(d + d^*)^2 = \nabla^*\nabla - 2W + \frac{s}{3}$$

where ∇ *is the Riemannian connection and* W *is the Weyl curvature tensor.*

Theorem 6.3.7. *The smooth manifold* $K3/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *does not admit any anti-self-dual(ASD) metric of positive scalar curvature(PSC).*

Proof. Again we are going to show this only for $K3$ as in (6.3.4). Suppose we have a metric of positive scalar curvature.

Anti-self-duality reduces our Weitzenbock Formula (6.3.6) to the form

$$(d+d^*)^2 = \nabla^*\nabla - 2W_- + \frac{s}{3}$$

because $W = W_-$ or $W_+ = 0$.

We have already explained in (6.2.2) and in Appendix (6.5) that b_2^+ of the $K3$ surface is nonzero. So take a nontrivial harmonic self-dual 2-form φ. $W_- : \Gamma(\Lambda^-) \to \Gamma(\Lambda^-)$ only acts on anti-self-dual forms, so it takes φ to zero. Applying the formula

$$0 = \nabla^*\nabla\varphi + \frac{s}{3}\varphi,$$

and taking the inner product with φ and integrating over the manifold yields similarly

$$0 = (\nabla^*\nabla\varphi, \varphi)_{L^2} + (\frac{s}{3}\varphi, \varphi)_{L^2} = (\nabla\varphi, \nabla\varphi)_{L^2} + \frac{s}{3}(\varphi, \varphi)_{L^2} = \int_M (|\nabla\varphi|^2 + \frac{s}{3}|\varphi|^2) dV$$

and $s > 0$ implies that $|\nabla\varphi| = |\varphi| = 0$ everywhere, hence $\varphi \equiv 0$, a contradiction. $\square$

6.4 Other Examples

In this section, we will go through some examples. We begin with the case $b^+ = 1$.

Theorem 6.4.1 ([KimLePon],[RS-SFK]). *For all integers $k \geq 10$, the con-*

nected sum

$\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$ *admits scalar-flat-Kähler(SFK) metrics.*[2]

The case $k \geq 14$ is achieved in [KimLePon]. They start with blow ups of $\mathbb{CP}_1 \times \Sigma_2$ the cartesian product of rational curve and genus-2 curve, which already have a SFK metric via the hyperbolic ansatz of [LeExp]. After applying an isometric involution, they get a SFK orbifold, which has isolated singularities modelled on $\mathbb{C}^2/\mathbb{Z}_2$. Replacing these singular models with smooth ones, they obtain the desired metric.

For the case $k \geq 10$, Rollin and Singer first construct a related SFK orbifold with isolated and cyclic singularities of which the algebra $\mathfrak{a}_0$ of non-parallel holomorphic vector fields is zero. This is done by an argument analogous to that of [Burns-Bart]. The target manifold is the minimal resolution of this orbifold. To obtain the target metric, they glue some suitable local models of SFK metrics to the orbifold. These models are asymptotically locally Euclidean(ALE) scalar flat Kähler metrics constructed in [Cal-Sing].

When a metric is Kähler, from the decomposition of the Riemann Curvature operator, scalar-flatness turns out to be equivalent to being anti-self-dual. So these metrics are SF-ASD.

Since these manifolds have $b^+ \neq 0$ Weitzenböck Formulas apply as in section §6.3, so automatically the scalar curvature can not change sign. These examples show why the case $b^+ = 0$ we focussed on, is interesting.

A second type of example is

[2]It is a curious fact that $k = 10$ is the minimal number for these type of metrics(SF-ASD) on $\mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2$, known by [LeSD] long before these constructions made. See [LeOM] for a survey.

Example 6.4.2. *Let Σ_g be the genus-$g(>1)$ surface with Kähler metric of constant curvature $\kappa = -1$, and S^2 be the 2-sphere with the round $\kappa = +1$ metric. Consider the product metric on $S^2 \times \Sigma_g$ which is Kähler with zero scalar curvature. So this metric is anti-self-dual. Then we have fixed point free, orientation reversing, isometric involutions of both surfaces obtained by antipodal maps. Combination of these involutions yield an isometric involution on the product and the metric pushes down to a metric on*

$$(S^2 \times \Sigma_g)/\mathbb{Z}_2 = \mathbb{R}P^2 \times (\underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{(g+1)-many})$$

which is SF-ASD as these properties are preserved under isometry. This is an example with all the key properties ($b^+ = 0$)where the metric is completely explicit. Note that this manifold is non-orientable.

One has to be careful about the involution on Σ_g though. There are many hyperbolic metrics on Σ_g which do not have an isometric involution satisfying our conditions. Involution must be conformal. One way to achieve this is as follows. We take a conformal structure on the $(g+1)\mathbb{R}P^2$, and pull this back to its orientable double cover Σ_g. By the uniformization theorem of Riemann Surfaces, there is a unique hyperbolic($\kappa = -1$) metric of Σ_g in this conformal class. Since this metric is unique in its conformal class, it is automatically invariant under the involution and pushes down to a hyperbolic metric on $(g+1)\mathbb{R}P^2$. It is known that the moduli space of hyperbolic metrics on Σ_g is $6g-6$ real dimensional, on the other hand $3g-3$ real dimensional on $(g+1)\mathbb{R}P^2$. So it is appearent that there are many hyperbolic metrics on Σ_g which are not coming from the quotient. So that they do not have the isometric *involution of the kind we use.*

Another way to construct this involution can be to begin with a surface in $\mathbb{R}^3$ *which is symmetric about the origin, e.g. add symmetric handles to a sphere or a torus about the origin. Then take the conformal structure induced from Euclidean* $\mathbb{R}^3$*. There is a unique hyperbolic metric that induces this conformal structure, so proceed as before.*

