Chow Motive of Fulton-MacPherson configuration spaces and wonderful compactifications

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Abstract of the Dissertation

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We study the Chow groups and the Chow motives of the so-called wonderful compactifications of arrangements of subvarieties. Given a variety Y and a "building set" $\mathcal{G}$ associated to an arrangement of subvarieties of Y, the wonderful compactification $Y_{\mathcal{G}}$ can be constructed by a sequence of blow-ups of Y along the subvarieties of the arrangement. Our main result is that the Chow motive of $Y_{\mathcal{G}}$ can be decomposed into a direct sum of the motive associated with Y and the twisted motives associated with the subvarieties of the arrangement. The decomposition obtained is canonical, for we

prove it to be independent of the order of the blow-ups. Moreover, the correspondences that give the motivic decomposition are explicitly expressed in terms of the exceptional divisors in $Y_{\mathcal{G}}$ and of the Chern classes of the normal bundles of the subvarieties of the arrangement.

In the special case of the Fulton-MacPherson configuration space $X[n]$, we prove a stronger result expressing the Chow group and the Chow motive in terms of X and n only. We provide a generating function for the Chow groups and for the Chow motive of $X[n]$. In the last chapter, we prove that the cobordism class of $X[n]$ depends only on n and on the cobordism class of X.

To my parents and my wife.

Contents

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Chapter 1

Introduction

The purpose of this thesis is to study the so-called wonderful compactifications of arrangements of subvarieties, in particular the Fulton-MacPherson configuration spaces. We focus on the decomposition of their Chow groups and of their Chow motives.

The theory of motives is built by considering algebraic cycles modulo suitable equivalence relations, e.g. rational equivalence (Chow motives), numerical equivalence (Grothendieck motives), homological equivalence.

An important idea in this theory is the motivic decomposition of the diagonal of a projective variety into pairwise orthogonal projectors. We consider the following simple example of a decomposition of the diagonal of the projective space $\mathbb{P}^n$ into pairwise orthogonal projectors (modulo rational equivalence) and the associated motivic decomposition. In §2.1.4 We shall give the definition of $h(X)$ (the Chow motive of X) and of $h(X)(i)$ (the twisted Chow motive of X). Let pt be a fixed point in $\mathbb{P}^n$. We have

$$[\Delta] = [\mathbb{P}^n \times pt] + [\mathbb{P}^{n-1} \times \mathbb{P}^1] + ... + [pt \times \mathbb{P}^n] \quad \text{(up to rational equivalence)},$$

$$h(\mathbb{P}^n) \cong h(pt) \oplus h(pt)(1) \oplus ... \oplus h(pt)(n).$$

Motivic decompositions are interesting because of the following

Principle: *A result proved for Chow motives is valid if we replace them by homological/numerical motives, Chow groups $A^*_{\mathbb{Q}}$, cohomology groups $H^*_{\mathbb{Q}}$, Grothendieck groups, Hodge structures, etc. (the aforementioned groups are taken with $\mathbb{Q}$-coefficients.)*

By this principle, the above motivic decomposition of $\mathbb{P}^n$ immediately implies the familiar cohomological decomposition

$$H^k_{\mathbb{Q}}(\mathbb{P}^n) = H^k_{\mathbb{Q}}(pt) \oplus H^{k-2}_{\mathbb{Q}}(pt) \oplus ... \oplus H^{k-2n}_{\mathbb{Q}}(pt), \quad \forall k \in \mathbb{Z}$$

as well as the analogous decompositions of the Chow groups, Hodge structures, Grothendieck groups, etc.

In [dCM02], de Cataldo and Migliorini consider the Hilbert-Chow morphism, $\pi : X^{[n]} \to X^{(n)}$, from the Hilbert scheme $X^{[n]}$ of n points on a surface X to the n-fold symmetric product $X^{(n)} := X^n/\mathbb{S}_n$ of X. They show that the motive of $X^{[n]}$ can be decomposed into motives of products of symmetric products of X (where they use a generalized notion of motives for possibly singular quotient varieties). More precisely, let $\mathfrak{P}(n)$ denote the set of partitions of n; For any $v = 1^{a_1} \cdots n^{a_n} \in \mathfrak{P}(n)$, denote by $l(v)$ its length (=$a_1 + \cdots + a_n$) and define $X^{(v)} = X^{(a_1)} \times \cdots \times X^{(a_n)}$; We have

$$h(X^{[n]}) \cong \bigoplus_{\nu \in \mathfrak{P}(n)} h(X^{(v)})(n - l(\nu)).$$

As per our principle, this motivic decomposition implies the analogous decompositions for the Chow groups, for singular cohomology, for mixed Hodge structures, for Grothendieck groups, etc. Moreover, de Cataldo and Migliorini give a Chow motive decomposition for any semismall algebraic map of complex algebraic varieties [dCM04].

Our aim is to get similar motivic decompositions for the wonderful compactifications of arrangements of subvarieties, in particular for the Fulton-MacPherson configuration spaces.

First we point out two differences between the Hilbert scheme $X^{[n]}$ and the Fulton-MacPherson scheme $X[n]$:

- The Hilbert-Chow morphism $X^{[n]} \to X^{(n)}$ is not a blow-up along a smooth center (or the composition of a sequence of blow-ups along smooth centers), thus the formula of blowup along smooth center (Theorem 2.1.5) cannot be applied; On the other hand, the formula can be applied to the wonderful compactifications in the thesis, since the compactifications can be constructed by a sequence of blow-ups along smooth centers.
- The Hilbert-Chow morphism is semismall, consequently the correspondences which give the motivic decomposition can be expressed canonically at the level of cycles. On the other hand, the morphism $X[n] \to X^n$ is not a semismall map, hence a priori the motivic decomposition of $X[n]$ depends on the order of the blow-ups and a canonical decomposition might not exist at all.

In the thesis we shall find the correspondences (on the level of rational equivalent classes of cycles) which give the motivic decomposition of a wonderful compactification (in particular $X[n]$) and show that they are independent of the order of the blow-ups, therefore the motivic decomposition is canonical.

The thesis is at first written exclusively for the Fulton-MacPherson configuration spaces $X[n]$. The motivic decomposition of $X[n]$ is established and shown to be independent of the order of blow-ups. We state the theorem with no attempt to go into details (see Theorems 4.1.1 and 4.1.2 for a precise discussion). The point is that the motive of $X[n]$ can be decomposed into a direct sum of (twisted) motives of the cartesian products of X.

Main Theorem of Fulton-MacPherson configuration spaces. *Let X be a nonsingular variety over $\mathbb{C}$. There is an isomorphism of Chow groups*

$$A^*(X[n]) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_{\mathcal{S}}} A^{*-\|\underline{\mu}\|}(X^{c(\mathcal{S})}).$$

When X is complete, there is also a natural isomorphism of Chow motives

$$h(X[n]) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_{\mathcal{S}}} h(X^{c(\mathcal{S})})(\|\underline{\mu}\|).$$

Then the author realize that the theorem can be generalized to a more general setting: the wonderful compactification of *an arrangement of subvarieties*. Here we give a brief review on this compactification.

The notion of an *arrangement of subvarieties* used in this thesis (Definition 2.3.5) is taken from [Hu03]. Briefly speaking, we consider a collection of nonsingular subvarieties whose mutual intersections satisfy certain properties.

The inspiring paper by De Concini and Procesi [DP95] gives a thorough discussion of an arrangement of linear subspaces of a vector space. Let Y be a vector space and $\mathcal{S}$ be an arrangement of subspaces. De Concini and Procesi give a condition for a subset $\mathcal{G} \subseteq \mathcal{S}$ such that there exists a so called *wonderful model* $Y_{\mathcal{G}}$ of the arrangement, i.e. the elements in $\mathcal{G}$ are replaced by a simple normal crossing divisor, and $Y_{\mathcal{G}}$ can be obtained from Y by a sequence of blow-ups along smooth subvarieties. A $\mathcal{G}$ satisfying the condition is called a *building set*. The paper also gives a criterion of whether the intersection of a collection of such divisors is nonempty. This brings in the notion of a *nest*.

Later, this idea has been generalized to nonsingular varieties over $\mathbb{C}$ with conical stratifications by MacPherson and Procesi [MP98]. The language of stratifications replaces the use of local coordinates in [DP95]. The notion of *building set* and *nest* is also generalized in this setting.

On the other hand, to the author's knowledge, the wonderful compactifications of arrangements of subvarieties, which should also be thought of as a natural generalization of the wonderful models of arrangements of subspaces, do not seem to be adequately discussed in the literature. Since a general arrangement of subvarieties may be far away from a conical stratification, the results for conical wonderful compactifications do not imply immediately the ones of arrangements of subvarieties.

In the thesis, we give the definition of arrangements of subvarieties, building sets and nests. The wonderful compactifications are shown to have analogues properties as the ones in [DP95] or [MP98]. The proof is a combination of local coordinate discussion and algebro-geometric methods (e.g. ideal sheaves, residue schemes) to overcome the difficulty that an arrangement of subvarieties may not induce a conical stratification. The idea of the induction used in the proof is inspired by [MP98].

After the setting of arrangements of subvarieties being fully established, we prove the main theorems about the Chow group and Chow motive decomposition of the wonderful compactification.

Let Y be a nonsingular projective variety endowed with an arrangement of subvarieties (Definition 2.3.5). Suppose $\mathcal{G}$ is a building set (Definition 2.3.6) and $Y_{\mathcal{G}}$ is the wonderful compactification with respect to $\mathcal{G}$ (Definition 2.3.16). We now state the main theorem, with no attempt to go into details (see Theorems 3.1.1 and 3.1.2 for a precise discussion).

The point is that the motive of $Y_{\mathcal{G}}$ can be decomposed into a direct sum of the motive of Y and the twisted motives of the subvarieties of the arrangement. Moreover, this decomposition is independent of the order in which the sequence of blow-ups $Y_{\mathcal{G}} \to Y_{N-1} \to \cdots \to Y_1 \to Y$ is carried out. The word 'canonical' in the following theorem means this independency.

Main Theorem. *There is a canonical Chow group decomposition*

$$A^*Y_{\mathcal{G}} \cong A^*Y \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu} \in M_{\mathcal{T}}} A^{*-\|\underline{\mu}\|}(Y_0\mathcal{T})$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests.

Moreover, when Y is complete, there is a canonical Chow motive decomposition

$$h(Y_{\mathcal{G}}) \cong h(Y) \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu} \in M_{\mathcal{T}}} h(Y_0\mathcal{T})(||\underline{\mu}||)$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests.

The outline of the thesis is as follows.

Chapter 2 is devoted to background material. Section 2.1 gives a brief review of the definition of motives. In §2.1.5 we prove a formula (Theorem 2.1.5) for the motive of a blow-up which is used to prove the main theorem. The formula proved here is, as far as we know, slightly more precise than the ones we could find in the literature (see Remark 2.1.7). Section 2.2 is an introduction to the Fulton-MacPherson configuration spaces. Section 2.3 is devoted to the definitions and proofs of properties of the so-called wonderful compactifications of an arrangement of subvarieties. This could be seen as a natural generalization of wonderful model of subspace arrangement. §2.4 briefly introduces several special examples of wonderful compactifications: the wonderful models of subspace arrangements given by De Concini and Procesi (§2.4.1), Ulyanov's polydiagonal compactification and Hu's compactification (§2.4.2) and Kuperberg-Thurston's construction (§2.4.3).

In Chapter 3, we state and prove the main theorems (Theorems 3.1.1 and 3.1.2) in the most general setting, i.e. for the wonderful compactification of an arrangement of subvarieties. The proof requires keeping track of the changes of subvarieties occurring at each blow-up (Proposition 2.3.20). The blow-up formula (Theorem 2.1.5) plays an important role in this context.

In Chapter 4, we prove more precise results for the Chow groups (Theorem 4.1.1) and the Chow motives (Theorem 4.1.2) of the Fulton-MacPherson configuration space $X[n]$; they depend only on X and n. We give a generating function which can be used to calculate the Chow groups and the Chow motives recursively (Theorem 4.2.1). Examples of Chow groups and Chow motives of $X[n]$ for $n = 2, 3, 4$ are given in Section 4.3.

Chapter 5 is independent of the previous chapters. We show that for certain kinds of wonderful models of X^n (e.g. the Fulton-MacPherson configuration spaces), the cobordism classes of the wonderful models depend only on n and the cobordism class of X.

Chapter 2

Background material

This chapter contains two quite different parts. The first part, §2.1, gives a brief review of the definition of motives. The second part, §2.2, §2.3 and §2.4, present various wonderful compactifications. §2.2 is an introduction to the Fulton-MacPherson configuration spaces, which people might be more familiar with. Then §2.3 shows the definitions and properties of the general setting: the so-called wonderful compactifications of an arrangement of subvarieties. As examples of other wonderful compactification, §2.4 reviews the wonderful models of subspace arrangements given by De Concini and Procesi, Ulyanov's polydiagonal compactification and Hu's compactification and Kuperberg-Thurston's construction.

2.1 Motives

Our understanding of the theory of motives has greatly increased since it was introduced in the middle 1960's by Alexander Grothendieck. His original idea was to attempt to give a universal cohomology theory encompassing existent cohomology theories for projective manifolds. Though the main part of the theory is still conjectural, a lot of developments and applications have been produced. We shall give a brief review here. For beautiful introductions to the subject see [Fu98] §16, [Man68], [Maz04],[Mu04]. For developments, see [JKS94].

2.1.1 Chow groups

A *variety* is a reduced irreducible algebraic scheme over a fixed algebraically closed field. Let X be a variety. Consider the cycle group

$$\mathcal{Z}_k X = \Big\{ \sum n_i V_i : V_i \subseteq X \text{ irreducible subvariety of dimension } k, \text{ and } n_i \in \mathbb{Z} \Big\}.$$

A codimension one cycle in a variety W is called *rationally equivalent to zero* if it is equal to $div(r)$, the divisor of the zeroes minus the poles of a non-zero rational function r on W. Two k-cycles α, β are called *rationally equivalent* if their difference can be expressed as the sum of divisors of some $(k+1)$-subvarieties, i.e., there are a finite number of $(k+1)$-dimensional subvarieties W_i of X and non-zero rational functions r_i on each W_i, such that

$$\alpha - \beta = \sum div(r_i).$$

Denote by $\mathrm{Rat}_k X$ the group of k-cycles in X which are rationally equivalent to zero. The *Chow group of dimension k* is defined as

$$A_k X := \mathcal{Z}_k X / \mathrm{Rat}_k X.$$

A Chow group with $\mathbb{Q}$-coefficient is defined as

$$(A_k X)_{\mathbb{Q}} := A_k X \otimes_{\mathbb{Z}} \mathbb{Q}.$$

2.1.2 Push-forward, pull-back, and intersection product

For a proper morphism $f : X \to Y$, there is a natural *push-forward* homomorphism $f_* : \mathcal{Z}_k X \to \mathcal{Z}_k Y$, which induces a *push-forward* homomorphism

$$f_* : A_k X \to A_k Y.$$

For a flat morphism $f : X \to Y$, the natural *pull-back* homomorphism $f^* : \mathcal{Z}_k Y \to \mathcal{Z}_{k+n} X$ $(n = \dim X - \dim Y)$ induces a *pull-back* homomorphism of Chow groups:

$$f^* : A_k Y \to A_{k+n} X.$$

Using techniques in intersection theory, we can drop the flatness condition and define $f^* : A_k Y \to A_{k+n} X$ for any morphism f whenever Y is non-singular (see [Fu98], §8).

When a variety X is non-singular, there is an *intersection product* on its Chow group:

$$A_i X \otimes A_j X \to A_{i+j-\dim X} X$$

and we define $A^k X := A_{\dim X - k} X$. We define the *Chow ring* of X to be the graded ring $A(X) := \oplus A^k X$. Define

$$A^k_{\mathbb{Q}} X := A^k X \otimes_{\mathbb{Z}} \mathbb{Q}, \quad A(X)_{\mathbb{Q}} := A(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

2.1.3 Correspondences

Let X, Y be complete and non-singular varieties. A *correspondence* from X to Y is a rational equivalence class of cycles on $X \times Y$. The *group of correspondences* (with integer coefficient) from X to Y is defined as

$$Corr(X, Y) := A(X \times Y).$$

The group of correspondences of *degree* r from X to Y is defined as

$$Corr^r(X, Y) := A^{\dim X + r}(X \times Y).$$

Define

$$Corr_{\mathbb{Q}}(X, Y) := A(X \times Y)_{\mathbb{Q}}, \quad Corr^r_{\mathbb{Q}}(X, Y) := A^{\dim X + r}_{\mathbb{Q}}(X \times Y).$$

The *composition* of two correspondences $f \in Corr(X_1, X_2)$, $g \in Corr(X_2, X_3)$ is defined as

$$g \circ f := \pi_{13*}(\pi_{12}^* f \cdot \pi_{23}^* g).$$

Here π_{ij} is the projection from $X_1 \times X_2 \times X_3$ to $X_i \times X_j$. The product $\pi_{12}^* f \cdot \pi_{23}^* g$ is the intersection product on the non-singular variety $X_1 \times X_2 \times X_3$. If we specify the degrees, then the composition gives

$$Corr^r(X, Y) \otimes Corr^s(Y, Z) \xrightarrow{\circ} Corr^{r+s}(X, Z).$$

A correspondence $f \in Corr^r(X, Y)$ induces a *push-forward* $f_* : A^i(X) \to$

$A^{i+r}(Y)$ and a *pull-back* $f^* : A^i(Y) \to A^{i+r+\dim X - \dim Y}(X)$ as follows

$$f_*(a) := \pi_{Y*}(f \cdot \pi_X^*(a)), \quad f^*(b) := \pi_{X*}(f \cdot \pi_Y^*(b)).$$

It is easy to see that $f^* = (f^t)_*$, where $f^t \in Corr(Y, X)$ is the transpose of f.

Remark: The notion of *correspondence* is an important generalization of the notion of *morphism*. It includes, as a special case, "multi-valued" maps. It is associative, i.e. given $\alpha \in Corr(X, Y), \beta \in Corr(Y, Z), \gamma \in Corr(Z, W)$, we have $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$. Sending a morphism $f : X \to Y$ to its graph Γ_f is functorial, in the sense that $\Gamma_g \circ \Gamma_f = \Gamma_{gf}$. By considering *correspondences* in place of *morphisms*, we can deal with (co)homology or Chow groups more effectively. The following two examples explain this idea.

Example: Let $\mathbb{P}^1$ be the projective line over the complex field $\mathbb{C}$. Consider the following three correspondences in $\mathbb{P}^1 \times \mathbb{P}^1$: the diagonal Δ, $\alpha = \mathbb{P}^1 \times pt$ and $\beta = pt \times \mathbb{P}^1$, where pt is a fixed point in $\mathbb{P}^1$. We have $\Delta = \alpha + \beta$ and

$$\Delta^2 = \Delta, \quad \alpha^2 = \alpha, \quad \beta^2 = \beta, \quad \alpha \circ \beta = \beta \circ \alpha = 0.$$

Moreover, Δ_* and Δ^* both induce the identity on $A(\mathbb{P}^1)$; α_* and β^* send $pt \mapsto pt, \mathbb{P}^1 \mapsto 0$; α^* and β_* send $pt \mapsto 0, \mathbb{P}^1 \mapsto \mathbb{P}^1$.
(Indeed, Δ is the graph of identity map $id : \mathbb{P}^1 \to \mathbb{P}^1$, so $\Delta^2 := \Delta \circ \Delta = \Gamma_{(id)\circ(id)} = \Gamma_{id} = \Delta$. Similarly, $\alpha^2 = \alpha$ since α is the graph of a constant morphism sending every point in $\mathbb{P}^1$ to pt. We can calculate β^2 using the

definition:

$$\beta^2 = \pi_{13*}(\pi_{12}^*\beta \cdot \pi_{23}^*\beta) = \pi_{13*}(pt\times\mathbb{P}^1\times\mathbb{P}^1 \cdot \mathbb{P}^1\times pt\times\mathbb{P}^1)$$
$$= \pi_{13*}(pt\times pt\times\mathbb{P}^1) = pt\times\mathbb{P}^1 = \beta.$$

The proof of the other statements is similar.)

2.1.4 Motives

A reference is [CH00].

A correspondence $p \in Corr^0(X, X)$ is called a *projector* of X if $p^2(:= p \circ p) = p$.

Let $\mathcal{V}$ denote the category of (not necessarily connected) non-singular projective varieties over a field k. The *category of Chow motives over* k (denoted by $CH\mathcal{M}$) is defined as follows:

Definition 2.1.1. *An object of $CH\mathcal{M}$ is a triple (X, p, r), where X is a non-singular projective variety, p is a projector of X, $r \in \mathbb{Z}$.*
Morphisms are defined as

$$Hom_{CH\mathcal{M}}\big((X, p, r), (Y, q, s)\big) := q \circ Corr^{s-r}(X, Y) \circ p.$$

The composition of morphisms is defined as the composition of correspondences.

Remark:

- The original definition of morphisms in $CH\mathcal{M}$ by Grothendieck is

$$Hom_{CH\mathcal{M}}((X,p,r),(Y,q,s)) :=$$
$$\{f \in Corr^{s-r}(X,Y) : f \circ p = q \circ f\}/\{f : f \circ p = 0\}.$$

  This definition is equivalent to the one given here.

- In some literature (e.g. Grothendieck), a motive (X,p,r) is also written as $(X,p)(r)$, while in others (e.g. Manin [Man68]) as $(X,p)(-r)$. We use the latter notation.

The functor $h : \mathcal{V}^{opp} \to CH\mathcal{M}$. There is a contravariant functor from the category of non-singular projective variety over a field k to the category of Chow motives over k, which sends X to $(X, id_X, 0)$ and sends a morphism $f : X \to Y$ to $\Gamma_f^t : h(Y) \to h(X)$, the transpose of the graph of f.

The Lefschetz and Tate motives. The Lefschetz motive is defined as $\mathbb{L} := (\text{Spec } k, id, -1)$. Tate motive is defined as $\mathbb{T} := (\text{Spec } k, id, 1)$. We denote $\mathbb{1} = (\text{Spec } k, id, 0)$.

It can be show that $\mathbb{L} \cong (\mathbb{P}^1, \mathbb{P}^1 \times p, 0)$ in $CH\mathcal{M}$, which corresponds to

Tensor product in $CH\mathcal{M}$. For $f \in Corr(X_1, X_2)$, $g \in Corr(X_3, X_4)$, we define the tensor product

$$f \otimes g := s_{23*}(\pi_{12}^* f \cdot \pi_{34}^* g) \in Corr(X_1 \times X_3, X_2 \times X_4),$$

where $s_{23} : X_1 \times X_2 \times X_3 \times X_4 \to X_1 \times X_3 \times X_2 \times X_4$ is the isomorphism which switches the second and third factors. This tensor product satisfies $(f_1 \otimes f_2) \circ$

$(g_1 \otimes g_2) = (f_1 \circ g_1) \otimes (f_2 \circ g_2)$.

The *tensor product* of two Chow motives is defined as

$$(X, p, r) \otimes (Y, q, s) := (X \times Y, p \otimes q, r + s).$$

Example:

- Define $\mathbb{L}^r := \mathbb{L}^{\otimes r}$. We have $\mathbb{L}^r \cong (\text{Spec } k, id, -r)$.

  Moreover, $(X, p, r) = (X, p, 0) \otimes \mathbb{L}^{-r}$.

- Let X be a projective space $\mathbb{P}^n$ or a product of projective spaces with $\dim X = n$. Let pt be a fixed point in X. Then $(X, X \times pt, 0) \cong \mathbb{L}^n$. In particular,

  $$(\mathbb{P}^1, \mathbb{P}^1 \times pt, 0) \cong \mathbb{L}.$$

  (See [Man68], §6.)

$CH\mathcal{M}$ **is a pseudo-abelian category.** A category is pseudo-abelian if it is additive and projectors have kernels and images. (See [Man68] §5.) To be precise:

Definition 2.1.2. *An additive category $\mathcal{D}$ is called* pseudo-abelian *if for any projector $p \in Hom(M, M)$, $M \in \mathcal{OB}(\mathcal{D})$, there exists a kernel* $\ker p$*, and the canonical homomorphism* $\ker p \oplus \ker(id_X - p) \to M$ *is an isomorphism.*

The category $CH\mathcal{M}$ is pseudo-abelian. The direct sum is defined as follows:

for any $t \geq r$ and s, let $r' = t - r$, $s' = t - s$, $\alpha = \mathbb{P}^1 \times pt$;

if $M = (X, p, t) \otimes \mathbb{L}^{r'} = (X, p, t) \otimes ((\mathbb{P}^1)^{r'}, \alpha^{\otimes r'}, 0) = (X \times (\mathbb{P}^1)^{r'}, p \otimes \alpha^{\otimes r'}, t)$,

and $N = (Y, q, s) \otimes \mathbb{L}^{s'} = (Y \times (\mathbb{P}^1)^{s'}, q \otimes \alpha^{\otimes s'}, t)$,

then define

$$M \oplus N = \big(X \times (\mathbb{P}^1)^{r'} \amalg Y \times (\mathbb{P}^1)^{s'}, p \otimes \alpha^{\otimes r'} \oplus q \otimes \alpha^{\otimes s'}, t\big).$$

The direct sums defined by different choices of t are canonically isomorphic, so that the direct sums are well-defined.

Manin's Identity Principle. The principle asserts that an identity between correspondences holds if and only if a collection of certain identities between morphisms of Chow groups hold. To be precise,

Fact 2.1.3. *(Manin's Identity Principle) Given $\varphi, \psi \in A(X \times Y)$, define $\varphi_T : A(T \times X) \to A(T \times Y)$ by $\varphi_T(g) = \varphi \circ g$ as composition of correspondences and define ψ_T similarly. Then the following are equivalent:*

(i) $\varphi = \psi$;

(ii) $\varphi_T = \psi_T$ for all smooth complete schemes T;

(iii) $(id_T \otimes \varphi)_* = (id_T \otimes \psi)_*$ for all smooth complete schemes T.

Proof. For $(i) \Leftrightarrow (ii)$, see [Fu98] §16. For $(ii) \Leftrightarrow (iii)$, see the first lemma of [Man68] §3 . □

2.1.5 A Formula for the motive of a blow-up

Suppose $f : \widetilde{Y} \to Y$ is the blow-up of a smooth algebraic variety Y along a smooth subvariety V, and P is the exceptional divisor. Denote by i, j, f, g the morphisms as in the following fibre square

$$\begin{array}{ccc} P & \xrightarrow{j} & \widetilde{Y} \\ {\scriptstyle g}\downarrow & \square & \downarrow{\scriptstyle f} \\ V & \xrightarrow[i]{} & Y \end{array}$$

Denote by $N := N_V Y$ the normal bundle of V in Y. Let $h := c_1(\mathcal{O}_N(1)) \in A^1(P)$. Let $r := \mathrm{codim}_V Y$ be the codimension of V in Y.

