Topics in Algebraic Cycles

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Abstract of the Dissertation
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The goal of this work is to explore the space $\mathcal{Z}^p(\mathbb{P}^n)$ of algebraic cycles of codimension p in $\mathbb{P}^n$. In the first chapter generalizations of the classical Gauss map for projective hypersurfaces are constructed. They are algebraic maps which associate to every point x a hypersurface of degree d in $\mathbb{P}^n$ which is a good approximation at x. Some geometric properties of these maps are described and applications are given.

The second chapter shows an explicit homotopy between two different H-space structures on the infinite projective space. One presentation has the advantage of being commutative and an infinite loop space structure while the other has a nice description in terms of line bundles and it has an explicit homotopy inverse.

The third chapter is concerned with the existence of an extension of the map classifying the tensor product of bundles to the space of all algebraic cycles of arbitrary degree in $\mathbb{P}^n$. Such an extension

is constructed in codimension 1 and a topological obstruction is exhibited for higher codimension assuming compatibility with the additive structure on the space of algebraic cycles.

Contents

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Chapter 1

Introduction

In the 1950's algebraic topologists realized that the homology and cohomology groups can be represented as homotopy classes of maps from one space into another. Dold and Thom constructed in [DT58] particularly beautiful models for the Eilenberg-MacLane spectrum. Their representations of homology and cohomology were

$$H_i(X,\mathbb{Z}) = [\mathbb{S}^i, \mathbb{Z}\cdot X] = \pi_i(\mathbb{Z}\cdot X) \qquad H^i(X,\mathbb{Z}) = [X, \mathbb{Z}\cdot \mathbb{S}^i]$$

where X is any compact, connected, CW-complex and $\mathbb{Z}\cdot X$ is the free abelian group generated by the points of X.

From the point of view of algebraic geometry, the free abelian group can be interpreted as the group of 0-dimensional cycles, thus it is natural to consider the group of p-dimensional algebraic cycles $\mathcal{Z}_p(X)$. This is something which Blaine Lawson did in his foundational paper [LJ89]. The consequences of these investigations have been far reaching for both the study of invariants of algebraic varieties and the study of the geometry of spectra.

The inclusion of the cycles of degree 1 (i.e. linear subspaces) in $\mathbb{P}^n$ into the space of all cycles stabilizes to give a map $c : BU \to \mathcal{Z}(\mathbb{P}^\infty)$ which represents the total Chern class map from topological K-theory into the cohomology ring $1 \times \prod_{k\geq 0} H^{2k}$ thought of as a group with respect to the cup product pairing. The existence of such a map was first observed by Grothendieck, and it was conjectured by Segal in [Seg75] that this map extends to a map of cohomology theories. In [BLLF$^+$93] this conjecture was settled by showing that the map c is actually a map of E_∞-spectra.

This dissertation is concerned with the development of the ideas arising from the representation of the Eilenberg-MacLane spectrum as the space of algebraic cycles in projective space.

Given the homotopy equivalence $\mathcal{Z}^p(\mathbb{P}^n) \simeq \prod_{n=0}^{p} K(\mathbb{Z}, 2n)$ we can think of a total cohomology class $[\Psi] \in H^0(X;\mathbb{Z}) \times \cdots \times H^p(X;\mathbb{Z})$ as the homotopy class of a map $\Psi : X \to \mathcal{Z}^p(\mathbb{P}^n)$. This is very appealing geometrically since we now have a new *picture* of cohomology classes. Namely, we can think of cohomology classes as a generalization of vector bundles:

- A vector bundle is a **continuous** choice of a projective subspace at every point $x \in X$. Analogously,
- A cohomology class is a **continuous** choice of a projective algebraic cycle at every point $x \in X$

The precise meaning of continuous is provided by the topology that we impose on the space of algebraic cycles. Some remarks about this idea are important

1. Since a vector space can be thought of as a linear space in projective space, it should define a cohomology class. This class is precisely the total chern class of the bundle

2. The total chern class homomorphism is an isomorphism if we take rational classes, i.e. the chern class induces an isomorphism

$$c : K(X) \otimes \mathbb{Q} \to H^{\times}(X) \otimes \mathbb{Q}$$

Therefore every rational cohomology class can be represented by a vector bundle.

3. The new "bundles" that we get when we choose continuously an algebraic cycle for every point $x \in X$ are far more complicated than vector bundles. Consider the following example: Let $Y \subset \mathbb{P}^n$ be a polarized variety and let $\pi : Y \to X$ be a flat map. Then the assignment

$$x \mapsto \pi^{-1}(x) \subset \mathbb{P}^n$$

defines a total cohomology class in X [1]. It then follows that we loose essential properties of vector bundles such as local triviality. The fibers are no longer homeomorphic but only cycles which lie in the same connected component of the Chow variety.

From this point of view we can think of the space $\mathcal{Z}^p(\mathbb{P}^n)$ as completing the space $\mathcal{Z}^p_1(\mathbb{P}^n) \cong \mathcal{G}^p(\mathbb{P}^n)$ of cycles of degree 1.

[1]Actually this defines an *algebraic cocycle* which is a more refined object taking values in a bi-valued theory defined by Lawson and Friedlander cf.[FLJ92]

The first chapter of this dissertation presents an instance of such a family of cycles which are importantly tied to the geometry of hypersurfaces: the higher degree Gauss maps.

One of the first examples of vector bundles we encounter is the Gauss map which associates to every point x of a smooth variety X the tangent space T_xX. A natural question to ask is: *Is there some natural cohomology class associated to X using cycles of higher degree?* The higher degree Gauss maps are good candidates for answering the question in the case of hypersurfaces.

These higher degree Gauss maps turn out to be well-defined maps even for singular hypersurfaces (possibly reducible) for sufficiently high degrees. They reflect both the local and the global structure of the variety, locally because each cycle is tangent to the variety at smooth points and globally because the assignment defines a cohomology class and the assignment is "aware" of the existence of singularities or flexes cf. Example 2.2.7 and Theorem 2.4.1. In other words, the higher gauss maps select distinguished points of the Chow variety corresponding to geometrically meaningful points. This continuous selection of points cannot be obtained using only the local structure of the variety.

The next two chapters address topological questions concerning the spaces $\mathcal{Z}^p(\mathbb{P}^n)$. Notice that taking the linear join $\eta \# p$ of an algebraic cycle $\eta \subset \mathbb{P}^n$ with a point $p \in \mathbb{P}^{n+1}$ such that $p \notin \mathbb{P}^n$ we get an inclusion $\mathcal{Z}^p(\mathbb{P}^n) \hookrightarrow \mathcal{Z}^p(\mathbb{P}^{n+1})$

which is compatible with the usual inclusion of the corresponding grassmannian. In this way we get the stable spaces $\mathbf{BU}(p)$ and $\mathcal{Z}^p$. Analogously we get inclusions $\mathcal{Z}^p \subset \mathcal{Z}^{p+1}$ and we obtain the space $\mathcal{Z}$ of all cycles of all possible codimensions. The map c between the double colimits $\mathbf{BU}$ and $\mathcal{Z}$ is the total chern class map. These double colimits are very rich in structure:

1. The space $\mathbf{BU}$ has two different H-space structures which can be enriched to infinite loop space structures (or they are induced by two different infinite loop space structures, if preferred). The first H-space structure on $\mathbf{BU}$ is obtained via the direct sum $\oplus$ of vector bundles. The second H-space structure is obtained via the tensor product $\otimes$ of vector bundles. These two structures are compatible and actually define an E_∞-ring space structure on $\mathbf{BU}$, that is, we obtain a ring spectrum, namely the spectrum defining topological K-theory.

2. The space $\mathcal{Z}$ also has two different H-space structures which can be enriched to infinite loop space structures. The first structure is induced by the sum of cycles $+$ in each $\mathcal{Z}^p$. The second structure is induced by the linear join of cycles $\#$. These two structures are compatible and they define an E_∞-ring structure on $\mathcal{Z}$ which defines a cohomology theory M.

3. The inclusion $c : \mathbf{BU} \to \mathcal{Z}$ is a map of E_∞-loop spaces

$$(\mathbf{BU}, \oplus) \to (\mathcal{Z}, \#)$$

The operations on these spaces are therefore very important from a topological point of view. Different operations will yield different infinite loop space structures and thus we will obtain different cohomology theories. Notice that the third item says that the chern class map is *only* a map of infinite loops spaces, it is *not* a map of E_∞-ring spectra. Actually, Totaro proved in [Tot93] that the spectrum $(\mathcal{Z}, \#)$ cannot be enhanced into a ring spectrum.

The authors of [BLLF$^+$93] noted that it would be nice to have an extension of the tensor product operation to the space of all algebraic cycles. The result of Totaro in [Tot93] implies that if such construction exists, then it is not compatible with the structure $\#$. The third chapter of this dissertation proves up to what extent there is such an extension with respect to the structure $+$. The result is affirmative and it is constructed explicitly for the case of cycles of codimension 1, i.e. there is a continuous biadditive product $\hat{\otimes}$ which makes the following diagram commute

$$\begin{tikzcd}
\mathcal{G}^1(\mathbb{P}^\infty) \times \mathcal{G}^1(\mathbb{P}^\infty) \arrow[r] \arrow[d, "c_1 \times c_1"'] & \mathcal{G}^1(\mathbb{P}^\infty) \arrow[d, "c_1"] \\
\mathcal{Z}^1_0(\mathbb{P}^\infty) \times \mathcal{Z}^1_0(\mathbb{P}^\infty) \arrow[r, "\hat{\otimes}"] & \mathcal{Z}^1_0(\mathbb{P}^\infty)
\end{tikzcd}$$

This construction gives us a working definition of what is the tensor product of two hypersurfaces. The geometry of this product is not completely understood, though some results in the case when one of the factors is a linear space are proved in chapter 3. Homotopically there is a better understanding of this product, and we recover the chern class formula for the tensor product of line bundles.

The extension of the tensor product to higher codimension is not possible.

The key result in proving this negative result is the factorization of the inclusion of the grassmannian $\mathcal{G}^p(\mathbb{P}^n)$ into the space of cycles $\mathcal{Z}^P(\mathbb{P}^n)$ through the free subgroup generated by the points of the grassmannian $\mathbb{Z}\mathcal{G}^p(\mathbb{P}^n)$. The two inclusions are characterized homotopically in the case $p = 1$ and then, using the Chern class formula for the tensor product of bundles it is proved that there is a topological obstruction for having such an extension of the tensor product pairing in higher codimensions.

The idea of the proof stems from the geometric picture mentioned at the beginning of this introduction: *The space of all algebraic cycles "completes" the space of linear cycles.* But now we regard this completion not only in the topological sense, but also in the algebraic sense as H-spaces. Explicitly, there is an H-space lying in between the two H-spaces $\mathbf{BU}$ and $\mathcal{Z}$: the free group $\mathbb{Z}\mathbf{BU}$ generated by the points of $\mathbf{BU}$. We can factor the total chern class map through the subgroup $\mathbb{Z}\mathbf{BU}$. It is possible to compute in rational cohomology what happens with this factorization, and we get a contradiction from the chern class formula for the tensor product of bundles.

