Neighborly Properties of Simple Convex Polytopes

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Abstract of the Dissertation

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We are interested in simple polytopes with all facets being pervasive and powerful, which will be defined in the first chapter. It is easy to see that simplexes and product of simplexes have these properties. In the paper "`Polytopes with Mass Linear Functions`" [1], Dusa McDuff and Susan Tolman proved that in dimension 4 the only simple convex polytopes with all facets pervasive and powerful are the 4-simplex and $\Delta_2 \times \Delta_2$. In this thesis we prove that there are no such pervasive and powerful polytopes with 8 or 9 facets in dimension 5, there is one in dimension 6 but still being a product of simplexes, and there is a non product one in dimension 7.

Contents

Chapter 1

Introduction

1.1 Definitions and Examples

We begin with the definitions of the two properties we are interested in. We shall consider compact polytopes in $\mathbf{R}^n$ that are convex and simple, that is, each vertex is the intersection of n facets. The facets are co dimension one faces. The standard n simplex is denoted as Δ_n.

Definition 1.1.1. *A facet F of a polytope Δ is said to be* **pervasive** *if it has nonempty intersection with every other facet of Δ. A polytope is called* **pervasive** *if all its facets are pervasive.*

Definition 1.1.2. *A facet F of a polytope Δ is said to be* **powerful** *if it is connected by an edge to each vertex which does not belong to F.*
Similarly, a vertex is said to be **powerful** *if it is connected by an edge to each facet to which it does not belong.*
We call a polytope **powerful** *if all its facets are powerful, which is equivalent to a polytope with all vertices being powerful.*

One can easily see that simplexes are both pervasive and powerful. Products of simplexes like $\Delta_m \times \Delta_n$ where m, $n \geq 2$ are non trivial examples of polytopes that are both pervasive and powerful. If one tries to form products with Δ_1, which is a line segment, that will destroy the pervasive property since the two facets at both ends of the line segment do not intersect with each other.

These two properties do not always come together. In particular a 2-dimensional square is powerful but not pervasive. Later on we will see a lot of ways to create pervasive polytopes which are not powerful by cutting products of simplexes with a hyperplane. There is also an interesting example from cyclic polytopes, which will be discussed in chapter 2.

The interest in pervasive and powerful polytopes arose from Dusa McDuff and Susan Tolman [1]. In order to use the strongest results in their paper, when we find pervasive and powerful polytopes we will examine them further to see if they are smooth.

Definition 1.1.3. *Let ℓ be a lattice. A polytope Δ is* **rational** *if we can choose the outward conormals η_i to lie in ℓ; in this case, we always choose η_i to be* **primitive***, that is, not a positive integer multiple of any other lattice element. A rational polytope is* **smooth** *if the primitive outward conormals η_i to the facets which meet at each vertex form a basis for ℓ.*

1.2 Main Results

In [1], Dusa McDuff and Susan Tolman proved that in dimension 4 there are no pervasive and powerful polytopes except for the 4-simplex and the product

$\Delta_2 \times \Delta_2$. In this thesis we discuss mainly the pervasive and powerful polytopes in dimension 5, 6, and 7, and we have the following results.

Theorem 1.2.1. *There are no pervasive and powerful polytopes with 8 or 9 facets in dimension 5.*

This will be proved in chapter 4 and 7.

In order to have a powerful polytope in dimension 5, the most facets a polytope can have is 10. That is because each vertex is the intersection of 5 facets, and it only has 5 edges to connect to other facets. We have not ruled out the possibility of a 10-facet pervasive and powerful polytope in dimension 5 yet, but we have some partial results in chapter 8.

Theorem 1.2.2. *In dimension 6, there is only one 9-facet pervasive and powerful polytope, which is combinatorially equivalent to* $\Delta_2 \times \Delta_2 \times \Delta_2$.

This will be proved in chapter 5.

Theorem 1.2.3. *In dimension 7, there exists a 10-facet pervasive and powerful polytope, which is not a product of simplexes.*

This will be proved in chapter 6. We have some partial results regarding to the smoothness of this polytope in the same chapter.

Chapter 2

Cyclic Polytopes

2.1 Definition and Dual properties

Recall that an n-dimensional cyclic polytope is the convex hull of vertices of the form,

$$(t_i, t_i^2, t_i^3, \cdots, t_i^n)$$

where $t_i \in \mathbf{R}$, $i = 1, 2, \cdots, k$, and $k \geq n$, and we always assume $t_1 < \cdots < t_k$ when we list them in this way. Cyclic polytopes are simplicial, so the dual polytopes are simple. It has been proved very early that when dimension ≥ 4, cyclic polytopes have the property called 2-neighborly, which means that there exists an edge connecting any 2 randomly chosen vertices. Consider the dual polytopes of cyclic polytopes. The dual property of 2-neighborly now becomes "*Given 2 randomly chosen facets, there exists an $(n-2)$-face that connects them.*", which is equivalent to our definition of pervasive polytopes. So the duals of cyclic polytopes are pervasive.

If these dual polytopes are also powerful, then the cyclic polytopes should have the dual property of being powerful, which is "*Given any facet F_0 and* *any vertex v outside of F_0, there exists another facet F_1 containing v such that the intersection of F_0 and F_1 is nonempty.*" The next proposition shows that the duals of 4-dimensional cyclic polytopes with 7 vertices are not powerful. We need some terminologies before actually proving it.

2.2 Duals of Cyclic Polytopes Are Not Powerful

In dimension 5, let P be the cyclic polytope with 8 vertices. Let $x(t_i)$, $i = 1, 2, \cdots, 8$ denote the vertices of P. Let X be a non-empty subset of the vertices of P. By a *component* of X we shall mean a non-empty subset Y of X such that Y contains vertices $x(t_i)$ with consecutive indices. By a *proper component* of X we shall mean a component Y of X such that neither $x(t_1)$ nor $x(t_8)$ is in Y. A component containing an even number of points is called an *even component.*

Lemma 2.2.1. *(Gale's Evenness Condition)*

A component X is the set of vertices of a facet of P if and only if all its proper components are even.

The readers can find the proof on p.87 of Arne Brondsted's book [2].

Proposition 2.2.2. *Let Δ be a 5-dimensional cyclic polytope with 8 vertices spanned by $t_1 < t_2 < \cdots < t_8$. Let v be the vertex $(t_2, t_2^2, t_2^3, t_2^4, t_2^5)$ and let F_0 be the facet spanned by the vertices of t_1, t_4, t_5, t_6 and t_7. Then none of the facets that contain v intersect with F_0 as a 3-face.*

Proof. Assume there is one such facet F. Since F contains v and intersects with F_0 as a 3-face, F is spanned by v and 4 other vertices from the set t_1, t_4, t_5, t_6 and t_7. If we omit t_1, then v is itself is a proper component which is not even. If we omit t_4 or t_7, then the rest three in 4 to 7 form a proper component which is not even. If we omit t_5 or t_6, then either t_4 or t_7 end up as a proper component which is not even. Hence there is no such facet. □

Although the proposition above is stated in the case of 8 vertices in dimension 5, we can extend the proof easily to 5-dimensional cyclic polytopes with 9 or more vertices similarly, or even to higher dimensions, because all we need is one single vertex that is not dually powerful.

Chapter 3

Known Results and Some New Tools

3.1 Quoted Lemmas

In Proposition 2.2.2 we proved that the duals of cyclic polytopes are pervasive but not powerful, now we want to see these properties in general polytopes. McDuff and Tolman [1] proved the only non-simplex pervasive and powerful polytopes in dimension 4 are products of simplexes. We are going to use some of their tools together with new ones on polytopes in dimension 5.

In dimension 5, the polytopes with 6 facets are combinatorially equivalent to Δ_5. By Prop 1.1.1 in V. Timorin's paper[3], the polytopes with 7 facets are combinatorially equivalent to $\Delta_3 \times \Delta_2$. We are going to create polytopes with 8 or more facets by cutting the 7 facet $\Delta_3 \times \Delta_2$ with hyperplanes, while making sure the newly created facets are pervasive to the old facets, and the intersections of old facets are not cut out completely.

