Phi-critical Submanifolds

and

Convexity in Calibrated Geometries

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Abstract of the Dissertation

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Plurisubharmonic functions in calibrated geometries are defined by Harvey and Lawson. These functions generalize the classical plurisubharmonic functions from complex geometry and enjoy their important properties. Harvey and Lawson extend their results to ϕ-critical submanifolds where ϕ is a calibration. These submanifolds are the generalization of calibrated submanifolds and they are also minimal. In this thesis, we find the first examples of ϕ-

critical submanifolds in $\mathbf{H}^n$ where ϕ is the quaternion calibration and we prove that they have a rich geometry, despite the lack of interesting calibrated submanifolds.

Secondly, we study strictly ϕ-convex domains which are also introduced by Harvey and Lawson. These are generalizations of Stein manifolds in complex geometry to calibrated manifolds. By using Morse Theory, we prove results about the topology of strictly ϕ-convex domains in $\mathbf{H}^n$ with quaternion calibration, in $\mathbf{R}^7$ with associative or coassociative calibration, and in $\mathbf{R}^8$ with Cayley calibration, similar to the result proved by Andreotti and Frankel for Stein manifolds. We use ϕ-free submanifolds which are the analogues of totally real submanifolds to find examples of strictly ϕ-convex domains with every topological type allowed by Morse Theory.

To my family,

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Chapter 1

Introduction

The theory of *Calibrated Geometries* was invented by Harvey and Lawson [HL_1]. It is about special kind of minimal submanifolds of a Riemannian manifold, called *calibrated submanifolds* which are defined by a closed form called a *calibration.*These submanifolds have recently been of great interest to physicists and mathematicians alike because of their appearances in gauge theories [T], mirror symmetry [SYZ], and string theory in Physics. Understanding these special geometries and their examples will help mathematicians and physicists very much. In this chapter, we review the basic definitions and results related to this work. (cf. [HL_1], [**?**], [**?**])

1.1 Calibrated Geometries

Let (X, g) be a Riemannian manifold. An oriented k-plane ξ on X is a vector subspace of some tangent space T_pX with dim(ξ)= k equipped with an orientation. $g_{|_\xi}$ with an orientation on ξ gives a natural *volume form* on ξ which is a k-form.

Definition 1.1.1. *Let ϕ be a closed k-form on X. ϕ is called a calibration if for every k-plane ξ on X we have :*

$$\phi_{|\xi} \leq vol_{|\xi} \tag{1.1}$$

Here we have $\phi_{|\xi} = \alpha vol_{|\xi}$ for some $\alpha \leq 1$. If $\phi_{|\xi} = vol_{|\xi}$, then ξ is called a ϕ-plane. If $\xi = e_1 \wedge .... \wedge e_k$, i,e,if ξ is given by a unit simple vector $e_1 \wedge .... \wedge e_k$ where e_i's form an orthonormal basis for ξ, then ξ is a ϕ-plane if and only if $\phi(\xi) = 1$. The set of all ϕ-planes is called *the contact set* of the calibration ϕ and denoted by $G(\phi)$.

Let M be a k-dimensional manifold of (X, ϕ). M is a *calibrated submanifold* or *ϕ-manifold* if $\phi_{|T_xM} = vol_{|T_xM}$ for all $x \in M$.

The fundamental observation about these submanifolds is the following:

Lemma 1.1.2 (The Fundamental Lemma of Calibration Theory)**.** *Let (X, g) be a Riemannian manifold, and ϕ a calibration on X. If M is a compact calibrated subanifold with boundary ∂M (possibly empty) of X, then M is volume minimizing in its homology class.*

Proof : Let M be a calibrated submanifold with dimension k and boundary ∂M. Let $[M] \in H_k(X, \mathbf{R})$ and $[\phi] \in H^k(X, \mathbf{R})$ be the homology and cohomology classes of M and ϕ, respectively. Then, we will have

$$[\phi] \cdot [M] = \int_{x \in M} \varphi_{|T_xM} = \int_{x \in M} vol_{T_xM} = Vol(M)$$

Since M is a calibrated submanifold and we have $\phi_{|T_xM} = vol_{|T_xM}$ for all $x \in M$. If M' is another compact k-submanifold of X with $\partial M = \partial M'$ and

$[M' - M] = 0$ in $H_k(X, \mathbf{R})$, then we will have

$$[\phi] \cdot [M] = [\phi] \cdot [M'] = \int_{x \in M'} \phi_{|_{T_x M'}} \leq \int_{x \in M'} vol_{T_x M} = Vol(M')$$

since $\phi_{|_{T_x M'}} \leq vol_{|_{T_x M'}}$ because ϕ is a calibration. As a result of the integrals above, we have $Vol(M) \leq Vol(M')$. Therefore, M is volume minimizing in its homology class. Also, we will have the equality if and only if M' is calibrated by ϕ. $\square$

As a result of the lemma above, we see that every calibrated submanifold is minimal i.e. mean curvature vanishes. Whether $M \subset X$ is a calibrated submanifold depends on its tangent space, so being calibrated with respect to ϕ is a first order equation whereas being a minimal submanifold is a second order equation. Hence, the calibrated equations are easier then the minimal submanifold equation.

The theory of calibrations is closely connected to theory of Riemannian holonomy groups since Riemannian manifolds with special holonomy usually come with one or more canonical calibrations. We will explain these relation below.

Let $G \subset O(n)$ be a holonomy group. Suppose ϕ_0 is a G-invariant closed k-form on $\mathbf{R}^n$. By rescaling ϕ_0, for each k-plane $\xi \subset \mathbf{R}^n$, we can get $\phi_{0|_\xi} \leq vol_{|_\xi}$ and $\phi_{0|_\xi} = vol_{|_\xi}$ for one ξ. Hence, $G(\phi)$ is nonempty. Since ϕ_0 is invariant under G, if $\xi \in G(\phi)$, then $g \cdot \xi$ will be in $G(\phi)$. This means that $G(\phi)$ is very large

Let X be a n-dimensional manifold with Riemannian metric g, Levi-Civita connection ∇ and holonomy G. Since ϕ_0 is G-invariant by using parallel translation we can extend ϕ_0 from $T_pX \cong \mathbf{R}^n$ for $p \in X$ to all of X with $\nabla\phi = 0$. Hence, $d\phi = 0$ and $\phi_{0_{|\xi}} \leq vol_{|\xi}$ for all oriented k-planes of $\mathbf{R}^n$ implies $\phi_{|\xi} \leq vol_{|\xi}$ for all oriented tangent k-planes ξ of X. Therefore, ϕ is a calibration on X.

By using Berger's classification of holonomy groups, we can give some very important examples. Let X be a n-dimensional Riemannian manifold with metric g

(1)$n = 2m$ and $Hol(g) \subseteq U(m)$

A Riemannian 2m-manifold (X, g) with holonomy in $U(m)$ is called a Kähler manifold. It is a complex manifold with Kähler form ω. ω is closed and it is a calibration on X. In this case, the ω-submanifolds are complex submanifolds of dimension one i.e. complex curves in X. In fact, $\phi = w^p/p!$ for some p, $1 \leq p \leq n$, on a Kähler manifold of dimension n is a calibration, and ϕ-submanifolds are complex submanifolds of dimension p.

(2)$n = 2m$ and $Hol(g) \subseteq SU(m)$

A Riemannian 2m-manifold (X, g) with holonomy SU(m) is called a *Calabi-Yau m-fold*. A *Calabi-Yau m-fold* is a quadruple (X, J, g, Ω) such that (X, J) is a complex m-dimensional manifold, g is a Kähler metric with holonomy contained in $SU(m)$, and Ω a parallel (m,0)-form on X called *holomorphic volume form*, which satisfies :

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega} \tag{1.2}$$

where ω is the Kähler form of g. The real part $Re\Omega$ is a calibration on X, and the corresponding calibrated submanifolds of real dimension m are called *special Lagrangian submanifolds*. An equivalent condition for a submanifold M being *special Lagrangian* is that

$$\omega_{|_M} = 0 \qquad and \qquad Im(\Omega)_{|_M} = 0 \tag{1.3}$$

(3)$n = 7$ and $Hol(g) \subseteq G_2$

A Riemannian 7-manifold (X, g) with holonomy in G_2 is called a G_2-manifold.(G_2 is the group automorphisms of $\mathbf{O}$ where $\mathbf{O}$ is the set of octonions.) It comes with a natural 3-form φ and a 4-form $*\varphi$ which are both calibrations. The corresponding calibrated submanifolds are called *associative 3-folds*, and *coassociative 4-folds*, respectively.

In $\mathbf{R}^7 \cong \mathrm{Im}(\mathbf{O})$, a 3-plane $\xi = x \wedge y \wedge z$ is an associative plane, if the associator $[x, y, z] = (x(yz) - (xy)z) = 0$. Moreover, if ξ is an associative 3-plane, then $\xi^{\perp}$ is an coassociative 4-plane. Another characterization of associative planes is given as :

If $u, v \in \mathbf{R}^7 \cong \mathrm{Im}(\mathbf{O})$, then $\{u, v, u \times v\}$ forms a basis for an associative 3-plane. Here $\times : \mathrm{Im}(\mathbf{O}) \times \mathrm{Im}(\mathbf{O}) \longrightarrow \mathrm{Im}(\mathbf{O})$ is the cross product defined as follow :$u \times v = \mathrm{Im}(\bar{v} \cdot u)$. A similar characterization for coassociative planes is given as follow :

If $u, v, w \in \mathbf{R}^7 \cong \mathrm{Im}(\mathbf{O})$, then $\{u, v, w, u \times v \times w\}$ forms a basis for an coassociative 4-plane.

(4)$n = 8$ and $Hol(g) \subseteq Spin(7)$

A Riemannian 8-manifold (X, g) with holonomy in $Spin(7)$ is called a

$Spin(7)$-manifold. $Spin(7)$ is the double cover of $SO(7)$. It comes with a 4-form Ω which is a calibration. The corresponding Ω-submanifolds are called *Cayley 4-folds*. A similar characterization for Cayley 4-folds can be given as follow:

If $u, v, w \in \mathbf{R}^8 \cong (\mathbf{O})$, then $\{u, v, w, u \times v \times w\}$ forms a basis for a Cayley 4-plane. Here $\times$ is the cross product defined on $\mathbf{O}$.

1.2 Plurisubharmonic Functions on Calibrated Manifolds

Analysis and geometry have always been very difficult on general calibrated manifolds because of lack of analogues of the holomorphic functions, and holomorphic curves in Kähler geometry. The ϕ-plurisubharmonic functions have been introduced by Harvey and Lawson in [HL_2] to study geometry and analysis on calibrated manifolds.

Let (X, ϕ) be a n-dimensional calibrated manifold with calibration ϕ of degree p and contact set $G(\phi)$ (the set of ϕ-planes).

Definition 1.2.1. *The d^ϕ-**operator** is defined by*

$$d^\phi f = \nabla f \lrcorner \phi$$

for all smooth functions f on X.

Hence,

$$d^\phi : \mathcal{E}^0(X) \longrightarrow \mathcal{E}^{p-1}(X) \qquad \textit{and} \qquad dd^\phi : \mathcal{E}^0(X) \longrightarrow \mathcal{E}^p(X)$$

where $\mathcal{E}^p(X)$ denotes the space of C^∞ p-forms on X.

If $\phi = \omega$ on a Kähler manifold, then $d^\omega = d^c = -J \circ d = -i(\partial - \bar{\partial})$. Hence , dd^ϕ generalizes the dd^c-operator in complex geometry.

Definition 1.2.2. *Suppose $\nabla\phi = 0$,i.e. ϕ is parallel. A function $f \in C^\infty(X)$ is ϕ-**plurisubharmonic** if*

$$(dd^\phi)(\xi) \geq 0 \qquad\qquad for \ \ all \ \ \xi \in G(\phi)$$

The set of ϕ-plurisubharmonic functions on X will be donated by $\mathcal{PSH}(X, \phi)$. If $(dd^\phi)(\xi) > 0$ for all $\xi \in G(\phi)$, then f is **strictly ϕ-plurisubharmonic**. If $(dd^\phi)(\xi) = 0$ for all $\xi \in G(\phi)$, then f is **ϕ-pluriharmonic**.

If ϕ is not parallel but just closed, then ϕ-plurisubharmonic functions are defined by replacing $dd^\phi f$ by

$$\mathcal{H}^\phi f = dd^\phi f - \nabla_{\nabla f}(\phi)$$

In the general case where ϕ is just a p-form, ϕ-plurisubharmonic are defined by a second order differential operator called the **ϕ- Hessian**, which is defined by

$$\mathcal{H}^\phi : C^\infty(X) \ \rightarrow \ \mathcal{E}^p(X)$$

$$\mathcal{H}^\phi(f) \ = \ \lambda_\phi(\mathrm{Hess} f)$$

where Hess$f$ is the Riemannian Hessian of f and $\lambda_\phi : \mathrm{End}(TX) \to \Lambda^p T^*X$ is the bundle map given by $\lambda_\phi(A) = D_{A^*}(\phi)$ where $D_{A^*} : \Lambda^p T^*X \longrightarrow \Lambda^p T^*X$ is the natural extension of $A^* : T^*X \to T^*X$ as a derivation.

A fundamental result about ϕ-plurisubharmonic functions is

Theorem 1.2.3 ([HL$_2$])**.** *Let (X,ϕ) be a calibrated manifold. If a function $f \in C^\infty(X)$ is ϕ-plurisubharmonic , then the restriction of f to a ϕ-submanifold $M \subset X$ is subharmonic in the induced metric.*

This is the result of the following theorem.

Theorem 1.2.4. *[HL$_2$] Suppose (X,ϕ) is a calibrated manifold. For each $f \in C^\infty(X)$,*

$$\mathcal{H}^\phi(f) = tr_\xi(Hessf) \qquad\qquad if \quad \xi \in G(\phi)$$

For a submanifold $M \subset X$, if we write the Laplace-Beltrami operator $\bar{\triangle}$,

$$\bar{\triangle}(f) = \triangle_M f = tr_{TM} Hessf - H(f)$$

where H is the mean curvature vector field on M.

If M is a calibrated submanifold of X,then M is minimal , so $H \equiv 0$. Hence, we get

$$\mathcal{H}^\phi(f)|_M = (\triangle_M f)vol_M$$

Therefore, we get the result in Theorem 1.2.3.

Since $dd^\omega = dd^c$, strictly ω-plurisubharmonic functions are the usual plurisubharmonic functions in complex geometry.

1.3 Convexity in Calibrated Geometries

Let (X, ϕ) be a non-compact connected calibrated manifold.

Definition 1.3.1. *If K is a compact subset of X, we define the* ϕ**-convex hull** *of of K by*

$$\widehat{K} \;=\; \{x \in X : f(x) \leq \sup_K f \quad \text{for all } f \in \mathcal{PSH}(X, \phi)\}$$

If a subset $K \subset$ X satisfies $K = \widehat{K}$, then K is called ϕ-**convex**. A calibrated manifold is called ϕ**-convex** if for every compact subset $K \subset X$, the hull $\widehat{K}$ is also compact. We have an equivalent condition for (X, ϕ) to be ϕ**-convex**

Theorem 1.3.2. *[HL_2] A calibrated manifold (X, ϕ) is ϕ-convex if and only if it admits a ϕ-plurisubharmonic proper exhaustion function $f : X \to \mathbf{R}$.*

The manifold is called **strictly** ϕ**-convex** if it admits a strictly ϕ-plurisubharmonic proper exhaustion function $f : X \to \mathbf{R}$. We will use this as the definition of a strictly ϕ-convex manifold.

In complex geometry, where $\phi = \omega$, strictly ϕ-convex manifolds are nothing but Stein manifolds. This can be proved by using the following theorem of Oka

Theorem 1.3.3. *[O] Let M be a complex manifold, admitting a smooth, proper exhaustion by a strictly plurisubharmonic function. Then M is Stein. Conversely, any Stein manifold admits such a function.*

and knowing that strictly plurisubharmonic functions in complex geometry are in fact strictly ω-plurisubharmonic functions.

By using Morse Theory [M] Andreotti and Frankel proved in [AF] that a Stein manifold M of complex dimension n has the homotopy type of a CW-complex of dimension n. This is proved by Harvey and Wells in [HW] by showing that the real Hessian of a strictly plurisubharmonic exhaustion function has at most n negative eigenvalues at a critical point. In this thesis, we will prove similar results for strictly ϕ-convex manifolds, also.