We use the notation $j \boxtimes g : P \to \widetilde{Y} \times V$ for the composition of the diagonal map $P \to P \times P$ with $j \times g : P \times P \to \widetilde{Y} \times V$ ($g \boxtimes j$ is defined similarly). Given $a \in A(P)$, denote by $\{a\}_i$ the image of the projection $A(P) \to A^i(P)$ of the Chow ring to its degree i direct summand. For $1 \leq k \leq r-1$, define

$$\left\{ \begin{array}{l} \alpha_k := -(j \boxtimes g)_* \Big(\sum_{l=0}^{r-1-k} g^* c_{r-1-k-l}(N) h^l \Big) \\ \qquad = -(j \boxtimes g)_* \Big\{ g^* c(N) \frac{1}{1-h} \Big\}_{r-1-k} \in Corr^{-k}(\widetilde{Y}, V), \\ \beta_k := (g \boxtimes j)_* h^{k-1} \in Corr^k(V, \widetilde{Y}), \\ p_k := \beta_k \circ \alpha_k \in Corr^0(\widetilde{Y}, \widetilde{Y}), \\ \alpha_0 := \Gamma_f \in Corr^0(\widetilde{Y}, Y), \\ \beta_0 := \Gamma_f^t \in Corr^0(Y, \widetilde{Y}), \\ p_0 := \beta_0 \circ \alpha_0 \quad \big(= \Gamma_f^t \circ \Gamma_f = (f \times f)^* \Delta_Y \big) \in Corr^0(\widetilde{Y}, \widetilde{Y}). \end{array} \right. \tag{2.1}$$

Proposition 2.1.4.

(1) $\alpha_0 \beta_0 = \Delta_Y$, $\alpha_k \beta_k = \Delta_V$ *for* $1 \leq k \leq r-1$, $\alpha_i \beta_j = 0$ *for* $i \neq j$.

(2) $p_0, p_1, p_2, ..., p_{r-1}$ *are pairwise orthogonal projectors of* $\widetilde{Y}$*, and*

$$\sum_{i=0}^{r-1} p_i = \Delta_{\widetilde{Y}} \quad \textit{in } A(\widetilde{Y} \times \widetilde{Y}),$$

i.e. equality holds up to rational equivalence.

(3) We have the following isomorphisms of motives,

$$\alpha_0 : (\widetilde{Y}, p_0, 0) \simeq h(Y), \textit{with inverse morphism } \beta_0,$$
$$\alpha_k : (\widetilde{Y}, p_k, 0) \simeq h(V)(k), \textit{with inverse morphism } \beta_k, \ \textit{for } 1 \leq k \leq r-1.$$

Define $\Gamma := \overset{r-1}{\underset{i=0}{\oplus}} \alpha_i$, $\Gamma' := \sum_{i=0}^{r-1} \beta_i$, then Proposition 2.1.4 can be conveniently reformulated as follows:

Theorem 2.1.5. *The correspondence* Γ *gives a canonical isomorphism in* $CH\mathcal{M}$*,*

$$\Gamma : h(\widetilde{Y}) \cong h(Y) \oplus \bigoplus_{k=1}^{r-1} h(V)(k).$$

with inverse isomorphism given by Γ'.

Remark 2.1.6. When the normal bundle N of V in Y is trivial (for example, when V is a point), P is isomorphic to a product space $V \times \mathbb{P}^{r-1}$ and $h = c_1(\mathcal{O}_P(1))$ can be represented (not canonically) by a product space $H = V \times \mathbb{P}^{r-2}$ in P. In this case, we have simple forms for the projectors:

$$p_k = -(j \times j)_*(H^{r-1-k} \times_V H^{k-1}), \text{ for } 1 \leq k \leq r-1;$$
$$p_0 = \Delta + \sum_{k=1}^{r-1} (j \times j)_*(H^{r-1-k} \times_V H^{k-1}).$$

In general, for a nontrivial normal bundle N, more terms involving the Chern classes of N are needed, and the correspondences cannot be represented by explicit and natural algebraic cycles.

Remark 2.1.7. The isomorphism of motives in Theorem 2.1.5 is also a consequence of "Theorem on the additive structure of the motif" of $\widetilde{Y}$ in [Man68] §9, which states, in our notation, that there is a split exact sequence

$$0 \longrightarrow h(V)(r) \xrightarrow{a} h(Y) \oplus h(P)(1) \xrightarrow{b} h(\widetilde{Y}) \longrightarrow 0 .$$

The correspondences given in our theorem are not given, at least not explicitly, in Manin's paper.

In order to clarify this point, define

$$\Phi = c_{r-1}(g^*N/\mathcal{O}_N(-1)) \in A^{r-1}(P), c_\Phi = \delta_{P*}(\Phi) \in Corr(P,P),$$

$$a = (i_*, c_\Phi \circ g^*), a' = g_*,$$

$$b = f^* + j_*,\ b' \text{ its right inverse},$$

$$d = \Delta_{Y \times P} - aa', d' = \Delta_Y \otimes (\Delta_P - p_0^P) \text{ (where } p_0^P = c_{h^{r-1}} \circ g^* \circ g_*),$$

denote by $e : \oplus_{k=1}^{r-1} V(k) \to (P, \Delta_P - p_0^P)$ the isomorphism implicitly defined in [Man68] §7, and denote by e' the inverse of e.

We have the following isomorphisms

$$h(Y) \oplus \bigoplus_{k=1}^{r-1} h(V)(k) \quad \underset{\Delta_Y \otimes e'}{\overset{\Delta_Y \otimes e}{\rightleftarrows}} \quad (Y \sqcup P, (\Delta_Y, \Delta_P - p_0^P)) \quad \underset{d'}{\overset{d}{\rightleftarrows}}$$

$$(Y \sqcup P, \Delta_{Y \sqcup P} - aa') \quad \underset{b'}{\overset{b}{\rightleftarrows}} \quad (\widetilde{Y}, \Delta_{\widetilde{Y}}) \ .$$

Hence the following is an isomorphism of Chow motives,

$$(\Delta_Y \otimes e') \circ d' \circ b' : h(\widetilde{Y}) \cong h(Y) \oplus \bigoplus_{k=1}^{r-1} h(V)(k).$$

with inverse $b \circ d \circ (\Delta_Y \otimes e)$.

Therefore, to write down the correspondence $(\Delta_Y \otimes e') \circ d' \circ b'$, we need to find explicitly the right inverse b' of b. However, in [Man68] the construction of b' is based on the surjectivity of $\gamma : A(\widetilde{Y} \times (Y \sqcup P)) \to A(\widetilde{Y} \times \widetilde{Y})$ as follows: by the surjectivity of γ, there is a cycle class $c \in A(\widetilde{Y} \times (Y \sqcup P))$ (which is not given, at least explicitly, in [Man68]) such that $\gamma(c) = \Delta_{\widetilde{Y}} \in A(\widetilde{Y} \times \widetilde{Y})$. Then b' is defined to be $(1 - aa')c$.

On the other hand, the correspondences Γ and Γ' we have constructed in Theorem 2.1.5 give an explicit construction of b'. Indeed, $b' = d \circ (\Delta_Y \otimes e) \circ \Gamma$.

Idea of proof of Proposition 2.1.4 (The proof, based on a series of preliminary results, is given at the end of this section.) It is well known that the Chow groups of the blow-up space $\widetilde{Y}$ can be naturally decomposed in terms of the Chow groups of Y and of the center V. The exact version we need is listed as Lemma 2.1.8. Then in Lemma 2.1.9 we will study the morphisms α_{i*}, β_{i*} and p_{i*} of Chow groups induced by the correspondences α_i, β_i and p_i. As a consequence, the identities of morphisms of Chow groups which are induced by the identities in Proposition 2.1.4 (1) (2) hold, see Corollary 2.1.10. On the other hand, Manin's Identity Principle (Fact 2.1.3) asserts that the identities of morphisms of Chow groups imply the identities of correspondences, providing that the correspondences are universal in some sense. Hence we conclude (1) and (2). The proof of Proposition 2.1.4 (3) is standard.

Lemma 2.1.8. *Using the notation at the beginning of this section, the following group morphism*

$$\gamma : \quad \bigoplus_{i=1}^{r-1} A_{k-r+i}(V) \bigoplus A_k(Y) \to A_k(\widetilde{Y})$$
$$(a_1, a_2, ..., a_{r-1}, y) \mapsto \sum_{i=1}^{r-1} j_*(g^* a_i \cdot h^{i-1}) + f^* y$$

is an isomorphism.

Proof. See [Vo03] Theorem 9.27. □

Remark: For convenience, we also write the above isomorphism in terms of degrees of Chow rings as

$$\bigoplus_{i=1}^{r-1} A^{*-i}(V) \bigoplus A^*(Y) = A^*(\widetilde{Y}).$$

From now on to the end of this section, we assume

$$1 \leq k \leq r-1, \quad 0 \leq i, j \leq r-1.$$

In the following lemma we compute the morphisms of Chow groups induced by the correspondences α_i, β_i:

Lemma 2.1.9. *Let $\tilde{y} \in A(\widetilde{Y})$ be expressed as according to Lemma 2.1.8 as:*

$$\tilde{y} = \sum_{i=1}^{r-1} j_*(g^*a_i \cdot h^{i-1}) + f^*y.$$

Then we have

(1) The morphism $\alpha_{k*} : A(\widetilde{Y}) \to A(V)$ maps $\tilde{y} \mapsto a_k$.

(2) The morphism $\beta_{k*} : A(V) \to A(\widetilde{Y})$ maps $x \mapsto j_*(g^*x \cdot h^{k-1})$.

(3) The morphism $\alpha_{0*} : A(\widetilde{Y}) \to A(Y)$ maps $\tilde{y} \mapsto y$.

(4) The morphism $\beta_{0*} : A(Y) \to A(\widetilde{Y})$ maps $y \mapsto f^*y$.

Proof. Denote by π_2 the projection $V \times \widetilde{Y} \to \widetilde{Y}$. In the following calculation we will use the fact that $\forall x \in A(X)$,

$$\delta_{X*}(x) = x \times 1 \cdot \Delta_P = 1 \times x \cdot \Delta_P.$$

For (2),

$$\begin{aligned}\beta_{k*}(x) &= \pi_{2*}[x{\times}1 \cdot (g{\times}j)_*\delta_{P*}h^{k-1}] = \pi_{2*}(g{\times}j)_*[g^*x{\times}1 \cdot h^{k-1}{\times}1 \cdot \Delta_P] \\ &= \pi_{2*}(g{\times}j)_*\delta_{P*}(g^*x \cdot h^{k-1}) = j_*(g^*x \cdot h^{k-1}).\end{aligned}$$

For (3), denote $a_0 = -i^*y$ for simplicity of notation, and note that $j^*j_*z = -h \cdot z$ for $\forall z \in A(X)$, we have

$$j^*\tilde{y} = \sum_{i=1}^{r-1} j^*j_*(g^*a_i \cdot h^{i-1}) + j^*f^*y = -\sum_{i=1}^{r-1} g^*a_i \cdot h^i + g^*i^*y = -\sum_{i=0}^{r-1} g^*a_i \cdot h^i.$$

By definition (see [Fu98] §3), the i-th Segre class of N is

$$s_i(N) := g_*(h^{i+r-1}),$$

hence

$$\begin{aligned}
\alpha_{k*}(\tilde{y}) &= -g_*\big(j^*\tilde{y} \cdot \sum_{l=0}^{r-1-k} g^* c_{r-1-k-l}(N) \cdot h^l\big) \\
&= -g_*\Big((-\sum_{i=0}^{r-1} g^* a_i \cdot h^i) \cdot \Big(\sum_{l=0}^{r-1-k} g^* c_{r-1-k-l} \cdot h^l\Big)\Big) \\
&= g_*\Big(\sum_{i=0}^{r-1}\sum_{l=0}^{r-1-k} g^*(a_i c_{r-1-k-l}) h^{i+l}\Big) = \sum_{i=0}^{r-1} a_i \Big(\sum_{l=0}^{r-1-k} c_{r-1-k-l} s_{i+l+1-r}\Big),
\end{aligned}$$

.

Since we have the relation $c(N)s(N) = 1$, where $c(N) := \sum c_i(N)$ is the total Chern class and $s(N) := \sum s_i(N)$ is the total Segre class, then

$$\sum_{l=0}^{r-1-k} c_{r-1-k-l} s_{i+l+1-r} = \sum_{l=-\infty}^{+\infty} c_{r-1-k-l} s_{i+l+1-r} = \{c(N)s(N)\}_{i-k} = \delta_{ik},$$

where the first equality is because $s_{i+l+1-r} = 0$ for $l < 0$, and $c_{r-1-k-l} = 0$ for $l > r-1-k$. It follows that $\alpha_{k*}(\tilde{y}) = a_k$.

Since $\alpha_{0*} = (\Gamma_f)_* = f_*$, $\beta_{0*} = (\Gamma_f^t)_* = f^*$ (see [Fu98] Proposition 16.1.2(c)), we immediately have (4). For (3), to calculate

$$\alpha_{0*}(\tilde{y}) = f_*(\tilde{y}) = f_*\Big(\sum_{i=1}^{r-1} j_*(g^* a_i \cdot h^{i-1}) + f^* y\Big),$$

Notice that

$$
\begin{aligned}
f_*(f^*y) &= y, \quad \text{by the projection formula;}\\
f_*j_*\big((g^*a_i \cdot h^{i-1})\big) &= i_*g_*(g^*a_i \cdot h^{i-1}) = i_*\big(a_i \cdot g_*(h^{i-1})\big)\\
&= i_*(a_i \cdot s_{i-r}) = 0, \quad \text{since } i-r<0.
\end{aligned}
$$

Then $\alpha_{0*}(\tilde{y}) = y$. □

Corollary 2.1.10. *We have the following identities:*

$$
\begin{aligned}
&\alpha_{k*}\beta_{k*} = id_{A(V)}, \quad \alpha_{0*}\beta_{0*} = id_{A(Y)}, \quad \alpha_{i*}\beta_{j*} = 0 \textit{ for } i \neq j,\\
&(p_ip_j)_* = \delta_{ij}p_{i*}, \quad \sum_{i=0}^{r-1} p_{i*} = id_{A(\widetilde{Y})}.
\end{aligned}
$$

Proof. They are deduced immediately from the above lemma. Indeed,

$$
\begin{aligned}
&\forall x \in A(V), \quad \alpha_{k*}\beta_{k*}x = \alpha_{k*}j_*(g^*x \cdot h^{k-1}) = x, \quad \text{so} \quad \alpha_{k*}\beta_{k*} = id_{A(V)};\\
&\forall y \in A(Y), \quad \alpha_{0*}\beta_{0*}y = \alpha_{0*}f^*y = y, \quad \text{so} \quad \alpha_{0*}\beta_{0*} = id_{A(Y)};
\end{aligned}
$$

The proof of $\alpha_{i*}\beta_{j*} = 0$ for $i \neq j$ is similar. For the proof of $(p_ip_j)_* = \delta_{ij}p_{i*}$, notice that

$$
(p_ip_j)_* = \beta_{i*}\alpha_{i*}\beta_{j*}\alpha_{j*} = \begin{cases} 0, & \text{if } i \neq j;\\ \beta_{i*}id_{A(V)}\alpha_{i*} = (p_i)_*, & \text{if } i = j > 0;\\ \beta_{i*}id_{A(Y)}\alpha_{i*} = (p_i)_*, & \text{if } i = j = 0. \end{cases}
$$

Finally,

$$\forall \tilde{y} = \sum_{i=1}^{r-1} j_*(g^* a_i \cdot h^{i-1}) + f^* y \in A(\widetilde{Y}),$$

since

$$\begin{aligned} p_{0*}(\tilde{y}) &= \beta_{0*}\alpha_{0*}(\tilde{y}) = f^* y, \\ p_{k*}(\tilde{y}) &= \beta_{k*}\alpha_{k*}(\tilde{y}) = \beta_{k*} a_k = j_*(g^* a_k \cdot h^{k-1}), \\ \text{then } \sum_{i=0}^{r-1} p_{i*}(\tilde{y}) &= \sum_{i=1}^{r-1} j_*(g^* a_i \cdot h^{i-1}) + f^* y = \tilde{y}. \end{aligned}$$

Therefore $\sum_{i=0}^{r-1} p_{i*} = id_{A(\widetilde{Y})}$. □

We now prove Proposition 2.1.4:

Proof of Proposition 2.1.4. For any smooth scheme T, $T \times \widetilde{Y}$ is the blow-up of $T \times Y$ along the smooth subvariety $T \times V$. Denote $j' = id_T \times j$, $g' = id_T \times g$, $f' = id_T \times f$, $i' = id_T \times i$, we have the following fiber square:

$$\begin{array}{ccc} T \times P & \xrightarrow{j'} & T \times \widetilde{Y} \\ \downarrow g' & \square & \downarrow f' \\ T \times V & \xrightarrow{i'} & T \times Y \end{array}$$

We can construct the correspondences $\alpha'_i, \beta'_i, p'_i$ for this fiber square as we did in (2.1). We have

$$\alpha'_i = id_T \otimes \alpha_i, \beta'_i = id_T \otimes \beta_i, p'_i = id_T \otimes p_i.$$

Indeed, the normal bundle N' of $T \times V$ in $T \times Y$ is the pullback of N under the morphism $T \times V \to V$, therefore $c(N') = 1_T \times c(N)$ and $h' := \mathcal{O}_{N'}(1) = 1_T \times h$.

The proof of the above three identities are similar and we only show the first identity,

$$\begin{aligned}
\alpha_i' &:= -(j' \boxtimes g')_* \{ g'^* c(N') \frac{1}{1-h'} \}_{r-1-k} \\
&= -(j' \boxtimes g')_* \{ 1_T \times \big(g^* c(N) \frac{1}{1-h} \big) \}_{r-1-k} \\
&= (s_{23})_* (\Delta_T \times (j \boxtimes g))_* \Big(1_T \times \big\{ 1_T \times g^* c(N) \frac{1}{1-h} \big\}_{r-1-k} \Big) \\
&= (s_{23})_* \Big(\Delta_T \times (j \boxtimes g))_* \{ 1_T \times g^* c(N) \frac{1}{1-h} \}_{r-1-k} \Big) \\
&= id_T \otimes \alpha_i.
\end{aligned}$$

Then Corollary 2.1.10 and Manin's Identity Principle imply (1) and (2) of Proposition 2.1.4.

For (3), to show that α_k gives an isomorphism $(\widetilde{Y}, p_k, 0) \simeq h(V)(k)$ with inverse β_k, we need to show that $p_k = p_k \circ \beta_k \circ \alpha_k$ and $id = id \circ \alpha_k \circ \beta_k$. but they are direct consequences of the fact that $\alpha_k \circ \beta_k = \Delta_V$ from (1). The proof for $(\widetilde{Y}, p_0, 0) \simeq h(Y)$ is similar. □

2.2 Fulton-MacPherson configuration spaces

Fulton and MacPherson have constructed in [FM94] a compactification of the configuration space of n distinct labeled points in a non-singular algebraic variety X. It is related to several areas of mathematics. In their original paper, Fulton and MacPherson use it to construct a differential graded algebra which is a model for $F(X, n)$ in the sense of Sullivan [FM94]. Axelrod-Singer constructed the compactification in the setting of smooth manifolds. $\mathbb{P}^1[n]$

is related to the Deligne-Mumford compactification $\overline{M}_{0,n}$ of the moduli space of nonsingular genus-0 projective curves. Now we give a brief review of this compactificaton.

For each subset $I \in [n] := \{1, \ldots, n\}$ with at least two elements, let $\mathrm{Bl}_\Delta(X^I)$ denote the blow-up of the corresponding cartesian product X^I along its small diagonal. Denote by Δ_I the diagonal in X^n where $x_i = x_j$ if $i, j \in I$.

The *configuration space* $F(X, n)$ is the complement of all diagonals in X^n, i.e.,

$$F(X, n) = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j, \forall i \neq j\}.$$

Fulton and MacPherson give two constructions of their compactification $X[n]$ as follows.

I. Construction as a closure. There is a natural locally closed embedding

$$i : F(X, n) \hookrightarrow X^n \times \prod_{|I| \geq 2} \mathrm{Bl}_\Delta(X^I).$$

The closure of this embedding is the Fulton-MacPherson compactification $X[n]$.

Remark 2.2.1. *This definition is equivalent to define $X[n]$ as the closure of*

$$i' : F(X, n) \hookrightarrow X^n \times \prod_{|I| \geq 2} Bl_{\Delta_I}(X^n).$$

Indeed, denote by I^c the complement of I in $[n]$. There is a natural isomorphism

$$Bl_{\Delta_I}(X^n) \cong X^{I^c} \times Bl_\Delta(X^I),$$

and there is a natural closed embedding $X^n \hookrightarrow X^n \times \prod_{|I|\geq 2} X^{I^c}$, *therefore a natural closed embedding*

$$j : X^n \times \prod_{|I|\geq 2} Bl_\Delta(X^I) \hookrightarrow X^n \times \prod_{|I|\geq 2} X^{I^c} \times \prod_{|I|\geq 2} Bl_{\Delta_I}(X^n) \cong X^n \times \prod_{|I|\geq 2} Bl_{\Delta_I}(X^n).$$

Then one can factor i' *through* $j \circ i$. *So the closure of the image of* i *is isomorphic to the closure of the image of* i'.

II. Construction by a sequence of blow-ups. The construction is inductive. $X[2]$ is the blow-up of X^2 along the diagonal Δ_{12}. $X[3]$ is a sequence of blow-ups of $X[2]\times X$ along non-singular subvarieties corresponding to $\{\Delta_{123}; \Delta_{13}, \Delta_{23}\}$. More specifically, denote by π the blow-up $X[2]\times X \rightarrow X^3$, we blow up first along $\pi^{-1}(\Delta_{123})$, then along the strict transforms of Δ_{13} and Δ_{23} (the two strict transforms are disjoint, so they can be blown up in any order). In general, $X[n+1]$ is a sequence of blow-ups of $X[n]\times X$ along smooth subvarieties corresponding to all diagonals Δ_I where $|I| \geq 2$ and $(n+1) \in I$.

A symmetric construction of $X[n]$ has been given by several people: De Concini and Procesi [DP95], MacPherson and Procesi [MP98], and Thurston [Th99]. We will discuss this in detail in the next section. For now we only mention that to get $X[n]$ we can blow up along diagonals by the order of ascending dimension, which is different from the non-symmetric order of the original construction. For example, $X[4]$ is the blow-up of X^4 along diagonals corresponding to:

$$1234; 123, 124, 134, 234; 12, 13, \ldots, 34.$$

Compare it with the order in [FM94]:

$$12; 123; 13, 23; 1234; 124, 134, 234; 14, 24, 34.$$

Geometrical description. We may say very roughly that Fulton-MacPherson compactification records the relative directions when points collide. The precise description is using *screens* (see [FM94]) to record the limiting configurations. Each screen is a tangent space at some point $x \in X$, with several points in it, modulo translation and homothety. For each configuration we may need several screens which satisfy certain compatibility condition.

Example: *Figure 1* gives a point corresponding to a degenerate configuration in $X[4]$ which can be described by three screens.

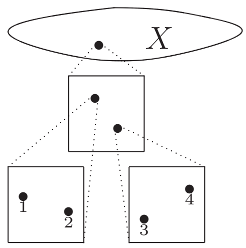


FIGURE 1. A point in $X[4]$

Stratification. The set of degenerate configurations $X[n]\backslash F(X, n)$ is a simple normal crossing divisor. To describe the intersection of divisors, the notion of a *nest* (of $[n]$) is introduced.

Definition 2.2.2. *A set $\mathcal{S}$ of subsets of $[n] := \{1, 2, \ldots, n\}$ is called a nest if any two elements $I, J \in \mathcal{S}$ are either disjoint or one contains the other, and all singletons $\{1\}, \ldots, \{n\}$ are in $\mathcal{S}$.*

Remark: : The definition of a nest (of $[n]$) we give here is a little different from the one in [FM94]: we require all singletons to be in $\mathcal{S}$ for convenience in this thesis, while in [FM94] a nest is defined to contain no singletons. The difference is not essential.

The following property has been proved in [FM94]:

Theorem 2.2.3. *For each $I \subseteq [n]$ that $|I| \geq 2$, there is a non-singular divisor D_I of $X[n]$, such that*

$$X[n] \setminus F(X, n) = \bigcup_{I \subset [n],\ |I| \geq 2} D_I.$$

Any set of these divisors meets transversally. The intersection of divisors $D_{I_1}, \ldots, D_{I_r}$ is non-empty if and only if $I_1, \ldots, I_r$ and all singletons form a nest of $[n]$.

This property immediately gives a stratification of $X[n]$.

Each divisor D_I also has a geometric description: D_I consists of those degenerate configurations that have a screen containing exactly points $\{x_i\}_{i \in I}$. (For example, a point as in *Figure 1* is contained in three divisors D_{12}, D_{34}, D_{1234}.)

Each stratum one-one corresponds to an oriented forest with n labeled leaves such that each 'father' should have more than one 'son'. (For example, in the following figure, the forest on the left hand side is allowed and corresponds to the stratum consists of points of configuration as in *Figure 1*, the forest on the right hand side is not allowed since one of the nodes has exactly one 'son'.)

2.3 Arrangements of subvarieties and the wonderful compactifications

This section is devoted to the fundamental material on the arrangements of subvarieties. First (§2.3.1), the definitions and criterions of transversal intersections and clean intersections are given. Secondly, in §2.3.2, the definition of an arrangement of subvarieties (Definition 2.3.5, which is adapted from [Hu03]) is given. For simplicity, we focus on simple arrangement of subvarieties. The notion of a building set (Definition 2.3.6) is explained. Then in §2.3.3, inspired by the idea of [MP98], we show that a building set of an arrangement of Y induces a building set of an arrangement of the blow-up $\widetilde{Y}$, therefore an inductive construction of blow-ups can keep on going until all elements in the building set become divisors at which point one stops. This processing of successively blowing up is one way to construct the wonderful compactification. We then show this construction coincides with a construction as a closure (Proposition 2.3.17). Finally, we show that a nest induces a nest after a blow-up.

2.3.1 Transversal intersections and Clean intersections

Transversal intersections and clean intersections play important roles here. We give the definitions and compare them at the beginning of this section.

Definition 2.3.1 (of transversal intersection). *Let Y be a nonsingular variety. Let A and B be two nonsingular subvarieties of Y. we call A and B intersect transversally, denoted by $A \pitchfork B$, if they intersect transversally at every point $y \in A \cap B$, i.e. their tangent spaces $T_{A,y}$ and $T_{B,y}$ at y generate the tangent space T_y of the ambient variety at y; equivalently, $T_{A,y}^{\perp} \oplus T_{B,y}^{\perp}$ form a direct sum in the dual space T_y^* of T_y.*

More generally, a finite collection of nonsingular subvarieties $A_1, \ldots, A_k$ intersect transversally, denoted by $A_1 \pitchfork A_2 \pitchfork \cdots \pitchfork A_k$, if their tangent spaces $\{T_{A_1,y}, \ldots, T_{A_k,y}\}$ at each point $y \in Y$ induce a direct sum $T_{A_1,y}^{\perp} \oplus \cdots \oplus T_{A_k,y}^{\perp}$ in T_y^*.

