Bi-invariant Norms on the Group of Symplectomorphisms

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Abstract of the Dissertation

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The group of diffeomorphisms of a symplectic manifold (M, ω) preserving the symplectic form is called the symplectomorphism group of (M, ω) and denoted by $\mathrm{Symp}(M, \omega)$. It has a very important subgroup $\mathrm{Ham}(M, \omega)$ called the Hamiltonian diffeomorphism group. The group $\mathrm{Ham}(M, \omega)$ admits a natural bi-invariant norm, known as the Hofer norm. In this thesis we study several aspects of bi-invariant norms on $\mathrm{Symp}(M, \omega)$, including the bounded isometry conjecture of Lalonde and Polterovich. In particular, we prove the conjecture for the Kodaira-Thurston manifold and for the 4-torus with all linear symplectic forms. Relatedly, we observe that there is an obstruction to extending the Hofer norm bi-invariantly to the identity component $\mathrm{Symp}_0(M,\omega)$ of $\mathrm{Symp}(M,\omega)$. This obstruction is shown to be non-trivial in some cases. We also prove that no Finsler norm on $\mathrm{Ham}(\mathbb{T}^{2n},\omega)$ satisfying a strong form of the invariance condition can extend to a bi-invariant norm on $\mathrm{Symp}_0(\mathbb{T}^{2n},\omega)$. Other bi-invariant norms on $\mathrm{Symp}_0(M,\omega)$ are studied as well and the induced topologies are discussed.

To my beloved wife, Yujen Shu.

Contents

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Chapter 1

Introduction and main results

1.1 The bounded isometry conjecture

Let (M, ω) be a closed symplectic manifold. There is a natural bi-invariant norm, called the Hofer norm ρ, defined on the Hamiltonian diffeomorphism group $\mathrm{Ham}(M, \omega)$. That is, $\rho(f)$ is the Hofer distance between the identity map id and f for all $f \in \mathrm{Ham}(M, \omega)$, see Section 2.3 for details. Lalonde and Polterovich [14] have studied the full symplectomorphism group $\mathrm{Symp}(M, \omega)$ within the framework of Hofer's geometry. We first recall the notion of bounded and unbounded symplectomorphisms. Namely, for each $\phi \in \mathrm{Symp}(M, \omega)$, define

$$r(\phi) := \sup \{ \rho([\phi, f]) \mid f \in \mathrm{Ham}(M, \omega) \},$$

where $[\phi, f] := \phi f \phi^{-1} f^{-1}$ is the commutator of ϕ and f.

Definition 1.1.1. *An element $\phi \in \mathrm{Symp}(M, \omega)$ is bounded if $r(\phi) < \infty$, and is unbounded if $r(\phi) = \infty$.*

Denote by $BI_0(M,\omega)$ the set of all bounded elements in the identity component $\mathrm{Symp}_0(M,\omega)$ of $\mathrm{Symp}(M,\omega)$. Since ρ is bi-invariant, it follows from the inequality $\rho([\phi, f]) \leqslant 2\rho(\phi)$ that $\mathrm{Ham}(M,\omega)$ is a subgroup of $BI_0(M,\omega)$. The converse is the following conjecture in [14].

Conjecture 1.1.2 (Bounded isometry conjecture). *For all symplectic manifolds (M,ω), $BI_0(M,\omega) = \mathrm{Ham}(M,\omega)$.*

This conjecture was proved in [14] for closed surfaces with area form and for arbitrary products of closed surfaces of genus greater than 0 with product symplectic form; Lalonde and Pestieau [15] confirmed it for product symplectic manifolds $M = N \times W$ with N being any product of closed surfaces and W being any closed symplectic manifold of first real Betti number equal to zero. We give a positive answer here to this conjecture for the Kodaira-Thurston manifold with the standard symplectic form. The detailed descriptions and proofs are contained in Chapter 2.

Theorem 1.1.3. *The bounded isometry conjecture holds for the Kodaira-Thurston manifold M with the standard symplectic form ω.*

1.2 Extending the Hofer norm

While studying the bounded isometry conjecture, we realized that this conjecture is closely related to another interesting question of extending the Hofer norm from $\mathrm{Ham}(M,\omega)$ to $\mathrm{Symp}_0(M,\omega)$. Since the Hofer norm is a bi-invariant norm defined on $\mathrm{Ham}(M,\omega)$, it is natural to consider its extensions to $\mathrm{Symp}_0(M,\omega)$. Banyaga and Donato [2] constructed such an extension in two different ways, but neither of the resulting norms is bi-invariant on $\mathrm{Symp}_0(M,\omega)$. By bi-invariant, we mean that the corresponding distance function d satisfies

$$d(\theta\phi,\theta\psi) = d(\phi\theta,\psi\theta) = d(\phi,\psi)$$

for all $\phi,\psi,\theta \in \mathrm{Symp}_0(M,\omega)$, see Section 3.1.

It turns out that the existence of unbounded symplectomorphisms serves as an obstruction to bi-invariant extensions of the Hofer norm to $\mathrm{Symp}_0(M,\omega)$. The following theorem follows almost immediately from the definition of unbounded symplectomorphisms.

Theorem 1.2.1. *Let (M,ω) be a closed symplectic manifold. Assume that there exists some $\phi \in \mathrm{Symp}_0(M,\omega)$ which is unbounded in the sense of Definition 1.1.1. Then the Hofer norm ρ on $\mathrm{Ham}(M,\omega)$ does not extend to a bi-invariant norm on $\mathrm{Symp}_0(M,\omega)$.*

Proof. Assume ρ extends to a bi-invariant norm on $\mathrm{Symp}_0(M,\omega)$, which we still denote by ρ. Then given $\phi \in \mathrm{Symp}_0(M,\omega)$, using the properties of a bi-invariant norm listed in Section 3.1, we have

$$\rho([\phi,f]) = \rho(\phi f\phi^{-1}f^{-1}) \leqslant \rho(\phi) + \rho(f\phi^{-1}f^{-1}) = \rho(\phi) + \rho(\phi^{-1}) = 2\rho(\phi)$$

for all $f \in \mathrm{Ham}(M,\omega)$. Taking the supremum over all $f \in \mathrm{Ham}(M,\omega)$ gives $r(\phi) \leqslant 2\rho(\phi) < \infty$. It follows then from Definition 1.1.1 that all elements $\phi \in \mathrm{Symp}_0(M,\omega)$ are bounded, which contradicts our assumption. $\square$

In particular, Theorem 1.2.1 applies to all symplectic manifolds where the bounded isometry conjecture holds. However, it is in general difficult to prove the bounded isometry conjecture, since one has to show that all nonHamiltonian symplectomorphisms are unbounded. On the other hand, it is often easier to find one single unbounded element. This is sufficient to show the Hofer norm does not extend. For instance, combining Theorem 1.2.1 with Theorem 2.5.2 below, one has the following

Corollary 1.2.2. *Let $L \subset M$ be a closed Lagrangian submanifold admitting a Riemannian metric with non-positive sectional curvature, and whose inclusion in M induces an injection on fundamental groups. If there exists some $\phi \in$* $\mathrm{Symp}_0(M, \omega)$ *such that $\phi(L) \cap L = \emptyset$, then the Hofer norm ρ on* $\mathrm{Ham}(M, \omega)$ *does not extend to a bi-invariant norm on* $\mathrm{Symp}_0(M, \omega)$.

Proof. In view of Theorem 2.5.2, such ϕ must be unbounded. The corollary follows from Theorem 1.2.1. □

Thus, we propose the following conjecture which seems more accessible than the bounded isometry conjecture.

Conjecture 1.2.3. *For any symplectic manifold (M, ω) such that* $\mathrm{Symp}_0(M, \omega)$ *is not identical to* $\mathrm{Ham}(M, \omega)$*, the Hofer norm on* $\mathrm{Ham}(M, \omega)$ *does not extend to a bi-invariant norm on* $\mathrm{Symp}_0(M, \omega)$.

Remark 1.2.4. Besides all manifolds mentioned above, Conjecture 1.2.3 also holds for $M = \Sigma \times W$ with Σ being any closed surface of genus > 0 and W being any closed symplectic manifold. One can simply argue, using the stable energy-capacity inequality as in Lalonde and Pestieau [15], that $\phi \times id \in \mathrm{Symp}_0(M)$ is unbounded with ϕ being any nonHamiltonian symplectomorphism in $\mathrm{Symp}_0(\Sigma)$ and id being the identity map on W. Note that we are able to drop the assumption on W, i.e. W has first real Betti number zero, from Lalonde and Pestieau's result (cf. Theorem 1.3 [15]) simply because the bounded isometry conjecture is stronger than Conjecture 1.2.3.

1.3 Extending χ-invariant norms

In this section, we state one of our main results concerning the so-called χ-invariant (which is necessarily bi-invariant) Finsler norms on $\mathrm{Ham}(\mathbb{T}^{2n}, \omega)$, see Section 3.2. The detailed formulation of this result and its generalizations, together with their proofs, are contained in Chapter 3.

Theorem 1.3.1. *Let* $(\mathbb{T}^{2n}, \omega)$ *be the torus with the standard symplectic form* ω*, and* ρ *be a* χ*-invariant Finsler norm on* $\mathrm{Ham}(\mathbb{T}^{2n}, \omega)$*, i.e. a Finsler norm induced by a* χ*-invariant norm* $||\cdot||$*. Then* ρ *does not extend to a bi-invariant norm on* $\mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$*.*

Question 1.3.2. We shall see in Chapter 3 that we are using the fact in our proof that the diameter of $\mathrm{Ham}(\mathbb{T}^{2n}, \omega)$ with respect to any χ-invariant Finsler norm ρ is infinite. Actually if the diameter with respect to ρ is finite, one can always extend ρ bi-invariantly to $\mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$ by giving a sufficiently large constant value for all nonHamiltonian symplectomorphisms (cf. Remark 4.3.3). The question is, will the infiniteness of the diameter of $\mathrm{Ham}(\mathbb{T}^{2n}, \omega)$ be sufficient to prove that ρ does not extend to a bi-invariant norm on $\mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$?

Chapter 2

The bounded isometry conjecture

In this chapter, we study the bounded isometry conjecture proposed by Lalonde and Polterovich in [14], i.e. Conjecture 1.1.2. We begin with some preparations and related results. Then in Section 2.5 we prove Theorem 1.1.3 which asserts that the bounded isometry conjecture holds for the Kodaira-Thurston manifold M with the standard symplectic form ω. We also prove in Section 2.6 the bounded isometry conjecture holds for the 4-torus with all linear symplectic forms. In Section 2.7 we study the same conjecture for the Kodaira-Thurston manifold M with all linear symplectic forms. While we are not able to prove the conjecture, some partial results are provided and the difficulties are discussed.

2.1 The flux subgroup

The flux homomorphism is best defined on the universal cover $\widetilde{\mathrm{Symp}_0}(M, \omega)$

of $\mathrm{Symp}_0(M,\omega)$,

$$\mathrm{flux} : \widetilde{\mathrm{Symp}}_0(M,\omega) \to H^1(M,\mathbb{R}).$$

Let $\{\phi_t\} \in \widetilde{\mathrm{Symp}}_0(M,\omega)$, i.e. ϕ_t is a smooth isotopy in $\mathrm{Symp}_0(M,\omega)$. There exists a unique family of vector fields X_t which generates the flow ϕ_t, i.e.

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t.$$

Define

$$\mathrm{flux}(\{\phi_t\}) := \int_0^1 \iota(X_t)\omega\, dt.$$

In particular, if $\{\phi_t\}$ is the flow of the time-independent symplecitc vector field X on the time interval $0 \leqslant t \leqslant 1$, then

$$\mathrm{flux}(\{\phi_t\}) = \iota(X)\omega. \tag{2.1}$$

This fact will often be used in later calculations.

The flux subgroup $\Gamma := \Gamma_\omega$ is the image

$$\mathrm{flux}(\pi_1(\mathrm{Symp}_0(M,\omega)) \subset H^1(M,\mathbb{R})$$

of the fundamental group of $\mathrm{Symp}_0(M,\omega)$ under the flux homomorphism. Thus there is an induced map from $\mathrm{Symp}_0(M,\omega)$, still denoted by flux,

$$\mathrm{flux} : \mathrm{Symp}_0(M,\omega) \to H^1(M,\mathbb{R})/\Gamma.$$

It is well known that this map is surjective, and its kernel is equal to $\mathrm{Ham}(M,\omega)$.

In other words, we have the following exact sequence of groups

$$0 \longrightarrow \mathrm{Ham}(M,\omega) \longrightarrow \mathrm{Symp}_0(M,\omega) \xrightarrow{\text{flux}} H^1(M,\mathbb{R})/\Gamma \longrightarrow 0.$$

We refer to [17] Chapter 10 for more details.

Since whether or not the flux is equal to 0 distinguishes a Hamiltonian diffeomorphism from a nonHamiltonian symplectomorphism, one main step in our applications is to understand the flux subgroup Γ.

For this, we denote as in [9] by $C(M)$ the space of continuous maps from M to M with the compact open topology. Given $p \in M$, we define the evaluation map $ev_c : C(M) \to M$ by $ev_c(f) = f(p)$. Denote by ev_s the restriction of ev_c to $\mathrm{Symp}_0(M,\omega)$. We will use the same notation for the induced maps on the fundamental groups. By $\widetilde{ev}_s$ we denote the homomorphism from $\pi_1(\mathrm{Symp}_0(M,\omega))$ to $H_1(M,\mathbb{Z})$, which is the composition of ev_s with the Hurewitz map from $\pi_1(M)$ to $H_1(M,\mathbb{Z})$.

The following commutative diagram due to Lalonde, McDuff and Polterovich [13] plays a crucial role in the calculation of the flux subgroup Γ.