Remark 6.4.3. *The other side of the story discussed here is that we have ASD, conformally flat deformations to negative scalar curvature metrics e.g. on* $M = \Sigma_g \times S^2$*. It is obtained by deforming*

$$\rho : \pi_1(M) \longrightarrow SL(2, \mathbb{H}),$$

the representation of the fundamental group in $SL(2, \mathbb{H})$*. This is the group of conformal transformations of* S^4*. It contains the isometry group* $SL(2, \mathbb{R})$ *due to the fact that* $\mathcal{H}^2 \times S^2$ *is conformally flat and conformally diffeomorphic to* $S^4 - S^1$*. This is the universal cover of* $\Sigma_g \times S^2$*, and its fundamental group acts by conformal transformations in* $SL(2, \mathbb{R}) \subset SL(2, \mathbb{H})$*.* $\pi_1(M) = \pi_1(\Sigma_g)$ *is generated by* $2g$ *elements* $\{a_1, b_1 \cdots a_g, b_g\}$ *and these are subject to the single relation* $\prod[a_j, b_j] = 1$ *the product of the commutators. So a representation in* $SL(2, \mathbb{H})$ *corresponds to a choice of* $2g$ *elements and a relation. Since this Lie group is* 15 *dimensional and we have to subtract the change of basis conjugation and the relations , this kind of representations depend on* $15 \times 2g - 15 \times 1 - 15 \times 1 = 30g - 30$ *parameters. On the other hand, a twisted product metric on* $\Sigma_g \times S^2$ *provides a representation in* $SL(2, \mathbb{R}) \times SO(3)$*, which is a* $3 + 3 = 6$ *dimensional Lie group. So we have* $6 \times 2g - 6 \times 1 - 6 \times 1 = 12g - 12$ *parameters* *for this representation. This difference means that the generic conformally flat structure on M does not come from a twisted product metric. We refer [Pon] for further details.*

Using the Weitzenböck Formula (6.3.6), LeBrun [LeSD] shows that a conformally flat metric on M of zero scalar curvature must be Kähler with respect to both orientations, and by a holonomy argument he further shows that the metric is of twisted product type.[3]

Similar parameter counts are valid for $M/\mathbb{Z}_2$ and this shows that the generic conformally flat metric on this manifold has negative Yamabe constant.

Also, by further investigation, it might be possible to get examples which are doubly covered by e.g. the simply connected examples of [KimLePon].

6.5 b^+ of the $K3$ Surface

In this appendix we are going to prove that the b^+ i.e. b_2^+ of the $K3$ Surface is nonzero in a fancy way.

Proposition 6.5.1 (Wu Formula with integers[GS]p30)**.** *Let M be an oriented 4-manifold and $\alpha \in H_2(M, \mathbb{Z})$. Then we have*

$$Q_M(\alpha, \alpha) \equiv \langle w_2(M), \alpha \rangle \quad (mod\ 2)$$

That is, the self intersection number may be computed modulo 2 by multiplication with the second Steifel-Whitney class.

[3] Thanks to C.LeBrun for this discussion.

Remark 6.5.2. *Here in the product $\langle w_2(M), \alpha\rangle$, α is taken to be the mod 2 reduction of its integral homology class.*

Proof. As for any of them, $\alpha \in H_2(M, \mathbb{Z})$ may be represented by an embedded oriented surface $\Sigma \subset M$. Then

$$\langle w_2(M), \alpha\rangle = w_2(TM|_\Sigma)[\Sigma] = w_2(T\Sigma \oplus N\Sigma)[\Sigma] = w_2(T\Sigma)[\Sigma] + w_1(T\Sigma) \smile w_1(N\Sigma)[\Sigma]$$

$$+w_2(N\Sigma)[\Sigma] = w_2(N\Sigma)[\Sigma] \equiv e(N\Sigma)[\Sigma] = Q_M([\Sigma], [\Sigma]) = \alpha^2$$

where $w_2(T\Sigma)[\Sigma] \equiv e(T\Sigma)[\Sigma] \equiv \chi(\Sigma) \equiv 2 - 2g \equiv 0 \ (mod\ 2)$, $w_1[\Sigma] = 0$ because of orientability.

□

Theorem 6.5.3 (Donaldson Theorem[GS]p16)**.** *If the intersection form Q_M of a smooth, simply connected, closed 4-manifold is negative definite, then it is equivalent to $n\langle -1\rangle$.*

Theorem 6.5.4 (Freedman Theorem[GS]p15)**.** *For any unimodular symmetric bilinear form Q, there exist a simply connected, closed, topological 4-manifold with an equivalent intersection form. If Q is even, this manifold is unique up to homeomorphism. If Q is odd, the manifold is not unique, there are exactly two different homeomorphism types with the given intersection form, at most one of which carries a smooth structure.*
Consequently, simply connected, smooth 4-manifolds are determined up to homeomorphism by their intersection forms.

Now we are ready to prove

Proposition 6.5.5. *b^+ of the $K3$ Surface is nonzero*

Proof. If $K3$ has a negative definite intersection form, then by the Donaldson's Theorem(6.5.3) the intersecton form $Q_M \approx n\langle -1\rangle$ for such smooth , simply connected and closed 4-manifolds. And by the Wu's formula with integer coefficients, $Q_{K3}(\alpha, \alpha) = \langle w_2(K3), \alpha\rangle = \langle 0, \alpha\rangle = 0 \pmod 2$, for oriented 4-manifolds and $\alpha \in H_2(K3, \mathbb{Z})$, implying that the intersection form is even. Then $n = 0$ and Q_M equals to the zero matrix. Since the simply connected smooth 4-manifolds are determined by their intersection forms upto homeomorphism via the Freedman's Theorem (6.5.4), $K3$ supposed to be homeomorphic to S^4, which is not the situation. Hence the intersection form is not negative definite.

$\square$

Chapter 7

Geometric Invariant Theory and Einstein-Weyl Geometry

We saw that the central theme in Twistor theory is the correspondence between certain complex 3-folds and self-dual Riemannian 4-manifolds. Similar to this, we have another correspondence called the Hitchin's correspondence. In the following sections, we will describe this new pair of manifolds following [JT], and address a problem involved.

In §7.1-7.2 we review the basics of Einstein-Weyl geometry and minitwistorspaces. In §7.3 and §7.4 we give a review of GIT and Toric Varieties, finally in §7.5 we present our application.