Chapter 2

Examples of ϕ-Critical Submanifolds

Calibrated geometry will be interesting and rich if we have lots of calibrated submanifolds. We have a very rich geometry for the calibrations mentioned before, but this is not always the case. For lots of calibrations, calibrated submanifolds are nothing but ϕ-planes. In their seminal paper [HL_2] Harvey and Lawson extended their definition of ϕ-submanifolds to ϕ-critical submanifolds. It was a very natural extension since every ϕ-submanifold or calibrated submanifold is also a ϕ-critical submanifold. In the lack of non-trivial examples of ϕ-submanifolds, we may get a very rich geometry if we look at the ϕ-critical submanifolds. Also, if we restrict a ϕ-plurisubharmonic function to a ϕ-critical submanifold, it will be subharmonic. (cf. [HL_2]). J.Zhou in [Z] showed that most of the well-known calibrations (Kähler,Special Lagrangian, Associative, Coassociative, Cayley) don't have critical values other than 1 and -1. Hence, in these cases, ϕ-critical geometry is the same as calibrated geometry, so all ϕ-critical submanifolds are in fact calibrated submanifolds. I will give the first examples of ϕ-critical submanifolds in $\mathbf{H}^n$ with quaternion calibration in this thesis.

2.1 ϕ-Critical Submanifolds

Let V be an inner product space, and $G_k(V)$ be the Grassmannian of k-planes in V and consider $G_k(V)$ as the unit simple vectors in $\bigwedge_k V$. In [HL$_2$] Harvey and Lawson defined the *ϕ-critical planes* of a calibration ϕ as the following :

Definition 2.1.1. *Given $\phi \in \bigwedge^k V^*$, an element $\xi \in G_k(V)$ is said to be a ϕ-critical plane if $\xi \in G(k, V)$ is a critical point of the function $\phi_{|G_k(V)}$. Equivalently, ϕ must vanish on $T_\xi G_k(V) \subset \bigwedge_k V$. $G^{cr}(\phi)$ will denote the set of ϕ-critical planes.*

If ϕ is a calibration, then $sup\ \phi|_{G_k(V)} = 1$ since for every *ϕ-plane* in $G_k(V)$, ϕ attains its maximum value 1 by (1.1). In particular, 1 is a critical value and all *ϕ-planes* are critical points of $\phi|_{G_k(V)}$. Moreover, 1 is the global maximum, and this is the reason why The Fundamental Lemma of Calibration Theory is true.

Let (X, ϕ) be a calibrated manifold with calibration ϕ. Then any oriented k-manifold $M \subset X$ will be called a *ϕ-critical submanifold* with critical value c if $T_x M \in G_c^{cr}(\phi)$,$\forall\ x \in M$ where

$$G_c^{cr}(\phi) = \{\xi \in G^{cr}(\phi) : \phi(\xi) = c\}$$

Let $(M, \partial M)$ be a compact p-dimensional manifold with boundary and $f : M \hookrightarrow X$ be an immersion. Now, we set $\psi = f_{|\partial M}$ and denote immersions from M to X by $\mathcal{I}mm(M, X)$. We define

$$\mathcal{I}mm(M, X; \psi) = \{g \in \mathcal{I}mm(M, X) : g_{|\partial M} = \psi\}$$

We will state our main theorem about ϕ-critical manifolds when $X = \mathbb{R}^n$ but under a weak assumption in fact it will be true for any calibrated manifold (X, ϕ).

Theorem 2.1.2. *Suppose f is ϕ-critical immersion of a compact manifold M with boundary ∂M into $\mathbb{R}^n$ where ϕ is a calibration. If the critical value is a local maximum of ϕ , then there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{C}^1$-topology on $\mathcal{I}mm(M, \mathbb{R}^n; \psi)$ such that*

$$Vol(f) \leq Vol(g) \qquad \forall g \in \mathcal{U}$$

and $Vol(f) = Vol(g)$ if and only if g is also ϕ-critical.

Proof : Let $G_c^{cr}(\phi)$ be the $\phi-$critical set with critical value c and $\mathcal{V} \subset G_k(V)$ be an open neighborhood of $G_c^{cr}(\phi)$ such that $\phi(\xi) < c \,\forall\, \xi \in \mathcal{V} \backslash G_c^{cr}(\phi) \subset G_k(V)$

If we look at the immersions in $\mathcal{I}mm(M, \mathbb{R}^n; \psi)$ whose image under the Gauss map is contained in $\mathcal{V}$, we can find a neighborhood $\mathcal{U}$ of f in the $\mathcal{C}^1-$topology of $\mathcal{I}mm(M, \mathbb{R}^n; \psi)$ such that the image of $\mathcal{U}$ under the Gauss map will be contained in $\mathcal{V} \subset G_k(\mathbb{R}^n)$. Then we will have :

$$\int_M f^*\phi = \int_{f(M)} \phi = \int_{f(M)} c \cdot vol_{f(M)} = c \cdot Vol(f)$$

since f is a ϕ-critical immersion. And for all $g \in \mathcal{U}$

$$\int_M g^*\phi = \int_{g(M)} \phi \leq \int_{g(M)} c \cdot vol_{g(M)} = c \cdot Vol(g)$$

since for any $x \in g(M)$, we have $T_x g(M) \in \mathcal{V}$ and this gives us, $\phi_{T_x g(M)} \leq c \cdot vol_{T_x g(M)}$. Also, we will have the equality when $T_x g(M) \in G_c^{cr}(\phi)$ i.e. g is

ϕ-critical.

Moreover, we know that $f(M) - g(M)$ is a cycle in $\mathbb{R}^n$ and therefore a boundary since the homology is zero. Hence, by Stokes Theorem, we get

$$c \cdot Vol(f) = \int_M f^*\phi = \int_{f(M)} \phi = \int_{g(M)} \phi = \int_M g^*\phi \leq c \cdot Vol(g) \qquad \forall g \in \mathcal{U}$$

By canceling c from both sides, we get $Vol(f) \leq Vol(g)$, and we have the equality if and only if g is also ϕ-critical. $\square$

Remark : In the proof above, if we replace $\mathbb{R}^n$ with any calibrated manifold (X, ϕ), the proof will still work if $f(M) - g(M)$ is a boundary in X. But, this can be achieved if we can choose $\mathcal{U}$ small enough, since for all $g \in \mathcal{U}$ close to f, we can find a smooth homotopy $F : M \times [0,1] \longrightarrow X$ between f and g which is fixed on ∂M. Then, the image of F gives the chain whose boundary is $f(M) - g(M)$.

However, ϕ-critical submanifolds will not be homologically volume minimizing in general, unless c=1.

2.2 The Quaternion 4-form on $\mathbf{H}^n$

In this section, we will review some definitions and results about the calibrated manifolds we will consider to give examples of ϕ-critical submanifolds.(cf. [BH], [K])

We will consider $\mathbf{H}^n$, the set of columns of height n of quaternions, as a (right) quaternion vector space (n-dimensional right module over the quaternions $\mathbf{H}$). We will denote the set of $n \times n$ matrices over $\mathbf{H}$ by $M_n(\mathbf{H})$ which are the $\mathbf{H}$-linear maps of $\mathbf{H}^n$ i.e. $End_{\mathbf{H}}(\mathbf{H}^n)$. If $A \in M_n(\mathbf{H})$, then it will act on $\mathbf{H}^n$ from left. We define the *standard quaternionic hermitian bilinear form* on $\mathbf{H}^n$,

$$\varepsilon(x, y) = \bar{x}^t y \equiv \sum_{i=1}^{n} \bar{x}^i y^i \tag{2.1}$$

The *special quaternionic unitary group* is defined as the set of all endomorphisms of $\mathbf{H}^n$ which preserves the "*standard quaternionic hermitian bilinear form*" i.e.

$$Sp(n) \equiv \{A \in M_n(\mathbf{H}) : \varepsilon(Ax, Ay) = \varepsilon(x, y)\} = \{A \in M_n(\mathbf{H}) : A \cdot A^\dagger = Identity\}$$

where $A^\dagger$ is the conjugate transpose of A.

It is also called the *symplectic group*. If we look at real part of $\varepsilon(x, y)$, $Re(\varepsilon(x, y))$ ($\equiv < x, y >$), it is equal to the standard Euclidean inner product on $\mathbf{R}^{4n} \cong \mathbf{H}^n$.

Right multiplication $R_u x \equiv xu$ by any imaginary unit quaternion $u = ai + bj + ck \in S^2 \subset Im\mathbf{H}$, $(a^2 + b^2 + c^2 = 1)$, defines an orthogonal complex structure on $\mathbf{R}^{4n}$ and this complex structure gives us a Kähler form,

$$\omega_u(x, y) \equiv < u, \varepsilon(x, y) > = < R_u x, y > = < xu, y >$$

Since $Sp(n)$ fixes the *standard quaternionic hermitian bilinear form*, it also fixes ω_u. Let us consider the three complex structures defined by i,j,and k on $\mathbf{H}^n$, which are $I = R_i$, $J = R_j$, and $K = R_k$, respectively, and their Kähler forms ω_I, ω_J,and ω_K, respectively. A simple calculation will give us

$$\varepsilon(x,y) = < x, y > + i\omega_I(x,y) + j\omega_I(x,y) + k\omega_I(x,y)$$

The group $Sp(1) = \{a \in \mathbf{H} : a \cdot a^{\dagger} = 1\} = \{a \in \mathbf{H} : \|a\| = 1\}$. In other words, $Sp(1)$ is the set of all unit quaternions. $Sp(1)$ acts on $\mathbf{H}^n$ from right, and we have an induced action on $\mathrm{Im}\mathbf{H} \cong \{\omega_u : u \in \mathbf{H}\}$ given by

$$R_v^*\omega_u = \omega_{vu\bar{v}} \qquad u \to vu\bar{v}$$

which has an easy proof :

$$R_v^*\omega_u(x,y) = \omega_u(xv, yv) = < xvu, yv > = < xvu\bar{v}, y > = \omega_{vu\bar{v}}(x,y)$$

This proof also shows us that right multiplication by a unit quaternion doesn't always preserve the Kähler form.

The group $Sp(1) \equiv S^3 \subset \mathbf{H}$ of unit scalars does not belong to $Sp(n)$.The intersection of $Sp(1)$ and $Sp(n)$ is $\{\pm Id\} \cong \mathbb{Z}_2$. Actually, $Sp(n)$ is the centralizer of $Sp(1)$ if we think both of them as subgroups of $SO(4n)$, by considering $\mathbf{H}^n = \mathbf{R}^{4n}$, and $Sp(n)$ acting on the left and $Sp(1)$ acting on the right on $\mathbf{H}^n = \mathbf{R}^{4n}$. Hence, these two subgroups generate a proper subgroup of $SO(4n) \subset GL_{\mathbf{R}}(\mathbf{H}^n)$

$$Sp(n) \cdot Sp(1) \equiv Sp(n) \times_{\mathbb{Z}_2} Sp(1) \qquad (2.2)$$

i.e. the subgroup of $SO(4n)$ generated by $Sp(n) \times Sp(1)$. This group is referred as *quaternionic unitary group.*

Considering the three complex structure i,j,and k and their Kähler forms ω_I,ω_J,and ω_K, by a simple calculation, we have

$$\omega_I(xi, yi) = -\omega_I(xj, yj) = -\omega_I(xk, yk) = \omega_I(x, y)$$
$$\omega_J(xj, yj) = -\omega_J(xk, yk) = -\omega_I(xi, yi) = \omega_J(x, y)$$
$$\omega_K(xk, yk) = -\omega_K(xi, yi) = -\omega_K(xi, yi) = \omega_K(x, y)$$

Now, if $u \in Sp(1)$, and let $u = a + bi + cj + dk$, then the right action of u on the forms ω_I,ω_J,ω_K will give us the following equations by a straight calculation.

$$R_u^*\omega_I = (a^2 + b^2 - c^2 - d^2)\omega_I + 2(ad + bc)\omega_J + 2(bd - ac)\omega_K$$
$$R_u^*\omega_J = 2(bc - ad)\omega_I + (a^2 - b^2 + c^2 - d^2)\omega_J + 2(ab + cd)\omega_K$$
$$R_u^*\omega_I = 2(ac + bd)\omega_I + 2(cd - ab)\omega_J + (a^2 - b^2 - c^2 + d^2)\omega_K$$

Define the action of the group $Sp(n) \cdot Sp(1)$ on $\mathbf{H}^n$ as follows : let $P \in \mathbf{H}^n$ and $(\sigma, u) \in Sp(n) \times Sp(1)$, then $(\sigma, u)P = \sigma P u$, i.e. appy σ from left and multiply on the right by the unit quaternion u.

Theorem 2.2.1. $\Omega = \omega_I^2 + \omega_J^2 + \omega_K^2$ *is invariant under the action of* $Sp(n) \cdot Sp(1)$.

Proof : As we know from the definition of $Sp(n)$, Ω is invariant under the action of $Sp(n)$ on the left. Now let $u \in Sp(1)$, then

$$R_u^*\Omega = R_u^*\omega_I \wedge R_u^*\omega_I + R_u^*\omega_J \wedge R_u^*\omega_J + R_u^*\omega_K \wedge R_u^*\omega_K.$$

By substituting the values we have calculated before, we get $R_u^*\Omega = \Omega$. Hence, Ω is invariant under the action of $Sp(1)$. $\square$

In fact, Bryant and Harvey [BH] proved that if $A \in SO(4n) \subset GL_{\mathbf{R}}(\mathbf{H}^n)$ fixes $\phi \equiv \frac{1}{3}\{\frac{w_I^2}{2}+\frac{w_J^2}{2}+\frac{w_K^2}{2}\}$, then $A \in Sp(n)\cdot Sp(1)$. So, this 4-form ϕ determines the *quaternionic unitary group* . This 4-form $\phi \equiv \frac{1}{3}\{\frac{w_I^2}{2}+\frac{w_J^2}{2}+\frac{w_K^2}{2}\} \in \bigwedge^4(\mathbf{H}^n)^*$ is a calibration on $\mathbf{H}^n$([HL$_1$], [BH]) and the set of calibrated 4-planes are quaternionic lines in $\mathbf{H}^n$. That is, the contact set of ϕ, $G(\phi) \equiv \{\xi \in G_4(\mathbf{H}^n) = G_{\mathbf{R}}(4, \mathbf{H}^n) : \phi(\xi) = 1\}$ is equal to $\mathbf{H}P^{n-1}$, quaternionic projective space. This is the 4-form of primary interest in this chapter.

2.3 $\frac{1}{3}$ is a critical value of ϕ

First of all, we will show that $\frac{1}{3}$ is a critical value of the function $\tilde{\phi} = \phi_{|_{G_4(\mathbf{H}^2)}}$: $G_4(\mathbf{H}^2) \cong G_4(\mathbf{R}^8) \longrightarrow \mathbf{R}$ where ϕ is the 4-form $\frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$ defined on $\bigwedge^4 \mathbf{H}^2 \cong \bigwedge^4 \mathbf{R}^8 \supset G_4(\mathbf{R}^8)$ To do this, we will show that gradient of $\tilde{\phi}$ is zero at $\xi = e_1 \wedge e_2 \wedge e_5 \wedge e_6 \in G_4(\mathbf{R}^8)$ which is in the preimage of $\frac{1}{3}$.

Now, let ξ be any point in $G_4(\mathbf{R}^8)$, and V be any tangent vector at ξ, i.e. $V \in T_\xi G_4(\mathbf{R}^8)$. Let $\xi(t)$ with $|t| < \epsilon$ be a C^∞ curve on $G_4(\mathbf{R}^8)$ with $\xi(0) = \xi$ and $\frac{d}{dt}\xi(t)\Big|_{t=0} = V \in T_\xi G_4(\mathbf{R}^8)$, then we will have

$$\begin{aligned} V \cdot \tilde{\phi} &= \frac{d}{dt}\tilde{\phi}(\xi(t))_{|t=0} \\ &= \frac{d}{dt} < \phi, \xi(t) >_{|t=0} \\ &= < \phi, V > \end{aligned}$$

So, we get $(grad\tilde{\phi})_\xi = 0$ if and only if $< \phi, V >= 0$ for all $V \in T_\xi G_4(\mathbf{R}^8)$.