Lemma 2.1.1 (LMP). *Let (M,ω) be a closed symplectic manifold of dimension $2n$. Then the following diagram commutes.*

$$\begin{array}{ccccc}
\pi_1(\mathrm{Symp}_0(M,\omega)) & \xrightarrow{\widetilde{ev}_s} & H_1(M,\mathbb{Z}) & \xrightarrow{\mathrm{PD}} & H^{2n-1}(M,\mathbb{Z}) \\
{\scriptstyle id}\downarrow & & & & \downarrow{\scriptstyle \cdot(n-1)!\mathrm{vol}(M)} \\
\pi_1(\mathrm{Symp}_0(M,\omega)) & \xrightarrow{\text{flux}} & H^1(M,\mathbb{R}) & \xrightarrow{\wedge[\omega]^{n-1}} & H^{2n-1}(M,\mathbb{R}).
\end{array}$$

□

2.2 The Kodaira-Thurston manifold

Let G be the group $(\mathbb{Z}^4, \cdot)$ where

$$(m_1, n_1, k_1, \ell_1) \cdot (m_2, n_2, k_2, \ell_2) = (m_1 + m_2, n_1 + n_2, k_1 + k_2 + m_1\ell_2, \ell_1 + \ell_2).$$

G acts on $\mathbb{R}^4$ via

$$G \longrightarrow \mathrm{Diff}(\mathbb{R}^4) : (m, n, k, \ell) \mapsto \rho_{mnk\ell}$$

where

$$\rho_{mnk\ell}(s, t, x, y) = (s + m, t + n, x + k + my, y + \ell).$$

Note that $\rho_{mnk\ell}$ preserves the symplectic form $\omega = ds \wedge dt + dx \wedge dy$ on $\mathbb{R}^4$. Hence the quotient $(M := \mathbb{R}^4/G, \omega)$ is a closed symplectic manifold, known as the Kodaira-Thurston manifold, see [26]. It was the first known example of a closed symplectic manifold which admits no kähler structure, since its first betti number $b_1 = 3$, see [17] Example 3.8.

The manifold $M = \mathbb{R}^4/G$ can also be described as a torus bundle over a torus, that is $M = \mathbb{R}^2 \times_{\mathbb{Z}^2} \mathbb{T}^2$. Here $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ in the usual way, and it acts on $\mathbb{T}^2$ via

$$(m, n) \longrightarrow A_{mn} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Therefore $M = \mathbb{R} \times S^1 \times \mathbb{T}^2/\sim$, where

$$(s, t, x, y) \sim (s + 1, t, x + y, y).$$

Our first task is to understand the flux subgroup Γ of the Kodaira-Thurston manifold described above. In particular, we have

Theorem 2.2.1. *The flux subgroup $\Gamma \subset H^1(M, \mathbb{R})$ of the Kodaira-Thurston manifold with the standard symplectic form $\omega = ds \wedge dt + dx \wedge dy$ has rank 2 over $\mathbb{Z}$. Namely, $\Gamma = \mathbb{Z}\langle ds, dy \rangle$.*

To prove Theorem 2.2.1, we need the following result on the cohomology groups of the Kodaira-Thurston manifold.

Lemma 2.2.2. *The cohomology groups of the Kodaira-Thurston manifold M described above are as follows: $H^1(M, \mathbb{R})$ is of rank 3, generated by ds, dt and dy; $H^2(M, \mathbb{R})$ is of rank 4, generated by $\gamma \wedge ds$, $\gamma \wedge dy$, $ds \wedge dt$ and $dy \wedge dt$; and $H^3(M, \mathbb{R})$ is of rank 3, generated by $\gamma \wedge dy \wedge dt$, $\gamma \wedge dy \wedge ds$ and $\gamma \wedge ds \wedge dt$, where $\gamma = dx - sdy$.*

Proof. It follows from an easy calculation. □

Proof of Theorem 2.2.1. We use the commutative diagram in Lemma 2.1.1. For manifolds of dimension 4, the diagram reads as

$$\begin{array}{ccccc}
\pi_1(\mathrm{Symp}_0(M,\omega)) & \xrightarrow{\widetilde{ev}_s} & H_1(M,\mathbb{Z}) & \xrightarrow{\mathrm{PD}} & H^3(M,\mathbb{Z}) \\
\Big\downarrow{\scriptstyle id} & & & & \Big\downarrow{\scriptstyle \cdot\mathrm{vol}(M)} \\
\pi_1(\mathrm{Symp}_0(M,\omega)) & \xrightarrow{\mathrm{flux}} & H^1(M,\mathbb{R}) & \xrightarrow{\wedge[\omega]} & H^3(M,\mathbb{R}).
\end{array}$$

Denote by $C_0(M)$ the identity component of $C(M)$. It was proved in Gottlieb [5] (Theorem III.2) that for all aspherical manifolds M,

$$ev_c : \pi_1(C_0(M)) \cong Z(\pi_1(M))$$

is a group isomorphism, where $Z(\pi_1(M))$ stands for the center of $\pi_1(M)$. For the Kodaira-Thurston manifold $M = \mathbb{R}^4/G$, we have $\pi_1(M) = G$. It is easy to check that $Z(\pi_1(M)) = \mathbb{Z}\langle\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\rangle$, and the commutator group $[\pi_1(M), \pi_1(M)] = \mathbb{Z}\langle\frac{\partial}{\partial x}\rangle$, see Example 3.8 in [17]. Thus the image of $\widetilde{ev}_s$ in $H_1(M, \mathbb{Z})$ is contained in

$$Z(\pi_1(M))/[\pi_1(M), \pi_1(M)] = \mathbb{Z}\langle\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\rangle/\mathbb{Z}\langle\frac{\partial}{\partial x}\rangle = \mathbb{Z}\langle\frac{\partial}{\partial t}\rangle.$$

Note that $PD(\frac{\partial}{\partial t}) = -dx \wedge dy \wedge ds = -\gamma \wedge dy \wedge ds$, where $\gamma = dx - sdy$. Now look at the map $\wedge\omega : H^1(M, \mathbb{R}) \to H^3(M, \mathbb{R})$,

$$ds \mapsto ds \wedge \omega = \gamma \wedge dy \wedge ds \neq 0,$$

$$dt \mapsto dt \wedge \omega = \gamma \wedge dy \wedge dt \neq 0,$$

$$dy \mapsto dy \wedge \omega = dy \wedge ds \wedge dt = 0.$$

Here we have used the fact that the 3-form $dy \wedge ds \wedge dt = d(\gamma \wedge dt)$ is exact, so it vanishes on the cohomology level. Since $\mathrm{vol}(M) = 1$, we conclude from the above commutative diagram that the flux subgroup $\Gamma \subset H^1(M, \mathbb{R})$ is contained in $\mathbb{Z}\langle ds, dy\rangle$. An explicit construction shows that Γ is actually equal to $\mathbb{Z}\langle ds, dy\rangle$. Namely, we take two elements $\{\phi_\theta\}$ and $\{\psi_\theta\}$ in $\pi_1(\mathrm{Symp}_0(M, \omega))$ such that

$$\phi_\theta(s, t, x, y) = (s, t - \theta, x, y), 0 \leqslant \theta \leqslant 1,$$

$$\psi_\theta(s, t, x, y) = (s, t, x + \theta, y), 0 \leqslant \theta \leqslant 1.$$

Using (2.1) in Section 2.1, one can show that $\mathrm{flux}(\{\phi_\theta\}) = ds$ and $\mathrm{flux}(\{\psi_\theta\}) =$

dy. This completes the proof of Theorem 2.2.1. □

2.3 The Hofer norm

Let (M,ω) be a closed symplectic manifold of dimension $2n$. Denote by $\mathcal{A}$ the space of all normalized smooth functions on M with respect to the volume form ω^n, i.e.

$$\mathcal{A} := \{F \in C^\infty(M) \mid \int_M F\,\omega^n = 0\}.$$

It is well known that $\mathcal{A}$ can be identified with the space of Hamiltonian vector fields, which is the Lie algebra[1] of the ∞-dimensional Lie group $\mathrm{Ham}(M,\omega)$.

The L_∞ norm on $\mathcal{A}$

$$||F||_\infty = \max F - \min F$$

gives rise to the Hofer metric d on $\mathrm{Ham}(M,\omega)$ in the following way: we define the Hofer length of a smooth Hamiltonian path $\alpha : [0,1] \to \mathrm{Ham}(M,\omega)$ as

$$\mathrm{length}(\alpha) := \int_0^1 ||\dot{\alpha}(t)||_\infty dt = \int_0^1 ||F_t||_\infty dt,$$

where $F_t(x) = F(t,x)$ is the time-dependent Hamiltonian function generating the path α. The Hofer distance d between two Hamiltonian diffeomorphisms

[1] As a vector space, the Lie algebra is by definition the tangent space to the Lie group at the identity. The tangent spaces to the Lie group at other points are identified with the Lie algebra with the help of right shifts of the group.

f and g is defined by

$$d(f,g) := \inf \{ \operatorname{length}(\alpha) \},$$

where the infimum is taken over all Hamiltonian paths α connecting f and g. The Hofer norm $\rho(f)$ is the Hofer distance between the identity map id and f, i.e.

$$\rho(f) := d(id, f).$$

It is easy to check that d is bi-invariant in the sense that

$$d(fh, gh) = d(hf, hg) = d(f, g)$$

for all $f, g, h \in \operatorname{Ham}(M, \omega)$. The fact that d is nondegenerate is highly non-trivial. This was proved by Hofer [8] for the case of $\mathbb{R}^{2n}$, then generalized by Polterovich [21] to some larger class of symplectic manifolds, and finally proved in the full generality by Lalonde and McDuff [11] using the following energy-capacity inequality

$$e(S) \geqslant \frac{1}{2}\operatorname{capacity}(S)$$

for a subset S of M. Here the capacity of S is equal to πr^2 when S is a symplectically embedded ball of radius r, and is defined in general as the supremum of the capacities of all symplectically embedded balls in S. The displacement energy $e(S)$ is defined to be the infimum of the Hofer norms of all $f \in \operatorname{Ham}(M, \omega)$ such that $f(S) \cap S = \emptyset$.

Note that the energy-capacity inequality provides a lower bound for the Hofer norm. Namely, we have

$$f(S) \cap S = \emptyset, \ \text{capacity}(S) > c \Longrightarrow \rho(f) > c/2.$$

This fact will be crucial in our proof of Theorem 1.1.3.

Recall in Definition 1.1.1 that an element $\phi \in \text{Symp}(M, \omega)$ is called unbounded if

$$r(\phi) := \sup \{ \rho([\phi, f]) \mid f \in \text{Ham}(M, \omega)\} = \infty.$$

Note that all Hamiltonian diffeomorphisms are bounded since $r(g) \leqslant 2\rho(g) < \infty$ for all $g \in \text{Ham}(M, \omega)$, where $\rho(g)$ is the Hofer norm of g. According to Proposition 1.2.A in [14], r satisfies the triangle inequality $r(\phi\psi) \leqslant r(\phi)+r(\psi)$. Since $\text{Ham}(M, \omega)$ is the kernel of the flux homomorphism, two symplectomorphisms ϕ and ψ have the same flux if and only if they differ by a Hamiltonian diffeomorphism. Combining these facts, we have the following

Observation A. [14] In order to prove $BI_0(M, \omega) = \text{Ham}(M, \omega)$, it suffices to show that for each nonzero value $v \in H^1(M, \mathbb{R})/\Gamma$, there exists some unbounded element $\phi \in \text{Symp}_0(M, \omega)$ with $\text{flux}(\phi) = v$.

2.4 The admissible lift

To prove an element $\phi \in \text{Symp}_0(M, \omega)$ is unbounded, one has to show that $\rho([\phi, f])$ can be arbitrarily large by choosing different $f \in \text{Ham}(M, \omega)$. Hence the energy-capacity inequality will not work directly for closed manifolds since the capacity of the manifold itself is finite. To go around this difficulty, we recall the notion of admissible lifts which was first introduced by Lalonde and Polterovich [14]. We shall point out that our definition is slightly different from theirs, but the two definitions are equivalent.

Let $\pi : (\widetilde{M}, \widetilde{\omega}) \to (M, \omega)$ be a symplectic covering map, i.e. a covering map π between two symplectic manifolds such that $\widetilde{\omega} = \pi^*\omega$.

Definition 2.4.1. *For every $g \in \mathrm{Ham}(M, \omega)$, assume g is the time-1 map of the Hamiltonian flow generated by time-dependent Hamiltonian function H_t. An admissible lift $\widetilde{g} \in \mathrm{Ham}(\widetilde{M}, \widetilde{\omega})$ of g with respect to π is defined to be the time-1 map of the Hamiltonian flow generated by $H_t \circ \pi$.*

Lemma 2.4.2 (existence and uniqueness of admissible lifts). *For all $g \in \mathrm{Ham}(M, \omega)$, such an admissible lift $\widetilde{g} \in \mathrm{Ham}(\widetilde{M}, \widetilde{\omega})$ exists and is unique.*

Proof. The existence follows from the definition. For the uniqueness, it suffices to show that the admissible lift $\widetilde{g}$ of g is independent of the choice of the Hamiltonian function H_t.

Note that the choice of H_t is equivalent to the choice of the Hamiltonian isotopy g_t connecting id to g. For every point $p \in M$, let

$$\widetilde{ev}_p : \pi_1(\mathrm{Ham}(M, \omega), id) \to \pi_1(M, p)$$

be the map induced by the evaluation map $ev_p : \mathrm{Ham}(M, \omega) \to M$ which takes g to $g(p)$. It follows from Floer theory that for all symplectic manifolds (M, ω), the induced map $\widetilde{ev}_p$ is trivial, see Chapter 11 [17] for instance. This deep result implies that for any two different paths g_t^1 and g_t^2 in $\mathrm{Ham}(M,\omega)$ connecting id to g, $g_t^1(p)$ and $g_t^2(p)$ must be homotopic paths in M. Therefore, for every point $\widetilde{p} \in \widetilde{M}$, the image $\widetilde{g}(\widetilde{p})$ of $\widetilde{p}$ under $\widetilde{g}$, being the endpoint of the lift of the path $g_t(p)$, is independent of the choice of the Hamiltonian isotopy g_t. This proves the uniqueness of admissible lifts. □

For our purposes, we consider the universal cover $\widetilde{M}$ of M. Note that $\widetilde{M}$ is not necessarily compact, and the admissible lift $\widetilde{g}$ of $g \in \mathrm{Ham}(M,\omega)$ is not necessarily compactly supported in $\widetilde{M}$. Instead, it belongs to $\mathrm{Ham}_b(\widetilde{M},\widetilde{\omega})$ of time-1 maps of bounded Hamiltonians $\widetilde{M} \times [0,1] \to \mathbb{R}$. The Hofer norm is still well defined and the same energy-capacity inequality still holds for this setting. This idea is due to Lalonde and Polterovich [14]. We shall spell out some details here for the sake of clarity.