7.1 Hitchin Correspondence

In this section we will describe the *Hitchin's Correspondence*, which tells us the following. For a Riemannian Manifold M^n the geodesics in a geodesically convex set are parametrized by a $2n-2$ dimensional manifold. We follow [MonGeo82], see also [CxEin82].

The tangent space is by definition, the paths of geodesics, in our case the variation of geodesics through geodesics. The variational vector fields in this case are the *Jacobi fields*. By definition they are the solutions of the Jacobi equation [dC]

$$\frac{D^2V}{dt^2} + R(\dot\gamma(t), V(t))\dot\gamma(t) = 0$$

where $DV/dt = \nabla_{\dot\gamma} V$ is the covariant derivative of V along the curve γ. For a geodesic variation $\gamma(t,s)$ of γ, so that $\gamma(t,0) = \gamma(t)$, the variational vector field $\partial\gamma/\partial s|_{s=0}$ is characterized by this equation. Now take an orhonormal frame at some point of γ, and extend it to the parallel, orthonormal fields $e_1(t)\cdots e_n(t)$ on the geodesic by parallel transportation. Write the vector field as $V = \sum v^i(t)e_i(t)$, the covariant differentiation becomes

$$\frac{DV}{dt} = \nabla_{\dot\gamma} v^i(t)e_i(t) = \dot\gamma(v^i(t))e_i(t) + v^i(t)\nabla_{\dot\gamma} e_i(t) = \dot v^i(t)e_i(t)$$

just the usual differentiation using the parallelism of e_i's. Denoting

$$a_{ij} = \langle R(\dot\gamma(t), e_i(t))\dot\gamma(t), e_j(t)\rangle,$$

we have the expression

$$R(\dot\gamma, V)\dot\gamma = \sum_j \langle R(\dot\gamma, V)\dot\gamma, e_j\rangle e_j = \sum_j v^i \langle R(\dot\gamma, e_i)\dot\gamma, e_j\rangle e_j = \sum_j v^i a_{ij} e_j.$$

Plugging these into the Jacobi equation, we obtain the following linear system of n-equations of second order

$$\ddot{v}^j + \sum_i a_{ij} v^i = 0 \quad \text{for} \quad j = 1 \cdots n.$$

Given the initial conditions $V(t_0)$ and $\frac{DV}{dt}(t_0)$, we have a unique C^∞ solution to the system. Each initial condition has n-degrees of freedom. Therefore we have a $2n$-dimensional solution space for the Jacobi equation. In this solution space, the vector fields $\dot{\gamma}$ and $t\dot{\gamma}$ are also included. $\dot{\gamma}$ trivially satisfies the equation because of self-parallelism. To see that $t\dot{\gamma}$ is a solution, we take $e_1 = \dot{\gamma}$, then we have

$$v^1 = t, \quad v^2 \cdots v^n = 0,$$

also

$$a_{11} = \langle R(\dot{\gamma}, t\dot{\gamma})\dot{\gamma}, t\dot{\gamma}\rangle = \langle \nabla_{t\dot{\gamma}} \nabla_{\dot{\gamma}} \dot{\gamma} - \nabla_{\dot{\gamma}} \nabla_{t\dot{\gamma}} \dot{\gamma} + \nabla_{[\dot{\gamma}, t\dot{\gamma}]} \dot{\gamma}, t\dot{\gamma}\rangle = 0$$

since all three terms are zero because of self-parallelism, for instance the third one:

$$\nabla_{[\dot{\gamma}, t\dot{\gamma}]} \dot{\gamma} = \nabla_{(\dot{\gamma}t)\dot{\gamma}} \dot{\gamma} = \dot{\gamma}t \nabla_{\dot{\gamma}} \dot{\gamma} = 0.$$

This trick will not work for $t^2\dot{\gamma}$ for the obvious reason that $\ddot{v}^1 = 2$. The variation of the geodesic on itself is clearly a kind of variation through geodesics. So the possible variational fields are of the form $v^1(t)\dot{\gamma}(t)$. The computations above show us that $a_{ij} = 0$, and the equation reduces to $\ddot{v}^1 = 0$. So that $\dot{\gamma}, t\dot{\gamma}$ are the generators of the self-variations. Taking the self-variations out, we obtain $2n-2$ dimensional space of nontrivial geodesic variations. The map

$$V \mapsto V - \langle V, \dot{\gamma} \rangle \, \dot{\gamma}$$

is an isomorphism from the tangent space to the space of geodesics at γ to the space of Jacobi fields orthogonal to γ.

If we let $n = 3$ from this point on, we have a 4-dimensional space of oriented geodesics, say G. We fix a geodesic γ with a specified direction. Then taking the advantage of 3-dimensions we can define the cross product and consequently we can define a linear map

$$\mathrm{J} : T_\gamma G \to T_\gamma G \quad , \quad \mathrm{J}(V) = \dot{\gamma} \times V$$

where we take the orthogonal Jacobi fields as the tangent space and assume that $\dot{\gamma} \times V$ is another Jacobi field. Taking the cross product yields a vector field orthogonal to $\dot{\gamma}$. To make it a Jacobi field we can make the following assumption

$$R(\dot{\gamma}, \dot{\gamma} \times V)\dot{\gamma} = \dot{\gamma} \times R(\dot{\gamma}, V)\dot{\gamma} \tag{7.1}$$

since then the Jacobi equation is satisfied as follows

$$\nabla^2_{\dot{\gamma}}(\dot{\gamma} \times V) + R(\dot{\gamma}, \dot{\gamma} \times V)\dot{\gamma} = \dot{\gamma} \times \nabla^2_{\dot{\gamma}} V + R(\dot{\gamma}, \dot{\gamma} \times V)\dot{\gamma} = \dot{\gamma} \times (\nabla^2_{\dot{\gamma}} V + R(\dot{\gamma}, V)\dot{\gamma}) = 0$$

because $\dot{\gamma}$ is constant along the geodesic and V is a Jacobi field. This map satisfies

$$\mathrm{J}^2(V) = \dot{\gamma} \times (\dot{\gamma} \times V) = \langle \dot{\gamma}, V \rangle \dot{\gamma} - \langle \dot{\gamma}, \dot{\gamma} \rangle V = 0 \cdot \dot{\gamma} - 1 \cdot V = -V,$$

because of the basic property that relates the cross product to the inner product. So taking the cross product with $\dot{\gamma}$ defines an almost complex structure on the 4-dimensional space of geodesics.