Definition 2.3.2 (of clean intersection). *Let Y be a nonsingular variety. Let A and B be two nonsingular subvarieties of Y. We call A and B intersect cleanly if their intersection is nonsingular and the tangent bundles satisfy*

$$T_{(A\cap B)} = T_A|_{(A\cap B)} \cap T_B|_{(A\cap B)}.$$

Remark: Transversal intersection must be clean. Indeed, if A and B intersect transversally, then there exist local coordinates containing defining functions of A and B. Therefore one can think of A and B locally as two linear subspaces, so A and B intersect cleanly.

Examples:

1. Suppose $Y = \operatorname{Spec} \mathbb{C}[u, v] \cong \mathbb{C}^2$, A is defined by $v = 0$, and B is defined by $v = u^2$. Then the intersection of A and B is neither clean nor transversal.

2. Suppose $Y = \mathbb{C}^3$, and A, B are two lines in Y intersecting at a point. Then A and B intersect cleanly but not transversally.

3. Suppose $Y = \mathbb{C}^3$, and A is a smooth surface containing a smooth curve B. Then A and B intersect cleanly but not transversally.

The following lemma states that, to verify the transversality of two subvarieties which already intersect cleanly along a connected subvariety, it is enough to check the transversality at a point of the intersection.

Lemma 2.3.3. *Let A and B be two nonsingular closed subvarieties of Y that intersect cleanly along a closed nonsingular subvariety C. If A and B intersect transversally at a point $y_0 \in C$, then they intersect transversally (at every point $y \in C$).*

Proof. By dimension counting,

$$\dim\big((T_A)_y + (T_B)_y\big) + \dim\big((T_A)_y \cap (T_B)_y\big) = \dim(T_A)_y + \dim(T_B)_y.$$

On the other hand, A and B intersecting cleanly implies

$$(T_A)_y \cap (T_B)_y = (T_C)_y, \quad \forall y \in C.$$

Hence $\dim\big((T_A)_y \cap (T_B)_y\big) = \dim(T_C)_y = \dim C$. So

$$\dim\big((T_A)_y + (T_B)_y\big) = \dim(A) + \dim(B) - \dim(C)$$

does not depend on the choice of $y \in C$. Now since A and B intersect transversally at y_0, so

$$(T_A)_{y_0} + (T_B)_{y_0} = T_{y_0},$$

then

$$\dim(T_A)_{y_0} + \dim(T_B)_{y_0} = \dim T_{y_0} = \dim Y,$$

which implies

$$\dim(T_A)_y + \dim(T_B)_y = \dim Y,$$

so $(T_A)_y + (T_B)_y = T_y$. Then A intersects B transversally at every point $y \in C$. □

We now give a criterion for clean intersection by ideal sheaves. Denote by $\mathcal{I}_U$ the ideal sheaf of U.

Lemma 2.3.4. *If U, V and $W = U \cap V$ are all nonsingular closed (not necessarily irreducible) subvarieties of Y, then*

$$\mathcal{I}_U + \mathcal{I}_V = \mathcal{I}_W \quad \textit{if and only if } U \textit{ and } V \textit{ intersect cleanly.}$$

Proof. Both $\mathcal{I}_U + \mathcal{I}_V$ and $\mathcal{I}_W$ are subsheaves of the structure sheaf $\mathcal{O}$ of Y. An equality between these two subsheaves is equivalent to an equality between their germs for every point $y \in Y$. i.e., the condition $\mathcal{I}_U + \mathcal{I}_V = \mathcal{I}_W$ is equivalent to

$$(\mathcal{I}_U)_y + (\mathcal{I}_V)_y = (\mathcal{I}_W)_y, \quad \forall y \in Y. \tag{2.2}$$

On the other hand, U and V intersecting cleanly means

$$(T_U)_y \cap (T_V)_y = (T_W)_y, \quad \forall y \in W. \tag{2.3}$$

Since

$$(T_U)_y = \{v \in T_y | df(v) = 0, \forall f \in (\mathcal{I}_U)_y\}.$$

Define $\phi : \mathfrak{m}_y \to \mathfrak{m}_y/\mathfrak{m}_y^2$ to be the natural quotient. It is well known that $\mathfrak{m}_y/\mathfrak{m}_y^2$ is the dual of T_y, in this sense we have $(T_U)_y = \phi((\mathcal{I}_U)_y)^\perp$. Therefore the equation (2.3) is equivalent to

$$\phi((\mathcal{I}_U)_y)^\perp \cap \phi((\mathcal{I}_V)_y)^\perp = \phi((\mathcal{I}_W)_y)^\perp,$$

which is equivalent to

$$\phi((\mathcal{I}_U)_y) + \phi((\mathcal{I}_V)_y) = \phi((\mathcal{I}_W)_y).$$

Notice that $\phi((\mathcal{I}_U)_y) = ((\mathcal{I}_U)_y + \mathfrak{m}_y^2)/\mathfrak{m}_y^2$, so the above equality is equivalent to

$$(\mathcal{I}_U)_y + (\mathcal{I}_V)_y + \mathfrak{m}_y^2 = (\mathcal{I}_W)_y + \mathfrak{m}_y^2, \quad \forall y \in W. \tag{2.4}$$

Obviously (2.2) $\Rightarrow$ (2.4). To see (2.4) $\Rightarrow$ (2.2), observe first that $(\mathcal{I}_U)_y + (\mathcal{I}_V)_y \subseteq (\mathcal{I}_W)_y$. We also have $(\mathcal{I}_W)_y \cap \mathfrak{m}_y^2 = (\mathcal{I}_W)_y \mathfrak{m}_y$, which can be easily checked using local coordinates. Equality (2.4) implies the surjection

$$(\mathcal{I}_U)_y + (\mathcal{I}_V)_y \twoheadrightarrow ((\mathcal{I}_W)_y + \mathfrak{m}_y^2)/\mathfrak{m}_y^2 \xrightarrow{\cong} (\mathcal{I}_W)_y/((\mathcal{I}_W)_y \cap \mathfrak{m}_y^2) \xrightarrow{\cong} (\mathcal{I}_W)_y/(\mathcal{I}_W)_y\mathfrak{m}_y.$$

Hence $(\mathcal{I}_U)_y + (\mathcal{I}_V)_y + (\mathcal{I}_W)_y\mathfrak{m}_y = (\mathcal{I}_W)_y$. Now apply Nakayama's lemma (see [AM69] Corollary 2.7), we have $(\mathcal{I}_U)_y + (\mathcal{I}_V)_y = (\mathcal{I}_W)_y$. For a point $y \notin W$, both sides of (2.2) are $\mathcal{O}_y$ hence the equality trivially holds. Therefore the lemma has been proved. □

2.3.2 Arrangements of subvarieties

Let Y be a nonsingular algebraic variety over $\mathbb{C}$.

Definition 2.3.5. *An* arrangement of subvarieties *of Y is a finite set $\mathcal{S} = \{S_i\}$ of nonsingular closed irreducible subvarieties of Y satisfying the following conditions*

(1) S_i and S_j intersect cleanly *(i.e. their intersection is nonsingular and the tangent bundles satisfy $T(S_i \cap S_j) = T(S_i)|_{(S_i \cap S_j)} \cap T(S_j)|_{(S_i \cap S_j)}$),*

(2) $S_i \cap S_j$ is either empty or a disjoint union of some S_k's.

If instead of satisfying condition (2), $\mathcal{S}$ satisfies a stronger condition that $S_i \cap S_j$ is either empty or one S_k, then we call $\mathcal{S}$ a *simple* arrangement.

For simplicity of notation only, here we discuss simple arrangements. A general arrangement is locally simple, so that all the following discussion will apply.

Definition 2.3.6. *Let $\mathcal{S}$ be an arrangement of subvarieties of Y. A subset $\mathcal{G} \subseteq \mathcal{S}$ is called a building set (with respect to $\mathcal{S}$) if $\forall S \in \mathcal{S}$, the minimal elements in $\mathcal{G}$ which are $\geq S$ intersect transversally and their intersection is S (this condition is always satisfied if $S \in \mathcal{G}$). These minimal elements are called the $\mathcal{G}$-factors of S.*

Remark 2.3.7. *Fix a point $y \in Y$. Let $\mathcal{S}^*$ be the set $\{(T_{S_i})_y^{\perp}\}_{S_i \in \mathcal{S}}$, and $\mathcal{G}^* \subseteq \mathcal{S}^*$ be the set $\{(T_{S_i})_y^{\perp}\}_{S_i \in \mathcal{G}}$. The definition of arrangement of subvarieties (Definition 2.3.5) asserts that the set $\mathcal{S}^*$ is a finite set of nonzero subspaces of T_y^* closed under sum, and that each element of $\mathcal{S}^*$ is equal to $(T_{S'})_y^{\perp}$ for a unique $S' \in \mathcal{S}$.*

Definition 2.3.6 just says that $\mathcal{G}$ is a building set if the following holds:

$\forall S' \in \mathcal{S}$, $\forall y \in S'$, let $T_1^\perp, \ldots, T_k^\perp$ be the maximal elements of $\mathcal{G}^*$ contained in $(T_{S'})_y^\perp$, then they form a direct sum

$$T_1^\perp \oplus T_2^\perp \oplus \cdots \oplus T_k^\perp = (T_{S'})_y^\perp.$$

Remark 2.3.8. *[DP95] §2.3 Theorem (2) asserts that the above condition implies the following: If $S'' \in \mathcal{S}$ that $S'' \supseteq S'$, then*

$$(T_{S''})_y^\perp = \bigoplus_{i=1}^{k} \big((T_{S''})_y^\perp \cap T_i^\perp\big).$$

Moreover, if $(T_{S''})_y^\perp = T_1'^\perp \oplus \cdots \oplus T_s'^\perp$ where $T_1'^\perp, \ldots, T_s'^\perp$ are the maximal elements in $\mathcal{G}^*$ contained in $(T_{S''})_y^\perp$, then each term $(T_{S''})_y^\perp \cap T_i^\perp$ is a direct sum of some $T_j'^\perp$.

In the following two facts, assume $\mathcal{S}$ is a simple arrangement of subvarieties of Y and $\mathcal{G}$ is a building set.

Fact 2.3.9. *Suppose $S \in \mathcal{S}$ and let $G_1, \ldots, G_k$ be all the $\mathcal{G}$-factors of S (see Definition 2.3.6). Then*

i) For any $1 \leq m \leq k$, let $S' = G_1 \cap \cdots \cap G_m$. Then $G_1, \ldots, G_m$ are all the $\mathcal{G}$-factors of S'.

ii) Suppose $G \in \mathcal{G}$ is minimal, $G \cap S \neq \emptyset$, $G \subseteq G_1, \ldots, G_m$ and $G \nsubseteq G_{m+1}, \ldots, G_k$. Then $G, G_{m+1}, \ldots, G_k$ are all the $\mathcal{G}$-factors of $G \cap S$.

Proof. It is convenient to prove using the dual tangent space T_y^*.

i) Fix a point $y \in S$ and consider the tangent spaces at y. The statement i) is equivalent to the following:

Suppose $T_1^\perp, \ldots, T_k^\perp$ are all the maximal elements in $\mathcal{G}^*$ which are $\subseteq (T_S)_y^\perp$. For $1 \le m \le k$, let $T^\perp = (T_1)^\perp \oplus \cdots \oplus (T_m)^\perp$. Then $(T_1)^\perp, \ldots, (T_m)^\perp$ are all the maximal elements in $\mathcal{G}^*$ which are $\subseteq T^\perp$.

The proof is as follows: notice that $T^\perp \subseteq T_1^\perp \oplus \cdots \oplus T_k^\perp = (T_S)_y^\perp$. So any element $V \in \mathcal{G}^*$ which is $\subseteq T^\perp$ is also a subspace of $(T_S)_y^\perp$. By the maximality of $T_1^\perp, \ldots, T_k^\perp$, we know V lies inside $T_i^\perp$ for some $1 \le i \le k$. But $V \subseteq T^\perp$, so $i \le m$. The statement follows.

ii) Fix a point $y \in G \cap S$ and consider the tangent spaces at y. The statement ii) is equivalent to the following:

Suppose $T_1^\perp, \ldots, T_k^\perp$ are all the maximal elements in $\mathcal{G}^*$ that are $\subseteq (T_S)_y^\perp$, $T^\perp \in \mathcal{G}^*$ is maximal, $T^\perp \supseteq T_1^\perp, \ldots, T_m^\perp$ and $T^\perp \not\supseteq T_{m+1}^\perp, \ldots, T_k^\perp$. Then $T^\perp, T_{m+1}^\perp, \ldots, T_k^\perp$ are all the maximal elements in $\mathcal{G}^*$ which are $\subseteq T^\perp + (T_S)_y^\perp$.

The proof is as follows: Let $C_1, \ldots, C_s$ be the maximal elements in $\mathcal{G}^*$ that are $\subseteq T^\perp + (T_S)_y^\perp$. Since $\mathcal{G}$ is a building set, we have

$$T^\perp + (T_S)_y^\perp = C_1 \oplus \cdots \oplus C_s. \tag{2.5}$$

Then $T^\perp$ is contained in one of C_i. With out loss of generality, assume $T^\perp \subseteq C_1$. By the maximality of $T^\perp$, we have $T^\perp = C_1$. The equality (2.5) becomes

$$T^\perp + (T_{m+1}^\perp \oplus \cdots \oplus T_k^\perp) = T^\perp \oplus C_2 \oplus \cdots \oplus C_s.$$

Since each T_i $(m+1 \le i \le k)$ is contained in C_j for some $2 \le j \le s$, it follows that each C_j $(2 \le j \le s)$ is the direct sum of some T_i's $(m+1 \le i \le k)$. But then $C_j \subseteq (T_S)_y^\perp$, hence the maximality of $T_1^\perp, \ldots, T_k^\perp$ implies that $C_j \subseteq T_i^\perp$

for some $1 \leq i \leq k$. Therefore $C_j = T_i^\perp$ for some $m+1 \leq i \leq k$. Then $\{T^\perp, T_{m+1}^\perp, \ldots, T_k^\perp\} = \{C_1, C_2, \ldots, C_s\}$. The statement is proved. □

Fact 2.3.10. *Suppose $G \in \mathcal{G}$ is minimal. Then*

i) Any $G' \in \mathcal{G}$ either contains G or intersects transversally with G.

ii) Every $S \in \mathcal{S}$ satisfying $S \cap G \neq \emptyset$ can be uniquely expressed as $A \cap B$ where $A, B \in \mathcal{S}$ satisfy $A \supseteq G$ and $B \pitchfork G$ (hence $A \pitchfork B$). We call this expression the G-factorization of S.

iii) Suppose the $\mathcal{G}$-factors of S are $G_1, \ldots, G_k$, where $G_1, \ldots, G_m$ $(0 \leq m \leq k)$ contain G. ($m = 0$ means that no $\mathcal{G}$-factors of S contain G.)

Then in the G-factorization of S, $\{G_i\}_{i=1}^m$ are all the $\mathcal{G}$-factors of A (so $A = \cap_{i=1}^m G_i$) and $\{G_i\}_{i=m+1}^k$ are all the $\mathcal{G}$-factors of B (so $B = \cap_{i=m+1}^k G_i$). (Here we assume $A = Y$ if $m = 0$, and assume $B = Y$ if $m = k$.)

iv) Suppose $S' \in \mathcal{S}$ also intersect G and the G-factorization of S' is $A' \cap B'$. Then $G \pitchfork (B \cap B')$, so the G-factorization of $S \cap S'$ is $(A \cap A') \cap (B \cap B')$.

Proof. i) is induced directly from the definition of the building set: if G is disjoint from G' then of course $G \pitchfork G'$; otherwise G' contains some $\mathcal{G}$-factor of $G \cap G'$. But a $\mathcal{G}$-factor of $G \cap G'$ is either G or is transversal to G (which implies $G' \pitchfork G$).

ii) and iii). We prove A and B as defined in iii) satisfy $A \supseteq G$ and $B \pitchfork G$. $A = \cap_{i=1}^m G_i \supseteq G$ is because of the definition of m. Fact 2.3.9 ii) asserts $G \pitchfork G_{m+1} \pitchfork \cdots \pitchfork G_k$, so G is transversal to $(G_{m+1} \cap \cdots \cap G_k) = B$.

Statement iii) follows from Fact 2.3.9 i).

Now we show the uniqueness of G-factorization in ii). Assume $S = A' \cap B'$ such that $A' \supseteq G$ and $B' \pitchfork G$. Since $B' \supseteq G \cap B' = G \cap S$ and the $\mathcal{G}$-factors of $G \cap S$ are $G, G_{m+1}, \ldots, G_k$ by Fact 2.3.9 ii), so each $\mathcal{G}$-factor G' of B' contains G or G_i for some $m+1 \leq i \leq k$. But $B' \pitchfork G$ implies $G' \pitchfork G$, hence $G' \not\supseteq G$. So $G' \supseteq G_i$ for some $m+1 \leq i \leq k$. Take the intersection of all G', we have $B' \supseteq \cap_{i=m+1}^{k} G_i = B$. Fix a point $y \in G \cap S$, we have

$$(T_G)_y^{\perp} \oplus (T_B)_y^{\perp} = (T_G)_y^{\perp} \oplus (T'_B)_y^{\perp}$$

and $(T_B)_y^{\perp} \supseteq (T'_B)_y^{\perp}$, therefore $(T_B)_y^{\perp} = (T'_B)_y^{\perp}$ hence $B = B'$. Similarly $A = A'$.

iv). Suppose the G-factorization of $S \cap S'$ is $A'' \cap B''$. Then $G \cap B''$ is the G-factorization of the intersection. Since $B \supseteq (G \cap s) = (G \cap B'')$ but $B \pitchfork G$, so $B \supseteq B''$. Similarly $B' \supseteq B''$. So $B \cap B' \supseteq B''$. By an analogues argument using the dual of tangent space as above, $B \cap B' = B''$. So $G \pitchfork (B \cap B')$. □

2.3.3 Wonderful compactifications

We show that, if $G \in \mathcal{G}$ is minimal, then there is a simple natural arrangement $\mathcal{S}'$ of subvarieties in $Y' = Bl_G Y$; moreover, the induced $\mathcal{G}'$ in $Y' = Bl_G Y$ is again a building set.

Denote by E the exceptional divisor of the blow-up $\pi : Y' = Bl_G Y \to Y$.

Step 1: The arrangement $\mathcal{S}'$ of subvarieties in $Y' = Bl_G Y$.

Definition 2.3.11 (Definition of $\sim$). *For any irreducible nonsingular subvariety V in Y, we define $\widetilde{V} \subseteq Bl_G Y$ to be the strict transform of V if $V \nsubseteq G$*

and to be $\pi^{-1}(V)$, the preimage of V, if $V \subseteq G$. For a reducible nonsingular subvariety $V = \cup V_i$ where V_i are the connected components of V, we define $\widetilde{V} = \cup \widetilde{V_i}$.

Proposition 2.3.12. *The collection of subvarieties*

$$\mathcal{S}' := \{\widetilde{S}\}_{S \in \mathcal{S}} \cup \{\widetilde{S} \cap E\}_{\emptyset \subsetneq S \cap G \subsetneq S}$$

is a (simple) arrangement of subvarieties in $Bl_G Y$.

Moreover, $\mathcal{G}' := \{\widetilde{G}_i\}_{G_i \in \mathcal{G}}$ is a building set (with respect to the arrangement $\mathcal{S}'$).

Lemma 2.3.13. *(i) Let A be a nonsingular closed subvariety of Y that contains G as a proper subvariety. Then $\widetilde{A} \cap E$ intersect transversally (hence cleanly).*

(ii) Let A_1 and A_2 be two nonsingular closed subvarieties of Y that intersect cleanly. Suppose $A_1 \nsubseteq A_2$, $A_2 \nsubseteq A_1$, and $A_1 \cap A_2 = G$. Then $\widetilde{A}_1 \cap \widetilde{A}_2 = \emptyset$.

(iii) Let A_1 and A_2 be two nonsingular closed subvarieties of Y that intersect cleanly, and suppose G is a proper subvariety of a connected component of $A_1 \cap A_2$. Then $\widetilde{A}_1 \cap \widetilde{A}_2 = \widetilde{A_1 \cap A_2}$. Moreover $\widetilde{A}_1$ and $\widetilde{A}_2$ intersect cleanly.

(iv) Let B_1 and B_2 be two nonsingular closed subvarieties of Y that intersect cleanly, and assume G is a nonsingular closed subvariety that intersects transversally with B_1, B_2 and $B_1 \cap B_2$ respectively. Then $\widetilde{B}_1 \cap \widetilde{B}_2 = \widetilde{B_1 \cap B_2}$. Moreover $\widetilde{B}_1$ and $\widetilde{B}_2$ intersect cleanly.

(v) *Let A and B be two nonsingular closed subvarieties of Y that intersect transversally, and $G \subseteq A$, $G \pitchfork B$. Then $\widetilde{A} \cap \widetilde{B} = \widetilde{A \cap B}$. Moreover, $\widetilde{A} \pitchfork \widetilde{B}$, $(E \cap \widetilde{A}) \pitchfork \widetilde{B}$.*

Proof. (i) and (v) can be easily checked using local coordinates, which we omit here.

(ii) In the complement of the exceptional divisor E, we have

$$(\widetilde{A}_1 \cap \widetilde{A}_2) \setminus E \cong (A_1 \setminus G) \cap (A_2 \setminus G) = (A_1 \cap A_2) \setminus G = \emptyset.$$

Inside E, we have

$$\begin{aligned}(\widetilde{A}_1 \cap \widetilde{A}_2) \cap E &= \mathbb{P}(N_G A_1) \cap \mathbb{P}(N_G A_2) = \mathbb{P}(T_{A_1}/T_G) \cap \mathbb{P}(T_{A_2}/T_G) \\ &= \mathbb{P}((T_{A_1} \cap T_{A_2})/T_G) = \mathbb{P}(T_{A_1 \cap A_2}/T_G) = \mathbb{P}(T_G/T_G) = \emptyset.\end{aligned}$$

Hence $\widetilde{A}_1 \cap \widetilde{A}_2 = \emptyset$.

(iii) In the complement of the exceptional divisor E, we have

$$(\widetilde{A}_1 \cap \widetilde{A}_2) \setminus E \cong (A_1 \setminus G) \cap (A_2 \setminus G) = (A_1 \cap A_2) \setminus G = \widetilde{A_1 \cap A_2} \setminus E.$$

Inside E, we have

$$\begin{aligned}(\widetilde{A}_1 \cap \widetilde{A}_2) \cap E &= \mathbb{P}(N_G A_1) \cap \mathbb{P}(N_G A_2) = \mathbb{P}(T_{A_1}/T_G) \cap \mathbb{P}(T_{A_2}/T_G) \\ &= \mathbb{P}((T_{A_1} \cap T_{A_2})/T_G) = \mathbb{P}(T_{A_1 \cap A_2}/T_G) = \mathbb{P}(N_S(A_1 \cap A_2)) \\ &= \widetilde{A_1 \cap A_2} \cap E,\end{aligned}$$

where the fourth equality is because A_1 and A_2 intersect cleanly.

Hence $\widetilde{A}_1 \cap \widetilde{A}_2 = \widetilde{A_1 \cap A_2}$.

According to Lemma 2.3.4, $\widetilde{A}_1$ and $\widetilde{A}_2$ intersect cleanly if and only if

$$\mathcal{I}_{\widetilde{A}_1} + \mathcal{I}_{\widetilde{A}_2} = \mathcal{I}_{\widetilde{A_1 \cap A_2}}. \tag{2.6}$$

But $\widetilde{A}_1 = \mathcal{R}(E, \pi^{-1}(A_1))$, the residue scheme to E in $\pi^{-1}(A_1)$ (see [Ke93] Theorem 1, [Fu98] §9.2). By a property of residue scheme, we have

$$\mathcal{I}_{\mathcal{R}(E,\pi^{-1}(A_1))} \cdot \mathcal{I}_E = \mathcal{I}_{\pi^{-1}(A_1)},$$

which is same as

$$\mathcal{I}_{\widetilde{A}_1} \cdot \mathcal{I}_E = \mathcal{I}_{\pi^{-1}(A_1)}.$$

Similarly, we have

$$\mathcal{I}_{\widetilde{A}_2} \cdot \mathcal{I}_E = \mathcal{I}_{\pi^{-1}(A_2)},$$

$$\mathcal{I}_{\widetilde{A_1 \cap A_2}} \cdot \mathcal{I}_E = \mathcal{I}_{\pi^{-1}(A_1 \cap A_2)}.$$

Since A_1 and A_2 intersect cleanly, so $\mathcal{I}_{A_1} + \mathcal{I}_{A_2} = \mathcal{I}_{A_1 \cap A_2}$, which implies

$$\pi^{-1}\mathcal{I}_{A_1} \cdot \mathcal{O}_{Y'} + \pi^{-1}\mathcal{I}_{A_2} \cdot \mathcal{O}_{Y'} = \pi^{-1}\mathcal{I}_{A_1 \cap A_2} \cdot \mathcal{O}_{Y'}.$$

This is equivalent to

$$\mathcal{I}_{\pi^{-1}(A_1)} + \mathcal{I}_{\pi^{-1}(A_2)} = \mathcal{I}_{\pi^{-1}(A_1 \cap A_2)}.$$

Thus we get an equality

$$\mathcal{I}_{\widetilde{A}_1} \cdot \mathcal{I}_E + \mathcal{I}_{\widetilde{A}_2} \cdot \mathcal{I}_E = \mathcal{I}_{\widetilde{A_1 \cap A_2}} \cdot \mathcal{I}_E.$$

Since $\mathcal{I}_E$ is an invertible sheaf, the above equality implies (2.6), hence (iii) is proved.

(iv) In the complement of the exceptional divisor E, we have

$$(\widetilde{B}_1 \cap \widetilde{B}_2) \setminus E \cong (B_1 \setminus G) \cap (B_2 \setminus G) = (B_1 \cap B_2) \setminus G = \widetilde{B_1 \cap B_2} \setminus E.$$

If a nonsingular closed (not necessarily irreducible) subvariety B intersects transversally with S, then we have the following standard fact:

$$\widetilde{B} \cap E = \mathbb{P}(N_{G \cap B} B) = \mathbb{P}(N_G|_{G \cap B}).$$

By this fact,

$$\begin{aligned}(\widetilde{B}_1 \cap \widetilde{B}_2) \cap E &= \mathbb{P}(N_G|_{G \cap B_1}) \cap \mathbb{P}(N_G|_{G \cap B_2}) = \mathbb{P}(N_G|_{G \cap B_1 \cap B_2}) \\ &= \widetilde{B_1 \cap B_2} \cap E.\end{aligned}$$

Hence $\widetilde{B}_1 \cap \widetilde{B}_2 = \widetilde{B_1 \cap B_2}$.