Denote by (N,σ) a noncompact symplectic manifold without boundary. We do not often consider the group $\mathrm{Ham}(N,\sigma)$ of all Hamiltonian diffeomorphisms with arbitrary support. One reason in our context is that it would not be possible to define the Hofer norm on $\mathrm{Ham}(N,\sigma)$ using the L_∞ norm on the space $\mathcal{A}$ of all Hamiltonian functions with arbitrary support, since not all elements in $\mathcal{A}$ have finite L_∞ norms.

One may consider the group $\mathrm{Ham}_c(N,\sigma)$ of Hamiltonian diffeomorphisms with compact support. The Hofer norm ρ is well defined on $\mathrm{Ham}_c(N,\sigma)$, and the energy-capacity inequality

$$e_c(S) \geqslant \frac{1}{2}\mathrm{capacity}(S)$$

is valid as usual, where

$$e_c(S) := \inf\{\,\rho(f) \mid f \in \mathrm{Ham}_c(N,\sigma),\ f(S) \cap S = \emptyset\}.$$

As we have already pointed out, however, this setting is not sufficient for our purposes since the admissible lift is usually not compactly supported. Hence we need to consider the larger group $\mathrm{Ham}_b(N,\sigma)$ of Hamiltonian diffeomorphisms which are time-1 maps of bounded Hamiltonians $H : N \times [0,1] \to \mathbb{R}$. Note that one can not use an arbitrary bounded Hamiltonians H, since the Hamiltonian flow generated by H need not be integrable. Instead, we only restrict to those bounded Hamiltonians whose flows are integrable.

The Hofer norm can be defined on $\mathrm{Ham}_b(N,\sigma)$ exactly the same way as in Section 2.3. For a subset S of N, define also the bounded displacement energy $e_b(S)$ as

$$e_b(S) := \inf\{\,\rho(f) \mid f \in \mathrm{Ham}_b(N,\sigma),\ f(S) \cap S = \emptyset\}.$$

Note that $\mathrm{Ham}_c(N,\sigma) \subset \mathrm{Ham}_b(N,\sigma)$ implies $e_b(S) \leqslant e_c(S)$ for any subset $S \subset N$. In fact, for any compact subset S, we have

$$e_b(S) = e_c(S).$$

To prove the other inequality, note that if $f \in \mathrm{Ham}_b(N,\sigma)$ displaces a compact subset S from itself, one can easily construct some cut-off $f_{cut} \in \mathrm{Ham}_c(N,\sigma)$ of f which still displaces S from itself, and the Hofer norm satisfies $\rho(f) \geqslant \rho(f_{cut})$. Taking the infimum implies $e_b(S) \geqslant e_c(S)$.

The above argument implies that the energy-capacity inequality still holds for the bounded displacement energy. That is

$$e_b(S) \geqslant \frac{1}{2}\text{capacity}(S).$$

Now back to our discussion about the admissible lift. Note that the admissible lift $\widetilde{g}$ of $g \in \text{Ham}(M,\omega)$ belongs to $\text{Ham}_b(\widetilde{M},\widetilde{\omega})$. And it follows from the definition of the admissible lift that

$$\rho(g) \geqslant \rho(\widetilde{g})$$

for all $g \in \text{Ham}(M,\omega)$ and the admissible lift $\widetilde{g} \in \text{Ham}_b(\widetilde{M},\widetilde{\omega})$ of g. Here the two ρ's are the Hofer norms on $\text{Ham}(M,\omega)$ and $\text{Ham}_b(\widetilde{M},\widetilde{\omega})$ respectively. The readers may find an argument for this in Lemma 3.4.2 and Remark 3.4.3. Combining all the above discussions, we have

Observation B. [14] To construct $g \in \text{Ham}(M,\omega)$ of arbitrarily large Hofer norm, it suffices to make sure that the unique admissible lift $\widetilde{g} \in \text{Ham}_b(\widetilde{M},\widetilde{\omega})$ of g displaces from itself a symplectic ball in $\widetilde{M}$ of arbitrarily large capacity.

2.5 Proof of Theorem 1.1.3

In this section, we prove Theorem 1.1.3. Recall that (M,ω) is the Kodaira-Thurston manifold with the standard symplectic form $\omega = ds \wedge dt + dx \wedge dy$. Recall also that $H^1(M,\mathbb{R}) = \mathbb{R}\langle ds, dy, dt\rangle$ and the flux subgroup $\Gamma = \mathbb{Z}\langle ds, dy\rangle$ by Lemma 2.2.2 and Theorem 2.2.1. In view of Observation A, to prove

$BI_0(M,\omega) = \mathrm{Ham}(M,\omega)$, it suffices to show that for every nonzero element $v \in H^1(M,\mathbb{R})/\Gamma = \mathbb{R}/\mathbb{Z}\langle ds, dy\rangle \oplus \mathbb{R}\langle dt\rangle$, there exists some unbounded symplectomorphism with flux equal to v. We begin with an explicit construction of symplectomorphisms with given fluxes.

Lemma 2.5.1. *Let v be an element in $H^1(M,\mathbb{R})/\Gamma = \mathbb{R}/\mathbb{Z}\langle ds, dy\rangle \oplus \mathbb{R}\langle dt\rangle$, say $v = \alpha ds + \beta dy + c dt$ where $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$ and $c \in \mathbb{R}$. Then there exists an element $\phi_{\alpha\beta c} \in \mathrm{Symp}_0(M,\omega)$ with $\mathrm{flux}(\phi_{\alpha\beta c}) = v$. Namely,*

$$\phi_{\alpha\beta c}(s,t,x,y) = (s+c, t-\alpha, x+\beta, y).$$

Proof. First $\phi_{\alpha\beta c}$ is well-defined. For instance, since (s,t,x,y) and $(s+1,t,x+y,y)$ represent the same point on M, one has to show that

$$\phi_{\alpha\beta c}(s,t,x,y) \sim \phi_{\alpha\beta c}(s+1,t,x+y,y).$$

This is true since

$$\phi_{\alpha\beta c}(s,t,x,y) = (s+c, t-\alpha, x+\beta, y),$$

and

$$\phi_{\alpha\beta c}(s+1,t,x+y,y) = (s+1+c, t-\alpha, x+y+\beta, y).$$

It is easy to see that $\phi_{\alpha\beta c}$ preserves ω, and the obvious isotopy from id to $\phi_{\alpha\beta c}$ implies that $\phi_{\alpha\beta c} \in \mathrm{Symp}_0(M,\omega)$. The calculation for $\mathrm{flux}(\phi_{\alpha\beta c}) = v$ is straightforward using (2.1) in Section 2.1. □

The following theorem due to Lalonde and Polterovich [14] is an important criteria for unbounded symplectomorphisms.

Theorem 2.5.2 (Theorem 1.4.A [14]). *Let $L \subset M$ be a closed Lagrangian submanifold admitting a Riemannian metric with non-positive sectional curvature, and whose inclusion in M induces an injection on fundamental groups. Let ϕ be an element in $\mathrm{Symp}_0(M, \omega)$ such that $\phi(L) \cap L = \emptyset$. Then ϕ is unbounded.*

For the proof, one passes to the universal cover $\widetilde{M}$ of M. The hypothesis implies that the lift of a neighbourhood U of L has infinite capacity. One then constructs a Hamiltonian isotopy f_τ supported in U so that the admissible lift $\widetilde{[\phi, f_\tau]}$ of the commutator $[\phi, f_\tau]$ will displace a symplectic ball of arbitrarily large capacity as τ goes to infinity. This implies ϕ is unbounded according to Observation B. See [14] for details.

Proof of Theorem 1.1.3. In view of Observation A, it suffices to show that the symplectomorphisms $\phi_{\alpha\beta c}$ constructed in Lemma 2.5.1 are unbounded in all cases, as long as the flux $v = \alpha ds + \beta dy + cdt$ does not vanish. We argue case by case. In the first two cases, this is a direct consequence of Theorem 2.5.2.

Case 1. $\alpha \neq 0 \in \mathbb{R}/\mathbb{Z}$.

Let $L \subset M$ be the subset of M defined by

$$L := \{(s, t, x, y) \in M \mid t = 0, y = 0\}.$$

It is easy to check that L is a Lagrangian torus satisfying the hypothesis of Theorem 2.5.2, and $\phi_{\alpha\beta c}$ displaces L from itself. Thus $\phi_{\alpha\beta c}$ is unbounded.

Case 2. $\alpha = 0 \in \mathbb{R}/\mathbb{Z}$, $\beta \neq 0 \in \mathbb{R}/\mathbb{Z}$ and $c = 0 \in \mathbb{R}$.

In this case, $\phi_\beta := \phi_{\alpha\beta c}$ maps (s,t,x,y) to $(s,t,x+\beta,y)$. As in the first case, ϕ_β displaces from itself a Lagrangian torus L of M defined by

$$L := \{(s,t,x,y) \in M \mid s = 0, x = 0\}.$$

We again get ϕ_β is unbounded in view of Theorem 2.5.2.

Case 3. $\alpha = 0 \in \mathbb{R}/\mathbb{Z}$ and $c \neq 0 \in \mathbb{R}$.

We write $\phi_{\beta c}$ for $\phi_{\alpha\beta c}$ in this case,

$$\phi_{\beta c} := \phi_{\alpha\beta c} : (s,t,x,y) \mapsto (s+c, t, x+\beta, y).$$

Consider two different situations, one of which is simple, while the other is more complicated.

3A. $\alpha = 0 \in \mathbb{R}/\mathbb{Z}$ and $c \notin \mathbb{Z}$.

As in case 1 and 2, $\phi_{\beta c}$ is unbounded as it displaces from itself

$$L := \{(s,t,x,y) \in M \mid s = 0, x = 0\}.$$

3B. $\alpha = 0 \in \mathbb{R}/\mathbb{Z}$ and $c \in \mathbb{Z}\backslash\{0\}$.

Note that $(s+c,t,x+\beta,y) \sim (s,t,x+\beta-cy,y)$. So the map $\phi_{\beta c} : M \to M$ can also be expressed as

$$\phi_{\beta c}(s,t,x,y) = (s,t,x+\beta-cy,y).$$

In contrast to all previous cases where we used the same argument, here we are facing a difficulty. The trouble is that in this case we are unable to find a Lagrangian torus of M which is disjoined from itself by the map $\phi_{\beta c}$. Thus the above argument breaks down.

To resolve this difficulty, we take f_τ to be the Hamiltonian isotopy whose support is in the subset

$$U := \{(s,t,x,y) \in M \mid |s| < \epsilon, |x| < \epsilon\}$$

of M. We require f_τ to flow only along y and t direction in U and its restriction to

$$V := \{(s,t,x,y) \in M \mid |s| < \epsilon/2, |x| < \epsilon/2\}$$

is defined by

$$f_\tau(s,t,x,y) = (s,t,x,y-\tau).$$

In the discussion below, $[f,g] := fgf^{-1}g^{-1}$ stands for the commutator of f and g. Our goal is to show that the unique admissible lift $\widetilde{[\phi_{\beta c}, f_\tau]}$ of $[\phi_{\beta c}, f_\tau]$ still displaces from itself a subset of $\mathbb{R}^4$ of arbitrarily large capacity when τ goes to infinity. For this, we need the following

Lemma 2.5.3. *Let $\phi \in \mathrm{Symp}_0(M,\omega)$, and f_τ be a Hamiltonian isotopy of M. Let $\widetilde{\phi} : \widetilde{M} \to \widetilde{M}$ be any lift of ϕ, and $\widetilde{[\phi, f_\tau]}$ and $\widetilde{f}_\tau$ be the unique admissible lift of $[\phi, f_\tau]$ and f_τ respectively. Then*

$$\widetilde{[\phi, f_\tau]} = [\widetilde{\phi}, \widetilde{f}_\tau].$$

Proof. Note that f_τ is Hamiltonian implies $[\phi, f_\tau]$ is Hamiltonian. So both admissible lifts $\widetilde{[\phi, f_\tau]}$ and $\widetilde{f}_\tau$ make sense. To simplify notation, denote

$$A_\tau := \widetilde{[\phi, f_\tau]} \text{ and } B_\tau := [\widetilde{\phi}, \widetilde{f}_\tau].$$

We want to show $A_\tau = B_\tau$, which is equivalent to $A_\tau B_\tau^{-1} = id$. Since A_τ and B_τ are both lifts of $[\phi, f_\tau]$, $A_\tau B_\tau^{-1}$ is the deck transformation of the covering map $\pi : \widetilde{M} \to M$. Now $A_0 B_0 = id$, and $\tau \to A_\tau B_\tau^{-1}$ is a continuously parametrized path into the discrete set of all deck transformations. Thus $A_\tau B_\tau^{-1} = id$ for all τ. □

Now back to the proof of Theorem 1.1.3. To prove $\phi_{\beta c}$ is unbounded, we need to show that the commutator $[\phi_{\beta c}, f_\tau]$ has arbitrarily large Hofer norm when τ goes to infinity. Let $V_0 \subset \mathbb{R}^4$ be the subset of $\mathbb{R}^4$ defined by

$$V_0 := \{(s, t, x, y) \in \mathbb{R}^4 \mid |s| < \epsilon/2, t \in \mathbb{R}, |x| < \epsilon/2, 0 < y < \tau/2\}.$$

Since V_0 has arbitrarily large capacity as τ goes to infinity, according to Observation B, it suffices to show that the admissible lift $\widetilde{[\phi_{\beta c}, f_\tau]}$ of $[\phi_{\beta c}, f_\tau]$ displaces V_0 from itself.

