Conformal dimension of Cantor sets and curve families in the plane. Central sets of Hausdorff dimension 2.

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Abstract of the Dissertation

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The infimal Hausdorff dimension among all quasisymmetric images of a metric space (X, d_X) is known as *conformal dimension.* This is an important quasisymmetric invariant which was used to distinguish certain Gromov hyperbolic groups. One usually finds lower bounds for the conformal dimension by finding appropriate "thick" families of curves in X. In this thesis we define a quasiconformally invariant notion of "thickness" of a family of (not necessarily rectifiable) curves. This is done by introducing a notion of a "conformal dimension of a curve family" which unlike the Hausdorff dimension is a quasiconformal invariant. We investigate the general properties

of these notions their connection to the conformal dimension and apply these ideas to distinguish quasiconformal classes of certain fractal spaces in the plane.

A subset of a line E is said to be *quasisymmetrically thick* if its every image $f(E)$ under a qs self map of a line has positive length. A part of the thesis is related to the following question of Bishop and Tyson. Is there a set on the line which is not quasisymmetrically thick but has conformal dimension 1? We answer this affirmatively by showing that uniformly perfect regular Cantor sets are minimal for conformal dimension iff they have Hausdorff dimension one. The corollaries are: 1) characterization of middle interval Cantor sets which are minimal for conformal dimension; 2) existence of "rigid" subsets of a line whose every quasisymmetric image has dimension 1 and 0 length; 3) existence of subsets of a line of conformal dimension 0 which are minimal for quasisymmetric self maps of the line. We also extend the main result to higher dimensions.

The third part of the dissertation answers the following question of Fremlin: Is there a planar domain D s.t. the central set of D has Hausdorff dimension greater than 1? Here the central set is the collection of centers of maximal discs included in D. We give an example of a simply connected domain which has central set of Hausdorff dimension 2.

To my family

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Introduction

A central problem of the classical complex analysis is uniformization: classification of simply connected Riemann surfaces up to conformal isomorphism. The modern version of this is known as quasisymmetric uniformization problem: classification of metric spaces up to quasisymmetric isomorphisms. Quasisymmetry is a notion introduced by Tukia and Vaisala which generalizes the concept of quasiconformality in $\mathbb{R}^n, n \geq 2$ (see below for the definitions).

Roughly speaking this problem asks for geometric or rather metric conditions on a usually nonsmooth or fractal metric space which imply its equivalence to some nice model. This is a hard problem and only for very few cases an explicit characterization is know. For instance in [38] Tukia and Vaisala give a necessary and sufficient condition for a metric space to be quasisymmetric to the unit circle. The important case of the 2 sphere has been considered by Bonk and Kleiner in [8] where a necessary and sufficient condition was given for an Ahlfors 2 regular metric sphere to be quasisymmetric to the $\mathbb{S}^2$ with the standard metric. Recall that a metric space (X, d_X) is Ahlfors Q regular if there is a constant $1 < C < \infty$ s.t. for every ball B_r of radius $r \leq \text{diam}X$

the following inequalities hold:

$$\frac{1}{C} r^Q \leq H_Q(B_r) \leq C r^Q, \tag{1}$$

where H_Q denotes the Hausdorff Q measure.

Given a metric space (X, d_X) one can consider the "snowflake" metric on X given by $d_X^\varepsilon(x,y)$, $0 < \varepsilon < 1$, which is quasisymmetric to the original metric but the Hausdorff dimension of X with respect to d_X^ε is strictly larger than with respect to d_X. On the other hand it is not always possible to "compress" the space to make the Hausdorff dimension go down. Indeed, since quasisymmetric maps are homeomorphisms they cannot map $\mathbb{R}^n$ onto a space of Hausdorff dimension less than n. More interesting examples arise when one considers "fractal" spaces for which the Hausdorff dimension is larger than topological dimension, e.g. Cantor sets, Sierpinski carpets etc.

In order to find a "nice" model for a metric space it would be natural to "minimize the Hausdorff dimension in an attempt to optimize the shape" (see [28], section 7).

Definition 0.0.1. *Let (X, d_X) be a metric space.* ***Conformal dimension*** *of X is*

$$\mathrm{Conf\,dim\,X} = \inf \dim_{\mathrm{H}} \mathrm{Y}, \tag{2}$$

where the infimum is taken over all spaces Y which are quasisymmetrically equivalent to (X, d_X).

This quantity was introduced by Pansu in [34] in the context of the boundaries at infinity of Gromov hyperbolic metric spaces (see also [27]) and since then several other quasisymmetric invariants have been called conformal dimension.

Besides the fact that finding the conformal dimension of a space is an attempt to find a nice metric on the space it could also be used to distinguish "conformal gauges" - quasisymmetric classes of metric spaces (in [26] this terminology is attributed to Dennis Sullivan).

Clearly one has

$$\dim_{top} X \leq \mathrm{Conf\,dim\,X} \leq \mathrm{dim_H\,X}.$$

Definition 0.0.2. *X is* ***minimal for conformal dimension*** *if conformal and Hausdorff dimensions of X coincide*

$$\mathrm{Conf\,dim\,X} = \mathrm{dim_H\,X}.$$

For every $\alpha \in [1, \infty]$ there are sets minimal for the conformal dimension of dimension α. These were constructed in [33], [41]. Moreover in [7] it was shown that there are Cantor sets of conformal dimension α. It was conjectured by Jeremy Tyson and proved recently by Leonid Kovalev [29] that if $\dim_H X < 1$ then the conformal dimension of X is 0. The important example of a space minimal for conformal dimension is $X \times (0, 1)$ with the product metric. This means that if two spaces have different Hausdorff dimensions then their product with an interval cannot represent the same conformal gauge.

Just like in the last example one usually proves that X is minimal by exhibiting a "thick" family of paths in the space or its tangent spaces. This means there is a family of rectifiable paths in X of nonzero modulus with respect to a suitable measure on X. In chapter 2 we introduce a *quasiconformally invariant notion of dimension* or "thickness" for path families. The idea is to combine the notions of conformal modulus and Hausdorff measures to obtain a one parameter family of conformally invariant measures $m(\cdot, t), 0 \leq t \leq 1$, on the space of path families of X. As a result one obtains a quasiconformal invariant: "conformal dimension" of a curve family. This is analogous to the fact that Lipshitz maps can distort the Hausdorff measure of a set but cannot change its Hausdorff dimension (in this case Lipshitz maps correspond to quasiconformal maps while isometries correspond to conformal maps). The usual modulus corresponds to $t = 1$. The case of the plane is particularly interesting since the definitions have geometric description in terms of conformal maps. This allows one to guess the conformal dimension of certain subsets of the plane and show that certain sets are not quasi-conformally equivalent using classical extremal length estimates.

In chapter 1 we characterize a class of Cantor sets on a line which are minimal for conformal dimension and therefore have conformal dimension 1 by the above mentioned theorem of Kovalev. Unlike the usual proofs of minimality, where one argues by contradiction, we construct a measure of dimension 1 on any quasisymmetric image of the set. By doing this we answer a question of Bishop and Tyson [7] and give the first examples of subsets of a line of zero Lesbegue measure and conformal dimension 1. It is known that a set of the form $X \times (0, 1)$ is minimal. So it is natural to ask whether $X \times E$ is minimal if E is. Even though we are not able to show this we are able to show that the products of these minimal Cantor sets with themselves are minimal.

In chapter 3 we investigate the following problem about domains in the plane. Let $D \subset \mathbb{C}$ be a domain. A disc is said do be maximal for D if it is not contained in a larger disc D' in the domain. Consider the set of the centers of maximal discs inscribed in D. In many cases this is also the non-differentiability locus of the distance function to the boundary of D. Fremlin proved in [22] that this set has zero Lesbegue measure for any domain of the plain and asked whether it can have Hausdorff dimension strictly larger than 1. We show that this is possible and construct simply connected domains with central sets of Hausdorff dimension 2 (also any number between 1 and 2). Moreover, given any gauge function $\phi(t)$ s.t. $\phi(t)/t^2 \to \infty$ as $t \to 0$ one can take the central set so that it has positive H_ϕ measure. This is done by constructing a class of domains which we call disc trees for which the central set is equal to the closure of a tree. The boundary of this domains can actually be taken to be Lipschitz and given any $\varepsilon > 0$ the new domain can be ε close to the unit disc in the Hausdorff topology. This shows how discontinuous the dependence of the central set for the domain could be.

Chapter 1

Conformal dimension of Cantor sets

1.1 Introduction

Definition 1.1.1. *Given $M \geq 1$, a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is said to be M-**quasisymmetric** if for every pair of adjacent intervals I and J of the same length*

$$\frac{1}{M} \leq \frac{|f(I)|}{|f(J)|} \leq M \tag{1.1}$$

(here and in sequel $|\cdot|$ stands for the 1-dimensional Lebesgue measure). A map is quasisymmetric if it is M-quasisymmetric for some $M \geq 1$.

We denote by QS and $QS(M)$ the set of all quasisymmetric and M-quasisymmetric homeomorphisms of $\mathbb{R}$, respectively. This class of maps was introduced in [3], where it was shown that a homeomorphism of a line is the boundary map of a quasiconformal mapping of the upper half plane if and only if it satisfies (1.1). Recall, that a homeomorphism $F : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal if:

(1) $F \in ACL$, i.e. F is absolutely continuous with repsect to Hausdorff

1-measure on almost every horizontal and vertical line;

(2) $|F_{\bar{z}}| \leq k|F_z|$ almost everywhere in $\mathbb{C}$, with $k = \frac{K-1}{K+1} < 1$.

This is one of the many definitions of quasiconformality (we will give other definitions as we need them). The Ahlfors' M-condition (1.1) was then used to show that sets of harmonic measure 0 are not preserved by quasiconformal maps. Namely, there are quasisymmetric maps of a line which are totally singular: they can map a set of full measure onto a set of zero measure. Even though this seemed like a very rare phenomenon, later it became apparent that singular quasisymmetric mapping arise naturally in Teichmuller theory and Holomorphic dynamics (see e.g. [30], [13]). On the other hand according the work of Gehring, Vaisala and Reshetnyak, quasiconformal maps of $\mathbb{R}^n, n \geq 2$ are absolutely continuous in the sense that a set has zero n-measure if and only if its image has zero n-measure (an important fact which was used for instance in the proof of Mostow rigidity theorem). Moreover qc maps ($n \geq 2$) preserve sets of Hausdorff dimension n.