The second chapter is a transition between chapters 1 and 3. In this chapter we show how two different presentations of the H-space structure of $\mathbb{P}^\infty$ are homotopic. One presentation is very appealing from the point of view of geometry: it is the operation induced by a stabilized version of the Segre product. The other presentation is appealing from the topological point of view: it is the H-space structure obtained from the free group $\mathbb{Z}_0\mathbb{P}^1$ of divisors of degree zero in $\mathbb{P}^1$. Moore showed how this is advantageous from a topological point of view: for a topological group all the higher Postnikov invariants vanish, hence we have up to homotopy a product of Eilenberg MacLane spaces, in this case $\mathbb{Z}_0\mathbb{P}^1 \simeq K(\mathbb{Z}, 2)$.

Chapter 2

Higher Degree Gauss Maps

2.1 Introduction

For any smooth variety $X \subset \mathbb{P}^n$ of codimension q the classical gauss map is the map

$$g^1 : X \to \mathcal{G}^q(\mathbb{P}^n)$$

which associates to each point ξ the projective linear subspace of codimension q tangent to X at ξ in $\mathbb{P}^n$:

$$\xi \mapsto \overline{T_\xi X}$$

If X is a hypersurface defined by the set of zeros of a homogeneous polynomial F, $X = V(F) := \{x \in \mathbb{P}^n \mid F(x) = 0\} \subset \mathbb{P}^n$, then the gauss map has the following coordinate expression

$$\xi \mapsto V\left(\sum \frac{\partial F}{\partial x_i}(\xi) x_i\right)$$

If X has singularities we no longer have a map which is regular, but only a rational map.

In this chapter higher degree Gauss maps will be defined

$$g^k : X \to \mathcal{C}^1_k(\mathbb{P}^n)$$

which associate to each point ξ the effective algebraic cycle of degree k and codimension 1 which approximates X at ξ. In analogy with the above, the degree k Gauss map associates to the point ξ a projective hypersurface defined by the k-th partial derivatives of F at ξ.

2.2 Definition and Euler Formula

In this section X will be a projective hypersurface (not necessarily irreducible) of dimension $n-1$ defined by the zero set of a homogeneous polynomial $F \in \mathbb{C}[x_0, \ldots, x_n]$ of degree d.

Let us recall the Euler relation:

$$d \cdot F = \sum_{i=0}^{n} \frac{\partial F}{\partial x_i} x_i \tag{2.1}$$

The Euler relation can be iterated, i.e., the following holds:

$$(d-1) \cdot \frac{\partial F}{\partial x_i} = \sum_{k=0}^{n} \frac{\partial^2 F}{\partial x_k \partial x_i} x_k \tag{2.2}$$

If we substitute equation (2.2) into equation (2.1) we get the following relation for F:

$$d \cdot (d-1) \cdot F = \sum_{i=0}^{n} \sum_{k=0}^{n} \frac{\partial^2 F}{\partial x_k \partial x_i} x_k x_i \tag{2.3}$$

In general, if $s \leq d = \deg(F)$ we have the following equation:

$$\boxed{d(d-1)\cdots(d-s+1)F = \sum_{|\alpha|=s} \frac{\partial^{|\alpha|} F}{\partial x^\alpha} x^\alpha} \tag{2.4}$$

where α runs over all multi-indices of length s, i.e.

$$\alpha = (\alpha_0, \ldots, \alpha_n) \text{ with } \alpha_i \in \mathbb{N}$$

and

$$|\alpha| = \alpha_0 + \cdots + \alpha_n$$

$$\frac{\partial^{|\alpha|} F}{\partial x^\alpha} = \frac{\partial^{|\alpha|} F}{\partial x_{\alpha_0} \partial x_{\alpha_1} \cdots \partial x_{\alpha_n}}$$

2.2.1 Remark. One of the consequences of the Euler formula is that the systems $\{\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}, F\}$ and $\{\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n}\}$ have the same set of solutions, more precisely, they define the same scheme since the ideals they generate are equal. Recursively we obtain the following lemma.

2.2.2 **Lemma**. *Let ξ be a point in $\mathbb{P}^n$ such that $\frac{\partial^s F}{\partial x_\alpha}(\xi) = 0$ for all α with $|\alpha| = s$. Then $F(\xi) = 0$ and $\frac{\partial^{|\beta|} F}{\partial x^\beta}(\xi) = 0$ for all β with $|\beta| \leq s$*

Now we will define the higher degree Gauss maps and we will derive some consequences from the generalized Euler formulas given above.

2.2.3 Definition. For every $k \leq d$, the *degree k Gauss map*

$$g^k : X \dashrightarrow \mathcal{C}^1_k(\mathbb{P}^n) \tag{2.5}$$

is the rational map defined by

$$\xi \mapsto V\left(\sum_{|\alpha|=k} \frac{\partial^k F}{\partial x_\alpha}(\xi) x^\alpha\right)$$

The space $\mathcal{C}^1_k(\mathbb{P}^n)$ of cycles of codimension 1 and degree k in $\mathbb{P}^n$ can be identified with $\mathbb{P}^{\binom{n+k}{k}-1}$, this identification is via the Chow coordinates. Every codimension 1 cycle is determined by a multivariable homogeneous polynomial of degree k. If a cycle is defined by a polynomial $\sum a_\alpha x^\alpha$ then its Chow coordinates are $[a_0 : \cdots : a_\alpha : \cdots]$. Using the Chow coordinates in $\mathcal{C}^1_k(\mathbb{P}^n) \cong \mathbb{P}^{\binom{n+k}{k}-1}$ the degree k Gauss map is just

$$\xi \mapsto \left[\frac{\partial^k F}{\partial x_0 \cdots \partial x_0}(\xi) : \ldots : \frac{\partial^k F}{\partial x^\alpha}(\xi) : \ldots : \frac{\partial^k F}{\partial x_n \cdots \partial x_n}\right] \tag{2.6}$$

Notice that the first gauss map g^1 coincides with the classical projective gauss map.

As in the case of the first Gauss map, the higher degree Gauss maps are only rational in general. Interestingly however, they can still be regular in the presence of certain singularities. More precisely, the following is true:

2.2.4 **Theorem**. *If a hypersurface of degree d has a regular Gauss map of degree p, it also has regular Gauss maps of degree q for $p \leq q \leq d$*

Proof. This follows immediately from the Euler relation: If the degree p gauss map is regular, then for every $\xi \in X$ some p-th partial derivative $\frac{\partial^p F}{\partial x_\alpha}(\xi)$ is not zero. On the other hand, if all the q-th partial derivatives are zero at ξ then lemma (2.2.2) implies that $\frac{\partial^p F}{\partial x^\alpha}(\xi) = 0$ for all α ! $\square$

2.2.5 Remark. These higher degree Gauss maps are **not** the higher order Gauss maps which define the Higher Fundamental Forms studied by Griffiths and Harris, Landsberg et. al. Furthermore, the higher degree Gauss maps are not osculating hypersurfaces at the point as defined by Landsberg.

2.2.6 Remark. Landsberg defined in [Lan96] what it means for a hypersurface V to be an *osculating hypersurface of order* k to a variety X at a point p. The definition is that if $V = V(F)$ and T is a local parametrization of X around p such that $F(T_0) = p$ then F should vanish to order k at T_0. The higher degree gauss maps are osculating hypersurfaces of order 1 at smooth points, but they do **not** define osculating hypersurfaces of higher order in general. They define approximations which are tangent to the hypersurface at smooth points, but they are "aware" of the global structure of the hypersurface. Note that the osculating hypersurfaces are not unique in general.

2.2.7 Example. Let $V \subset \mathbb{P}^2$ be the nodal plane cubic defined by $F(x_0, x_1, x_2) = x_2x_1^2 - x_0^3 - x_0^2x_2$, then V does not have a regular Gauss map of degree 1, but it has a well defined Gauss map of degree 2 (see figure 2.1):

$$\overline{\xi} \mapsto V_{\overline{\xi}} = V(-(3\xi_0 + \xi_2)x_0^2 - 2\xi_0x_0x_2 + \xi_2x_1^2 + 2\xi_1x_1x_2)$$

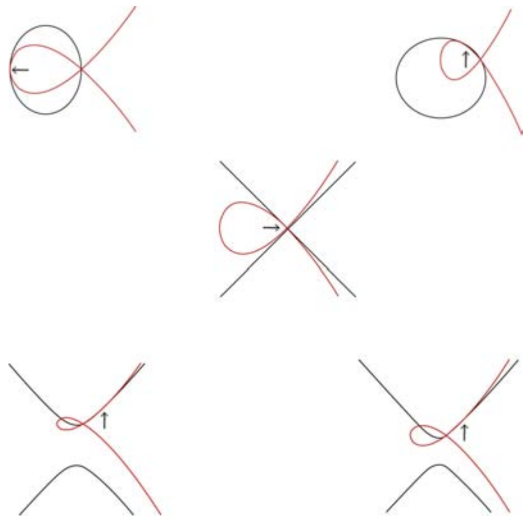

Figure 2.1: Second degree approximations to the nodal cubic: The red curve is the nodal cubic, the black curves are the conics approximating the curve at the point signaled by the arrow.

That is, to every point v we associate a quadric which approximates the curve at v. Notice that at the node $[0 : 0 : 1]$ we do have a well defined second order approximation: the union of the two possible tangents.

2.3 Degree and Dimension

The closure of the image of the first Gauss map defines the dual variety of the hypersurface X. We could ask what are the general properties of the images of these higher degree Gauss maps. The following results extend some classic results of projective geometry. We shall assume throughout that X is a hypersurface of degree d.

2.3.1 **Theorem**. *If X is not a cone, then the $(d-1)$ Gauss map $g^{(d-1)}$is an isomorphism from X into its image. (This extends the well known result for the duals of smooth quadrics. Note that any singular quadric is a cone.)*

In order to prove this theorem we will first prove a characterization of cones. Recall that a cone is the linear join $Y\#p$ of a variety $Y \in \mathbb{P}^n$ with a point $p \in \mathbb{P}^n$ such that $p \notin Y$.

2.3.2 **Lemma**. *$X \subset \mathbb{P}^n$ is a cone if and only if there is some $\xi \in X$ such that* $\mathrm{mult}_\xi X = d$.

Proof. If X is a cone $X = Y\#p$, then p is a point such that $\mathrm{mult}_p X = d$. Conversely, if ξ is a point of multiplicity d, let l be any line passing through ξ. If l intersects X at any other point q then $\overline{\xi q} = l \subset X$ (otherwise the degree of X would be greater than d). $\square$

Now we can prove Theorem 2.3.1.

Proof. Notice that using the Chow representation (2.6) , $g^{(d-1)}$ is a linear map. That is, it is the map induced by a linear map $\tilde{L} : \mathbb{P}^n \to \mathbb{P}^{\binom{n+k}{k}-1}$ by restriction to X. This means that $\tilde{L}$ itself is induced by a linear map $L : \mathbb{C}^{n+1} \to \mathbb{C}^{\binom{n+k}{k}}$.

If $g^{(d-1)}$ is not an injection then L is certainly not an injection. We will show that this leads to a contradiction.

If L is not an injection then it has a non-zero kernel. Let $\bar{\xi}$ be a non-zero vector in that kernel. This means that all the $(d-1)$-partial derivatives of F vanish at $\bar{\xi}$. But the generalized Euler relation implies then that $\xi \in X$. This is a contradiction because we get a point $\xi \in X$ such that $\mathrm{mult}_{\xi} X = d$ (because all the $(d-1)-$partial derivatives vanish at ξ), i.e. X is a cone (by the previous lemma).