For this, denote by $\widetilde{\phi}_{\beta c} : \mathbb{R}^4 \to \mathbb{R}^4$ the preferred lift of the map $\phi_{\beta c}$ such that

$$\widetilde{\phi}_{\beta c}(s, t, x, y) = (s, t, x + \beta - cy, y).$$

By the above lemma, it suffices to show that $[\widetilde{\phi}_{\beta c}, \widetilde{f}_\tau](V_0) \cap V_0 = \emptyset$, which is equivalent to

$$\widetilde{\phi}_{\beta c}^{-1}\widetilde{f}_{\tau}^{-1}(V_0) \cap \widetilde{f}_{\tau}^{-1}\widetilde{\phi}_{\beta c}^{-1}(V_0) = \emptyset.$$

Note that the restriction of $\widetilde{f}_{\tau}$ to

$$\widetilde{V} := \{(s,t,x,y) \in \mathbb{R}^4 \mid |s| < \epsilon/2, t \in \mathbb{R}, |x| < \epsilon/2, y \in \mathbb{R}\}$$

is defined by

$$\widetilde{f}_{\tau}(s,t,x,y) = (s,t,x,y-\tau).$$

We have

$$\widetilde{f}_{\tau}^{-1}(V_0) = \{|s| < \epsilon/2, t \in \mathbb{R}, |x| < \epsilon/2, \tau < y < 3\tau/2\}.$$

Hence

$$\widetilde{\phi}_{\beta c}^{-1}\widetilde{f}_{\tau}^{-1}(V_0) = \{|s| < \epsilon/2, t \in \mathbb{R}, |x + \beta - cy| < \epsilon/2, \tau < y < 3\tau/2\}.$$

On the other hand,

$$\widetilde{\phi}_{\beta c}^{-1}(V_0) = \{|s| < \epsilon/2, t \in \mathbb{R}, |x + \beta - cy| < \epsilon/2, 0 < y < \tau/2\}.$$

Note that in the set $\widetilde{\phi}_{\beta c}^{-1}\widetilde{f}_{\tau}^{-1}(V_0)$ we have

$$|x| > |cy| - |\beta| - \epsilon/2 > |c|\tau - |\beta| - \epsilon/2,$$

and in $\widetilde{\phi}_{\beta c}^{-1}(V_0)$ we have

$$|x| < |cy| + |\beta| + \epsilon/2 < |c|\tau/2 + |\beta| + \epsilon/2.$$

Thus for sufficiently large τ, these two sets do not share the same values in x coordinates. Since the flow $\widetilde{f}_\tau^{-1}$ only changes the y and t-coordinates when restricted to $\widetilde{\phi}_{\beta c}^{-1}(V_0)$, we conclude

$$\widetilde{\phi}_{\beta c}^{-1}\widetilde{f}_\tau^{-1}(V_0) \cap \widetilde{f}_\tau^{-1}\widetilde{\phi}_{\beta c}^{-1}(V_0) = \emptyset.$$

As we have already mentioned above, this implies $\phi_{\beta c}$ is unbounded in case 3B, which completes the proof of Theorem 1.1.3. □

2.6 Bounded isometries for $(\mathbb{T}^4, \omega)$

In this section we study bounded isometries for the 4-torus with all linear symplectic forms. We have already mentioned in Chapter 1 that the bounded isometry conjecture holds for the torus with the standard symplectic form. The following theorem generalizes this result.

Theorem 2.6.1. *The bounded isometry conjecture holds for the* 4*-torus* $(\mathbb{T}^4, \omega)$ *with any linear symplectic form* $\omega := \sum_{i<j} a_{ij}\, dx_i \wedge dx_j$*. That is,* $BI_0(\mathbb{T}^4, \omega) = \mathrm{Ham}(\mathbb{T}^4, \omega)$.

Remark 2.6.2. The 2-form $\omega = \sum_{i<j} a_{ij}\, dx_i \wedge dx_j$ is symplectic, i.e. nondegenerate if and only if $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} \neq 0$.

For each $1 \leqslant i \leqslant 4$, let $\{\phi^i_\theta\} \in \pi_1(\mathrm{Symp}_0(\mathbb{T}^4, \omega))$ be the loop of rotations of $\mathbb{T}^4$ along x_i direction. Let $\xi_i \in H^1(\mathbb{T}^4, \mathbb{R})$ be the image of $\{\phi^i_\theta\}$ under the flux homomorphism. Using (2.1) in Section 2.1, one easily gets

$$\xi_i := \mathrm{flux}(\{\phi^i_\theta\}) = \sum_{j=1}^{4} a_{ij}\, dx_j.$$

Here we take the convention that $a_{ij} = -a_{ji}$. In particular, $a_{ii} = 0$.

Lemma 2.6.3. *For the 4-torus $(\mathbb{T}^4, \omega)$ with the linear symplectic form $\omega := \sum_{i<j} a_{ij}\, dx_i \wedge dx_j$, the flux subgroup $\Gamma \subset H^1(\mathbb{T}^4, \mathbb{R})$ is generated by the above ξ_i's over $\mathbb{Z}$. That is, $\Gamma = \mathbb{Z}\langle \xi_1, \xi_2, \xi_3, \xi_4 \rangle$.*

Proof. According to Lemma 2.1.1, we have the following commutative diagram for the manifold $(\mathbb{T}^4, \omega)$.

$$\begin{array}{ccccc}
\pi_1(\mathrm{Symp}_0(\mathbb{T}^4, \omega)) & \xrightarrow{\widetilde{ev}_s} & H_1(\mathbb{T}^4, \mathbb{Z}) & \xrightarrow{\mathrm{PD}} & H^3(\mathbb{T}^4, \mathbb{Z}) \\
\Big\downarrow id & & & & \Big\downarrow \cdot\mathrm{vol}(\mathbb{T}^4) \\
\pi_1(\mathrm{Symp}_0(\mathbb{T}^4, \omega)) & \xrightarrow{\mathrm{flux}} & H^1(\mathbb{T}^4, \mathbb{R}) & \xrightarrow{\wedge[\omega]} & H^3(\mathbb{T}^4, \mathbb{R}).
\end{array}$$

Note that $\widetilde{ev}_s$ is surjective, and $\wedge[\omega] : H^1(\mathbb{T}^4, \mathbb{R}) \to H^3(\mathbb{T}^4, \mathbb{R})$ is an isomorphism. Note also that $\mathrm{vol}(\mathbb{T}^4) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$. It follows from a similar argument as in the proof of Theorem 2.2.1 that $\xi_i (1 \leqslant i \leqslant 4)$ span the flux subgroup Γ over $\mathbb{Z}$. $\square$

Now let $\phi \in Symp_0(\mathbb{T}^4, \omega)$ such that

$$\phi(x_1, x_2, x_3, x_4) = (x_1 + \alpha_1, x_2 + \alpha_2, x_3 + \alpha_3, x_4 + \alpha_4)$$

where $\alpha_i \in \mathbb{R}/\mathbb{Z}$ for $1 \leqslant i \leqslant 4$. Then

$$\text{flux}(\phi) = \sum_{i=1}^{4} \alpha_i \, \xi_i.$$

Recall that in view of Observation A in Section 2.3, to prove Theorem 2.6.1, it suffices to show ϕ is unbounded as long as at least one $\alpha_i \in \mathbb{R}/\mathbb{Z}$ is nonzero. One may attempt to apply Theorem 2.5.2 by showing ϕ disjoins some Lagrangian torus $L \subset \mathbb{T}^4$ from itself. For a general symplectic form ω, however, there may not exist any such Lagrangian torus in $\mathbb{T}^4$. Nevertheless, we can still prove ϕ is unbounded using the following

Lemma 2.6.4. *Let (M, ω) be an aspherical symplectic manifold. Let $f_\tau \in \mathrm{Ham}(M, \omega)$ be the flow generated by an autonomous Hamiltonian which has no nonconstant contractible orbits. Then the Hofer norm $\rho(f_\tau)$ goes to infinity as τ goes to infinity.*

This result can be found in Oh [19], Schwarz [25] and Kerman-Lalonde [10]. The main idea of the argument is that the Hofer norm is bounded from below by the spectral norm (cf. Remark 3.3.3), while the spectral norm of such f_τ grows linearly with respect to τ.

Proof of Theorem 2.6.1. Let $\phi \in Symp_0(\mathbb{T}^4, \omega)$ such that

$$\phi(x_1, x_2, x_3, x_4) = (x_1 + \alpha_1, x_2 + \alpha_2, x_3 + \alpha_3, x_4 + \alpha_4).$$

As discussed above, it suffices to show ϕ is unbounded when at least one $\alpha_i \in \mathbb{R}/\mathbb{Z}$ is nonzero. Assume $\alpha_1 \neq 0$ without loss of generality. Thus $\phi(U) \cap U = \emptyset$

where $U \subset \mathbb{T}^4$ is defined by

$$U := \{(x_1, x_2, x_3, x_4) \in \mathbb{T}^4 \mid |x_1| < \epsilon\}.$$

for sufficiently small ϵ.

Let H be a time-independent Hamiltonian function of $\mathbb{T}^4$ supported in U. Denote by f_τ the (autonomous) Hamiltonian flow generated by H. Since $\phi(U) \cap U = \emptyset$, we know that $[\phi, f_\tau] := \phi f_\tau \phi^{-1} f_\tau^{-1}$ is also an autonomous Hamiltonian flow supported in the union of two disjoint sets $U \cup \phi(U)$. If we further require that H depend only on the first coordinate x_1, using the fact that ω is a linear symplectic form, we conclude that $[\phi, f_\tau]$ has no nonconstant contractible orbits. Thus it follows from Lemma 2.6.4 that the Hofer norm $\rho([\phi, f_\tau])$ goes to infinity as τ goes to infinity. Hence ϕ is unbounded in the sense of Definition 1.1.1. □

2.7 The Kodaira-Thurston manifold with linear symplectic forms

So far we have studied bounded isometries for the Kodaira-Thurston manifold with the standard symplectic form and for the 4-torus with all linear symplectic forms. In particular, we have shown that the bounded isometry conjecture holds in both cases. In this section we will study the same question for the Kodaira-Thurston manifold with all linear symplectic forms.

Question 2.7.1. *Does the bounded isometry conjecture hold for the Kodaira-Thurston manifold with all linear symplectic forms?*

We expect the answer to be positive. Although we are not able to give a complete proof yet at this time, we shall provide some partial results including the following

Theorem 2.7.2. *For the Kodaira-Thurston manifold (M, ω) with any linear symplectic form ω, the Hofer norm ρ on $\mathrm{Ham}(M, \omega)$ does not extend to a bi-invariant norm on $\mathrm{Symp}_0(M, \omega)$.*

We begin by describing the linear symplectic forms on the Kodaira-Thurston manifold M. Recall that it follows from Lemma 2.2.2 that $H^2(M, \mathbb{R})$ is of rank 4, generated by $\gamma \wedge ds$, $\gamma \wedge dy$, $ds \wedge dt$, and $dy \wedge dt$ where $\gamma = dx - sdy$. We consider linear 2-forms

$$\omega_{abef} := a\gamma \wedge ds + b\gamma \wedge dy + eds \wedge dt + fdy \wedge dt.$$

Note that ω_{abef} is a symplectic form if and only if $be - af \neq 0$. In particular, the standard symplectic form corresponds to $b = e = 1$ and $a = f = 0$. The following lemma on the flux subgroup generalizes Theorem 2.2.1.

Lemma 2.7.3. *The flux subgroup $\Gamma \subset H^1(M, \mathbb{R})$ of the Kodaira-Thurston manifold with the linear symplectic form ω_{abef} has rank 2 over $\mathbb{Z}$. More precisely, we have $\Gamma = \mathbb{Z}\langle eds + fdy,\ ads + bdy\rangle$.*

Proof. The proof follows the same lines as that of Theorem 2.2.1. According to Lemma 2.1.1, we have the following commutative diagram.

$$\begin{array}{ccccc}
\pi_1(\mathrm{Symp}_0(M,\omega_{abef})) & \xrightarrow{\widetilde{ev}_s} & H_1(M,\mathbb{Z}) & \xrightarrow{\mathrm{PD}} & H^3(M,\mathbb{Z}) \\
{\scriptstyle id}\downarrow & & & & \downarrow{\scriptstyle .\mathrm{vol}(M)} \\
\pi_1(\mathrm{Symp}_0(M,\omega_{abef})) & \xrightarrow{\mathrm{flux}} & H^1(M,\mathbb{R}) & \xrightarrow{\wedge[\omega_{abef}]} & H^3(M,\mathbb{R}).
\end{array}$$

As in the proof of Theorem 2.2.1, the image of $\widetilde{ev}_s$ in $H_1(M,\mathbb{Z})$ is contained in $\mathbb{Z}\langle\frac{\partial}{\partial t}\rangle$. Note that $PD(\frac{\partial}{\partial t}) = -\gamma \wedge dy \wedge ds$, where $\gamma = dx - sdy$. Now look at the map $\wedge\omega_{abef} : H^1(M,\mathbb{R}) \to H^3(M,\mathbb{R})$,

$$ds \mapsto ds \wedge \omega_{abef} = b\gamma \wedge dy \wedge ds - f dy \wedge ds \wedge dt = b\gamma \wedge dy \wedge ds,$$

$$dt \mapsto dt \wedge \omega_{abef} = a\gamma \wedge ds \wedge dt + b\gamma \wedge dy \wedge dt,$$

$$dy \mapsto dy \wedge \omega_{abef} = -a\gamma \wedge dy \wedge ds + e dy \wedge ds \wedge dt = -a\gamma \wedge dy \wedge ds.$$

Here we have used the fact that the 3-form $dy \wedge ds \wedge dt = d(\gamma \wedge dt)$ is exact, so it vanishes on the cohomology level. Since $\mathrm{vol}(M) = be - af \neq 0$, we conclude by tracing the diagram that the flux subgroup $\Gamma \subset H^1(M,\mathbb{R})$ is contained in $\mathbb{Z}\langle eds + fdy, ads + bdy\rangle$. Note that the fact $be - af \neq 0$ implies that $eds + fdy$ and $ads + bdy$ are linearly independent. An explicit construction shows that Γ is actually equal to $\mathbb{Z}\langle eds + fdy, ads + bdy\rangle$. Namely, we take two elements $\{\phi_\theta\}$ and $\{\psi_\theta\}$ in $\pi_1(\mathrm{Symp}_0(M,\omega_{abef}))$ such that

$$\phi_\theta(s,t,x,y) = (s, t - \theta, x, y), 0 \leqslant \theta \leqslant 1,$$

$$\psi_\theta(s,t,x,y) = (s, t, x + \theta, y), 0 \leqslant \theta \leqslant 1.$$

A straightforward calculation using (2.1) in Section 2.1 shows that $\mathrm{flux}(\{\phi_\theta\}) = eds + fdy$ and $\mathrm{flux}(\{\psi_\theta\}) = ads + bdy$. □

As in Lemma 2.5.1, we explicitly construct symplectomorphisms below with given fluxes.