Hitchin claims in [CxEin82, MonGeo82] that the curvature condition (7.1) producing Jacobi fields by the cross product is satisfied for a Riemannian manifold if the traceless Ricci curvature tensor

$$\overset{\circ}{Ric} := Ric - \frac{s}{n} g$$

identically vanishes. In this case the metric is Einstein which amounts to being constant sectional curvature in 3-dimension. Moreover the almost complex structure is integrable. So that G, the space of geodesics is a complex surface.

7.2 Einstein-Weyl Geometry

Let (M,g) be a self-dual Riemannian 4-manifold. Suppose it admits a free isometric circle(S^1) action. Then the quotient manifold M/S^1 is naturally equipped with a so-called *Einstein-Weyl Geometry*. That is to say we have a triple $(M/S^1, [h], D)$ where $[h]$ is a conformal class, here for the induced metric of the quotient, and D is a torsion-free affine connection. The conditions

$Ric_{(ij)} = \lambda h_{ij}$ (Einstein-like)

more precisely $Ric(u, v) + Ric(v, u) = 2\lambda h(u, v)$, and

$Dh = \alpha \otimes h$ for some 1-form α (Weyl Connection)

are to be satisfied. More interestingly, this action can naturally be extended to a holomorphic $\mathbb{C}^*$-action over the twistor space. And we call the corresponding quotient $Z/\mathbb{C}^*$ as the *Minitwistorspace* of the self-dual manifold. If the twistor space is algebraic or Moishezon this quotient space becomes a complex surface with singularities in general.

So, whenever one has a self-dual metric with an isometric circle action, it is a very natural question to ask what is the minitwistor space.

In the march of 2004, Honda gave an explicit description for the twistor space of some self-dual metrics on $3\mathbb{CP}_2$ admitting a free isometric circle action, equivalently a nowhere zero Killing Field as follows :

Theorem 7.2.1 (Nobuhiro Honda,[Ho04]). *Let g be a self-dual metric on $3\mathbb{CP}_2$ which admits a non-trivial Killing Field. Suppose further that it is of positive scalar curvature type, and not conformally equivalent to the explicit self-dual metrics constructed by LeBrun's hyperbolic ansatz[LeExp].*
Then the twistor space is bimeromorphic to(or small resolution of) the double cover of $\mathbb{CP}_3$ branched along a quartic whose equation according to some homogeneous coordinates is given by

$$(Z_2 Z_3 + Q(Z_0, Z_1))^2 - Z_0 Z_1 (Z_0 + Z_1)(Z_0 - a Z_1) = 0$$

where $Q(Z_0, Z_1)$ is a quadratic form of Z_0 and Z_1 with real coefficients, and $a \in \mathbb{R}^+$
moreover, the naturally induced real structure on $\mathbb{CP}_3$ is given by

$$\sigma(Z_0 : Z_1 : Z_2 : Z_3) = \left(\bar{Z}_0 : \bar{Z}_1 : \bar{Z}_3 : \bar{Z}_2\right),$$

and the naturally induced $U(1)$-action on $\mathbb{CP}_3$ is given by

$$(Z_0 : Z_1 : Z_2 : Z_3) \mapsto \big(Z_0 : Z_1 : e^{i\theta} Z_2 : e^{-i\theta} Z_3\big) \quad for \quad e^{i\theta} \in U(1).$$

To construct the minitwistor space of the Honda metrics, we appeal to the Geometric Invariant Theory(GIT) for Toric Varieties. This celebrated theory is developed by D. Mumford around 1970's to understand the quotients of group actions on manifolds.

We will try to compute the image under the double branched cover, so that we could be able to recover the original minitwistor space by taking a double cover along the related branch locus. GIT computes the quotients according to some linearizations. It takes out some bad orbits(unstable) and give a toric variety as a result.

7.3 Action of a torus on an affine space

In this section we will analyze the actions of the group $T = \mathbb{C}^{*r}$ on the affine space $\mathbb{C}^n$ and understand the quotients arisen this way. We call $T = \mathbb{C}^{*r} = (\mathbb{C}^*)^r$ as an *algebraic torus* for each positive integer r. We begin by

Proposition 7.3.1 ([Do]p73,[Mu]p119)**.** *Any character*

$$\chi : T \longrightarrow \mathbb{C}^*$$

is given by

$$\chi(t) = \chi(t_1 \cdots t_r) = t_1^{a_1} t_2^{a_2} \cdots t_r^{a_r} = \prod_{i=1}^{r} t_i^{a_i}$$

for $t_i \in \mathbb{C}$ and $a_i \in \mathbb{Z}$. So we have the isomorphism $\chi(T) \approx \mathbb{Z}^r$.

Recall that a character χ of a group with values in a field is a homomorphism from the group to the multiplicative group, i.e. satisfying $\chi(gh) = \chi(g)\chi(h)$. Moreover $\chi(T)$ stands for the group of characters of T.

Consequently, after diagonalizing, a T action on $\mathbb{C}^n$ is written as

$$t \cdot \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} \chi_1(t)Z_1 \\ \vdots \\ \chi_n(t)Z_n \end{pmatrix} = \begin{pmatrix} t^{a_1}Z_1 \\ \vdots \\ t^{a_n}Z_n \end{pmatrix} = \begin{pmatrix} t_1^{a_{11}} \cdots t_r^{a_{r1}} Z_1 \\ \vdots \\ t_1^{a_{1n}} \cdots t_r^{a_{rn}} Z_n \end{pmatrix}.$$

So the matrix $A = [a_{ij}] \in M_{r\times n}(\mathbb{Z})$ encodes the action.