Similarly to (iii), to show $\widetilde{B}_1$ intersect cleanly with $\widetilde{B}_2$, it is enough to show that

$$\mathcal{I}_{\widetilde{B}_1} + \mathcal{I}_{\widetilde{B}_2} = \mathcal{I}_{\widetilde{B_1 \cap B_2}}. \tag{2.7}$$

Since B_1 intersect transversally with the center G of the blow-up, it can be easily checked (using local coordinates) that

$$\mathcal{I}_{\widetilde{B}_i} = \pi^{-1}\mathcal{I}_{B_i}, \text{ for } i = 1, 2.$$

By the assumption that B_1 and B_2 intersect cleanly, we have

$$\mathcal{I}_{B_1} + \mathcal{I}_{B_2} = \mathcal{I}_{B_1 \cap B_2},$$

hence

$$\pi^{-1}\mathcal{I}_{B_1} \cdot \mathcal{O}_{Y'} + \pi^{-1}\mathcal{I}_{B_2} \cdot \mathcal{O}_{Y'} = \pi^{-1}\mathcal{I}_{B_1 \cap B_2} \cdot \mathcal{O}_{Y'},$$

and (2.7) follows.

□

Proof. (of Proposition 2.3.12)

Suppose $S, S' \in \mathcal{S}$. By Fact 2.3.10, consider the G-factorization of $S = A \cap B$ and $S' = A' \cap B'$. Then the G-factorization of $S \cap S'$ is $(A \cap A') \cap (B \cap B')$.

Lemma 2.3.13(v) asserts that $\widetilde{S} = \widetilde{A} \cap \widetilde{B}$, $\widetilde{S}' = \widetilde{A}' \cap \widetilde{B}'$. We prove first that $\widetilde{S}$ and $\widetilde{S}'$ intersect cleanly along some element in $\mathcal{S}'$. There are three cases:

1) $G \subsetneq A \cap A'$. In this case we have $(S \cap S')^{\sim} = (A \cap A')^{\sim} \cap (B \cap B')^{\sim}$, and

$$\widetilde{S} \cap \widetilde{S}' = (\widetilde{A} \cap \widetilde{A}') \cap (\widetilde{B} \cap \widetilde{B}') = (A \cap A')^{\sim} \cap (B \cap B')^{\sim} = (S \cap S')^{\sim}.$$

Moreover, the tangent bundles satisfy

$$T_{\widetilde{S}} \cap T_{\widetilde{S}'} = T_{\widetilde{A}} \cap T_{\widetilde{B}} \cap T_{\widetilde{A}'} \cap T_{\widetilde{B}'} = T_{(A \cap A')^{\sim}} \cap T_{(B \cap B')^{\sim}} = T_{(S \cap S')^{\sim}}.$$

Thus $\widetilde{S}$ intersects $\widetilde{S}'$ cleanly along $(S \cap S')^{\sim} \in \mathcal{S}'$.

2) $G = A \cap A'$ but $G \neq A$ and $G \neq A'$. By Lemma 2.3.13 (ii), $\widetilde{A} \cap \widetilde{A}' = \emptyset$, hence

$$\widetilde{S} \cap \widetilde{S} = (\widetilde{A} \cap \widetilde{A}') \cap (\widetilde{B} \cap \widetilde{B}') = \emptyset.$$

3) $S = A$ or A'. Without loss of generality, we assume $S = A$.

$$\widetilde{S} \cap \widetilde{S}' = (\widetilde{A} \cap \widetilde{A}') \cap (\widetilde{B} \cap \widetilde{B}') = E \cap (\widetilde{A}' \cap \widetilde{B} \cap \widetilde{B}') = E \cap (A' \cap B \cap B')^{\sim}.$$

Moreover, by Lemma 2.3.13 (i) and (v), the tangent bundles satisfy

$$T_{\widetilde{S}} \cap T_{\widetilde{S}'} = (T_E \cap T_{\widetilde{B}}) \cap (T_{\widetilde{A}'} \cap T_{\widetilde{B}'}) = (T_E \cap T_{\widetilde{A}'}) \cap T_{(B \cap B')^{\sim}} = T_{E \cap (A' \cap B \cap B')^{\sim}}.$$

Thus $\widetilde{S}$ intersects $\widetilde{S}'$ cleanly along $E \cap (A \cap B \cap B')^{\sim} \in \mathcal{S}'$.

Next we show that $\forall \widetilde{S}, (\widetilde{S}' \cap E) \in \mathcal{S}'$, they intersect cleanly along some element in $\mathcal{S}'$. There are again three cases, similar as above:

1') $G \subsetneq A \cap A'$. In this case

$$\widetilde{S} \cap (\widetilde{S}' \cap E) = (S \cap S')^{\sim} \cap E,$$

and the tangent bundles satisfy

$$T_{\widetilde{S}} \cap T_{\widetilde{S}' \cap E} = T_{\widetilde{A}} \cap T_{\widetilde{B}} \cap T_{\widetilde{A}'} \cap T_{\widetilde{B}'} \cap T_E = T_{(A \cap A')^{\sim} \cap E} \cap T_{(B \cap B')^{\sim}} = T_{(S \cap S')^{\sim} \cap E}.$$

Thus $\widetilde{S}$ intersects $\widetilde{S}' \cap E$ cleanly along $(S \cap S')^{\sim} \cap E \in \mathcal{S}'$.

2') $G = A \cap A'$ but $G \neq A$ and $G \neq A'$. We have $\widetilde{S} \cap (\widetilde{S}' \cap E) = \emptyset$.

3') $G = A$. (Notice that $G \neq A'$ by the definition of $\mathcal{S}'$.) We have

$$\widetilde{S} \cap (\widetilde{S}' \cap E) = E \cap (A' \cap B \cap B')^{\sim}$$

and the equality of tangent bundles

$$T_{\widetilde{S}} \cap T_{\widetilde{S}' \cap E} = (T_E \cap T_{\widetilde{B}}) \cap (T_{\widetilde{A}'} \cap T_{\widetilde{B}'} \cap T_E) = T_{E \cap (A' \cap B \cap B')^{\sim}}.$$

Then we show that $\forall (\widetilde{S} \cap E), (\widetilde{S}' \cap E) \in \mathcal{S}'$, they intersect cleanly along some element in $\mathcal{S}'$. There are two cases:

1") $G \subsetneq A \cap A'$. In this case

$$(\widetilde{S} \cap E) \cap (\widetilde{S}' \cap E) = (S \cap S')^{\sim} \cap E,$$

and the tangent bundles satisfy

$$T_{\widetilde{S} \cap E} \cap T_{\widetilde{S}' \cap E} = T_{\widetilde{A}} \cap T_{\widetilde{B}} \cap T_{\widetilde{A}'} \cap T_{\widetilde{B}'} \cap T_E = T_{(A \cap A')^{\sim} \cap E} \cap T_{(B \cap B')^{\sim}} = T_{(S \cap S')^{\sim} \cap E}.$$

Thus $\widetilde{S} \cap E$ intersects $\widetilde{S}' \cap E$ cleanly along $(S \cap S')^{\sim} \cap E \in \mathcal{S}'$.

2") $G = A \cap A'$ but $G \neq A$ and $G \neq A'$. We have $(\widetilde{S} \cap E) \cap (\widetilde{S}' \cap E) = \emptyset$.

Finally we show that $\mathcal{G}' := \{\widetilde{G}_i\}_{G_i \in \mathcal{G}}$ is a building set, that is, $\forall \widetilde{S}$ (resp. $(\widetilde{S} \cap E)$) $\in \mathcal{S}'$, the $\mathcal{G}'$-factors of $\widetilde{S}$ (resp. of $(\widetilde{S} \cap E)$)intersect transversally along $\widetilde{S}$ (resp. along $(\widetilde{S} \cap E)$).

By Fact 2.3.10, we can assume $S = (G_1 \pitchfork \cdots \pitchfork G_m) \pitchfork (G_{m+1} \pitchfork \cdots \pitchfork G_k) = A \pitchfork B$, $G \subseteq G_1, \ldots, G_m$, and $G \pitchfork G_{m+1}, \ldots, G_k$. Then $\widetilde{S} = \widetilde{A} \cap \widetilde{B}$ by Lemma 2.3.13 (v).

Case I: $G \subsetneq A$. Lemma 2.3.13 implies that

$$\widetilde{S} = \widetilde{G}_1 \pitchfork \cdots \pitchfork \widetilde{G}_k.$$

Moreover, $\widetilde{G}_1, \ldots, \widetilde{G}_k$ are all the $\mathcal{G}'$-factors of$\widetilde{S}$. (Indeed, if $\widetilde{G}' \in \mathcal{G}'$ satisfies $\widetilde{G}' \supseteq \widetilde{S}$, then $\pi(\widetilde{G}') \supseteq \pi(\widetilde{S})$, i.e. $G' \supseteq S$. Since $G_1, \ldots, G_k$ are all the minimal elements in $\mathcal{G}$ that are $\supseteq S$, so $G' \supseteq G_r$ for some $1 \leq r \leq k$. Then their strict transforms still have the inclusion relation $\widetilde{G}' \supseteq \widetilde{G}_r$.) Therefore the $\mathcal{G}'$-factors of $\widetilde{S}$ intersect transversally.

Next we show that the $\mathcal{G}'$-factors of $(\widetilde{G}' \cap E)$ intersect transversally.

$$\widetilde{S} \cap E = E \pitchfork \widetilde{A} \pitchfork \widetilde{B} = E \pitchfork \widetilde{G}_1 \pitchfork \cdots \pitchfork \widetilde{G}_k.$$

We show that $E, \widetilde{G}_1, \cdots, \widetilde{G}_k$ are all the $\mathcal{G}'$-factors of $(\widetilde{S} \cap E)$. It is enough to show that $\forall \widetilde{G}' \in \mathcal{G}'$ satisfying $\widetilde{G}' \supseteq (\widetilde{S} \cap E)$, we have either $\widetilde{G}' = E$ or $\widetilde{G}' \supseteq \widetilde{G}_r$ for some $1 \leq r \leq k$.

$\widetilde{G}' \supseteq (\widetilde{S} \cap E)$ implies $G' \supseteq (S \cap G)$ by taking the image of π. By Fact 2.3.9 (ii), we know that G, $G_{m+1}, \ldots, G_k$ are all the $\mathcal{G}$-factors of $(S \cap G)$. Therefore G' contains either G or one of G_r for $m + 1 \leq r \leq k$. In the latter case, we immediate get the conclusion. So we assume that G' contains G.

If $G' = G$, then $\widetilde{G}' = E$ and we get the conclusion. Thus in the following we assume $G' \supsetneq G$. Since

$$\widetilde{G}' \cap E = \mathbb{P}(T_{G'}/T_G),$$

$$\widetilde{S} \cap E = \mathbb{P}((T_A/T_G)|_{G \cap B})$$

and $\widetilde{G'} \cap E \supseteq \widetilde{S} \cap E$, so $\forall y \in G \cap B$, we have $(T_{G'})_y \supseteq (T_A)_y$, which implies $G' \supseteq A = G_1 \cap \cdots \cap G_m$. But $G_1, \ldots, G_l$ are the $\mathcal{G}$-factors of A by Fact 2.3.9 (i). Therefore $G' \in \mathcal{G}$ contains G_r for some $1 \leq r \leq l$.

Case II: $G = A$. By Fact 2.3.9 (ii), $S_1, \ldots, S_m$ are all the $\mathcal{G}$-factors of A. But $A = G$ is already in $\mathcal{G}$, so $m = 1$ and $G_1 = G$. Then $S = G \pitchfork B$ and hence $\widetilde{S} = E \pitchfork \widetilde{B}$.

Now we show that $E, \widetilde{G}_2, \ldots, \widetilde{G}_k$ are all the $\mathcal{G}'$-factors of $(E \pitchfork \widetilde{B})$. Suppose $\widetilde{S} \subseteq \widetilde{G'} \in \mathcal{G}'$. Take the image under π, we have $S \subseteq G'$, hence $G' \supseteq G$ or $G' \supseteq G_r$ for some $2 \leq r \leq k$. The latter case is the expected conclusion. In the former case, if $G' = G$ then $\widetilde{G'} = E$ which also gives the conclusion. So we can assume $G' \supsetneq G$. Fix a point $y \in S$,

$$\widetilde{G'} \cap \pi^{-1}(y) = \mathbb{P}((T_{G'})_y/(T_G)_y),$$
$$\widetilde{S} \cap \pi^{-1}(y) = \mathbb{P}(T_y/(T_G)_y).$$

Then $\widetilde{G'} \supseteq \widetilde{S}$ implies

$$\mathbb{P}((T_{G'})_y/(T_G)_y) \supseteq \mathbb{P}(T_y/(T_G)_y),$$

hence $(T_{G'})_y \supseteq T_y$, contradicts the fact that G' is a proper nonsingular subvariety of Y.

Therefore, $\mathcal{G}'$ is a building set with respect to the arrangement $\mathcal{S}'$.

□

Lemma 2.3.14. *Let Y be a nonsingular algebraic variety over $\mathbb{C}$. G and V are two nonsingular subvarieties of Y either intersect transversally or one* *contains the other. Let $f : Y_1 \to Y$ (resp. $g : Y_2 \to Y$) be the blow-up of Y along G (resp. V). Let $\widetilde{V}$ be the the $f^{-1}(V)$ if $V \subseteq G$, be the strict transform of V with respect to f otherwise. Let $g' : Y_3 \to Y_1$ be the blow-up of Y_1 along $\widetilde{V}$. Then there exists a morphism $f' : Y_3 \to Y_2$ such that the following diagram commutes:*

$$\begin{array}{ccc} Y_3 & \xrightarrow{f'} & Y_2 \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y_1 & \xrightarrow{f} & Y \end{array}$$

Moreover, $g' \boxtimes f' : Y_3 \to Y_1 \times Y_2$ is a closed immersion.

Proof. Because of the universal property of blowing up (see [Ha77] Proposition 7.14), to show the existence of f', we need only to show that $(fg')^{-1}\mathcal{I}_V \cdot \mathcal{O}_{Y_3}$ is an invertible sheaf of ideals on Y_3. This is true because

$$f^{-1}\mathcal{I}_V \cdot \mathcal{O}_{Y_1} = \mathcal{I}_{\widetilde{V}} \text{ or } \mathcal{I}_{\widetilde{V}} \cdot \mathcal{I}_E$$

where E is the exceptional divisor of the blow-up $f : Y_1 \to Y$. Hence

$$(fg')^{-1}\mathcal{I}_V \cdot \mathcal{O}_{Y_3} = g'^{-1}(f^{-1}\mathcal{I}_V \cdot \mathcal{O}_{Y_1}) \cdot \mathcal{O}_{Y_3} = g'^{-1}\mathcal{I}_{\widetilde{V}} \cdot \mathcal{O}_{Y_3} \text{ or } g'^{-1}(\mathcal{I}_{\widetilde{V}} \cdot \mathcal{I}_E) \cdot \mathcal{O}_{Y_3}.$$

$g'^{-1}\mathcal{I}_{\widetilde{V}}$ is invertible by the construction of g', therefore the above ideal sheaf is invertible.

The fact that $g' \boxtimes f'$ is a closed immersion can be checked using local coordinates. □

Lemma 2.3.15. *Suppose $X_1, X_2, X_3, Y_1, Y_2, Y_3$ are nonsingular varieties such* *that the following diagram commutes,*

$$\begin{array}{ccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} \\ Y_1 & \xrightarrow{h_1} & Y_2 & \xrightarrow{h_2} & Y_3 \end{array}$$

If $g_1 \boxtimes f_1 : X_1 \to Y_1 \times X_2$ *and* $g_2 \boxtimes f_2 : X_2 \to Y_2 \times X_3$ *are closed immersions, then* $g_1 \boxtimes (f_2 f_1) : X_1 \to Y_1 \times X_3$ *is also a closed immersion.*

As a consequence, if we have the following commutative diagram

$$\begin{array}{ccccccc} X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{k-1}} & X_k \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & & & \downarrow{\scriptstyle g_k} \\ Y_1 & \xrightarrow{h_1} & Y_2 & \xrightarrow{h_2} & \cdots & \xrightarrow{h_{k-1}} & Y_k \end{array}$$

and $g_i \boxtimes f_i : X_i \to Y_i \times X_{i+1}$ *is closed immersion for all* $1 \leq i \leq k-1$*, then* $g_1 \boxtimes (f_{k-1} \cdots f_1) : X_1 \to Y_1 \times X_k$ *is also a closed immersion.*

Proof. The composition of two closed immersion is still a closed immersion, so

$$g \boxtimes (g_2 f_1) \boxtimes (f_2 f_1) : X_1 \to Y_1 \times Y_2 \times X_3$$

is a closed immersion. Denote by X_1' the image (which is a closed subvariety of $Y_1 \times Y_2 \times X_3$). Consider the projection $\pi_{13} : Y_1 \times Y_2 \times X_3 \to Y_1 \times X_3$, and the morphism $\Gamma_{h_1} \times 1_{X_3} : Y_1 \times X_3 \to Y_1 \times Y_2 \times X_3$. Notice that $\pi_{13} \circ (\Gamma_{h_1} \times 1_{X_3})$ is the identity automorphism of $Y_1 \times X_3$, and $(\Gamma_{h_1} \times 1_{X_3}) \circ \pi_{13}|_{X'}$ is the identity automorphism of X'. The conclusion that $g_1 \boxtimes (f_2 f_1) : X_1 \to Y_1 \times X_3$ is also a closed immersion follows from this. □

Step 2: Construction of $Y_{\mathcal{G}}$.

Let Y be a nonsingular algebraic variety over $\mathbb{C}$ with an arrangement of subvarieties $\mathcal{S}$ (see Definition 2.3.5). Let $\mathcal{G}$ be a building set with respect to $\mathcal{S}$ (see Definition 2.3.6).

Similar to the Fulton-MacPherson compactification, we give two constructions of $Y_{\mathcal{G}}$ and show that they coincide.

I. Construction as a closure.

Definition 2.3.16. *Define $Y^\circ = Y \setminus \cup_{G\in\mathcal{G}} G$. There is a natural locally closed embedding*

$$Y^\circ \hookrightarrow Y \times \prod_{G\in\mathcal{G}} Bl_G Y.$$

The closure of this embedding is called the wonderful compactification with respect to $\mathcal{G}$, denoted by $Y_{\mathcal{G}}$.

II. Construction by a sequence of blow-ups. Suppose

$$\mathcal{G} = \{G_1, \ldots, G_N\}$$

is indexed in an order compatible with inclusion relations, i.e. $i \leq j$ if $G_i \subseteq G_j$. We give an inductive construction of $Y_{\mathcal{G}}$, by defining Y_k with an arrangement of subvarieties $\mathcal{S}^{(k)}$, and a building set $\mathcal{G}^{(k)} = \{G_i^{(k)}\}_{i=1}^N$ with respect to $\mathcal{S}^{(k)}$.

For $k = 0$, Let $Y_0 = Y$, $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{G}^{(0)} = \mathcal{G}$.

Assume Y_{k-1} is already constructed.

In Y_{k-1}, $G_k^{(k-1)}$ is minimal in the building set $\mathcal{G}^{(k-1)}$ (because $G_i^{(k-1)}$ for $i < k$ become divisors hence are not contained in $G_k^{(k-1)}$).

Let Y_k be the blow-up of Y_{k-1} along the nonsingular subvariety $G_k^{(k-1)}$. Define $G^{(k)} := \widetilde{G^{(k-1)}}$ for $\forall G \in \mathcal{G}$ (cf. Definition 2.3.11). By Proposition 2.3.12, let $\mathcal{S}^{(k)}$ be the induced arrangement $(\mathcal{S}^{(k-1)})'$, and let $\mathcal{G}^{(k)}$ be the induced building set $(\mathcal{G}^{(k-1)})' = \{G^{(k)}\}_{G\in\mathcal{G}}$.

Finally, we get a nonsingular variety Y_N where all elements in the building set $\mathcal{G}^{(N)}$ are divisors. We show that Y_N is isomorphic to $Y_{\mathcal{G}}$ defined in Definition 2.3.16.

Proposition 2.3.17. *Y_N is isomorphic to $Y_{\mathcal{G}}$, the closure of the inclusion*

$$Y^\circ \hookrightarrow Y \times \prod_{G\in\mathcal{G}} Bl_G Y.$$

Proof. We prove by induction that Y_k is the closure of the inclusion

$$Y^\circ \hookrightarrow Y \times \prod_{i=1}^{k} Bl_{G_i} Y.$$

The proposition is the special case $k = N$.

Let $0 \le i \le k-1$. Since $G_{i+1}^{(i)}$ is minimal in $\mathcal{G}^{(i)}$, Fact 2.3.10 (i) asserts that there are only two possible relations between the nonsingular subvarieties $G_k^{(i)}$ and $G_{i+1}^{(i)}$ of Y_i: either $G_k^{(i)} \supseteq G_{i+1}^{(i)}$ or $G_k^{(i)} \pitchfork G_{i+1}^{(i)}$.

Therefore Lemma 2.3.14 applies. Since $G_k^{(i+1)} = \widetilde{G_k^{(i)}}$, then there exists a morphism f' such that following diagram commutes,

$$\begin{array}{ccc} Bl_{G_k^{(i+1)}} Y_{i+1} & \xrightarrow{f'} & Bl_{G_k^{(i)}} Y_i \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y_{i+1} & \xrightarrow{f} & Y_i \end{array}$$

meanwhile, the morphism $g'\boxtimes f' : Bl_{G_k^{(i+1)}} Y_{i+1} \to Y_{i+1} \times Bl_{G_k^{(i)}} Y_i$ is a closed immersion. Use Lemma 2.3.15 on the following diagram

$$\begin{array}{ccccccc} Bl_{G_k^{(k-1)}} Y_{k-1} & \longrightarrow & Bl_{G_k^{(k-2)}} Y_{k-2} & \longrightarrow \cdots \longrightarrow & Bl_{G_k^{(0)}} Y_0 \\ \downarrow & & \downarrow & & \downarrow \\ Y_{k-1} & \longrightarrow & Y_{k-2} & \longrightarrow \cdots \longrightarrow & Y_0 \end{array}$$

and notice that $Y_k = Bl_{G_k^{(k-1)}} Y_{k-1}$, $G_k^{(0)} = G_k$, $Y_0 = Y$, we have

$$Y_k \to Y_{k-1} \times Bl_{G_k} Y \text{ is closed immersion.}$$

Since composition of closed immersions is still a closed immersion, so

$$Y_k \to Y \times \prod_{i=1}^{k} Bl_{G_i} Y$$

is a closed immersion by the inductive assumption. (Actually, the only case we need the factor Y is when $\mathcal{G} = \emptyset$.) Then since Y° is an open subset of Y_k and Y_k is irreducible, the following composition

$$Y^\circ \hookrightarrow Y_k \hookrightarrow Y \times \prod_{i=1}^{k} Bl_{G_i} Y$$

implies that the closure of Y° is Y_k. □

Step 3: $\mathcal{G}$-nests.

Definition 2.3.18. *A subset $\mathcal{T} \subseteq \mathcal{G}$ is called a* $\mathcal{G}$-nest *(or* $\mathcal{G}$-nested*) if it satisfies one of the following equivalent relations:*

1. *There is a flag of elements in $\mathcal{S}$: $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$, such that*

$$\mathcal{T} = \bigcup_{i=1}^{k} \{A : \ A \text{ is a } \mathcal{G}\text{-factor of } S_i\}.$$

   (We call $\mathcal{T}$ is induced by the flag $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$.)

2. *Let $A_1, \ldots, A_k$ be the minimal elements of $\mathcal{T}$, then they are all the $\mathcal{G}$-factors of certain element in $\mathcal{S}$, and each set $\{A \in \mathcal{T} : A \supseteq A_i\}$ is also $\mathcal{G}$-nested defined by induction.*

Now we show that after blowing up Y along a minimal element $G \in \mathcal{G}$, a $\mathcal{G}$-nest $\mathcal{T}$ gives a $\mathcal{G}'$-nest $\mathcal{T}'$.

Proposition 2.3.19. *Let $\mathcal{T}$ be a subset of $\mathcal{G}$. Define $\mathcal{T}' := \{\widetilde{A}\}_{A \in \mathcal{T}} \subseteq \mathcal{G}'$. Then $\mathcal{T}$ is a $\mathcal{G}$-nest if and only if $\mathcal{T}'$ is a $\mathcal{G}'$-nest.*

Proof. "$\Rightarrow$": Suppose $\mathcal{T}$ is induced by the flag $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$. If $S_1 \nsubseteq G$ or $S_k \subseteq G$, then $\mathcal{T}'$ is induced by the flag $\widetilde{S}_1 \subseteq \widetilde{S}_2 \subseteq \cdots \subseteq \widetilde{S}_k$; otherwise there is $1 \leq m \leq k-1$ where $S_m \subseteq G$ but $S_{m+1} \nsubseteq G$. In this case $\mathcal{T}'$ is generated by the flag

$$(\widetilde{S}_1 \cap \widetilde{S}_{m+1}) \subseteq \cdots \subseteq (\widetilde{S}_m \cap \widetilde{S}_{m+1}) \subseteq (\widetilde{S}_{m+1} \cap E) \subseteq \cdots \subseteq (\widetilde{S}_k \cap E). \tag{2.8}$$

Indeed, for $1 \leq i \leq m$, take the G-factorization $S_i = S \cap B_i$ and $S_{m+1} = A \cap B$ (see Fact 2.3.10). Notice that B is the intersection of all the $\mathcal{G}$-factors of S_{m+1} which are transversal to G, and each G' of these $\mathcal{G}$-factors contains S_i, hence contains some $\mathcal{G}$-factor of S_i. But G' is transversal to G hence does not contain G, so G' must contain some $\mathcal{G}$-factor of B_i. Therefore $B \supseteq B_i$, which implies

$\widetilde{B} \supseteq \widetilde{B}_i$. Then

$$\widetilde{S}_1 \cap \widetilde{S}_{m+1} = (E \cap \widetilde{B_i}) \cap (\widetilde{A} \cap \widetilde{B}) = (\widetilde{A} \cap E) \cap \widetilde{B}_i = E \cap (A \cap B_i)^{\sim}.$$

In the proof of Proposition 2.3.12, we have shown that the $\mathcal{G}'$-factors of $E \cap (A \cap B_i)^{\sim}$ are E and all the $\mathcal{G}'$-factors of $(A \cap B_i)^{\sim}$. Equivalently, the set of $\mathcal{G}'$-factors of $E \cap (A \cap B_i)^{\sim}$ consists of all $\mathcal{G}'$-factors of $\widetilde{A}$, all $\mathcal{G}'$-factors of $\widetilde{B}_i$, and E. For $m+1 \le i \le k$, the $\mathcal{G}'$-factors of $(\widetilde{S}_i \cap E)$ are E and all the $\mathcal{G}'$-factors of $\widetilde{S}_i$. Hence the flag (2.8) induces a $\mathcal{G}'$-nest consists of E $(= \widetilde{G})$, all the $\mathcal{G}'$-factors of $\widetilde{B}_i$ (which are the strict transforms of the $\mathcal{G}$-factors of B_i) for $1 \le i \le m$, and all the $\mathcal{G}'$-factors of $\widetilde{S}_i$ (which are the strict transforms of the $\mathcal{G}$-factors of S_i) for $m+1 \le i \le k$. This $\mathcal{G}'$-nest is exactly $\mathcal{T}'$.