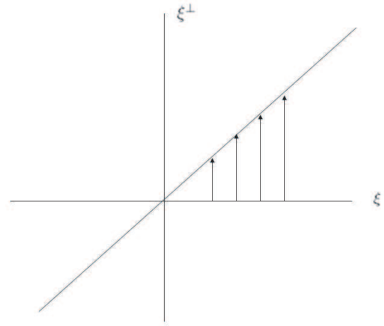

Figure 2.1: Tangent Space of $\xi \in G_k(\mathbf{R}^n)$

A tangent vector V to $G_k(\mathbf{R}^n)$ at a plane ξ in $\mathbf{R}^n$ may be regarded as an infinitesimal motion of the plane ξ (Figure 2.1). Such a motion corresponds to a linear map from ξ to the quotient space $\mathbf{R}^n/\xi$, which can be represented by $\xi^\perp$ relative to the metric.Thus, we have

$$\bigwedge^k \mathbf{R}^n \supset T_\xi G_k(\mathbf{R}^n) = \mathrm{Hom}(\xi, \xi^\perp) \cong \xi \bigotimes (\xi^\perp)^* \cong \mathrm{Span}\{\text{First Cousins of } \xi\ \} \subset \bigwedge^k \mathbf{R}^n$$

Thus a basis of $T_\xi G_k(\mathbf{R}^n)$ is given by 1^{st} **cousins** as follows :

Let $\xi = v_1 \wedge \ldots \wedge v_k$ and $v_1, \ldots, v_n$ be an orthonormal basis for $\mathbf{R}^n$, then the 1^{st} cousins η_{ij}

$$\eta_{i,j} \equiv v_j \wedge (v_i \llcorner \xi) \equiv v_1 \wedge \ldots \wedge v_{i-1} \wedge v_j \wedge v_{i+1} \wedge \ldots \wedge v_k$$

span $T_\xi G_k(\mathbf{R}^n)$ and because of the dimension, they form a basis.

Before passing to prove that $\frac{1}{3}$ is a critical value, since we are talking about 1^{st} cousins, it is the best time to state the ϕ-critical geometry version of the " First Cousin Principle" which has been used in most of the papers on calibrations, starting with [HL_1]. Our version will of course encompasses the usual version since every calibrated submanifold is also ϕ-critical. This principle is a simple consequence of elementary calculus, and we will include its proof here for the sake of completeness.

Lemma 2.3.1 (The First Cousin Principle for ϕ-Critical Submanifolds)**.** *If $\phi \in \bigwedge^p V^*$ is a calibration and $\xi = v_1 \wedge \ldots \wedge v_p \in G_{cr}(\phi)$, where $v_1, \ldots, v_n$ is an orthonormal basis for V, then ϕ vanishes on all the first cousins of ξ,*

$$\eta_{j,k} \equiv v_k \wedge (v_j \llcorner \xi) \equiv v_1 \wedge \ldots \wedge v_{j-1} \wedge v_k \wedge v_{j+1} \wedge \ldots \wedge v_p$$

where $1 \leq j \leq p$ and $p < k \leq n$.

Proof : Let $f(\theta) \equiv \phi(v_1 \wedge \ldots \wedge (\cos(\theta)v_j + \sin(\theta)v_k) \wedge \ldots \wedge v_p)$. Since $\theta = 0$ is a critical point for f, we have that $f'(0) = \phi(\eta_{jk})$ must vanish. $\square$

Let's consider $\mathbf{H}^2$ as a right-$\mathbf{H}$ vector space and denote the standard basis for $\bigwedge^1 \mathbf{H}^2$ by $\omega^1, ..., \omega^n$ where $\omega^1 = (1,0)^*, \omega^2 = (i,0)^*, ...\omega^8 = (0,k)^*$.(Here * indicates the dual) Right multiplication by i, j, k will give us the complex structures I, J, K , and each will determine a Kähler form w_I, w_J, w_K respectively.

$$\omega_I(x,y) =< xi, y >, \qquad \omega_J(x,y) =< xj, y >, \qquad \omega_K(x,y) =< xk, y >$$

$$
\begin{aligned}
\omega_I &= \omega^{12} - \omega^{34} + \omega^{56} - \omega^{78} \\
\omega_J &= \omega^{13} - \omega^{42} + \omega^{57} - \omega^{86} \\
\omega_K &= \omega^{14} - \omega^{23} + \omega^{58} - \omega^{67}
\end{aligned}
$$

where $\omega^{ij} = \omega^i \wedge \omega^j$. Therefore, we get;

$$
\begin{aligned}
\phi &= \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\} \\
&= -\omega^{1234} - \omega^{5678} - \frac{1}{3}\omega^{1278} - \frac{1}{3}\omega^{3456} - \frac{1}{3}\omega^{1467} - \frac{1}{3}\omega^{2358} + \frac{1}{3}\omega^{3478} \\
&+ \frac{1}{3}\omega^{1357} + \frac{1}{3}\omega^{1368} + \frac{1}{3}\omega^{2457} + \frac{1}{3}\omega^{2468} + \frac{1}{3}\omega^{1458} + \frac{1}{3}\omega^{2367} + \frac{1}{3}\omega^{1256}
\end{aligned}
$$

where $\omega^{ijkl} = \omega^i \wedge \omega^j \wedge \omega^k \wedge \omega^l$.

Now, we can state one of the main resul of this chapter.

Theorem 2.3.2. $\pm\frac{1}{3}$ *are critical values of* $\phi_| : G_4(\mathbf{H}^2) \cong\longrightarrow \mathbf{R}$ *where* ϕ *is the 4-form* $\frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$ *defined on* $\bigwedge^4 \mathbf{H}^2 \supset G_4(\mathbf{R}^8)$.

Proof : To show this, we need to find a critical point whose image is equal to $\pm\frac{1}{3}$. From the explicit form of the form ϕ an easy candidate to try will be $\xi = e_1 \wedge e_2 \wedge e_5 \wedge e_6$. Hence, we get $\xi^\perp = e_3 \wedge e_4 \wedge e_7 \wedge e_8$, then a basis of $T_\xi G_4(\mathbf{H})^2 \cong T_\xi G_4(\mathbf{R})^8$ by 1^{st} cousins is given as follows:

$$
\begin{array}{c|c}
\eta_{1,1} = e_3 \wedge e_2 \wedge e_5 \wedge e_6 & \eta_{1,2} = e_4 \wedge e_2 \wedge e_5 \wedge e_6 \\
\eta_{2,1} = e_1 \wedge e_3 \wedge e_5 \wedge e_6 & \eta_{2,2} = e_1 \wedge e_4 \wedge e_5 \wedge e_6 \\
\eta_{3,1} = e_1 \wedge e_2 \wedge e_3 \wedge e_6 & \eta_{3,2} = e_1 \wedge e_2 \wedge e_4 \wedge e_6 \\
\eta_{4,1} = e_1 \wedge e_2 \wedge e_5 \wedge e_3 & \eta_{4,2} = e_1 \wedge e_2 \wedge e_5 \wedge e_4
\end{array}
$$

$$
\begin{array}{c|c}
\eta_{1,3} = e_7 \wedge e_2 \wedge e_5 \wedge e_6 & \eta_{1,4} = e_8 \wedge e_2 \wedge e_5 \wedge e_6 \\
\eta_{2,3} = e_1 \wedge e_7 \wedge e_5 \wedge e_6 & \eta_{2,4} = e_1 \wedge e_8 \wedge e_5 \wedge e_6 \\
\eta_{3,3} = e_1 \wedge e_2 \wedge e_7 \wedge e_6 & \eta_{3,4} = e_1 \wedge e_2 \wedge e_8 \wedge \\
\eta_{4,3} = e_1 \wedge e_2 \wedge e_5 \wedge e_7 & \eta_{4,4} = e_1 \wedge e_2 \wedge e_5 \wedge e_8 \\
e_6 &
\end{array}
$$

If we reorganize them to more easily see their images under ϕ, we get the following list of basis vectors.

$$
\begin{array}{c|c}
-e_2 \wedge e_3 \wedge e_5 \wedge e_6 & -e_2 \wedge e_4 \wedge e_5 \wedge e_6 \\
e_1 \wedge e_3 \wedge e_5 \wedge e_6 & e_1 \wedge e_4 \wedge e_5 \wedge e_6 \\
e_1 \wedge e_2 \wedge e_3 \wedge e_6 & e_1 \wedge e_2 \wedge e_4 \wedge e_6 \\
-e_1 \wedge e_2 \wedge e_3 \wedge e_5 & -e_1 \wedge e_2 \wedge e_4 \wedge e_5
\end{array}
$$

$$
\begin{array}{c|c}
-e_2 \wedge e_5 \wedge e_6 \wedge e_7 & -e_2 \wedge e_5 \wedge e_6 \wedge e_8 \\
e_1 \wedge e_5 \wedge e_6 \wedge e_7 & e_1 \wedge e_5 \wedge e_6 \wedge e_8 \\
-e_1 \wedge e_2 \wedge e_6 \wedge e_7 & e_1 \wedge e_2 \wedge e_6 \wedge e_8 \\
e_1 \wedge e_2 \wedge e_5 \wedge e_7 & e_1 \wedge e_2 \wedge e_5 \wedge e_8
\end{array}
$$

It can be seen very easily from the above list that, ϕ vanishes on all of the first cousins of $\xi = e_1 \wedge e_2 \wedge e_5 \wedge e_6$ and this proves that $\xi = e_1 \wedge e_2 \wedge e_5 \wedge e_6$

is a critical point of the critical value $\frac{1}{3}$. By changing the orientation of ξ, we can also prove that $-\frac{1}{3}$ is a critical value of ϕ. $\square$

2.4 Orbit

In this section, we will calculate the orbit of $\xi = e_1 \wedge e_2 \wedge e_5 \wedge e_6$ under the action of the group $G = Sp(2) \cdot Sp(1)$ and the dimension of the orbit. This will at least give us an idea of how big the set of ϕ-critical planes in $G_4(\mathbf{H}^2) = G_4(\mathbf{R}^8)$ is. We know that the bigger the dimension of the G^{cr} is, the more chance to have an interesting ϕ-critical geometry.

First of all, we know that ϕ is G-invariant where $G = Sp(2) \cdot Sp(1)$ i.e. the following diagram commutes for every $\Psi \in Sp(2) \cdot Sp(1)$.

$$\begin{tikzcd}
G_4(\mathbb{R}^8) \arrow[r, "\Psi"] \arrow[d, "\phi"'] & G_4(\mathbb{R}^8) \arrow[d, "\phi"] \\
{[-1,1]} \arrow[r, "Id"'] & {[-1,1]}
\end{tikzcd}$$

Figure 2.2: Diagram commutes for every $\Psi \in Sp(2) \cdot Sp(1)$

So, since ξ is a critical point, and the commuting diagram above shows us that any η in the orbit of ξ,i.e. $\eta \in G \cdot \xi \equiv \{g \cdot \xi : g \in Sp(2) \cdot Sp(1)\}$ will be contained in the the critical set $G^{cr}_{\frac{1}{3}}$, as a result $G \cdot \xi \subseteq G^{cr}_{\frac{1}{3}}$.

Remark : After this, the Lie group G will stand for $Sp(2) \cdot Sp(1)$ and $\mathfrak{g}$ will stand for Lie algebra of $Sp(2) \cdot Sp(1)$ which is $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$.

The orbit of $\xi = e_1 \wedge e_2 \wedge e_5 \wedge e_6$ is isomorphic to G/G_ξ where G_ξ is the isotropy group of ξ in G. That is,

$$G_\xi = \{g \in Sp(2) \cdot Sp(1) : g \cdot \xi = \xi\}$$

Hence, to find the orbit of ξ we need to find its isotropy group under the given action. This seems a little bit hard. So, we will start with its dimension first. We already know that the dimension of $Sp(2) \cdot Sp(1)$ is 13. So, if we can calculate the dimension of G_ξ, then we will learn the dimension of the orbit which will give us an idea about the dimension of the critical set.

To find the dimension of G_ξ, we will calculate the dimension of its Lie algebra $\mathfrak{g}_\xi$. That is,

$$\mathfrak{g}_\xi = \{X \in \mathfrak{sp}(2) \oplus \mathfrak{sp}(1) : X \cdot \xi = 0\} \tag{2.3}$$

Here the action is on the Lie algebra level. We will find this action first i.e the action of $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ on $\xi \in G_4(\mathbf{R}^8) \subset \bigwedge^4 \mathbf{R}^8$.

We know that $Sp(2) \cdot Sp(1)$ acts on $\mathbf{H}^2 = \mathbf{R}^8$, so $Sp(2) \cdot Sp(1)$ will act on $\bigwedge^4 \mathbf{R}^8$ linearly also. Let $g = (\sigma, u) \in Sp(2) \cdot Sp(1)$. We have $g \cdot e_k = \sigma e_k u$, for any $e_k \in \mathbf{R}^8$. Therefore,

$$\begin{aligned} g \cdot \xi &= g \cdot e_1 \wedge g \cdot e_2 \wedge g \cdot e_5 \wedge g \cdot e_6 \\ &= (\sigma e_1 u) \wedge (\sigma e_2 u) \wedge (\sigma e_5 u) \wedge (\sigma e_6 u) \end{aligned}$$

Let $X = (S, U) \in \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ where $S \in \mathfrak{sp}(2)$ and $U \in \mathfrak{sp}(1)$, then $g_t = exp(tX) = e^{tX} = exp(t(S,U)) = e^{tS} \cdot e^{tU}$ will be a curve in $Sp(2) \cdot Sp(1)$

where exp is **exponential map** from Lie algebra $\mathfrak{g} = \mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ to Lie group $G = Sp(2) \cdot Sp(1)$. Then,

$$g_t \cdot e_k = e^{tS} e_k e^{tU}$$

If we take the derivative of g_t at $t = 0$, we get

$$\frac{d}{dt}(g_t \cdot e_k)_{t=0} = S \cdot e_k + e_k \cdot U \quad for \quad S \in \mathfrak{sp}(2), \quad U \in \mathfrak{sp}(1) \tag{2.4}$$

Now, if we do the same thing for the action of g_t on ξ, and take the derivative at $t = 0$,

$$
\begin{aligned}
\frac{d}{dt}(g_t\xi)_{t=0} &= \frac{d}{dt}\Big(g_te_1\wedge g_te_2\wedge g_te_5\wedge g_te_6\Big)_{t=0} \\
&= \frac{d}{dt}\Big(g_te_1\Big)_{t=0}\wedge e_2\wedge e_5\wedge e_6 + e_1\wedge\frac{d}{dt}\Big(g_te_2\Big)_{t=0}\wedge e_5\wedge e_6 \\
&+ e_1\wedge e_2\wedge\frac{d}{dt}\Big(g_te_5\Big)_{t=0}\wedge e_6 + e_1\wedge e_2\wedge e_5\wedge\frac{d}{dt}\Big(g_te_6\Big)_{t=0} \\
&= (Se_1+e_1U)\wedge e_2\wedge e_5\wedge e_6 + e_1\wedge(Se_2+e_2U)\wedge e_5\wedge e_6 \\
&+ e_1\wedge e_2\wedge(Se_5+e_5U)\wedge e_6 + e_1\wedge e_2\wedge e_5\wedge(Se_6+e_6U) \\
&= Se_1\wedge e_2\wedge e_5\wedge e_6 + e_1\wedge Se_2\wedge e_5\wedge e_6 \\
&+ e_1\wedge e_2\wedge Se_5\wedge e_6 + e_1\wedge e_2\wedge e_5\wedge Se_6 \\
&+ e_1U\wedge e_2\wedge e_5\wedge e_6 + e_1\wedge e_2U\wedge e_5\wedge e_6 \\
&+ e_1\wedge e_2\wedge e_5U\wedge e_6 + e_1\wedge e_2\wedge e_5\wedge e_6U \\
&= S\xi+\xi U
\end{aligned}
$$