Lemma 2.7.4. *Let v be an element in*

$$H^1(M,\mathbb{R})/\Gamma = \mathbb{R}/\mathbb{Z}\langle eds + fdy,\ ads + bdy\rangle \oplus \mathbb{R}\langle dt\rangle,$$

say

$$v = \alpha(eds + fdy) + \beta(ads + bdy) + c(be - af)dt$$

where $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$ and $c \in \mathbb{R}$. Then there exists $\phi_{\alpha\beta c} \in \mathrm{Symp}_0(M, \omega_{abef})$ with $\mathrm{flux}(\phi_{\alpha\beta c}) = v$. *Namely,*

$$\phi_{\alpha\beta c}(s,t,x,y) = (s + bc, t - \alpha, x + \beta - acs, y - ac).$$

Proof. First $\phi_{\alpha\beta c}$ is well-defined. For instance, since (s,t,x,y) and $(s+1,t,x+y,y)$ represent the same point in M, one has to show that

$$\phi_{\alpha\beta c}(s,t,x,y) \sim \phi_{\alpha\beta c}(s+1,t,x+y,y).$$

This is true since

$$\phi_{\alpha\beta c}(s,t,x,y) = (s + bc, t - \alpha, x + \beta - acs, y - ac)$$

and

$$\phi_{\alpha\beta c}(s+1,t,x+y,y) = (s + 1 + bc, t - \alpha, x + y + \beta - ac(s+1), y - ac)$$

also represent the same point. One can check that $\phi^*_{\alpha\beta c}\omega_{abef} = \omega_{abef}$, and the obvious isotopy from id to $\phi_{\alpha\beta c}$ implies that $\phi_{\alpha\beta c} \in \mathrm{Symp}_0(M, \omega_{abef})$.

It remains to show that $\mathrm{flux}(\phi_{\alpha\beta c}) = v$. Note that $\phi_{\alpha\beta c}$ is the time-1 map of the flow generated by the time-independent symplectic vector field

$$X := bc\frac{\partial}{\partial s} - \alpha\frac{\partial}{\partial t} + (\beta - acs)\frac{\partial}{\partial x} - ac\frac{\partial}{\partial y}.$$

Using (2.1) in Section 2.1, we have

$$\begin{aligned}
\mathrm{flux}(\phi_{\alpha\beta c}) &= \iota(X)\,\omega_{abef} \\
&= \iota(bc\frac{\partial}{\partial s} - \alpha\frac{\partial}{\partial t} + (\beta - acs)\frac{\partial}{\partial x} - ac\frac{\partial}{\partial y})\,\omega_{abef} \\
&= -abc(dx - sdy) + bcedt + \alpha eds + \alpha f dy \\
&+ a(\beta - acs)ds + b(\beta - acs)dy + a^2csds + abcdx - acf dt \\
&= \alpha(eds + fdy) + \beta(ads + bdy) + c(be - af)dt \\
&= v.
\end{aligned}$$

□

Proof of Theorem 2.7.2. In view of Theorem 1.2.1, it suffices to find some unbounded symplectomorphism. Note that since $be - af \neq 0$ implies a and b can not both vanish, there exists some constant c such that ac and bc are not both integers. Thus $\phi_{\alpha\beta c}$ disjoins a Lagrangian torus

$$L := \{(s, t, x, y) \in M \mid s = 0,\, y = 0\}.$$

It then follows from Theorem 2.5.2 that $\phi_{\alpha\beta c}$ is unbounded. This completes the proof. □

Note that in the above proof, we have showed that most elements in $\mathrm{Symp}_0(M, \omega_{abef})$ are unbounded for the Kodaira-Thurston manifold M with any linear symplectic form ω_{abef}. To answer Question 2.7.1, however, one has to check whether $\phi_{\alpha\beta c}$ constructed in Lemma 2.7.4 is always unbounded whenever its flux v is nonzero in $H^1(M, \mathbb{R})/\Gamma$. As we have mentioned already, this is in general a very hard question. In the remaining of this section, we will study this question. In particular, we will give a proof for some known cases. For the unknown cases, we will try to point out what difficulty is involved.

Case 1: $\alpha \neq 0 \in \mathbb{R}/\mathbb{Z}$. In this case we will prove $\phi_{\alpha\beta c}$ is always unbounded. Note that $\phi_{\alpha\beta c}(U) \cap U = \emptyset$ where $U \subset M$ is defined by

$$U := \{(s, t, x, y) \in M \mid |t| < \epsilon\}$$

for sufficiently small ϵ. We will apply Lemma 2.6.4 as in the proof of Theorem 2.6.1. Recall that the only thing we need to do is to construct time-independent Hamiltonian H supported in U whose flow has no nonconstant contractible orbits. This follows from a tedious but straightforward calculation which asserts that

$$\iota(X)\,\omega_{abef} = dt$$

where

$$X := \frac{1}{be - af}\left(-as\frac{\partial}{\partial x} - a\frac{\partial}{\partial y} + b\frac{\partial}{\partial s}\right).$$

Note that this is actually a special case of the construction in Lemma 2.7.4.

And the fact that X is a well defined vector field on M follows from the equivalence relation $(s,t,x,y) \sim (s+1,t,x+y,y)$. Since a and b can not be both zero, if we further require H to depend only on the t-coordinates, we know that the Hamiltonian flow generated by H will have no nonconstant contractible orbits. Therefore $\phi_{\alpha\beta c}$ is always unbounded in this case.

Case 2: $\alpha = 0 \in \mathbb{R}/\mathbb{Z}$ and $c \neq 0 \in \mathbb{R}$. First we assume ac and bc are not both integers. Note that this is always the case when the ratio $a:b$ is irrational. Under this assumption, $\phi_{\beta c} := \phi_{\alpha\beta c}$ is unbounded in view of Theorem 2.5.2 as it disjoins a Lagrangian torus

$$L := \{(s,t,x,y) \in M \mid s=0,\ y=0\}.$$

If the ratio $a:b$ is rational, then there exists $c \neq 0$ such that both ac and bc are integers. In this case, using the equivalence relation $(s,t,x,y) \sim (s+1,t,x+y,y)$, we can write the map

$$\phi_{\beta c} : (s,t,x,y) \mapsto (s+bc, t, x+\beta-acs, y-ac)$$

as

$$\phi_{\beta c} : (s,t,x,y) \mapsto (s,t,x+\beta-acs-bcy,y).$$

It is natural to attempt the admissible lift argument as in Case 3B of Theorem 1.1.3 for the standard Kodaira-Thurston manifold. One would try to construct a Hamiltonian isotopy $\widetilde{f}_\tau$ on $\mathbb{R}^4$ supported in

$$\widetilde{U} := \{(s,t,x,y) \in \mathbb{R}^4 \mid |es+fy| < \epsilon, |x| < \epsilon\}$$

which flows only along s and y directions, and whose restriction to

$$\widetilde{V} := \{(s,t,x,y) \in \mathbb{R}^4 \mid |es + fy| < \epsilon/2, |x| < \epsilon/2\}$$

is defined by

$$\widetilde{f}_\tau(s,t,x,y) = (s + f\tau, t, x, y - e\tau).$$

Note that the above construction allows us to show that the lift

$$\widetilde{\phi}_{\beta c} : (s,t,x,y) \mapsto (s,t,x+\beta - acs - bcy, y)$$

of $\phi_{\beta c}$ is unbounded on the universal cover level. For this, one would argue as in Case 3B of Theorem 1.1.3, that the commutator $[\widetilde{\phi}_{\beta c}, \widetilde{f}_\tau]$ displaces some subset $V_0 \subset \mathbb{R}^4$ of arbitrarily large capacity with respect to the symplectic form $\widetilde{\omega}_{abef} := \pi^*\omega_{abef}$. Namely,

$$V_0 := \{|es + fy| < \epsilon/2,\ t \in \mathbb{R},\ |x| < \epsilon/2, 0 < as + by < |be - af|\tau/2\}.$$

The problem here is that $\widetilde{f}_\tau$ does not descend to a Hamiltonian isotopy on M. Note that in proving $\phi_{\beta c}$ itself is unbounded, it is crucial to have such a Hamiltonian isotopy on M, not just on the universal cover $\mathbb{R}^4$. Hence this case is still unsolved.

Case 3: $\alpha = 0 \in \mathbb{R}/\mathbb{Z}$, $c = 0 \in \mathbb{R}$ and $\beta \neq 0 \in \mathbb{R}/\mathbb{Z}$. In this case, the map $\phi_\beta := \phi_{\alpha\beta c}$ has the simple form

$$\phi_\beta : (s,t,x,y) \mapsto (s,t,x+\beta, y).$$

We do not know in general how to prove ϕ_β is unbounded for this seemingly easy case. The difficulty in applying Theorem 2.5.2 is that the obvious torus

$$L := \{(s,t,x,y) \in M \mid s = 0, x = 0\}$$

displaced by ϕ_β is not necessarily Lagrangian with respect to all symplectic forms ω_{abef}. If we assume $f = 0$, then L is actually a Lagrangian torus, and ϕ_β will be unbounded in view of Theorem 2.5.2.

Note also that Lemma 2.6.4 does not work here either since our situation here is different from Case 1 above. The main reason is that

$$U := \{(s,t,x,y) \in M \mid |x| < \epsilon\}$$

is not a well defined set in M. Thus one can no longer apply Lemma 2.6.4 by constructing a time-independent Hamiltonian H supported in U whose flow has no nonconstant contractible orbits.

Chapter 3

Extending bi-invariant norms

In Chapter 1 we have already seen that for a number of symplectic manifolds (M, ω), the Hofer norm on $\mathrm{Ham}(M, \omega)$ does not extend to a bi-invariant norm on $\mathrm{Symp}_0(M, \omega)$. In this chapter we will study the question of extending more general bi-invariant norms on $\mathrm{Ham}(M, \omega)$. In order to formulate our main result, we begin with some preliminaries.

3.1 Bi-invariant Finsler norms

Let G be either $\mathrm{Ham}(M, \omega)$ or $\mathrm{Symp}_0(M, \omega)$. Let $d : G \times G \to \mathbb{R}$ be a bi-invariant distance function in the sense that d satisfies the following properties.

(a) $d(f, g) \geqslant 0$, and $d(f, g) = 0 \Longleftrightarrow f = g$.

(b) $d(f, g) = d(g, f)$.

(c) $d(f, h) \leqslant d(f, g) + d(g, h)$.

(d) $d(hf, hg) = d(fh, gh) = d(f, g)$.

Define ρ to be the function on G such that for all $f \in G$,

$$\rho(f) := d(id, f).$$

Then the function ρ satisfies the corresponding properties below.

(a) $\rho(f) \geqslant 0$, and $\rho(f) = 0 \Longleftrightarrow f = id$.

(b) $\rho(f) = \rho(f^{-1})$.

(c) $\rho(fg) \leqslant \rho(f) + \rho(g)$.

(d) $\rho(fgf^{-1}) = \rho(g)$.

Such a function ρ is called a bi-invariant norm[1]. If ρ satisfies all the above properties except the nondegeneracy, i.e. the second part of (a), then ρ is called a bi-invariant pseudo-norm. For our purposes, it is sometimes more convenient to deal with a bi-invariant norm ρ than to deal with a bi-invariant distance function d, although they are certainly equivalent.

Now we recall the notion of Finsler norms on $\mathrm{Ham}(M, \omega)$. As in Section 2.3, since the tangent spaces to the group $\mathrm{Ham}(M, \omega)$ are identified with the space $\mathcal{A}$ of all normalized smooth functions on M, every choice of norm $||\cdot||$ on $\mathcal{A}$ gives rise to a pseudo-metric d on $\mathrm{Ham}(M, \omega)$ in the same manner as the Hofer metric. Namely, we define the length of a smooth Hamiltonian path $\alpha : [0,1] \to \mathrm{Ham}(M, \omega)$ as

$$\mathrm{length}(\alpha) := \int_0^1 ||\dot{\alpha}(t)||dt = \int_0^1 ||F_t||dt,$$

[1] In view of property (d), it might be more consistent to call ρ a conjugate-invariant norm. However, we shall call it bi-invariant to emphasize that the corresponding distance function is bi-invariant.

where $F_t(x) = F(t, x)$ is the time-dependent Hamiltonian function generating the path α. This is the usual notion of Finsler length. The distance between two Hamiltonian diffeomorphisms f and g is defined by

$$d(f, g) := \inf \{ \text{length}(\alpha) \},$$

where the infimum is taken over all Hamiltonian paths α connecting f and g. It is easy to verify that d is a pseudo-distance function. Denote by $\rho(f)$ the distance between the identity map id and f, i.e.

$$\rho(f) := d(id, f).$$

Then ρ is a pseudo-norm. Such a pseudo-norm is called a Finsler pseudo-norm, and the corresponding pseudo-metric is called a Finsler pseudo-metric.

The adjoint action of Lie group $\text{Ham}(M, \omega)$ on its Lie algebra $\mathcal{A}$ is the standard action of diffeomorphisms on functions, i.e.

$$Ad_f G = G \circ f^{-1}$$

for all $G \in \mathcal{A}$ and $f \in \text{Ham}(M, \omega)$. We say a norm $|| \cdot ||$ on $\mathcal{A}$ is $\text{Ham}(M, \omega)$-invariant if $|| \cdot ||$ is invariant under the adjoint action of $\text{Ham}(M, \omega)$, i.e.

$$||G \circ f^{-1}|| = ||G||$$

for all $G \in \mathcal{A}$ and $f \in \text{Ham}(M, \omega)$. Note that $|| \cdot ||$ is $\text{Ham}(M, \omega)$-invariant implies that the induced Finsler pseudo-norm is bi-invariant.