"$\Leftarrow$": Suppose $\mathcal{T}'$ is induced by the flag $S'_1 \subseteq S'_2 \subseteq \cdots \subseteq S'_k$. If $S'_1 \nsubseteq E$, then $\mathcal{T}$ is induced by the flag $\pi(S'_1) \subseteq \pi(S'_2) \subseteq \cdots \subseteq \pi(S'_k)$. Now let m be the maximal integer satisfying $S'_m \subseteq E$. Since E is both minimal and maximal in $\mathcal{G}'$, we have the E-factorization $S'_i = E \cap C'_i$ for $1 \le i \le m$. Then $\mathcal{T}$ is induced by the following flag

$$(G \cap \pi(C'_1)) \subseteq \pi(C'_1) \subseteq \cdots \subseteq \pi(C'_m) \subseteq \pi(S'_{m+1}) \subseteq \cdots \subseteq \pi(S'_k). \qquad (2.9)$$

We first show this is really a flag by showing that $\pi(C'_m) \subseteq \pi(S'_{m+1})$. Since $E \cap C'_m \subseteq S'_{m+1}$, and each $\mathcal{G}'$-factor G' of S'_{m+1} contains either E or C'_m. But G' cannot contain E, otherwise G' must be E (since E is maximal) and hence $S'_{m+1} \subseteq E$ which contradicts our assumption. So every $\mathcal{G}'$-factor of S'_{m+1} contains C'_m. This implies $C'_m \subseteq S'_{m+1}$, hence $\pi(C'_m) \subseteq \pi(S'_{m+1})$.

Then we show that the flag (2.9) induces $\mathcal{T}$. By Proposition 2.3.12, for $1 \le i \le m$, there exists $C_i \in \mathcal{S}$ where $C_i \nsubseteq G$, such that $C_i' = \widetilde{C}_i$. Then $\pi(C_i') = C_i$, and the $\mathcal{G}'$-factors of C_i' are the strict transforms of the $\mathcal{G}$-factors of C_i. Similarly, the $\mathcal{G}'$-factors of S_i' are the strict transforms of the $\mathcal{G}$-factors of $\pi(S_i)$. Moreover, suppose the G-factorization of C_1 is $A_1 \cap B_1$ (see Fact 2.3.10), then

$$G \cap \pi(C_1') = G \cap A_1 \cap B_1 = G \cap B_1.$$

Hence the $\mathcal{G}$-factors of $(G \cap \pi(C_1'))$ are G and all $\mathcal{G}$-factors of B_1 (notice that their strict transforms are $\mathcal{G}'$-factors of C_1', hence are $\mathcal{G}'$-factors of S_1'). Therefore, the flag (2.9) induces the nest consists of G, all $\mathcal{G}$-factors of C_i $(1 \le i \le m)$ and all $\mathcal{G}$-factors of $\pi(S_i')$ $(m+1 \le i \le k)$. This is exactly $\mathcal{T}$. □

For any subset $\mathcal{T} \subseteq \mathcal{G}$, define $Y_k\mathcal{T} = \bigcap_{G \in \mathcal{T}} G^{(k)}$.

Proposition 2.3.20. *Let $0 \le k \le N-2$ and let $\mathcal{T} \subseteq \{G_{k+2}, G_{k+3}, \ldots, G_N\}$ be a $\mathcal{G}$-nest. Then $Y_{k+1}\mathcal{T}$ is an irreducible nonsingular subvariety of Y_{k+1} with the following property:*

If $\mathcal{T} \cup \{G_{k+1}\}$ is not a $\mathcal{G}$-nest, then $G_{k+1}^{(k)} \cap Y_k\mathcal{T} = \emptyset$ and $Y_{k+1}\mathcal{T} \cong Y_k\mathcal{T}$; otherwise, $Y_{k+1}\mathcal{T}$ is isomorphic to the blow-up of $Y_k\mathcal{T}$ along $G_{k+1}^{(k)} \cap Y_k\mathcal{T}$, and the exceptional divisor is $G_{k+1}^{(k+1)} \cap Y_{k+1}\mathcal{T}$. In the latter case, the codimension of $G_{k+1}^{(k)} \cap Y_k\mathcal{T}$ in $Y_k\mathcal{T}$ is equal to

$$\begin{cases} \dim \cap_{G_{k+1} \subsetneq G \in \mathcal{T}} G - \dim G_{k+1}, & \textit{if } \{G : G_{k+1} \subsetneq G \in \mathcal{T}\} \neq \emptyset; \\ \dim Y - \dim G_{k+1}, & \textit{otherwise.} \end{cases}$$

Proof. Use induction. The case $k = 0$ is obvious. Assume the proposition is true for k.

1) We show that if $\mathcal{T} \cup \{G_{k+1}\}$ is not a $\mathcal{G}$-nest, then $G_{k+1}^{(k)} \cap Y_k\mathcal{T} = \emptyset$. The conclusion $Y_{k+1}\mathcal{T} \cong Y_k\mathcal{T}$ follows immediately.

Since $\mathcal{T}$ is a $\mathcal{G}$-nest, then $\{G^{(k)}\}_{G\in\mathcal{T}}$ is a $\mathcal{G}^{(k)}$-nest by applying Proposition 2.3.19 k times. Suppose the nest $\{G^{(k)}\}_{G\in\mathcal{T}}$ is induced by a flag

$$S_1' \subseteq S_2' \subseteq \cdots \subseteq S_l'$$

where $S_i' \in \mathcal{S}^{(k)}$. We assert that if $G_{k+1}^{(k)} \cap Y_k\mathcal{T} \neq \emptyset$ then by adding one more subvariety $G_{k+1}^{(k)}$ to the nest $\{G^{(k)}\}_{G\in\mathcal{T}}$ we still get a nest. More precisely, we assert

$$\{G_{k+1}^{(k)}\} \cup \{G^{(k)}\}_{G\in\mathcal{T}} \subseteq \mathcal{G}^{(k)}$$

is a $\mathcal{G}^{(k)}$-nest induced by the flag

$$G_{k+1}^{(k)} \cap S_1' \subseteq S_1' \subseteq S_2' \subseteq \cdots \subseteq S_l'. \tag{2.10}$$

(Indeed, $Y_k\mathcal{T} = S_1'$, so $G_{k+1}^{(k)} \cap S_1' \neq \emptyset$. By Fact 2.3.9 (ii), the $\mathcal{G}^{(k)}$-factors of $G_{k+1}^{(k)} \cap S_1'$ are $G^{(k)}$ and some $\mathcal{G}^{(k)}$-factors of S_1'. Therefore, the flag 2.10 induces the nest $\{G_{k+1}^{(k)}\} \cup \{G^{(k)}\}_{G\in\mathcal{T}} \subseteq \mathcal{G}^{(k)}$.)

Proposition 2.3.19 asserts that $\{G_{k+1}^{(k)}\} \cup \{G^{(k)}\}_{G\in\mathcal{T}}$ is a $\mathcal{G}^{(k)}$-nest if and only if $\{G_{k+1}\} \cup \mathcal{T}$ is a $\mathcal{G}$-nest. Then 1) follows.

2) Suppose the $\mathcal{G}^{(k)}$-factors of $Y_k\mathcal{T}$ are $G_1', \ldots, G_r'$, they are also minimal elements in the $\mathcal{G}^{(k)}$-nest $\{G^{(k)}\}_{G\in\mathcal{T}}$ by the definition of a nest (see Definition 2.3.18). Assume without loss of generality that the first m subvarieties contain $G_{k+1}^{(k)}$. Define $A = \cap_{i=1}^m G_i'$, $B = \cap_{i=m+1}^r G_i'$, then $Y_k\mathcal{T} = A \cap B$ is the $G_{k+1}^{(k)}$-

factorization of $Y_k\mathcal{T}$ by Fact 2.3.10.

$Y_{k+1}\mathcal{T} = \cap_{G\in\mathcal{T}} G^{(k+1)}$ by definition. Notice that for $p, q \geq k+2$ and $G_p^{(k)} \subseteq G_q^{(k)}$, we have $G_p^{(k+1)} \subseteq G_q^{(k+1)}$ because strict transforming keeps the containing relation. Moreover, $G'_1, \ldots, G'_r$ are the minimal elements in $\mathcal{G}^{(k)}$ which contain $Y_k\mathcal{T}$. Therefore $Y_{k+1}\mathcal{T} = \cap_{i=1}^r \widetilde{G'_i}$. Then

$$\widetilde{A} = \cap_{i=1}^m \widetilde{G'_i},$$

$$\widetilde{B} = \cap_{i=m+1}^r \widetilde{G'_i},$$

$$\widetilde{A\cap B} = \widetilde{A} \cap \widetilde{B} = \cap_{i=1}^m \widetilde{G'_i}$$

by Fact 2.3.13. Thus $Y_{k+1}\mathcal{T} = \widetilde{Y_k\mathcal{T}}$. We also know that $Y_k\mathcal{T}$ and $G_{k+1}^{(k)}$ intersect cleanly, so $Y_{k+1}\mathcal{T}$ is the blow-up of $Y_k\mathcal{T}$ along the center $Y_k\mathcal{T} \cap G_{k+1}^{(k)}$. The exceptional divisor is the preimage of the center, hence is $Y_{k+1}\mathcal{T} \cap G_{k+1}^{(k+1)}$.

The codimension of the center $Y_k\mathcal{T} \cap G_{k+1}^{(k)}$ in $Y_k\mathcal{T}$ is

$$codim_{A\cap B\cap G_{k+1}^{(k)}} A\cap B = codim_{G_{k+1}^{(k)}\cap B} A\cap B = codim_{G_{k+1}^{(k)}} A,$$

where the second equality is because of the transversality of the intersection $G_{k+1}^{(k)} \cap B$. If no elements in $\mathcal{T}$ contain G_{k+1}, then $A = Y$ and

$$codim_{G_{k+1}^{(k)}} A = \dim Y - \dim G_{k+1};$$

otherwise $codim_{G_{k+1}^{(k)}} A$ is equal to

$$codim_{G_{k+1}^{(k)}} \bigcap_{i=1}^m G'_i = codim_{G_{k+1}} \bigcap_{G_{k+1}\subsetneq G\in\mathcal{T}} G = \dim \bigcap_{G_{k+1}\subsetneq G\in\mathcal{T}} G - \dim G_{k+1}.$$

Thus the proof is complete.

□

2.4 Examples of wonderful compactifications

In this section, several examples of wonderful compactifications are given. Namely, the wonderful models of subspace arrangements given by De Concini and Procesi (§2.4.1), Ulyanov's polydiagonal compactification and Hu's compactification (§2.4.2) and Kuperberg-Thurston's construction (§2.4.3).

2.4.1 Wonderful model of subspace arrangements

Given a finite collection of subspaces of a vector space V, there are many ways to construct a smooth variety birational to V which is unchanged in the complement of those subspaces and replace those subspaces by a normal crossing divisor. In the paper [DP95], De Concini and Procesi gave a combinatorial condition saying that, one can blow up a set of subspaces satisfying this condition (which is then called a *building set*, and coincides with the Definition 2.3.6) using certain orders, and the resulting space will not depend on the chosen order. More precisely, if $\mathcal{G}$ is a building set, we get a smooth variety $Y_{\mathcal{G}}$ by blowing up all elements in $\mathcal{G}$ in the order of ascending dimensions. This variety $Y_{\mathcal{G}}$ is isomorphic to the closure of the natural locally closed embedding

$$i : V \setminus \bigcup_{W \in \mathcal{G}} W \hookrightarrow V \times \prod_{W \in \mathcal{G}} \mathbb{P}(V/W).$$

Therefore $Y_{\mathcal{G}}$ does not depend on the order of blow-ups.

Remark: The ambient space $V\times\prod_{W\in\mathcal{G}}\mathbb{P}(V/W)$ can be replaced by a larger space $V\times\prod_{W\in\mathcal{G}} Bl_W V$ without changing the closure $Y_{\mathcal{G}}$ (up to isomorphism). To see this, fix a projection of $\pi_W : V \to W$ for each $W \in \mathcal{G}$, hence an isomorphism $V \cong W \times V/W$ and therefore a closed immersion

$$V \hookrightarrow V \times \prod_{W\in\mathcal{G}} (V/W).$$

The exceptional divisor of the blow-up $Bl_W V \to V$ is isomorphic to $W \times \mathbb{P}(V/W)$. So there is a closed immersion $W \times \mathbb{P}(V/W) \hookrightarrow Bl_W V$ (which depends on the projection π_W). Apply a similar argument as in Remark 2.2.1 to the following factorization

$$V \setminus \bigcup_{W\in\mathcal{G}} W \overset{i}{\hookrightarrow} V\times \prod_{W\in\mathcal{G}} \mathbb{P}(V/W) \hookrightarrow \Big(V\times\prod_{W\in\mathcal{G}} W\Big)\times\prod_{W\in\mathcal{G}}\mathbb{P}(V/W) \cong$$
$$\cong V\times\prod_{W\in\mathcal{G}}\Big(W\times\mathbb{P}(V/W)\Big) \hookrightarrow V\times\prod_{W\in\mathcal{G}} Bl_W V$$

and the statement follows.

Remark: De Concini and Procesi's work on subspace arrangements can be easily adapted to arrangements of subvarieties once the subvarieties 'locally' appear as a collection of subspaces. However, it does not cover the general cases of arrangement of subvarieties discussed in this paper, since in general it is impossible to find a local coordinate that all the subvarieties in the arrangement are linear subspaces.

2.4.2 Ulyanov's polydiagonal compactification and Hu's compactification

Ulyanov's compactification. After Fulton and MacPherson's paper ([FM94]), Ulyanov has discovered another compactification of the configuration space $F(X, n)$, which he denoted by $X\langle n\rangle$ ([Ul02]). The construction consists of blowing up more subvarieties in X^n than Fulton-MacPherson's construction. Namely, we blow up not only diagonals but also intersections of diagonals (those intersections are called *polydiagonals*, each of which corresponds to a partition of $\{1, \ldots, n\}$). The order of the blow-ups is the ascending order of the dimensions of polydiagonals. For example, $X\langle 4\rangle$ is the blow-up of X^4 along polydiagonals in the following order:

$$(1234); (123), (124), (134), (234), (12, 34), (13, 24), (14, 23); (12), \ldots, (34).$$

Those sets separated with commas will be disjoint before being blown up, so they can be blown up in any order.

This polydiagonal compactification shares many similar properties with the Fulton-MacPherson's compactification. Moreover, in the case of characteristic 0, under the action of S_n on $X\langle n\rangle$, the isotropy group of any point in $X\langle n\rangle$ is abelian, while the isotropy group in $X[n]$ is only solvable.

The polydiagonal compactification has a geometric description similar to the Fulton-MacPherson compactification. The screens are leveled. Screens at the same level need extra data called *scale factors* to record their relative speed.

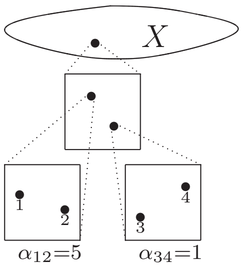


FIGURE 2. A point in $X\langle 4\rangle$

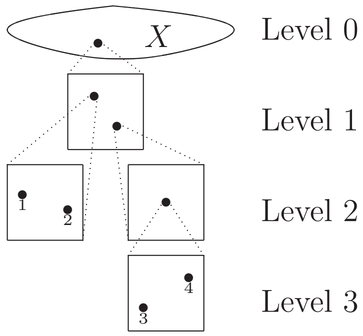


FIGURE 3. Another point in $X\langle 4\rangle$

Example: *Figure 2* gives a point in $X\langle 4\rangle$. Notice that the scales $\alpha_{12} = 5$ and $\alpha_{34} = 1$ gives us the ratio of speed of point x_1, x_2 approaching together and the speed of x_3, x_4 approaching together. so only the pair $(\alpha_{12} : \alpha_{34}) = (5 : 1) \in \mathbb{P}^1$ matters. If we change the scale to $\alpha_{12} = 10$ and $\alpha_{34} = 2$, it will give the same point. *Figure 3* gives a case when $\alpha_{34} = 0$, so the screen containing x_3, x_4 descends to level 3.

Hu's compactification. We now consider the general situation where Y is nonsingular with an arrangement of subvarieties $\mathcal{S}$. By blowing up all $S \in \mathcal{S}$ in the order of ascending dimesions, we get a nonsingular variety $Bl_{\mathcal{S}}Y$ ([Hu03]). Define $Y^\circ := Y \setminus \cup_{S\in\mathcal{S}} S$, the open strata of Y. It is isomorphic to an open set of $Bl_{\mathcal{S}}Y$. Then

1. *The boundary $Bl_{\mathcal{S}}Y \setminus Y^\circ = \cup_{S\in\mathcal{S}} D_S$ is a simple normal crossing divisor.*
2. *For any $S_1, ..., S_n \in \mathcal{S}$, the intersection of $D_{S_1} \dots D_{S_k}$ is nonempty if and only if $\{S_i\}$ form a chain, i.e., $S_1 \subseteq \cdots \subseteq S_k$ with a rearrangement of indices if necessary.*

Hu's compactification generalized Ulyanov's polydiagonal compactification. It is a special case of the wonderful compactification of arrangement of subvarieties given in this paper where the building set $\mathcal{G} = \mathcal{S}$. (In this special case, a $\mathcal{G}$-nest is simply a chain of subvarieties.)

2.4.3 Kuperberg-Thurston's compactification

In their paper [KT99], Kuperberg and Thurston construct an interesting compactification of configuration space $F(X, n)$. They did it in real field $\mathbb{R}$ and we adapt here their compactification to complex field $\mathbb{C}$. We give a brief introduction here.

Let Γ be a connected graph with n labeled vertices (assume without loss of generality that Γ has no self-loops and multiple edges). If Γ' is a subgraph of Γ, denote by $\Delta_{\Gamma'}$ the diagonal in X^n where $x_i = x_j$ if i, j are connected in Γ'. We call a graph Γ' *vertex-2-connected* if the graph is connected and will still be connected if we remove any vertex (In particular, a single edge is vertex-2-connected).

It is mentioned with a sketched proof in [KT99] that blowing up along $\Delta_{\Gamma'}$ for all vertex-2-connected subgraphs $\Gamma' \subseteq \Gamma$ gives a compactification X^Γ. When Γ is the full graph with n vertices (i.e. any two vertices is joint with an edge), the compactification X^Γ is exactly the Fulton-MacPherson compactication $X[n]$.

The Kuperburg-Thurston's compactification X^Γ is also a special case of the wonderful compactification of arrangement of subvarieties given in this paper. Indeed, let $Y = X^n$ and let $\mathcal{S}$ be the set of all polydiagonals of X^n. Then X^Γ

is canonically defined without depending on the order of the blow-ups, once

$$\mathcal{G} := \{\Delta_{\Gamma'} : \Gamma' \text{ is vertex-2-connected}\}$$

is a building set with respect to $\mathcal{S}$. $\mathcal{G}$ is indeed a building set because of the following observation:

1. We call a subgraph $\Gamma'' \subseteq \Gamma$ is *full* if the following is satisfied:

   (a) It contains all vertices in Γ.

   (b) For any edge $e \in \Gamma$, if its endpoints p and q are connected in Γ'', then $e \in \Gamma''$.

   There is a one-one correspondences between the set of all full subgraphs of Γ and the set $\mathcal{S}$, which maps Γ'' to $\Delta_{\Gamma''}$.

2. Any full subgraph Γ'' has a unique decomposition into vertex-2-connected subgraphs $\Gamma_1, \ldots, \Gamma_k$. Notice that $\Delta_{\Gamma_1}, \ldots, \Delta_{\Gamma_k}$ are the minimal elements in $\mathcal{G}$ which are $\geq \Delta_{\Gamma''}$, and they intersect transversally with the intersection $\Delta_{\Gamma''}$. Therefore $\mathcal{G}$ is a building set by Definition 2.3.6.

It is also easy to describe a general $\mathcal{G}$-nest: it corresponds to a set of vertex-2-connected subgraphs of Γ, where any two subgraphs should be either "disjoint" or "intersect at one vertex" or "one contains the other".

Chapter 3

Main theorems on wonderful compactification of arrangements of subvarieties

This chapter is devoted to the Chow groups and Chow motives decomposition of the wonderful compactification of arrangement of subvarieties.

3.1 Statement of the theorems

Notations:

- Let Y be a nonsingular algebraic variety over $\mathbb{C}$ with an arrangement of subvarieties $\mathcal{S}$ (see Definition 2.3.5). Let $\mathcal{G}$ be a building set with respect to $\mathcal{S}$ (see Definition 2.3.6). Let $Y_{\mathcal{G}}$ be the wonderful compactification of the arrangement $\mathcal{S}$ associated to $\mathcal{G}$ (see Definition 2.3.16). Let $\mathcal{T}$ denote a $\mathcal{G}$-nest (see Definition 2.3.18).

- Denote D_T to be the divisor in $Y_{\mathcal{G}}$ that corresponds to $T \in \mathcal{G}$. When no confusion arise, we use the same notation D_T for its restriction to a subvariety of $Y_{\mathcal{G}}$.

- $Y_0 := Y$, $Y_0\mathcal{T} := \bigcap_{T\in\mathcal{T}} T$, $Y_\mathcal{G}\mathcal{T} := \bigcap_{T\in\mathcal{T}} D_T$.

  Denote $j_\mathcal{T} : Y_\mathcal{G}\mathcal{T} \to Y_\mathcal{G}$ to be the natural imbedding.

  Denote $g_\mathcal{T} : Y_\mathcal{G}\mathcal{T} \to Y_0\mathcal{T}$ to be the restriction of the natural morphism $Y_\mathcal{G} \to Y$.

- Suppose $j : B \to C$ and $g : B \to D$ are two morphisms of varieties. Denote by $j\boxtimes g : B \to C\times D$ the composition of the diagonal map Δ with $f\times g$:

$$j\boxtimes g : B \overset{\Delta}{\to} B\times B \overset{f\times g}{\to} C\times D.$$

- (We assume $\bigcap_{G\subsetneq T\in\mathcal{T}} T = Y$ if no T satisfies $G \subsetneq T \in \mathcal{T}$.)

  Define $r_G := \dim(\bigcap_{G\subsetneq T\in\mathcal{T}} T) - \dim G$.

  Define $N_G := N_G(\bigcap_{G\subsetneq T\in\mathcal{T}} T)|_{Y_0\mathcal{T}}$, the restriction to $Y_0\mathcal{T}$ of the normal bundle of G in the ambient space $(\bigcap_{G\subsetneq T\in\mathcal{T}} T)$.

  Define

$$M_\mathcal{T} := \left\{\underline{\mu} = \{\mu_G\}_{G\in\mathcal{G}} : 1 \le \mu_G \le r_G - 1\right\}$$

  and define $||\underline{\mu}|| := \sum_{G\in\mathcal{G}} \mu_G$ for $\underline{\mu} \in M_\mathcal{T}$.

Theorem 3.1.1. *We have the Chow group decomposition*

$$A^*Y_\mathcal{G} = A^*Y \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu}\in M_\mathcal{T}} A^{*-||\underline{\mu}||}(Y_0\mathcal{T})$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests.

Moreover, when Y is complete, we have the Chow motive decomposition

$$h(Y_{\mathcal{G}}) = h(Y) \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu} \in M_{\mathcal{T}}} h(Y_0\mathcal{T})(||\underline{\mu}||)$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nests.

Theorem 3.1.2. *The correspondence that gives each of the above direct summand can be explicitly expressed as follows,*

$$\alpha : h(Y_{\mathcal{G}}) \to h(Y_0\mathcal{T})(\|\underline{\mu}\|)$$
$$\alpha = (j_{\mathcal{T}} \boxtimes g_{\mathcal{T}})_* \prod_{G \in \mathcal{T}} \left\{ c\Big(g_{\mathcal{T}}^*(N_G) \otimes \mathcal{O}\big(- \sum_{(\bigstar)} D_{G'}\big)\Big) \frac{1}{1 + D_G} \right\}_{r_G - 1 - \mu_G},$$

where the condition $(\bigstar)$ is: $G' \subsetneq G$ and $\mathcal{T} \cup \{G'\}$ is a $\mathcal{G}$-nest.

The inverse correspondence is

$$\beta : h(Y_0\mathcal{T})(\|\underline{\mu}\|) \to h(Y_{\mathcal{G}})$$
$$\beta = (g_{\mathcal{T}} \boxtimes j_{\mathcal{T}})_* \prod_{G \in \mathcal{T}} \big(- D_G\big)^{\mu_G - 1}.$$

3.2 Proof of the theorems

Apply the formula for the motive of a blow-up(Theorem 2.1.5) to Proposition 2.3.20 immediately gives the following lemma:

Lemma 3.2.1. *Given a $\mathcal{G}$-nest $\mathcal{T} \subseteq \{G_{k+2}, \ldots, G_N\}$. Suppose $\mathcal{T}' := \mathcal{T} \cup$*

$\{G_{k+1}\}$ is also a $\mathcal{G}$-nest. Define $r_{k,\mathcal{T}}$ (or r if no confusion arises) to be

$$\begin{cases} \dim \cap_{G_{k+1} \subsetneq G \in \mathcal{T}} G - \dim G_{k+1}, & \text{if } \{G : G_{k+1} \subsetneq G \in \mathcal{T}\} \neq \emptyset; \\ \dim Y - \dim G_{k+1}, & \text{otherwise.} \end{cases}$$

Then the following Chow group decomposition holds:

$$A^*(Y_{k+1}\mathcal{T}) = A^*(Y_k\mathcal{T}) \oplus \bigoplus_{t=1}^{r-1} A^{*-t}(Y_k\mathcal{T}').$$

When Y is complete, we also have the motivic decomposition

$$h(Y_{k+1}\mathcal{T}) = h(Y_k\mathcal{T}) \oplus \bigoplus_{t=1}^{r-1} h(Y_k\mathcal{T}')(t).$$

Applying inductively the above lemma gives the proof of the Chow group and Chow motivic decompositions of the wonderful compactification $Y_{\mathcal{G}}$ in Theorem 3.1.1 as follows,

Proof of Theorem 3.1.1. Define

$$M_{\mathcal{T}}^{(k)} = \{\underline{\mu} = \{\mu_G\}_{G\in\mathcal{G}} : 1 \leq \mu_G \leq \dim(\bigcap_{\substack{T\in\mathcal{T} \\ G^{(k)}\subsetneq T^{(k)}}} T^{(k)}) - \dim G^{(k)} - 1\}$$

and define $||\underline{\mu}|| := \sum_{G\in\mathcal{G}} \mu_G$ for $\underline{\mu} \in M_{\mathcal{T}}^{(k)}$.

We prove the following statement using the downward induction on k :

$$A^*Y_{\mathcal{G}} = A^*Y_k \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu}\in M_{\mathcal{T}}^{(k)}} A^{*-||\underline{\mu}||}(Y_k\mathcal{T}). \tag{3.1}$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nest such that $\mathcal{T} \subseteq \{G_{k+1}, G_{k+2}, \ldots, G_N\}$.