Hence, for finding the dimension of $\mathfrak{g}_\xi$, we will compute the dimension of

$$\{(S,U)\in\mathfrak{sp}(2)\oplus\mathfrak{sp}(1) : S\xi+\xi U=0\}$$

As we know from the properties of $Sp(2)$, for an I-complex 4-plane ξ, the resulting 4-plane after the action of an element σ of $Sp(2)$ will be again I-complex, but for the action of an element u of the group $Sp(1)$ on ξ, by considering the equation 2.2 we can see that the resulting plane will be a 4-plane with a different complex structure unless $u=\pm i$. Actually, as it can be seen from the equation 2.2 that, it will be a $(ui\bar{u})$-complex plane. Hence, we see that these two actions are independent in their nature. So, we will calculate the dimension of following two Lie subalgebras :

$$\mathfrak{g}^1_\xi \equiv \{S \in \mathfrak{sp}(2) : S \cdot \xi = 0\}$$
$$\mathfrak{g}^2_\xi \equiv \{U \in \mathfrak{sp}(1) : \xi \cdot U = 0\}$$

Now, $\mathfrak{sp}(2) \equiv \{S \ M_2(\mathbf{H}) : S + S^\dagger = 0\}$. Then any $S \in \mathfrak{sp}(2)$ will be of the form

$$S = \begin{bmatrix} a_2 i + a_3 j + a_4 k & b_1 + b_2 i + b_3 j + b_4 k \\ -b_1 + b_2 i + b_3 j + b_4 k & d_2 i + d_3 j + d_4 k \end{bmatrix}$$

Here, $a_i, b_i, c_i, d_i \in \mathbf{R}$ for i=1 to 4 Let's show $e_1, e_2, e_5, e_6 \in \mathbf{H}^2$ as

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad e_2 = \begin{bmatrix} i \\ 0 \end{bmatrix} \qquad e_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad e_6 = \begin{bmatrix} 0 \\ i \end{bmatrix}$$

Then we get

$$
\begin{aligned}
Se_1 &= \begin{bmatrix} a_2 i + a_3 j + a_4 k & b_1 + b_2 i + b_3 j + b_4 k \\ -b_1 + b_2 i + b_3 j + b_4 k & d_2 i + d_3 j + d_4 k \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} (a_2) i + (a_3) j + (a_4) k \\ -b_1 + b_2 i + b_3 j + b_4 k \end{bmatrix} \\
&= (a_2) \begin{bmatrix} i \\ 0 \end{bmatrix} + (a_3) \begin{bmatrix} j \\ 0 \end{bmatrix} + (a_4) \begin{bmatrix} k \\ 0 \end{bmatrix} + (-b_1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&+ (b_2) \begin{bmatrix} 0 \\ i \end{bmatrix} + (b_3) \begin{bmatrix} 0 \\ j \end{bmatrix} + (b_4) \begin{bmatrix} 0 \\ k \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
Se_2 &= \begin{bmatrix} a_2 i + a_3 j + a_4 k & b_1 + b_2 i + b_3 j + b_4 k \\ -b_1 + b_2 i + b_3 j + b_4 k & d_2 i + d_3 j + d_4 k \end{bmatrix} \begin{bmatrix} i \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -a_2 + a_4 j - a_3 k \\ -b_2 - b_1 i + b_4 j - b_3 k \end{bmatrix} \\
&= (-a_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (a_4) \begin{bmatrix} j \\ 0 \end{bmatrix} + (-a_3) \begin{bmatrix} k \\ 0 \end{bmatrix} + (-b_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&+ (-b_1) \begin{bmatrix} 0 \\ i \end{bmatrix} + (b_4) \begin{bmatrix} 0 \\ j \end{bmatrix} + (-b_3) \begin{bmatrix} 0 \\ k \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
Se_5 &= \begin{bmatrix} a_2i + a_3j + a_4k & b_1 + b_2i + b_3j + b_4k \\ -b_1 + b_2i + b_3j + b_4k & d_2i + d_3j + d_4k \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} b_1 + b_2i + b_3j + b_4k \\ d_2i + d_3j + d_4k \end{bmatrix} \\
&= (b_1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (b_2) \begin{bmatrix} i \\ 0 \end{bmatrix} + (b_3) \begin{bmatrix} j \\ 0 \end{bmatrix} + (b_4) \begin{bmatrix} k \\ 0 \end{bmatrix} \\
&+ (d_2) \begin{bmatrix} 0 \\ i \end{bmatrix} + (d_3) \begin{bmatrix} 0 \\ j \end{bmatrix} + (d_4) \begin{bmatrix} 0 \\ k \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
Se_6 &= \begin{bmatrix} a_2i + a_3j + a_4k & b_1 + b_2i + b_3j + b_4k \\ -b_1 + b_2i + b_3j + b_4k & d_2i + d_3j + d_4k \end{bmatrix} \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&= \begin{bmatrix} -b_2 - b_1i + b_4j - b_3k \\ -d_2 + d_4j + -d_3k \end{bmatrix} \\
&= (-b_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-b_1) \begin{bmatrix} i \\ 0 \end{bmatrix} + (b_4) \begin{bmatrix} j \\ 0 \end{bmatrix} + (-b_3) \begin{bmatrix} k \\ 0 \end{bmatrix} \\
&+ (-d_2) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + (d_4) \begin{bmatrix} 0 \\ j \end{bmatrix} + (-d_3) \begin{bmatrix} 0 \\ k \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
Se_1 \wedge e_2 \wedge e_5 \wedge e_6 &= \begin{bmatrix} a_2 i + a_3 j + a_4 k \\ -b_1 + b_2 i + b_3 j + b_4 k \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&= a_3 \begin{bmatrix} j \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&+ a_4 \begin{bmatrix} k \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&+ b_3 \begin{bmatrix} 0 \\ j \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&+ b_4 \begin{bmatrix} 0 \\ k \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
e_1 \wedge Se_2 \wedge e_5 \wedge e_6 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} -a_2 + a_4 j + a_3 k \\ -b_2 - b_1 i + b_4 j - b_3 k \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&= a_4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} j \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&+ a_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} k \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&+ b_4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ j \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&- b_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ k \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
e_1 \wedge e_2 \wedge Se_5 \wedge e_6 \;&=\; \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} b_1 + b_2 i + b_3 j + b_4 k \\ d_2 i + d_3 j + d_4 k \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&=\; b_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} j \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&+\; b_4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} k \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&+\; d_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ j \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&+\; d_4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ k \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
e_1 \wedge e_2 \wedge e_5 \wedge Se_6 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} -b_2 - b_1 i + b_4 j - b_3 k \\ d_2) + d_4 j + -d_3 k \end{bmatrix} \\
&= b_4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} j \\ 0 \end{bmatrix} \\
&- b_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} k \\ 0 \end{bmatrix} \\
&+ d_4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ j \end{bmatrix} \\
&- d_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ k \end{bmatrix}
\end{aligned}
$$

As a result of these, we have

$$\begin{aligned}
0 &= Se_1\wedge e_2\wedge e_5\wedge e_6 + e_1\wedge Se_2\wedge e_5\wedge e_6\\
&+ e_1\wedge e_2\wedge Se_5\wedge e_6 + e_1\wedge e_2\wedge e_5\wedge Se_6\\
&= -\mathbf{a_3}e_2\wedge e_3\wedge e_5\wedge e_6 + \mathbf{a_4}e_2\wedge e_4\wedge e_5\wedge e_6\\
&- \mathbf{b_3}e_2\wedge e_5\wedge e_6\wedge e_7 - \mathbf{b_4}e_2\wedge e_5\wedge e_6\wedge e_8\\
&+ \mathbf{a_4}e_1\wedge e_3\wedge e_5\wedge e_6 + \mathbf{a_3}e_1\wedge e_4\wedge e_5\wedge e_6\\
&+ \mathbf{b_4}e_1\wedge e_5\wedge e_6\wedge e_7 - \mathbf{b_3}e_1\wedge e_5\wedge e_6\wedge e_8\\
&+ \mathbf{b_3}e_1\wedge e_2\wedge e_3\wedge e_6 + \mathbf{b_4}e_1\wedge e_2\wedge e_4\wedge e_6\\
&- \mathbf{d_3}e_1\wedge e_2\wedge e_6\wedge e_7 - \mathbf{d_4}e_1\wedge e_2\wedge e_6\wedge e_8\\
&+ \mathbf{b_4}e_1\wedge e_2\wedge e_3\wedge e_5 - \mathbf{b_3}e_1\wedge e_2\wedge e_4\wedge e_5\\
&- \mathbf{d_4}e_1\wedge e_2\wedge e_5\wedge e_7 - \mathbf{d_3}e_1\wedge e_2\wedge e_5\wedge e_8
\end{aligned}$$

and this gives us

$$(1)a_3 = 0 \quad (2)a_4 = 0 \quad (3)b_3 = 0 \quad (4)b_4 = 0 \quad (5)d_3 = 0 \quad (6)d_4 = 0$$

Now, $\mathfrak{sp}(1) \equiv \mathrm{Im}\mathbf{H} \equiv \{U \in \mathbf{H} : U + \bar{U} = 0\}$. Then any $U \in \mathfrak{sp}(1)$ will be of the form

$$U = \begin{bmatrix} u_2 i + u_3 j + u_4 k \end{bmatrix}$$

Here, $u_j \in \mathbf{R}$ for j=2 to 3. If we calculate the right action, we get

$$e_1 U = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} u_2 i + u_3 j + u_4 k \end{bmatrix} = \begin{bmatrix} u_2 i + u_3 j + u_4 k \\ 0 \end{bmatrix}$$

$$e_2U = \begin{bmatrix} i \\ 0 \end{bmatrix} \begin{bmatrix} u_2 i + u_3 j + u_4 k \end{bmatrix} = \begin{bmatrix} -u_2 - u_3 k + u_4 j \\ 0 \end{bmatrix}$$

$$e_5U = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} u_2 i + u_3 j + u_4 k \end{bmatrix} = \begin{bmatrix} 0 \\ u_2 i + u_3 j + u_4 k \end{bmatrix}$$

$$e_6U = \begin{bmatrix} 0 \\ i \end{bmatrix} \begin{bmatrix} u_2 i + u_3 j + u_4 k \end{bmatrix} \begin{bmatrix} 0 \\ -u_2 - u_3 k + u_4 j \end{bmatrix}$$

By using these, we get

$$\begin{aligned} e_1U \wedge e_2 \wedge e_5 \wedge e_6 &= \begin{bmatrix} u_2 i + u_3 j + u_4 k \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\ &= u_3 \begin{bmatrix} j \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\ &+ u_4 \begin{bmatrix} j \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
e_1 \wedge e_2 U \wedge e_5 \wedge e_6 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} -u_2 - u_3 k + u_4 j \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&= -u_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} k \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&+ u_4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} j \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
e_1 \wedge e_2 \wedge e_5 U \wedge e_6 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ u_2 i + u_3 j + u_4 k \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&= u_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ j \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix} \\
&+ u_4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ k \end{bmatrix} \wedge \begin{bmatrix} 0 \\ i \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
e_1 \wedge e_2 \wedge e_5 \wedge e_6 U &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ -u_2 - u_3 k + u_4 j \end{bmatrix} \\
&= -u_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ k \end{bmatrix} \\
&+ u_4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge \begin{bmatrix} i \\ 0 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} 0 \\ j \end{bmatrix}
\end{aligned}
$$

As a result of these, we have

$$\begin{aligned}
0 &= e_1U \wedge e_2 \wedge e_5 \wedge e_6 + e_1 \wedge e_2U \wedge e_5 \wedge e_6 \\
&+ e_1 \wedge e_2 \wedge e_5U \wedge e_6 + e_1 \wedge e_2 \wedge e_5 \wedge e_6U \\
&= \mathbf{u_3}e_3 \wedge e_2 \wedge e_5 \wedge e_6 + \mathbf{u_4}e_4 \wedge e_2 \wedge e_5 \wedge e_6 \\
&- \mathbf{u_3}e_1 \wedge e_4 \wedge e_5 \wedge e_6 + \mathbf{u_4}e_1 \wedge e_3 \wedge e_5 \wedge e_6 \\
&+ \mathbf{u_3}e_1 \wedge e_2 \wedge e_7 \wedge e_6 + \mathbf{u_4}e_1 \wedge e_2 \wedge e_8 \wedge e_6 \\
&- \mathbf{u_3}e_1 \wedge e_2 \wedge e_5 \wedge e_8 + \mathbf{u_4}e_1 \wedge e_2 \wedge e_5 \wedge e_7
\end{aligned}$$

and this gives us

$$(7) u_3 = 0 \quad (8) u_4 = 0$$

Hence, we have 8 equations which give us the codimension of $\mathfrak{g}_\xi = \mathfrak{g}_\xi^1 \oplus \mathfrak{g}_\xi^2$ in $\mathfrak{sp}(2) \oplus \mathfrak{sp}(1)$ But, this is actually the dimension of the orbit of ξ. Hence, we get the following result.

Lemma 2.4.1. *The dimension of the orbit of ξ under the action of the group $Sp(2) \cdot Sp(1)$ is 8.*

$dim(G \cdot \xi) = 8$.

In fact, this is what we expected. We know that $\xi = e_1 \wedge e_2 \wedge e_5 \wedge e_6$ is an I-complex plane which is Lagrangian with respect to ω_J and ω_K i.e. $\omega_{J_{|\xi}} = 0$ $\omega_{K_{|\xi}} = 0$. Moreover, $Sp(2)$ acts transitively on complex Lagrangian planes with isotropy subgroup equal to $U(2)$(cf [Weinstein]). That is, for a fixed complex structure, the complex Lagrangian Grassmannian is isomorphic to

$Sp(2)/U(2)$.

We also have that $Sp(1)$ acts on ξ from right and acts on the Kähler forms ω_J and ω_K. As we know its action on Kähler forms, the resulting 4-plane will be Lagrangian with respect to the resulting Kähler forms we get after the action of $Sp(1)$. By the following proposition, the resulting 4-plane will be a complex plane with respect to a different complex structure. Hence, the resulting 4-plane will be a complex Lagrangian plane.

Proposition 2.4.2. *Let I_1, I_2, and I_3 be complex structures on $\mathbf{H}^n$ with $I_1 \cdot I_2 = I_3$. If $\omega_{I_1|_V} = 0$ and $\omega_{I_2|_V} = 0$ for a n-plane V, then V is I_3-complex.*

Proof : Since $\omega_{I_1|_V} = 0$ we have $I_1(V) \perp V$ and since $\omega_{I_2|_V} = 0$ we have $I_2(V) \perp V$. dim($\mathbf{H}^n$)=2n, so we have $I_1(V) = I_2(V)$. Then $I_2^{-1} \cdot I_1(V) = V$. Since $I_2 \cdot I_2 = -$Identity, $I_2^{-1} = -I_2$. Hence, $I_2^{-1} \cdot I_1 = -I_2 \cdot I_1 = -(-I_3) = I_3$. Since, $I_3(V) = V$, V is an I_3-complex plane. □

Since the isotropy subgroup of $Sp(1)$ which fixes ξ is 1-dimensional, we will have all possible complex Lagrangian planes for every complex structure determined by a unit imaginary quaternion. Hence, $G \cdot \xi \cong Sp(2)/U(2) \times S^2$ where S^2 is the 2-sphere of complex structures which is the same as imaginary unit quaternions. Hence, we get our main result :

Theorem 2.4.3. *Let $\phi = \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$ be the quaternion calibration on $\mathbf{H}^2$. Then $\pm\frac{1}{3}$ are critical values and the ϕ-critical submanifolds with critical value $\pm\frac{1}{3}$ include all complex Lagrangian submanifolds for any complex structure defined by right multiplication by a unit imaginary quaternion.*

In $\mathbf{H}^n$, it can easily be showed that ξ is a critical value of $\phi_{|_{G_4(\mathbf{H}^n)}}$ with critical value $\frac{1}{3}$. (By changing the orientation, we can also prove that $-\frac{1}{3}$ is a critical value of $\phi_{|_{G_4(\mathbf{H}^n)}}$ Since $Sp(2)\cdot Sp(1)$ is a subgroup of $Sp(n)\cdot Sp(1)$, $(Sp(2)\cdot Sp(1))\cdot\xi$ will be contained in the $\frac{1}{3}$-critical set. In this case, because of the degree of ϕ, we will have the following result.