□

2.3.3 **Lemma**. *If the degree p Gauss Map is regular, then*

$$(g^p)^*(O(1)) = O_X(d-p)$$

Proof. First of all, notice that if g^p is regular, then it extends to a regular map defined on *all* of $\mathbb{P}^n$:

$$\tilde{g}^p : \mathbb{P}^n \to \mathbb{P}^N$$

We see this as follows. Note that the extension is given by the same coordinate functions cf. (2.6). This extension is regular for the following reason: If there is some point $\xi \in \mathbb{P}^n$ where all the coordinate functions vanish simultaneously, Lemma 2.2.2 then implies that $F(\xi) = 0$, i.e. $\xi \in X$, but this would imply that g^p is not regular!

Now, since the coordinate functions of $\tilde{g}^p$ can be interpreted as sections of the bundle $(\tilde{g}^p)^*(O(1))$ and they are polynomials of degree $d-p$, we get that the pullback of $O(1)$ under this map is $O(d-p)$. Since g^p is just the restriction of $\tilde{g}^p$ we get the lemma.

□

2.3.4 **Theorem**. *If the degree p Gauss map is regular, the dimension of the p-th Gauss image variety is $n-1$ and the degree of the p-th Gauss image variety is $d(d-p)^{n-1}$. (This extends the fact that the dual of a smooth non-linear hypersurface is a hypersurface and the classical formulas for the degree of the dual of a smooth hypersurface).*

Proof. Both statements follow from calculating the Kronecker pairing of the image variety with a linear subspace of the appropriate codimension. Using the previous lemma, this becomes just a chern class calculation. We will denote with $H_{\mathbb{P}^s}$ the class of a hyperplane in $H^2(\mathbb{P}^s, \mathbb{Z})$ and with $H^k_{\mathbb{P}^s}$ its k-fold cup product. Recall that $\dim X = n-1$. Using the previous lemma we obtain:

$$(g^p)^*(H_{\mathbb{P}^N}) = (d-p)H_{\mathbb{P}^n}$$

therefore

$$(g^p)^*(H^{n-1}_{\mathbb{P}^N}) = (d-p)^{(n-1)}H^{(n-1)}_{\mathbb{P}^n}$$

If $\langle , \rangle$ denotes the Kronecker pairing, then the following calculation proves the result:

$$\begin{aligned}
\deg g^p(X) &= \langle g^p(X), H_{\mathbb{P}^N}^{(n-1)} \rangle \\
&= \langle (\tilde{g}^p)_*(X), H_{\mathbb{P}^N}^{(n-1)} \rangle \\
&= \langle X, \tilde{g}^*(H_{\mathbb{P}^N}^{(n-1)}) \rangle \\
&= \langle X, (d-p)^{(n-1)} H_{\mathbb{P}^n}^{(n-1)} \rangle \\
&= d(d-p)^{(n-1)}
\end{aligned}$$

□

Lemma 2.3.3 also allows us to prove the following calculation. Recall that $f : X \to \mathbb{P}^N \subset \mathbb{P}^\infty = K(\mathbb{Z}, 2)$ defines a cohomology class $[f] \in H^2(X, \mathbb{Z})$.

2.3.5 **Theorem**. *If X is a smooth hypersurface, the cohomology class defined by the p-th Gauss map $[g^p] \in H^2(X)$ satisfies*

$$[g^p] = \frac{d-p}{d-1}[g^1] = \frac{d-p}{d-1} c(N_X(-1))$$

More generally,

$$[g^p] = c_1(O_X(d-p))$$

Proof. We note that the cohomology class $[g^p]$ coincides with $c_1((g^p)^*(O(1)))$. Lemma 2.3.3 provides this last calculation:

$$c_1((g^p)^*(O(1))) = c_1(O_X(d-p))$$

To prove the first claim we recall the adjunction formula:

$$O_X(d) = [X]|_X = N_X$$

But we can also write

$$O_X(d) = O_X(d-1) \otimes O_X(1) = [g^1] \otimes O_X(1)$$

therefore

$$[g^1] = N_X(-1)$$

$\square$

In the next chapter an alternative proof of Theorem 2.3.3 will be provided using the $H-$space structure of $\mathbb{P}^\infty$.

2.4 Examples and Applications

THE higher degree gauss maps encode information about the underlying variety. For example if the p-th Gauss map is regular then the variety cannot have singularities of order greater than or equal to p. The next theorem recovers the classical calculation of the number of flexes of a smooth plane curve. Recall that a flex of a plain curve C is a point $p \in C$ where the tangent line has contact of order higher than 2, that is, the local intersection number at p of the tangent line at p and the curve is greater than 2.

2.4.1 **Theorem**. *Let C be a smooth plane curve $C \subset \mathbb{P}^2$ of degree $d \geq 2$. Then C has $3d(d-2)$ flexes (counted with multiplicity).*

Proof. Let C be defined by a homogeneous polynomial F. A point p is a flex if and only if the determinant of the Hessian matrix H_p is zero, where

$$H_p = \begin{pmatrix} \frac{\partial^2 F}{\partial x_0^2}(p) & \frac{\partial^2 F}{\partial x_0 x_1}(p) & \frac{\partial^2 F}{\partial x_0 x_2}(p) \\ \frac{\partial^2 F}{\partial x_0 x_1}(p) & \frac{\partial^2 F}{\partial x_1^2}(p) & \frac{\partial^2 F}{\partial x_1 x_2}(p) \\ \frac{\partial^2 F}{\partial x_0 x_2}(p) & \frac{\partial^2 F}{\partial x_1 x_2}(p) & \frac{\partial^2 F}{\partial x_2^2}(p) \end{pmatrix}$$

(cf. [**?**, EGAC]emma 13.2) but H_p is the quadratic form which defines the second gauss map at p, i.e.

$$g^2(p) = \xi^T H_p \xi \tag{2.7}$$

Notice that since C is smooth, the second gauss map is regular. So the condition of p being a flex is exactly the same as $g^2(p)$ being a singular quadric. Now, singular quadrics form a hypersurface Δ of degree 3 in the space $\mathbb{P}^5$ of all degree 2 homogeneous polynomials in three variables. This hypersurface Δ is given by the vanishing of the determinant of the matrix defining a quadratic form.

Thus we are interested in computing the number of intersection points of $g^2(C)$ with Δ. But $\deg(g^2(C)) = d(d-2)$ and $\deg(\Delta) = 3$, therefore Bezout's theorem implies that the number of flexes is $3d(d-2)$. $\square$

The next example shows how it is possible to have a hypersurface X with degenerate gauss map (the image of the gauss map will be a curve) and nevertheless the second gauss map has the same dimension as the hypersurface. It would be interesting to find examples of hypersurfaces X with degenerate higher gauss images such that X is not a cone.

2.4.2 Example. Let $\Sigma \in \mathbb{P}^3$ be the rational normal curve. That is, Σ is the image of the rational parametrization

$$t \mapsto [1 : t : t^2 : t^3]$$

It is known that the dual variety $\Sigma^\vee$ is a hypersurface defined by the discriminant of the general single variable polynomial of degree 3 (cf [GKZ94] ch. 1.), namely, the equation defining the dual hypersurface is:

$$\Delta = x_1^2 x_2^2 - 4x_1^3 x_3^2 - 27x_0^2 x_3^2 + 18x_0 x_1 x_2 x_3$$

Now, $\Sigma^\vee$ must necessarily be singular, since otherwise the dual variety would be a hypersurface. But the singularities of $\Sigma^\vee$ actually have order 1, therefore the second gauss map is regular and using our calculations we can conclude that $g^2(\Sigma^\vee)$ is a surface of degree $4(4-2)^2 = 16$ in $\mathbb{P}^9$.

Chapter 3

H-space structures on $\mathbb{P}^\infty$

The infinite complex projective space $\mathbb{P}^\infty$ is an Eilenberg-MacLane space of type $K(\mathbb{Z}, 2)$. Therefore it is an infinite loop space. In this section we will present two equivalent H-space structures on $\mathbb{P}^\infty$ which coincide with its H-space structure determined by the infinite loop space structure. One presentation has the advantage of being strictly commutative while the other has a nice geometric interpretation in terms of the classification of line bundles as well as an explicit description of the H-inverse of the operation.

3.1 Addition of points

There is an H-space structure on $\mathbb{P}^\infty$ obtained via the homeomorphism

$$\phi : \mathbb{P}^\infty \to \mathbf{SP}^\infty \mathbb{P}^1$$

Given by

$$[p_0 : p_1 : \ldots : p_t : 0 : \ldots : 0 : \ldots] \mapsto Z(\sum p_i x^i y^{t-i})$$

With inverse

$$\sum_{i=0}^{n}[x_i : y_i] \mapsto [a_0 : a_1 : \ldots : a_n : 0 : \ldots]$$

where $\sum a_j x^j y^{n-j}$ is any polynomial having the divisor $\sum[x_i : y_i]$ as its roots (we use the term *divisor* to emphasize that we are counting multiplicities).

The operation is just the formal addition of cycles in $\mathbb{P}^1$:

$$\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j = \sum_{k=1}^{n+m} z_k$$

where $z_k = x_k$ if $k \leq n$ and $z_k = y_{k-n}$ if $k > n$.

The induced H-space structure on $\mathbb{P}^\infty$ via the homeomorphism above is

$$\begin{array}{c} ([x_0 : x_1 : \ldots : x_n : \ldots], [y_0 : y_1 : \ldots : y_m : \ldots]) \\ \big\downarrow \\ [x_0 y_0 : x_0 y_1 + x_1 y_0 : \ldots : \sum x_i y_{k-i} : \ldots] \end{array} \tag{3.1}$$

Notice that from the description 3.1 it follows immediately that this operation is strictly commutative (it also follows from the fact that the infinite symmetric product is just the free *abelian* monoid on points in $\mathbb{P}^1$).

3.2 Classification of Line Bundles

It follows from the exponential sequence or from the general theory of classification of vector bundles that $\mathbb{P}^\infty$ is the classifying space for line bundles. Isomorphism classes of line bundles form a group under the tensor product operation. Therefore it is natural to look for the map classifying the tensor product of line bundles. Restricting our attention to projective spaces of finite dimension, this map is the Segré product map:

$$\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{mn+m+n}$$

$$[x_0 : \ldots : x_n], [y_0 : \ldots : y_m] \mapsto [x_0 y_0 : \ldots : x_i y_j : \ldots : x_n y_m]$$

In order to get a well defined map in the colimit, we have to order consistently the entries of the Segré product. This can be achieved in the following way:

$$\mathbb{P}^\infty \times \mathbb{P}^\infty \xrightarrow{\otimes} \mathbb{P}^\infty \tag{3.2}$$

$$([x_0 : \ldots : x_i : \ldots], [y_0 : \ldots : y_j : \ldots]) \tag{3.3}$$

$$\Big\downarrow \otimes$$

$$[x_0 y_0 : \underbrace{x_0 y_1 : x_1 y_0} : \ldots : \underbrace{x_0 y_k : \ldots : x_k y_0}_{\text{subindices add up to } k} : \ldots]$$

This product has an H-inverse j, i.e. a map which makes the following compositions homotopic to the identity:

$$\mathbb{P}^\infty \xrightarrow{(id,j)} \mathbb{P}^\infty \times \mathbb{P}^\infty \xrightarrow{\otimes} \mathbb{P}^\infty \tag{3.4}$$

$$\mathbb{P}^\infty \xrightarrow{(j,id)} \mathbb{P}^\infty \times \mathbb{P}^\infty \xrightarrow{\otimes} \mathbb{P}^\infty \tag{3.5}$$

The inverse map j is defined by

$$j[x_0 : \ldots : x_i : \ldots] = [\bar{x}_0 : \ldots : \bar{x}_i : \ldots]$$

where $\bar{x}$ is the complex conjugate of x. The composition (3.4) then becomes

$$[x_0 : \ldots : x_i : \ldots] \mapsto [\|x_0\|^2 : x_0\bar{x}_1 : x_1\bar{x}_0 : \ldots : x_i\bar{x}_j : \ldots]$$

Next we apply a homotopy to this composition. This homotopy multiplies by $(1-t)$ all coordinates where the subindices are not equal:

$$H([\ldots : x_i : \ldots], t) = [\|x_0\|^2 : (1-t)x_0\bar{x}_1 : (1-t)x_1\bar{x}_0 : \underbrace{(1-t)x_i\bar{x}_j}_{\text{if } i \neq j} : \ldots] \quad (3.6)$$

Notice that the image of $H(-,1)$ lies within a convex subspace of $\mathbb{P}^\infty$, namely the subspace

$$C^+ := \{[\ldots : x_i : \ldots] \in \mathbb{P}^\infty \mid x_i \geq 0\,\forall i \text{ , and } x_j > 0 \text{ for some } j\}$$

Therefore the composition (3.4) is nullhomotopic. Analogously, the composition (3.5) is nullhomotopic.