The assertion for $k = N$ is trivial because all $G^{(N)}$ are divisors in $Y_{\mathcal{G}}$ hence of codimension 1 and $M_{\mathcal{T}}^{(k)} = \emptyset$.

Assume (3.1) has been proved for $k+1$, i.e.,

$$A^* Y_{\mathcal{G}} = A^* Y_{k+1} \oplus \bigoplus_{\mathcal{T}} \bigoplus_{\underline{\mu} \in M_{\mathcal{T}}^{(k+1)}} A^{*-\|\mu\|}(Y_{k+1}\mathcal{T})$$

where $\mathcal{T}$ runs through all $\mathcal{G}$-nest such that $\mathcal{T} \subseteq \{G_{k+2}, G_{k+3}, \ldots, G_N\}$. Apply Lemma 3.2.1, we have

$$\begin{aligned} A^* Y_{\mathcal{G}} =& A^* Y_k \oplus \left(\bigoplus_{t=1}^{s-1} A^{*-t}(G_{k+1}^{(k)}) \right) \\ & \oplus \left(A^{*-\|\mu\|}(Y_k \mathcal{T}) \right) \oplus \left(\bigoplus_{t=1}^{r_{k+1,\mathcal{T}}-1} A^{*-\|\mu\|-t}\big(Y_k(\{G_{k+1}\} \cup \mathcal{T})\big) \right) \end{aligned}$$

where $s = codim_{G_{k+1}} Y$. This immediately gives the Chow group decomposition (3.1) for k. Indeed, any $\mathcal{G}$-nest contained in $\{G_{k+1}, G_{k+2}, \ldots, G_N\}$ must be one of the three: $\{G_{k+1}\}$, $\mathcal{G}$-nest $\mathcal{T}$ contained in $\{G_{k+2}, G_{k+3}, \ldots, G_N\}$, or $\{G_{k+1}\} \cup \mathcal{T}$. They correspond to the second, third and last summands respectively.

Therefore, the Chow group decomposition (3.1) holds for all k, in particular the case $k = 0$ gives the desired Chow group decomposition. The proof of the Chow motive decomposition is almost an repetition of the above proof and been omitted here. □

Our next step is to explicitly express the correspondences that give the Chow motive decomposition i.e. to prove Theorem 3.1.2. We first introduce some notations.

Fix a $\mathcal{G}$-nest $\mathcal{T}$.

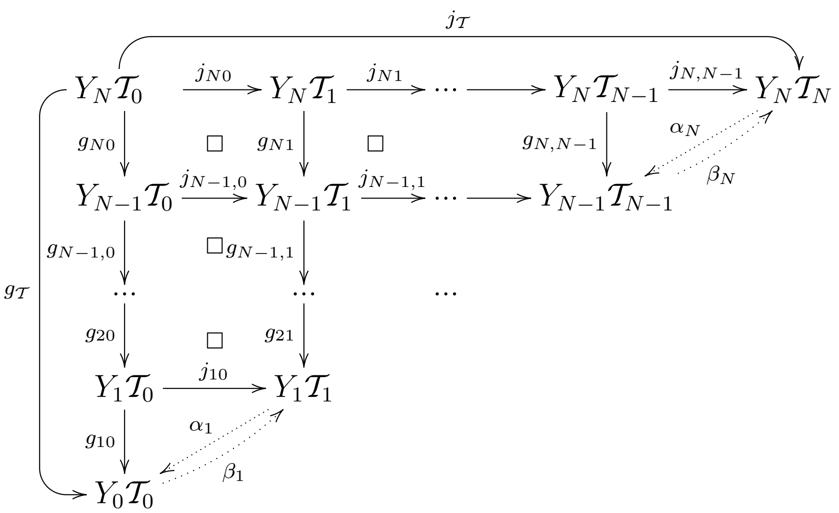
- Define $\mathcal{T}_k := \mathcal{T} \bigcap \{G_{k+1}, G_{k+2}, \ldots, G_N\}$ for $0 \leq k \leq N$. Then we have a chain

$$(\mathcal{T} =)\mathcal{T}_0 \supseteq \mathcal{T}_1 \supseteq \cdots \supseteq \mathcal{T}_N (= \emptyset).$$

- For $1 \leq i \leq N$, define

$$\mu_i := \begin{cases} \mu_{T_i}, & \text{if } T_i \in \mathcal{G}; \\ 0, & \text{otherwise.} \end{cases}$$

- j_{kl} and g_{kl} ($N \geq k > l \geq 0$) are the natural morphisms as in the following diagram



(3.2)

Lemma 3.2.2. *Denote by $g : Y_k \to Y_{k-1}$ the natural morphism. Then for $l \leq k-1$, we have*

$$g^{-1}(G_l^{(k-1)}) = G_l^{(k)}.$$

Proof. First, notice that $G_l^{(k-1)} \not\supseteq G_k^{(k-1)}$ since their image under the natural morphism $Y_k \to Y_0$ are G_l and G_k, respectively. By the assumption of the order of $\{G_i\}$, we have $G_l \not\supseteq G_k$. On the other hand, $G_l^{(k-1)}$ is a divisor, so $G_l^{(k-1)} \not\subseteq G_k^{(k-1)}$. We also know that $G_l^{(k-1)}$ and $G_k^{(k-1)}$ intersect cleanly, therefore they must intersect transversally. Then it is standard to show by local coordinates calculation that the following isomorphism between ideal sheaves holds:

$$g^{-1}\mathcal{I}(G_l^{(k-1)}) \cdot \mathcal{O}_{Y_k} \cong \mathcal{I}(G_l^{(k)}).$$

The desired conclusion follows from this. □

Proposition 3.2.3. *In Diagram (3.2), all squares are fiber squares. Moreover, for any $N \geq k > l \geq 0$, j_{kl} is injective; g_{kl} is the projection of a projective bundle with fiber of dimension $r_{k,\mathcal{T}} - 1$ if $G_k \in \mathcal{T}$ (see Lemma 3.2.1 for definition of $r_{k,\mathcal{T}}$); g_{kl} is the blow-up of $Y_{k-1}\mathcal{T}_l$ along $G_k^{(k-1)} \cap Y_{k-1}\mathcal{T}_l$ if $G_k \notin \mathcal{T}$ but $\{G_k\} \cup \mathcal{T}_l$ is a $\mathcal{G}$-nest; g_{kl} is an isomorphism if $\{G_k\} \cup \mathcal{T}_l$ is not a $\mathcal{G}$-nest.*

Proof. It is obvious that j_{kl} is injective.

To show that g_{kl} is the projection of a projective bundle if $G_k \in \mathcal{T}$, we apply Proposition 2.3.20: $Y_k\mathcal{T}_k$ is the blow-up of $Y_{k-1}\mathcal{T}_k$ along the center $Y_{k-1}\mathcal{T}_{k-1}$, and the exceptional divisor is $Y_k\mathcal{T}_{k-1}$. Therefore $g_{k,k-1} : Y_k\mathcal{T}_{k-1} \to Y_{k-1}\mathcal{T}_{k-1}$ is a projective bundle, and the dimension of a fibre is $r_{k,\mathcal{T}} - 1$. Next we show that for any $l \leq k-1$, g_{kl} is the restriction of $g_{k,k-1}$ to a smaller base $Y_{k-1}\mathcal{T}_l$, then g_{kl} is also a projective bundle with fiber of the same dimension $r_{k,\mathcal{T}} - 1$. Fix k and use downward induction on l. By inductive assumption, $g_{k,l+1}$ is a restriction of $g_{k,k-1}$. Since

$$g_{k,l+1}^{-1}(G_{l+1}^{(k-1)} \cap Y_{k-1}\mathcal{T}_l) = G_{l+1}^{(k)} \cap Y_k\mathcal{T}_l$$

by Lemma 3.2.2, the restriction of the projective bundle $g_{k,l+1}$ to a smaller base space $Y_{k-1}\mathcal{T}_l = Y_{k-1}\mathcal{T}_{l+1} \cap G_{l+1}^{(k-1)}$ is exactly g_{kl}.

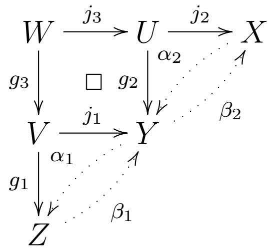
Next, we show g_{kl} is birational if $G_k \notin \mathcal{T}$. This is again implied by Proposition 2.3.20. Notice that $G_k^{(k-1)}$ is minimal in

$$\mathcal{T}' := \{G_k^{(k-1)}\} \cup \{G^{(k-1)}\}_{G \in \mathcal{T}_l}.$$

If $\mathcal{T}'$ is a $\mathcal{G}^{(k-1)}$-nest, then $g_{kl} : Y_k\mathcal{T}_l \to Y_{k-1}\mathcal{T}_l$ is a blow-up along the center $G_k^{(k-1)} \cap Y_{k-1}\mathcal{T}_l$; otherwise, g_{kl} is an isomorphism. In both cases, g_{kl} is birational.

Finally, all squares in Diagram (3.2) are fiber squares since $\forall l \leq k-2$, g_{kl} is a restriction of $g_{k,l+1}$. The proof is complete. $\square$

Proposition 3.2.4. *Suppose* W, U, V, X, Y, Z *are nonsingular varieties, the square in the following diagram is a fiber square, and* $\dim W - \dim V = \dim U - \dim Y$,



Also assume $j_k, g_k (1 \leq k \leq 3)$ *are proper and l.c.i. (local complete intersection) morphisms (cf. [Fu98]). If there exists* $\gamma_1, \gamma_1' \in A(V), \gamma_2, \gamma_2' \in A(U)$, *such that the correspondences*

$$\alpha_k = (j_k \boxtimes g_k)_*\gamma_k, \beta_k = (g_k \boxtimes j_k)_*\gamma'_k \quad \textit{for } k = 1, 2$$

then we have

$$\alpha_1\alpha_2 = (j_2j_3 \boxtimes g_1g_3)_*(j_3^*\gamma_2 \cdot g_3^*\gamma_1),$$
$$\beta_2\beta_1 = (g_1g_3 \boxtimes j_2j_3)_*(g_3^*\gamma'_1 \cdot j_3^*\gamma'_2).$$

Proof. Denote by π_{13}^{XYZ} the projection $X \times Y \times Z \to X \times Z$.

$$\begin{aligned}\alpha_1\alpha_2 &= (\pi_{13}^{XYZ})_*[(j_2 \boxtimes g_2)_*\gamma_2 \times 1_Z \cdot 1_X \times (j_1 \boxtimes g_1)_*\gamma_1] \\ &= (\pi_{13}^{XYZ})_*[(j_2 \times g_2 \times 1_Z)_*(\delta_{U*}\gamma_2 \times 1_Z) \cdot 1_X \times (j_1 \times g_1)_*\delta_{V*}\gamma_1].\end{aligned}$$

By projection formula, the expression in the bracket equals to

$$\begin{aligned}&(j_2 \times g_2 \times 1_Z)_*[\delta_{U*}\gamma_2 \times 1_Z \cdot (j_2 \times g_2 \times 1_Z)^*(1_X \times (j_1 \times g_1)_*\delta_{V*}\gamma_1)] \\ =&(j_2 \times g_2 \times 1_Z)_*[\delta_{U*}\gamma_2 \times 1_Z \cdot 1_U \times (g_2 \times 1_Z)^*(j_1 \times g_1)_*\delta_{V*}\gamma_1].\end{aligned}$$

Next, we show that $(g_2 \times 1_Z)^*(j_1 \times g_1)_* = (j_3 \times g_1)_*(g_3 \times 1_V)^*$. Indeed, notice that in the following fiber square, the relative dimensions of the left arrow $(g_3 \times 1_V)$ and right arrow $(g_2 \times 1_Z)$ are equal,

$$\begin{array}{ccc} W \times V & \xrightarrow{j_3 \times g_1} & U \times Z \\ {\scriptstyle g_3 \times 1_V}\downarrow & \square & \downarrow{\scriptstyle g_2 \times 1_Z} \\ V \times V & \xrightarrow{j_1 \times g_1} & Y \times Z \end{array}$$

Apply the push-forward formula and excess intersection formula for l.c.i. morphisms gives the desired equality. ([Fu98] Proposition 6.6, asserts that Theorem 6.2 and Theorem 6.3 in [Fu98] are also valid for l.c.i. morphisms.)

$$\alpha_1\alpha_2 = (\pi_{13}^{XYZ})_*(j_2 \times g_2 \times 1_Z)_*[\delta_{U*}\gamma_2 \times 1_Z \cdot 1_U \times (j_3 \times g_1)_*(g_3 \times 1_V)^*\delta_{V*}\gamma_1]$$
$$= (j_2 \times 1_Z)_*(\pi_{13}^{UUZ})_*(1_U \times j_3 \times g_1)_*[(1_U \times j_3)^*\delta_{U*}\gamma_2 \times 1_V \cdot 1_U \times (g_3 \times 1_V)^*\delta_{V*}\gamma_1]$$

The expression in the bracket equals to

$$(1_U \times j_3)^*(\Delta_U \cdot (1_U \times \gamma_2)) \times 1_V \cdot 1_U \times (g_3 \times 1_V)^*(\Delta_V \cdot (\gamma_1 \times 1_V))$$
$$= (1_U \times j_3)^*\Delta_U \times 1_V \cdot 1_U \times j_3^*\gamma_2 \times 1_V \cdot 1_U \times (g_3 \times 1_V)^*\Delta_V \cdot 1_U \times g_3^*\gamma_1 \times 1_V]$$

Since $(1_U \times j_3)^*\Delta_U = \Gamma_{j_3}^t = (j_3 \times 1_W)_*\Delta_W$ and $(g_3 \times 1_V)^*\Delta_V = \Gamma_{g_3} = (1_W \times g_3)_*\Delta_W$, the above expression equals to

$$(j_3 \times 1_W)_*\Delta_W \times 1_V \cdot 1_U \times (1_W \times g_3)_*\Delta_W \cdot 1_U \times (j_3^*\gamma_2 \cdot g_3^*\gamma_1) \times 1_V$$
$$= (j_3 \times 1_W \times g_3)_*[\Delta_W \times 1_W \cdot 1_W \times \Delta_W \cdot 1_W \times (j_3^*\gamma_2 \cdot g_3^*\gamma_1) \times 1_V]$$

Then, because of

$$(j_2 \times 1_Z)_*(\pi_{13}^{UUZ})_*(1_U \times j_3 \times g_1)_*(j_3 \times 1_W \times g_3)_* = (j_2 j_3 \times g_1 g_3)_*(\pi_{13}^{WWW})_*,$$

and

$$\Delta_W \times 1_W \cdot 1_W \times \Gamma \times 1_W = \Delta_W \times 1_W \cdot \Gamma \times 1_W \times 1_W = \Delta_W \times 1_W \cdot \pi_{13}^*(\Gamma \times 1_W),$$

We have

$$\begin{aligned}
\alpha_1\alpha_2 &= (j_2j_3{\times}g_1g_3)_*\pi_{13*}[\Delta_W \times 1_W \cdot 1_W \times \Delta_W \cdot \pi_{13}^*(\Gamma \times 1_W)] \\
&= (j_2j_3{\times}g_1g_3)_*[\pi_{13*}(\Delta_W \times 1_W \cdot 1_W \times \Delta_W) \cdot \Gamma \times 1_W] \\
&= (j_2j_3{\times}g_1g_3)_*[\Delta_W \cdot \Gamma \times 1_W] \\
&= (j_2j_3{\times}g_1g_3)_*\delta_{W*}\Gamma \\
&= (j_2j_3 \boxtimes g_1g_3)_*\Gamma.
\end{aligned}$$

Where $\Gamma = j_3^*\gamma_2 \cdot g_3^*\gamma_1$. □

Notice a simple fact: *If A, B_i, C_{ij} are motives such that*

1. *$\bigoplus_i \alpha_i : A \cong \bigoplus_i B_i$ is an isomorphism with inverse $\sum_i \beta_i$,*
2. *$\bigoplus_j \alpha_{ij} : B_i \cong \bigoplus_j C_{ij}$ is an isomorphism with inverse $\sum_j \beta_{ij}$,*

then $\bigoplus_{i,j} \alpha_{ij} \circ \alpha_i : A \cong \bigoplus_{i,j} C_{ij}$ is also an isomorphism with inverse $\sum_{i,j} \beta_i \circ \beta_{ij}$.

For $G_k \in \mathcal{T}$, define $h_k \in A^1(Y_k\mathcal{T}_{k-1})$ to be first chern class of the invertible sheaf $\mathcal{O}(1)$ of the projective bundle $g_{k,k-1}$. Define

$$\alpha_k = \begin{cases} (j_{k,k-1}\boxtimes g_{k,k-1})_*1, & \text{if } G_k \notin \mathcal{T}; \\ (j_{k,k-1}\boxtimes g_{k,k-1})_*\{g_{k,k-1}^*c(N_k)\frac{1}{1-h_k}\}_{r_k-1-\mu_k}, & \text{if } G_k \in \mathcal{T}, \end{cases}$$

where $N_k := N_{Y_{k-1}\mathcal{T}_{k-1}}Y_{k-1}\mathcal{T}_k$, and define

$$\beta_k = \begin{cases} (g_{k,k-1}\boxtimes j_{k,k-1})_*1, & \text{if } G_k \notin \mathcal{T}; \\ (g_{k,k-1}\boxtimes j_{k,k-1})_*h_k^{\mu_k-1}, & \text{if } G_k \in \mathcal{T}. \end{cases}$$

Thanks to the formula of the motive decomposition of a blow-up (Theorem 2.1.5), the correspondence

$$a_k : h(Y_k\mathcal{T}_k)(\sum_{i=k+1}^{N} \mu_i) \to h(Y_{k-1}\mathcal{T}_{k-1})(\sum_{i=k}^{N} \mu_i)$$

expresses $h(Y_{k-1}\mathcal{T}_{k-1})(\sum_k^N \mu_i)$ as a direct summand of $h(Y_k\mathcal{T}_k)(\sum_{k+1}^N \mu_i)$ with right inverse β_k.

By the above simple fact, the correspondence

$$\alpha_{\mathcal{T},\underline{\mu}} : h(Y_{\mathcal{G}}) \to h(Y_0\mathcal{T})(\|\underline{\mu}\|)$$

that gives the direct summand $h(Y_0\mathcal{T})(\|\underline{\mu}\|)$ in Theorem 3.1.1 can be expressed as the composition $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_N$, with right inverse $\beta_N \circ \cdots \circ \beta_1$.

Now combine Proposition 3.2.3 and Proposition 3.2.4 with the above discussion, we arrive at the following proposition:

Proposition 3.2.5. *Denote by $f_k : Y_N\mathcal{T}_0 \to Y_k\mathcal{T}_{k-1}$ the natural map in Diagram (3.2). (i.e. $g_{k+1,k-1} \cdots \circ g_{N,k-1} \circ j_{N,k-2} \circ \cdots \circ j_{N0}$.) Then*

$$\alpha_1 \circ \cdots \circ \alpha_N = (j_{\mathcal{T}} \boxtimes g_{\mathcal{T}})_* \prod_{G_k \in \mathcal{T}} \{f_k^* g_{k,k-1}^* c(N_k) \frac{1}{1 - f_k^* h_k}\}_{r_k - 1 - \mu_k},$$

$$\beta_N \circ \cdots \circ \beta_1 = (g_{\mathcal{T}} \boxtimes j_{\mathcal{T}})_* \prod_{G_k \in \mathcal{T}} f_k^* h_k^{\mu_k - 1}.$$

The following two standard facts about normal bundles of subvarieties are used in the proof of Theorem 3.1.2.

Fact 3.2.6. *Let Y, W be nonsingular proper subvarieties of Z and assume Y*

intersects transversally with W. Let $\pi : \widetilde{Z} \to Z$ be the blow-up of Z along W and let $\widetilde{Y}$ be the strict transform of Y. Then

$$N_{\widetilde{Y}}\widetilde{Z} \simeq \pi^* N_Y Z.$$

Fact 3.2.7. *Let $W \subsetneq Y \subsetneq Z$ be nonsingular varieties and $\pi : \widetilde{Z} \to Z$ be the blow-up of Z along W. Denote by $\widetilde{Y}$ the strict transform of Y, and denote by E the exceptional divisor on $\widetilde{Y}$. Then*

$$N_{\widetilde{Y}}\widetilde{Z} \simeq \pi^* N_Y Z \otimes \mathcal{O}(-E).$$

Proof of the above two facts. Prove by local coordinates. Or see [Fu98]. $\square$

Proof of Theorem 3.1.2. The proof contains three steps.

Step 1: Show $f_k^* h_k = -D_{G_k}|_{Y_N \mathcal{T}_0}$.

Recall that for $G_k \in \mathcal{T}$, h_k is first chern class of the invertible sheaf $\mathcal{O}(1)$ of the projective bundle $g_{k,k-1}$.

Consider the following diagram (not necessary a fiber square) where π and j are the natural morphisms:

$$\begin{array}{ccc} Y_N\mathcal{T}_0 & \xrightarrow{j_{\mathcal{T}}} & Y_N \\ \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle \pi} \\ Y_k\mathcal{T}_{k-1} & \xrightarrow{j} & Y_k \end{array}$$

By Proposition 2.3.20, $Y_k\mathcal{T}_{k-1}$ is the exceptional divisor of the blow-up $g_{k,k-1} : Y_k\mathcal{T}_{k-1} \to Y_{k-1}\mathcal{T}_{k-1}$. So $h_k = -j_{k,k-1}^*[Y_k\mathcal{T}_{k-1}]$. Since $Y_k\mathcal{T}_{k-1}$ is the transversal intersection $Y_k\mathcal{T}_k \cap G_k^{(k)}$, $h_k = -j^*[G_k^{(k)}]$. so

$$f_k^* h_k = -f_k^* j^*[G_k^{(k)}] = -j_{\mathcal{T}}^* \pi^*[G_k^{(k)}] = -j_{\mathcal{T}}^* D_{G_k} = -D_{G_k}|_{Y_N \mathcal{T}_0}.$$

where the third equality is by successively applying Lemma 3.2.2.

Step 2: Let $0 \le s < k \le N$. Denote $g_{sk} : Y_s\mathcal{T}_k \to Y_{s-1}\mathcal{T}_k$ to be the natural map induced from $Y_s \to Y_{s-1}$. We show that, if $G_k \in \mathcal{T}$ (hence $\mathcal{T}_{k-1} = \mathcal{T}_k \cup \{G_k\}$), then the normal bundle $N_{Y_s\mathcal{T}_{k-1}} Y_s\mathcal{T}_k$ is isomorphic to

$$\begin{cases} g_{s,k-1}^*\big(N_{Y_{s-1}\mathcal{T}_{k-1}} Y_{s-1}\mathcal{T}_k\big) \otimes (-[G_s^{(s)}]|_{Y_s\mathcal{T}_{k-1}}), & \text{if (**) holds;} \\ \\ g_{s,k-1}^*\big(N_{Y_{s-1}\mathcal{T}_{k-1}} Y_{s-1}\mathcal{T}_k\big), & \text{otherwise.} \end{cases}$$

where condition (**) is: $G_s \subsetneq G_k$ and $\mathcal{T}_k \cup \{G_s\}$ is a $\mathcal{G}$-nest.

For the proof, we discuss three cases.

Case I: when (**) holds. It is a direct conclusion of Fact 3.2.7. To apply this Fact, we need

$$Y_{s-1}\mathcal{T}_k \cap G_s^{(s-1)} \subsetneq Y_{s-1}\mathcal{T}_k \cap G_k^{(s-1)} \subsetneq Y_{s-1}\mathcal{T}_k.$$

The second inequality is obvious. The first inclusion is strict because of the following reason. $G_s^{(s-1)}$ is a $\mathcal{G}^{(s-1)}$-factor of $Y_{s-1}\mathcal{T}_k \cap G_s^{(s-1)}$, therefore $G_k^{(s-1)}$ is not a $\mathcal{G}^{(s-1)}$-factor because it strictly contains $G_s^{(s-1)}$. On the other hand, $G_k^{(s-1)}$ is a $\mathcal{G}^{(s-1)}$-factor of $Y_{s-1}\mathcal{T}_k \cap G_k^{(s-1)}$. So the first inclusion is strict.

Case II: $\mathcal{T}_k \cup \{G_s\}$ is not $\mathcal{G}$-nested. In this case, $G_s^{(s-1)} \cap Y_{s-1}\mathcal{T}_k = \emptyset$ by Proposition 2.3.20. Hence no twisting is needed for the normal bundle.

Case III: $\mathcal{T}_k \cup \{G_s\}$ is $\mathcal{G}$-nested but G_s is not strictly contained in G_k. If $\mathcal{T}_{k-1} \cup \{G_s\}$ is not a $\mathcal{G}$-nest, then $G_s^{(s-1)} \cap Y_{s-1}\mathcal{T}_{k-1} = \emptyset$ by Proposition 2.3.20. Hence blowing up along $G_s^{(s-1)}$ will not affect the normal bundle of $Y_{s-1}\mathcal{T}_{k-1}$, so no twisting is needed. Otherwise, assume $\mathcal{T}_{k-1} \cup \{G_s\}$ is a $\mathcal{G}$-nest. Both G_s and G_k are minimal in the $\mathcal{G}$-nest $\mathcal{T}_{k-1} \cup \{G_s\}$. Then $G_s^{(s-1)}$ and $G_k^{(s-1)}$ are minimal in a nest and neither one contains the other, therefore they intersect transversally by the definition of *nest*. Thus, $Y_{s-1}\mathcal{T}_k \cap G_k^{(s-1)}$ and $Y_{s-1}\mathcal{T}_k \cap G_s^{(s-1)}$, regarded as subvarieties of ambient space $Y_{s-1}\mathcal{T}_k$, intersect transversally. Therefore Fact 3.2.6 applies, hence no twisting is needed for the normal bundle.