Theorem 2.4.4. *Let $\phi = \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$ be the quaternion calibration on $\mathbf{H}^n$. Then $\pm\frac{1}{3}$ are critical values and the ϕ-critical submanifolds with critical value $\pm\frac{1}{3}$ include all complex isotropic submanifolds for any complex structure defined by right multiplication by a unit imaginary quaternion.*

2.5 Calculation of the Hessian

We will now calculate the Hessian of the function $\phi :\ G_4(\mathbf{R}^8) \longrightarrow [-1,1] \subset \mathbf{R}$ at $\xi = e_1\wedge e_2\wedge e_5\wedge e_6$. Before that we will recall some facts from the differential geometry of submanifolds, which will help us for the calculation of the Hessian. Given a submanifold $\overline{M} \subset M$, for each $x \in M$, the inner product on T_xM splits into the direct sum

$$T_xM = T_x\overline{M} \oplus (T_x\overline{M})^{\perp}$$

where $(T_x\overline{M})^{\perp}$ is the orthogonal complement of $T_x\overline{M}$ in T_xM. If $V \in T_xM, x \in M$, we can write V in the following form.

$$V = V^T + V^N$$

Here, $V^T \in T_x\overline{M}$ is the tangential component, $V^N \in (T_x\overline{M})^\perp$ is the normal component

$(\bullet)^T$ and $(\bullet)^N$ are the orthogonal projections of T_xM onto $T_x\overline{M}$ and $(T_x\overline{M})^\perp$,respectively. The Riemannian connection ∇ on M will induce a connection $\overline{\nabla}$ on $\overline{M}$ which is the Riemannian connection of the induced metric on $\overline{M}$. The connection $\overline{\nabla}$ is defined as

$$\overline{\nabla}_V W = (\nabla_V W)^T, \ V \text{ and } W \text{ are tangent vector fields on } \overline{M}$$

We define the **second fundamental form** as

$$B(V, W) = \nabla_V W - \overline{\nabla}_V W$$

for tangent vector fields V and W on $\overline{M}$. The fundamental property of the second fundamental form is given in the following proposition.

Proposition 2.5.1. *The second fundamental form is bilinear and symmetric with values in the normal space.*

Proof : The properties of a connection will directly imply that B is linear in the first slot and it is additive in the second slot. Hence, it remains to show that

$$B(V, fW) = fB(V, W), \quad f \in C^\infty(M)$$

for the tangent vector fields V and W on $\overline{M}$.

$$\begin{aligned} B(V, fW) &= \nabla_V fW - \overline{\nabla}_V fW \\ &= V(f)W + f\nabla_V W - V(f)W - f\overline{\nabla}_V W \\ &= f\nabla_V W - f\overline{\nabla}_V W \\ &= fB(V, W) \end{aligned}$$

To prove that B is symmetric, we use the properties of the Riemannian connection.

$$B(V, W) = \nabla_V W - \overline{\nabla}_V W = \nabla_W V + [V, W] - \overline{\nabla}_W V - [V, W] = B(W, V)$$

Since B is bilinear, the value of $B(V, W)(x)$ depends only on the values of $V(x)$ and $W(x)$. The trace of the second fundamental form, $H = traceB$ is the **mean curvature vector field** of the submanifold $\overline{M}$, and $\overline{M}$ is called **minimal submanifold** if $H \equiv 0$.

For a smooth map $f : M \longrightarrow \mathbf{R}$, the **gradient vector field**of f, ∇f is defined as the vector field satisfying $< \nabla f, v >= D_v f(v) = df(v)$ The **Hessian** $\nabla^2 f$ is defined as the (1,1)-tensor $\nabla(\nabla f)$, $\nabla^2 f(V) = \nabla_V \nabla f$ for a smooth vector field V. This tensor is self-adjoint, or symmetric, since

$$\begin{aligned} < \nabla^2(V), W > &= < \nabla_V \nabla f, W > \\ &= V\cdot < \nabla f, W > - < \nabla f, \nabla_V W > \\ &= V \cdot (W \cdot f) - df(\nabla_V W) \\ &= V \cdot (W \cdot f) - df(\nabla_W V) - df([V, W]) \\ &= V \cdot (W \cdot f) - V \cdot (W \cdot f) + W \cdot (V \cdot f) - df(\nabla_W V) \\ &= < \nabla^2 f(W), V > \end{aligned}$$

This will also tell us that $\nabla^2 f$ can be interpreted as the symmetric (0,2)-tensor $\nabla^2 f(V, W) =< \nabla^2 f(V), W >= V \cdot (W \cdot f) - \nabla_{\nabla_V W} f$. This way of thinking is in fact the most familiar, and we will use this one in our calculations. We will denote $\nabla^2 f$ as Hess(f). Also, on $\overline{M} \subset M$, the Hessian will be denoted by $\overline{\text{Hess}}(f)$. The relation between Hess(f) and $\overline{\text{Hess}}(f)$ is given by the following equation.

$$\overline{\text{Hess}}(f)(V, W) = \text{Hess}(f)(V, W) - B_{V,W} \cdot f \tag{2.5}$$

The proof of (2.5) is straightforward:

$$
\begin{aligned}
\mathrm{Hess}(f)(V,W) &= \nabla^2 f(V,W) \\
&= <\nabla_V \nabla f, W> \\
&= V\cdot <\nabla f, W> + <\nabla f, \nabla_V W> \\
&= V\cdot W\cdot f + (\nabla_V W)\cdot f \\
&= V\cdot W\cdot f + (\overline{\nabla}_V W + B_{V,W})\cdot f \\
&= V\cdot W\cdot f + (\overline{\nabla}_V W)\cdot f + B_{V,W})\cdot f \\
&= \overline{\mathrm{Hess}}(f)(V,W) + B_{V,W}\cdot f
\end{aligned}
$$

We now calculate the Hessian of $\phi : G_4(\mathbf{R}^8) \longrightarrow \mathbf{R}$ at $\xi = e_1\wedge e_2\wedge e_5\wedge e_6$. Here we have $\phi : \bigwedge^4 \mathbf{R}^8 \longrightarrow \mathbf{R}$ which is linear and $G_4(\mathbf{R}^8) \subset \bigwedge^4 \mathbf{R}^8 \cong \mathbf{R}^{70}$. Hence, by the help of (2.5) we get :

$$
\begin{aligned}
\overline{\mathrm{Hess}}_\xi(\phi)(V,W) &= \mathrm{Hess}_\xi(\phi|_{G_4(\mathbf{R}^8)})(V,W) \\
&= \mathrm{Hess}_\xi(\phi|_{\bigwedge^4 \mathbf{R}^8})(V,W) - B_{V,W}(\xi)\cdot\phi \qquad (2.6)
\end{aligned}
$$

But, $\phi : \bigwedge^4 \mathbf{R}^8 \longrightarrow \mathbf{R}$ is linear, so we have $\mathrm{Hess}_\xi(\phi|_{\bigwedge^4 \mathbf{R}^8}) \equiv 0$. Hence, (2.6) becomes

$$
\overline{\mathrm{Hess}}_\xi(\phi)(V,W) = \mathrm{Hess}_\xi(\phi|_{G_4(\mathbf{R}^8)})(V,W) = -B_{V,W}(\xi)\cdot\phi \qquad (2.7)
$$

By definition, $B_{V,W} = \nabla_V W - \overline{\nabla}_V W$, and $\overline{\nabla}_V W = (\nabla_V W)^T$. Since $\xi = e_1\wedge e_2\wedge e_5\wedge e_6$ is a critical point of $\phi : G_4(\mathbf{R}^8) \longrightarrow \mathbf{R}$, we have $\overline{\nabla}_V W\cdot\phi = (\nabla_V W)^T\cdot\phi =<\phi, \nabla_V W>\equiv 0$. Hence, we get $B_{V,W}\cdot\phi = \nabla_V W\cdot\phi$.

Here, ∇ is the Riemannian connection on $\bigwedge^4 \mathbf{R}^8 \cong \mathbf{R}^{70}$, i.e. it is the flat connection. This will make our calculations easier.

As we know from the previous sections, $T_\xi G_4(\mathbf{R}^8) \cong Hom(\xi, \xi^\perp)$

$\cong$ linear span of $\{1^{st}$ cousins of $\xi\}$ =linear span of $\{f \wedge (e \llcorner \xi) : e \in \xi, f \in \xi^\perp\}$.

For given $V,\ W \in Hom(\xi, \xi^\perp)$, to calculate $\nabla_V W$, we need to write a curve $\xi(t)$ and a tangent vector field $W(t)$ along $\xi(t)$ such that

$$\begin{array}{rclcl} \xi(t) & \in & G_4(\mathbf{R}^8) & \qquad & W(t) \in Hom(\xi(t), \xi(t)^\perp) \\ \xi(0) & = & \xi & \qquad & W(0) = W \\ \xi^{'}(0) & = & V & & \end{array}$$

Then, we will have $\frac{dW}{dt}_{|t=0} = \nabla_V W$.

Now, we can start the calculation of the Hessian at $\xi = e_1 \wedge e_2 \wedge e_5 \wedge e_6$.Let $V = f \wedge (e \llcorner \xi)$ and $W = f' \wedge (e' \llcorner \xi)$ where e and $e' \in \xi$ and f and $f' \in \xi^\perp$. To calculate the Hessian, we will need e and $e' \in \{e_1, e_2, e_5, e_6\}$ and f and $f' \in \{e_3, e_4, e_7, e_8\}$.If we choose, $\xi(t) = (e\cos(t) + f\sin(t)) \wedge (e \llcorner \xi)$, then it will satisfy all the conditions we wanted from $\xi(t)$. We will define $W(t)$ to be a 1^{st} cousin of $\xi(t)$ with $W(0) = W$. The definition of $W(t)$ will basically drop down into four cases depending on $W(0) = W$.

Case I : $e' = e$ and $f' = f$

In this case, we have W=V.

$$V(t) = \frac{d\xi(t)}{dt} = (-e\sin(t) + f\cos(t)) \wedge (e \llcorner \xi)$$

Now, set W(t)=V(t), then

$$\begin{aligned}\frac{dW(t)}{dt}_{|t=0} &= -(e\cos(t)+f\sin(t))\wedge(e\llcorner\xi)_{|t=0}\\ &= -e\wedge(e\llcorner\xi)\\ &= -\xi\end{aligned}$$

So, $\nabla_V W = -\xi$, and $\overline{\text{Hess}}(V,W) = -B_{V,W}\cdot\phi = -(\nabla_V W - \overline{\nabla}_V W)\cdot\phi$ $= -\nabla_V W\cdot\phi = -(-\xi)\cdot\phi = <\phi,\xi> = \frac{1}{3}$. We must have $\overline{\nabla}_V W\cdot\phi = 0$, since $\overline{\nabla}_V W$ is a tangent vector to $G_4(\mathbf{R}^8)$ at $\xi = e_1\wedge e_2\wedge e_5\wedge e_6$ which is a critical point of ϕ.

Case II : $e' = e$ and $f'\perp f$ (also,$f'\perp\xi$)

If we set

$$W(t) = f'\wedge(e\llcorner\xi)$$

, then W(0)=W obviously and W(t) is a 1^{st} cousin of $\xi(t)$ since $e\in\xi(t)$ and $f'\perp\xi(t)$. As a result, we get $\frac{dW(t)}{dt}_{|t=0} = 0$ and therefore

$\nabla_V W = 0$, which implies that we have $\overline{\text{Hess}}(V,W) = 0$ in this case.

Case III : $e'\perp e$ and $f' = f$

We have $\xi = e_1\wedge e_2\wedge e_5\wedge e_6$, without loss of generality, let's take $e = e_1$ and $e' = e_2$, then we have

$$\xi(t) = (e\cos(t)+f\sin(t))\wedge e_2\wedge e_5\wedge e_6$$

and

$$e' \llcorner \xi(t) = -(e\cos(t) + f\sin(t)) \wedge e_5 \wedge e_6$$

If we set $W(t) = (f\cos(t) - e\sin(t)) \wedge (e' \llcorner \xi(t))$, then $W(0) = W$, and $W(t)$ is a 1^{st} cousin of $\xi(t)$ since $e' \in \xi(t)$ and $f\cos(t) - e\sin(t)$ is perpendicular to $\xi(t)$(It is obvious that $f\cos(t) - e\sin(t)$ is $90°$-rotation of $e\cos(t) + f\sin(t)$).In this case, we'll have :

$$\begin{aligned}
W(t) &= (f\cos(t) - e\sin(t)) \wedge (e' \llcorner \xi(t)) \\
&= (f\cos(t) - e\sin(t)) \wedge (e\cos(t) + f\sin(t)) \wedge e_5 \wedge e_6 \\
&= -e \wedge f \wedge e_5 \wedge e_6 (\cos^2(t) + \sin^2(t)) \\
&= -e \wedge f \wedge e_5 \wedge e_6 \\
&= f \wedge e \wedge e_5 \wedge e_6
\end{aligned}$$

So, $\frac{dW(t)}{dt}|_{t=0} = 0$. We get the same result for other choices of $e\, e'$. Similar to **Case II**, we will have $\overline{\text{Hess}}(V, W) = 0$ in this case, too.

Case IV : $e' \perp e$ and $f' \perp f$ (also,$f' \perp \xi$)

Similar to the previous case, without loss of generality, let's take $e = e_1$ and $e' = e_2$.If we set $W(t) = f' \wedge (e' \llcorner \xi(t))$, then we get $W(0) = W$ and $W(t)$ is a 1^{st} cousin of $\xi(t)$ since $e' \in \xi(t)$ and $f' \perp \xi(t)$ according to the definition of

$\xi(t)$.In this case, we'll have :

$$\begin{aligned} W(t) &= f' \wedge (e' \llcorner \xi(t)) \\ &= f' \wedge (e\cos(t) + f\sin(t)) \wedge e_5 \wedge e_6 \\ &= -(e\cos(t) + f\sin(t)) \wedge f' \wedge e_5 \wedge e_6 \end{aligned}$$

So, we get $\frac{dW(t)}{dt}_{|t=0} = f \wedge f' \wedge e_5 \wedge e_6$

For other choices of e, e', we will have similar results. In this case, we will have

$\overline{\text{Hess}}(V, W) = -B_{V,W} \cdot \phi = -(\nabla_V W - \overline{\nabla}_V W) \cdot \phi = -\nabla_V W \cdot \phi$

$= - < \phi, \nabla_V W >$. As we can see from the explicit form of ϕ in page 21, ± 1 and $\pm\frac{1}{3}$ are the only possible choices.

Hence, we get $\mathbf{Hess}(\phi)_{G_4(\mathbf{R}^8)}$ equal to

$$
\begin{bmatrix}
\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} \\
0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & 0 & -1 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} \\
0 & 0 & 0 & -1 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & -1 & 0 & 0 & 0 \\
-\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \\
0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 & -1 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 1 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\
0 & \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 \\
-\frac{1}{3} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{3}
\end{bmatrix}
$$

By using a computer program, we found that the Eigenvalues are $\lambda_1 = \frac{4}{3}$, $\lambda_2 = -\frac{4}{3}$, and $\lambda_3 = 0$ and the dimensions of the corresponding Eigenspaces are :

$$dim(W_{\lambda_1}) = 6 \qquad dim(W_{\lambda_2}) = 2 \qquad dim(W_{\lambda_3}) = 8$$

Now, $\mathbf{Hess}(\phi|_{G_4(\mathbf{R}^8)})(\xi)$ is positive definite on W_{λ_1}, negative definite on W_{λ_2} and zero on W_{λ_3}. Moreover, we want to denote W_{λ_1} by D^+(positive definite), W_{λ_2} by D^-(negative definite), and W_{λ_3} by N(null). Then we will have $T_\xi G_4(\mathbb{R}^8) = D^+ \oplus N \oplus D^-$. Now, we will prove the following theorem which gives us the dimension of the critical set.

Theorem 2.5.2. *The critical set $G^{cr}_{\frac{1}{3}}$ is equal to the orbit of ξ*

Proof : First of all, we know that ϕ is G-invariant where $G = Sp(2)\cdot Sp(1)$ i.e. the following diagram commutes for every $\Psi \in Sp(2)\cdot Sp(1)$.

$$
\begin{tikzcd}
G_4(\mathbb{R}^8) \arrow[r, "\Psi"] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & G_4(\mathbb{R}^8) \arrow[d, "\phi"] \\
{[-1,1]} \arrow[r, "Id"'] & {[-1,1]}
\end{tikzcd}
$$

Figure 2.3: Diagram commutes for every $\Psi \in Sp(2)\cdot Sp(1)$

So, since ξ is a critical point, then the orbit of ξ,i.e. $G\cdot\xi$ will contained in the the critical set $G^{cr}_{\frac{1}{3}}$ i.e. $G\cdot\xi \subseteq G^{cr}_{\frac{1}{3}}$.