Before doing that, we shall list all the vertices of $\Delta_3 \times \Delta_2$ in the following 3×4 grid.

O O O O

O O O O

O O O O

We will label these vertices by their positions in the matrix, so the top left one will be $v_{1,1}$ while the bottom right one will be $v_{3,4}$.

The facet spanned by

$v_{2,1}$ $v_{2,2}$ $v_{2,3}$ $v_{2,4}$

$v_{3,1}$ $v_{3,2}$ $v_{3,3}$ and $v_{3,4}$

will be denoted by F_1, because we get these 8 vertices by "omitting" the first row of vertices. Similarly, the facets spanned by omitting the second and third row of vertices will be denote as F_2 and F_3.

The facet spanned by

$v_{1,2}$ $v_{1,3}$ $v_{1,4}$

$v_{2,2}$ $v_{2,3}$ $v_{2,4}$

$v_{3,2}$ $v_{3,3}$ and $v_{3,4}$

will be denoted as F_1', because we get these 9 vertices by omitting the first column of vertices. Similarly, we'll have F_2', F_3' and F_4' as well.

Now, let H_ℓ be the hyperplane that cuts through $\Delta_3 \times \Delta_2$ and creates the ℓ-th facet. Let V_ℓ^+ be the set of vertices of $\Delta_3 \times \Delta_2$ that are being kept in the newly created polytope after the cut H_ℓ. $V_\ell^+ \neq \emptyset$. Let V_ℓ^- be the set of vertices of $\Delta_3 \times \Delta_2$ that are being discarded after the cut H_ℓ. Let F_ℓ denote the new facet created in this way. On the matrix grid, we'll use O to denote the vertices that are in V_ℓ^+ (being kept), and X to denote the vertices that are in V_ℓ^- (being cut out). Since $\Delta_3 \times \Delta_2$ has 7 facets, we'll start our cut by H_8.

Here's an example of a typical cut.

O O O X

O O O X

O O X X

We use Q_ℓ to denote the resulting polytope after the single cut H_ℓ, and let $P_\ell = \bigcap_{k=8}^{\ell} Q_k$.

Definition 3.1.1. *We say a cut by H_ℓ is a* **valid cut** *if P_ℓ is pervasive and powerful.*

The following facts were used in McDuff and Tolman [1], but they did not discuss them in detail. Since they are very useful when we do the cuts later, we shall list them as lemmas and give them proofs.

By *local grid* we mean the four vertices of a 2-face spanned by the 2 by 2 sub grid. For example the 4 O below,

O . O .

. . . .

O . O .

Lemma 3.1.2. *In any cut, V_ℓ^+ cannot just have a diagonal pair of vertices in any local 2×2 grid. That is,* $\begin{matrix} O & X \\ X & O \end{matrix}$ *is not allowed.*

Proof. This is a result of the convex property. For the 2-face spanned by $\begin{matrix} \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{matrix}$, there is no way to cut the diagonal pair of vertices with a hyperplane. ☐

Lemma 3.1.3. *The old facets intersect with each other in Q_ℓ if and only if V_ℓ^+ contains at least one vertex in each row, and at least one vertex in each pair of columns.*

Proof. If V_ℓ^+ contains no vertex in the first row, then the 3-face spanned by those 4 vertices is completely cut off. Since that 3-face is the intersection of F_2 and F_3, these two facets will no longer be pervasive in Δ. So V_ℓ^+ must contain at least one vertex in the first row, and similarly, second and third row as well.

If V_ℓ^+ contains no vertex in the first two columns, then the 3-face spanned by those 6 vertices is completely cut off. Since that 3-face is the intersection of F_3' and F_4', these two facets will no longer be pervasive in Δ. So V_ℓ^+ must contain at least one vertex in these two columns, and similarly, in each pair of columns.

On the other hand, if V_ℓ^+ contains at least one vertex in each row, and at least one vertex in each pair of columns, then all the intersections of old facets have something survived after the cut, so the facets will still intersect with each other. ☐

For example,

O O X X
O O X X
O O X X

is not a valid cut, since the intersection of F_1' and F_2' is cut out completely.

Lemma 3.1.4. *The new facet F_ℓ meets all the original facets if and only if the cut contains a 2×2 grid like $\begin{matrix} & X \\ X & X \end{matrix}$ locally, where the top left vertex can be either O or X.*

Proof. Suppose that the newly created facet F_ℓ intersects all the old facets. Then H_ℓ must cut out at least one vertex from F_j for all j, and H_ℓ must cut out at least one vertex from $F_k^{'}$ for all k. Together with the convex property above, V_ℓ^- must contain at least 2 vertices in the same column for F_ℓ to meet all F_j. Similarly, for F_ℓ to intersect with all $F_k^{'}$, V_ℓ^- must contain at least 2 vertices in the same row. Without losing generality, we can relabel the vertices and get the desired grid stated in the lemma.

On the other hand, if the cut contains a 2×2 grid like $\begin{matrix} & \text{X} \\ \text{X} & \text{X} \end{matrix}$ locally, then H_ℓ cut through each old facet without missing a single one. That is, the new facet F_ℓ will meet all the old facets, hence pervasive. □

Lemma 3.1.5. *Old vertices are still connected to old facets if and only if P_ℓ does not contain exactly one vertex in any row or column.*

Proof. Suppose that after the cut, $v_{i,j}$ is the only vertex left on the grid in the i-th row. Then $v_{i,j}$ is not powerful, since there is no edge connecting $v_{i,j}$ to $F_j^{'}$. Similarly, $v_{i,j}$ cannot be the only vertex left on the grid in the j-th column.

On the other hand, if P_ℓ does not contain exactly one vertex in any row or column, then if $v_{i,j}$ is in P_ℓ, we must have $v_{i,k}$ and $v_{m,j}$ are also in P_ℓ for some $k \neq j$ and some $m \neq i$. Hence there will be edges connecting $v_{i,j}$ to F_i and $F_j^{'}$. □

For example,

O O O X

O O X X

O O X X

is not a valid cut, since $v_{1,3}$ will not be connected to any vertex in F_1.

Lemma 3.1.6. *For the new facet F_ℓ to connect to old vertices, if $v_{i,j}$ is in Q_ℓ, V_ℓ^- must contain some vertex which either lies in the same row or same column as $v_{i,j}$. This condition is also sufficient.*

Proof. If not, all vertices on i-th row and j-th column are in Q_ℓ. All 5 edges connecting to $v_{i,j}$ are also in Q_ℓ without being touched by H_ℓ, so none of them will connect $v_{i,j}$ to the new facet F_ℓ.

On the other hand, if $v_{i,j}$ is in Q_ℓ, and there is at least 1 vertex q on the i-th row or the j-th column lies in V_ℓ^-, that means H_ℓ cuts through the edge connecting $v_{i,j}$ and q, hence F_ℓ is connected to $v_{i,j}$. □

3.2 New Tools

The above lemmas were enough to rule out all possible cuts in dimension 4. However in dimension 5 and higher, we will need one new lemma which examines the new vertices created by the hyperplane H_ℓ.

When we have to record the newly created vertices from H_ℓ into the grid, we use small dots next to a cut off vertex X to indicate the newly created vertices caused by cutting that X off. For example,

```
O   O      O      O
                   .
O   O      O   ...X
           :       .
O   O   . .X    ..X
        p q
```

records all the new vertices created by the cut H_8. Let p denote the vertex created between $v_{3,1}$ and $v_{3,3}$.
Let q denote the vertex created between $v_{3,2}$ and $v_{3,3}$.
These two are newly created by H_8. We get p because H_8 cut through the edge connecting $v_{3,1}$ and $v_{3,3}$. We get q because H_8 cut through the edge connecting $v_{3,2}$ and $v_{3,3}$. Similar idea applies to the dots above $v_{3,3}$, and other vertices as well.

In the grid above, the vertices p and q are not in $F_3^{'}$, because they are not in the span of the 1st, 2nd and 4th columns of vertices. Similarly, the two newly created vertices above $v_{3,3}$ are not in F_3, and so on.