As mentioned in Section 2.3, it is highly non-trivial to check whether such a pseudo-norm ρ is non-degenerate. When it is, ρ will be called a Finsler norm, and $\rho(f)$ will be referred to as the Finsler norm of f. On one hand, it is now well known that the L_∞ norm on $\mathcal{A}$ gives rise to a nondegenerate bi-invariant norm known as the Hofer norm on $\mathrm{Ham}(M, \omega)$. On the other hand, Eliashberg and Polterovich showed in [3] that for $1 \leqslant p < \infty$, the Finsler pseudo-norm on $\mathrm{Ham}(M, \omega)$ induced by the L_p norm on $\mathcal{A}$ vanishes identically. Thus the following question arises in [3] and [22].

Question 3.1.1. *Which invariant norms on $\mathcal{A}$ give rise to genuine bi-invariant Finsler metrics? Is it true that such norms are always bounded below by $C||\cdot||_\infty$?*

This question was studied by Ostrover and Wagner in [20]. One of their main results is the following

Theorem 3.1.2 (Theorem 1.3 [20]). *Let $||\cdot||$ be a $\mathrm{Ham}(M, \omega)$-invariant norm on $\mathcal{A}$ such that $||\cdot|| \leqslant C||\cdot||_\infty$ for some constant C, but the two norms are not equivalent. Then the associated pseudo-distance function on $\mathrm{Ham}(M, \omega)$ vanishes identically.*

In general, Question 3.1.1 is still open, although the above theorem suggests that the answer to the second question is likely to be positive. If this was the case, it would imply that all bi-invariant Finsler norms are bounded below by a constant multiple of the Hofer norm. Therefore all the nonextension results concerning the existence of unbounded symplectomorphisms would still be valid for all bi-invariant Finsler norms on $\mathrm{Ham}(M, \omega)$. However, since all of these questions are not completely understood yet at this time, we find it interesting to have some kind of nonextension result for general Finsler norms. In particular, we consider Finsler norms induced by χ-invariant norms which we shall define in the next section.

3.2 χ-invariant Finsler norms

Definition 3.2.1. *Let H be a normalized Hamiltonian function in $\mathcal{A}$. The characteristic function $\chi_H : \mathbb{R} \to [0,1]$ of H is defined by*

$$\chi_H(c) := \frac{\mathrm{vol}(\{p \in M \mid H(p) < c\}, \omega)}{\mathrm{vol}(M, \omega)}.$$

Definition 3.2.2. *For $F, G \in \mathcal{A}$, we say F is χ-equivalent to G if $\chi_F = \chi_G$.*

For instance, if $F = G \circ \phi$ for some volume preserving diffeomorphism ϕ, then F and G are χ-equivalent. Also, let $\pi : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ be a covering map of $\mathbb{T}^{2n}$ over itself, and H be a smooth Hamiltonian function on $\mathbb{T}^{2n}$. Then H is χ-equivalent to $H \circ \pi$.

Definition 3.2.3. *A norm $||\cdot||$ on $\mathcal{A}$ is said to be χ-invariant if all χ-equivalent Hamiltonian functions have the same norm, i.e. $||F|| = ||G||$ if $\chi_F = \chi_G$.*

For example, L_p norm and L_∞ norm are χ-invariant. Observe that a χ-invariant norm $||\cdot||$ on $\mathcal{A}$ is necessarily $\mathrm{Ham}(M,\omega)$-invariant. Hence the induced Finsler norm ρ on $\mathrm{Ham}(M,\omega)$ must be bi-invariant. The following proposition, which follows from a result by Ostrover and Wagner [20], explains why the χ-invariance hypothesis in Theorem 1.3.1 on the norm $||\cdot||$ is of interest.

Proposition 3.2.4. *Any* $\mathrm{Ham}(M, \omega)$*-invariant norm* $||\cdot||$ *on* $\mathcal{A}$ *which is bounded from above by* $||\cdot||_\infty$ *is* χ*-invariant.*

Proof. It is proved in [20] (Theorem 1.4) that such a norm $||\cdot||$ can be extended to a (semi)norm $||\cdot|| \leqslant C||\cdot||_\infty$ on $L_\infty(M)$ which is invariant under all measure preserving bijections on M.

Let F, $G \in \mathcal{A}$ such that $\chi_F = \chi_G$. Then there exist two sequences of step functions $F_n, G_n \in L_\infty(M)$ converging to F and G respectively in L_∞ norm, and a sequence of measure preserving bijections ϕ_n on M such that for each n, F_n coincides with $G_n \circ \phi_n$ outside of a set of measure zero. Thus we have $||F_n|| = ||G_n \circ \phi_n|| = ||G_n||$. The first equality holds because F_n and $G_n \circ \phi_n$ only differ at a measure zero set and $||\cdot|| \leqslant ||\cdot||_\infty$; the second one holds since $||\cdot||$ is invariant under measure preserving bijections. Note that it also follows from $||\cdot|| \leqslant ||\cdot||_\infty$ that $||F_n|| \to ||F||$ and $||G_n|| \to ||G||$. Therefore we conclude that $||F|| = ||G||$, which completes the proof. □

The main result of this chapter is Theorem 1.3.1. In the coming sections, we will give a proof to this theorem and discuss its possible generalizations.

3.3 A generalization of Theorem 1.3.1

In this section, we formulate and prove Theorem 3.3.1, a generalization of Theorem 1.3.1 which works for all (not necessarily Finsler) bi-invariant norms.

Then we use it to prove Theorem 1.3.1 in the next section.

Let π be the symplectic covering map

$$\pi : (\mathbb{T}^{2n}, 2\omega) \longrightarrow (\mathbb{T}^{2n}, \omega),\ (x, y) \mapsto (x, 2y)$$

where $(x, y) := (x_1, \cdots, x_n; y_1, \cdots, y_n)$, and

$$\omega := dx \wedge dy = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n.$$

Theorem 3.3.1. *Let $(\mathbb{T}^{2n}, \omega)$ be the torus with the standard symplectic form ω, and ρ be a bi-invariant norm on $\mathrm{Ham}(\mathbb{T}^{2n}, \omega)$. Assume that $\exists$ some $\lambda > 1$ s.t. $\rho(g) \geqslant \lambda\rho(\widetilde{g})$ for all $g \in \mathrm{Ham}(\mathbb{T}^{2n}, \omega)$, where $\widetilde{g}$ is the admissible lift of g with respect to the covering map π defined above. Then ρ does not extend to a bi-invariant norm on $\mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$.*

Remark 3.3.2. For all $g \in \mathrm{Ham}(\mathbb{T}^{2n}, \omega)$, the admissible lift $\widetilde{g}$ of g with respect to π is by definition an element in $\mathrm{Ham}(\mathbb{T}^{2n}, 2\omega)$. Since $\mathrm{Ham}(\mathbb{T}^{2n}, 2\omega) = \mathrm{Ham}(\mathbb{T}^{2n}, \omega)$ as sets, one can think of $\widetilde{g}$ as an element in $\mathrm{Ham}(\mathbb{T}^{2n}, \omega)$. Thus it makes sense to talk about the norm $\rho(\widetilde{g})$ of $\widetilde{g}$.

Remark 3.3.3. The above theorem should also apply to the spectral norm γ on $\mathrm{Ham}(\mathbb{T}^{2n}, \omega)$ such that

$$\gamma(g) := c([1]; g) - c([\omega^n]; g)$$

for all $g \in \mathrm{Ham}(\mathbb{T}^{2n}, \omega)$. Here $c([1]; \cdot)$ and $c([\omega^n]; \cdot)$ denote the section of the action spectrum bundle over $\mathrm{Ham}(\mathbb{T}^{2n}, \omega)$ associated to the cohomology classes

$[1] \in H^0(\mathbb{T}^{2n})$ and $[\omega^n] \in H^{2n}(\mathbb{T}^{2n})$ respectively. For details, the readers are referred to Schwarz [25] for the case of symplectically aspherical manifolds, and to Oh [19] for general symplectic manifolds. We expect to prove that γ does not extend to a bi-invariant norm on $\mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$ by showing that γ satisfies the hypothesis of Theorem 3.3.1. This will be studied elsewhere.

We shall recall the following notion of displacement energy which will be used in the final step of the proof of Theorem 3.3.1. Let ρ be any bi-invariant pseudo-norm on $\mathrm{Ham}(M, \omega)$. For each subset U of M, recall that its displacement energy with respect to ρ is defined to be

$$de(U, \rho) := \inf \{ \rho(f) \mid f \in \mathrm{Ham}(M, \omega),\ f(U) \cap U = \emptyset \}.$$

If the set of such f is empty, we say $de(U, \rho) = \infty$. The following result is due to Eliashberg and Polterovich [3].

Theorem 3.3.4 (Theorem 1.3.A [3]). *Let ρ be a bi-invariant pseudo-norm on $\mathrm{Ham}(M, \omega)$. Then ρ is nondegenerate if and only if $de(U, \rho) > 0$ for every non-empty open subset U.*

We refer to [3] for the proof. One can also find the same argument in the proof of Lemma 4.2.2 below which is an analogy of the above theorem for bi-invariant norms on $\mathrm{Symp}_0(M, \omega)$.

Proof of Theorem 3.3.1. Let ρ be any bi-invariant norm on $\mathrm{Ham}(\mathbb{T}^{2n}, \omega)$. As for the Hofer norm in Definition 1.1.1, one can define bounded and unbounded symplectomorphisms with respect to ρ. More precisely, for each

$\phi \in \mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$, we define

$$r_\rho(\phi) := \sup \{ \rho([\phi, f]) \mid f \in \mathrm{Ham}(\mathbb{T}^{2n}, \omega)\},$$

where $[\phi, f] := \phi f \phi^{-1} f^{-1}$ is the commutator of ϕ and f. We say ϕ is bounded with respect to ρ if $r_\rho(\phi) < \infty$, and unbounded otherwise.

If ρ can extend to a bi-invariant norm on $\mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$, still denoted by ρ, then given $\phi \in \mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$, we have $\rho([\phi, f]) \leqslant 2\rho(\phi)$ for all $f \in \mathrm{Ham}(\mathbb{T}^{2n}, \omega)$. This implies that all elements $\phi \in \mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$ are bounded with respect to ρ. Thus as in Theorem 1.2.1, to show ρ does not extend to a bi-invariant norm on $\mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$, it suffices to find some symplectomorphism unbounded with respect to ρ.

Let $\phi \in \mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$ be the halfway rotation of $\mathbb{T}^{2n}$ along x_1-axis, i.e.

$$\phi(x_1, \cdots, x_n; y_1, \cdots, y_n) = (x_1 + \frac{1}{2}, \cdots, x_n; y_1, \cdots, y_n).$$

We want to show that ϕ is unbounded with respect to any bi-invariant norm ρ satisfying the hypothesis of Theorem 3.3.1. For this, we denote by V the subset of $\mathbb{T}^{2n}$ defined by $\{|x_1| < \frac{1}{4}\}$ which is obviously displaced by ϕ. It is easy to construct a smooth family $f_s \in \mathrm{Ham}(\mathbb{T}^{2n}, \omega)$ supported in V such that the restriction of f_s to the subset $\{|x_1| < \frac{1}{8}\}$ is defined by

$$f_s(x_1, \cdots, x_n; y_1, \cdots, y_n) = (x_1, \cdots, x_n; y_1 + s, \cdots, y_n).$$

Denote by g_s the commutator of ϕ and f_s, i.e. $g_s := [\phi, f_s] = \phi f_s \phi^{-1} f_s^{-1}$. In order to prove ϕ is unbounded with respect to ρ, it suffices to show that

$$\limsup_{s\to\infty} \rho(g_s) = \infty.$$

For this, we need to consider the admissible lift $\widetilde{g}_s$ of g_s. Let $m = 2^k$ for some positive integer k. Consider the symplectic covering map

$$\pi_m : (\mathbb{T}^{2n}, m\omega) \to (\mathbb{T}^{2n}, \omega),\ (x, y) \mapsto (x, my).$$

For each $g_s \in \mathrm{Ham}(\mathbb{T}^{2n}, \omega)$ constructed above, denote by $\widetilde{g}_s \in \mathrm{Ham}(\mathbb{T}^{2n}, m\omega)$ the unique admissible lift of g_s with respect to π_m. Since $\mathrm{Ham}(\mathbb{T}^{2n}, m\omega) = \mathrm{Ham}(\mathbb{T}^{2n}, \omega)$, as in Remark 3.3.2, we think of $\widetilde{g}_s$ as an element in $\mathrm{Ham}(\mathbb{T}^{2n}, \omega)$. Thus the norm $\rho(\widetilde{g}_s)$ makes sense.

It follows from the definition of g_s and $\widetilde{g}_s$ that the restriction of $\widetilde{g}_s$ to the subset $\{|x_1| < \frac{1}{8}\}$ is defined by

$$\widetilde{g}_s(x_1, \cdots, x_n; y_1, \cdots, y_n) = (x_1, \cdots, x_n; y_1 - \frac{s}{m}, \cdots, y_n).$$

Thus when $\frac{m}{4} < s < \frac{m}{2}$, $\widetilde{g}_s$ displaces an open subset $U \subset \mathbb{T}^{2n}$ defined by

$$U := \{(x_1, \cdots, x_n; y_1, \cdots, y_n) \in \mathbb{T}^{2n} \mid -\frac{1}{8} < x_1 < \frac{1}{8},\ 0 < y_1 < \frac{1}{4}\}.$$

Using the definition of the displacement energy, we get $\rho(\widetilde{g}_s) \geqslant de(U, \rho) > 0$. The second inequality holds because of Theorem 3.3.4.

Since $\pi_m = \pi^k$ where π is the covering map in the theorem, it follows from the hypothesis on ρ that $\rho(g_s) \geqslant \lambda^k \rho(\widetilde{g}_s)$. Here $\lambda > 1$ is the same constant as in the theorem. We conclude that $\rho(g_s)$ can be arbitrarily large when k is arbitrarily large. This completes the proof of Theorem 3.3.1. □

Remark 3.3.5. The choice of the symplectic covering map π in Theorem 3.3.1 is a very subtle question. One might expect the same result to hold when choosing different covering maps. This is the case for all

$$\pi_m : (\mathbb{T}^{2n}, m\omega) \longrightarrow (\mathbb{T}^{2n}, \omega),\ (x, y) \mapsto (x, my)$$

with $m \geqslant 2$ a positive integer. The proof goes exactly as in the $m = 2$ case above. On the other hand, the argument breaks down if we choose for instance, the covering map

$$p : (\mathbb{T}^{2n}, 4\omega) \longrightarrow (\mathbb{T}^{2n}, \omega),\ (x, y) \mapsto (2x, 2y).$$

The reason is as follows.