For arbitrary metric spaces Tukia and Vaisala [38] introduced the following notion of *quasisymmetry* which generalizes both, quasiconformality in $\mathbb{R}^n, n \geq 2$ and quasisymmetry in $\mathbb{R}$.

Definition 1.1.2. *A homeomorphism f between metric spaces (X, d_X) and (Y, d_Y) is called η-**quasisymmetric** for a self-homeomorphism $\eta : [0,\infty) \to [0,\infty)$ if for all distinct triples $x, y, z \in X$ and $t > 0$*

$$\frac{d_X(x,y)}{d_X(y,z)} \leq t \quad \Rightarrow \quad \frac{d_Y(f(x),f(y))}{d_Y(f(y),f(z))} \leq \eta(t). \tag{1.2}$$

Definition 1.1.3. *A quasisymmetric map f is* ***power quasisymmetric*** *if f is η-quasisymmetric where η is such that there are constants $C > 0$, $\alpha > 1$ such that for every $t > 0$*

$$\eta(t) \leq C \max\{t^{1/\alpha}, t^{\alpha}\}. \tag{1.3}$$

It is a deep property of the Euclidean spaces ($n \geq 2$) that the spaces of quasisymmetric and quasiconformal maps coinside. Therefore, below when we refer to a quasiconformal map of $\mathbb{R}^n$ we will mean (1.2).

Given a compact metric space X we are interested in the distortion of the Hausdorff dimension of X under quasisymmetric maps. If $X \subset \mathbb{R}^n$ then one can consider the distortion under quasiconformal selfmaps of the ambient space. In this case if $\dim_H(X) = 0$ then $\dim_H(f(X)) = 0$ since quasiconformal maps of $\mathbb{R}^n$ are Hölder continuous (see [1] for $n = 2$, [23] for $n > 2$). In [4] Bishop showed that if $\dim_H(X) > 0$ then for any $\varepsilon > 0$ one can find a quasiconformal map of $\mathbb{R}^n$ such that $\dim_H(f(E)) > 1 - \varepsilon$. As noted in the introduction this could be done for a general metric space by considering the "snowflake" metric on it, the point is that this map may not extend to the ambient space. In the opposite direction Tukia [37] proved that for any $\varepsilon > 0$ there is a set $E \subset \mathbb{R}$ and $f \in QS$ such that $\dim_H(\mathbb{R} \setminus E) < \varepsilon$ and $\dim_H(f(E)) < \varepsilon$. To explain the results of this chapter we need the following definition from [36].

Definition 1.1.4. *Given a sequence $\mathbf{c} = \{c_i\}_{i=1}^{\infty}$, such that $0 < c_i < 1$, we will denote by $E(\mathbf{c})$ the corresponding $\{c_i\}$* ***middle-interval Cantor*** *set, which is constructed as follow. From $[0,1]$, which we will denote by $E_{0,1}$, remove the* *middle interval of length* c_1 *centered at* $1/2$. *Call the removed interval* $J_{1,1}$ *and the components of the remaining set* $E_{1,1}$ *and* $E_{1,2}$. *From the middle of* $E_{1,i}$ *remove the interval* $J_{2,i}$ *of length* $c_2|E_{1,i}|$, $i = 1, 2$. *Continue in the same fashion. Let* $E = \bigcap_{n=0}^{\infty} \bigcup_{j=1}^{2^{n-1}} E_{n,j}$ *where* $E_{n,j} = E_{n+1,2j-1} \cup J_{n+1,j} \cup E_{n+1,2j}$, $|J_{n,j}| = c_n|E_{n-1,j}|$ *and* $|E_{n+1,j}| = |E_{n+1,j'}|$ *for any* j *and* j'.

The Lebesgue measure of E is $|E| = \prod_{i=1}^{\infty}(1 - c_i)$ which is positive if and only if $\sum_{i=1}^{\infty} c_i < \infty$.

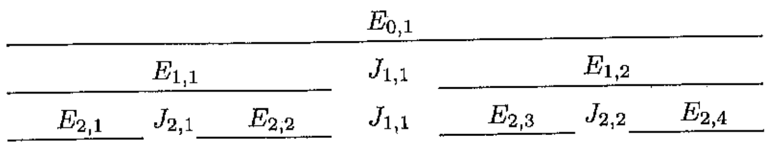


Figure 1.1: A middle interval Cantor set

Theorem 1.1.5. *If* E *is a middle interval Cantor set,* $\dim_H E = 1$ *and* f *is a power quasisymmetric map then* $\dim_H f(E) \geq 1$.

The middle interval Cantor sets were considered in [36], where a sufficient condition was given, in terms of the defining sequence $\mathbf{c}$, for $E(\mathbf{c})$ to be **quasisymmetrically thick**: every image under a quasisymmetric homeomorphism of a line has positive length. In [14] quasisymmetrically thick middle interval Cantor sets are characterized.

Theorem 1.1.6 (Buckley, Hanson, McManus.). *Let* $E = E(\mathbf{c})$ *be a middle* *interval Cantor set. Then*

$$
\begin{aligned}
\boldsymbol{c} \in \bigcap_{0<p<1} l^p \quad &\Leftrightarrow \quad |f(E(\boldsymbol{c}))| > 0, \forall f \in QS, \\
\boldsymbol{c} \notin \bigcup_{0<p<1} l^p \quad &\Leftrightarrow \quad \forall M > 1, \exists f \in QS(M) \; s.t. \; |f(E(\boldsymbol{c}))| = 0, \\
\exists p_1, p_2 \in (0,1) \, s.t. \, \boldsymbol{c} \in l^{p_1} \setminus l^{p_2} \quad &\Leftrightarrow \quad \forall f \in QS(M_1), |f(E(\boldsymbol{c}))| > 0 \quad and \\
&\qquad \exists f \in QS(M_2) \, s.t. |f(E(\boldsymbol{c}))| = 0.
\end{aligned}
$$

So it is natural to ask whether there is a middle interval Cantor set of positive measure or even Hausdorff dimension 1 which could be mapped by a quasisymmetry of a line to a set of dimension < 1. The following corollary of the theorem 1.1.5 answers this.

Corollary 1.1.7. *If $E \in MIC$ and $\dim_H E = 1$ then $\dim_H f(E) = 1, \forall f \in QS$.*

The reason this follows from theorem 1.1.5 is that every quasisymmetric map of the line is power quasisymmetric. This interesting fact is true for the much more general class of uniformly perfect spaces, which we define later. As mentioned in the introduction Kovalev proved that for every set E of Hausdorff dimension < 1 there is a quasisymmetric map of the ambient space such that $\dim_H f(E) < \varepsilon$ for any $\varepsilon > 0$. To this end one can define the following quasisymmetric invariant of a subset of a Euclidean space.

$$\operatorname{QConf}\dim E = \inf\{\, \dim_H(f(E)) \mid f : \mathbb{R}^n \to \mathbb{R}^n \text{quasisymmetric}\}.$$

This is called **quasiconformal dimension** of E. Therefore combining the previous corollary with the theorem of Kovalev we have

Corollary 1.1.8. *A middle interval Cantor set E has quasiconformal dimension 1 if and only if* $\dim_H E = 1$.

Given $E \subset \mathbb{R}^n$ the following inequalities follow directly from the definitions

$$\dim_{top} E \leq \mathrm{Conf\,dim\,E} \leq \mathrm{QConf\,dim\,E} \leq \mathrm{dim_H\,E}.$$

Strict inequalities can occur for each of these cases. Let us comment on the case of conformal and quasiconformal dimension. It is well known that there are "wild" embeddings of the Cantor set into $\mathbb{R}^3$ (e.g. Antoine's necklace) such that the complement of these sets are not simply connected. Since every set of Hausdorff dimension < 1 has a simply connected complement this means that the quasiconformal dimension of these wild Cantor sets is ≥ 1. On the other hand it is not hard to construct a set like that which is quasisymmetric (through a map which doesn't extend to a quasiconformal map of $\mathbb{R}^3$) to the standard middle third Cantor set which has conformal dimension 0.

In the examples above $\mathrm{Conf\,dim\,E} \leq \mathrm{QConf\,dim\,E}$ holds for topological reasons. So it would be interesting to know wether there are geometric obstructions for lowering dimension by global quasiconformal maps. Namely, suppose $E \subset \mathbb{R}^n$ and there exists a homeomorphism f of $\mathbb{R}^n$ s.t. $\dim_H f(E) = \mathrm{Conf\,dim\,E}$ is it then true that $\mathrm{QConf\,dim}\,E = \mathrm{Conf\,dim\,E}$? In particular is it possible for Cantor subsets of the plane or a line to have different conformal and quasiconformal dimensions?

Theorem 1.1.9. *$E(\mathbf{c}) \in MIC$ is minimal for conformal dimension if and only if* $\dim_H E(\mathbf{c}) = 1$ *and* $\limsup_{i\to\infty} c_i < 1$.

One of the directions of this theorem follows from the fact that the quasisymmetric embeddings of uniformly perfect spaces are power quasisymmetric, see section 1.3. The other direction is obtained from the fact that if a space is not uniformly perfect then quasisymmetric maps of that space can be very irregular, for instance they do not have to be Holder continuous, see [26].

Combining this with Corollary 1.1.8 we obtain

Corollary 1.1.10. *There are middle interval Cantor sets of conformal dimension 0 and quasiconformal dimension 1.*

These examples will have to be non-uniformly perfect. Which raises the questions.

Question 1. *Is there a connected set $E \subset \mathbb{R}^n$ s.t.* $\mathrm{Conf\,dim\,E} < \mathrm{QConf\,dim\,E}$*? Is there a similar uniformly perfect example in the plane or on the line?*

Bishop and Tyson constructed many examples of Cantor sets minimal for conformal dimension and asked the following question in [7] (see page 370). If $E \subset \mathbb{R}$ is not a quasisymmetrically thick set, then is there a quasisymmetric image of E of Hausdorff dimension < 1? In other words: If a set is not quasisymmetrically thick then does it necessarily have conformal dimension < 1? Theorem 1.1.5 shows that every uniformly perfect middle interval Cantor set of zero length and Hausdorff dimension one is an example of a set which is not quasisymmetrically thick but has conformal dimension 1.

One could also ask: Is there a "rigid" set whose every QS-image has dimension 1 and length 0? In [36] a set is called *quasisymmetrically null* if all its

QS-images have zero length. In [43] Wu had shown that if $\mathbf{c} = c_i \notin l^p, \forall p \geq 1$ then $E = E(\mathbf{c})$ is null. She noticed that in particular null sets can have Hausdorff dimension 1. Therefore combining Theorem 1.1.5 with Wu's theorem we get an affirmative answer to the question above.