Lemma 3.2.1. *In order to have the newly created vertices be powerful, if H_ℓ cuts through the edge joining $v_{i,j}$ and $v_{i,j+1}$, then H_ℓ must cut through the edge joining $v_{k,j}$ and $v_{k,j+1}$ for some $k \neq i$ as well. Similarly, if H_ℓ cut through the edge joining $v_{i,j}$ and $v_{i+1,j}$, then H_ℓ must cut through the edge joining $v_{i,m}$ and $v_{i+1,m}$ for some $m \neq j$ as well. Moreover, if H_ℓ is the first cut, these* *conditions are also sufficient.*

It's easier to understand this lemma if we look at the grid. The lemma states that if we have a cut on the grid looking like

O X,

then we must have

O X

O X

Similar situation for the vertical case.

Proof. It's easier for the readers to understand this lemma by looking at the example provided after the proof. Without losing generality, assume $v_{i,j}$ is in V_ℓ^- in the i-th row, with i $\neq$ 1. The edge between $v_{i,j}$ and $v_{i-1,j}$ is being cut in half and creating a new vertex q in F_ℓ. If the edge between $v_{i,j}$ and $v_{i-1,j}$ is the only edge being cut through by H_ℓ between the i-th row and the row above it, on F_ℓ there is no other newly created vertex that also lies on $F_j^{'}$. Since q is not connected to any old vertices on $F_j^{'}$ either, q is not powerful.

If H_ℓ is the first cut and P_ℓ contains a grid listed in the lemma, the newly created vertices will have edges connecting to the newly created vertices at the parallel positions, hence they are powerful. □

For example,

O	O		O	.	X
				p	
O	O		O	.	X
				r	
O	O	.	X		X
		q			

is not a valid cut, since the new vertex q created between $v_{3,2}$ and $v_{3,3}$ is not connected to F_3. However, the new vertices p and r connect each other, so these 2 vertices are powerful.

With the help of these lemmas, we can now actually do the cuts, and see if the resulting polytopes can be both pervasive and powerful.

Chapter 4

Making One Cut in Dimension 5

4.1 8-facet Polytopes in Dimension 5

In dimension 5, the polytopes with 6 facets are combinatorially equivalent to Δ_5. The polytopes with 7 facets are combinatorially equivalent to $\Delta_3 \times \Delta_2$. We are going to create polytopes with 8 or more facets by cutting the 7 facet $\Delta_3 \times \Delta_2$ with hyperplanes, while making sure the newly created facets are pervasive to the old facets, and the intersections of old facets are not cut out completely.

The polytope $\Delta_3 \times \Delta_2$ is being cut by H_8 to create the 8th facet. We will be doing a lot of re-labellings if possible, to consolidate the vertices in V_8^- to the lower right corner of the grid. According to Lemma 3.1.4, we'll need at least 3 vertices in V_8^- to make sure F_8 is pervasive, so we'll start from there.

Proposition 4.1.1. *If V_8^- contains exactly 3 vertices, then H_8 is not a valid cut.*

Proof. By Lemma 3.1.4, the grid after the cut must be like this:

O O O O

O O O X

O O X X

But then $v_{1,4}$ is the only vertex in the 4th column, a contradiction to Lemma 3.1.5. □

Proposition 4.1.2. *If V_8^- contains exactly 4 vertices, then H_8 is not a valid cut.*

Proof. V_8^- must contain the 3 vertices shown as in the proof of the proposition above for the same reason of Lemma 3.1.4. After relabelling, we'll have 3 cases to discuss.

1. $v_{3,2}$ cut out as well:

   O O O O

   O O O X

   O X X X

   Then $v_{3,1}$ is the only vertex in the 3rd row, a contradiction to Lemma 3.1.5.

2. $v_{2,3}$ cut out as well:

   O O O O

   O O X X

   O O X X

   Then both $v_{1,3}$ and $v_{1,4}$ are the only vertices left in their columns, a contradiction to Lemma 3.1.5.

3. $v_{1,4}$ cut out as well:

   O O O X

   O O O X

   O O X X

   At the first glance this seems fine, satisfying all the first 5 rules. However, Lemma 3.2.1 is not satisfied at $v_{3,3}$.

□

Proposition 4.1.3. *If V_8^- contains exactly 5 vertices, then H_8 is not a valid cut.*

Proof. As before, we'll still have to start from the 3 vertices cut

O O O O

O O O X

O O X X

to make sure F_8 is pervasive. With the experience from the previous proposition, we know that $v_{1,4}$ must be cut as well, otherwise it will not be powerful when left in P_8. According to the convex property of Lemma 3.1.2, we cannot cut the 5th vertex arbitrarily. For example, $v_{2,2}$ cannot be cut because of Lemma 3.1.2. That leaves us with the following cases,

1. $v_{3,2}$ cut out as well:

   O O O X

   O O O X

   O X X X

   Then $v_{3,1}$ is the only vertex left in the 3rd row, a contradiction to Lemma 3.1.5.

2. $v_{2,3}$ cut out as well:

   O O O X

   O O X X

   O O X X

   Then $v_{1,3}$ is the only vertex left in the 3rd column, a contradiction to Lemma 3.1.5.

Since there are no other possible ways, this completes the proof. □

Proposition 4.1.4. *If V_8^- contains 6 or more vertices, then H_8 is not a valid cut.*

Proof. When cutting out 6 vertices, similar to the proofs above, we'll either end up with a single uncut vertex in its own row or column, or we'll end up with the following two grids:

O O X X

O O X X

O O X X

O O O X

O O O X

X X X X

In the first case we cut off the 3rd and 4th columns completely, a contradiction to Lemma 3.1.3. In the later case we cut off the 3rd row completely, a contradiction to Lemma 3.1.3 as well. One can see easily that there are not enough vertices left for us to cut without violating any of the rules. This completes the proof. □

Combining all the propositions above, we have proven

Theorem 4.1.5. *There is no 8-facet polytope with all facets being pervasive and powerful in dimension 5.*

As the reader may have noticed, the invalid cuts discussed above may have contradicted more than one lemma mentioned in the previous chapter. However since our purpose is to prove there are no valid cuts, a single contradiction is sufficient.

The reason we failed to find a valid cut after H_8 in dimension 5 is simply because we don't have enough vertices to work with. We need to cut out more to have new vertices being powerful. But in the meantime, we are left with single vertex on the grid, or even no vertex left in the intersections of old facets. If we had more vertices to work with, this would have worked.

Chapter 5

Making One Cut in Dimension 6

In dimension 6, the polytopes with 7 facets are combinatorially equivalent to Δ_6. The polytopes with 8 facets are combinatorially equivalent to either $\Delta_4 \times \Delta_2$ or $\Delta_3 \times \Delta_3$. Let's discuss the case of $\Delta_3 \times \Delta_3$ first.

5.1 Cutting $\Delta_3 \times \Delta_3$

We can apply all the lemmas we used before to dimension 6, with some modifications. First of all, in the case of $\Delta_3 \times \Delta_3$, the vertex grid now becomes

O O O O

O O O O

O O O O

O O O O

The 8 facets will be labelled as F_1 through F_4, and F_1' through F_4' as the same way we did in dimension 5. Since the intersections of F_is are now pairs of rows, Lemma 3.1.3 will be modified to

Lemma 5.1.1. *The old facets intersect with each other in P_ℓ if and only if*

V_ℓ^+ contains at least one vertex in each pair of rows, and at least one vertex in each pair of columns.

The proof is similar to the proof of Lemma 3.1.3.

The polytope $\Delta_3 \times \Delta_3$ is being cut by H_9 to create the 9th facet. As before, we will be doing a lot of re-labellings if possible, to consolidate the vertices in V_9^- to the lower right corner of the grid. According to Lemma 3.1.4, we'll need at least 3 vertices in V_9^- to make sure F_9 is pervasive, so we'll start from there.