In view of Theorem 3.3.4, the displacement energy $de(U, \rho)$ of any open subset $U \subset \mathbb{T}^{2n}$ with respect to any bi-invariant norm ρ is always positive. This is sufficient for the proof of Theorem 3.3.1 since we have been able to show that, by carefully choosing s and m, the admissible lift of g_s with respect to $\pi_m = \pi^k$ displaces some fixed subset U of $\mathbb{T}^{2n}$ despite of the rescaling (enlarging the symplectic form) of the torus. One the other hand, for the covering map p, the admissible lift of g_s with respect to $p_m = p^k$ can only be arranged to displace a shrinking portion U_m of $\mathbb{T}^{2n}$. One would still be able to get the same result by carefully analyzing the effect on the capacity of the rescaling process if as for the Hofer norm, the energy-capacity inequality

$$de(U, \rho) \geqslant c \cdot \text{capacity}(U, \omega)$$

holds for our bi-invariant norm ρ. However, we do not know if this is true, nor do we have any counter-examples. It would be interesting to have an answer in either direction.

3.4 Proof of Theorem 1.3.1

Now back to Theorem 1.3.1 for χ-invariant Finsler norms. We begin with a remark on the χ-invariance hypothesis.

Remark 3.4.1. We have already mentioned that a χ-invariant norm $||\cdot||$ is necessarily $\text{Ham}(\mathbb{T}^{2n}, \omega)$-invariant. Hence a χ-invariant Finsler norm ρ must be bi-invariant. Moreover, for any χ-invariant norm, $||H|| = ||H \circ \pi||$ for all $H \in \mathcal{A}$, where $\pi : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ is any covering map of $\mathbb{T}^{2n}$ over itself. The latter will be crucial for the proof of Theorem 1.3.1.

The following lemma gives a relation between the Finsler norms of a Hamiltonian diffeomorphism and its admissible lift.

Lemma 3.4.2. *Let $\pi : (\widetilde{M}, \widetilde{\omega}) \to (M, \omega)$ be a covering map such that $\widetilde{\omega} = \pi^*\omega$. Let $\widetilde{||\cdot||}$ and $||\cdot||$ be norms on $\widetilde{\mathcal{A}}$ and $\mathcal{A}$ respectively, such that $\widetilde{||H \circ \pi||} = ||H||$. Denote by $\widetilde{\rho}$ and ρ the induced Finsler pseudo-norms on $\text{Ham}(\widetilde{M}, \widetilde{\omega})$ and $\text{Ham}(M, \omega)$ respectively. Then $\rho(g) \geqslant \widetilde{\rho}(\widetilde{g})$, where $\widetilde{g} \in \text{Ham}(\widetilde{M}, \widetilde{\omega})$ is the admissible lift of $g \in \text{Ham}(M, \omega)$ with respect to π.*

Proof. By the definition of the admissible lift, if $g \in \text{Ham}(M, \omega)$ is the time-1 map of the Hamiltonian flow generated by time-dependent Hamiltonian function $H_t \in \mathcal{A}$, then $\widetilde{g} \in \text{Ham}(\widetilde{M}, \widetilde{\omega})$ is the time-1 map of the Hamiltonian flow generated by $H_t \circ \pi \in \widetilde{\mathcal{A}}$. Now $||H|| = \widetilde{||H \circ \pi||}$, by taking the infimum we get $\rho(g) \geqslant \widetilde{\rho}(\widetilde{g})$ since the first infimum is taken on a smaller set. □

Remark 3.4.3. For instance, Lemma 3.4.2 applies when both $\widetilde{||\cdot||}$ and $||\cdot||$ are L_∞ norms. This implies $\rho(g) \geqslant \rho(\widetilde{g})$ where $\widetilde{g}$ is the admissible lift of g and the two ρ's are both Hofer norms.

We continue with the following lemma. We will use it to deduce Theorem 1.3.1 from Theorem 3.3.1.

Lemma 3.4.4. *Let ρ be a Finsler norm on $\text{Ham}(\mathbb{T}^{2n}, \omega)$ induced by a χ-invariant norm $||\cdot||$. Then $\rho(g) \geqslant 2\rho(\widetilde{g})$ for all $g \in \text{Ham}(\mathbb{T}^{2n}, \omega)$ and the admissible lift $\widetilde{g}$ of g with respect to the symplectic covering map $\pi : (\mathbb{T}^{2n}, 2\omega) \to (\mathbb{T}^{2n}, \omega)$ such that $\pi(x, y) = (x, 2y)$.*

Proof. In view of Remark 3.3.2, $\text{Ham}(\mathbb{T}^{2n}, 2\omega) = \text{Ham}(\mathbb{T}^{2n}, \omega)$, so they share the same Lie algebra $\mathcal{A}$. Let $\widetilde{\rho}$ and ρ be the Finsler norms on $\text{Ham}(\mathbb{T}^{2n}, 2\omega)$ and $\text{Ham}(\mathbb{T}^{2n}, \omega)$ respectively, but both are induced by the same norm $||\cdot||$ on $\mathcal{A}$. We claim that $\widetilde{\rho} = 2\rho$. In fact, an element $g \in \text{Ham}(\mathbb{T}^{2n}, \omega)$ is the time-1 map of the Hamiltonian flow generated by some time-dependent Hamiltonian function H_t if and only if the same g considered as an element in $\text{Ham}(\mathbb{T}^{2n}, 2\omega)$ is the time-1 map of the Hamiltonian flow generated by $2H_t$. Taking the infimum gives the equality $\widetilde{\rho}(g) = 2\rho(g)$ for all g.

On the other hand, $||\cdot||$ is χ-invariant implies $||H \circ \pi|| = ||H||$ for all $H \in \mathcal{A}$, where π is the covering map in the lemma. Lemma 3.4.2 implies that $\rho(g) \geqslant \widetilde{\rho}(\widetilde{g})$. Combining this with the previous equality $\widetilde{\rho} = 2\rho$, we obtain $\rho(g) \geqslant \widetilde{\rho}(\widetilde{g}) \geqslant 2\rho(\widetilde{g})$ as desired. □

Proof of Theorem 1.3.1. It follows from the above lemma that any χ-invariant Finsler norm ρ on $\mathrm{Ham}(\mathbb{T}^{2n}, \omega)$ must satisfy the hypothesis of Theorem 3.3.1 with the constant $\lambda = 2$. Thus Theorem 1.3.1 follows. □

3.5 Further remarks

In this section, we try to see how we can push Theorem 3.3.1 to other symplectic manifolds such as $(\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$. Here $(\mathbb{T}^{2n}, \omega)$ is the torus with the standard symplectic form, and (M, σ) is any closed symplectic manifold.

Let $\pi : (\mathbb{T}^{2n} \times M, 2\omega \oplus \sigma) \to (\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$, $(x, y, p) \mapsto (x, 2y, p)$ be a symplectic covering map, i.e. $\pi^*(\omega \oplus \sigma) = 2\omega \oplus \sigma$. Thus for every $g \in \mathrm{Ham}(\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$, one can define the admissible lift $\widetilde{g} \in \mathrm{Ham}(\mathbb{T}^{2n} \times M, 2\omega \oplus \sigma)$ with respect to π. Let ρ be a bi-invariant norm on $\mathrm{Ham}(\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$. Note that $\mathrm{Ham}(\mathbb{T}^{2n} \times M, 2\omega \oplus \sigma) \neq \mathrm{Ham}(\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$. Therefore $\rho(\widetilde{g})$ is not defined for all $\widetilde{g}$. Hence the assumption $\rho(g) \geqslant \lambda\rho(\widetilde{g})$ for all g and $\widetilde{g}$ in Theorem 3.3.1 makes no sense in this context.

However, both groups $\mathrm{Ham}(\mathbb{T}^{2n} \times M, 2\omega \oplus \sigma)$ and $\mathrm{Ham}(\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$ contain the product Hamiltonian diffeomorphisms $f \times g$ where $f \in \mathrm{Ham}(\mathbb{T}^{2n}, \omega) = \mathrm{Ham}(\mathbb{T}^{2n}, 2\omega)$ and $g \in \mathrm{Ham}(M, \sigma)$, and the admissible lift $\widetilde{f \times g} = \widetilde{f} \times g$ of

$f \times g$ with respect to π is also a product Hamiltonian diffeomorphism. So we can think of $\widetilde{f \times g}$ as an element in $\mathrm{Ham}(\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$. Thus the norm $\rho(\widetilde{f \times g})$ does make sense.

The following theorem slightly generalizes Theorem 3.3.1.

Theorem 3.5.1. *Let ρ be a bi-invariant norm on $\mathrm{Ham}(\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$. Assume that $\exists\ \lambda > 1$ s.t. $\rho(f \times id) \geqslant \lambda \rho(\widetilde{f \times id})$ for all $f \times id \in \mathrm{Ham}(\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$ and $\widetilde{f \times id}$ the admissible lift of $f \times id$ with respect to the covering map π described above. Then ρ does not extend to a bi-invariant norm on $\mathrm{Symp}_0(\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$.*

Proof. The proof follows the same lines as that of Theorem 3.3.1. Let ϕ, f_s and $g_s := [\phi, f_s]$ be those maps as in the proof of Theorem 3.3.1. One has to show that $\phi \times id$ is unbounded with respect to ρ satisfying the hypothesis in this theorem. It suffices to show that $\rho(g_s \times id) = \rho([\phi \times id, f_s \times id])$ can be arbitrarily large. This can easily be achieved by considering the admissible lift of $g_s \times id$ as in Theorem 3.3.1. □

Remark 3.5.2. Theorem 3.3.1 is a special case of Theorem 3.5.1 with M being a point. We have already seen that Theorem 3.3.1 can be applied to χ-invariant Finsler norms including the Hofer norm. On the other hand, we are not able to find any application of Theorem 3.5.1 since it is very hard to check its hypothesis. In particular, we do not know if the hypothesis will be satisfied by the Hofer norm. In view of Remark 1.2.4, however, we already know that the Hofer norm on $\mathrm{Ham}(\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$ does not extend bi-invariantly to $\mathrm{Symp}_0(\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$.

Chapter 4

Bi-invariant norms on $\mathrm{Symp}_0(M, \omega)$

In this chapter, we first introduce the notion of C^k-continuous norms on $\mathrm{Ham}(M, \omega)$ or $\mathrm{Symp}_0(M, \omega)$. Then we will study the Hofer norm and other bi-invariant norms in this context. One of the main results is Theorem 4.2.1, which states that there exists no C^1-continuous bi-invariant norm on $\mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$. We also construct two families of bi-invariant norms on $\mathrm{Symp}_0(M, \omega)$ and study their topological properties.

4.1 C^k-continuous norms

We begin with the definition of C^k-continuous bi-invariant norms. Here we use the term ρ-topology for the topology induced by the norm ρ.

Definition 4.1.1. *Let $k = 0$ or 1, G be either* $\mathrm{Ham}(M, \omega)$ *or* $\mathrm{Symp}_0(M, \omega)$. *A bi-invariant norm ρ on G is said to be C^k-continuous if the C^k-topology on*

G is finer than ρ-topology, or in other words, if the identity map on G

$$\mathrm{Id}_G : (G, C^k\text{-}topology) \to (G, \rho\text{-}topology)$$

is a continuous map.

The following proposition is trivial from the definition of the Hofer norm.

Proposition 4.1.2. *For all (M, ω), the Hofer norm on $\mathrm{Ham}(M, \omega)$ is C^1-continuous.*

In general, the Hofer norm is not C^0-continuous. This seems obvious since the definition of the Hofer length of a smooth path uses the derivative of the path. To produce a counter-example, however, one is forced to use the following deep result by Polterovich [23].

Theorem 4.1.3 (Theorem 1.A [23]). *Let $L \subset S^2$ be an equator of the standard 2-sphere S^2 with area form ω. Let $f \in \mathrm{Ham}(S^2, \omega)$ be the time-1 map of the flow generated by a time-dependent Hamiltonian function F_t. Assume there exists $c > 0$ such that $F(x, t) \geqslant c$ for all $x \in L$ and $t \in [0, 1]$. Then the Hofer norm $\rho(f)$ satisfies $\rho(f) \geqslant c$.*

Using the above theorem, Polterovich was able to prove the following proposition in [24]. We include the proof below for the sake of completeness.

Proposition 4.1.4 (Polterovich [24]). *The Hofer norm on $\mathrm{Ham}(S^2, \omega)$ is not C^0-continuous.*

Proof. For any $c > 0$, take a sequence of autonomous Hamiltonian functions $F_n \in \mathcal{A}$ such that $F_n = c$ except for a small disc $B_n \subset S^2$ whose diameter goes to 0 as n goes to infinity. Let $f_n \in \mathrm{Ham}(S^2, \omega)$ be the time-1 map of the Hamiltonian flow generated by F_n. It follows from Theorem 4.1.3 that the Hofer norm $\rho(f_n) \geqslant c$. On the other hand, the C^0-limit of the sequence f_n is the identity map $id \in \mathrm{Ham}(S^2, \omega)$ with $\rho(id) = 0$. This shows ρ is not C^0-continuous. □

Remark 4.1.5. In view of [22] Theorem 7.2.C, the above construction works also for closed surfaces Σ of genus > 0 where any non-contractible closed curve L in Σ plays the same role as the equator in S^2. It also holds for $S^2 \times S^2$ with the split symplectic form $\omega \oplus \omega$ and $\mathbb{C}P^n$ endowed with the Fubini-Study form, using the Calabi quasimorphism constructed by Entov and Polterovich on the Hamiltonian diffeomorphism group of these manifolds (cf. Remark 1.10 [4]).