As it was mentioned in the introduction of this chapter the two products are homotopic.

3.2.1 **Theorem**. *The operations $\otimes$ and $+$ are homotopically equivalent*

Proof. Consider first the homotopy $H_1(x, y, t) : \mathbb{P}^\infty \times \mathbb{P}^\infty \times [0, 1] \to \mathbb{P}^\infty$ given by

$$([x_0 : \ldots : x_n : \ldots], [y_0 : \ldots : y_m : \ldots], t) \tag{3.7}$$
$$\downarrow$$
$$[x_0y_0 : x_0y_1 + tx_1y_0 : \ldots : x_iy_j + t \sum_{\substack{k \neq i \\ 0 \leq k \leq i+j}} x_ky_{i+j-k} : \ldots]$$

(Lemma 3.2.2 proves that H_1 is continuous) Notice that $H_1(x, y, 0) = x \otimes y$ and $H_1(-, -, 1)$ is the function given by

$$[x_0 : \ldots : x_n : \ldots], [y_0 : \ldots : y_m : \ldots] \tag{3.8}$$
$$\downarrow$$
$$[x_0y_0 : \underbrace{x_0y_1 + x_1y_0 : x_1y_0 + x_0y_1}_{\substack{\text{2 terms with indices} \\ \text{adding up to 1}}} : \ldots : \underbrace{\textstyle\sum_{0 \leq k \leq M} x_ky_{M-k}}_{\substack{M+1 \text{ terms with indices} \\ \text{adding up to } M}} : \ldots]$$

There are $M+1$ coordinates containing the term $\sum_{0 \leq k \leq M} x_ky_{M-k}$, if we multiply the last M coordinates by $(1-t)$ we obtain a homotopy between $H(x, y, 1)$ and the function given by

$$[x_0 : \ldots : x_n : \ldots], [y_0 : \ldots : y_m : \ldots] \tag{3.9}$$
$$\downarrow$$
$$[x_0y_0 : x_0y_1 + x_1y_0 : 0 : \ldots : 0 : \textstyle\sum_{0 \leq k \leq M} x_ky_{M-k} : \underbrace{0 : \ldots : 0}_{M \text{ terms}} : \ldots]$$

This last function differs from the operation $+$ just by a permutation of the coordinates. Since GL is connected, the result follows.

$\square$

3.2.2 **Lemma**. *The function $H_1 : \mathbb{P}^\infty \times \mathbb{P}^\infty \times [0, 1] \to \mathbb{P}^\infty$ is continuous*

Proof. Since the coordinate functions are continuous functions, and $\mathbb{P}^\infty$ has the compactly generated topology, it suffices to show that they cannot be all simultaneously equal to zero for any value of $t \in [0,1]$. If $t = 0$ then we get the operation $\otimes$ and if $t = 1$ we get the coordinate functions of the operation $+$. Therefore it suffices to consider $t \in (0,1)$.

Consider the $M+1$ entries involving the variables x_i and y_j with $i+j = M$. If all these entries are zero for some t, then we have the following system of $M+1$ equations:

$$x_M y_0 + t x_1 y_{M-1} + \ldots + t x_0 y_M = 0 \tag{3.10}$$

$$t x_M y_0 + x_1 y_{M-1} + \ldots + t x_0 y_M = 0$$

$$\vdots$$

$$t x_M y_0 + t x_1 y_{M-1} + \ldots + x_0 y_M = 0$$

This system is equivalent to the vanishing of the determinant of the following $(M+1) \times (M+1)$ matrix

$$A = \begin{pmatrix} 1 & t & \ldots & t \\ t & 1 & \ldots & t \\ \vdots & & & \vdots \\ t & t & \ldots & 1 \end{pmatrix} \tag{3.11}$$

But $\det A = (Mt+1)(1-t)^M$. Since the roots are $t = -\frac{1}{M}$ and $t = 1$ we are done (because $t \in (0,1)$).

$\square$

3.2.3 Remark. These products are not only $H-$space structures on $\mathbb{P}^\infty$ but also infinite loop space structures. It is known that the product $+$ coincides with the product of $K(\mathbb{Z},2)$ which gives $K(\mathbb{Z},2)$ its infinite loop space structure.

Using the operation $\otimes$ we can give an explicit description of the maps which classify all bundles obtained from a fixed line bundle by taking tensor products (either positive or negative powers).

3.2.4 **Lemma**. *Let L be a line bundle over a space X and let $f : X \to \mathbb{P}^\infty$ be its classifying map. If*

$$f(x) = [f_0(x) : f_1(x) : \cdots]$$

then for $m > 0$ the map f^m classifies $L^{\otimes m}$ where

$$f^m(x) = [f_0(x) \cdots f_0(x) : \ldots : f_{\alpha_1}(x) \cdots f_{\alpha_m}(x) : \ldots]$$

Also, f^{-1} classifies L^{-1} where

$$f^{-1}(x) = [\overline{f_0(x)} : \cdots]$$

As an application of the flexibility provided by the homotopy between these two operations, an alternative proof of theorem (2.3.4) will be presented.

3.2.5 **Lemma**. *If the degree p Gauss Map is regular, then*

$$(g^p)^*(O(1)) = O_X(d-p)$$

Proof. If g^p is regular, then g^p extends to a map of $\mathbb{P}^n$ which we will denote with the same letter. The statement is equivalent to the following:

$$[g^p] = [L^{\otimes(d-p)}] \in H^2(\mathbb{P}^n, \mathbb{Z})$$

where L is the class of a linear embedding of $\mathbb{P}^n$ in $\mathbb{P}^\infty$ (the inclusion, for example).

The case $p = d-1$ follows from the fact that if g^{d-1} is regular, then it is a linear embedding. To see this note first that it is a linear map and if it had a non-zero kernel, then any non-zero vector ξ in this kernel would satisfy $g^{d-1}(\xi) = 0$ and therefore $\xi \in X$. But then g^{d-1} would not be regular!

Also, if g^p is regular then g^q is regular for any $q \geq p$. Therefore we can proceed by induction. Assume that $[g^{p-1}] = [L^{\otimes(d-(p-1))}]$. Now just observe that $g^p \otimes i \simeq g^{p-1}$ where $i : X \to \mathbb{P}^N$ is the canonical linear embedding of X in $\mathbb{P}^N$. This is because the product $\otimes$ of any function f with i is homotopically equivalent to the integration of each of the coordinate functions with respect to the variables $x_0, \ldots, x_n$ (as long as the integration with respect to the variables is a well-defined function). Since $i \simeq L$ we get the statement of the theorem.

$\square$

Chapter 4

(co-)Tensor Product of codimension 1 cycles

4.1 Introduction

Boyer, Lawson, Lima-Filho, Mann and Michelsohn settled the Segal conjecture in [BLLF+93]. One of the fundamental results which motivated the proof is that there is a geometric construction which extends the map classifying the direct sum of vector bundles in **BU** to the space $\mathcal{Z}$ of all algebraic cycles, namely, the linear join # of cycles, i.e. the following diagram commutes:

$$\begin{tikzcd} \mathbf{BU} \times \mathbf{BU} \arrow[r, "\oplus"] \arrow[d, "c"'] & \mathbf{BU} \arrow[d, "c"] \\ \mathcal{Z} \times \mathcal{Z} \arrow[r, "\#"] & \mathcal{Z} \end{tikzcd}$$

In this paper the authors mention that one would like to have a geometric construction on the space of algebraic cycles which extends the tensor product in the level of **BU** (i.e. degree-one cycles). Segal proved in [Seg74] that **BU** has an infinite loop space structure where the H-space structure is induced by the map classifying the tensor product of vector bundles. Therefore the construction requested by the authors of [BLLF$^+$93] would give yet another infinite loop space structure on $\mathcal{Z}$. My results provide an idea of the extent to which such construction is possible:

4.1.1 **Theorem**. *There is an algebraic biadditive pairing $\hat{\otimes}$ which extends the tensor product to all effective divisors:*

$$\mathcal{C}^1_d(\mathbb{P}^n) \times \mathcal{C}^1_e(\mathbb{P}^m) \xrightarrow{\hat{\otimes}} \mathcal{C}^1_{de}(\mathbb{P}^{mn+m+n})$$

This product is constructed via an algebraic pairing in the corresponding rings of polynomials which may be of interest in its own right. The formula obtained from stabilizing and group-completing the pairing to the stabilized space $\mathcal{Z}^1_0(\mathbb{P}^\infty)$ of algebraic cycles of codimension 1 and degree 0 yields a commutative diagram

$$\begin{tikzcd}
\mathcal{G}^1(\mathbb{P}^\infty) \times \mathcal{G}^1(\mathbb{P}^\infty) \arrow[r] \arrow[d, "c_1 \times c_1"'] & \mathcal{G}^1(\mathbb{P}^\infty) \arrow[d, "c_1"] \\
\mathcal{Z}^1_0(\mathbb{P}^\infty) \times \mathcal{Z}^1_0(\mathbb{P}^\infty) \arrow[r, "\hat{\otimes}"] & \mathcal{Z}^1_0(\mathbb{P}^\infty)
\end{tikzcd}$$

which recovers the group structure in the second cohomology group given by the tensor product of line bundles

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \in H^2(BU_1)$$

The Hurewicz map is the main tool in proving that a general pairing does not exist. The following theorem calculates the classes pulled back by the Hurewicz map.