Proposition 5.1.2. *If V_9^- contains exactly 3 vertices, then H_9 is not a valid cut.*

Proof. By Lemma 3.1.4, the grid after the cut must be like this:

O O O O

O O O O

O O O X

O O X X

But then at $v_{1,1}$, there is no vertex from the first column or first row in V_9^-, a contradiction to Lemma 3.1.6. □

Proposition 5.1.3. *If V_9^- contains exactly 4 vertices, then H_9 is not a valid cut.*

Proof. V_9^- must contain the 3 vertices shown as in the proof of the proposition above for the same reason of Lemma 3.1.4. After relabelling, we'll have 3 cases to discuss.

1. $v_{4,2}$ cut out as well:

   O O O O

   O O O O

   O O O X

   O X X X

   Then $v_{4,1}$ is the only vertex in the 4th row, a contradiction to Lemma 3.1.5.

2. $v_{2,4}$ cut out as well:

   O O O O

   O O O X

   O O O X

   O O X X

   Then $v_{1,4}$ is the only vertices left in the first row, a contradiction to Lemma 3.1.5.

3. $v_{3,3}$ cut out as well:

   O O O O

   O O O O

   O O X X

   O O X X

   But then at $v_{1,1}$, there is no vertex from the first column or first row in V_9^-, a contradiction to Lemma 3.1.6.

□

Proposition 5.1.4. *If V_9^- contains exactly 5 vertices, then H_9 is not a valid* *cut.*

Proof. As before, we still have to start from the 3 vertices cut

O O O O

O O O O

O O O X

O O X X

to make sure F_9 is pervasive. That gives us with the following cases,

1. $v_{4,1}$ and $v_{4,2}$ cut out as well:

   O O O O

   O O O O

   O O O X

   X X X X

   Then Lemma 3.2.1 is not satisfied at $v_{3,4}$.

2. $v_{3,3}$ and $v_{4,2}$ cut out as well:

   O O O O

   O O O O

   O O X X

   O X X X

   Then $v_{4,1}$ is the only vertex left in the 4th row, a contradiction to Lemma 3.1.5.

3. $v_{2,4}$ and $v_{4,2}$ cut out as well:

O	O	O	O
O	O	O	X
O	O	O	X
O	X	X	X

Then $v_{4,1}$ is the only vertex left in the 4th row, a contradiction to Lemma 3.1.5.

4. $v_{2,4}$ and $v_{3,3}$ cut out as well:

O	O	O	O
O	O	O	X
O	O	X	X
O	O	X	X

Then $v_{1,4}$ is the only vertex left in the 4th column, a contradiction to Lemma 3.1.5.

5. $v_{1,4}$ and $v_{2,4}$ cut out as well:

O	O	O	X
O	O	O	X
O	O	O	X
O	O	X	X

Then Lemma 3.2.1 is not satisfied at $v_{4,3}$.

Since there are no other possible ways, this completes the proof. □

Proposition 5.1.5. *If V_9^- contains exactly 6 vertices, then H_9 is not a valid cut.*

Proof. As before, we still have to start from the 3 vertices cut

O O O O

O O O O

O O O X

O O X X

to make sure F_9 is pervasive. That gives us with the following cases,

1. $v_{4,1}$, $v_{4,2}$ and $v_{3,3}$ cut out as well:

   O O O O

   O O O O

   O O X X

   X X X X

   Then Lemma 3.2.1 is not satisfied at $v_{3,3}$, when viewed horizontally.

2. $v_{4,2}$, $v_{3,2}$ and $v_{3,3}$ cut out as well:

   O O O O

   O O O O

   O X X X

   O X X X

   Then $v_{3,1}$ is the only vertex left in the 3rd row, a contradiction to Lemma 3.1.5.

3. $v_{4,1}$, $v_{4,2}$ and $v_{2,4}$ cut out as well:

   O O O O

   O O O X

   O O O X

   X X X X

Then $v_{1,4}$ is the only vertex left in the 4th column, a contradiction to Lemma 3.1.5.

4. $v_{4,2}$, $v_{3,3}$ and $v_{2,4}$ cut out as well:

   O O O O

   O O O X

   O O X X

   O X X X

   Then $v_{1,4}$ is the only vertex left in the 4th column, a contradiction to Lemma 3.1.5.

5. $v_{2,3}$, $v_{3,3}$ and $v_{2,4}$ cut out as well:

   O O O O

   O O X X

   O O X X

   O O X X

   Then $v_{1,4}$ is the only vertex left in the 4th column, a contradiction to Lemma 3.1.5.

6. $v_{4,2}$, $v_{1,4}$ and $v_{2,4}$ cut out as well:

   O O O X

   O O O X

   O O O X

   O X X X

   Then $v_{4,1}$ is the only vertex left in the 4th row, a contradiction to Lemma 3.1.5.

7. $v_{3,3}$, $v_{1,4}$ and $v_{2,4}$ cut out as well:

   O O O X

   O O O X

   O O X X

   O O X X

   Then Lemma 3.2.1 is not satisfied at $v_{3,3}$, when viewed vertically.

Since there are no other possibilities, this completes the proof. □

Unlike dimension 5, it is possible to have a valid cut when we cut out 7 vertices in dimension 6 in a certain way.

Proposition 5.1.6. *The following cut by H_9 is a valid cut.*

O O O X

O O O X

O O O X

X X X X

Proof. Since no pairs of 2 rows or 2 columns are cut out, part of the intersections of the original facets still remain in P_9. By Lemma 3.1.4 F_9 intersects with all the original facets. Hence P_9 is pervasive. To see P_9 is powerful, we start with original vertices that are not cut by H_9. Since none of them are the only vertex in their rows or columns, they connect to all the original facets that do not contain them. Since H_9 cuts out one vertex in the first three rows, all the old vertices will also connect to the new facet F_9. Hence all the old vertices are powerful.

The last issue is to see if all the new vertices are powerful. We will examine the following two new vertices, the one created between $v_{1,3}$ and $v_{1,4}$, and the one created between $v_{2,3}$ and $v_{2,4}$. They are marked as vertices p and q in the grid below.

```
O  O  O  .. .X
             p
O  O  O  .. .X
             q
O  O  O   ...X
X  X  X      X
```

By Lemma 3.2.1, p and q connect each other. The only concern about p was the connection to F_1, but obviously q is on F_1. Hence p is powerful. Since all other new vertices created by cutting out the 4th column and 4th row can be proved to be powerful in the similar way, all the new vertices are powerful. Hence P_9 is powerful, and H_9 shown in the grid is a valid cut. □

We can see from the above cut, 9 old vertices left after the cut, 9 new vertices are created by cutting out the 4th column, and 9 other new vertices are created by cutting out the 4th row. The cut is parallel to $F_4 \cap F_4'$, which is equivalent to $\Delta_2 \times \Delta_2$. One can easily see the resulting polytope is combinatorially equivalent to the product of simplexes $\Delta_2 \times \Delta_2 \times \Delta_2$.

Proposition 5.1.7. *If V_9^- contains exactly 7 vertices, then H_9 is not a valid cut except the one in the last proposition.*

Proof. To distribute 7 vertices into 4 rows, we have the following ways.

$(0,0,3,4)$, $(0,1,2,4)$, $(0,1,3,3)$, $(0,2,2,3)$, $(1,1,1,4)$, $(1,1,2,3)$, and $(1,2,2,2)$. $(1,1,1,4)$ is the valid cut in the previous proposition. We know that we cannot have exactly 3 vertices cut out in a row because that will end up with a single vertex left in the row, so we only have the following two cases left to discuss.

1. $(0,1,2,4)$:

   O O O O

   O O O X

   O O X X

   X X X X

   Then $v_{1,4}$ is the only vertex left in the 4th column, a contradiction to Lemma 3.1.5.

2. $(1,2,2,2)$:

   O O O X

   O O X X

   O O X X

   O O X X

   Then $v_{1,3}$ is the only vertex left in the 3rd column, a contradiction to Lemma 3.1.5.

□

Proposition 5.1.8. *If V_9^- contains 8 or more vertices, then H_9 is not a valid cut.*

Proof. To distribute 8 vertices into 4 rows, we have the following ways.

$(0,0,4,4)$, $(0,1,3,4)$, $(0,2,2,4)$, $(0,2,3,3)$, $(1,1,2,4)$, $(1,1,3,3)$, $(1,2,2,3)$, and $(2, 2, 2, 2)$.