Step 3: Apply the result of Step 2 successively for $s = 1, 2, \ldots, k-1$. The normal bundle $N_{Y_{k-1}\mathcal{T}_{k-1}} Y_{k-1}\mathcal{T}_k$ is isomorphic to

$$\Big(g^*_{k-1,k-1} \cdots g^*_{1,k-1}\big(N_{Y_0\mathcal{T}_{k-1}} Y_0\mathcal{T}_k\big)\Big) \otimes \big(-\sum_{(**)} [G_s^{(k-1)}]|_{Y_{k-1}\mathcal{T}_{k-1}}\big)$$

where the sum is over all s that satisfying condition (**) holds. (Here we use Lemma 3.2.2.) Therefore

$$\begin{aligned} & f_k^* g^*_{k,k-1} c(N_{Y_{k-1}\mathcal{T}_{k-1}} Y_{k-1}\mathcal{T}_k) \\ &= c\Big(g^*_{\mathcal{T}}\big(N_{Y_0\mathcal{T}_{k-1}} Y_0\mathcal{T}_k|_{Y_0\mathcal{T}}\big) \otimes \mathcal{O}\big(-\sum_{(**)} [D_{G_s}]|_{Y_N\mathcal{T}_{k-1}}\big)\Big). \end{aligned}$$

Notice that

$$\big(N_{Y_0\mathcal{T}_{k-1}} Y_0\mathcal{T}_k\big)|_{Y_0\mathcal{T}} = N_{G_k}\big(\bigcap_{G_k \subsetneq G \in \mathcal{T}} G\big)|_{Y_0\mathcal{T}}$$

which is denoted by N_{G_k} by our notation. (The proof is as follows: Suppose $T_1, ..., T_m, T_{m+1}, ..., T_r$ are the minimal elements of the nest $\mathcal{T}_k$, where the first m elements contain G_k. Then the minimal element of the nest $\mathcal{T}_{k-1}$ are $G_k, T_{m+1}, ..., T_r$. By the definition of nest, $Y_0\mathcal{T}_k$ is the transversal intersection $T_1 \cap \cdots \cap T_m \cap T_{m+1} \cap \cdots \cap T_r$, and $Y_0\mathcal{T}_{k-1}$ is the transversal intersection $G_k \cap T_{m+1} \cap \cdots \cap T_r$. Therefore the normal bundle

$$N_{Y_0\mathcal{T}_{k-1}} Y_0\mathcal{T}_k = N_{G_k}(T_1 \cap \cdots \cap T_m)|_{Y_0\mathcal{T}_{k-1}}.$$

Since $T_1 \cap \cdots \cap T_m = \bigcap_{G_k \subsetneq G \in \mathcal{T}} G$, the conclusion follows immediately.)

Now put everything into Corollary 3.2.5, we have

$$\begin{aligned}
&\alpha_1 \circ \cdots \circ \alpha_N \\
&= (j_{\mathcal{T}} \boxtimes g_{\mathcal{T}})_* \prod_{G_k \in \mathcal{T}} \{c(g_{\mathcal{T}}^*(N_{G_k}) \otimes \mathcal{O}(-\sum_{(*')} [D_{G_s}]|_{Y_N\mathcal{T}}) \frac{1}{1 + D_{G_k}|_{Y_N\mathcal{T}}}\}_{r_k - 1 - \mu_k}, \\
&\beta_N \circ \cdots \circ \beta_1 = (g_{\mathcal{T}} \boxtimes j_{\mathcal{T}})_* \prod_{G_k \in \mathcal{T}} (-D_{G_k})^{\mu_k - 1}|_{Y_N\mathcal{T}}.
\end{aligned}$$

Finally, we show that the condition (**) can be replaced by the following condition:

($\bigstar$) : $G_s \subsetneq G_k$ and $\mathcal{T} \cup \{G_s\}$ is a $\mathcal{G}$-nest.

Indeed, ($\bigstar$) is stronger than (**). However, for those G_s satisfying (**) but not ($\bigstar$), the divisor $[D_{G_s}]|_{Y_N\mathcal{T}}$ would be trivial because $D_{G_s} \cap Y_N\mathcal{T} = \emptyset$. Therefore, replacing (**) by ($\bigstar$) will not change the result.

Hence the proof is complete. $\square$

we write a direct conclusion from Step 3 for later usage:

Corollary 3.2.8. *Denote* $\pi : G_{k+1}^{(k)} \to G_{k+1}$. *Then*

$$c(N_{G_{k+1}^{(k)}} Y_k) = c\Big(\pi^* N_{(G_{k+1})} Y \otimes \sum_{G_{k+1} \supsetneq G \in \mathcal{T}} (-[D_G])|_{G_{k+1}^{(k)}}\Big).$$

Proof. Apply Step 3 to the nest $\mathcal{T} = \{G_{k+1}\}$. □

Chapter 4

Theorems on the Fulton-MacPherson configuration spaces

In this chapter, we prove more precise results for the Chow groups (Theorem 4.1.1) and the Chow motives (Theorem 4.1.2) of the Fulton-MacPherson configuration space $X[n]$. We also give a generating function which can be used to calculate the Chow groups and the Chow motives recursively (Theorem 4.2.1). Examples of Chow groups and Chow motives of $X[n]$ for $n = 2, 3, 4$ are given in Section 4.3.

4.1 Statements and proofs

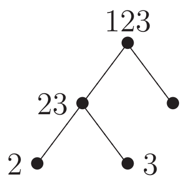
Notation:

1. We call two subsets $I, J \subseteq [n] := \{1, 2, \ldots, n\}$ are *overlapped* if $I \cap J$ is a nonempty proper subset of I and of J. For a set $\mathcal{S}$ of subsets of $[n]$, we call I is compatible with $\mathcal{S}$ (denote by $I \sim \mathcal{S}$) if I does not overlap any element in $\mathcal{S}$.

A *nest* $\mathcal{S}$ is a set of subsets of $[n]$ such that any two elements $I \neq J \in \mathcal{S}$ are not overlapped, and all singletons $\{1\}, \ldots, \{n\}$ are in $\mathcal{S}$. Notice that the nest defined here, unlike the one defined in [FM94], is allowed to contain singletons.

Given a nest $\mathcal{S}$, define $\mathcal{S}^\circ = \mathcal{S} \setminus \big\{\{1\}, \ldots, \{n\}\big\}$. In the description of nests by forests below, $\mathcal{S}^\circ$ correspond to the forest $\mathcal{S}$ cutting of all leaves.

A nest $\mathcal{S}$ naturally corresponds to a not necessarily connected tree (which is also called a *forest*), each node of which is labeled by an element in $\mathcal{S}$. For example, the following forest corresponds to a nest $\mathcal{S} = \{1, 2, 3, 23, 123\}$.



Denote by $c(\mathcal{S})$ the number of connected components of the forest, i.e., the number of maximal elements of $\mathcal{S}$. Denote by $c_I(\mathcal{S})$ (or c_I if no ambiguity arise) the number of maximal elements of the set $\{J \in \mathcal{S} | J \subsetneq I\}$, i.e. the number of sons of the node I. In the above example, $c(\mathcal{S}) = 1$, $c_{123} = c_{23} = 2$.

2. Let X be a nonsingular variety of dimension d.

   It is shown in [FM94] that $X[n] \setminus F(X, n) = \cup D_I$, where I runs through all subsets of $[n]$ with at least two elements. $\cup D_I$ is a simple normal crossing divisor. For every nest $\mathcal{S}$, $X(\mathcal{S}) := \cap_{I \in \mathcal{S}} D_I$ is a nonsingular subvariety of $X[n]$.

Define $j_{\mathcal{S}} : X(\mathcal{S}) \hookrightarrow X[n]$ to be the natural inclusion.

Define $\Delta_{\mathcal{S}} := \cap_{I\in\mathcal{S}}\Delta_I$. Define $g_{\mathcal{S}} : X(\mathcal{S}) \to \Delta_{\mathcal{S}}$ to be the restriction of the morphism $\pi : X[n] \to X^n$ to the subvariety $X(\mathcal{S})$.

3. Let $p_I : X[n] \to X$ be the composition of $\pi : X[n] \to X^n$ with the projection $X^n \to X$ to the i-th factor for an arbitrary $i \in I$. (The choice of $i \in I$ is not essential: indeed, the only place we need p_I is in the formulation of $\alpha_{\mathcal{S},\mu}$ below, where need the composition $j_{\mathcal{S}}^* p_I^*$. By the following diagram

$$\begin{array}{ccc} X(\mathcal{S}) & \xrightarrow{j_{\mathcal{S}}} & X[n] \\ \downarrow{\scriptstyle g_{\mathcal{S}}} & & \downarrow{\scriptstyle p_i} \\ \Delta_{\mathcal{S}} & \xrightarrow{q_i} & X \end{array}$$

where $i \in I$, we have $j_{\mathcal{S}}^* p_i^* = g_{\mathcal{S}}^* q_i^*$, but q_i is independent of the choice of $i \in I$ since $\Delta_{\mathcal{S}} \subseteq \Delta_I$, so $j_{\mathcal{S}}^* p_I^*$ is independent of the choice of $i \in I$ for p_I.)

4. For a nest $\mathcal{S} \neq \{\{1\}, \ldots, \{n\}\}$ (i.e. $\mathcal{S}^\circ \neq \emptyset$), define

$$M_{\mathcal{S}} := \big\{\underline{\mu} = \{\mu_I\}_{I\in\mathcal{S}^\circ} : 1 \leq \mu_I \leq d(c_I - 1) - 1\big\}.$$

(recall that $d = \dim X$, $c_I = c_I(\mathcal{S})$ is defined in Notation 1) and define $||\underline{\mu}|| := \sum_{I\in\mathcal{S}^\circ} \mu_I$, $\forall \underline{\mu} \in M_{\mathcal{S}}$.

For $\mathcal{S} = \{\{1\}, \ldots, \{n\}\}$, assume $M_{\mathcal{S}} = \{\underline{\mu}\}$ with $\|\underline{\mu}\| = 0$.

Define function $\zeta(x) := \sum_{i=0}^{d}(1+x)^{d-i}c_i(T_X)$.

Define $\alpha_{\mathcal{S},\underline{\mu}} \in Corr^{-||\underline{\mu}||}(X[n], \Delta_{\mathcal{S}})$, $\beta_{\mathcal{S},\underline{\mu}} \in Corr^{||\underline{\mu}||}(\Delta_{\mathcal{S}}, X[n])$, $p_{\mathcal{S},\underline{\mu}} \in$

$Corr^0(X[n], X[n])$ as follows,

$$\alpha_{\mathcal{S},\underline{\mu}} = (j_\mathcal{S} \boxtimes g_\mathcal{S})_* j_\mathcal{S}^* \Bigg(\prod_{I \in \mathcal{S}^\circ} \big\{ -p_I^* \zeta (-\sum_{\substack{J \sim \mathcal{S} \\ J \supsetneq I}} D_J)^{c_I - 1} \frac{1}{1 + D_I} \big\}_{d(c_I - 1) - 1 - \mu_I} \Bigg),$$

$$\beta_{\mathcal{S},\underline{\mu}} = (g_\mathcal{S} \boxtimes j_\mathcal{S})_* j_\mathcal{S}^* \Bigg(\prod_{I \in \mathcal{S}^\circ} D_I^{\mu_I - 1} \Bigg),$$

$$p_{\mathcal{S},\underline{\mu}} = \beta_{\mathcal{S},\underline{\mu}} \circ \alpha_{\mathcal{S},\underline{\mu}}.$$

(In the above definition of $\alpha_{\mathcal{S},\underline{\mu}}$ and $\beta_{\mathcal{S},\underline{\mu}}$, the products are assumed to be $1_{X(\mathcal{S})} \in A^0\big(X(\mathcal{S})\big)$ if $\mathcal{S}^\circ = \emptyset$.)

The following are the main theorems on the Chow groups and Chow motives of Fulton-MacPherson configuration spaces.

Theorem 4.1.1. *Let X be a (not necessarily complete) nonsingular variety. There is an isomorphism of Chow groups:*

$$A^*(X[n]) = \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_\mathcal{S}} A^{* - \|\underline{\mu}\|}(X^{c(\mathcal{S})}).$$

where $\mathcal{S}$ runs through all nests of $[n]$.

Theorem 4.1.2. *Let X be a complete nonsingular variety defined over an algebraically closed field. Then there is a natural isomorphism of Chow motives*

$$\bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_\mathcal{S}} \alpha_{\mathcal{S},\underline{\mu}} : h(X[n]) \cong \bigoplus_{\mathcal{S}} \bigoplus_{\underline{\mu} \in M_\mathcal{S}} h(\Delta_\mathcal{S})(\|\underline{\mu}\|)$$

with the inverse $\sum_{\mathcal{S}}\sum_{\underline{\mu}\in\mathcal{S}}\beta_{\mathcal{S},\underline{\mu}}$. *Equivalently, we have*

$$h(X[n]) \cong \bigoplus_{\mathcal{S}}\bigoplus_{\underline{\mu}\in M_{\mathcal{S}}} h(X^{c(\mathcal{S})})(\|\underline{\mu}\|).$$

Remark 4.1.3. *Observe that the two sets of correspondences* $\{\alpha_{\mathcal{S},\underline{\mu}}\}$, $\{\beta_{\mathcal{S},\underline{\mu}}\}$ *are* $\mathbb{S}_n$*-symmetric, in the sense that for any* $\sigma\in\mathbb{S}_n$*,*

$$\alpha_{\sigma(\mathcal{S}),\sigma(\underline{\mu})} = \sigma(\alpha_{\mathcal{S},\underline{\mu}}), \quad \beta_{\sigma(\mathcal{S}),\sigma(\underline{\mu})} = \sigma(\beta_{\mathcal{S},\underline{\mu}}),$$

where the actions of σ *are the obvious ones.*

Thus the $\mathbb{S}_n$*-action on* $X[n]$ *is compatible with* $\mathbb{S}_n$*-action on the motive decomposition in the sense that the following diagram commutes, where* $\Gamma = \bigoplus_{\mathcal{S},\underline{\mu}}\alpha_{\mathcal{S},\underline{\mu}}$*:*

$$\begin{array}{ccc} h(X[n]) & \xrightarrow{\Gamma} & \bigoplus_{\mathcal{S},\underline{\mu}} h(\Delta_{\mathcal{S}})(\|\underline{\mu}\|) \\ \downarrow{\sigma} & & \downarrow{\sigma} \\ h(X[n]) & \xrightarrow{\Gamma} & \bigoplus_{\mathcal{S},\underline{\mu}} h(\Delta_{\mathcal{S}})(\|\underline{\mu}\|) \end{array}$$

Proof of Theorem 4.1.1. Apply Theorem 3.1.1 with the ambient space $Y = X^n$ and the building set

$$\mathcal{G} = \{\Delta_I\}_{I\subseteq[n],|I|\geq 2}$$

First notice that a nest $\mathcal{S}$ of $[n]$ gives a $\mathcal{G}$-nest $\mathcal{T} = \{\Delta_I\}_{I\in\mathcal{S}^\circ}$. Moreover, the inverse is also true: a $\mathcal{G}$-nest will give a nest of $[n]$. Indeed, given a partition $\Pi = (I_1,\ldots,I_t)$ of $[n]$, a $\mathcal{G}$-factor of Δ_Π by definition is a minimal element in $\mathcal{G}$ that is $\supseteq \Delta_\Pi$, so $\{\Delta_{I_1},\ldots,\Delta_{I_t}\}$ are all the $\mathcal{G}$-factors of Δ_Π. By the definition of $\mathcal{G}$-nest (Definition 2.3.18), a $\mathcal{G}$-nest $\mathcal{T}$ is induced from a flag of strata

$$\Delta_{\Pi_1} \supseteq \Delta_{\Pi_2} \supseteq \cdots \supseteq \Delta_{\Pi_t}.$$

Then

$$\Pi_1 \geq \Pi_2 \geq \cdots \geq \Pi_k.$$

(Here $\Pi \geq \Pi'$ means Π is a finer partition than Π', e.g.$(12,3,4) \geq (123,4)$.) The nest $\mathcal{T}$ is induced by "taking the union of all factors of each Δ_Π", which corresponds to "take all I's that appears in any of the partition Π_i". Since the partitions is totally ordered, the set of I's forms a nest of $[n]$.

Next we prove the range of $\underline{\mu}$ is as stated. Theorem 3.1.1 says

$$1 \leq \mu_G \leq r_G - 1.$$

Now $G = \Delta_I$ is a diagonal, by definition

$$\begin{aligned} r_G &:= \dim(\bigcap_{G \subsetneq T \in \mathcal{T}} T) - \dim G \\ &= \dim(\bigcap_{I \supsetneq I' \in \mathcal{S}} \Delta_{I'}) - \dim \Delta_I \\ &= d(c_I - 1). \end{aligned}$$

Finally, observe that

$$Y_0\mathcal{T} = \bigcap_{G \in \mathcal{T}} G = \bigcap_{I \in \mathcal{S}} \Delta_I = \Delta_{\mathcal{S}} \cong X^{c(\mathcal{S})}.$$

The proof is complete. □

Proof of Theorem 4.1.2. The statement of the motive decomposition is proved exactly as the above proof.

The correspondences are induced from Theorem 3.1.2. The improvement of this theorem than Theorem 3.1.2 is: we can say more about the chern classes appeared in the correspondence $\alpha_{\mathcal{S},\underline{\mu}}$ in Theorem 3.1.2.

First, for $G = \Delta_I$, let $\Pi = (I_1, \dots, I_{c_I})$ be the partition containing all sons of I in $\mathcal{S}$. We calculate the normal bundle $N_G := N_{\Delta_I}\Delta_\Pi$. Without loss of generality, assume $I = (12\dots m)$, where $m \le n$.

Denote $p_i : \Delta_I \to X$, $q_i : \Delta_\Pi \to X$ be the projections induced from the projection of X^n to the i-th factor. For each $1 \le i \le c_I$, pick an $a_i \in I_i$.

$$\begin{aligned} T_{\Delta_I} &= p_1^* T_X \oplus p_{m+1}^* T_X \oplus \cdots \oplus p_n^* T_X \\ T_{\Delta_\Pi} &= q_{a_1}^* T_X \oplus \cdots \oplus q_{a_{c_I}}^* T_X \oplus q_{m+1}^* T_X \oplus \cdots \oplus q_n^* T_X \\ T_{\Delta_\Pi}|_{\Delta_I} &= p_1^* T_X \oplus \cdots \oplus p_1^* T_X \oplus q_{m+1}^* T_X \oplus \cdots \oplus q_n^* T_X \end{aligned}$$

Therefore, $c(N_G) = p_1^* c(T_X)^{c_I - 1}$.

To calculate the chern classes of N_G twisted by a line bundle L, we use the chern root technique. For any vector bundle N on X, define the chern polynomial as

$$c_y(N) := c_0(N) + c_1(N)y + c_2(N)y^2 + \dots.$$

Define $x = c_1(L)$. Recall that the rank of N_G is $r_G = d(c_I - 1)$. Then

$$\begin{aligned} c(N_G \otimes L) &= c_{r_G}(N_G) + c_{r_G-1}(N_G)(1+x) + ... + c_0(N_G)(1+x)^{r_G} \\ &= (x+1)^{r_G} c_{\frac{1}{x+1}}(N_G) \\ &= (x+1)^{d(c_I-1)} p_1^* c_{\frac{1}{x+1}}(T_X)^{c_I-1} \\ &= p_1^*[(x+1)^d c_{\frac{1}{x+1}}(T_X)]^{c_I-1} = p_1^* \zeta(x)^{c_I-1}. \end{aligned}$$

Finally, by restricting to $\Delta_{\mathcal{S}}$ and pulling back to $X(\mathcal{S})$ we get the expected formula for correspondences $\alpha_{\mathcal{S},\underline{\mu}}$. □

4.2 A formula for the generating function of Chow groups and Chow motive of $X[n]$

In this section, we show that the decompositions of the Chow groups (Theorem 4.1.1) and the Chow motive (Theorem 4.1.2) can be expressed using exponential generating functions.

Define $[\frac{x^i t^n}{n!}]$ to be a function to pick up the coefficient of $\frac{x^i t^n}{n!}$ from a power series with two variables x and t, i.e.,

$$[\frac{x^i t^n}{n!}] \sum_{j,m} a_{jm} \frac{x^j t^m}{m!} := a_{in}.$$

The main theorem of this section is the following:

Theorem 4.2.1. *Define $f_i(x)$ to be the polynomials whose exponential gener-*

ating function $N(x,t) = \sum_{i\geq 1} f_i(x)\frac{t^i}{i!}$ *satisfies the identity*

$$(1-x)x^d t + (1-x^{d+1}) = \exp(x^d N) - x^{d+1}\exp(N).$$

where $d = \dim X$*. Then*

$$A^*(X[n]) = \bigoplus_{\substack{1\leq k\leq n\\ i\geq 0}} A^{*-i}(X^k)^{\oplus[\frac{x^i t^n}{n!}]\frac{N^k}{k!}}.$$

Moreover, if X *is complete, then we have the motive decomposition*

$$\begin{aligned} h(X[n]) &= \bigoplus_{\substack{\Pi=(I_1,\ldots,I_k)\\ \text{partition of } [n]}} \big(h(\Delta_\Pi)(i)\big)^{\oplus[x^i](f_{|I_1|}(x)\ldots f_{|I_k|}(x))} \\ &= \bigoplus_{\substack{1\leq k\leq n\\ i\geq 0}} \big(h(X^k)(i)\big)^{\oplus[\frac{x^i t^n}{n!}]\frac{N^k}{k!}}. \end{aligned}$$

Remark 4.2.2. *One can write down by hand the first several terms of* N*. Define* $\sigma_j = \sum_{i=1}^{dj-1} x^i$ *(when* $d=1$*, define* $\sigma_1 = 0$*). Then*

$$\begin{aligned} N = t + \sigma_1\frac{t^2}{2!} + (\sigma_2 + 3\sigma_1^2)\frac{t^3}{3!} + (\sigma_3 + 10\sigma_1\sigma_2 + 15\sigma_1^3)\frac{t^4}{4!} \\ + (\sigma_4 + 15\sigma_1\sigma_3 + 10\sigma_2^2 + 105\sigma_1^2\sigma_2 + 105\sigma_1^4)\frac{t^5}{5!} + \ldots. \end{aligned}$$

Proof of Theorem 4.2.1. We prove only the statement for motives, since the statement for Chow groups can be proved by exactly the same method.

By Theorem 4.1.2, we want to count for any given i and k, how many possible $\mathcal{S}$ and $\underline{\mu} \in \mathcal{S}$ satisfy $c(\mathcal{S}) = k$ and $\|\underline{\mu}\| = i$. First, consider the case when $c(\mathcal{S}) = 1$, i.e. $\mathcal{S}$ is a connected forest.

Define

$$f_n(x) := \sum_{\mathcal{S}:c(\mathcal{S})=1} \sum_{\underline{\mu}\in M_{\mathcal{S}}} x^{||\underline{\mu}||},$$

and define $f_1(x) = 1$.

For a nest $\mathcal{S}$ of $[n]$ with $c(\mathcal{S}) = 1$, we have

$$\sum_{\underline{\mu}\in M_{\mathcal{S}}} x^{\|\underline{\mu}\|} = \prod_{I\in\mathcal{S}^\circ} \sigma_{(c_I-1)},$$

i.e., I goes through all non-leaves of $\mathcal{S}$ (if $n = 1$, then the sum is assumed to be 1). Since the sons of the root of $\mathcal{S}$ correspond to a partition $\{I_1, \ldots, I_k\}$ of $[n]$, we have following formula for $n \geq 2$,

$$f_n(x) = \sum_{\{I_1,\ldots,I_k\}\text{partition of } [n]} f_{|I_1|} f_{|I_2|} \ldots f_{|I_k|} \sigma_{k-1}.$$

where $\sigma_k = \sum_{i=1}^{dk-1} x^i$ for $k > 0$, and $\sigma_0 = 0$. Since the equality does not hold for $n = 1$ where $f_1(x) = 1$ but the right side is 0, so one define

$$\tilde{f}_n(x) = \begin{cases} f_n(x), & \text{if } n > 1; \\ 0, & \text{if } n = 1. \end{cases}$$

Then the following holds for any $n \geq 1$:

$$\tilde{f}_n(x) = \sum_{\{I_1,\ldots,I_k\}\text{partition of } [n]} f_{|I_1|} f_{|I_2|} \ldots f_{|I_k|} \sigma_{k-1}.$$

Recall the Compositional Formula of exponential generating functions (cf.

[St99], Theorem 5.1.4), which asserts that if an equation as above holds, then

$$E_{\tilde{f}}(t) = E_\sigma(E_f(t)),$$

where

$$\begin{aligned}
E_{\tilde{f}}(t) &= 1 + \tilde{f}_1 t + \tilde{f}_2 t^2/2! + \tilde{f}_3 t^3/3! + \dots \\
E_\sigma(t) &= 1 + \sigma_0 t + \sigma_1 t^2/2! + \sigma_2 t^3/3! + \dots \\
E_f(t) &= \qquad f_1 t + f_2 t^2/2! + f_3 t^3/3! + \dots
\end{aligned}$$

By the definition of $\tilde{f}$, $E_{\tilde{f}} = E_f - t + 1$. Denote $N = E_f$, one has

$$N - t + 1 = E_g(N),$$

A standard Calculation shows

$$E_g(N) = 1 + N + \frac{1}{x-1}\Big[\frac{1}{x^d}(e^{x^d N - 1} - 1) - xe^N + x\Big].$$

Therefore

$$(1-x)x^d t + (1 - x^{d+1}) = \exp(x^d N) - x^{d+1}\exp(N).$$

Now consider the case when $c(\mathcal{S})$ is not necessarily 1, i.e., the forest $\mathcal{S}$ is not necessarily connected. For a partition $\Pi = \{I_1, ..., I_k\}$ of $[n]$, the number of times that $h(\Delta_\Pi)(i)$ appears in the decomposition of $h(X[n])$ is equal to $[x^k](f_{|I_1|}(x)...f_{|I_k|}(x))$, the coefficient of x^k in the product. Denote by $a_{k,i}$ the sum of these numbers for all partitions with k blocks. Then $a_{k,i}$ is the number of times that $h(X^k)(i)$ appears in the decomposition of $H(X[n])$.

Define

$$F_n(y) = \sum_{\{I_1,\dots,I_k\}\text{partition of }[n]} f_{|I_1|} f_{|I_2|} \dots f_{|I_k|} y^k.$$

Then the coefficient $[y^k]F_n(y) = \sum a_{k,i}x^i$. Use the Compositional Formula again,

$$F_n = [\frac{t^n}{n!}] \exp(yN).$$

Therefore

$$\begin{aligned}[y^k]F_n(y) &= [y^k][\frac{t^n}{n!}] \exp(yN) \\ &= [\frac{t^n}{n!}][y^k] \exp(yN) \\ &= [\frac{t^n}{n!}] \frac{N^k}{k!}.\end{aligned}$$

This yields the formula for the decomposition of the Chow motive $h(X[n])$. □

4.3 Description of $X[n]$ for small n

In this section we explain the previous Theorems (4.1.1, 4.1.2, and 4.2.1) about Fulton-MacPherson configuration space $X[n]$ for small $n = 2, 3, 4$.

For unification of notation, assume $d > 1$ in the following examples (1), (2) and (3). (The case $d = 1$ is simpler but needs a revise of notation.)

1. $n = 2$. The morphism $\pi : X[2] \to X^2$ is a blow-up along the diagonal

Δ_{12}. Theorem 4.2.1 asserts

$$h(X[2]) \cong h(X^2) \oplus \bigoplus_{i=1}^{d-1} h(\Delta_{12})(i) \cong h(X^2) \oplus \bigoplus_{i=1}^{d-1} h(X)(i).$$

There are 2 possible nests: $\mathcal{S} = \{1, 2\}$ and $\mathcal{S} = \{1, 2, 12\}$. Theorem 4.1.2 asserts the follows:

For the first nest, $M_{\mathcal{S}} = \{\underline{\mu}\}$ with $\|\underline{\mu}\| = 0$. Therefore $\alpha = \Gamma_\pi$, $\beta = \Gamma_\pi^t$, $p = \Gamma_\pi^t \circ \Gamma_\pi$. They give the first direct summand in the above decomposition.