More generally, let T be a group acting on the complex manifold X by the map $\sigma : T \times X \to X$. For a holomorphic line bundle $\pi : L \to X$, we define

Definition 7.3.2. *A* linearization *of L with respect to the action of T is an action $\overline{\sigma} : T \times L \to L$ satisfying*

(1) The following diagram commutes

$$\begin{array}{ccc} T \times L & \xrightarrow{\overline{\sigma}} & L \\ {\scriptstyle id\times\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ T \times X & \xrightarrow{\sigma} & X \end{array}$$

(2) The zero section $X \approx L_0 \subset L$ is T- invariant.

So this is the extension of the action σ to L, preserving the fibers, i.e. all point on a fiber maps onto the same fiber under the action of an element. It follows from the definition that this action on a fiber $\overline{\sigma}_t : L_p \to L_{tp}$ for any $t \in T$ and any $p \in X$ is a linear isomorphism.

In our case, the action of $\mathbb{C}^{r*}$ on $\mathbb{C}^n$ is given by the matrix $A = (a_1 \cdots a_r) \in M_{n\times r}(\mathbb{Z})$. Consider the trivial line bundle $\underline{\mathbb{C}} \to \mathbb{C}^{n*}$. Fix $\alpha = (\alpha_1 \cdots \alpha_r) \in \mathbb{Z}^r$. Extend the action over to the bundle $\underline{\mathbb{C}}$ as

$$t \cdot (Z, W) = (t \cdot Z, t^\alpha W) = (t \cdot Z, t_1^{\alpha_1} t_2^{\alpha_2} \cdots t_r^{\alpha_r} W) \quad \text{where} \quad Z \in \mathbb{C}^n, W \in \mathbb{C}$$

We denote this linearized line bundle by L_α. So any $a \in \mathbb{Z}^r$ gives an extension or a linearization.

Recall that the sections of the trivial line bundle are identified with the polynomials $F \in \mathbb{C}[Z_1 \cdots Z_n]$, like the homogenous polynomials for bundles over $\mathbb{P}^n$. A section F is an invariant section of L_α if

$$t \cdot (Z, F(Z)) = (t \cdot Z, t^\alpha \cdot F(Z)) = (t \cdot Z, F(t \cdot Z))$$

$$\text{i.e. } \ t^\alpha \cdot F(Z) = F(t \cdot Z)$$

$$t_1^{\alpha_1} \cdots t_r^{\alpha_r} F(Z_1 \cdots Z_r) = F(t^{a_1} Z_1 \cdots t^{a_r} Z_r)$$

The action of $\overline{\sigma}$ on L induces an action on $L^{\otimes d}$ as for a decomposable $l \in L_p$, $\ \overline{\sigma}_t(l) = \overline{\sigma}_t(l_1 \otimes \cdots \otimes l_d) = \overline{\sigma}_t(l_1) \otimes \cdots \otimes \overline{\sigma}_t(l_d) \in L_{tp}$

Likewise, G is an invariant section of $L_\alpha^{\otimes d}$ if for $G = F_1 \cdots F_d$

$$G(t \cdot Z) = F_1(t \cdot Z) \cdots F_d(t \cdot Z) = (t^\alpha \cdot F_1) \cdots (t^\alpha \cdot F_d)$$

$$= t^{\alpha d} \cdot F_1 \cdots F_d = t^{\alpha d} \cdot G(Z)$$

Proposition 7.3.3. *$G \in H^0(\mathbb{C}^n, L_\alpha^{\otimes d})^T$ i.e. G is an invariant section of the linearized line bundle $L_\alpha^{\otimes d}$ iff it is a linear combination of monomials $Z^m (= Z_1^{m_1} \cdots Z_n^{m_n})$*

such that

$$[A \ , -\alpha \ ] \begin{bmatrix} m \\ d \end{bmatrix} = \Theta$$

where $A \in M_{r\times n}(\mathbb{Z})$ is the action matrix, $\alpha \in \mathbb{Z}^r$ is the tuple for the extension

Proof. Say $G = Z^m$, then

$$\begin{aligned}
G(t \cdot Z) &= t^{\alpha d} \cdot G(Z) \\
G(t^{a_1} Z_1 \cdots t^{a_n} Z_n) &= (t_1^{\alpha_1} \cdots t_n^{\alpha_n})^d Z^m \\
(t^{a_1} Z_1)^{m_1} \cdots (t^{a_n} Z_n)^{m_n} &= \quad '' \\
t^{a_1 m_1} \cdots t^{a_n m_n} Z^m &= \quad '' \\
(t_1^{a_{11}} \cdot\cdot t_r^{a_{r1}})^{m_1} \cdots (t_1^{a_{1n}} \cdot\cdot t_r^{a_{rn}})^{m_n} Z^m &= \quad ''
\end{aligned}$$

comparing the powers of t_i's from both sides we obtain the equality

$$\begin{aligned}
a_{i1} m_1 + \cdots + a_{in} m_n &= \alpha_i d \\
[a_{i1} \cdots a_{in}] \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} &= [\alpha_i] d \quad \text{for any} \quad 1 \le i \le r \\
Am &= \alpha d
\end{aligned}$$

□

Example 7.3.4. *Consider the following action of* $\mathbb{C}^{*2}$ *on* $\mathbb{C}^n$ *:*

$$(t_1, t_2) \cdot \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} = \begin{pmatrix} t_1 X \\ t_1^{-n} t_2 Y \\ t_1 Z \\ t_2 W \end{pmatrix} \quad , \quad \alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The action matrix is $A = \begin{bmatrix} 1 & -n & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ *and the monomials for the invariant sections are obtained from the equation*

$$\left[\begin{array}{cccc|c} 1 & -n & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 \end{array}\right] \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ d \end{bmatrix} = \Theta.$$

Next we are going to give some definitions in the Geometric Invariant Theory(GIT). GIT deals with the actions of groups on manifolds, and figuring out their corresponding quotients.