Corollary 1.1.11. *There are quasisymmetrically null sets which are minimal.*

It's enough to take the sequence

$$c_i = \begin{cases} (1/i)^{\frac{1}{i}} & \text{if } i = 2^m \\ (1/i)^i & \text{if } i \neq 2^m \end{cases}$$

and construct the corresponding Cantor set $E(\mathbf{c})$. We leave it to the reader to verify that the set has dimension 1 and that Wu's condition is also satisfied.

Given a Borel set E in [41] it was shown that $E \times (0,1)$ is minimal for conformal dimension. Also there are some quasisymmetrically thick Cantor sets with the similar property. So a natural question is: If M is minimal for the conformal dimension is it true that $E \times M$ is also minimal? Even though we are not able to answer this question we can show the following.

Theorem 1.1.12. *Let $E \subset \mathbb{R}$ be a middle interval Cantor set s.t.* $\dim_H E = 1$ *and let $E^m = E \times \ldots \times E$ be the m-fold product of E. Then* $\dim_H f(E^m) \geq m$ *for every power quasisymmetric map f.*

Corollary 1.1.13. *If E is as above then*

$$\operatorname{QConf} \dim E^m = m \quad \Leftrightarrow \quad \dim_H E^m = m.$$

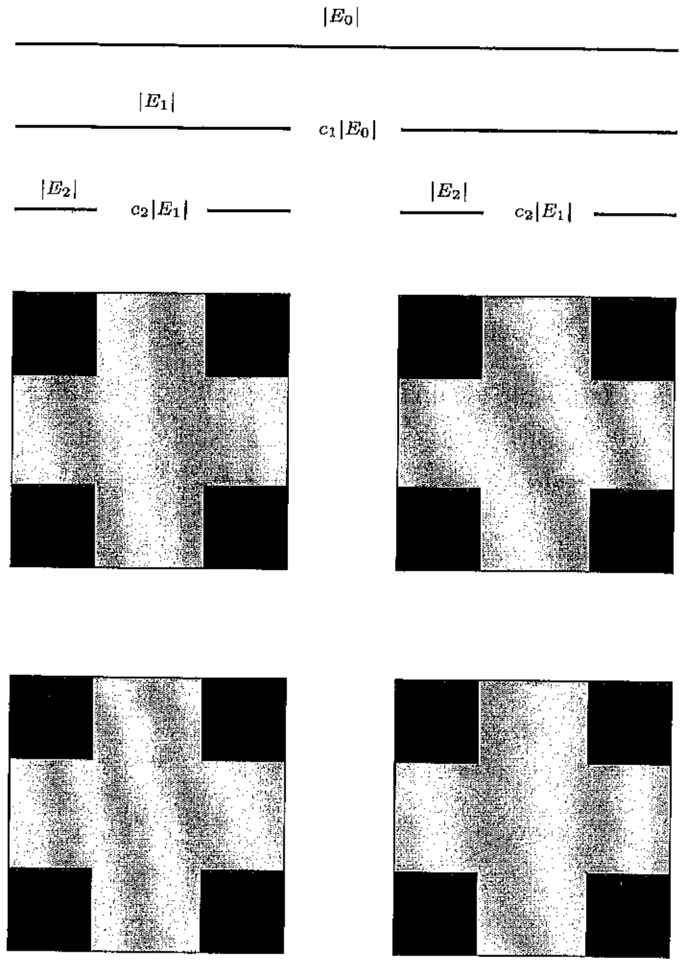


Figure 1.2: $E(\mathbf{c})$ and $E^2(\mathbf{c})$

1.2 Background Material

Recall that the Hausdorff t-measure of a metric space (X, d_X) is defined by

$$H^t(X) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} (\mathrm{diam} U_i)^t : X \subset \bigcup_{i=1}^{\infty} U_i,\ \mathrm{diam} U_i < \varepsilon \right\},$$

where $\{U_i\}_{i=1}^{\infty}$ is an open cover of X. The Hausdorff dimension of X is

$$\begin{aligned}\dim_H(X) &= \inf\{\, t \,:\, H^t(X) = 0\} \\ &= \sup\{\, t \,:\, H^t(X) = \infty\}\end{aligned}$$

One usually gives an upper bound for the Hausdorff dimension of a set by finding explicit covers for it. Lower bounds can be obtained from the following:

Lemma 1.2.1 (Mass distribution principle). *If the metric space (X, d_X) supports a positive Borel measure μ satisfying $\mu(U) \leq C(\mathrm{diam} U)^d$, for some fixed constant $C > 0$ and every $U \subset X$ then $\dim_H(E) \geq d$.*

Proof. For every cover $\{U_i\}_{i=1}^{\infty}$ of X we have

$$\sum_i (\mathrm{diam} U_i)^d \geq \frac{1}{C} \sum_i \mu(U_i) \geq \frac{1}{C}\mu(X).$$

Therefore $H^d(X) \geq \frac{\mu(X)}{C} > 0$. □

Let $N(X, \varepsilon)$ be the minimal number of ε balls needed to cover X.

Definition 1.2.2. *Upper and lower Minkowski dimensions of X are*

$$\begin{aligned}\overline{\dim}_M(X) &= \limsup_{\varepsilon \to 0} \frac{\log N(X,\varepsilon)}{\log 1/\varepsilon}, \\ \underline{\dim}_M(X) &= \liminf_{\varepsilon \to 0} \frac{\log N(X,\varepsilon)}{\log 1/\varepsilon}\end{aligned}$$

respectively. When this two numbers are the same the common value is called the Minkowski dimension of X and is denoted by $\dim_M X$.

Generally one has

$$\dim_H(X) \leq \underline{\dim}_M(X) \leq \overline{\dim}_M(X)$$

(see [19]). Therefore if $X \subset \mathbb{R}$ and $\dim_H(X) = 1$ then the Minkowski dimension of X exists and is equal 1.

Lemma 1.2.3. *If $E = E(\mathbf{c})$ and $\dim_M(E) = 1$ then*

$$\begin{aligned} \sqrt[n]{\prod_{i=1}^{n}(1-c_i)} &\rightarrow 1, & (1.4)\\ \frac{1}{n}\sum_{i=1}^{n} c_i &\rightarrow 0. & (1.5) \end{aligned}$$

Proof. From the definition of Minkowski dimension we get

$$\begin{aligned} \dim_M(E) &= \lim_{n\to\infty} \frac{\log 2^n}{\log \frac{2^n}{\prod_{i=1}^{n}(1-c_i)}} \\ &= \lim_{n\to\infty} \frac{1}{1 - \frac{1}{\log 2}\log \sqrt[n]{\prod_{i=1}^{n}(1-c_i)}} = 1. \end{aligned}$$

Therefore (1.4) holds. Now, from the usual inequality between geometric and arithmetic means $\sqrt[n]{\prod_{i=1}^{n}(1-c_i)} \leq \frac{1}{n}\sum_{i=1}^{n}(1-c_i) \leq 1$ we get that $\frac{1}{n}\sum_{i=1}^{n}(1-c_i) \to 1$ or, equivalently, $\frac{1}{n}\sum_{i=1}^{n} c_i \to 0$. □

Note that (1.5) implies the following.

Corollary 1.2.4. *For a given $a > 0$ let $S = S_a(\mathbf{c}) = \{i \in \mathbb{N} \mid c_i < a\}$, and $s_n = \#(S_a \cap \{i \leq n\})$. If $\dim_H(E(\mathbf{c})) = 1$ then*

$$\frac{s_n}{n} \to 1. \tag{1.6}$$

1.3 Quasisymmetric maps and uniformly perfect spaces

Our main tool for proving theorem 1.1.5 will be the following lemma from [26].

Lemma 1.3.1. *If $f : X \to Y$ is η-quasisymmetric and if $A \subset B \subset X$ are such that* $0 < \mathrm{diam}A \leq \mathrm{diam}B < \infty$, *then* $\mathrm{diam}f(B)$ *is finite and*

$$\frac{1}{2\eta\left(\frac{\mathrm{diam}B}{\mathrm{diam}A}\right)} \leq \frac{\mathrm{diam}f(A)}{\mathrm{diam}f(B)} \leq \eta\left(\frac{2\mathrm{diam}A}{\mathrm{diam}B}\right). \tag{1.7}$$

By distance between sets below we mean the following: if $Y, Z \subset X$ then

$$\mathrm{dist}_X(Y, Z) = \inf\{d_X(y, z) \,|\, y \in Y, z \in Z\}.$$

We will need a different version of (1.7).

Lemma 1.3.2. *Suppose* $X = X_1 \cup X_2$, *with* X_1, X_2 *compact and* $\mathrm{dist}(X_1, X_2) > 0$. *Then*

$$\frac{1}{2\eta\left(\frac{\mathrm{diam}X}{\mathrm{dist}(X_1,X_2)}\right)} \leq \frac{\mathrm{dist}(f(X_1), f(X_2))}{\mathrm{diam}f(X)} \leq \eta\left(2\frac{\mathrm{dist}(X_1, X_2)}{\mathrm{diam}X}\right). \tag{1.8}$$

Proof. Suppose $x_1 \in X_1$ and $x_2 \in X_2$ are such that $\mathrm{dist}(X_1, X_2) = d_X(x_1, x_2)$. This is possible since X_1 and X_2 are compact. Let $A = \{x_1, x_2\}$ then right hand inequality in (1.7) implies

$$\begin{aligned}\frac{\operatorname{dist}(f(X_1), f(X_2))}{\operatorname{diam} f(X)} &\leq \frac{\operatorname{dist}(f(x_1), f(x_2))}{\operatorname{diam} f(X)} \leq \eta\left(\frac{2\operatorname{dist}(x_1, x_2)}{\operatorname{diam} X}\right) \\ &= \eta\left(2\frac{\operatorname{dist}(X_1, X_2)}{\operatorname{diam} X}\right).\end{aligned}$$

To obtain the other inequality of (1.8) take $y_1 \in f(X_1), y_2 \in f(X_2)$ in such a way that $\operatorname{dist}(f(X_1), f(X_2)) = d_Y(y_1, y_2)$. Let $x_i' = f^{-1}(y_i)$. Now take $A = \{x_1', x_2'\}$. Then again using 1.7 we get

$$\frac{\operatorname{dist}(f(X_1), f(X_2))}{\operatorname{diam} f(X)} \geq \frac{1}{2\eta\left(\frac{\operatorname{diam} X}{\operatorname{dist}(x_1', x_2')}\right)}$$

Since $d_X(x_1', x_2') \geq \operatorname{dist}(X_1, X_2)$ and since η is increasing we obtain

$$\frac{1}{2\eta\left(\frac{\operatorname{diam} X}{\operatorname{dist}(x_1', x_2')}\right)} \geq \frac{1}{2\eta\left(\frac{\operatorname{diam} X}{\operatorname{dist}(X_1, X_2)}\right)}.$$

Combining this with the previous inequality gives (1.8). □

Definition 1.3.3. *A metric space is* ***uniformly perfect*** *if there is a constant $C \geq 1$ so that for each $x \in X$ and for all $r > 0$*

$$X \setminus B(x, r) \neq \emptyset \quad \Longrightarrow \quad B(x, r) \setminus B(x, \frac{r}{C}) \neq \emptyset. \tag{1.9}$$

This condition in a sense rules out "large gaps" in the space. Examples of uniformly perfect sets are connected sets as well as many totally disconnected sets, like middle third Cantor set or many sets arising in conformal dynamics.