Let $T = exp_\xi(D^+ \oplus D^-)$ be a normal slice.(Figure 2.4) If we restrict ϕ onto T, then ξ is a non-degenerate critical point of ϕ. So, by Morse Lemma [M] there exists coordinates (x,y) where $x = (x_1, x_2, x_3, x_4, x_5, x_6)$ and $y = (y_1, y_2)$ such that

$$\tilde{\phi} = \frac{1}{3} + |x|^2 - |y|^2 = \frac{1}{3} + {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2 + {x_5}^2 + {x_6}^2 - {y_1}^2 - {y_2}^2$$

Hence, $d(\phi_{|T}) \neq 0$ for $(x, y) \neq (0, 0)$. Since $\tilde{\phi}$ is G-invariant, this holds for all $g\cdot\xi \in G$. As a result, we see that $dim(G^{cr}_{\frac{1}{3}}) \leq dim(N) = 8$. But, $dim(G\cdot\xi)$, the dimension of the orbit of ξ under group $G = Sp(2)\cdot Sp(1)$ is also 8. So, we get that $G^{cr}_{\frac{1}{3}} = G\cdot\xi$ where $G = Sp(2)\cdot Sp(1)$. $\square$.

An alternative proof can be given if we can directly prove that

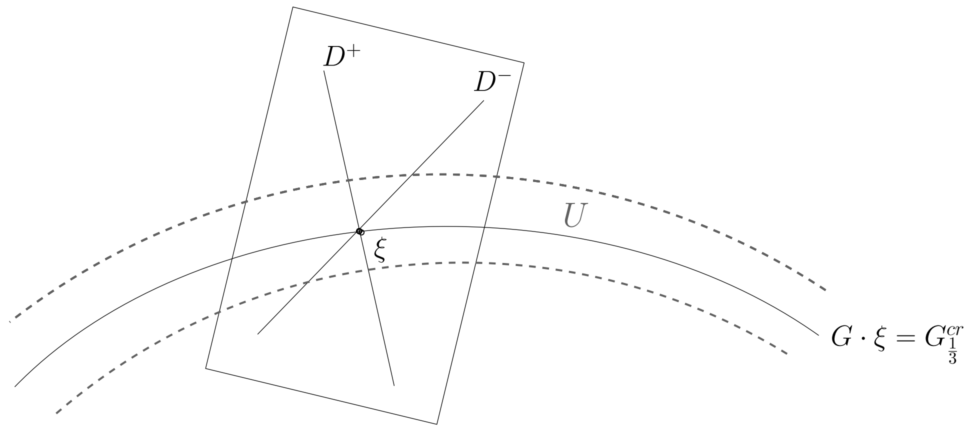

Figure 2.4: Definite Eigenspaces of Hessian at $\xi \in G_k(\mathbf{R}^n)$

$T_\xi G^{cr}_{\frac{1}{3}} \subset Null(\mathbf{Hess}(\phi_{|_{G_4(\mathbf{R}^8)}})(\xi))$ Again, this will give us an upper bound for the dimension of the critical set. Since, we have
dim (Null($\mathbf{Hess}(\phi_{|_{G_4(\mathbf{R}^8)}})(\xi))) = 8 =$dim $(G \cdot \xi)$, we will get $G^{cr}_{\frac{1}{3}} = G \cdot \xi$.

Lemma 2.5.3. *The tangent space to $G^{cr}_{\frac{1}{3}}$ at ξ will be contained in the null space of $\boldsymbol{Hess}(\phi_{|_{G_4(\mathbf{R}^8)}})(\xi)$.*

Proof : Let $X \in T_\xi G^{cr}_{\frac{1}{3}}$, and $Y \in T_\xi G_4(\mathbb{R}^8)$. We'll show that

$$\mathbf{Hess}_\xi(\phi_{|_{G_4(\mathbf{R}^8)}})(X, Y) = < Y, \nabla_X \nabla \phi_{G_4(\mathbb{R}^8)} >= 0$$

First of all, let us look at $< Y, \nabla \phi_{G_4(\mathbb{R}^8)} >$. If we extend Y a tangent vector field, this will define a function on $G_4(\mathbb{R}^8)$. We will try to calculate its directional derivative in the direction of X. To do this, we will need a curve $\xi(t)$

such that

$$\begin{aligned}
\xi(t) &\in G_4(\mathbf{R}^8) \\
\xi(0) &= \xi \\
\xi^{'}(0) &= X
\end{aligned}$$

Since $X \in T_\xi G^{cr}_{\frac{1}{3}}$, we can always find such a curve which lies on $G^{cr}_{\frac{1}{3}} \subset G_4(\mathbf{R}^8)$. Then, we will have $X\cdot < Y, \nabla\phi_{G_4(\mathbb{R}^8)} > (\xi) = \frac{d}{dt}(< Y(\xi(t)), \nabla\phi_{G_4(\mathbb{R}^8)}(\xi(t)) >)_{|t=0}$. But, $\nabla\phi_{G_4(\mathbb{R}^8)}(\xi(t))$ is zero. So, we get that $< Y(\xi(t)), \nabla\phi_{G_4(\mathbb{R}^8)}(\xi(t)) >$ is zero, and this gives us that $X\cdot < Y, \nabla\phi_{G_4(\mathbb{R}^8)} >$ at (ξ) is zero. By using the properties of Riemannian connection on $G_4(\mathbb{R}^8)$, we will have

$$0 = X\cdot < Y, \nabla\phi_{G_4(\mathbb{R}^8)} >=< \nabla_X Y, \nabla\phi_{G_4(\mathbb{R}^8)} > + < Y, \nabla_X\nabla\phi_{G_4(\mathbb{R}^8)} > \quad at \;\; \xi$$

But, $< \nabla_X Y, \nabla\phi_{G_4(\mathbb{R}^8)} >$ is again zero at ξ since $\nabla\phi_{G_4(\mathbb{R}^8)}$ is zero at ξ. Hence, we get $< Y, \nabla_X\nabla\phi_{G_4(\mathbb{R}^8)} >= 0$ at ξ. □

Hence, we proved the following result.

Theorem 2.5.4. *The critical set $G^{cr}_{\frac{1}{3}}$ is equal to all complex Lagrangian planes in $\mathbf{H}^2$ for each complex structure defined by right multiplication by a unit imaginary quaternion.*

As a result of this, we get the following corollary.

Corollary 2.5.5. *Let $\phi = \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$ be the quaternion calibration on $\mathbf{H}^2$. Then $\pm\frac{1}{3}$ is a critical value and the ϕ-critical submanifolds with critical value $\pm\frac{1}{3}$ include all complex Lagrangian submanifolds for each complex structure defined by right multiplication by a unit imaginary quaternion.*

2.6 Examples

In this section, we will give some examples of 4-dimensional manifolds which are ϕ-critical submanifolds of $\mathbf{H}^2$.As we know from the previous section, these are the complex Lagrangian submanifolds of $\mathbf{H}^2 \cong \mathbf{C}^4 \cong \mathbf{R}^8$. In the general case, where we have $\mathbf{H}^n$, we know that 2-dimensional complex isotropic submanifolds of $\mathbf{H}^n$ for any complex structure defined by a imaginary unit quaternion will be ϕ-critical, too.

We begin our list with the most obvious examples. If $V \subset \mathbb{H}^2$ is any I-complex 2-plane where we have $\omega_{J|_V} = \omega_{K|_V} = 0$ (i.e. I-complex and J and K lagrangian), then $\phi_{|_V} = \frac{1}{3}$ and $\nabla\phi(V) = 0$. In fact, if we take any real 4-plane in $\mathbf{H}^2$ which is Lagrangian with respect to the Kähler forms ω_{J_1} and ω_{J_2} defined by two different orthogonal complex structures J_1 and J_2, then it will be a $J_2^{-1} \cdot J_1$-complex plane, so it will a ϕ-critical manifold.

Suppose M is a complex lagrangian submanifold of $\mathbf{H}^2$. Locally, M can be described as the graph of a function. Now, we will derive the differential equation which must be satisfied by a function so that its graph will be a complex Lagrangian submanifold of $\mathbf{H}^2$.

Let us choose I complex coordinates z_1,z_2,w_1,w_2 on $\mathbf{H}^2 \cong \mathbf{C}^4$. If we define $\sigma = \omega_J - \sqrt{-1}\omega_K$, then σ will be a *I-holomorphic* 2-form. With respect to I complex coordinates :

$$\sigma = dz_1 \wedge dw_1 + dz_2 \wedge dw_2 \tag{2.8}$$

Let $f : \Omega \subset \mathbf{C}^2 \longrightarrow \mathbf{C}^2$ be a I-holomorphic map given by :

$$f(z_1, z_2) = (w_1, w_2) = (f_1(z_1, z_2), f_2(z_1, z_2)) \tag{2.9}$$

Its graph

$$\Gamma_f \equiv \{(z_1, z_2, w_1, w_2) \in \mathbf{C}^4 : w_1 = f_(z_1, z_2), w_2 = f_2(z_1, z_2) \forall (z_1, z_2) \in \Omega\}$$

will be a I-complex submanifold of $\mathbf{H}^2 \cong \mathbf{C}^4$.

Proposition 2.6.1. Γ_f *will satisfy* $f^*\sigma = 0$ *i.e. Lagrangian with respect to* ω_J *and* ω_K *if and only if* f *satisfies the single first order equation*

$$\frac{\partial f_1}{\partial z_2} = \frac{\partial f_2}{\partial z_1} \tag{2.10}$$

Proof : If $w_1 = f_1(z_1, z_2)$ and $w_2 = f_2(z_1, z_2)$, then

$$\begin{aligned} dw_1 &= \frac{\partial f_1}{\partial z_1} dz_1 + \frac{\partial f_1}{\partial z_2} dz_2 \\ dw_2 &= \frac{\partial f_2}{\partial z_1} dz_1 + \frac{\partial f_2}{\partial z_2} dz_2 \end{aligned}$$

Then

$$\begin{aligned} \sigma_{\Gamma_f} &= dz_1 \wedge dw_1 + dz_2 \wedge dw_2 \\ &= \frac{\partial f_1}{\partial z_2} dz_1 \wedge dz_2 + \frac{\partial f_2}{\partial z_1} dz_2 \wedge dz_1 \\ &= (\frac{\partial f_1}{\partial z_2} - \frac{\partial f_2}{\partial z_1}) dz_1 \wedge dz_2 \end{aligned}$$

Hence, $f^*\sigma = \sigma_{\Gamma_f} = 0$ if and only if $\frac{\partial f_1}{\partial z_2} = \frac{\partial f_2}{\partial z_1}$. $\square$

This equation has many solutions. We can find some of these solutions by picking an $f \in \mathcal{O}(\Omega)$ and setting the following equations :

$$f_1 = \frac{\partial f}{\partial z_1} \qquad\qquad f_2 = \frac{\partial f}{\partial z_2}$$

If Ω is simply connected, then this is all solutions.

In the general case, if we choose I-complex coordinates $z_1, ...z_n, w_1, ...w_n$ on $\mathbf{H}^n \cong \mathbf{C}^{2n}$, then

$$\sigma = \omega_J - \sqrt{-1}\omega_K = dz_1 \wedge dw_1 + dz_2 \wedge dw_2 + ... + dz_n \wedge dz_n$$

If $f : \Omega \subset \mathbf{C}^2 \longrightarrow \mathbf{C}^{2n}$ is a holomorphic map, then $f^*\sigma = 0$ if and only if $f = (z_1(u,v), .., z_n(u,v), w_1(u,v), .., w_n(u,v))$ satisfies the first order equation

$$\sum_{i=1}^{n} \frac{\partial z_i}{\partial u} \cdot \frac{\partial w_i}{\partial v} - \frac{\partial z_i}{\partial v} \cdot \frac{\partial w_i}{\partial u} = 0 \tag{2.11}$$

Again, for a simply connected Ω, this equation will have many solutions, too.

Chapter 3

ϕ-Free Submanifolds and Topology

Let (X, ϕ) be a non-compact, connected calibrated manifold with calibration ϕ and let $\Omega \subset X$ be a strictly ϕ-convex domain. Similar to the Stein case in Kähler geometry, we have an integer bound for the homotopy type of Ω. We will give these integer bounds for certain calibrated manifolds in **3.1** and will discuss the techniques to get examples of strictly ϕ-convex domains in **3.2** and will give some examples of these domains in certain calibrated manifolds in **3.3**.

3.1 Topology of Strictly ϕ-Convex Manifolds

We start with a definition :

Definition 3.1.1. *The free dimension, denoted* ***hd(ϕ)****, of a calibrated manifold (X,ϕ) is the maximum dimension of a linear subspace in TX which contains no ϕ-planes.Subspaces with this property are called ϕ-free.*

After this definition, we state the main result which is due to Harvey and Lawson [HL_2]

Theorem 3.1.2. *[HL_2] Suppose (X, ϕ) is a strictly ϕ-convex manifold. Then X has the homotopy type of a CW-complex of dimension $\leq$ **hd(ϕ)**.*

Proof : Since (X, ϕ) is strictly ϕ-convex, we have a strictly ϕ-plurisubharmonic proper exhaustion function on X. Let it be $f : X \longrightarrow \mathbf{R}^+$. By perturbing we may assume that f has non-degenerate critical points (Being strictly ϕ-plurisubharmonic is an open condition). If we can show that each critical point has Morse index (the number of negative eigenvalues of the Hessian at the critical point) $\leq \mathbf{hd}(\phi)$, then the result follows from Morse Theory.(cf. [M]). Suppose that f has a non-degenerate critical point x where $Hess_x f$ has at least $\mathbf{hd}(\phi) + 1$ negative eigenvalues. In particular, this will give us a subspace $W \subset T_x X$ of dimension $\mathbf{hd}(\phi) + 1$ with $Hess_x f_{|W} < 0$. By the definition of $\mathbf{hd}(\phi)$ given above, W must contain a ϕ-plane $\xi \in G(\phi)$, and this will give us $tr_\xi Hess_x f < 0$. So, we get a contradiction since f is strictly ϕ-convex and $tr_\xi Hess_x f > 0$ for any $\xi \in G(\phi)$ $\square$

EXAMPLES

(1) Let (X, ω) be a Kähler manifold of real dimension 2n, then $\mathbf{hd}(\phi) = n$. As we know that $G(\omega)$ is the set of complex lines in $\mathbf{C}^n$. If V is a real subspace of dimension $n + 1$ in $\mathbf{C}^n$, and J is the almost complex structure, then codimension of $V \cap J(V) \leq$ (codimension(V)+ codimension($J(V)$) = $(n - 1 + n - 1) = 2n - 2$. So, $L = V \cap J(V)$ is a complex line contained in $\mathbf{C}^n$. If we take $\{e_1, Je_1, ...e_n, Je_n\}$ as a basis of $\mathbf{C}^n$, then the subspace generated by $\{e_1, e_2, ...e_n\}$ doesn't contain any complex line.

In Kähler geometry, strictly ω-convex manifolds are nothing but **Stein** manifolds. So, we actually prove the following important theorem which is proved first by Andreotti and Frankel ([AF]) and then by Harvey and Wells ([HW]) by using a different Morse function.

Theorem 3.1.3. *Let X be a Stein manifold of dimension n. Then X has the homotopy type of a CW-complex of dimension n. Hence,*

$$H_i(X, \mathbb{Z}) = 0 \text{ for } i > n$$

(2) If (X, ϕ) is a quaternion Kähler manifold or hyperKähler manifold of real dimension 4n with the quaternionic calibration $\phi = \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$, then $\mathbf{hd}(\phi)$=$3n$.

In this case, $G(\phi)$ is the set of quaternion lines in $\mathbf{H}^n$. If V is a real dimension of dimension $3n+1$, and I, J, K are the standard almost complex structures, then codimension of $V \cap I(V) \cap J(V) \cap K(V) \leq$ codimension(V) + codimension($I(V)$) + codimension($J(V)$) + codimension($K(V)$)= $4n-4$. Hence, $L = V \cap I(V) \cap J(V) \cap K(V)$ contains a quaternion line. If we take $\{e_1, Ie_1, Je_1, Ke_1, ...e_n, Ie_n, Je_n, Ke_n\}$ as a basis of $\mathbf{H}^n$, then the subspace generated by $\{e_1, Ie_1, Je_1, ..., e_n, Ie_n, Je_n\}$ doesn't contain any quaternion line.

In particular, as a result of this we prove the following theorem.