For the second nest, $\mathcal{S}^\circ = \{12\}$, $1 \leq \mu_{12} \leq d-1$, so there are $d-1$ direct summands for this nest. Denote $j : D_{12} \hookrightarrow X[2]$, $g : D_{12} \to \Delta_{12}$ as the natural map, we have

$$\alpha_{\mathcal{S},\underline{\mu}} = -(j \boxtimes g)_* j^* \Big(\sum_{i=0}^{d-1-\mu_{12}} p_1^* c_i(T_X)(-D_{12})^{d-1-\mu_{12}-i} \Big),$$

$$\beta_{\mathcal{S},\underline{\mu}} = (g \boxtimes j)_* j^* \big(D^{\mu_{12}-1} \big),$$

$$p_{\mathcal{S},\underline{\mu}} = \beta_{\mathcal{S},\underline{\mu}} \circ \alpha_{\mathcal{S},\underline{\mu}}.$$

They give the direct summand $h(\Delta_{12})(\mu_{12})$.

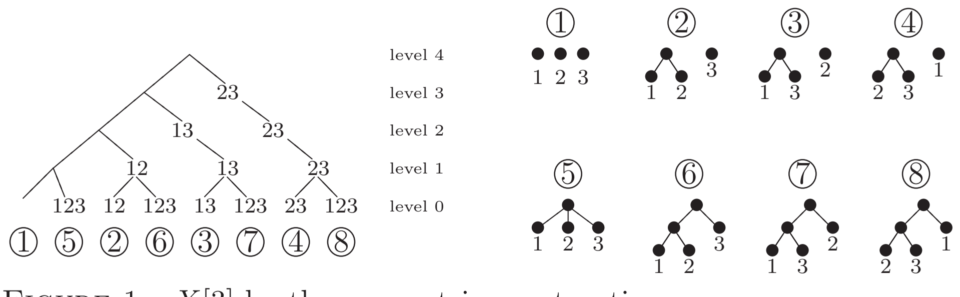


Figure 1. $X[3]$ by the symmetric construction.

2. $n = 3$. Apply Theorem 4.2.1,

$$\begin{aligned} h(X[3]) &\cong h(X^3) \oplus \bigoplus_{i=1}^{d-1} h(\Delta_{12})(i) \oplus \bigoplus_{i=1}^{d-1} h(\Delta_{13})(i) \oplus \bigoplus_{i=1}^{d-1} h(\Delta_{23})(i) \\ &\quad \oplus \bigoplus_{i=1}^{2d-1} \big(h(\Delta_{123})(i)\big)^{\oplus min\{3i-2, 6d-3i-2\}} \\ &\cong h(X^3) \oplus \bigoplus_{i=1}^{d-1} \big(h(X^2)(i)\big)^{\oplus 3} \oplus \bigoplus_{i=1}^{2d-1} \big(h(X)(i)\big)^{\oplus min\{3i-2, 6d-3i-2\}} \end{aligned}$$

Now we write out all the correspondences that give the decomposition of motives. There are 8 possible nests, correspond to 8 trees (see the right side of Figure 1).

The tree on the left side of Figure 1 helps us to understand the relation between subvarieties of different Y_i's (i.e. at different levels): each node with label I at level k correspond to the subvariety $Y_k I := (\Delta_I)^{(k)}$ in Y_k. The node at level k without label correspond to Y_k. For example, the root at level 4 corresponds to Y_4, its two successors correspond to Y_3 and $Y_3(23)$, and the relation is that Y_4 is the blow-up of Y_3 along $Y_3(23)$.

We list below those correspondences α, β, p for the 8 trees:

① gives $\alpha = \Gamma_\pi, \beta = \Gamma_\pi^t, p = \Gamma_\pi^t \circ \Gamma_\pi$.

② (and ③, ④ are similar) gives

$$\begin{aligned} \alpha_{\mathcal{S},\underline{\mu}} &= (j_{\mathcal{S}} \boxtimes g_{\mathcal{S}})_* j_{\mathcal{S}}^* \big(\{-p_1^* \zeta(-D_{123}) \frac{1}{1+D_{12}}\}_{d-1-\mu_{12}}\big), \\ \beta_{\mathcal{S},\underline{\mu}} &= (g_{\mathcal{S}} \boxtimes j_{\mathcal{S}})_* j_{\mathcal{S}}^* \big(D_{12}^{\mu_{12}-1}\big). \end{aligned}$$

where $X(\mathcal{S}) = D_{12}$, $1 \leq \mu_{12} \leq d-1$.

⑤ gives

$$\alpha_{\mathcal{S},\underline{\mu}} = (j_{\mathcal{S}} \boxtimes g_{\mathcal{S}})_* j_{\mathcal{S}}^* \big(\{-p_1^*\zeta(\mathcal{O})^2 \frac{1}{1+D_{123}}\}_{2d-1-\mu_{123}}\big),$$
$$\beta_{\mathcal{S},\underline{\mu}} = (g_{\mathcal{S}} \boxtimes j_{\mathcal{S}})_* j_{\mathcal{S}}^* \big(D_{123}^{\mu_{123}-1}\big).$$

where $X(\mathcal{S}) = D_{123}$, $1 \leq \mu_{123} \leq 2d-1$.

⑥ (and ⑦, ⑧ are similar) gives

$$\begin{aligned}\alpha_{\mathcal{S},\underline{\mu}} =&(j_{\mathcal{S}} \boxtimes g_{\mathcal{S}})_* j_{\mathcal{S}}^* \\ &\big(\{p_1^*\zeta(-D_{123})\frac{1}{1+D_{12}}\}_{d-1-\mu_{12}}\{p_1^*\zeta(\mathcal{O})\frac{1}{1+D_{123}}\}_{d-1-\mu_{123}}\big),\\ \beta_{\mathcal{S},\underline{\mu}} =&(g_{\mathcal{S}} \boxtimes j_{\mathcal{S}})_* j_{\mathcal{S}}^* \big(D_{12}^{\mu_{12}-1}D_{123}^{\mu_{123}-1}\big).\end{aligned}$$

where $X(\mathcal{S}) = D_{12} \cap D_{123}$, $1 \leq \mu_{12}, \mu_{123} \leq d-1$.

Remark 4.3.1. *If we use Fulton and MacPherson's nonsymmetric construction of $X[3]$, we would get another set of correspondences which also gives a decomposition of the motive $h(X[n])$. This set of correspondences turns out to be different than the ones given above: a straightforward calculation shows that, by the nonsymmetric construction of $X[3]$, the correspondence that gives the direct summand $h(\Delta_{12})(\mu_{12})$ is*

$$\alpha: \quad h(X[3]) \to h(\Delta_{12})(\mu_{12}),$$
$$\alpha = (j_{12} \boxtimes g_{12})_* j_{12}^* \big(\{p_1^*\zeta(\mathcal{O})\frac{1}{1+D_{12}}\}_{d-1-\mu_{12}}\big).$$

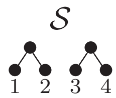
where $j_{12}: D_{12} \hookrightarrow X[3]$ and $g_{12}: D_{12} \to \Delta_{12}$ are the natural morphisms.

However, the correspondence giving the direct summand $h(\Delta_{13})(\mu_{13})$ *is*

$$\alpha' : \quad h(X[3]) \to h(\Delta_{13}) \otimes \mathbb{L}^{\mu_{13}},$$
$$\alpha' = (j_{13} \boxtimes g_{13})_* j_{13}^* \big(\{p_1^* \zeta(-D_{123}) \frac{1}{1+D_{13}}\}_{d-1-\mu_{13}}\big).$$

where $j_{13} : D_{13} \hookrightarrow X[3], g_{13} : D_{13} \to \Delta_{13}$ *are the natural morphisms. Notice that* α *and* α' *are not of similar forms (Compare* $\zeta(\mathcal{O})$ *with* $\zeta(-D_{123})$*). Therefore the non-symmetry of the construction of* $X[3]$ *induces the non-symmetry of correspondences. Actually, this is a reason why we choose the symmetric construction of* $X[n]$ *(cf. Remark 4.1.3).*

3. For $n = 4$, we just look at one nest $\mathcal{S}$:



We have $X(\mathcal{S}) = D_{12} \cap D_{34}$, $1 \le \mu_{12}, \mu_{34} \le d-1$ and

$$\alpha_{\mathcal{S},\underline{\mu}} = (j_{\mathcal{S}} \boxtimes g_{\mathcal{S}})_* j_{\mathcal{S}}^* \big(\{p_1^* \zeta(-D_{1234}) \frac{1}{1+D_{12}}\}_{d-1-\mu_{12}} \{p_3^* \zeta(-D_{1234}) \frac{1}{1+D_{34}}\}_{d-1-\mu_{34}}\big),$$
$$\beta_{\mathcal{S},\underline{\mu}} = (g_{\mathcal{S}} \boxtimes j_{\mathcal{S}})_* j_{\mathcal{S}}^* \big(D_{12}^{\mu_{12}-1} D_{34}^{\mu_{34}-1}\big).$$

Since Δ_{12} and Δ_{34} intersect and would not be disjoint in the procedure of blow-ups, so a priori we have to make a choice of order that whether blow up along (the strict transform of) Δ_{12} first, or along (the strict transform of) Δ_{34} first. Although an order is chosen to calculate the correspondences, it turns out that the correspondences (hence projectors) which give the motive decomposition in Theorem 4.1.2 are actually independent of the choice. This independence is a special case of Remark 4.1.3: for $\sigma = (13)(24) \in \mathbb{S}_4$, the above correspondences is invariant under the action induced by σ.

4. An application of Theorem 4.2.1 is: we can calculate the rank of $A(X[n])$ (as an abelian group) once given the ranks of $A(X^k)$ for all $1 \le k \le n$ (assuming that the ranks of $A(X^k)$'s are finite).

   Let us take $\mathbb{P}^d[5]$ for example. Since the rank of $A((\mathbb{P}^d)^k)$ is $(d+1)^k$, Theorem 4.2.1 implies that the rank of $A(\mathbb{P}^d[5])$ is

$$\sum_{1 \le k \le 5} (d+1)^k \Big([\frac{t^5}{t!}] (\frac{N^k}{k!}|_{x=1}) \Big).$$

   By Remark 4.2.2, we can calculate the following

$$\begin{aligned}
\frac{N^2}{2!} &= \frac{t^2}{2!} + 3\sigma_1 \frac{t^3}{3!} + (15\sigma_1^2 + 4\sigma_2)\frac{t^4}{4!} + (105\sigma_1^3 + 60\sigma_1\sigma_2 + 5\sigma_3)\frac{t^5}{5!} + ... \\
\frac{N^3}{3!} &= \frac{t^3}{3!} + 6\sigma_1 \frac{t^4}{4!} + (45\sigma_1^2 + 10\sigma_2)\frac{t^5}{5!} + ... \\
\frac{N^4}{4!} &= \frac{t^4}{4!} + 10\sigma_1 \frac{t^5}{5!} + ... \\
\frac{N^5}{5!} &= \frac{t^5}{5!} + ...
\end{aligned}$$

   Now plug in $x = 1$, we have $\sigma_j = dj - 1$. The above sum is a polynomial of d as follows

$$\begin{aligned}&(d+1)^5+(d+1)^4 10\sigma_1+(d+1)^3(45\sigma_1^2+10\sigma_2)\\&+(d+1)^2(105\sigma_1^3+60\sigma_1\sigma_2+5\sigma_3)\\&+(d+1)(\sigma_4+15\sigma_1\sigma_3+10\sigma_2^2+105\sigma_1^2\sigma_2+105\sigma_1^4).\end{aligned}$$

In particular, the rank of $A(\mathbb{P}^1[5])$ is 178, the rank of $A(\mathbb{P}^2[5])$ is 7644.

Remark: For the example $X=\mathbb{P}^d$, since $X[n]$ has an affine cell decomposition, the rank of the Chow group $A_k(X[n])$ coincides with the $2k$-th Betti number of $X[n]$. Therefore we could also get the above rank by the Poincaré polynomial of $X[n]$ calculated in [FM94]. However, the rank of $A(X[n])$ for a general variety X is not implied by the Poincaré polynomial of $X[n]$.

Chapter 5

On the cobordism class

Consider the complex cobordism ring with rational coefficients $\Omega = \Omega^U \otimes \mathbb{Q}$. The complex cobordism class of a stably complex manifold is completely determined by the collection of its Chern numbers. In this chapter we show that for certain wonderful compactifications of X^n, in particular the Fulton-MacPherson configuration spaces, their cobordism classes depend only on the cobordism class $[X] \in \Omega_{\dim X}$.

The proof of our theorem is based on a well-known blow-up theorem of Chern classes (see Theorem 5.2.2), which asserts that the Chern classes of the blow-up variety are determined by the Chern classes of the original variety, the ones of the center and the ones of the normal bundle. We apply the theorem inductively since the wonderful compactification is constructed by a sequence of blow-ups. The key ingredient is: during the procedure of blow-ups, each center is again a wonderful compactification of $X^{n'}$ for some $n' < n$, and the normal bundle of each center can be expressed using the Chern classes of X and certain exceptional divisors.

The discussion is inspired by the result on the cobordism class of the Hilbert

Schemes of a surface [EGL01].

5.1 Theorems

Denote $d = \dim X$. Let $Y = X^n$. Fix an arrangement of subvarieties of Y, where every subvariety is a polydiagonal of X^n. (Notice here the arrangement does not necessarily contain all polydiagonals. For example, the arrangement $\{\Delta_{12}, \Delta_{123}\}$ in X^3 is allowed.)

Let $\mathcal{G} = \{\Delta_\Pi\}_{\Pi \in \mathcal{H}}$ be a building set of Y with respect to the fixed arrangement. The set $\mathcal{H}$ of partitions of $[n]$ is independent of X. Denote $X^{\mathcal{H}} := Y_{\mathcal{G}}$.

Before we state the main theorem, recall that for a partition $\lambda = \{i_1, \ldots, i_t\}$ of $\dim Y$, the Chern number $c_\lambda(Y)$ is defined as

$$c_\lambda(Y) = \int_Y c_{i_1}(Y) c_{i_2}(Y) \cdots c_{i_t}(Y).$$

Theorem 5.1.1. *For any partition λ of $dn(= \dim Y)$, there is a universal polynomial P_λ (depends on λ, d, n, and set $\mathcal{H}$, but does not depend X) such that the Chern number*

$$c_\lambda Y_{\mathcal{G}} = P_\lambda\big(c_1(X)^d, c_1(X)^{d-2} c_2(X), \ldots, c_d(X)\big).$$

In particular, for Fulton-MacPherson configuration spaces, we have

Corollary 5.1.2. *Define*

$$H(X) := \sum_{n=0}^{\infty} \big[X[n]\big] \frac{z^n}{n!},$$

then $H(X)$ *depends only on the cobordism class* $[X] \in \Omega_d$*. Therefore, if* $a_1, a_2 \in \mathbb{Q}$ *such that* $[X] = a_1[X_1] + a_2[X_2]$*, then*

$$H(X) = H(X_1)^{a_1} H(X_2)^{a_2}.$$

Remark: This is true in general for any $H(X) := \sum_{n=0}^{\infty} [X^{\mathcal{H}}] \frac{z^n}{n!}$. In particular, it applies to Ulyanov's polydiagonal compactification $X\langle n \rangle$, Kuperberg-Thurson's compactification X^{Γ}.

Proof of Corollary 5.1.2. Consider the case $X = X_1 \sqcup X_2$. Obviously

$$X[n] = \bigsqcup_{I \subseteq [n]} X_1[I] X_2[I^c].$$

($I^c := [n] \setminus I$). Since $X[I] \cong X[|I|]$, $H(X) = H(X_1)H(X_2)$ follows from the following fact:

If a power series $H(x) := \sum_{n=0}^{\infty} c_n(x) \frac{z^n}{n!}$ *satisfies*

$$c_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} c_k(x) c_{n-k}(y),$$

then $H(x)H(y) = H(x+y)$.

Then use induction to show $H(X)^m = H(X_1)^{m_1} H(X_2)^{m_2}$ if $m[X] = m_1[X_1] + m_2[X_2]$ for positive integers m, m_1, m_2. The corollary follows. □

5.2 Proof

Fix $0 \leq k \leq k$. Denote by $p_l : Y_k \to X$ the composition of $Y_k \to X^n$ with $X^n \to X$, the projection to the l-th factor.

Theorem 5.1.1 is a special case of the following proposition:

Proposition 5.2.1. *Given a polynomial Q of $c_m(Y_k)$ $(\forall m)$, $p_l^* c_m(X)$ $(\forall l, m)$ and divisors $[G_m^{(k)}]$ $(\forall m \leq k)$. There exists a polynomial P depending only on Q such that*

$$\int_{Y_k} Q = P(c_1(X)^d, c_1(X)^{d-2}c_2(X), \ldots, c_d(X)).$$

Before the proof of the above Proposition, recall a blow-up theorem of chern class proved by Porteous [Po60], later been proved by Lascu and Scott [LS78] using a simpler method.

Suppose $\widetilde{Y}$ is the blow-up of a nonsingular algebraic variety Y along a nonsingular subvariety V, and P is the exceptional divisor. Denote by i, j, f, g the morphisms as in the following fibre square

$$\begin{array}{ccc} P & \xrightarrow{j} & \widetilde{Y} \\ {\scriptstyle g}\downarrow & \square & \downarrow{\scriptstyle f} \\ V & \xrightarrow[i]{} & Y \end{array}$$

Denote by $N := N_V Y$ the normal bundle of V in Y, define $h := c_1(\mathcal{O}_N(1)) \in A^1(P)$, define $r := \mathrm{codim}_V Y$ to be the codimension of V in Y. Denote

$$\overline{C}(t, N) := t^r + c_1(N)t^{r-1} + \cdots + c_r(N).$$

Theorem 5.2.2 (of Porteous [Po60], Lascu and Scott [LS78]).

$$
\begin{aligned}
&\overline{C}(t, \widetilde{Y}) - f^*\overline{C}(t, Y) \\
&= j_*\left[\frac{-1}{h} g^*\overline{C}(t, V)\left\{(1 - \frac{h}{t})(t+h) g^*\overline{C}(t+h, N) - t g^*\overline{C}(t, N)\right\}\right]
\end{aligned}
$$

In brief, the above theorem asserts that $c_i(\widetilde{Y}) - f^*c_i(Y) = j_*(R)$ where R is a polynomial of $g^*(c_l(V))(\forall l)$, $g^*(c_l(N))(\forall l)$ and h.

Proof of Proposition 5.2.1. Use induction first on n, then on k. When $k = 0$, we have $Y_k = Y_0 = X^n$, so Q is a polynomial of $c_m(X^n)(\forall m)$, $p_l^* c_m(X)(\forall l, m)$. Since

$$
T_{X^n} = p_1^* T_X \oplus \cdots \oplus p_n^* T_X,
$$

So Q can be expressed as $\sum(\text{const})(p_1^* c_{\lambda_1})(p_2^* c_{\lambda_2}) \ldots (p_n^* c_{\lambda_n})$, where each λ_i is a partition (of some integer). Then

$$
\begin{aligned}
\int_{X^n} Q &= \sum(\text{const}) \int_{X^n} (p_1^* c_{\lambda_1})(p_2^* c_{\lambda_2}) \ldots (p_n^* c_{\lambda_n}) \\
&= \sum(\text{const}) \int_X c_{\lambda_1} \int_X c_{\lambda_2} \ldots \int_X c_{\lambda_n}.
\end{aligned}
$$

Assume the case k has been proved, consider the case $k+1$. We have the following blow-up diagram

$$
\begin{array}{ccc}
G_{k+1}^{(k+1)} & \xrightarrow{j} & Y_{k+1} \\
{\scriptstyle g}\downarrow & \square & \downarrow{\scriptstyle f} \\
G_{k+1}^{(k)} & \xrightarrow[i]{} & Y_k
\end{array}
$$

Now Q is a polynomial of $c_m(Y_{k+1})(\forall m)$, $p_l^* c_m(X)$ $(\forall l, m)$, G_m^{k+1} $(\forall m \leq$

$k+1$). By Theorem 5.2.2, Q it is a linear combination of $P_1P_2P_3P_4$, where P_r ($1 \le r \le 4$) are of the form

$$P_1 = \prod f^* c_m(Y_k), P_2 = \prod [G_m^{(k+1)}], P_3 = \prod p_l^* c_m(X), P_4 = \prod m_*(R).$$

To distinct from $p_l : Y_{k+1} \to X$, we denote $p_l' : Y_k \to X$. Obviously $p_l = p_l' \circ f$.

The product $P_1P_2P_3P_4$ can be expressed as f^*P_1' or $(f^*P_1')(j_*P_2')$, where P_1' is a product of terms as $c_m(Y_k)$, $[G_m^{(k)}](m \le k)$, $p_l^{'*} c_m(X)$, and P_2' is a product of terms as $g^*(c_m(G_{k+1}^{(k)}))$, $g^*(c_m(N_{(G_{k+1}^{(k)})}))$ and $h = [-G_{k+1}^{(k+1)}]|_{G_{k+1}^{(k+1)}}$. (Indeed,

$$[G_m^{(k+1)}] = \begin{cases} f^*([G_m^{(k)}]), & \text{for } m < k+1; \\ j_*1, & \text{for } m = k+1. \end{cases}$$

Notice that $p_l^* c_m(X) = f^* \prod p_l^{'*} c_m(X)$,

$$j_*a \cdot j_*b = j_*(a \cdot j^* j_* b) = j_*(a \cdot b \cdot h),$$

the desired expression follows.)

Now discuss these two cases for the product $P_1P_2P_3P_4$.

Case I: $P_1P_2P_3P_4 = f^*P_1'$.

$$\int_{Y_{k+1}} P_1P_2P_3P_4 = \int_{Y_{k+1}} f^*P_1' = \int_{Y_k} P_1'.$$

This can be expressed as a polynomial of Chern numbers of X by inductive assumption on k.

Case II: $P_1P_2P_3P_4 = (f^*P_1')(j_*P_2')$.

$$\int_{Y_{k+1}} P_1P_2P_3P_4 = \int_{Y_{k+1}} (f^*P_1')(j_*P_2') = \int_{Y_{k+1}} j_*[(j^*f^*P_1')P_2']$$

Use the fact that $\int_{Y_{k+1}} j_*\omega = \int_{G_{k+1}^{(k+1)}} \omega = \int_{G_{k+1}^{(k)}} g_*\omega$ for any form ω,

$$\text{above} = \int_{G_{k+1}^{(k)}} g_*[(g^*i^*P_1')P_2'] = \int_{G_{k+1}^{(k)}} (i^*P_1')(g_*P_2').$$

Now we claim that the last integral can be expressed by a polynomial of Chern numbers of X by inductive assumption on n. Indeed, $G_{k+1}^{(k)}$ is a wonderful compactification of $G_{k+1} \cong X^{n'}$ for some $n' < n$. Observe the following facts:

$$i^*c(Y_k) = c(Y_k|_{G_{k+1}^{(k)}}) = c(G_{k+1}^{(k)})c(N_{G_{k+1}^{(k)}}Y_k).$$

Similar to the proof of Theorem 4.1.2, $c(N_{G_{k+1}}Y)$ can be expressed as product of $p_l^*c_m(X)$. By Corollary 3.2.8 and Chern root technique, $c(N_{G_{k+1}^{(k)}}Y_k)$ can be expressed as a polynomial of the chern classes of $N_{G_{k+1}}Y$ and $i^*[D_{G_l}]$ $(l \le k)$. Moreover, $g_*(h^l) = s_l(N_{G_{k+1}^{(k)}}Y_k)$, but the Segre class s_l can be expressed in terms of chern classes. By routine check, the induction goes through.

From the proof we see that the polynomial we got depends only on the polynomial Q (so is independent of X). □

Bibliography

[AM69] M. Atiyah, I. Macdonald, Introduction to commutative algebra, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969 ix+128 pp.

[CH00] A. Corti, M. Hanamura, *Motivic decomposition and intersection Chow groups. I*, Duke Math. J. 103 (2000), no. 3, 459–522.

[dCM02] M. deCataldo, L. Migliorini, *The Chow groups and the motive of the Hilbert scheme of points on a surface*, J. Algebra 251 (2002), no. 2, 824–848.

[dCM04] M. deCataldo, L. Migliorini, *The Chow motive of semismall resolutions*, Math. Res. Lett. 11 (2004), no. 2-3, 151–170.

[DP95] C. De Concini, C. Procesi, *Wonderful models of subspace arrangements*, Selecta Mathematica 1 (1995), 459–494.

[EGL01] G. Ellingsrud, L.Göttsche, M. Lehn, *On the cobordism class of the Hilbert scheme of a surface*, J. Algebraic Geom. 10 (2001), no. 1, 81–100.

[Fu98] W. Fulton, Intersection theory, Second edition, Springer-Verlag, Berlin, 1998.

[FM94] W. Fulton, R. MacPherson, *A compactification of configuration spaces*, Ann. Math. **139** (1994), 183–225.

[Ha77] R. Hartshorne, Algebraic geometry, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977.

[Hu03] Y. Hu, *A compactification of open varieties*, Trans. Amer. Math. Soc. 355 (2003), no. 12, 4737–4753.

[Ke93] S. Keel, *Intersection theory of linear embeddings* Trans. Amer. Math. Soc. 335 (1993), no. 1, 195–212.

[KT99] G. Kuperberg, D. Thurston, *Perturbative 3-manifold invariants by cut-and-paste topology*, math.GT/9912167.

[LS78] A. Lascu, D. Scott, *A simple proof of the formula for the blowing up of Chern classes*, Amer. J. Math. 100 (1978), no. 2, 293–301.

[Man68] J.I. Manin, *Correspondences, motifs and monoidal transformations*, Math. USSR-Sbornik 6 (1968), 439–470.

[MP98] R. MacPherson, C. Procesi, *Making conical compactifications wonderful*, Selecta Math. (N.S.) **4** (1998), no. 1, 125–139.

[Maz04] B. Mazur, *What is … a motive?*, Notices Amer. Math. Soc. 51 (2004), no. 10, 1214–1216.

[JKS94] U. Jannsen, S. Kleiman, J.-P. Serre : eds., Motives, 55(1)(2), of Proceedings of Symposia in Pure Mathematics, 1994.

[Mu04] J. Murre, Lecture on motives. Transcendental aspects of algebraic cycles, 123–170, London Math. Soc. Lecture Note Ser., 313, Cambridge Univ. Press, Cambridge, 2004.

[Po60] I. Porteous, *Blowing up Chern classes*, Proc. Cambridge Philos. Soc. 56 1960 118–124.

[St99] R.P. Stanley, Enumerative combinatorics, Vol. 2., Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press, Cambridge, 1999.

[Th99] D. Thurston, *Integral Expressions for the Vassiliev Knot Invariants*, math.AG/9901110.

[Ul02] A. Ulyanov, *Polydiagonal compactification of configuration spaces*, J. Algebraic Geom. 11 (2002), no. 1, 129–159.

[Vo03] C. Voisin, Hodge theory and complex algebraic geometry. II. Translated from the French by Leila Schneps. Cambridge Studies in Advanced Mathematics, 77. Cambridge University Press, Cambridge, 2003.