The following theorem is of importance for us (see [26] for the proof).

Theorem 1.3.4. *A quasisymmetric embedding f of a uniformly perfect space X is η-quasisymmetric with η of the form*

$$\eta(t) = C \max\{t^{\alpha}, t^{1/\alpha}\}, \tag{1.10}$$

where $C \geq 1$ and $\alpha \in (0,1]$ depend on f and X.

In particular, quasiconformal maps of $\mathbb{R}^n$ are power quasisymmetric and are Holder continuous. A middle interval Cantor set $E(\mathbf{c})$ is uniformly perfect if and only if there is no subsequence of $\mathbf{c}$ converging to 1. In particular this means that if $|E(\mathbf{c})| > 0$ then it is uniformly perfect and hence by Theorem 1.1.5 has conformal dimension 1. This in particular proves Theorem 1.1.9.

1.4 Proof of Theorem 1.1.5

Proof. Fix $d < 1$ and suppose $f : E \to Y$ is a power-quasisymmetric homeomorphism. We will construct a measure μ on Y satisfying

$$\mu(B(y,r)) \leq C r^d$$

for some constant $C > 0$ all $r > 0$ and all $y \in Y$. It would follow from the mass distribution principle that $\dim_H(f(E)) \geq d$. Since d is arbitrary we would have shown that $\dim_H(Y) \geq 1$.

First note that E (as well as Y) has a structure of a binary tree. Given an interval $E = E_{n,j}$ as in Definition 1.1.4 there is a unique interval of generation $n-1$ containing E, which we will call the parent of E and will denote by

$\widetilde{E}$. Each interval has exactly two "children" the collection of which will be denoted by $\mathcal{C}(E)$. All the above notations will also be used for Y: for an $I \subset Y$ of the form $I = I_{n,j} = f(E_{n,j})$ we will denote by $\tilde{I}$ its parent and by I' its only "sibling", i.e. the other interval with the same parent as I.

Now define μ as follows. Let $\mu(Y) = 1$. For any $I \subset Y$ of the form $I = f(E_{n,j})$ let:

$$\mu(I) = \frac{\mathrm{diam}^d I}{\mathrm{diam}^d I + \mathrm{diam}^d I'}\mu(\tilde{I}). \tag{1.11}$$

First we will show the growth condition for the subsets of the form $I = I_{n,j}$. Given such an I there is a unique sequence of nested subsets

$$I = I_n \subset I_{n-1} \subset I_{n-2} \subset \ldots \subset I_2 \subset I_1 \subset I_0 = Y$$

containing it, so that $I_{k-1} = \tilde{I}_k$. By induction we have

$$\begin{aligned}\frac{\mu(I)}{\mathrm{diam}^d I} &= \frac{\mu(I_n)}{\mathrm{diam}^d I_n} \\ &= \frac{1}{\mathrm{diam}^d I_n + \mathrm{diam}^d I'_n} \cdot \frac{\mathrm{diam}^d I_{n-1}}{\mathrm{diam}^d I_{n-1} + \mathrm{diam}^d I'_{n-1}} \\ &\quad \cdot \ldots \cdot \frac{\mathrm{diam}^d I_1}{\mathrm{diam}^d I_1 + \mathrm{diam}^d I'_1}\mu(I_0).\end{aligned} \tag{1.12}$$

Since $\mathrm{diam}(A \cup B) \leq \mathrm{diam}A + \mathrm{dist}(A, B) + \mathrm{diam}B$ we have

$$\frac{\mu(I)}{\mathrm{diam}^d I} \leq \prod_{i=1}^{n} \frac{(\mathrm{diam}I_i + \mathrm{dist}(I_i, I'_i) + \mathrm{diam}I'_i)^d}{\mathrm{diam}^d I_i + \mathrm{diam}^d I'_i}. \tag{1.13}$$

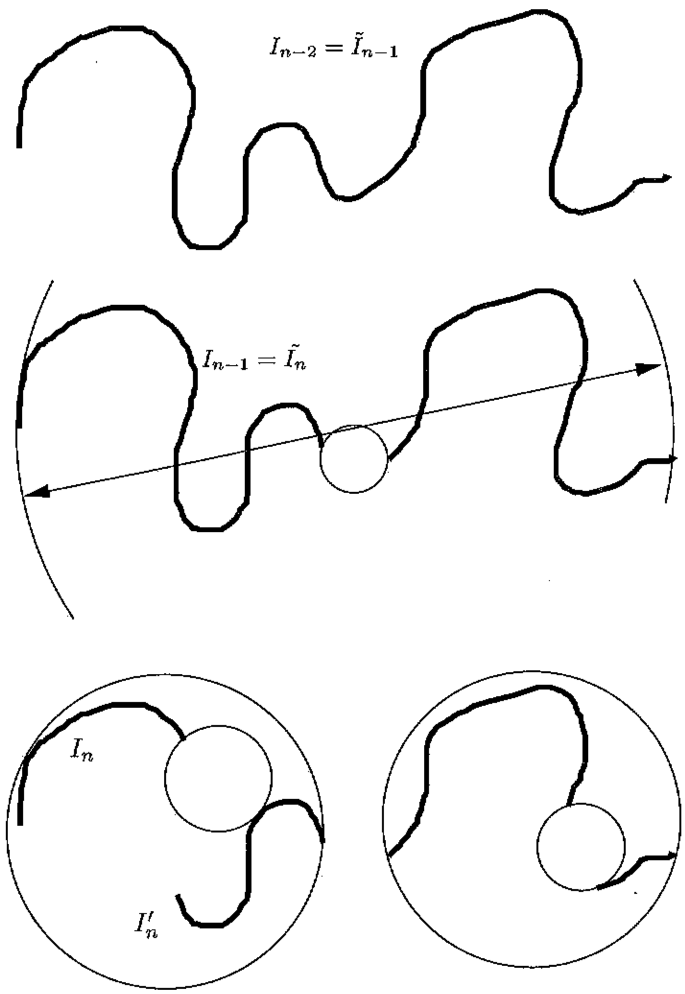


Figure 1.3: Image of the Cantor set.

Let

$$p_i = \frac{(\mathrm{diam} I_i + \mathrm{dist}(I_i, I'_i) + \mathrm{diam} I'_i)^d}{\mathrm{diam}^d I_i + \mathrm{diam}^d I'_i} \tag{1.14}$$

To prove the theorem it is sufficient to show that $\prod_{i=1}^n p_i \to 0$ as $n \to \infty$. Indeed, if this is the case then $\exists C < \infty$ s.t. $\prod_{i=1}^n p_i < C, \forall n \in \mathbb{N}$ and μ satisfies the mass distribution principle. Now, to prove $\prod_{i=1}^n p_i \to 0$ we will need the following estimates.

Lemma 1.4.1 (Small gaps). $\exists a > 0, C_1 < 1 \ s.t \ c_i < a \Rightarrow p_i < C_1 < 1$.

Lemma 1.4.2 (Large gaps). $\exists C_2 > 1 \ s.t. \ p_i < \frac{C_2}{(1-c_i)^{d/\alpha}}, \forall i$.

Let us prove the theorem assuming these two lemmas. First of all

$$\begin{aligned}\prod_{i=1}^{n} p_i &\leq \prod_{\{i \leq n | c_i < a\}} C_1 \prod_{\{i \leq n | c_i \geq a\}} \frac{C_2}{(1-c_i)^{d/\alpha}} \qquad \text{(by the two lemmas)} \\ &\leq C_1^{s_n} \frac{C_2^{n-s_n}}{\prod_{i=1}^{n}(1-c_i)^{d/\alpha}} \qquad \text{(where } s_n \text{ is like in Corollary 1.2.4).}\end{aligned}$$

Now, if $C_1 < 1$ and $s_n/n \to 1$ then for every number $C_2 < \infty$ there is a $C_3 < 1$ and $N \in \mathbb{N}$ s.t. for $n > N$

$$C_1^{s_n} C_2^{n-s_n} \leq C_3^n.$$

Hence

$$\prod_{i=1}^{n} p_i \leq \left(\frac{C_3}{\sqrt[n]{\prod_{i=1}^{n}(1-c_i)^{d/\alpha}}} \right)^n.$$

Since $\sqrt[n]{\prod_{i=1}^{n}(1-c_i)} \to 1$ and $C_3 < 1$ it follows that $\prod_{i=1}^{n} p_i \to 0$. And hence the growth condition of the mass distribution principle for μ holds for the intervals I_n.

Next we prove the Lemmas (1.4.1) and (1.4.2).

1.4.1 Proof of lemma 1.4.1. Small gaps.

Recall that for a given $a > 0$ we had

$$S_a = \{i \in \mathbb{N} | \, c_i < a\}, S_n = S_a \cap \{i \leq n\}, s_n = \text{card}(S_n).$$

Without loss of generality we can assume $a < 1/2$.