4.1.2 **Theorem**. *The inclusion of the grassmannian* $\mathcal{G}^1(\mathbb{P}^n)$ *of hyperplanes in* $\mathbb{P}^n$ *into the space* $\mathcal{Z}^1(\mathbb{P}^n)$ *of all cycles in* $\mathbb{P}^n$ *factors through the free group* $\mathbb{Z}\mathcal{G}^1(\mathbb{P}^n)$*:*

$$\begin{tikzcd} \mathcal{G}^1(\mathbb{P}^n) \arrow[r, hook, "i"] & \mathbb{Z}\mathcal{G}^1(\mathbb{P}^n) \arrow[r, hook, "j"] \arrow[d, "="] & \mathcal{Z}^1(\mathbb{P}^n) \arrow[d, "="] \\ & \prod_{j=0}^{n} K(\mathbb{Z}, 2j) & K(\mathbb{Z},0) \times K(\mathbb{Z},2) \end{tikzcd} \tag{4.1}$$

With respect to the canonical product decomposition given in (4.1), the map i *classifies the cohomology class* $1 \times \omega \times \cdots \times \omega^n$ *where* ω *is the multiplicative generator of* $H^2(\mathcal{G}^1(\mathbb{P}^n))$ *and the map* j *is homotopic to the projection* $\pi_0 \times \pi_1$ *onto the first two factors.*

The pairing constructed for divisors in Theorem 4.1.1 cannot be extended to a continuous biadditive pairing on the space of cycles of higher codimension, but it does admit an extension if we restrict the second factor of the pairing to the subgroup $\mathbb{Z}\mathcal{G}^1(\mathbb{P}^m)$ of cycles which are unions of hyperplanes (possibly with multiplicities).

4.1.3 **Theorem**. *There is a continuous biadditive pairing* $\tilde{\otimes}$ *which makes the following diagram commute*

$$\begin{tikzcd} \mathcal{G}^p(\mathbb{P}^n) \times \mathcal{G}^1(\mathbb{P}^m) \arrow[r, "\otimes"] \arrow[d, "c\times h"'] & \mathcal{G}^p(\mathbb{P}^{nm+n+m}) \arrow[d, "c"] \\ \mathcal{Z}_0^p(\mathbb{P}^n) \times \mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^m) \arrow[r, "\tilde{\otimes}"] & \mathcal{Z}_0^p(\mathbb{P}^{nm+n+m}) \end{tikzcd}$$

The relevance of this diagram is twofold, on the one hand it provides a new way of calculating the formula for the total chern class of the tensor product of a vector bundle and a line bundle, namely

$$c_i(E \otimes L) = \sum_{j=0}^{i} \binom{rk(E) - j}{i - j} c_j(E) c_1(L)^{i-j} \tag{4.2}$$

and on the other hand it suggests a path for generalizing the Bott periodicity map which is related to the top arrow of this diagram (the problem for generalizing the Bott map is that there is no "orthogonal complement" in the space of cycles.)

The formula 4.2 does not describe completely the map induced in cohomology by the pairing $\tilde{\otimes}$, since $h^*(i_{2k}^m) = h^*(i_{2t}^s) = \omega^{km}$ if $km = ts$. This calculation can actually be obtained rationally:

4.1.4 **Theorem**. *The pairing*

$$\mathcal{Z}_0^p(\mathbb{P}^n) \times \mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^m) \to \mathcal{Z}^p(\mathbb{P}^{nm+n+m})$$

induces the following map in rational cohomology

$$\tilde{\otimes}^*(i_{2k}) = \sum_{j=0}^{k} \binom{p - j}{k - j} i_{2j} \otimes \tilde{i}_{2(k-j)}$$

where i_{2k} is the fundamental class of the k-th factor of $\mathcal{Z}_0^p(\mathbb{P}^n) \simeq \prod_{j=1}^{p} K(\mathbb{Z}, 2j)$ and $\tilde{i}_{2l}$ is the fundamental class in the l-th factor of $\mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^m) \simeq \prod_{j=1}^{m} K(\mathbb{Z}, 2j)$

Using theorems 4.1.2 and 4.1.4 the following theorem can be proved

4.1.5 **Theorem**. *There is no continuous biadditive pairing in the stabilized* *space of cycles which makes the following diagram commute*

$$\begin{tikzcd} \mathbf{BU} \times \mathbf{BU} \arrow[r, "\otimes"] \arrow[d, "c\times c"'] & \mathbf{BU} \arrow[d, "c"] \\ \mathcal{Z} \times \mathcal{Z} \arrow[r, "\tilde{\otimes}"] & \mathcal{Z} \end{tikzcd}$$

4.2 Tensor Pairing for divisors

In this section we will define a product

$$\tilde{\otimes} : \mathcal{C}^1(\mathbb{P}^{n-1}) \times \mathcal{C}^1(\mathbb{P}^{m-1}) \to \mathcal{C}^1(\mathbb{P}^{mn-1})$$

which is continuous and biadditive (each of the factors has the structure of a topological monoid). This pairing generalizes the pairing on cycles of degree 1 which classifies the tensor product of the universal quotient bundle.

Let $\mathbb{C}[\bar{x}]_d := \mathbb{C}[x_0, \ldots, x_{n-1}]_d$ denote the set of complex polynomials of degree d in the variables $x_0, \ldots, x_{n-1}$. This set is a complex vector space of dimension N. If we order the variables lexicographically, we get the following ordered basis for this vector space:

$$\mathcal{B} := \{x_0 x_0 \cdots x_0, \ldots, x_{j_1} x_{j_2} \cdots x_{j_{n-1}}, \ldots, x_{n-1} \cdots x_{n-1}\}$$

where $j_1 \leq j_2 \leq \ldots \leq j_{n-1}$, i.e. the set of all monomials of degree d ordered lexicographically. Also, throughout this section $\mathbb{C}[\bar{z}]$ will be the polynomial ring in the double-indexed variables zst.

4.2.1 Definition. Let

$$\Psi_{de} : \underbrace{\mathbb{C}[\bar{x}]_d \times \cdots \times \mathbb{C}[\bar{x}]_d}_{e-\text{times}} \times \underbrace{\mathbb{C}[\bar{y}]_e \times \cdots \times \mathbb{C}[\bar{y}]_e}_{d-\text{times}} \to \mathbb{C}[\bar{z}]_{de}$$

be the multilinear homomorphism defined on the elements of the bases $\mathcal{B}$ by

$$\Psi_{de}(x_{j_1^1} \cdots x_{j_d^1}, \ldots, x_{j_1^e} \cdots x_{j_d^e}, y_{k_1^1} \cdots y_{k_e^1}, \ldots, y_{k_1^d} \cdots y_{k_e^d}) = \\ z_{j_1^1 k_1^1} \cdots z_{j_d^1 k_1^d} \cdots z_{j_1^e k_e^1} \cdots z_{j_d^e k_e^d}$$

and extended multilinearly.

The idea is to define the map in the monomials which form a basis. The variables of the monomial that we obtain from two monomials will have two indices, these indices are computed from the original monomials.

Notice that this definition depends on the ordered bases $\mathcal{B}$, in particular, a different order on the elements would yield a different homomorphism.

The function Ψ_{1e} has a particularly nice expression when the degree 1 forms are monomials.

4.2.2 **Lemma**.

$$\Psi_{1e}(x_{i_1}, \ldots, x_{i_e}, g) = g(z_{i_1 j^1}, \ldots, z_{i_e j^e})$$

Proof. Let $g = \sum b_{j_1 \cdots j_e} y_{j_1} \cdots y_{j_e}$. Then, following the definition we get

$$\Psi(x_{i_1}, \ldots, x_{i_e}, g) = \Psi(x_{i_1}, \ldots, x_{i_e}, \sum b_{j_1 \cdots j_e} y_{j_1} \cdots y_{j_e}) =$$
$$\sum b_{j_1 \cdots j_e} \Psi(x_{i_1}, \ldots, x_{i_e}, y_{j_1} \cdots y_{j_e}) = \sum b_{j_1 \cdots j_e} z_{i_1 j_1} \cdots z_{i_e j_e} =$$
$$g(z_{i_1 j_1}, \ldots, z_{i_e j_e})$$

□

With the previous definition, we can now define the tensor product of divisors:

4.2.3 Definition. Given $f \in \mathbb{C}[x_0, \ldots, x_{n-1}]_d$ and $g \in \mathbb{C}[y_0, \ldots, y_{m-1}]_e$ we define $f \tilde{\otimes} g \in \mathbb{C}[\ldots, z_{jk}, \ldots]$ by

$$f \tilde{\otimes} g := \Psi_{de}(\underbrace{f, \ldots, f}_{e-\text{times}}, \underbrace{g, \ldots, g}_{d-\text{times}})$$

4.2.4 Example. Let $f = x_0^2 - 3x_1x_2$ and $g = y_5y_7$. Then

$$f \tilde{\otimes} g = \Psi(f, f, g, g) = \Psi(x_0^2 - 3x_1x_2, x_0^2 - 3x_1x_2, y_5y_7, y_5y_7) =$$
$$\Psi(x_0^2, x_0^2 - 3x_1x_2, y_5y_7, y_5y_7) - 3\Psi(x_1x_2, x_0^2 - 3x_1x_2, y_5y_7, y_5y_7) =$$
$$\Psi(x_0^2, x_0^2, y_5y_7, y_5y_7) - 3\Psi(x_0^2, x_1x_2, y_5y_7, y_5y_7)$$
$$- 3\Psi(x_1x_2, x_0^2, y_5y_7, y_5y_7) + 9\Psi(x_1x_2, x_1x_2, y_5y_7, y_5y_7) =$$
$$z_{05}z_{05}z_{07}z_{07} - 3z_{05}z_{05}z_{17}z_{27} - 3z_{15}z_{25}z_{07}z_{07} + 9z_{15}z_{25}z_{17}z_{27}$$

The next result is the first step toward proving that the pairing is indeed biadditive.

4.2.5 **Proposition**.

$$\Psi_{rm}(f_1,\ldots,f_m,g,\ldots,g)\Psi_{sm}(\phi_1,\ldots,\phi_m,g,\ldots,g) = \Psi_{(r+s)m}(f_1\phi_1,\ldots,f_m\phi_m,g,\ldots,g)$$

Proof. Suppose that

$$\Psi_{rm}(f_1,\ldots,f_m,g,\ldots,g)\Psi_{sm}(\phi_1,\ldots,\phi_m,g,\ldots,g) = \Psi_{(r+s)m}(f_1\phi_1,\ldots,f_m\phi_m,g,\ldots,g)$$

and

$$\Psi_{rm}(F,f_2,\ldots,f_m,g,\ldots,g)\Psi_{sm}(\phi_1,\ldots,\phi_m,g,\ldots,g) = \Psi_{(r+s)m}(F\phi_1,f_2\phi_2,\ldots,f_m\phi_m,g,\ldots,g),$$

then it follows from the multilinearity of Ψ_{ij} that

$$\Psi_{rm}(f_1+cF,f_2,\ldots,f_m,g,\ldots,g)\Psi_{sm}(\phi_1,\ldots,\phi_m,g,\ldots,g) =$$
$$[\Psi_{rm}(f_1,\ldots,f_m,g,\ldots,g)+\Psi_{rm}(cF,f_2,\ldots,f_m,g,\ldots,g)]\,\Psi_{sm}(\phi_1,\ldots,\phi_m,g,\ldots,g) =$$
$$\Psi_{(r+s)m}(f_1\phi_1,f_2\phi_2,\ldots,f_m\phi_m,g,\ldots,g)+\Psi_{(r+s)m}(f_1\phi_1+cF\phi_1,f_2\phi_2,\ldots,f_m\phi_m,g\ldots,g) =$$
$$\Psi_{(r+s)m}((f_1+cF)\phi_1,\ldots,f_m\phi_m,g,\ldots,g).$$

Therefore, it suffices to prove the statement in the case that f_1 is a monic monomial. Analogously, it suffices to prove the statement in the case that every f_i is a monomial and ϕ_i is a monomial. That is, we must show

$$\Psi_{rm}(x_{i_1^1}\cdots x_{i_r^1},\ldots,x_{i_1^m}\cdots x_{i_r^m},g,\ldots,g)\Psi_{sm}(x_{k_1^1}\cdots x_{k_s^1},\ldots,x_{x_1^m}\cdots x_{i_s^m},g,\ldots,g) =$$
$$\Psi_{(r+s)m}(x_{i_1^1}\cdots x_{i_r^1}x_{k_1^1}\cdots x_{k_s^1},\ldots,x_{i_1^m}\cdots x_{i_r^m}x_{k_1^m}\cdots x_{k_m^s},g,\ldots,g) \quad (4.3)$$

without loss of generality we may assume that $i_a^w \leq i_b^w$ if $a \leq b$ and $k_a^w \leq k_b^w$ if $a \leq b$. We will prove equation 4.3 by induction on r and s.