As before, we cannot have exactly 3 vertices cut out in a single row, that leaves us with the following cases to discuss.

1. $(0, 0, 4, 4)$:

   O O O O

   O O O O

   X X X X

   X X X X

   The 3rd and 4th rows are cut out, a contradiction to Lemma 5.1.1.

2. $(0, 2, 2, 4)$:

   O O O O

   O O X X

   O O X X

   X X X X

   Then $v_{1,3}$ is the only vertex left in the 3rd column, a contradiction to Lemma 3.1.5.

3. $(1, 1, 2, 4)$:

   O O O X

   O O O X

   O O X X

   X X X X

   Then Lemma 3.2.1 is not satisfied at $v_{3,3}$.

4. $(2, 2, 2, 2)$:

O	O	X	X
O	O	X	X
O	O	X	X
O	O	X	X

The 3rd and 4th columns are cut out, a contradiction to Lemma 5.1.1.

It is easy for the reader to see that if we cut out 9 or more vertices, we will end up with either a single vertex left in its row or column, or a pair of two rows or columns being cut out completely. Either way, H_9 is not a valid cut. $\square$

5.2 Cutting $\Delta_4 \times \Delta_2$

We can apply all the lemmas we used before as well, again with some modifications. The vertex grid now becomes

O	O	O	O	O
O	O	O	O	O
O	O	O	O	O

The 8 facets will be labelled as F_1 through F_3, and $F_1^{'}$ through $F_5^{'}$ as the same way we did in dimension 5. Since the intersections of $F_i^{'}$s are now groups of 3 columns instead of just pairs of two, Lemma 3.1.3 will be modified to

Lemma 5.2.1. *The old facets intersect with each other in P_ℓ if and only if V_ℓ^+ contains at least one vertex in each row, and at least one vertex in each group of three columns.*

The proof is similar to the original proof to Lemma 3.1.3.

The polytope $\Delta_4 \times \Delta_2$ is being cut by H_9 to create the 9th facet. As before, we will be doing a lot of re-labellings if possible, to consolidate the vertices in V_9^- to the lower right corner of the grid. According to Lemma 3.1.4, we'll need at least 3 vertices in V_9^- to make sure F_9 is pervasive, so we'll start from there.

Proposition 5.2.2. *If V_9^- contains exactly 3 vertices, then H_9 is not a valid cut.*

Proof. By Lemma 3.1.4, the grid after the cut must be like this:

O O O O O

O O O O X

O O O X X

But then $v_{1,5}$ is the only vertex in the 5th column, a contradiction to Lemma 3.1.5. □

Proposition 5.2.3. *If V_9^- contains exactly 4 vertices, then H_9 is not a valid cut.*

Proof. V_9^- must contain the 3 vertices shown as in the proof of the proposition above for the same reason of Lemma 3.1.4. After relabelling, we'll have 3 cases to discuss.

1. $v_{3,3}$ cut out as well:

   O O O O O

   O O O O X

   O O X X X

   Then $v_{1,5}$ is the only vertex in the 5th column, a contradiction to Lemma 3.1.5.

2. $v_{2,4}$ cut out as well:

   O O O O O

   O O O X X

   O O O X X

   Then both $v_{1,4}$ and $v_{1,5}$ are the only vertices left in their columns, a contradiction to Lemma 3.1.5.

3. $v_{1,5}$ cut out as well:

   O O O O X

   O O O O X

   O O O X X

   Lemma 3.2.1 is not satisfied at $v_{3,4}$.

□

Proposition 5.2.4. *If V_9^- contains exactly 5 vertices, then H_9 is not a valid cut.*

Proof. As before, we'll still have to start from the 3 vertices cut

O O O O O

O O O O X

O O O X X

to make sure F_9 is pervasive. With the experience from the previous proposition, we know that $v_{1,5}$ must be cut as well, otherwise it will not be powerful when left in P_9. That leaves us with the following cases,

1. $v_{3,3}$ cut out as well:

O O O O X

O O O O X

O O X X X

Then Lemma 3.2.1 is not satisfied at $v_{3,3}$ when viewed horizontally.

2. $v_{2,4}$ cut out as well:

   O O O O X

   O O O X X

   O O O X X

   Then $v_{1,4}$ is the only vertex left in the 4th column, a contradiction to Lemma 3.1.5.

Since there are no other possible ways, this completes the proof. □

Proposition 5.2.5. *The following cut by $H_9$is a valid cut.*

O O O ...X ...X

O O O ...X ...X

O O O ...X ...X

Proof. The intersections of original facets are not cut out completely, and by Lemma 3.1.4 F_9 intersects with all other facets, hence P_9 is pervasive. By cutting out the 4th and 5th columns, the old vertices are connected to F_9, and hence powerful since they still connect to all other facets that don't contain them. By Lemma 3.2.1 the new vertices are also powerful. Hence H_9 is a valid cut. □

As we can see from the grid in the above proposition, we have 9 old vertices left in P_9, 9 new vertices created by cutting the 4th column, and another 9 vertices created by cutting the 5th column. It is easy to see the that the resulting polytope P_9 is again combinatorially equivalent to $\Delta_2 \times \Delta_2 \times \Delta_2$.

Proposition 5.2.6. *If V_9^- contains exactly 6 vertices, then H_9 is not a valid cut except the one in the above proposition.*

Proof. When we try to divide 6 vertices into 3 rows, we have the following options, $(5,1,0)$, $(4,2,0)$, $(3,3,0)$, $(4,1,1)$, $(3,2,1)$ and $(2,2,2)$, where the last one $(2,2,2)$ is the valid cut in the previous proposition. For the first 3 cases, based on the experience we had before, we know that we cannot leave the first row uncut since that will end up with $v_{1,5}$ being a single vertex left in the 5th column, a contradiction to Lemma 3.1.5, so we rule out the cases $(5,1,0)$, $(4,2,0)$, and $(3,3,0)$. For $(4,1,1)$, $v_{3,1}$ will be the single vertex left in the 3rd row. For $(3,2,1)$, $v_{1,4}$ will be the single vertex left in the 4th column. Since both are also contradictions to Lemma 3.1.5 as well, this completes the proof. □

Proposition 5.2.7. *If V_9^- contains exactly 7 vertices, then H_9 is not a valid cut.*

Proof. When we try to divide 7 vertices into 3 rows, we have the following options, $(5,2,0)$, $(4,3,0)$, $(5,1,1)$, $(4,2,1)$, $(3,3,1)$ and $(3,2,2)$. Similarly, we cannot leave the first row uncut, that rules out the first two cases. Case $(5,1,1)$ cuts out the 3rd row completely, a contradiction to Lemma 3.1.3. For the rest 3 cases:

1. Case $(4,2,1)$

O O O O X

O O O X X

O X X X X

Then $v_{1,4}$ is the only vertex left in the 4th column, a contradiction to Lemma 3.1.5.

2. Case $(3, 3, 1)$

   O O O O X

   O O X X X

   O O X X X

   Again $v_{1,4}$ is the only vertex left in the 4th column, a contradiction to Lemma 3.1.5.

3. Case $(3, 2, 2)$

   O O O X X

   O O O X X

   O O X X X

   Then Lemma 3.2.1 is not satisfied at $v_{3,3}$ when viewed horizontally.

□

Proposition 5.2.8. *If V_9^- contains 8 or more vertices, then H_9 is not a valid cut.*

Proof. As before, we have to distribute 8 vertices into 3 rows.

1. Case $(5, 2, 1)$

O O O O X

O O O X X

X X X X X

Then the 3rd row is cut out completely, a contradiction to Lemma 3.1.3.

2. Case $(4, 2, 2)$

   O O O X X

   O O O X X

   O X X X X

   Then $v_{3,1}$ is the only vertex left in the 3rd row, a contradiction to Lemma 3.1.5.

3. Case $(4, 3, 1)$

   O O O O X

   O O X X X

   O X X X X

   Again $v_{3,1}$ is the only vertex left in the 3rd row, a contradiction to Lemma 3.1.5.