Suppose now $i \in S_a$. We find it easier to estimate p_i^{-1} from below.

$$\begin{aligned}
p_i^{-1} &= \frac{\operatorname{diam}^d I_i + \operatorname{diam}^d I_i'}{(\operatorname{diam} I_i + \operatorname{diam} I_i')^d} \cdot \frac{(\operatorname{diam} I_i + \operatorname{diam} I_i')^d}{(\operatorname{diam} I_i + \operatorname{dist}(I_i, I_i') + \operatorname{diam} I_i')^d} \\
&\geq \frac{\operatorname{diam}^d I_i + \operatorname{diam}^d I_i'}{(\operatorname{diam} I_i + \operatorname{diam} I_i')^d} \cdot \left(1 - \frac{\operatorname{dist}(I_i, I_i')}{\operatorname{diam} I_{i-1}}\right)^d \\
&\geq \frac{\operatorname{diam}^d I_i + \operatorname{diam}^d I_i'}{(\operatorname{diam} I_i + \operatorname{diam} I_i')^d} \cdot (1 - \eta(2c_i))^d \qquad \text{(by (1.8))} \\
&= \frac{1 + \left(\frac{\operatorname{diam} I_i'}{\operatorname{diam} I_i}\right)^d}{\left(1 + \frac{\operatorname{diam} I_i'}{\operatorname{diam} I_i}\right)^d} \cdot (1 - \eta(2c_i))^d.
\end{aligned}$$

We will show that the the first term in this product is bounded below by a constant strictly greater than 1. To do that, first note that there is a constant $1 < D(\eta) < \infty$ so that $D^{-1} < \operatorname{diam} I_i / \operatorname{diam} I_i' < D$. Indeed,

$$\begin{aligned}
\frac{\operatorname{diam} I_i}{\operatorname{diam} I_i'} &\geq \frac{\operatorname{diam} I_i}{\operatorname{diam} I_{i-1}} \\
&\geq \frac{1}{2\eta\left(\frac{\operatorname{diam} E_{i-1}}{\operatorname{diam} E_i}\right)} \qquad \text{(by(1.8))} \\
&= \frac{1}{2\eta\left(\frac{2}{1-c_i}\right)} \geq \frac{1}{2\eta\left(\frac{2}{1-1/2}\right)} = \frac{1}{2\eta(4)} \geq D^{-1} > 0.
\end{aligned}$$

The second inequality follows by symmetry.

Considering the function $x \mapsto \frac{1+x^d}{(1+x)^d}$ for $d < 1$ one can easily see that on an interval $[D^{-1}, D]$ its smallest value is attained at D and is strictly larger than 1. We will denote this value by $C_4 = C_4(\eta, d) > 1$. Therefore

$$p_i^{-1} \geq C_4(1 - \eta(2c_i))^d \geq C_4(1 - \eta(2a))^d. \tag{1.15}$$

Since η is increasing and $c_i < a$. Now, $\eta(t) \to 0$ as $t \to 0$. Therefore we can always choose a small enough so that $C_4(1-\eta(2a))^d > 1$. So finally we conclude that there is an a so that for $i \in S_a$ one has $p_i \geq C_5 > 1$. Equivalently p_i is bounded from above by a constant strictly less than 1.

Remark 1.4.3. *Note we haven't yet used the fact that f is power quasisymmetric.*

1.4.2 Proof of lemma 1.4.2. Large gaps.

Suppose $i \notin S_a$. Since $\text{diam} I_i, \text{diam} I_i', \text{dist}(I_i, I_i') < \text{diam} I_{i-1}$, we have

$$\begin{aligned}
p_i &= \frac{(\text{diam} I_i + \text{dist}(I_i, I_i') + \text{diam} I_i')^d}{\text{diam}^d I_i + \text{diam}^d I_i'} \\
&\leq \frac{3^d \text{diam}^d I_{i-1}}{\text{diam}^d I_i + \text{diam}^d I_i'} \\
&= 3^d \left[\left(\frac{\text{diam} I_i}{\text{diam} I_{i-1}} \right)^d + \left(\frac{\text{diam} I_i'}{\text{diam} I_{i-1}} \right)^d \right]^{-1} \\
&\leq 3^d \frac{\eta^d \left(\frac{\text{diam} E_{i-1}}{\text{diam} E_i} \right)}{2} \qquad \text{(by(1.8))} \\
&= C\eta^d \left(\frac{2}{1-c_i} \right).
\end{aligned}$$

Now, since $\eta(t) \leq C \max\{t^{1/\alpha}, t^\alpha\}$, the last inequality gives

$$p_i \leq \frac{C_2(d,\alpha)}{(1-c_i)^{d/\alpha}}.$$

As shown before, it follows that

$$\mu(I_n) \leq C \text{diam}^d I_n$$

for every I_n.

To complete the proof of the theorem we need to show that a similar estimate holds for any ball $B = B(y,r)$ with $y \in Y$. For that let $l_i = |E_{i,j}|, \forall i,j$. Clearly $l_i \searrow 0$ so there is an i such that $l_{i+1} \le |f^{-1}(B)| < l_i$. It follows then that there are at most 2 intervals of generation i which intersect $|f^{-1}(B)|$ and therefore at most 4 such intervals of generation $i+1$. Denoting the latter ones by $E_1, \ldots, E_4$ (some of these may be empty) we get

$$\mu(B) \le \mu(f(E_1)) + \ldots + \mu(f(E_4)) \le C(|f(E_1)|^d + \ldots + |f(E_4)|^d).$$

Now since $|E_i| \le |f^{-1}(B)|$ it follows that $E_1 \cup \ldots \cup E_4 \subset 3f^{-1}(B)$, where $3f^{-1}(B)$ is just the dilation of $f^{-1}(B))$ and hence

$$|f(E_1)|^d + \ldots + |f(E_4)|^d \le 4|f(3f^{-1}(B))|^d.$$

From the definition of quasisymmetry it follows that

$$|f(3f^{-1}(I))|^d \le (1+2M)^d |f(f^{-1}(I))|^d = C(M,d)|I|^d$$

and hence combining the last three inequalities we conclude the desired growth

$$\mu(I) \le C|I|^d$$

for some constant C and any ball $B(y,r)$. As we noted in the beginning if follows that $\dim_H(f(E)) = 1$ since d could be chosen as close to 1 as one would like. $\square$

1.5 Proof of Theorem 1.1.12.

Proof. The proof is similar to that of theorem 1.1.5 so we skip over the identical details.

Fix $d < 1$. We will construct a measure μ on Y satisfying

$$\mu(B(y,r)) \leq Cr^{md}$$

for some constant $C > 0$ all $r > 0$ and all $y \in Y$. This implies $\dim Y \geq m$.

First note that E^m also has a tree structure where each parent (which in this case is an m dimensional cube) has exactly 2^m children and the same is true for Y. For any $I \subset Y$, with I being an image of a generating cube of E^m, we will denote by $\tilde{I}$ its parent and by $C(I)$ the set of children of I. Now define μ as follows. Let $\mu(Y) = 1$ and for any $I \subset Y$ of the form $I = f(Q)$ let:

$$\mu(I) = \left(\frac{\operatorname{diam}^d I}{\sum_{J \in C(\tilde{I})} \operatorname{diam}^d J} \right)^m \mu(\tilde{I}).$$

Similarly to the previous proof for every I we have a sequence $I = I_n \subset I_{n-1} \subset \ldots \subset I_1 \subset I_0 = Y$ and

$$\frac{\mu(I)}{\operatorname{diam}^{md} I} = \prod_{i=1}^{n} {p_i}^m, \tag{1.16}$$

where now p_i is defined as follows

$$p_i = \frac{\operatorname{diam}^d I_{i-1}}{\sum_{J \in C_{i-1}} \operatorname{diam}^d J},$$

here C_i is the collection of children of I_i. Let S_a, S_n, and s_n be as before.

Small gaps. Suppose $i \in S_a$. Let J_i, J_i' be the cylinders where the points realizing $\mathrm{diam} I_{i-1}$ lie. In other words $\exists x_i \in J_i, x_i' \in J_i'$ s.t. $\mathrm{dist}(x_i, x_i') = \mathrm{diam} I_{i-1}$. Then

$$
\begin{aligned}
p_i &= \frac{\left(\sum_{J\in C_{i-1}} \mathrm{diam} J\right)^d}{\sum_{J\in C_{i-1}} \mathrm{diam}^d J} \cdot \frac{\mathrm{diam}^d I_{i-1}}{\left(\sum_{J\in C_{i-1}} \mathrm{diam} J\right)^d} \\
&\leq \frac{\left(\sum_{J\in C_{i-1}} \mathrm{diam} J\right)^d}{\sum_{J\in C_{i-1}} \mathrm{diam}^d J} \cdot \frac{\mathrm{diam}^d I_{i-1}}{(\mathrm{diam} J_i + \mathrm{diam} J_i' + \mathrm{dist}(J_i, J_i') - \mathrm{dist}(J_i, J_i'))^d} \\
&\leq \frac{\left(\sum_{J\in C_{i-1}} \mathrm{diam} J\right)^d}{\sum_{J\in C_{i-1}} \mathrm{diam}^d J} \cdot \left(1 - \frac{\mathrm{dist}(J_i, J_i')}{\mathrm{diam} I_{i-1}}\right)^{-d},
\end{aligned} \tag{1.17}
$$

where the last inequality holds because $\mathrm{diam} I_{i-1} = \mathrm{dist}(x_i, x_i') \leq \mathrm{diam} J_i + \mathrm{diam} J_i' + \mathrm{dist}(J_i, J_i')$. Furthermore, since

$$
\frac{\mathrm{dist}(J_i, J_i')}{\mathrm{diam} I_{i-1}} \leq \eta\left(2\frac{\mathrm{dist}(f^{-1}(J_i), f^{-1}(J_j'))}{\mathrm{diam} f^{-1}(I_{i-1})}\right) \leq \eta(2c_i),
$$

we have

$$
p_i \leq \frac{\left(\sum_{J\in C_{i-1}} \mathrm{diam} J\right)^d}{\sum_{J\in C_{i-1}} \mathrm{diam}^d J} \cdot (1 - \eta(2a))^{-d}.
$$

Now, note that there is a constant $1 < D = D(\eta, m) < \infty$ so that whenever J, J' are siblings $D^{-1} < \mathrm{diam} J/\mathrm{diam} J' < D$. Indeed, we show one of these inequalities and the other one follows by symmetry. Since $c_i < 1/2$ and η is increasing, we have

$$\begin{aligned}\frac{\mathrm{diam}J}{\mathrm{diam}J'} &\geq \frac{\mathrm{diam}J}{\mathrm{diam}I_{i-1}} \geq \frac{1}{2\eta\left(\frac{\mathrm{diam}f^{-1}(I_{i-1})}{\mathrm{diam}f^{-1}(J)}\right)}\\ &\geq \frac{1}{2\eta\left(\frac{2}{1-c_i}\right)} \geq \frac{1}{2\eta\,(4)} \geq D^{-1}.\end{aligned}$$

Next, consider the function

$$x = (x_1, ..., x_n) \mapsto \frac{(1 + \sum_{i=1}^{n} x_i)^d}{1 + \sum_{i=1}^{n} x_i^d}$$

for $d < 1$ and note that its largest value is strictly smaller than 1 provided $0 < 1/D < x_i < D < \infty, \forall i$. We will denote this value by $C_2' = C_2'(\eta, d, m) < 1$. Therefore (1.17) gives us the following estimate

$$p_i \leq C_2'(1 - \eta(2a))^{-d} = C_3'(\eta, d, a). \tag{1.18}$$

Since $C_2' > 1$ and $\eta(t) \to 0$ as $t \to 0$ we can always choose a small enough so that $C_3' > 1$. So finally we conclude that there is an a so that for $i \in S_a$ one has $p_i \geq C_3' > 1$.