Base: $r = 1$ and $s = 1$. Notice that by lemma 4.2.2

$$\Psi(x_{i_1^1},\ldots,x_{i_1^m},g) = g(z_{i_1^1 j_1^1},\ldots,z_{i_1^m j_m^1})$$

therefore,

$$\Psi(x_{i_1^1},\ldots,x_{i_1^m},g)\Psi(x_{k_1^1},\ldots,x_{k_1^m},g) =$$
$$g(z_{i_1^1 j_1^1},\ldots,z_{i_1^m j_m^1})g(z_{k_1^1 j_1^1},\ldots,z_{k_1^m j_m^1})$$

On the other hand, if

$$g(y_0,\ldots,y_{m-1}) = \sum a_{j_1\cdots j_r} y_{j_1}\cdots y_{j_r}$$

then

$$\Psi(x_{i_1^1}x_{k_1^1},\ldots,x_{i_1^m}x_{k_1^m},g,g)=$$
$$\Psi(x_{i_1^1}x_{k_1^1},\ldots,x_{i_1^m}x_{k_1^m},\sum a_{j_1^1\cdots j_r^1}y_{j_1^1}\cdots y_{j_r^1},\sum a_{j_1^2\cdots j_r^2}y_{j_1^2}\cdots y_{j_r^2})=$$
$$\sum a_{j_1^1\cdots j_r^1}a_{j_1^2\cdots j_r^2}\Psi(x_{i_1^1}x_{k_1^1},\ldots,x_{i_1^m}x_{k_1^m},y_{j_1^1}\cdots y_{j_r^1},y_{j_1^2}\cdots y_{j_r^2})=$$
$$\sum a_{j_1^1\cdots j_r^1}a_{j_1^2\cdots j_r^2}z_{\sigma_1^1 j_1^1}\cdots z_{\sigma_1^m j_m^1}z_{\tau_1^1 j_1^2}\cdots z_{\tau_1^m j_m^2} \quad (4.4)$$

where

$$\sigma_s^t=\begin{cases} i_s^t, & \text{if } i_s^t\le k_s^t \\ k_s^t, & \text{if } i_s^t> k_s^t\end{cases} \quad \text{and} \quad \tau_s^t=\begin{cases} k_s^t, & \text{if } i_s^t\le k_s^t \\ i_s^t, & \text{if } i_s^t> k_s^t\end{cases}$$

Now, notice that if we exchange the definition of σ and τ the sum on the right hand side of 4.4 remains unchanged. This happens because we are taking two copies of g. Therefore we may assume without loss of generality that $\sigma_s^t=i_s^t$ and $\tau_s^t=k_s^t$. In this case, equation 4.4 becomes

$$\Psi(x_{i_1^1}x_{k_1^1},\ldots,x_{i_1^m}x_{k_1^m},g,g)=$$
$$\sum a_{j_1^1\cdots j_r^1}a_{j_1^2\cdots j_r^2}z_{i_1^1 j_1^1}\cdots z_{i_1^m j_m^1}z_{k_1^1 j_1^2}\cdots z_{k_1^m j_m^2}=$$
$$\left(\sum a_{j_1^1\cdots j_r^1}z_{i_1^1 j_1^1}\cdots z_{i_1^m j_m^1}\right)\left(\sum a_{j_1^2\cdots j_r^2}z_{k_1^1 j_1^2}\cdots z_{k_1^m j_m^2}\right)=$$
$$g(z_{i_1^1 j_1^1},\ldots,z_{i_1^m j_m^1})g(z_{k_1^1 j_1^1},\ldots,z_{k_1^m j_m^1}) \quad (4.5)$$

and the base of the induction is proved.

Inductive step: Essentially the same argument proves that both sides of 4.3 are equal to

$$g(z_{i_1^1 j_1^1}, \dots, z_{i_1^m j_m^1}) \cdots g(z_{i_r^1 j_1^r}, \dots, z_{i_r^m j_m^r}) \cdot$$

$$g(z_{k_1^1 j_1^1}, \dots, z_{k_1^m j_m^1}) \cdots g(z_{k_s^1 j_1^s}, \dots, z_{k_s^m j_m^s}) \quad (4.6)$$

□

4.2.6 **Corollary**.

$$(f_1 f_2)\tilde{\otimes} g = (f_1 \tilde{\otimes} g)(f_2 \tilde{\otimes} g)$$

Proof. This follows immediately from the definitions and the proposition. □

Analogously the following is true

4.2.7 **Corollary**.

$$f\tilde{\otimes}(g_1 g_2) = (f\tilde{\otimes} g_1)(f\tilde{\otimes} g_2)$$

This theorem gives some insight into the geometry of the hypersurfaces obtained as $\tilde{\otimes}$ products. The next lemma provides the description in the case that one of the factors is a linear space.

4.2.8 **Lemma**. *If* $f = \sum a_i x_i$ *then*

$$f\tilde{\otimes} g = g\left(f\left(z_{00}, \dots, z_{(n-1)0}\right), \dots, f\left(z_{0(m-1)}, \dots, z_{(n-1)(m-1)}\right)\right)$$

That is, the codimension 1 *cycle defined by* $f\tilde{\otimes} g$ *is isomorphic to the linear join of a linear space and the cycle defined by* g *(Using Lawson's terminology, it is the iterated suspension of the cycle defined by* g*)*

Proof. Let $g = \sum a_{j_1 \cdots j_e} y_{j_1} \cdots y_{j_e}$. Then,

$$f \tilde{\otimes} g = \Psi(f, \ldots, f, g) = \Psi(f, \ldots, f, \sum a_{j_1 \cdots j_e} y_{j_1} \cdots y_{j_e}) =$$
$$\sum a_{j_1 \cdots j_e} \Psi(f, \ldots, f, y_{j_1} \cdots y_{j_e}) = \sum a_{j_1 \cdots j_e} f \tilde{\otimes} (y_{j_1} \cdots y_{j_e}) =$$
$$\sum a_{j_1 \cdots j_e} (f \tilde{\otimes} y_{j_1}) \cdots (f \tilde{\otimes} y_{j_e})$$

This last expression is exactly what we are looking for, it says that we should substitute the variable y_{j_i} in the polynomial g with the polynomial $f \tilde{\otimes} y_{j_i}$ which in turn is equal to the polynomial f evaluated in the variables $z_{0 j_i}, \ldots, z_{(n-1) j_i}$

□

Now let us recall that there is a one to one correspondence between homogeneous polynomials in the variables $x_0, \ldots, x_s$ and the codimension 1 algebraic cycles in $\mathbb{P}^s$. The correspondence is given in the following way: If f is a polynomial and it decomposes as a product $f_1^{\alpha_1} \cdots f_t^{\alpha_t}$ where each f_k is irreducible, then the corresponding cycle $\mathcal{C}(f)$ is given by $\sum \alpha_i V(f_i)$ where $V(f_i)$ is the (necessarily irreducible) variety defined by the polynomial f_i. The disjoint union of all codimension cycles $\mathcal{C}^1(\mathbb{P}^s)$ forms a monoid with respect to the formal addition of cycles. The following theorem expresses the results of this section in terms of cycles.

4.2.9 **Theorem**. *There is an algebraic pairing $\tilde{\otimes}$ in the space of codimension 1 cycles in projective space:*

$$\tilde{\otimes} : \mathcal{C}^1(\mathbb{P}^{n-1}) \times \mathcal{C}^1(\mathbb{P}^{m-1}) \to \mathcal{C}^1(\mathbb{P}^{mn-1})$$

which satisfies the following properties

1. $\tilde{\otimes}$ *coincides with the tensor product* $\otimes$ *on linear cycles.*

2. $\tilde{\otimes}$ *is biadditive:*

$$\eta_1\eta_2\tilde{\otimes}\xi = \eta_1\tilde{\otimes}\xi + \eta_2\tilde{\otimes}\xi$$

   and

$$\eta\tilde{\otimes}\xi_1\xi_2 = \eta\tilde{\otimes}\xi_1 + \eta\tilde{\otimes}\xi_2$$

3. $\tilde{\otimes}$ *stabilizes to a pairing*

$$\tilde{\otimes} : \mathcal{C}^1(\mathbb{P}^\infty) \times \mathcal{C}^1(\mathbb{P}^\infty) \to \mathcal{C}^1(\mathbb{P}^\infty)$$

Since the pairing is biadditive, it induces a pairing in the group completion

$$\tilde{\otimes} : \mathcal{Z}^1(\mathbb{P}^N) \times \mathcal{Z}^1(\mathbb{P}^M) \to \mathcal{Z}^1(\mathbb{P}^{NM+N+M})$$

Following the idea used for the join pairing in [LJM88] we construct the associated pairing $\hat{\otimes}$:

$$\hat{\otimes}(\eta, \xi) := \eta\tilde{\otimes}\xi + \eta\tilde{\otimes}\xi_0 + \eta_0\tilde{\otimes}\xi$$

where ξ_0 and η_0 are two fixed hyperplanes.

4.2.10 **Theorem**. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{G}^1(\mathbb{P}^\infty) \times \mathcal{G}^1(\mathbb{P}^\infty) & \longrightarrow & \mathcal{G}^1(\mathbb{P}^\infty) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} \\ \mathcal{Z}^1(\mathbb{P}^\infty) \times \mathcal{Z}^1(\mathbb{P}^\infty) & \xrightarrow{\hat{\otimes}} & \mathcal{Z}^1(\mathbb{P}^\infty) \end{array}$$

Lawson and Michelsohn also prove in [LJM88] that the space $\mathcal{Z}^1(\mathbb{P}^\infty)$ splits as $\mathbb{Z} \times \mathcal{Z}^1_0(\mathbb{P}^\infty)$, where $\mathcal{Z}^1_0(\mathbb{P}^\infty)$ is the subgroup of all cycles of degree zero.

Since we know that $\deg(\eta \tilde{\otimes} \xi) = \deg(\eta)\deg(\xi)$ we only have to calculate what happens with the pairing $\hat{\otimes}$ when we restrict it to cycles of degree 0. Lawson proved that $\mathcal{Z}^1_0(\mathbb{P}^\infty)$ is an Eilenberg-Maclane space of type $K(\mathbb{Z}, 2)$. Using this fact we will show that the pairing $\tilde{\otimes}$ restricted to the subgroup of cycles of degree zero is nullhomotopic.