Theorem 3.1.4. *Let $\phi = \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$ be the quaternion calibration on $\mathbf{H}^n$. If $\Omega \subset \mathbf{H}^n$ is a strictly ϕ-convex manifold, then Ω has the homotopy type of a CW-complex of dimension less than or equal to 3n; in particular $H_q(\Omega, \mathbf{Z}) = 0$ for $q >$ 3n.*

We will try to give the examples of strictly ϕ-convex manifolds in $\mathbf{H}^n$ with different homotopy types after learning enough machinery and techniques.

(3) If X is 7-manifold with an associative calibration ϕ, or coassociative calibration $\psi = *\phi$, then $\mathbf{hd}(\phi)$=3 and $\mathbf{hd}(\psi)$=4.

In the first case, $G(\phi)$ is the set of associative 3-planes of Im$\mathbb{O} \cong \mathbf{R}^7$ where Im$\mathbb{O}$ is the imaginary octonions. Let V be a real plane of dimension 4 in Im$\mathbb{O}$ So, $V^\perp$ will have dimension 3. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for $V^\perp$. Then $W = span\{e_1, e_2, e_3, e_1 \times e_2 \times e_3\}$ is coassociative. (Here, $\times$:Im$\mathbb{O}\times$ Im$\mathbb{O} \longrightarrow$ Im$\mathbb{O}$ is the cross product given by $u \times v$ =Im$(\bar{v} \cdot u)$). So, $W^\perp$ is associative and $W^\perp \subset V$. Since ϕ is a 3-form, not every 3-plane is associative. Hence, $\mathbf{hd(}\phi\mathbf{)}$=3.

In the second case, $G(\psi)$ is the set of coassociative 4-planes of $\mathbf{R}^7$. Let V be a real plane of dimension 5 in $\mathbf{R}^7 \cong$Im$\mathbb{O}$. Then, $V^\perp$ is 2-dimensional. Let $\{e_1, e_2\}$ be an orthonormal basis of $V^\perp$. Let W =span$\{e_1, e_2, e_1 \times e_2\}$, then W is associative. So, $W^\perp$ is coassociative and $W^\perp \subset V$. Since ψ is a 4-form, not every 4-plane is coassociative. Hence, $\mathbf{hd(}\psi\mathbf{)}$=4.

In particular, this result proves us the following theorem :

Theorem 3.1.5. *Let ϕ be the associative calibration and $\psi = *\phi$ be the coassociative calibration on Im $\mathbb{O} \cong \mathbf{R}^7$.*

i) If $\Omega \subset \mathbf{R}^7$ is a strictly ϕ-convex manifold, then Ω has the homotopy type of a CW-complex of dimension less than or equal to 3; in particular $H_q(\Omega, \mathbf{Z}) = 0$ for $q > 3$.

ii) If $\Omega \subset \mathbf{R}^7$ is a strictly ψ-convex manifold, then Ω has the homotopy type of a CW-complex of dimension less than or equal to 4; in particular $H_q(\Omega, \mathbf{Z}) = 0$ for $q > 4$.

We will give examples of strictly ϕ-convex and strictly ψ-convex manifolds of $\mathbf{R}^7$ in the following section.

(4) If (X, Φ) is an 8-manifold with a Cayley calibration Φ, then $\mathbf{hd}(\Phi)$=4. In this case, $G(\Phi)$ is the set of Cayley 4-planes of $\mathbb{O} \cong \mathbf{R}^8$. Assume $V \subset \mathbb{O}$ is a 5-dimensional real subspace. Then $V^\perp$ is 3-dimensional. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for $V^\perp$. Then $W = span\{e_1, e_2, e_3, e_1 \times e_2 \times e_3\}$ is Cayley (Here, again $\times$ is cross product given by $\times : \mathbb{O} \times \mathbb{O} \longrightarrow \mathbb{O}$, $u \times v = \mathrm{Im}(\bar{v} \cdot u)$). Hence $W^\perp \subset V$ is Cayley. Again, Φ is a 4-form, so not every 4-plane is Cayley. As a result, we get the following result :

Theorem 3.1.6. *Let Φ be the Cayley calibration on $\mathbb{O} \cong \mathbf{R}^8$. If $\Omega \subset \mathbf{R}^8$ is a strictly Φ-convex manifold, then Ω has a homotopy type of a CW-complex of dimension less than or equal to 4; in particular $H_q(\Omega, \mathbf{Z}) = 0$ for $q > 4$.*

Even though we have these homotopy restrictions, we have lots of examples of strictly ϕ-convex manifolds with different topologies. We will give these examples in **Section 3.3**. First of all, we will explain the techniques that we will use to give these examples in the following section. All of these techniques are found by Harvey and Lawson ([HL_2]).

3.2 ϕ-Free Submanifolds

Let (X, ϕ) be a calibrated manifold with calibration ϕ and with the contact set $G(\phi)$, the set of ϕ-planes.

Definition 3.2.1. *A p-plane ξ is said to be **tangential** to a submanifold $M \subset (X, \phi)$ if $span\xi \subset T_xM$ for some $x \in M$*

Definition 3.2.2. *A closed submanifold $M \subset X$ is ϕ-free if there are no ϕ-planes $\xi \in G(\phi)$ which are tangential to M. If $\phi_{|_M} \equiv 0$, then M is called ϕ-isotropic.*

Remarks: It is obvious that ϕ-isotropic submanifolds are ϕ-free, since $\phi_{|_M} \equiv 0$ tells us that T_xM doesn't contain any ϕ-plane in it for any $x \in M$. Moreover, it is also trivial that any submanifold whose dimension is less than the degree of the calibration ϕ is ϕ-free. Also, if ϕ is a calibration of degree p on a manifold X of dimension n, then considering the dimension of $G(\phi)$ in $G_p(\mathbf{R}^n)$, we can say that generic p-planes are ϕ-free, so locally, generic submanifolds of dimension p is ϕ-free. However, in the next section, we will find examples of ϕ-free submanifolds whose dimension is bigger than the degree of ϕ.

On a Kähler manifold, the Kähler form w is a calibration where the w-*planes* are complex lines. Hence, w-free submanifolds will be those with no complex lines contained in the tangent spaces, which are exactly the totally real submanifolds. In Special Lagrangian geometry where $\phi \equiv Re(\Theta)$ where Θ is the holomorphic volume form on the given Calabi-Yau manifold, ϕ-free submanifolds will include the complex submanifolds of all dimensions. This is because of the Lagrangian condition which tells us that any ϕ-plane V is totally perpendicular to $J(V)$ where J is the almost complex structure.

Now, we will state two very important theorems that will be used to give examples of ϕ-free submanifolds and examples of strictly ϕ-convex manifolds. The latter one is the result of the first one and that will be the main tool for us to find examples of strictly ϕ-convex manifolds with different topologies. Both of them are proved by Harvey and Lawson ([HL$_2$]).

Theorem 3.2.3 (Harvey-Lawson [HL$_2$]). *Suppose M is a closed submanifold of (X,ϕ) and let $f_M(x) \equiv \frac{1}{2}\mathrm{dist}(x,M)^2$ denote half of the square of the distance to M. Then M is ϕ-free if and only if the function f_M is strictly ϕ-plurisubharmonic at each point in M (and hence in a neighborhood of M).*

Theorem 3.2.4 (Harvey-Lawson [HL$_2$]). *Suppose M is a ϕ-free submanifold of (X,ϕ). Then there exists a fundamental neighborhood system $\mathcal{F}(M)$ of M such that:*

(a) M is a deformation retract of each $U \in \mathcal{F}(M)$.

(b) Each neighborhood $U \in \mathcal{F}(M)$ is strictly ϕ-convex .

(c) $\mathcal{PSH}(V,\phi)$ is dense in $\mathcal{PSH}(U,\phi)$if $U \subset V$ and , $V, U \in \mathcal{F}(M)$

(d) Each compact set $K \subset M$ is $\mathcal{PSH}(U,\phi)$-convex for each $U \in \mathcal{F}(M)$.

Remark : Before giving our examples of strictly ϕ-convex manifolds in the following section, at this point, Theorem **3.2.4** tells us the existence of a vast amount of strictly ϕ-convex domains in any calibrated manifold (X,ϕ). If $M \subset X$ is a submanifold of dimension $<$ the degree of the calibration ϕ, then by Theorem **3.2.4** M has a fundamental system of neighborhoods each of which is strictly ϕ-convex and homotopic to M.

3.3 Examples of Strictly ϕ-Convex Manifolds

In this section, we will give examples of ϕ-free submanifolds whose dimension is bigger than or equal to the degree of the calibration ϕ. As a result of Theorem **3.2.4**, these examples will give us strictly ϕ-convex domains with different homotopy types.

We want to use the term "domain" rather than the manifold since they are actually open neighborhoods in calibrated manifolds. If we restrict the calibration, they will be calibrated manifolds with the same calibration, too.

In this thesis, our main focus will be on $\mathbf{H}^n$ with quaternion calibration $\phi \equiv \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$, $\mathbf{R}^7$ with associative calibration ϕ and coassociative calibration ψ, and $\mathbf{R}^8$ with Cayley calibration Φ. In fact, these examples will give us information for the strictly ϕ-convex domains in quaternion Kähler or hyperKähler manifolds with quaternion calibration, G_2 manifolds with associative or coassociative calibration and $Spin(7)$-manifolds with Cayley calibration.

Remark : In complex geometry with Kähler calibration ω, ϕ-free manifolds are the totally real manifolds. There has been a vast amount of research for the topology of these manifolds, and topological obstructions for the embeddings to be ϕ-free. (cf. [HL_4], [L], [**?**]).

First of all, we will start with some definitions and some very important theorems that we will use in our proofs.

Definition 3.3.1. *An Euclidean motion of* $\mathbf{R}^n$ *is an affine transformation* $\mu : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ *given by*

$$\mu(x) = L(x) + w$$

where L is an orthogonal transformation and $w \in \mathbf{R}^n$.

- If L=Identity, then μ is called a translation.
- If $w = 0$, and det(L)= +1, then μ is called a rotation.
- If $w = 0$ and det(L)= -1, then μ is called a reflection.

For any k-dimensional submanifold $M \subset \mathbf{R}^n$, we define $\Phi : M \longrightarrow G_k(\mathbf{R}^n)$ to be the Gauss map of M, and $\tilde{\Phi} : \mu(M) \longrightarrow G_k(\mathbf{R}^n)$ to be the Gauss map of $\mu(M)$ where μ is an Euclidean motion. Let us denote the action of μ on $G_k(\mathbf{R}^n)$ by $\tilde{\mu}$ which is actually the action of differential of μ on tangent spaces. Then, we have :

Theorem 3.3.2. *The following diagram commutes.*

$$\begin{tikzcd} M \arrow[r, "\Phi"] \arrow[d, "\mu"'] \arrow[dr, phantom, "\circlearrowright"] & G_k(\mathbb{R}^n) \arrow[d, "\tilde{\mu}"] \\ \mu(M) \arrow[r, "\tilde{\Phi}"'] & G_k(\mathbb{R}^n) \end{tikzcd}$$

Proof : It is obvious that any translation will preserve the Gauss map, and its action on $G_k(\mathbf{R}^n)$ will be Identity, so the diagram will commute. For any orthogonal transformation L, $d(L_{|_M})(x) = dL_{|_{T_xM}} = L_{|_{T_xM}}$ for any $x \in M$ since L is linear. Hence, for any orthogonal transformation, the diagram will commute, too. $\square$

The following lemma, which is proved by Harvey and Lawson ([HL$_3$]) will be very useful for us. It will be used to prove that in certain cases, we can produce ϕ-free immersions or embeddings of submanifolds by composing the given immersions or embeddings with an Euclidean motion.

Lemma 3.3.3. *Let U, M, $\overline{M}$ be differentiable manifolds and $S \subset \overline{M}$ a differentiable submanifold; and let $F : U \times M \longrightarrow \overline{M}$ be a differentiable map. Assume that everything is of class C^r where $r >$ max $\{0, dim\ (M)\text{-}codim\ (S)\}$. Then if F is transverse to S, the mapping $F_u : M \longrightarrow M$, given by $F_u(x) = F(u, x)$, is also transverse to S for almost all $u \in U$.*

Let G be the group of Euclidean motions of $\mathbf{R}^n$. Actually, $G \cong O(n) \times \mathbf{R}^n$. For a k-dimensional submanifold M of $\mathbf{R}^n$, we define the following map which is central for our examples.

$$\begin{array}{ccccccc} F & : & G & \times & M & \longrightarrow & G_k(\mathbf{R}^n) \\ & & (g & , & x) & \longrightarrow & T_{g(x)}g(M) \end{array}$$

The most important property of this function for us is that F is a submersion. It is easy to see this. For fixed $x \in M$, let's look at the map

$$F_{|_x} : O(n) \longrightarrow G_k(\mathbf{R}^n)$$

That is, we restrict the map F to just orthogonal transformations. Then

$$F_{|_x}(g) = T_{g(x)}g(M) = g \cdot (T_x M)$$

by Theorem **3.3.2**. Since $O(n)$ acts transitively on $G_k(\mathbf{R}^n)$, the orbit of $T_x M$ under $O(n)$ will be $G_k(\mathbf{R}^n)$. Hence, $F_{|_x}$ is nothing but the quotient map from $O(n)$ to $\frac{O(n)}{O(k)\times O(n-k)} \cong G_k(\mathbf{R}^n)$ which is a submersion. Then, F will be transverse to any submanifold S of $G_k(\mathbf{R}^n)$. Now, we will start our examples.

Examples

(1) $(\mathbf{H}^n, \phi)$, $\phi \equiv \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$

First of all, we will give our examples for certain dimensions, and then state our result for the general dimension n. As we know from Theorem 3.1.2 that any manifold of real dimension bigger than 3n can not be ϕ-free. So, we will look for examples whose dimension is less than or equal to 3n. In this case, ϕ-planes are quaternion lines. Hence, we will look for examples whose tangent space doesn't contain any quaternion line.

We start with $(\mathbf{H}^2, \phi)$. ϕ is a differential form of degree 4, so any manifold of dimension less than 4 is ϕ-free. Let's look at dimension 4 first. Define the set

$$S \equiv \{P \in G_4(\mathbf{H}^2) = G_4(\mathbf{R}^8) : P \text{ contains a quaternion line } \}$$

Since the real dimension of a quaternion line is 4 , then any plane of real dimension 4 which contains a quaternion line must be a quaternion line. Hence, S is the Grassmannian of quaternion lines in $\mathbf{H}^2$ which is the quaternionic projective space $\mathbf{H}P^1$. So, we get that dim(S)=dim($\mathbf{H}P^1$) $= 4$. If we look at the map :

$$\begin{array}{ccccccc} F & : & G & \times & M^4 & \longrightarrow & G_4(\mathbf{R}^8) \\ & & (g & , & x) & \longrightarrow & T_{g(x)}g(M) \end{array}$$

for any submanifold M of dimension 4 in $\mathbf{R}^8$, F will be transverse to $S = \mathbf{H}P^1 \subset G_4(\mathbf{R}^8)$.(The usual notation for this is $F \bar{\pitchfork} S$) By Lemma 3.3.3, F_g will also be transverse to S for almost all $g \in G = O(8) \times \mathbf{R}^8$. But, dim($S$)+dim($M$) $=$ $4 + 4 = 8 <$ dim$(G_4(\mathbf{R}^8)) = 4.4 = 16$. So, for F_g to be transverse, we need $F_g(M) \cap S = \emptyset$. This tells us that $F_g(M)$ will not contain any quaternion line in its tangent space at any point, i.e. it will be ϕ-free for almost all $g \in G = O(8) \times \mathbf{R}^8$. Therefore, we get the following result.