Definition 7.3.5 (Stability[Do]p115)**.** *Let* L *be a* T*-linearized line bundle on the algebraic variety* X *and let* $x \in X$
(i) x *is called* semi-stable *with respect to* L *if it belongs to the set* $X_s = \{y \in X : s(y) \neq 0\} = X \backslash \{s = 0\} \subset \mathbb{C}^n$ *(affine) for some* $m > 0$ *and* $s \in H^0(X, L^m)^T$.
(ii) x *is called* unstable *with respect to* L *if it is not semi-stable.*

$X^{ss}(L)$ and $X^{us}(L)$ denotes respectively the set of semi-stable and unstable points in X.

Definition 7.3.6 (Categorical Quotient[Do]p92)**.** *A* categorical quotient *of a T-variety X is a T-invariant morphism $p : X \to Y$ such that for any T-invariant morphism $g : X \to Z$, there exist a unique morphism $\bar{g} : Y \to Z$ satisfying $\bar{g} \circ p = g$. Y is written sometimes as $X /\!\!/ T$ and also called the* categorical quotient *by the abuse of terminology.*

The GIT guarantees a (good) categorical quotient $X^{ss}(L_\alpha)/T$, see [Do]p118, denoted alternatively by $X(L)/\!\!/_\alpha T$. This is the quotient obtained by taking out the unstable orbits. So according to the GIT, semi-stable points has this well behaving quotient computable as follows.

Proposition 7.3.7 ([Do]p120)**.** *If X is projective and L is ample, we can compute the categorical quotient by*

$$X(L)/\!\!/_\alpha T = Proj\left(\bigoplus_{d \geq 0} \Gamma(X, L_\alpha^{\otimes d})^T\right).$$

7.4 Toric Varieties

We begin by the following definition

Definition 7.4.1. *Let $V \subset \mathbb{C}^n$ be an affine variety. We define its* Coordinate Ring *to be*

$$\mathbb{C}[V] = \mathbb{C}[z_1 \cdots z_n]|_V.$$

This is to say the coordinate ring is the ring of *regular* functions according to the terminology of [Sha]p24. If we look at the restriction map

$$restr : \mathbb{C}[z_1 \cdots z_n] \longrightarrow \mathbb{C}[z_1 \cdots z_n]|_V$$

we see that its kernel is equal to I_V, the vanishing ideal of V. So the coordinate ring becomes

$$\mathbb{C}[V] = \mathbb{C}[z_1 \cdots z_n]/I_V.$$

For any ring R define its *maximal spectrum* by

$$Specm(R) = \{I < R : I \text{ is a maximal ideal } \}.$$

For any affine variety $V \subset \mathbb{C}^n$, defining the Zariski Topology on each side we have the homeomorphism $V \approx Specm(\mathbb{C}[V])$ between an affine variety and its coordinate ring. As the trivial case, $\mathbb{C}^n \approx Specm\mathbb{C}[z_1 \cdots z_n]$, where a point $a \in \mathbb{C}^n$ corresponds to its vanishing ideal $I_{\{a\}} = \mathbb{C}[z](z_1 - a_1) + \cdots + \mathbb{C}[z](z_n - a_n) = (z_1 - a_1, \cdots, z_n - a_n)$. The maximal ideals of the latter type consumes the maximal ideals of the polynomial ring $\mathbb{C}[z_1 \cdots z_n]$[Mu]p82 which is referred as the Weak Nullstellensatz in the literature [JPB]p59. The full spectrum is the larger space of prime ideals with which we do not deal here.

We first go into the definition of an affine toric variety. For that purpose we take a cone $\sigma \in \mathbb{R}^n$ satisfying the conditions of the following definition for the canonical lattice $N \approx \mathbb{Z}^n \subset \mathbb{R}^n$

Definition 7.4.2 (polyhedral,lattice,strongly convex)**.** *Let* $A = \{x_1 \cdots x_n\} \subset \mathbb{R}^n$ *be a finite set of vectors. Then A cone* σ *is called*

- polyhedral *if it is of the form* $\{x \in \mathbb{R}^n : x = \lambda_1 x_1 \cdots \lambda_r x_r, \lambda_i \geq 0 \ \text{ and real}\}$

- lattice cone *if all the vectors* $x_i \in A$ *belong to* N

- strongly convex *if it does not contain any straight line going through the origin, i.e.* $\sigma \cap -\sigma = \{0\}$

then we define the *affine toric variety* corresponding to σ as

$$U_\sigma := Specm\mathbb{C}[\check{\sigma} \cap N^*]$$

where the duals are defined to be $\check{\sigma} = \{u \in \mathbb{R}^n : \langle u, \sigma \rangle \geq 0\}$ and $N^* = Hom_{\mathbb{Z}}(N, \mathbb{Z})$. One can abuse the notation and show it as $Spec\mathbb{C}[\check{\sigma}]$.

Similar to the way that the cones correspond to an affine toric variety, some collection of cones called fans correspond to a toric variety. More precisely

Definition 7.4.3 (Fan,[JPB])**.** *A* fan Δ *is a finite union of cones such that*

- *the cones are polyhedral,lattice and strongly convex*

- *every face of a cone of* Δ *is again a cone of* Δ

- $\sigma \cap \sigma'$ *is a common face of the cones* σ *and* σ' *in* Δ

Now for a fan Δ in N, we can naturally glue $\{U_\sigma : \sigma \in \Delta\}$ together to obtain a Hausdorff complex analytic space

$$X_\Delta := \bigcup_{\sigma \in \Delta} U_\sigma$$

which is irreducible and normal with dimension equal to $rk(N)$ and called the *Toric Variety*[Oda] associated to the fan (N, Δ). Topologically endowed with an open covering by the affine toric varieties $U_\sigma = Specm\mathbb{C}[\check{\sigma}]$.