Large gaps. For $i \notin S_a$ we use the first inequality of (1.7) to get

$$\begin{aligned}p_i &= \frac{\mathrm{diam}^d I_{i-1}}{\sum_{J\in C_{i-1}} \mathrm{diam}^d J} = \frac{1}{\sum_{J\in C_{i-1}} \left(\frac{\mathrm{diam}J}{\mathrm{diam}I_{i-1}}\right)^d}\\ &\leq \left(2 \max_{J\in C_{i-1}} \eta\left(\frac{\mathrm{diam}f^{-1}(J)}{\mathrm{diam}f^{-1}(I_{i-1})}\right)\right)^d = C_4'\eta^d\left(\frac{2}{1-c_i}\right).\end{aligned} \tag{1.19}$$

The rest of the proof is the same as for theorem 1.1.5. $\square$

Chapter 2

Dimension type quasiconformal invariants for curve families and sets in the plane

2.1 Modulus of a curve family and conformal dimension

Usually one obtains a lower bound for the conformal dimension of a metric space by showing that the space has sufficiently many curves. Recall from the introduction that (X, d_X) is an *Ahlfors Q-regular* metric space if there is a Borel measure μ on X and a constant $C > 1$ s.t. for every ball B_r of radius $r \leq \operatorname{diam}(X)$ the following holds

$$\frac{1}{C} r^Q \leq \mu(B_r) \leq C r^Q.$$

Let (X, μ) be a metric measure space, $p \geq 1$, Γ a collection of paths in X.

A Borel measurable function $\rho : X \to [0, \infty]$ is Γ-*admissible* if

$$\int_\gamma \rho ds \geq 1,$$

for every rectifiable $\gamma \in \Gamma$, where integration is with respect to arclength. Now, define the *p-modulus of* Γ:

$$\operatorname{mod}_p(\Gamma) = \inf_{\rho \in \mathcal{A}(\Gamma)} \int_X \rho^p d\mu, \tag{2.1}$$

where $\mathcal{A}(\Gamma)$ is the collection of Γ admissible functions.

The following two fact will be important for us:

I. mod_p *is an outer measure* on the space of curve families of X

II. quasisymmetric maps between locally compact, connected Ahlfors Q-regular spaces can change the Q-modulus only by a bounded factor

see [26], [40] for details. The importance of the modulus for the conformal dimension comes from the following fact.

Theorem 2.1.1 (Bonk-Tyson, [26] Thm. 15.10). *Suppose X is Ahlfors Q-regular and Γ is a path family in X s.t.* $\operatorname{mod}_Q(\Gamma) > 0$. *Then* $\operatorname{Conf dim} X = Q$.

So a "thick" curve family is an obstruction to making the dimension of a space smaller by a quasisymmetry. The problem is that given a curve family it is hard to guess in general what is the right measure μ on the space, or in other words, "how thick" Γ is. Moreover, if X doesn't have any rectifiable curves then one cannot apply Theorem 2.1.1. On the other hand if $\Gamma \subset X$

where X is Ahlfors Q-regular one can define a "dimension of a curve family" just using the conformal modulus, i.e. $\mathrm{mod}_Q(\cdot)$. This is done below in the case of a plane but can be generalized to spaces where **II** holds (e.g. $\mathbb{R}^n, n \geq 3$).

2.2 Extremal length and modulus of a curve family.

In this section we give the definitions and summarize some of the properties of extremal length and modulus (or extremal width) of a family of curves in the plane (see [1] and [2] for further details).

2.2.1 Definitions

Let Ω be a domain in the plane and Γ be a family of rectifiable curves in Ω. To obtain a conformally invariant version of length one considers the collection of all conformal Riemannian metrics $ds = \rho|dz|$. Here ρ is a positive Borel-measurable function on Ω. Each ρ allows one to measure the ρ-length of a curve $\gamma \in \Gamma$ and the ρ-area of Ω:

$$L(\gamma, \rho) = \int\limits_{\gamma} \rho|dz|,$$

$$A(\Omega, \rho) = \iint\limits_{\Omega} \rho^2 dx dy.$$

The ρ-length of the family Γ is

$$L(\Gamma, \rho) = \inf_{\gamma \in \Gamma} L(\gamma, \rho).$$

Definition 2.2.1. *The* ***extremal length*** *of* Γ *is*

$$\lambda_\Omega(\Gamma) = \frac{\sup_\rho L^2(\Gamma, \rho)}{A(\Omega, \rho)},$$

where sup *is over the family of all functions* ρ *chosen so that* $0 < A(\Omega, \rho) < \infty$.

Definition 2.2.2. ***Modulus*** *or extremal width of* Γ *is*

$$\mathrm{mod}(\Gamma, \Omega) = \inf_{\rho \in \mathcal{A}(\Gamma)} A(\Omega, \rho), \tag{2.2}$$

where $\mathcal{A}(\Gamma)$ *is the collection of all* ρ*-s for which* ρ*-length of every curve of* Γ *is at least* 1.

Modulus is usually defined as $\mathrm{mod}(\Gamma) = \lambda^{-1}(\Gamma)$, but this is equivalent to the definition given above.

2.2.2 Composition laws and other properties.

The following properties of the modulus are easily verified.

- $\mathrm{mod}(\Gamma) \geq 0$ and $\mathrm{mod}\emptyset = 0$
- $\mathrm{mod}\Gamma' \leq \mathrm{mod}\Gamma$ if $\Gamma' \subset \Gamma$.

We say Γ_1 and Γ_2 are *disjoint* if no curve of Γ_1 intersects a curve of Γ_2.

Theorem 2.2.3 (Subadditivity or parallel law.). *For every two curve families* Γ_1 *and* Γ_2

$$\text{mod}(\Gamma_1 \cup \Gamma_2) \leq \text{mod}\Gamma_1 + \text{mod}\Gamma_2.$$

If Γ_1 *and* Γ_2 *are disjoint then*

$$\text{mod}(\Gamma_1 \cup \Gamma_2) = \text{mod}\Gamma_1 + \text{mod}\Gamma_2.$$

The properties above imply that mod is an outer measure on the set of all curves in $\mathbb{C}$ (this is true in more generality, see [26]). The importance of this observation for us is that this allows the definitions in the next section. One can think of the mod being analogous to the Lebesgue measure on the line and wonder if there are analogues of Hausdorff measures and dimension in this context.

We list some more results of the classical modulus and the extremal length. We say Γ *overflows* Γ_1 if for every $\gamma \in \Gamma$ there is a curve $\gamma_1 \in \Gamma_1$ s.t. $\gamma_1 \subset \gamma$ (terminology is from [31]). So roughly speaking Γ overflows Γ_1 if there are fewer curves in Γ and they are longer than curves of Γ_1.

Theorem 2.2.4. *(Series law) Let* Γ_1 *and* Γ_2 *be disjoint disjoint curve families. If* Γ *overflows* Γ_1 *and* Γ_2 *then*

$$\lambda(\Gamma) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2), \quad \textit{or}$$
$$\text{mod}\Gamma \leq \frac{\text{mod}\Gamma_1 \text{mod}\Gamma_2}{\text{mod}\Gamma_1 + \text{mod}\Gamma_2}.$$

In particular if Γ overflows Γ_1 then $\lambda(\Gamma) \geq \lambda(\Gamma_1)$ or equivalently $\text{mod}(\Gamma) \leq \text{mod}(\Gamma_1)$.

Examples.

1. If $Q = [0, a] \times [0, b]$ then the extremal length of curves connecting the vertical sides is b/a. Euclidean metric is the extremal in this case.

2. If $A = A(r, R) = \{r < |z| < R\}$ then the extremal length of the curves connecting the boundary components of A is $\frac{1}{2\pi} \log \frac{R}{r}$.

3. An important example is the *extremal distance* of two subsets $E_1, E_2 \subset \partial\Omega$ denoted by $d_\Omega(E_1, E_2)$. This is the extremal length of the family of curves connecting E_1 and E_2 in Ω. The extremal length of this family can be calculated explicitly. Let $u = u_{E_1,E_2}$ be the harmonic function in Ω which is equal to 0 and 1 on E_1 and E_2 respectively, and the normal derivative of u vanishes on $\partial\Omega \setminus E_0 \cup E_1$. Then

$$d_\Omega(E_1, E_2) = \iint\limits_\Omega |\nabla u|^2 dxdy.$$

According to the geometric definition of quasiconformality, $f : \mathbb{C} \to \mathbb{C}$ is K-qc if

$$\frac{1}{K}\mathrm{mod}(\Gamma) \leq \mathrm{mod}(f(\Gamma)) \leq K\mathrm{mod}(\Gamma) \tag{2.3}$$

for a fixed constant $K \geq 1$ and every curve family $\Gamma \subset \mathbb{C}$ (see [1]). In particular modulus is a conformal invariant. It is known that in the plane (2.3) is equivalent to the quasisymmetry condition.

2.3 Conformal dimension of a curve family.

2.3.1 Definitions

Let $\Omega \subset \mathbb{C}$ be a Jordan domain. A *quadrilateral* $Q(E, F)$ in Ω is a Jordan domain $Q \subset \Omega$ with two distinguished disjoint arcs E and F on the boundary of Ω. Let $\mathcal{Q}(\Omega)$ denote the collection of the quadrilaterals in Ω. The collection of all curves in a quadrilateral Q connecting E and F will be denoted $(Q; E, F)$.

Next, given a domain Ω let Γ be a curve family such that that $\gamma(0), \gamma(1) \in \partial\Omega, \forall\gamma \in \Gamma$. We say Γ is a **disjoint curve family** if all curves in Γ are pairwise disjoint.

Definition 2.3.1. *An ε-**covering of a disjoint curve family** Γ is a collection of quadrilaterals $\{Q_i(E_i, F_i)\}_{i=1}^{\infty}$ such that:*

1. $Q_i(E_i, F_i) \in \mathcal{Q}(\Omega)$,

2. $\Gamma \subset \bigcup_i (Q_i; E_i, F_i)$,

3. $\mathrm{mod}(Q_i; E_i, F_i) < \varepsilon, \forall i$.