4.2.11 **Theorem**. *Any continuous biadditive pairing*

$$\tilde{\otimes} : \mathcal{Z}^1_0(\mathbb{P}^\infty) \times \mathcal{Z}^1_0(\mathbb{P}^\infty) \to \mathcal{Z}^1_0(\mathbb{P}^\infty)$$

is nullhomotopic.

Proof. Since the pairing is biadditive it factors through the smash product

$$\mathcal{Z}^1_0(\mathbb{P}^\infty) \wedge \mathcal{Z}^1_0(\mathbb{P}^\infty) \to \mathcal{Z}^1_0(\mathbb{P}^\infty)$$

homotopically this last function is equivalent to

$$K(\mathbb{Z}, 2) \wedge K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)$$

Now, notice that $K(\mathbb{Z},2)\wedge K(\mathbb{Z},2)$ is a CW-complex with cells only in dimension 4 and higher, therefore, the pullback in the second cohomology groups of the fundamental class in $K(\mathbb{Z},2)$ is zero. $\square$

This theorem allows us to calculate the class pulled back via the pairing $\hat{\otimes}$.

4.2.12 **Corollary**. *Let $\hat{\otimes}$ be the pairing*

$$\hat{\otimes}(\eta,\xi) := \eta\tilde{\otimes}\xi + \eta\tilde{\otimes}\xi_0 + \eta_0\tilde{\otimes}\xi$$

and let i_2 be the fundamental class in $H^2(\mathcal{Z}^1(\mathbb{P}^{NM+N+M});\mathbb{Z})$. Then

$$\hat{\otimes}^*(i_2) = i_2 \otimes i_0 + i_0 \otimes i_2$$

Proof. The formula follows at once from pulling back the last two summands of the pairing $\hat{\otimes}$, since by the previous theorem the first summand is nullhomotopic. $\square$

4.2.13 **Corollary**. *Let L_1 and L_2 be two line bundles. Then*

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

where c_1 denotes the first chern class.

Proof. Lawson and Michelsohn proved in [LJM88] that the inclusion i in theorem 4.2.10 classifies the chern class of the universal quotient bundle. Therefore the formula follows from the previous corollary. $\square$

4.3 Topological Obstruction for a General Pairing

The pairing constructed in the last section might be considered as a hint for a pairing in higher codimensions. We will prove that there is a topological obstruction for the existence of such a pairing. The general strategy is to factor the inclusion of the grassmannian $\mathcal{G}^p(\mathbb{P}^n)$ into the space $\mathcal{Z}^p(\mathbb{P}^n)$ of all codimension p cycles. This inclusion factors through the free abelian group $\mathbb{Z}\mathcal{G}^p(\mathbb{P}^n)$ generated by the points of the grassmannian. The existence of this factorization and the chern class formula for the tensor product of bundles will yield a contradiction if we assume the existence of a pairing.

Let us start by observing that the Dold-Thom theorem implies that the free abelian group $\mathbb{Z}\mathcal{G}^1(\mathbb{P}^n)$ is homotopically equivalent to the product $\prod_{i=0}^n K(\mathbb{Z}, 2i)$. Also, if we consider the subgroup $\mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^n)$ which is the kernel of the degree homomorphism i.e. the subgroup of 0-dimensional cycles of degree 0, we get the following homotopy equivalence

$$\mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^n) \simeq \prod_{i=1}^n K(\mathbb{Z}, 2i)$$

Dold and Thom also proved that the inclusion

$$i : \mathcal{G}^1(\mathbb{P}^n) \hookrightarrow \mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^n)$$

induces the Hurewicz map when the π_i functors are applied. The next theorem calculates the class pulled back in cohomology by the inclusion i.

4.3.1 **Theorem**. *The Hurewicz map*

$$h : \mathcal{G}^1(\mathbb{P}^n) \hookrightarrow \mathbb{Z}\mathcal{G}^1(\mathbb{P}^n) \simeq \prod_{i=0}^{n} K(\mathbb{Z}, 2i)$$

induces the following map in cohomology

$$h^*(i_{2k}) = \omega^k$$

where i_{2k} is the generator of $H^{2k}(K(\mathbb{Z}, 2k), \mathbb{Z})$ and ω is the generator of $H^2(\mathcal{G}^1(\mathbb{P}^n), \mathbb{Z})$.

Proof. By induction on n.

Base: The case $n = 1$ is a result of Lawson and Michelsohn in [LJM88]. Namely, they prove that the inclusion $i : \mathcal{G}^1(\mathbb{P}^1) \hookrightarrow \mathcal{Z}^1(\mathcal{G}^1(\mathbb{P}^1))$ classifies the total chern class of the universal quotient bundle, i.e. that $i \simeq 1 \times \omega$. But in this case $\mathcal{Z}^1(\mathcal{G}^1(\mathbb{P}^1)) = \mathbb{Z}\mathcal{G}^1(\mathbb{P}^1)$ and i is the Hurewicz map h. Thus $h \simeq 1 \times \omega$.

Inductive Step: Notice that $\mathcal{G}^1(\mathbb{P}^n) = \mathbb{P}^{n^\vee} \cong \mathbb{P}^n$ so we will substitute throughout $\mathcal{G}^1(\mathbb{P}^n)$ with $\mathbb{P}^n$ Suppose that $h^*(i_{2k}) = \omega^k$ for $h : \mathbb{P}^n \hookrightarrow \mathbb{Z}\mathbb{P}^n$. The inclusion of $\mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$ is a cofibration and the quotient $\mathbb{P}^{n+1}/\mathbb{P}^n$ is homeomorphic to the sphere $\mathbb{S}^{2(n+1)}$. Dold and Thom proved in [DT58] that a cofibration sequence induces a quasifibration sequence when taking the free abelian group functor. Hence we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, hook, "j"] \arrow[d, "h"'] & \mathbb{P}^{n+1} \arrow[r] \arrow[d, "h"] & \mathbb{S}^{2(n+1)} \arrow[d, "h"] \\
\mathbb{Z}\mathbb{P}^n \arrow[r, hook] & \mathbb{Z}\mathbb{P}^{n+1} \arrow[r, "p"] & \mathbb{Z}\mathbb{S}^{2(n+1)}
\end{tikzcd}$$

where j is a cofibration, p is a quasifibration and each h is the corresponding Hurewicz map. This diagram is equivalent to the following

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, hook, "j"] \arrow[d, "h"'] & \mathbb{P}^{n+1} \arrow[r] \arrow[d, "h"] & \mathbb{S}^{2(n+1)} \arrow[d, "h"] \\ \prod_{i=0}^{n} K(\mathbb{Z},2i) \arrow[r, hook] & \prod_{i=0}^{n+1} K(\mathbb{Z},2i) \arrow[r, "\pi_{n+1}"] & K(\mathbb{Z},2(n+1)) \end{tikzcd}$$

The induction hypothesis implies that the arrow on the left satisfies the condition $h^*(i_{2k}) = \omega^k$. Therefore we are only concerned with what happens to the pullback of $i_{2(n+1)}$. But this is determined by the Hurewicz map on the far right of the diagram. □

4.3.2 **Theorem**. *For $p > 1$ there is no continuous biadditive pairing*

$$\hat{\otimes} : \mathcal{Z}^1(\mathbb{P}^n) \times \mathcal{Z}^p(\mathbb{P}^m) \to \mathcal{Z}^p(\mathbb{P}^{nm+n+m})$$

such that $\eta \hat{\otimes} \xi = \eta \otimes \xi$ where η and ξ are linear spaces and $\otimes$ is the map which classifies the tensor product of bundles via the universal quotient bundle.

Proof. Suppose that such a pairing exists. Then it must necessarily satisfy the following relation in the degrees

$$\deg(\eta \hat{\otimes} \xi) = \deg(\eta) \deg(\xi)$$

This is because it is biadditive and continuous and it maps the degree one effective cycles into the degree one effective cycles. Thus it induces a continuous pairing in the subgroup $\mathcal{Z}_0$ of cycles of degree zero:

$$\mathcal{Z}_0^1(\mathbb{P}^n) \times \mathcal{Z}_0^p(\mathbb{P}^m) \to \mathcal{Z}_0^1(\mathbb{P}^{nm+n+m})$$

Let $\mu : \mathcal{Z}_0^1(\mathbb{P}^n) \times \mathcal{Z}^p(\mathbb{P}^m) \to \mathcal{Z}_0^1(\mathbb{P}^{nm+n+m})$ be the function defined by

$$\mu(\eta, \xi) = \eta \hat{\otimes} \xi + \eta_0 \hat{\otimes} \xi + \eta \hat{\otimes} \xi_0$$

Then the following diagram commutes:

$$\begin{tikzcd} \mathcal{G}^1(\mathbb{P}^n) \times \mathcal{G}^p(\mathbb{P}^m) \arrow[r, "\otimes"] \arrow[d, "c"'] & \mathcal{G}^p(\mathbb{P}^{nm+n+m}) \arrow[d, "c"] \\ \mathcal{Z}_0^1(\mathbb{P}^n) \times \mathcal{Z}_0^p(\mathbb{P}^m) \arrow[r, "\mu"] & \mathcal{Z}_0^p(\mathbb{P}^{nm+n+m}) \end{tikzcd} \tag{4.7}$$

where the vertical maps are the inclusions mapping a linear space η into $\eta - \eta_0$ where η_0 is a fixed subspace. Lawson and Michelsohn proved in [LJM88] that this inclusion classifies the total chern class map of the universal quotient bundle. Now, notice that we can restrict the pairing μ on the first factor to the subspaces $\mathbb{Z}_0\mathcal{G}^1$ of cycles generated by the points of the grassmannian, that is, to the cycles which are formal sums of linear hypersurfaces with coefficients adding up to zero. Let ρ be the restriction, then we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{G}^1(\mathbb{P}^n) \times \mathcal{G}^p(\mathbb{P}^m) \arrow[r, "\otimes"] \arrow[d, "i\times c"'] & \mathcal{G}^p(\mathbb{P}^{nm+n+m}) \arrow[d, "c"] \\
\mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^n) \times \mathcal{Z}_0^p(\mathbb{P}^m) \arrow[r, "\rho"] \arrow[d, "j\times id"'] & \mathcal{Z}_0^p(\mathbb{P}^{nm+n+m}) \arrow[d, "id"] \\
\mathcal{Z}_0^1(\mathbb{P}^n) \times \mathcal{Z}_0^p(\mathbb{P}^m) \arrow[r, "\mu"] & \mathcal{Z}_0^p(\mathbb{P}^{nm+n+m})
\end{tikzcd}$$

where i is the same map as before, $i(L) = L - L_0$ and j is just the natural inclusion. The previous theorem gives a description of what i and j are in terms of the homotopy equivalences with the products of Eilenberg-Maclane spaces, namely i is the Hurewicz map and j is the projection onto the first factor. Hence we have the following diagram:

$$\begin{tikzcd}
\mathcal{G}^1(\mathbb{P}^n) \times \mathcal{G}^p(\mathbb{P}^m) \arrow[r, "\otimes"] \arrow[d, "h\times c"'] & \mathcal{G}^p(\mathbb{P}^{nm+n+m}) \arrow[d, "c"] \\
\prod_{i=1}^n K(\mathbb{Z},2i) \times \prod_{i=1}^p K(\mathbb{Z},2i) \arrow[r, "\rho"] \arrow[d, "\pi_1\times id"'] & \prod_{i=1}^{nm+n+m} K(\mathbb{Z},2i) \arrow[d, "id"] \\
K(\mathbb{Z},2) \times \prod_{i=1}^p K(\mathbb{Z},2i) \arrow[r, "\mu"] & \prod_{i=1}^{nm+n+m} K(\mathbb{Z},2i)
\end{tikzcd} \tag{4.8}$$

Now, to fix ideas, let us examine the case $p = 2$. The chern class formula for the tensor product of a line bundle and a 2-dimensional bundle yields:

$$c_2(L \otimes E) = c_1^2(L) + c_1(E)c_1(L) + c_2(E)$$

The vertical arrows in the diagram 4.7 induce isomorphisms in rational cohomology. So the chern class formula implies that in the 4-th cohomology groups

ρ should induce the following map:

$$\rho^*(i_4) = 1 \otimes i_4 + i_2 \otimes i_2 + ai_4 \otimes 1 + bi_2^2 \otimes 1$$

where each i_k is the generator of $H^k(K(\mathbb{Z}, 2k); \mathbb{Q})$ and $a + b = 1$.