Summarizing what we did in high brow terms [Do]$p189$: we constructed the $U_\sigma = Specm\mathbb{C}[\check{\sigma} \cap N^*]$ as the affine variety with $\mathcal{O}(U_\sigma)$ isomorphic to $C[\check{\sigma} \cap N^*]$. Since for any $\sigma, \sigma' \in \Delta$, $\sigma \cap \sigma'$ is a face in both cones, we obtain that $\mathbb{C}[(\check{\sigma \cap \sigma'}) \cap N^*]$ is a localization of each algebra $C[\check{\sigma} \cap N^*]$ and $C[\check{\sigma'} \cap N^*]$. This shows that $Specm\mathbb{C}[(\check{\sigma \cap \sigma'}) \cap N^*]$ is isomorphic to an open subset of U_σ and $U_{\sigma'}$. Which allows us to glue together the varieties U_σ's to obtain the toric variety X_Δ

By definition, X_Δ has a cover by open affine subsets U_σ. Since each algebra $\mathbb{C}[\sigma \cap N^*]$ is a subalgebra of $\mathbb{C}[N^*] \approx \mathbb{C}[Z_1^{\pm 1} \cdots Z_n^{\pm 1}]$ we obtain a morphism $\mathbb{C}^n \to X_\Delta$, which is $\mathbb{C}^n$ equivariant for the action of $\mathbb{C}^n$ on itself by left translations and on X_Δ by means of the $\mathbb{Z}^n$-grading of each algebra $\mathbb{C}[\sigma \cap N^*]$. If no cone $\sigma \in \Delta$ contains a linear subspace, which is the case, the morphism $\mathbb{C}^n \to X_\Delta$ is an isomorphism onto an open orbit. In general, X_Δ always contains an open orbit isomorphic to a factor group of $\mathbb{C}^n$. All toric varieties X_Δ are normal and rational. So we obtain

Theorem 7.4.4 ([Do]p189). *Let Δ be the N-fan formed by the cones $\sigma_j, j = 1 \cdots s$. Then*

$$\mathbb{C}^n(L)/\!\!/_\alpha T = (\mathbb{C}^n)^{ss}(L_\alpha)/T \approx X_\Delta$$

Example 7.4.5. *The weighted projective space $\mathbb{P}_{1,1,2}$ is by definition the quotient of $\mathbb{C}^3 - 0$ by the $\mathbb{C}^*$-action given by the matrix $A = [1, 1, 2]$*
If we linearize the trivial bundle over $\mathbb{C}^3$ by $\alpha = 2$, the linear system $Am = \alpha$ is just $a + b + 2c = 2$, and nonnegative solutions for the triple (a, b, c) are

generated by

$$(2,0,0) \quad (1,1,0) \quad (0,2,0) \quad (0,0,1)$$

so that the coordinate rings are obtained as

$$\mathbb{C}[\mathbb{N}^4 \cap \pi^{-1}(2)] = \mathbb{C}[X^2, XY, Y^2, Z]$$

$$\mathbb{C}[U_1/\mathbb{C}^*] = \mathbb{C}[1, \tfrac{Y}{X}, \tfrac{Y^2}{X^2}, \tfrac{Z}{X^2}] = \mathbb{C}[\tfrac{Y}{X}, \tfrac{Z}{X^2}] = \mathbb{C}[a, b]$$

$$\mathbb{C}[U_2/\mathbb{C}^*] = \mathbb{C}[\tfrac{X}{Y}, 1, \tfrac{Y}{X}, \tfrac{Z}{XY}] = \mathbb{C}[\tfrac{X}{Y}, \tfrac{Y}{X}, \tfrac{Z}{XY}] = \mathbb{C}[a^{-1}, a, a^{-1}b]$$

$$\mathbb{C}[U_3/\mathbb{C}^*] = \mathbb{C}[\tfrac{X^2}{Y^2}, \tfrac{X}{Y}, 1, \tfrac{Z}{Y^2}] = \mathbb{C}[\tfrac{X}{Y}, \tfrac{Z}{Y^2}] = \mathbb{C}[a^{-1}, ba^{-2}]$$

$$\mathbb{C}[U_4/\mathbb{C}^*] = \mathbb{C}[\tfrac{X^2}{Z}, \tfrac{XY}{Z}, \tfrac{Y^2}{Z}, 1] = \mathbb{C}[\tfrac{X^2}{Z}, \tfrac{XY}{Z}, \tfrac{Y^2}{Z}] = \mathbb{C}[b^{-1}, ab^{-1}, a^2b^{-1}]$$

if we assign $a = \frac{Y}{X}$ *and* $b = \frac{Z}{X^2}$.

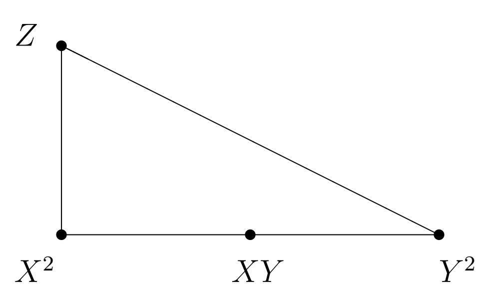
Then since

$$\bigcup_{i=1}^4 U_i = \mathbb{C}^3 \backslash \{\{X^2 = 0\} \cap \{XY = 0\} \cap \{Y^2 = 0\} \cap \{Z = 0\}\} =$$

$$\mathbb{C}^3 \backslash \{\{X = 0\} \cap \{XY = 0\} \cap \{Y = 0\} \cap \{Z = 0\}\} = \mathbb{C}^3 \backslash \{X = Y = Z = 0\}$$

these are the coordinate rings of the stated weighted projective space.