Definition 2.3.2 (Disjoint curve families). *Let Γ be a disjoint family of curves with endpoints in $\partial\Omega$ and $\mathfrak{S}_\varepsilon(\Gamma, \Omega)$ denote the collection of ε-coverings of Γ. Define*

$$m_\Omega(\Gamma, t, \varepsilon) = \inf_{\mathfrak{S}_\varepsilon(\Gamma,\Omega)} \left\{ \sum_{i=1}^{\infty} \mathrm{mod}^t(Q_i) \right\} \tag{2.4}$$

and

$$m_\Omega(\Gamma, t) = \lim_{\varepsilon \to 0} m_\Omega(\Gamma, t, \varepsilon). \tag{2.5}$$

The limit above exists since $m_\Omega(\Gamma, t, \varepsilon)$ is increasing in ε. Now we can extend this definition and define $m_\Omega(\Gamma, t)$ in general.

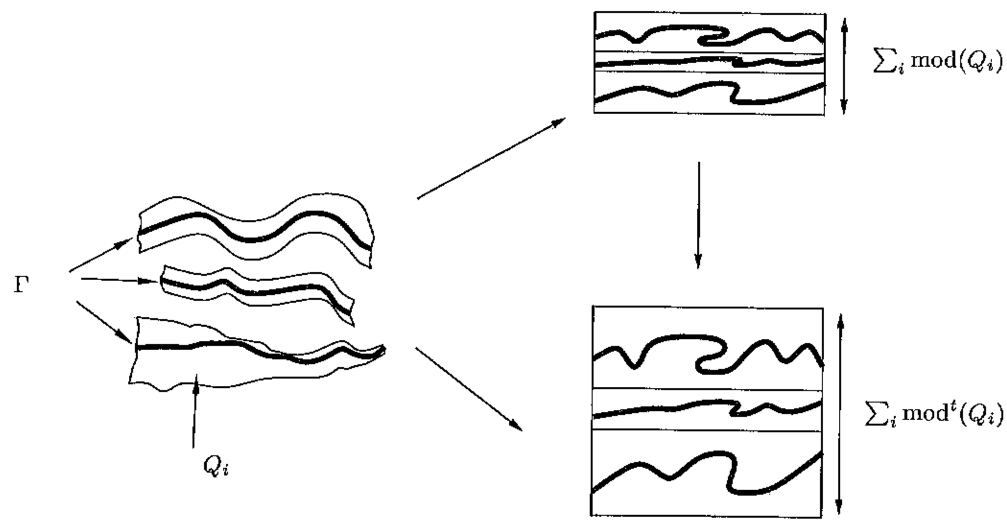


Figure 2.1: Covering of a curve family.

Definition 2.3.3 (General curve families). *For a curve family* Γ *such that* $\gamma(0), \gamma(1) \in \Omega, \forall g \in \Gamma$ *let* $\mathfrak{D}(\Gamma)$ *denote the collection of disjoint subfamilies of* Γ *and define*

$$m_\Omega(\Gamma, t) = \sup_{\Gamma' \in \mathfrak{D}(G)} m_\Omega(\Gamma', t). \tag{2.6}$$

As a function of t, $m_\Omega(\cdot, t)$ behaves like the usual Hausdorff measures.

Lemma 2.3.4. *Let* Γ *be a curve family. Then*

a. If $m_\Omega(\Gamma, t) > 0$ *then* $m_\Omega(\Gamma, s) = \infty, \forall 0 \leq s < t$,

b. If $m_\Omega(\Gamma, t) < \infty$ *then* $m_\Omega(\Gamma, s) = 0, \forall t < s \leq 1$.

Proof. Suppose $s < t$. From the definitions we have $m_\Omega(\Gamma, t, \varepsilon) \leq \varepsilon^{t-s} m_\Omega(\Gamma, s, \varepsilon)$. Taking $\varepsilon \to 0$ we obtain (a). (b) is proved similarly. $\square$

So it is natural to consider the following definition.

Definition 2.3.5 (Conformal dimension of a curve family). *For every* $\Gamma \subset \Omega$ *with endpoints in* $\partial\Omega$ *let*

$$\begin{aligned} \dim_{\mathrm{mod}}(\Gamma,\Omega) &= \inf\{\, t \,|\, m_\Omega(\Gamma,t) = 0\} \\ &= \sup\{\, t \,|\, m_\Omega(\Gamma,t) = \infty\}. \end{aligned} \tag{2.7}$$

Remark 2.3.6. *Note that the above definitions do not require* Γ *to contain any rectifiable curves. We will see below that* $m(\cdot,t)$, $\dim_{\mathrm{mod}}$ *give interesting information even in these cases.*

Remark 2.3.7. *One may also define a local version of* $D_{\mathrm{mod}}(X)$ *by considering the tangent spaces of* X *in the Gromov-Hausdorff sense. Then one may talk about* $D_{\mathrm{mod}}(X)$ *even for Cantor sets i.e. when* X *doesn't contain any curves.*

Remark 2.3.8. *In many cases it would be more useful to consider only closed curves. For example if* Γ *is the collection of all simple closed curves in a (round) Sierpinski carpet. In that case it would be more appropriate to start with the collection of all annuli instead of quadrilaterals.*

Let us consider the dependence of $m_\Omega(\Gamma,t)$ on the Jordan domain Ω.

Lemma 2.3.9. *Suppose* $\Gamma \subset \Omega \subset \Omega'$, *where* Ω,Ω' *are Jordan domains and* $\gamma(0),\gamma(1) \in \partial\Omega \cap \partial\Omega', \forall\gamma \in \Gamma$. *Then*

$$m_\Omega(\Gamma,t) \geq m_{\Omega'}(\Gamma,t). \tag{2.8}$$

Proof. Suppose $Q(E,F) \in \mathcal{Q}(\Omega)$. Consider a quadrilateral $Q'(E',F') \in \mathcal{Q}(\Omega')$ obtained by connecting the endpoints of $E,F \in \partial\Omega$ to endpoints of two disjoint arcs $E', F' \in \partial\Omega'$ by nonintersecting curves in such a way that $\Gamma \cap (Q; E, F) \subset \Gamma \cap (Q; E', F')$. Then $\mathrm{mod}(Q; E, F) \geq \mathrm{mod}(Q'; E', F')$ since the second family overflows the first one.

Now, suppose $\{Q_i\}_{i=1}^{\infty}$ is an ε-covering of Γ with $Q_i \in \mathcal{Q}(\Omega)$. Then there is an ε covering $\{Q_i'\}_{i=1}^{\infty}$ of Γ by quadrilaterals $Q_i' \in \mathcal{Q}(\Omega')$ such that $\mathrm{mod}Q_i \geq \mathrm{mod}Q_i', \forall i$. Hence $m_{\Omega}(\Gamma, t) \geq m_{\Omega'}(\Gamma, t)$. □

So we introduce the following quantities associated to mod. First let

$$m(\Gamma, t) = \sup_{\Omega} m_{\Omega}(\Gamma, t), \tag{2.9}$$

where the supremum is taken over all Jordan domains Ω in the plane which contain the endpoints of all the curves in Γ.

Definition 2.3.10 (Dimension type quasiconformal invariant of a set). *Let $X \subset \mathbb{C}$. Define*

$$D_{\mathrm{mod}}(X) = 1 + \sup_{\Gamma \subset X} \dim_{\mathrm{mod}} \Gamma, \tag{2.10}$$

where the supremum is taken over all non-degenerate curve families in X, i.e those which do not contain constant curves.

Another version of an invariant of a set $X \subset \mathbb{C}$ may be defined as follows. Suppose E and F are compact disjoint subsets of X. Let $\Gamma_X(E, F)$ be the collection of all curves in X connecting E and F. Define

$$\tilde{D}_{\mathrm{mod}}(X) = 1 + \sup_{\substack{E, F \subset K \\ E \cap F = \emptyset}} \dim_{\mathrm{mod}} \Gamma_K(E, F). \tag{2.11}$$

The reason for adding 1 in the definitions above is that $\dim_{\mathrm{mod}}$ measures mainly the "transverse direction of a curve family" while if we want a quantity which is related to the Hausdorff dimension of the set we have to consider "both directions".

Since Γ can also be thought of as a subset of the plane $D_{\text{mod}}(\Gamma)$ makes sense and is a number ≥ 1. From definitions it then follows that

$$D_{\text{mod}}(\Gamma) \geq 1 + \dim_{\text{mod}} \Gamma. \tag{2.12}$$

Example of "Cantor curtains" below shows that strict inequality can occur.

2.3.2 Properties

Theorem 2.3.11. *If f is a K-quasiconformal map of the plane then for every $t \in [0,1]$*

$$\frac{1}{K^t} m_\Omega(\Gamma, t) \leq m_{f(\Omega)}(f(\Gamma), t) \leq K^t m_\Omega(\Gamma, t). \tag{2.13}$$

Proof. Clearly it's enough to consider Γ to be a disjoint family. First, note that if $\{Q_i\}_i$ is an ε-cover of Γ then $\{f(Q_i)\}_i$ is a $K\varepsilon$-cover of $f(\Gamma)$. Therefore

$$\begin{aligned} m_{f(\Omega)}(f(\Gamma), t, K\varepsilon) &= \inf_{\mathfrak{S}_{K\varepsilon}(f\Gamma, f(\Omega))} \sum_i \operatorname{mod}^t(Q_i) \leq \inf_{\mathfrak{S}_\varepsilon(\Gamma,\Omega)} \sum_i \operatorname{mod}^t(f(Q_i)) \\ &\leq K^t \inf_{\mathfrak{S}_\varepsilon(\Gamma,\Omega)} \sum_i \operatorname{mod}^t(Q_i) = K^t m_\Omega(\Gamma, t, \varepsilon) \end{aligned}$$

For $\varepsilon \to 0$ we obtain

$$m_{f(\Omega)}(f\Gamma, t) \leq K^t m_\Omega(\Gamma, t) \tag{2.14}$$

The other inequality follows from the fact that if f is K-quasiconformal then so is f^{-1}. $\square$

Corollary 2.3.12. *If f is conformal then $m_{f(\Omega)}(f(\Gamma), t) = m_\Omega(\Gamma, t)$ for all $t \in [0,1]$.*

So $m_\Omega(\cdot, t)$ can be thought of as a conformally invariant transverse measure on Γ. From (2.13) and the definitions of the previous section we obtain the following.