4. Case $(3, 3, 2)$

   O O O X X

   O O X X X

   O O X X X

   Then $v_{1,3}$ is the only vertex left in the 3rd column, a contradiction to Lemma 3.1.5.

It is easy for the reader to see that if we cut out 9 or more vertices, we will end up with either a single vertex left in its row or column, or a group of three columns being cut out completely. Either way, H_9 is not a valid cut. □

Combine both sections, we have proved

Theorem 5.2.9. *In dimension 6, there is only one 9-facet pervasive and powerful polytope which is combinatorially equivalent to* $\Delta_2 \times \Delta_2 \times \Delta_2$.

Chapter 6

A Valid Cut in Dimension 7

6.1 Cutting from $\Delta_4 \times \Delta_3$

In this chapter we show that in dimension 7 there exists a *non-trivial* valid cut, which means the resulting polytope is not combinatorially equivalent to products of simplexes. We cut the 9 facet polytope $\Delta_4 \times \Delta_3$, so the vertex grid is the 4×5 matrix.

Proposition 6.1.1. *In dimension 7, consider the grid of vertices of* $\Delta_4 \times \Delta_3$.

$$\begin{matrix} O & O & O & O & X \\ O & O & O & O & X \\ O & O & X & X & X \\ O & O & X & X & X \end{matrix}$$

The cut by H_{10} *shown in the grid is a valid cut.*

Proof. Being pervasive is easy to check. None of the intersections of old facets are completely cut out, and obviously F_{10} is pervasive as well. All the old vertices left are powerful, because they are not the only ones in their rows or columns, and they all have at least an edge connecting them to the new facet F_{10}. For the new vertices, they all have another newly created vertex at the parallel position, like we mentioned in Lemma 3.2.1, to make them powerful. For example, the vertex q created between $v_{4,2}$ and $v_{4,3}$, our concern is connecting it to F_4. But there is an edge from q to the vertex created between $v_{3,2}$ and $v_{3,3}$, which is on F_4. One can easily check all the new vertices with similar method. □

Proposition 6.1.2. *The valid cut above is not combinatorially equivalent to product of simplexes.*

Proof. In dimension 7, since we cannot have Δ_1 in the product, the only 10 facet pervasive and powerful product of simplexes is $\Delta_2 \times \Delta_2 \times \Delta_3$, up to equivalence. The polytope $\Delta_2 \times \Delta_2 \times \Delta_3$ has 36 vertices, while the valid cut in the previous proposition has 40 vertices. So the two polytopes are not combinatorially equivalent. □

6.2 Smoothness

Now we've found a pervasive and powerful polytope which is not a product of simplexes. It will be interesting to see if it is also smooth. Let P be the polytope $\Delta_4 \times \Delta_3$.

Proposition 6.2.1. *The polytope in Proposition 6.1.1 is not smooth when P is viewed as a Δ_3 bundle over Δ_4.*

Proof. We write out the conormals of each facet in the following way:

$\eta_1 = (-1, 0, 0, 0, 0, 0, 0)$

$\eta_2 = (0, -1, 0, 0, 0, 0, 0)$

$\eta_3 = (0, 0, -1, 0, 0, 0, 0)$

$\eta_4 = (1, 1, 1, 0, 0, 0, 0)$

$\eta_1^{'} = (a_{1,1}, a_{1,2}, a_{1,3}, -1, 0, 0, 0)$

$\eta_2^{'} = (a_{2,1}, a_{2,2}, a_{2,3}, 0, -1, 0, 0)$

$\eta_3^{'} = (a_{3,1}, a_{3,2}, a_{3,3}, 0, 0, -1, 0)$

$\eta_4^{'} = (a_{4,1}, a_{4,2}, a_{4,3}, 0, 0, 0, -1)$

$\eta_5^{'} = (a_{5,1}, a_{5,2}, a_{5,3}, 1, 1, 1, 1)$

Assume the conormal of the new facet F_{10} is $\eta_{10} = (w_1, w_2, w_3, w_4, w_5, w_6, w_7)$, we will try to solve these variables by looking at the new vertices. Consider the following grid with all new vertices shown:

```
O  O   O   O  ....X
O  O   O   O  ....X
       :    :
O  O  ..X  ..X   ..X
       :    :
O  O  ..X  ..X   ..X
```

Let p be the vertex created between $v_{3,2}$ and $v_{3,3}$. Then p is on F_1, F_2, F_4, $F_1^{'}$,$F_4^{'}$, $F_5^{'}$, and F_{10}. That is, $\det(\eta_1, \eta_2, \eta_4, \eta_1^{'}, \eta_4^{'}, \eta_5^{'}, \eta_{10}) = \pm 1$. After simplifying the matrix we get the equation $w_5 - w_6 = \pm 1$.

Let q be the vertex created between $v_{1,2}$ and $v_{1,5}$. Then q is on F_2, F_3, F_4,

$F_1^{'}$,$F_3^{'}$, $F_4^{'}$, and F_{10}. That is, $\det(\eta_2, \eta_3, \eta_4, \eta_1^{'}, \eta_3^{'}, \eta_4^{'}, \eta_{10}) = \pm 1$. After simplifying the matrix we get the equation $w_5 = \pm 1$.

Let r be the vertex created between $v_{1,3}$ and $v_{1,5}$. Then q is on F_2, F_3, F_4, $F_1^{'}$,$F_2^{'}$, $F_4^{'}$, and F_{10}. That is, $\det(\eta_2, \eta_3, \eta_4, \eta_1^{'}, \eta_2^{'}, \eta_4^{'}, \eta_{10}) = \pm 1$. After simplifying the matrix we get the equation $w_6 = \pm 1$.

Now since both w_5 and w_6 are ± 1, we got a contradiction from the first equation $w_5 - w_6 = \pm 1$. Hence the cut cannot be smooth. □

On the other hand, when P is viewed as a Δ_4 bundle over Δ_3, we split the 9 facets of P into fiber facets and base facets as before. For fiber facets $F_1^{'}$ to $F_5^{'}$, we put the first four in the standard position and the fifth one as the slant facet:

$\eta_1^{'} = (0, 0, 0, -1, 0, 0, 0)$

$\eta_2^{'} = (0, 0, 0, 0, -1, 0, 0)$

$\eta_3^{'} = (0, 0, 0, 0, 0, -1, 0)$

$\eta_4^{'} = (0, 0, 0, 0, 0, 0, -1)$

$\eta_5^{'} = (0, 0, 0, 1, 1, 1, 1)$

For base facets F_1 to F_4, we put the first three in the standard position and the fourth one as the tilted one with 4 coordinates listed as linear combinations of fiber facet conormals:

$\eta_1 = (-1, 0, 0, 0, 0, 0, 0)$

$\eta_2 = (0, -1, 0, 0, 0, 0, 0)$

$\eta_3 = (0, 0, -1, 0, 0, 0, 0)$

$\eta_4 = (1, 1, 1, b_{4,1}, b_{4,2}, b_{4,3}, b_{4,4})$.

We have the following result.

Proposition 6.2.2. *If the polytope in Proposition 6.1.1 can be made smooth with the conormals listed above, then*

$\eta_{10} = (-1, -1, 0, -1, -1, 0, 0)$

$b_{4,1} = 0$, $b_{4,2} = 0$, *and* $b_{4,3} = -1$.

Proof. Assume $\eta_{10} = (w_1, w_2, w_3, w_4, w_5, w_6, w_7)$. We get these equations by checking all 40 new vertices and see the determinants. For example the vertex created between $v_{1,3}$ and $v_{3,3}$ is the intersection of facets F_2, F_4, $F_1^{'}$, $F_2^{'}$, $F_4^{'}$, $F_5^{'}$, and F_{10}. We assign the value $+1$ to $\det(\eta_2, \eta_4, \eta_1^{'}, \eta_2^{'}, \eta_4^{'}, \eta_5^{'}, \eta_{10})$. Then with the same orientation, the vertex created between $v_{2,3}$ and $v_{3,3}$ will have determinant value -1. After solving the 40 equations we have the solution listed in the proposition. □

The solution in the above proposition is a necessary condition. In fact, we do not know if this polytope can be made smooth yet.