The moment polytope looks like :



7.5 Minitwistor Space

The image of the Honda Minitwistor space (7.2.1) is the quotient of $\mathbb{CP}_3$ by the $\mathbb{C}^*$ action

$$(Z_0 : Z_1 : Z_2 : Z_3) \mapsto \left(Z_0 : Z_1 : \lambda Z_2 : \lambda^{-1} Z_3\right) \quad for \quad \lambda \in \mathbb{C}^*$$

On the other hand, to obtain $\mathbb{CP}_3$, we already have the classical $\mathbb{C}^*$ action

$$(Z_0 : Z_1 : Z_2 : Z_3) \mapsto (\lambda Z_0 : \lambda Z_1 : \lambda Z_2 : \lambda Z_3) \quad for \quad \lambda \in \mathbb{C}^*.$$

Combining the two, the image equals to the quotient of the $\mathbb{C}^{*2}$ action by the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

on $\mathbb{C}^4$. Now extend this action canonically to the trivial line bundle over $\mathbb{C}^4$. Among all the linearization, one of them proves to have minimal number of unstable orbits :

Theorem 7.5.1. *The categorical quotient of $\mathbb{C}^4$ under the $\mathbb{C}^{*2}$ action*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

alternatively, the categorical quotient of $\mathbb{CP}_3$ under the $\mathbb{C}^*$ action $\begin{bmatrix} 0 & 0 & 1 & -1 \end{bmatrix}$ linearized by $\alpha = (3, 1)$ is the weighted projective space $\mathbb{P}_{1,1,2}$

Proof. The linear system $Am = \alpha$ is

$$\left.\begin{array}{rcl} a+b+c+d &=& 3 \\ c-d &=& 1 \end{array}\right\} \quad or \quad \left\{\begin{array}{rcl} a+b+2d &=& 2 \\ c &=& d+1 \end{array}\right.$$

looking for nonnegative solutions, $1, 0$ are the only possibilities for d

$d = 1 : a = b = 0, c = 2$ yields the solution $\begin{pmatrix} 0 & 0 & 2 & 1 \end{pmatrix}$

$d = 0 : a + b = 2, c = 1$ yields the solutions $\begin{matrix} (2 & 0 & 1 & 0) \\ (1 & 1 & 1 & 0) \\ (0 & 2 & 1 & 0) \end{matrix}$

the coordinate rings are

$$\mathbb{C}[\mathbb{N}^4 \cap \pi^{-1}(3,1)] = \mathbb{C}[X^2Z, XYZ, Y^2Z, Z^2W]$$

$$\mathbb{C}[U_1/\mathbb{C}^{*2}] = \mathbb{C}[1, \tfrac{XYZ}{X^2Z}, \tfrac{Y^2Z}{X^2Z}, \tfrac{Z^2W}{X^2Z}] = \mathbb{C}[\tfrac{Y}{X}, \tfrac{Y^2}{X^2}, \tfrac{ZW}{X^2}] = \mathbb{C}[\tfrac{Y}{X}, \tfrac{ZW}{X^2}]$$

$$\mathbb{C}[U_2/\mathbb{C}^{*2}] = \mathbb{C}[\tfrac{X^2Z}{XYZ}, 1, \tfrac{Y^2Z}{XYZ}, \tfrac{Z^2W}{XYZ}] = \mathbb{C}[\tfrac{X}{Y}, \tfrac{Y}{X}, \tfrac{ZW}{XY}]$$

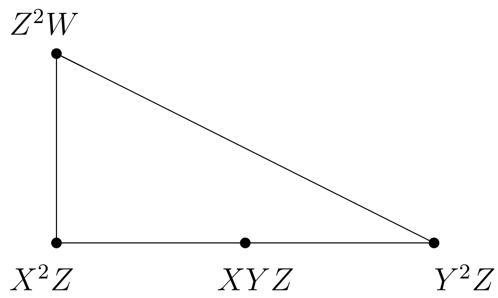
$$\mathbb{C}[U_3/\mathbb{C}^{*2}] = \mathbb{C}[\tfrac{X^2Z}{Y^2Z}, \tfrac{XYZ}{Y^2Z}, 1, \tfrac{Z^2W}{Y^2Z}] = \mathbb{C}[\tfrac{X^2}{Y^2}, \tfrac{X}{Y}, \tfrac{ZW}{Y^2}] = \mathbb{C}[\tfrac{X}{Y}, \tfrac{ZW}{Y^2}]$$

$$\mathbb{C}[U_4/\mathbb{C}^{*2}] = \mathbb{C}[\tfrac{X^2Z}{Z^2W}, \tfrac{XYZ}{Z^2W}, \tfrac{Y^2Z}{Z^2W}, 1] = \mathbb{C}[\tfrac{X^2}{ZW}, \tfrac{XY}{ZW}, \tfrac{Y^2}{ZW}]$$

and these coordinate rings are isomorphic to the ones for the $\mathbb{P}_{1,1,2}$ as in (7.4.5). Realize the isomorphism by assigning $a = \frac{Y}{X}, b = \frac{ZW}{X^2}$ so that the coordinate rings respectively becomes

$$\mathbb{C}[a,b] \ , \ \mathbb{C}[a, a^{-1}, a^{-1}b] \ , \ \mathbb{C}[a^{-1}, a^{-2}b] \ , \ \mathbb{C}[b^{-1}, ab^{-1}, a^2b^{-1}]$$

Besides, the moment polytope may help to visualize this isomorphism :



$\square$

Since

$$\bigcup_{i=1}^{4} U_i = \mathbb{C}^4 \backslash \{\{X^2Z = 0\} \cap \{XYZ = 0\} \cap \{Y^2Z = 0\} \cap \{Z^2W = 0\}\} = \mathbb{C}^4 \backslash \{Z = 0\}$$

so the orbits lying entirely on the hyperplane $\{Z = 0\} \subset \mathbb{CP}_3$ are not counted under the $\mathbb{C}^*$ action.

Remark 7.5.2. *If we take $\alpha = (1,1)$ to be the linearization, the quotient reduces to be a $\mathbb{P}^1$ as follows.*

$$[A|\alpha] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right] \approx \left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

produces the solution space $S = \langle(-1,1,1,0), (-2,0,2,1)\rangle$, so the coordinate rings are

$\mathbb{C}[\mathbb{N}^4 \cap \pi^{-1}(1,1)] = \mathbb{C}[X^{-1}YZ, X^{-2}Z^2T]$

$\mathbb{C}[U_1/\mathbb{C}^{*2}] = \mathbb{C}[1, \frac{X^{-2}Z^2T}{X^{-1}YZ}] = \mathbb{C}[1, X^{-1}Y^{-1}ZT] = \mathbb{C}[\beta]$

$\mathbb{C}[U_2/\mathbb{C}^{*2}] = \mathbb{C}[XYZ^{-1}T^{-1}, 1] = \mathbb{C}[\beta^{-1}]$

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