Proposition 3.3.4. *Let M be a 4-dimensional submanifold of $\mathbf{R}^8 \cong \mathbf{H}^2$. For almost all Euclidean motions g of $\mathbf{R}^8$, $g(M)$ will be ϕ-free. In particular, if M is compact, we can get a strictly ϕ-convex neighborhood Ω of M whose deformation retract is M, and $H_4(\Omega, \mathbb{Z}) \neq 0$.*

We will now look at dimension 5. And again , we define the set

$$S \equiv \{P \in G_5(\mathbf{H}^2) = G_5(\mathbf{R}^8) : P \text{ contains a quaternion line } \}$$

A 5-plane P contains a unique quaternion line. So, we can define a map from $\pi : S \longrightarrow \mathbf{H}P^1$ which maps each 5-plane P in S to the quaternion line that is in P. If we look at $\pi^{-1}(Q_0)$ for any quaternion line $Q_0 \in \mathbf{H}P^1$:

$$\pi^{-1}(Q_0) = \{P \in G_5(\mathbf{R}^8) : Q_0 \subset P\}$$

If P is a plane that contains Q_0, then $P = Q_0 \oplus \ell$ where ℓ is a line in $\mathbf{R}^8/Q_0 \cong \mathbf{R}^8/\mathbf{H} \cong \mathbf{R}^4$. So, $\pi^{-1}(Q_0) \cong \{\text{lines in } \mathbf{R}^4\} \cong \mathbf{R}P^3$, where $\mathbf{R}P^3$ is the 3-dimensional real projective space. As a result of this, we get the following fibration :

$$\begin{array}{ccc} \mathbb{R}P^3 & \longrightarrow & \mathbf{S} \\ & & \downarrow \\ & & \mathbf{H}P^1 \end{array}$$

This fibration gives us the dimension of S which is equal to the sum of dimension of base space which is the quaternionic projective space $\mathbf{H}P^1$ and dimension of the fiber which is $\mathbf{R}P^3$. Hence dim(S)= dim($\mathbf{H}P^1$) + dim($\mathbf{R}P^3$)=4

+ 3 =7. Again, if we define the map which is central for this section for any submanifold $M \subset \mathbf{R}^8$ of dimension 5 :

$$\begin{array}{ccccccc} F & : & G & \times & M^5 & \longrightarrow & G_5(\mathbf{R}^8) \\ & & (g & , & x) & \longrightarrow & T_{g(x)}g(M) \end{array}$$

Again, we have that F is transverse to S. Hence, by Lemma 3.3.3, F_g will also be transverse to S for almost all $g \in G = O(8) \times \mathbf{R}^8$. But, dim($S$)+dim($M$) $= 7 + 5 = 12 <$ dim$(G_5(\mathbf{R}^8) = 5.3 = 15$. So,again for F_g to be transverse, we need $F_g(M) \cap S = \emptyset$. This tells us that $F_g(M)$ will not contain any quaternion line in its tangent space at any point, i.e. it will be ϕ-free for almost all $g \in G = O(8) \times \mathbf{R}^8$. Hence, we get the following result.

Proposition 3.3.5. *Let M be a 5-dimensional submanifold of $\mathbf{R}^8 \cong \mathbf{H}^2$. For almost all Euclidean motions g of $\mathbf{R}^8$, $g(M)$ will be ϕ-free. In particular, if M is compact, we can get a strictly ϕ-convex neighborhood Ω of M whose deformation retract is M, and $H_5(\Omega, \mathbb{Z}) \neq 0$.*

Unfortunately, this method doesn't work for dimension 6. If we define our set S similarly :

$$S \equiv \{P \in G_6(\mathbf{H}^2) = G_6(\mathbf{R}^8) : P \text{ contains a quaternion line } \}$$

In this case, a 6-plane will again contain a unique quaternion line. So, again we can define a map $\pi : S \longrightarrow \mathbf{H}P^1$ which maps each 6-plane P in S to the quaternion line that is in P. If a plane P contains a quaternion line Q_0, then $P = Q_0 \oplus \lambda$ where λ is 2-plane in $\mathbf{R}^8/Q_0 \cong \mathbf{R}^4/\mathbf{H} \cong \mathbf{R}^4$. Hence, $\pi^{-1}(Q_0) \cong G_2(\mathbf{R}^4)$, and we have the following fibration :

$$\begin{array}{ccc} G_2(\mathbf{R}^4) & \longrightarrow & \mathbf{S} \\ & & \downarrow \\ & & \mathbf{H}P^1 \end{array}$$

As a result of this fibration, we get the dimension of S= dim($G_2(\mathbf{R}^4)$+ dim($\mathbf{H}P^1$)= 2.2+4=8. If we define the map F for any submanifold $M \subset \mathbf{R}^8$ of dimension 6 similarly :

$$\begin{array}{ccccccc} F & : & G & \times & M^6 & \longrightarrow & G_6(\mathbf{R}^8) \\ & & (g & , & x) & \longrightarrow & T_{g(x)}g(M) \end{array}$$

Again, by Lemma 3.3.3, for any $g \in G$, $F_g : M \longrightarrow G_6(\mathbf{R}^8)$ will be transverse to S. But, in this case, we have dim(M) + dim(S) = 6 + 8 =14> dim ($G_6(\mathbf{R}^8)$)=6.2 =12. So, for almost all Euclidean motions, the image of M will have a non-empty intersection with S. That is, the image of M will not be ϕ-free.

Actually, for $\mathbf{H}^2$ with quaternion calibration, 6 is the maximum possible dimension for a manifold to be ϕ-free, also it is the maximum possible dimension for non-zero homology group of a strictly ϕ-convex domain $\Omega \subset \mathbf{H}^2$.

We continued to try the same method for $(\mathbf{H}^3, \phi)$. In this case, our calculations showed the following result.

Proposition 3.3.6. *If M is a k-dimensional submanifold of $\mathbf{H}^3$ where k<7, then for almost all Euclidean motions g, $g(M)$ will be ϕ-free.*

For this case, 9 is the maximum possible dimension for a manifold $M \subset \mathbf{H}^3$ to be ϕ-free.

Now, we will prove the general condition on the dimension of the submanifold $M \subset \mathbf{H}^n \cong \mathbf{R}^{4n}$ for which M can be made ϕ-free by using Euclidean motions.

Theorem 3.3.7. *Suppose M is a k-dimensional submanifold of $(\mathbf{H}^n, \phi)$ where $\phi \equiv \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$. If $5k < 12n+4$, then for almost all Euclidean motions g of $\mathbf{H}^n \cong \mathbf{R}^{4n}$, the submanifold $\widetilde{M} = g(M)$ is ϕ-free.*

Proof : Let us define the sets S and S_0 as the following:

$$S \equiv \{P \in G_k(\mathbf{R}^{4n}) : P \text{ contains a quaternion line }\}$$
$$S_0 \equiv \{P \in S : P \text{ contains a quaternion plane }\}$$

If $P \in S - S_0$, then P will contain a unique quaternion line. If it contained quaternion lines more than one, then it must contain a quaternion plane. Then, we can define the map $\pi : S - S_0 \longrightarrow \mathbf{H}P^{n-1}$ which maps each k-plane $P \in S - S_0$ to the quaternion line that is in P. If we look at the fibers, i.e. $\pi^{-1}(Q_0)$ for any quaternion line in $\mathbf{H}P^{n-1}$, we see that any k-plane $P \in \pi^{-1}(Q_0)$ will be of the form $P = Q_0 \oplus \Lambda$ where Λ is a (k-4) plane in $\mathbf{H}^n/\mathbf{H} \cong \mathbf{R}^{4n}/\mathbf{R}^4$ but Λ doesn't contain a quaternion line. Hence,

$$\pi^{-1}(Q_0) \cong G_{k-4}(\mathbf{R}^{4n-4}) - S'$$

where $S' \equiv \{P \in G_{k-4}(\mathbf{R}^{4n-4}) : P \text{ contains a quaternion line }\}$. So, $\pi^{-1}(Q_0)$ is homeomorphic to an open set in $G_{k-4}(\mathbf{R}^{4n-4})$. As a result of this, $\dim(S - S_0) = \dim(G_{k-4}(\mathbf{R}^{4n-4})) + \dim(\mathbf{H}P^{n-1}) = (k-4)\cdot(4n-4-k+4) + 4(n-1)$ $= 4nk - k^2 - 12n + 4k - 4$. Also, away from S_0, the dimension of S will be the same as the dimension of $S - S_0$. We may think S_0 as a singular set of S whose dimension is less than the dimension of S.

Again, if we define our map F for any submanifold M of dimension k :

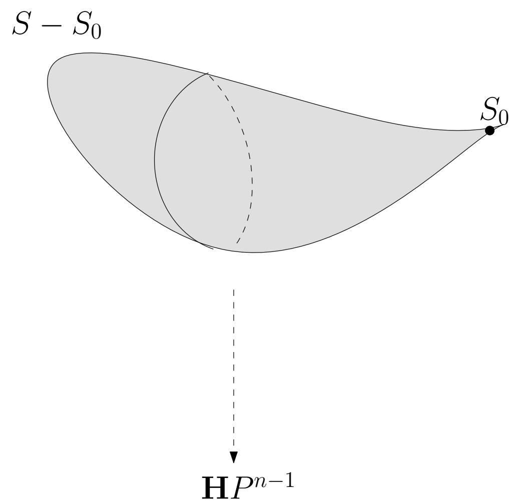

Figure 3.1: Fibration of $S - S_0$ over $\mathbf{H}P^{n-1}$

$$\begin{array}{ccccccc} F & : & G & \times & M^k & \longrightarrow & G_k(\mathbf{R}^{4n}) \\ & & (g & , & x) & \longrightarrow & T_{g(x)}g(M) \end{array}$$

Hence, by Lemma 3.3.3, for almost all Euclidean motions g, F_g will be transverse to S. If codimension of S in $G_k(\mathbf{R}^{4n})$ is bigger than k, then for almost all Euclidean motions g, since F_g is transverse to S, we will have $F_g(M) \cap S = \emptyset$. So, $F_g(M)$ will be ϕ-free. Now, if we solve the inequality $codim(S) > k$, we will have :

$$\begin{aligned} codim(S) &> k \\ (4n-k)\cdot k - 4nk + k^2 + 12n - 4k + 4 &> k \\ 4nk - k^2 - 4nk + k^2 + 12n - 4k + 4 &> k \\ 12n + 4 &> 5k \end{aligned}$$

Theorem 3.3.8. *Let $f : M \hookrightarrow \mathbf{R}^{3n}$ be an embedding of a k-dimensional manifold M where $k \leq 3n$ into $\mathbf{R}^{3n}$. We can find an embedding $\mu : M \hookrightarrow \mathbf{H}^n$*

such that $\mu(M)$ *is* ϕ*-free where* $\phi \equiv \frac{1}{6}\{w_I^2 + w_J^2 + w_K^2\}$.

Proof : Since $\mathbf{hd}(\phi)$=3n, we can find a 3n-plane $P \cong \mathbf{R}^{3n}$ which is ϕ-free. (An example can be the one given in **Section 3.1**). Hence, if M is a manifold of dimension k which embeds into $\mathbf{R}^{3n}$, we can embed it into P. Then, T_xM for any $x \in M$ will be a k-plane in P (By carrying it to the origin). Hence, it will not contain any quaternion line in it, and M will be ϕ-free. □

By using this result, we can get examples of strictly ϕ-convex domains Ω such that $H_k(\Omega, \mathbf{Z}) \neq 0$ for $k \leq 3n-1$. For example , if we take $M = S^k$, for $k = 1$ to $3n-1$, then we can find a strictly ϕ-convex domain Ω such that $M = S^k$ is a deformation retract of Ω, so $H_k(\Omega, \mathbf{Z}) \neq 0$. All of our techniques up to now give us strictly ϕ-convex domains Ω, where $H_{3n-1}(\Omega, \mathbf{Z}) \neq 0$ is the maximum non-zero homology group. Existence of a strictly ϕ-convex domain Ω with $H_{3n}(\Omega, \mathbf{Z}) \neq 0$ is still a conjecture.

(2)$(\mathbf{R}^7, \phi)$ $\phi \equiv$ associative calibration

We know that in this case $\mathbf{hd}(\phi)$=3. Hence, the dimension of ϕ-free submanifold $M \subset \mathbf{R}^7$ can be at most 3. Since ϕ is a 3-form, we already know that any submanifold $M \subset \mathbf{R}^7$ of dimension 1 or 2 is automatically ϕ-free. Also, locally, generic submanifolds of dimension 3 will be ϕ-free. We will actually prove more.

First of all, let's look at the codimension of $G(\phi)$ in $G_3(\mathbf{R}^7)$. As we know from [HL$_1$], $G(\phi) \cong G_2/SO(4)$. Hence, dimension of $G(\phi)$ is dim(G_2) - dim($SO(4)$) which is 14-6 = 8. Then, we get the codimension of $G(\phi)$ in $G_3(\mathbf{R}^7)$ equal to 12-8=4. As usual, if we define our map which is central for this section for a submanifold $M \subset \mathbf{R}^7$ of dimension 3,

$$
\begin{array}{ccccccc}
F & : & G & \times & M^3 & \longrightarrow & G_3(\mathbf{R}^7) \\
 & & (g & , & x) & \longrightarrow & T_{g(x)}g(M)
\end{array}
$$

Again, we see that F_g is transverse to $G(\phi) \subset G_3(\mathbf{R}^7)$ for almost all Euclidean motions $g \in G$. Since, codimension of $G(\phi)$ is bigger than 3, for F_g to be transverse to $G(\phi)$, we need to have $F_g(M) \cap G(\phi) = \emptyset$. Hence, the image of any submanifold of dimension 3 in $\mathbf{R}^7$ under an Euclidean motion will be ϕ-free.

Moreover, by the Whitney Embedding Theorem, any manifold M of dimension 3 can be embedded in $\mathbf{R}^{2\cdot 3+1} = \mathbf{R}^7$. Hence, we get the following result.

Theorem 3.3.9. *Every 3-manifold can be embedded in $(\mathbf{R}^7, \phi)$ as a ϕ-free submanifold, where ϕ is the associative calibration..*

As a result of this, we can find strictly ϕ-convex domains in $\mathbf{R}^7$ with any homotopy type allowed by Morse Theory.

(3) $(\mathbf{R}^7, \psi)$, $\psi \equiv$ the coassociative calibration $\psi = *\phi$

In this case, every k-dimensional submanifold $M \subset \mathbf{R}^7$ with $k \leq 3$ will be automatically ψ-free since ψ is a 4-form. Locally, generic 4-dimensional submanifold of $\mathbf{R}^7$ will be ψ-free. Moreover, since we have $\mathbf{hd}(\psi) = 4$, the maximum dimension of a ψ-free submanifold of $\mathbf{R}^7$ can be 4.

Unfortunately, our method of using Euclidean motions for any 4-dimensional submanifold $M \subset \mathbf{R}^7$ to get ψ-free manifolds doesn't work. Again, by [HL$_1$] we know that $G(\psi) \cong G_2/SO(4)$. Hence its dimension is equal to 8 again. Then its codimension in $G_4(\mathbf{R}^7)$ whose dimension is equal to $4 \cdot 3 = 12$ will be 4. This is also equal to the dimension of the submanifold M. Hence, for any

Euclidean motion $g \in G$, $F_g(M) \cap G(\psi)$ may not empty.

If a 4-dimensional oriented compact manifold is embedded in $\mathbf{R}^7$, then generically, it will have finitely many points with a coassociative tangent plane. Our research will continue with the problem that whether by counting these points with appropriately defined indices, we can produce a topological invariant. Finding such an invariant may also tell us to where we should look to find a *ψ-free submanifold*. Similar things were done for the Kähler case by Webster in [Webster] who found a topological invariant using the points with complex tangents of a compact surface M embedded in a complex surface. Higher dimensional analogues are done by Lai [L] and Harvey and Lawson [HL_4].

(4)$(\mathbf{R}^8, \Phi)$ $\Phi \equiv$ the Cayley calibration

In this case,every k-dimensional submanifold with $k \leq 3$ will be automatically Φ-free since Φ is a 4-form.

Unfortunately, this case is similar to the case (3). The maximum dimension of a Φ-free submanifold M of $\mathbf{R}^7$ can be 4 since $\mathbf{hd}(\Phi)$=4. By [HL_1], we know that $G(\Phi) \cong Spin(7)/K$ where $K = SU(2) \times SU(2) \times SU(2)/\mathbb{Z}_2$. As a result of this, dim($G(\Phi)$)= dim($Spin(7)$) - dim($SU(2) \times SU(2) \times SU(2)/\mathbb{Z}_2$) $= \frac{7 \cdot 6}{2} - (3+3+3) = 12$. Then its codimension in $G_4(\mathbf{R}^8)$ will be 16-12=4. For any 4-dimensional submanifold M of $\mathbf{R}^7$, similar to the case **(3)**, its image under F_g for any Euclidean motion g may have a non-empty intersection with $G(\Phi) \subset G_4(\mathbf{R}^8)$.

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