Chapter 7

Making Two Cuts in Dimension 5

7.1 9-facet Polytopes in Dimension 5

When we try to make the cut by H_8, even if we make sure that all facets including F_8 are pervasive, we proved in chapter 4 that the powerful condition cannot be fulfilled at the same time, so we will have some non-powerful vertices left. If we try to have a powerful and pervasive 9-facet polytope after two cuts, H_9 must cut out all the non-powerful vertices left in the previous cut H_8 without creating any new problems, while still keeping the pervasive properties.

We are going to discuss this topic case by case like before, sorted by how many vertices were cut out by H_8. Luckily for us there are not too many cases to worry about. Again, we need at least 3 vertices in V_8^- to make sure F_8 is pervasive, so we'll start from there.

Proposition 7.1.1. *If V_8^- contains exactly 3 vertices, then there is no way for H_9 to make a valid cut.*

Proof. By Lemma 3.1.4, this must be the grid after the first cut.

O O O O

O O O X

O O X X

The non-powerful old vertices are

$v_{1,4}$, because it is the only vertex left in the 4th column by Lemma 3.1.5.

$v_{1,1}$ and $v_{1,2}$, because they are not connected to F_8 by Lemma 3.1.6.

As a result, we'll have to cut out these 3 vertices by H_9, but then $v_{1,3}$ will be the only vertex left in the first row, so it has to be cut by Lemma 3.1.5 again. Then H_9 has to cut out the entire first row, a contradiction to Lemma 3.1.3. □

Proposition 7.1.2. *If V_8^- contains exactly 4 vertices, then there is no way for H_9 to make a valid cut.*

Proof. As in proposition 2.2, we'll divide this into 3 cases.

1. $v_{2,3}$ cut out as well:

   O O O O

   O O X X

   O O X X

   Both $v_{1,3}$ and $v_{1,4}$ are the only vertices left in their columns, so they have to be cut by H_9 according to Lemma 3.1.5. Both $v_{1,1}$ and $v_{1,2}$ are not connected to F_8, so they must be cut as well by Lemma 3.1.6. As a result, H_9 has to cut our the entire first row, a contradiction to Lemma 3.1.3.

2. $v_{3,2}$ cut out as well:

O	O	O	O
O	O	O	X
O	X	X	X

Although H_9 has to cut out $v_{3,1}$ by Lemma 3.1.5, it does not mean the 3-face spanned by the 3rd row, which is the intersection of F_1 and F_2, has to be cut out completely. It's possible to cut out all 4 vertices of that 3-face with 2 cuts, which is the case here, but still have something from it left. We need to examine it more closely here. H_8 cuts out $v_{3,2}$, $v_{3,3}$, and $v_{3,4}$, and also creates new vertices on the 3rd row.

Let q_1 denote the vertex created between $v_{3,1}$ and $v_{3,2}$.

Let q_2 denote the vertex created between $v_{3,1}$ and $v_{3,3}$.

Let q_3 denote the vertex created between $v_{3,1}$ and $v_{3,4}$.

See the grid below.

O	O	O	O
			.
O	O	O	...X
	:	:	.
O	.X	.X	.X
	q_1	q_2	q_3

For q_1 and q_2, there is no edge to connect them to F_3, so they must be cut out by H_9. Since $v_{3,1}$ has to be cut out by Lemma 3.1.5 as well, there is no vertices left for q_3 to connect to F_4', so q_3 has to be cut by H_9 too. As a result, since H_9 has to cut out all the old and new vertices on the 3rd row, it has to cut out the 3-face spanned by the 3rd row, a contradiction to Lemma 3.1.3.

3. $v_{1,4}$ cut out as well:

   O O O X

   O O O X

   O O X X

   This is the most complicated case, since only the vertices created by H_8 are non powerful. We mark those vertices out in the following grid.

O	O	O	. . .X
			p q r
O	O	O	. . .X
		:	$s_1 s_2 s_3$
O	O	..X	. .X
			$t_1 t_2$

   In the grid,

   let p denote the vertex created between $v_{1,1}$ and $v_{1,4}$.

   let q denote the vertex created between $v_{1,2}$ and $v_{1,4}$.

   let r denote the vertex created between $v_{1,3}$ and $v_{1,4}$.

   let s_1 denote the vertex created between $v_{2,1}$ and $v_{2,4}$.

   let s_2 denote the vertex created between $v_{2,2}$ and $v_{2,4}$.

   let s_3 denote the vertex created between $v_{2,3}$ and $v_{2,4}$.

   let t_1 denote the vertex created between $v_{3,1}$ and $v_{3,4}$.

let t_2 denote the vertex created between $v_{3,2}$ and $v_{3,4}$.

By Lemma 3.2.1, the newly created vertices to the left of $v_{3,3}$ and above $v_{3,3}$ in the grid are not powerful, so H_9 has to cut them out no matter what.

(a) Step 1: H_9 must keep $v_{3,1}$, $v_{3,2}$, t_1 and t_2:

On the 3rd row, H_9 cannot just cut $v_{3,1}$ but keep $v_{3,2}$, because $v_{3,1}$ is the only vertex that connects $v_{3,2}$ to $F_2^{'}$. Similarly, we cannot just cut $v_{3,2}$ but keep $v_{3,1}$. If we try to let H_9 cut out both $v_{3,1}$ and $v_{3,2}$, the vertices t_1 and t_2 will not be powerful to $F_4^{'}$, so they have to go as well. As a result, H_9 will have to cut out the entire 3rd row, a contradiction to Lemma 3.1.3. In short, H_9 has to keep both $v_{3,1}$ and $v_{3,2}$. Then vertices t_1 and t_2 cannot be cut either, because they are the only connection from $v_{3,1}$ and $v_{3,2}$ to F_8.

(b) Step 2: H_9 cannot just cut one of $v_{1,3}$ and $v_{2,3}$.

On the 3rd column, H_9 cannot just cut $v_{1,3}$ but keep $v_{2,3}$, because $v_{1,3}$ is the only vertex that connects $v_{2,3}$ to F_2. Similarly, we cannot just cut $v_{2,3}$ but keep $v_{1,3}$.

(c) Step 3: H_9 is not a valid cut if it keeps both $v_{1,3}$ and $v_{2,3}$.

H_9 has to cut something from $F_3^{'}$ to make sure F_9 is pervasive. But

H_9 cannot cut $v_{2,2}$, due to the convex property in the 2×2 grid spanned by $v_{2,2}$, $v_{2,3}$, $v_{3,2}$ and $v_{3,3}$. Hence H_9 cannot cut any of $v_{1,1}$, $v_{1,2}$, or $v_{2,1}$, because we can always re-labelling and make it be $v_{2,2}$. If we try to let H_9 to cut s_2 to make F_9 intersect with $F_3^{'}$, $v_{2,2}$ will lose its connection to F_8, a contradiction. Similar contradiction if we try to cut $v_{2,1}$ instead. In short, if H_9 keeps both $v_{1,3}$ and $v_{2,3}$, F_9 will not intersect with $F_3^{'}$.

(d) Step 4: H_9 is not a valid cut if it cuts both $v_{1,3}$ and $v_{2,3}$.

It is possible that H_9 cuts both $v_{1,3}$ and $v_{2,3}$ out to avoid the contradiction in step 3. Then vertex r has to be cut too because it loses the connection to $F_4^{'}$. However H_9 has to cut more to make sure F_9 is intersected with $F_3^{'}$. With relabelling, H_9 has to cut out at least one of the following 3 vertices: $v_{1,2}$, p or q, because these are the vertices in $F_3^{'}$.

If H_9 cuts out $v_{1,2}$, then $v_{1,1}$ is the only vertex left in the first row, so it has to be cut as well. Then p and q will lose their connections to $F_4^{'}$, so they have to be cut as well. Then H_9 has to cut out the first row completely, a contradiction to Lemma 3.1.3.

If H_9 cuts out p instead, then $v_{1,1}$ will lose its connection to F_8 and has to be cut. As a result, $v_{1,2}$ will be the only old vertex left in the first row and has to be cut as well by Lemma 3.1.5. Then vertex q will lose its connection to $F_4^{'}$ and has to be cut too. Then H_9 has to cut out the entire first row again, a contradiction to Lemma 3.1.3.