We claim that $a = 1$ and $b = 0$. This claim is the content of proposition 4.3.3. The argument to prove the theorem is then the following:

The existence of the product $\hat{\otimes}$ implies the existence of the function μ which in turn implies the existence of the restriction ρ. But then, the claim implies that the diagram 4.8 cannot commute!

This is because diagram 4.8 implies that

$$\rho^*(i_4) = (\pi_1 \times id)^* \mu^*(i_4)$$

but there is no element in $H^4(K(\mathbb{Z}, 2) \times \prod_{i=1}^{2} K(\mathbb{Z}, 2i); \mathbb{Q})$ which gets pulled back to $i_4 \otimes 1$ in $H^4(\prod_{i=1}^{n} K(\mathbb{Z}, 2i) \times \prod_{i=1}^{2} K(\mathbb{Z}, 2i); \mathbb{Q})$ because π_1 is the projection into the first factor:

$$\pi_1 : \prod_{i=1}^{n} K(\mathbb{Z}, 2i) \to K(\mathbb{Z}, 2)$$

therefore we can only pullback elements of the form $ai_2^p \otimes j$ where i_2 is the generator of

$$H^*(K(\mathbb{Z},2);\mathbb{Q}) = \mathbb{Q}[i_2] \subset H^*(\prod_{i=1}^{n} K(\mathbb{Z},2i);\mathbb{Q}) = \mathbb{Q}[i_2,\ldots,i_{2n}]$$

(These last equalities being a classical result of Serre).

$\square$

Now we prove the claim mentioned in theorem 4.3.2

4.3.3 **Proposition**. *Using the notation of theorem 4.3.2 we have the formula*

$$\rho^*(i_4) = 1 \otimes i_4 + i_2 \otimes i_2 + i_4 \otimes 1$$

Proof. The chern class formula for the tensor product of bundles and the commutativity of 4.8 implies that

$$\rho^*(i_4) = 1 \otimes i_4 + i_2 \otimes i_2 + a(i_4 \otimes 1) + b(i_2^2 \otimes 1)$$

with $a + b = 1$. Consider the diagram

$$\begin{array}{ccc} [\mathcal{G}^1(\mathbb{P}^n) \times \mathcal{G}^1(\mathbb{P}^n)] \times \mathcal{G}^2(\mathbb{P}^m) & \xrightarrow{\otimes_1 \times \otimes_2} & \mathcal{G}^2(\mathbb{P}^{mn+n+m}) \times \mathcal{G}^2(\mathbb{P}^{mn+n+m}) \\ \Big\downarrow{\scriptstyle \phi\times c} & & \Big\downarrow{\scriptstyle c+c} \\ \mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^n) \times \mathcal{Z}_0^2(\mathbb{P}^m) & \xrightarrow{\rho'} & \mathcal{Z}_0^2(\mathbb{P}^{nm+n+m}) \end{array} \tag{4.9}$$

where

- $\phi : \mathcal{G}^1(\mathbb{P}^n) \times \mathcal{G}^1(\mathbb{P}^n)] \to \mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^n)$ is given by

$$\phi(L_1, L_2) = (L_1 - L_0) + (L_2 - L_0)$$

where L_0 is a fixed linear space.

- $\otimes_1 \times \otimes_2$ is given by

$$(\otimes_1 \times \otimes_2)(L_1, L_2, E) = (L_1 \otimes E, L_2 \otimes E)$$

- $c + c$ is given by

$$(c + c)(E_1, E_2) = (E_1 - L_0 \otimes E_0) + (E_2 - L_0 \otimes E_0)$$

- $\rho' = \rho + \tau$ where

$$\tau(\eta, \xi) = L_0 \hat{\otimes} \xi$$

We will verify that diagram 4.9 commutes:

$$(c + c)(\otimes_1 \times \otimes_2)(L_1, L_2, E) = (c + c)(L_1 \otimes E, L_2 \otimes E) =$$
$$(L_1 \otimes E - L_0 \otimes E_0) + (L_2 \otimes E - L_0 \otimes E_0)$$

On the other hand:

$$\rho'(\phi \times c)(L_1, L_2, E) = \rho'((L_1 - L_0) + (L_2 - L_0), E - E_0) =$$
$$\rho((L_1 - L_0) + (L_2 - L_0), E - E_0)) + \tau((L_1 - L_0) + (L_2 - L_0), E - E_0) =$$
$$\rho((L_1 - L_0) + (L_2 - L_0), E - E_0)) + L_0 \hat{\otimes} (E - E_0) =$$
$$((L_1 - L_0) + (L_2 - L_0)) \hat{\otimes} (E - E_0) + 2(L_0 \hat{\otimes} (E - E_0)) + ((L_1 - L_0) + (L_2 - L_0)) \hat{\otimes} E_0 =$$
$$L_1 \hat{\otimes} E + L_2 \hat{\otimes} E - 2(L_0 \hat{\otimes} E_0)$$

Now recall that the space $\mathcal{Z}_0^2(\mathbb{P}^{nm+n+m})$ on the lower right corner of diagram 4.9 is homotopically equivalent to $K(\mathbb{Z},2) \times K(\mathbb{Z},4)$. We will compute the pullback through the whole diagram of the generator i_4 of the cohomology group $H^4(K(\mathbb{Z},4);\mathbb{Q})$, considered as a subgroup

$$H^4(K(\mathbb{Z},4);\mathbb{Q}) \subset H^4(K(\mathbb{Z},2) \times K(\mathbb{Z},4);\mathbb{Q}) = \mathbb{Q}i_4 \oplus \mathbb{Q}i_2^2$$

To simplify the notation we will denote by L_1, L_2 and E the universal quotient bundles on $\mathcal{G}^1$, $\mathcal{G}^1$ and $\mathcal{G}^2$ correspondingly. Then the chern class formula for the tensor product and the fundamental result of [LJM88] compute the composition $(\otimes_1 \times \otimes_2)^*(c+c)^*$:

$$\begin{aligned}(\otimes_1 \times \otimes_2)^*(c+c)^*(i_4) &= (\otimes_1 \times \otimes_2)^*(c_2(E) \otimes c_2(E)) = \\ c_1(L_1)^2 + c_1(L_1)c_1(E) + c_2(E) &+ c_1(L_2)^2 + c_1(L_2)c_1(E) + c_2(E)\end{aligned} \quad (4.10)$$

Now, notice that theorem 4.1.2 implies that in rational cohomology

$$\phi^*(i_2) = \omega \otimes 1 + 1 \otimes \omega \ \text{ and } \ \phi^*(i_4) = \omega^2 \otimes 1 + 1 \otimes \omega^2 \quad (4.11)$$

Hence, using the previous equation and the description that we have for ρ we get

$$\begin{aligned}(\phi \times id)^*(\rho')^*(i_4) = (\phi \times id)^*(i_2 \otimes i_2 + 1 \otimes i_4 + a(i_2^2 \otimes 1) + b(i_4 \otimes 1) + 1 \otimes i_4) = \\ c_1(L_1)c_1(E) + c_1(L_2)c_1(E) + c_2(E) + \\ a(c_1(L_1) + c_1(L_2))^2 + b(c_1(L_1)^2 + c_1(L_2)^2) + c_2(E)\end{aligned} \quad (4.12)$$

Setting equal the compositions 4.10 and 4.12 we get that $a = 0$ and therefore $b = 1$, since there is no term $2c_1(L_1)c_1(L_2)$ in 4.10.

$\square$

4.4 A Pairing for Higher Codimension

In the previous section we proved that there is no pairing in the space of cycles which extends the map which classifies the tensor product of bundles. The proof is based on the fact that a pairing on the space of cycles would induce by restriction a pairing in the free group generated by the points of the grassmannian.This induced pairing would in turn make it impossible to have a pairing in the space of cycles. In this section we will prove that there is indeed a pairing restricting the first factor to the subgroup of the space of cycles generated by the points of the grassmannian.

4.4.1 Definition. Let H be a linear hypersurface defined by a linear form

$$L = \sum a_i x_i \in \mathbb{C}[x_0, \dots, x_n]$$

.

Let X be an algebraic cycle defined by an ideal

$$I = \langle f_1, \dots, f_s \rangle \text{ with } f_i \in \mathbb{C}[y_0, \dots, y_m]$$

.

The *tensor product* of H and X, denoted by $H\tilde{\otimes}X$ is the algebraic cycle defined by the ideal

$$L\tilde{\otimes}I = \langle f_1\tilde{\otimes}L, \dots, f_s\tilde{\otimes}L \rangle$$

Lemma 4.2.8 implies that this definition does not depend on the choice of generators for the ideal defining X. Geometrically the cycle $H\tilde{\otimes}X$ is isomorphic to a suspension of X, the position of this suspension depends on the linear form defining H. This proves the following lemma

4.4.2 **Lemma**. *The function*

$$\tilde{\otimes} : \mathcal{G}^1(\mathbb{P}^n) \times \mathcal{Z}^p(\mathbb{P}^n) \to \mathcal{Z}^p(\mathbb{P}^{nm+n+m})$$

defined by

$$(H, \eta) \mapsto H\tilde{\otimes}\eta$$

is continuous and additive in the second variable.

This continuous function in turn, induces a continuous function

$$\mathcal{G}^1(\mathbb{P}^n) \to \mathbf{Hom}\left(\mathcal{Z}^p(\mathbb{P}^n), \mathcal{Z}^p(\mathbb{P}^{nm+n+m})\right)$$

defined by

$$H \mapsto \{X \mapsto L\tilde{\otimes}X\}$$

by the universal property of the free abelian topological group, this function factors through a function

$$\hat{\otimes}' : \mathbb{Z}\mathcal{G}^1(\mathbb{P}^n) \to \mathbf{Hom}\left(\mathcal{Z}^p(\mathbb{P}^n), \mathcal{Z}^p(\mathbb{P}^{nm+n+m})\right)$$

4.4.3 Definition. The pairing

$$\hat{\otimes} : \mathbb{Z}\mathcal{G}^1(\mathbb{P}^n) \times \mathcal{Z}^p(\mathbb{P}^n) \to \mathcal{Z}^p(\mathbb{P}^{nm+n+m})$$

is the continuous pairing defined by

$$\hat{\otimes}(\eta, \xi) = \hat{\otimes}'(\eta)(\xi)$$

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