4.2 A result on C^1-continuous norms

We have seen in Proposition 4.1.2 that C^1-continuous bi-invariant norms such as the Hofer norm always exist on $\mathrm{Ham}(M, \omega)$ for all (M, ω). However, this is not true in general for $\mathrm{Symp}_0(M, \omega)$. In particular, we have

Theorem 4.2.1. *For the standard torus there exists no C^1-continuous bi-invariant norm on* $\mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$.

To prove this theorem, we need the following lemma which is analogous to Theorem 3.3.4 (Theorem 1.3.A [3]). Let ρ be any bi-invariant pseudo-norm on $\mathrm{Symp}_0(M, \omega)$. For each subset U of M, we define its symplectic displacement energy with respect to ρ

$$de^s(U,\rho) := \inf \{ \rho(\phi) \mid \phi \in \mathrm{Symp}_0(M,\omega),\ \phi(U) \cap U = \emptyset \}.$$

If the set of such ϕ is empty, we say $de^s(U,\rho) = \infty$.

Lemma 4.2.2. *A bi-invariant pseudo-norm ρ on $\mathrm{Symp}_0(M,\omega)$ is nondegenerate if and only if $de^s(U,\rho) > 0$ for every non-empty open subset U.*

Proof. Our argument goes along the same lines as that of Theorem 1.3.A in [3]. Assume $de^s(U,\rho) > 0$ for all non-empty open subsets U. Since each nonidentity map $\phi \in \mathrm{Symp}_0(M,\omega)$ must displace some small ball $U \subset M$, we get that $\rho(\phi) \geqslant de^s(U,\rho) > 0$. For the converse, note that for any non-empty open set $U \subset M$, there exist $\phi, \psi \in \mathrm{Symp}_0(M,\omega)$ supported in U such that $[\phi,\psi] \neq id$. Since ρ is nondegenerate, $\rho([\phi,\psi]) > 0$. To complete our argument, it suffices to prove the following claim.

Claim: Let U be a non-empty open subset of M. For all $\phi, \psi \in \mathrm{Symp}_0(M,\omega)$ supported in U, $de^s(U,\rho) \geqslant \frac{1}{4}\rho([\phi,\psi])$.

For the argument of the claim, assume there exists $\eta \in \mathrm{Symp}_0(M,\omega)$ such that $\eta(U) \cap U = \emptyset$ (if such an η does not exist we are done because $de^s(U,\rho) = \infty$). Set

$$\theta := [\phi,\eta] = \phi\eta\phi^{-1}\eta^{-1}.$$

Using the fact that η displaces U and that ϕ, ψ are supported in U, one can easily verify that $[\phi,\psi] = [\theta,\psi]$. Therefore we get

$$\rho([\phi,\psi]) = \rho([\theta,\psi]) \leqslant 2\rho(\theta) = 2\rho([\phi,\eta]) \leqslant 4\rho(\eta).$$

Here we have used the bi-invariance of ρ and the triangle inequality. Since this holds for all $\eta \in \mathrm{Symp}_0(M, \omega)$ with $\eta(U) \cap U = \emptyset$, we obtain

$$de^s(U, \rho) \geqslant \frac{1}{4}\rho([\phi, \psi])$$

by taking the infimum over all such η's. □

Proof of Theorem 4.2.1. Let $\phi_\alpha : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ be the maps such that

$$\phi_\alpha(x_1, \cdots, x_n; y_1, \cdots, y_n) = (x_1 + \alpha, \cdots, x_n; y_1, \cdots, y_n).$$

Claim : For each $0 < \alpha < \frac{1}{8}$, there exists $\psi_\alpha \in \mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$ such that the conjugate $\psi_\alpha \phi_\alpha \psi_\alpha^{-1}$ of ϕ_α will displace an open set U of $\mathbb{T}^{2n}$ which is independent of α.

Assume the claim to be true for the moment. For any bi-invariant norm ρ on $\mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$, we have

$$\rho(\phi_\alpha) = \rho(\psi_\alpha \phi_\alpha \psi_\alpha^{-1}) \geqslant de(U, \rho),$$

where the last term $de(U, \rho)$ is a positive number by Lemma 4.2.2. Since the C^1-limit of ϕ_α as α approaches 0 is the identity map $id \in \mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$, we conclude that ρ is not C^1-continuous.

It suffices to prove the claim by direct construction. For each $0 < \alpha < \frac{1}{8}$, let $h_\alpha : S^1 \to \mathbb{R}$ be a smooth function such that for $\frac{1}{4} < x < \frac{3}{4}$,

$$h_\alpha(x + \alpha) - h_\alpha(x) = \frac{1}{2}.$$

Define $\psi_\alpha : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ such that

$$\psi_\alpha(x_1, \cdots, x_n; y_1, \cdots, y_n) = (x_1, \cdots, x_n; y_1 + h_\alpha(x_1), \cdots, y_n).$$

It is obvious that $\psi_\alpha \in \mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$. It also follows that $\psi_\alpha \phi_\alpha \psi_\alpha^{-1}$ maps $(x_1, \cdots, x_n; y_1, \cdots, y_n)$ to $(x_1 + \alpha, \cdots, x_n; y_1 + h_\alpha(x_1 + \alpha) - h_\alpha(x_1), \cdots, y_n)$. Thus, $\psi_\alpha \phi_\alpha \psi_\alpha^{-1}$ displaces an open set $U \subset \mathbb{T}^{2n}$ defined by

$$U = \{(x_1, \cdots, x_n; y_1, \cdots, y_n) \in \mathbb{T}^{2n} \mid \frac{1}{4} < x_1 < \frac{3}{4}, 0 < y_1 < \frac{1}{4}\}.$$

This completes the proof of the claim, hence the theorem. □

Remark 4.2.3. Theorem 4.2.1 can be generalized to $(\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$. That is, there exists no C^1-continuous bi-invariant norm on $\mathrm{Symp}_0(\mathbb{T}^{2n} \times M, \omega \oplus \sigma)$, where $(\mathbb{T}^{2n}, \omega)$ is the standard torus and (M, σ) is any closed symplectic manifold. This is true since one can show, as in the proof of Theorem 4.2.1, that the conjugate of $\phi_\alpha \times id$ will displace some fixed subset $U \times M$ of $\mathbb{T}^{2n} \times M$. Here ϕ_α denote the same rotation maps of $\mathbb{T}^{2n}$ as above.

4.3 Bi-invariant norms on $\mathrm{Symp}_0(M, \omega)$

In this section we give two explicit constructions of bi-invariant norms on $\mathrm{Symp}_0(M, \omega)$ and discuss their topological properties. In both constructions, we only consider closed symplectic manifolds (M, ω), and ρ stands for the Hofer norm on $\mathrm{Ham}(M, \omega)$. The first construction is due to Lalonde and Polterovich

[14]. For every positive number a, we define $r_a : \mathrm{Symp}_0(M,\omega) \to \mathbb{R}$ such that for all $\phi \in \mathrm{Symp}_0(M,\omega)$,

$$r_a(\phi) := \sup\{\, \rho([\phi, f]) \mid f \in \mathrm{Ham}(M,\omega),\ \rho(f) \leqslant a\},$$

where $[\phi, f] := \phi f \phi^{-1} f^{-1}$ is the commutator of ϕ and f.

Proposition 4.3.1 (Prop 1.2.A [14]). *For every $a \in (0,\infty)$, the function r_a is a bi-invariant norm on* $\mathrm{Symp}_0(M,\omega)$.

For the second construction, let $K > 0$. Define $\rho_K : \mathrm{Symp}_0(M,\omega) \to \mathbb{R}$ such that for all $\phi \in \mathrm{Symp}_0(M,\omega)$,

$$\rho_K(\phi) := \begin{cases} \min(\rho(\phi), K), & \text{if } \phi \in \mathrm{Ham}(M,\omega), \\ K, & \text{otherwise.} \end{cases}$$

Proposition 4.3.2. *For every $K \in (0,\infty)$, the function ρ_K is a bi-invariant norm on* $\mathrm{Symp}_0(M,\omega)$.

The proofs of both propositions are straightforward and therefore omitted.

Remark 4.3.3. r_a and ρ_K restrict to bi-invariant norms on $\mathrm{Ham}(M,\omega)$. One can think of r_a and ρ_K as bi-invariant extensions of their corresponding norms on $\mathrm{Ham}(M,\omega)$. Note that the diameter of $\mathrm{Ham}(M,\omega)$ with respect to these norms is finite, so one can always extend them bi-invariantly to $\mathrm{Symp}_0(M,\omega)$ by giving a sufficiently large constant value for all nonHamiltonian symplectomorphisms. Compare this with Question 1.3.2.

For the properties concerning these norms, first we shall see that for any (M,ω) such that $\mathrm{Symp}_0(M,\omega)$ is not identical to $\mathrm{Ham}(M,\omega)$, ρ_K is not C^1-continuous on $\mathrm{Symp}_0(M,\omega)$. This is true since $\rho_K(\phi) = K$ for all nonHamiltonian symplectomorphisms ϕ and $\rho_K(id) = 0$. We do not know as much for bi-invariant norms r_a. However, we do know that r_a is not C^1-continuous on $\mathrm{Symp}_0(\mathbb{T}^{2n},\omega)$ for the standard torus $(\mathbb{T}^{2n},\omega)$ in view of Theorem 4.2.1. This can also be proved directly, based on a direct calculation that $r_a(\phi) = 2a$ for every non-identity rotation ϕ and $r_a(id) = 0$. On the other hand, the restrictions of r_a and ρ_K to $\mathrm{Ham}(M,\omega)$ are C^1-continuous, since both are bounded from above by the Hofer norm. More precisely, we have $r_a(f) \leqslant 2\rho(f)$ and $\rho_K(f) \leqslant \rho(f)$ for all $f \in \mathrm{Ham}(M,\omega)$. Since the Hofer norm is C^1-continuous according to Proposition 4.1.2, r_a and ρ_K are also.

For bi-invariant norms ρ_K, we also have the following easy result.

Proposition 4.3.4. *For each $K > 0$, the identity component of* $\mathrm{Symp}_0(M,\omega)$ *with respect to the ρ_K-topology is* $\mathrm{Ham}(M,\omega)$.

Proof. For all $f \in \mathrm{Ham}(M,\omega)$ and $\phi \notin \mathrm{Ham}(M,\omega)$, we have the distance $d(f,\phi) = \rho(\phi f^{-1}) = K$ since $\phi f^{-1} \notin \mathrm{Ham}(M,\omega)$. On the other hand, $\mathrm{Ham}(M,\omega)$ is obviously path-connected with respect to ρ_K-topology. The proposition follows immediately. □

This leads us to the following question. We content ourselves with formulating the question only in terms of the standard torus $(\mathbb{T}^{2n},\omega)$.

Question 4.3.5. Is $\mathrm{Ham}(\mathbb{T}^{2n},\omega)$ the identity component of $\mathrm{Symp}_0(\mathbb{T}^{2n},\omega)$ with respect to the r_a-topology? Is it true for all bi-invariant norms on $\mathrm{Symp}_0(\mathbb{T}^{2n},\omega)$?

Proposition 4.3.4 gives a positive answer to the above question for bi-invariant norms ρ_K. For bi-invariant norms r_a, as a partial answer, we have the following theorem which states that an r_a-continuous smooth isotopy in $\mathrm{Symp}_0(\mathbb{T}^{2n},\omega)$ must lie entirely in $\mathrm{Ham}(\mathbb{T}^{2n},\omega)$.

Theorem 4.3.6. *Let $\gamma : [0,1] \to \mathrm{Symp}_0(\mathbb{T}^{2n},\omega)$ be a smooth isotopy, i.e. a C^1-continuous path starting from id. Then γ is r_a-continuous if and only if it is a smooth isotopy in* $\mathrm{Ham}(\mathbb{T}^{2n},\omega)$.

Proof. Let γ be a smooth isotopy in $\mathrm{Ham}(\mathbb{T}^{2n},\omega)$, i.e. γ is a C^1-continuous path with $\gamma_0 = id$. We have pointed out above that the bi-invariant norm r_a, when restricted to $\mathrm{Ham}(\mathbb{T}^{2n},\omega)$, is C^1-continuous in the sense of Definition 4.1.1. Thus γ is a C^1-continuous path implies that it is also r_a-continuous.

On the other hand, we have to show that if there exists some $t_0 \in [0,1]$ such that $\gamma_{t_0} \notin \mathrm{Ham}(\mathbb{T}^{2n},\omega)$, then γ is not r_a-continuous. For each $t \in [0,1]$, we have the unique decomposition $\gamma_t = \phi_t \circ f_t$, where ϕ_t is the unique rotation of the torus such that $f_t = \phi_t^{-1} \circ \gamma_t$ is in $\mathrm{Ham}(\mathbb{T}^{2n},\omega)$. Note that $\phi_0 = f_0 = id$, and the assumption $\gamma_{t_0} \notin \mathrm{Ham}(\mathbb{T}^{2n},\omega)$ for some t_0 implies $\phi_{t_0} \neq id$. Now γ is a C^1-continuous path, so are the paths ϕ and f. Since r_a is C^1-continuous when restricted to $\mathrm{Ham}(\mathbb{T}^{2n},\omega)$, f is a C^1-continuous path in $\mathrm{Ham}(\mathbb{T}^{2n},\omega)$ implies f is also r_a-continuous. If the path γ were r_a-continuous, it would imply that the path ϕ is also. However, as we already pointed out before, for each a, r_a only assumes two values on rotations, i.e. $r_a(id) = 0$, and $r_a(\psi) = 2a$ for all nonidentity rotations ψ. Since we have $\phi_0 = id$, and $\phi_{t_0} \neq id$ for some t_0, it is not possible for the path ϕ to be r_a-continuous, which is the contradiction. The proof is therefore completed. □

Remark 4.3.7. The proof of Theorem 4.3.6 implies that for $(\mathbb{T}^{2n}, \omega)$, the distance between Hamiltonian diffeomorphisms and nonHamiltonian symplectomorphisms with respect to r_a is bounded away from 0 by some constant if the two elements are C^1-close. If this remains true when they are not C^1-close, then the answer to Question 4.3.5 would be positive for bi-invariant norms r_a. However, we do not know this yet at this time. More study concerning the topology of the symplectomorphism group with respect to bi-invariant norms will be attempted in the future.

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