If H_9 cuts out q instead, the similar argument as cutting p above can be applied.

□

Proposition 7.1.3. *If V_8^- contains exactly 5 vertices, then there is no way for H_9 to make a valid cut.*

Proof. Again we have 3 cases to discuss, since H_8 cannot cut out 4 vertices in one row.

1. If V_8^- contains 5 vertices like the following grid:

   O O O O

   O O X X

   O X X X

   We can see that both $v_{1,3}$ and $v_{1,4}$ have to be cut by H_9 by Lemma 3.1.5, since they are both the only vertices in their columns. $v_{1,1}$ has to be cut as well by Lemma 3.1.6. As a result, $v_{1,2}$ will be the only vertex left in the first row and has to be cut too. Then H_9 has to cut out the entire first row, a contradiction to Lemma 3.1.3.

2. If V_8^- contains 5 vertices like the following grid:

```
O   O   O  ...X

O   O   O  ...X

    :   :

O  .X  .X    .X
   p   q    r
```

Let p denote the vertex created by H_8 between $v_{3,1}$ and $v_{3,2}$.

Let q denote the vertex created by H_8 between $v_{3,1}$ and $v_{3,3}$.

Let r denote the vertex created by H_8 between $v_{3,1}$ and $v_{3,4}$.

$v_{3,1}$ is not powerful and has to be cut by H_9 by Lemma 3.1.5. Vertices p and q are not powerful and have to be cut by H_9 by Lemma 3.2.1. Then vertex r will lose its connection to F_4' and has to be cut as well. As a result, H_9 has to cut out the entire 3rd row, a contradiction to Lemma 3.1.3.

3. If V_8^- contains 5 vertices like the following grid:

```
O  O    O  .. .X
               r
        p·

O  O  ..X    ..X

        q·

O  O  ..X    ..X
```

Let p denote the vertex created by H_8 between $v_{1,3}$ and $v_{2,3}$.

Let q denote the vertex created by H_8 between $v_{1,3}$ and $v_{3,3}$.

Let r denote the vertex created by H_8 between $v_{1,3}$ and $v_{1,4}$.

After the cut by H_8 the 3-face spanned by the 3rd and 4th columns, which is the intersection of $F_1^{'}$ and $F_2^{'}$, has only the small portion spanned by $v_{1,3}$, p, q, and r left. When we do the cut of H_9, $v_{1,3}$ has to be cut because of not powerful by Lemma 3.1.5. Both vertices p and q have to be cut because of not powerful by Lemma 3.2.1. Then the vertex r will lost its connection to $F_4^{'}$ and has to be cut as well. As a result, H_9 has to cut out all these 4 vertices, hence the 3-face spanned by these 4 vertices is gone, the intersection of $F_1^{'}$ and $F_2^{'}$ is gone too.

□

Proposition 7.1.4. *If V_8^- contains exactly 6 vertices, then there is no way for H_9 to make a valid cut.*

Proof. There are only two cases left to discuss.

1. If V_8^- contains 6 vertices like the following grid:

   O O O O

   O X X X

   O X X X

   Then H_9 has to cut out $v_{1,2}$, $v_{1,3}$ and $v_{1,4}$ by Lemma 3.1.5, and $v_{1,1}$ as well by Lemma 3.1.6. As a result, H_9 has to cut out the entire first row, a contradiction to Lemma 3.1.3.

2. If V_8^- contains 6 vertices like the following grid:

O O O ...X

r

$p\cdot$

O O ..X ..X

$q\cdot$

O .X .X .X

Let p denote the vertex created by H_8 between $v_{1,3}$ and $v_{2,3}$.

Let q denote the vertex created by H_8 between $v_{1,3}$ and $v_{3,3}$.

Let r denote the vertex created by H_8 between $v_{1,3}$ and $v_{1,4}$.

After the cut by H_8 the 3-face spanned by the 3rd and 4th columns, which is the intersection of $F_1^{'}$ and $F_2^{'}$, has only the small portion spanned by $v_{1,3}$, p, q, and r left. When we do the cut of H_9, $v_{1,3}$ has to be cut because of not powerful by Lemma 3.1.5. The vertices p and r have to be cut because of not powerful by Lemma 3.2.1. Then the vertex q will lost its connection to F_3 and has to be cut as well. As a result, H_9 has to cut out all these 4 vertices, hence the 3-face spanned by these 4 vertices is gone, the intersection of $F_1^{'}$ and $F_2^{'}$ is gone too.

□

Proposition 7.1.5. *If V_8^- contains exactly 7 vertices, then there is no way for H_9 to make a valid cut.*

Proof. There is only 1 case this time.

O O O . . .X

. . p q r

O .X .X .X

. .

O .X .X .X

Let p denote the vertex created by H_8 between $v_{1,1}$ and $v_{1,4}$.

Let q denote the vertex created by H_8 between $v_{1,2}$ and $v_{1,4}$.

Let r denote the vertex created by H_8 between $v_{1,3}$ and $v_{1,4}$.

The vertices $v_{1,2}$ and $v_{1,3}$ have to be cut by H_9, since they are not powerful by Lemma 3.1.5. Then $v_{1,1}$ has to be cut by Lemma 3.1.5 as well. The vertices q and r have to be cut by H_9, since they are not powerful by Lemma 3.2.1. Then the vertex p will lost its connection to $F_4^{'}$ and has to be cut too. As a result, H_9 has to cut out the entire first row, a contradiction to Lemma 3.1.3. $\square$

One can easily verify that if H_8 cut out 8 or more vertices, it must cut out either a row, or 2 columns completely, so the cut will not be pervasive. Combining with all the above propositions, we have proven

Theorem 7.1.6. *There is no 9-facet polytope with all facets being pervasive and powerful in dimension 5.*

Chapter 8

Making Three Cuts in Dimension 5

8.1 10-facet Polytopes in Dimension 5

In dimension 5, since the polytopes with 11 facets or more will not be powerful, the 10-facet polytopes is our last chance to find powerful and pervasive polytopes. We have the following partial results.

Proposition 8.1.1. *If P_{10} is a valid cut after H_8, H_9, and H_{10}, then P_{10} can contain at most 4 original vertices from $\Delta_3 \times \Delta_2$.*

Proof. Assume $v_{1,1}$ survived after the three cuts. By Lemma 3.1.5, the possible grid of P_{10} that contains the fewest original vertices from $\Delta_3 \times \Delta_2$ is

O O X X

O O X X

X X X X

If $v_{1,3}$ is also in P_{10}, then $v_{1,1}$ will have three edges connecting to old facets, leaving only two edges to connect to three new facets, hence it will not be powerful. Similarly, no other vertex can be in P_{10} except the four in the grid above. □

It is possible that the three cuts H_8, H_9, and H_{10} cut out the entire set of original vertices, as long as Lemma 3.1.3 holds. When the original vertices are cut out, most of our lemmas in chapter 3 become not so useful here, since they are mostly about the original vertices. The newly created vertices by those 3 cuts can be quite complicated, and currently we do not have enough information about them yet. However, we do have the following proposition.

Proposition 8.1.2. *If P_{10} is a valid cut after H_8, H_9, and H_{10}, then P_{10} does not contain any triangular 2-face.*

Proof. Assume P_{10} contains a triangular 2-face spanned by vertices p, q, and r. Then the line segment $\overline{pq}$ is an edge and the intersection of 4 facets. That is, vertices p and q will be in the exactly same 4 facets. So will p, r and q, r.
Assume

p is in facets A, B, C, D, and E.

q is in facets A, B, C, D, and F.

The vertex r cannot be in facets A, B, C, and D, since the intersection of those 4 facets is a 1-face not a 2-face. So r must be in E and F. Assume

r is in facets A, B, C, E, and F.

Then both the edge $\overline{pq}$ and $\overline{pr}$ give the vertex p the connection to the same facet F. However, in the 10-facet polytope, all 5 edges from the vertex p must connect to 5 different facets in order to be powerful. Hence P_{10} is not a valid cut. □

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