Theorem 2.3.13 (Invariance of dimension). *If f is a quasiconformal map of Ω then*

$$\begin{aligned} \dim_{\text{mod}}(\Gamma, \Omega) &= \dim_{\text{mod}}(f(\Gamma), \Omega), \\ D_{\text{mod}}(X, \Omega) &= D_{\text{mod}}(f(X), f(\Omega)). \end{aligned} \tag{2.15}$$

Just like for Hausdorff dimension the upper bounds for $\dim_{\text{mod}}$ are obtained by using explicit coverings, in this case by quadrilaterals. The lower bounds one can obtain by using the following analogue of mass distribution principle.

In the next lemma μ is a nonnegative, countably subadditive function on the set of curve families in the plane. If $\mu(\Gamma) > 0$ we say μ is a positive measure on Γ.

Lemma 2.3.14 (Conformal mass distribution principle). *Let Γ be a disjoint curve family. If there is a positive measure μ on Γ and a constant $C < \infty$ s.t.*

$$\mu(Q \cap \Gamma) \leq C \text{mod}^\alpha(Q)$$

for any quadrilateral Q then $\dim_{\text{mod}}(\Gamma) \geq \alpha$ and hence $D_{\text{mod}}(\Gamma) \geq 1 + \alpha$ by (2.12).

Proof. Take an ε-cover Q_i of Γ. Then

$$\sum_{i=1}^{\infty} \operatorname{mod}^{\alpha}(Q_i) \geq \frac{1}{C}\sum_{i=1}^{\infty} \mu(Q_i \cup \Gamma) > \frac{\mu(\Gamma)}{C}. \tag{2.16}$$

Since this is true for every covering of Γ it follows that $m(\Gamma, \alpha) > \frac{\mu(\Gamma)}{C} > 0$. $\square$

From the properties of mod one obtains the following analogues of the composition laws:

Lemma 2.3.15. *If* Γ *overflows* Γ' *then* $m(\Gamma, t) \leq m(\Gamma', t)$ *for every* $t \in [0, 1]$. *In particular* $\dim_{\text{mod}} \Gamma \leq \dim_{\text{mod}} \Gamma'$.

Lemma 2.3.16. *For every* Γ_1 *and* Γ_2

$$m(\Gamma_1 \cup \Gamma_2, t) \leq m(\Gamma_1, t) + m(\Gamma_2, t)$$

and if the families are disjoint one actually has an equality.

2.4 Comparing $\operatorname{Conf}\dim$ and $\dim_{\text{mod}}$. Examples.

In general we suspect the following inequality to hold for every compact subset of the plane:

$$D_{\text{mod}}(X) \leq \operatorname{Conf}\dim \mathrm{X}. \tag{2.17}$$

Moreover, even though we show below that strict inequality can occur here, we expect (2.17) to hold with an equality for the local version of $\dim_{\text{mod}}$ as explained in Remark 2.3.7.

We are not able to prove (2.17) but in this section we calculate $\dim_{\mathrm{mod}}$ for several examples for which (2.17) holds with equality as well as some for which there is a strict inequality. Main techniques used for calculating $\dim_{\mathrm{mod}}$ in these cases are Lemma 2.3.14 and estimates for the extremal length in the plane. Conf dim can be computed in these examples using Theorem 2.1.1.

Example 2.4.1. *If Γ is a countable collection of curves then* $\dim_{\mathrm{mod}} \Gamma = 0$.

Proof. It is enough to consider disjoint families. For each curve γ in the plane there is a quadrilateral of arbitrary small modulus for which γ is a "horizontal curve". Hence given ε and $\Gamma = \{\gamma\}_{i=1}^{\infty}$ we can find quadrilaterals Q_i s.t. $\mathrm{mod}(Q_i) < \varepsilon/2^i$. Therefore $m(\Gamma, t) \leq \varepsilon^t \sum_i 2^{-ti} \to 0$ as $\varepsilon \to 0$, for every $t > 0$. □

In [7] Bishop and Tyson construct Cantor sets of conformal dimension > 1. Consider such a Cantor set and let γ be a curve which contains it. By the Lemma above $\dim_{\mathrm{mod}} \gamma = 0$ and hence $D_{\mathrm{mod}}\gamma = 1 < \mathrm{Conf}\dim\gamma$. So there are examples of sets s.t. (2.17) holds with a strict inequality.

Lemma 2.4.2. *Let $\Gamma = \Gamma_E = E \times (0,1)$, with $E \subset [0,1]$. Then*

$$\dim_{\mathrm{mod}}(\Gamma, [0,1]^2) = \dim_H(E), \tag{2.18}$$

$$D_{\mathrm{mod}}(\Gamma, [0,1]^2) = \dim_H E + 1 = \mathrm{Conf}\dim\Gamma. \tag{2.19}$$

Proof. The upper bound on $\dim_{\mathrm{mod}}$ is easily obtained by considering rectangles with bases covering E. So $\dim_m(\Gamma_E) \leq \dim_H E$.

To obtain the lower bound we use Lemma 2.3.14. First recall that if the set E has positive Hausdorff t-measure then there is a positive measure ν_E on E

called Frostmann measure which satisfies $\nu_E(I \cap E) \leq C|I|^t$ for some constant $C < \infty$ and every interval $I \subset \mathbb{R}$. Suppose $E \subset \mathbb{R}$, $\dim_H(E) = \alpha$ and

$$\Gamma_E = \{\, \gamma \,|\, \gamma(0) \in E, \gamma(t) = \gamma(0) + it, 0 \leq t \leq 1\}.$$

Take $t < \alpha$. Then $H_t(E) = \infty$ and let ν_E be the corresponding Frostmann measure. For an interval $I \subset \mathbb{R}$ let

$$\mu(\Gamma_I) = \nu_E(\Gamma_I \cap \mathbb{R}).$$

Hence $\mu(\Gamma_I) \leq C|I|^t = C\mathrm{mod}^t(\Gamma_I)$. Now for an arbitrary quadrilateral Q define μ by

$$\mu(Q) = \sup_{I \subset \mathbb{R}} \mu(Q_h \cap \Gamma_I),$$

where Q_h is the collection of horizontal curves in the quadrilateral Q. Then $\mu(Q) \leq C\mathrm{mod}^t(Q)$ as well. From the Lemma 2.3.14 it follows that $m(\Gamma_E, t) > 0$, for every $t < \alpha$. Therefore $\dim_{\mathrm{mod}}(\Gamma_E) \geq \dim_H E$. Hence, $\dim_{\mathrm{mod}} \Gamma_E = \dim_H E$.

The equality $\mathrm{Conf\,dim\,E} \times (0,1) = \mathrm{dim_H\,E} \times (0,1)$ was shown for Ahlfors regular E in [41] and for general metric spaces in [7] (the proof is similar to the proof of the Theorem 2.1.1). $\square$

A similar example can be constructed as follows. Take two middle interval Cantor sets constructed so that the ratio of the lengths of the removed interval to the interval from which it is being removed doesn't change from step to step. So for every number $0 < r < 1$ there is a corresponding middle interval Cantor set E_r. It's easy to see that $\dim_H E_r = \log_{2/(1-r)} 2$. Now consider the curve

Figure 2.2: $\Gamma_{E_{r_1},E_{r_2}}$

family which connects $\{0\} \times E_{r_1}$ to $\{1\} \times E_{r_2}$ by straight lines. Denote this curve family by $\Gamma_{E_{r_1},E_{r_2}}$.

Lemma 2.4.3. *Let* $\Gamma = \Gamma_{E_{r_1},E_{r_2}}$. *Then*

$$\dim_{\text{mod}} \Gamma = \max\{\dim_H E_{r_1}, \dim_H E_{r_2}\}.$$

Proof. We suppose $r_1 < r_2$ and show that $\dim_{\text{mod}} = \dim_{E_{r_1}}$. We consider the covering of Γ by the quadrilaterals the vertical sides of which are the generating intervals of same generation for the two Cantor sets. So there are 2^i quadrilaterals, each with vertical sides of length $\left(\frac{1-r_1}{2}\right)^i$ and $\left(\frac{1-r_2}{2}\right)^i$. We can clearly write Γ as follows $\Gamma = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{2^i} \Gamma_{ij}$, where Γ_{ij} is the curve family connecting the vertical sides of the corresponding quadrilaterals Q_{ij}. Let us estimate the modulus of Γ_{ij}. Since for i large enough the ratio of the lengths of the vertical sides tends to infinity we will compare modΓ_{ij} to the modulus of truncated annuli (see the figure below). In this case the modulus can be explicitly calculated as $\alpha \log \frac{r}{R}$, where α is the corresponding central angle and

r and R are the inner and outer radii of the annulus. In our case

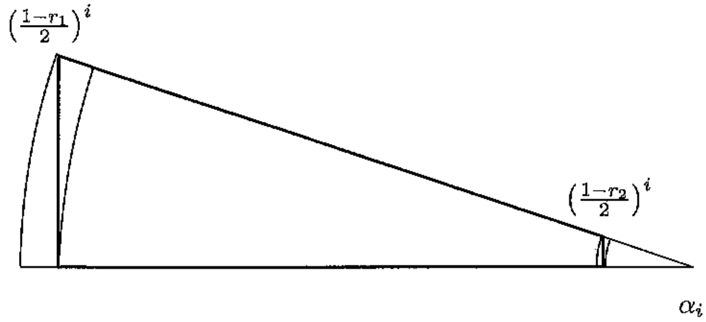


Figure 2.3: Truncated annuli

$\alpha_i \asymp \sin\alpha_i \asymp \left(\frac{1-r_1}{2}\right)^i$, and the ratio of the small and the large radii is of the order $\left(\frac{1-r_2}{1-r_1}\right)^i$. So

$$\mathrm{mod}\Gamma_{ij} \asymp \left(\frac{1-r_1}{2}\right)^i \log\left(\frac{1-r_2}{1-r_1}\right)^i.$$

From the definitions it is easy to see that

$$\dim_{\mathrm{mod}} \Gamma \leq \inf\{t : \lim_{i\to\infty} \sum_{j=1}^{2^i} \mathrm{mod}^t \Gamma_{ij} < \infty\}.$$

Hence $\dim_{\mathrm{mod}} \Gamma \leq \inf\{t : \lim_{i\to\infty} 2^i i^t \left(\frac{1-r_1}{2}\right)^{it} < \infty\} = \log_{2/(1-r_1)} 2 = \dim_H E_{r_1}$.

The opposite inequality is proved similarly to the previous Lemma by using Lemma 2.3.14. □

Remark 2.4.4. *Let Γ_x be the intersection of Γ with the vertical line which passes through the point x of the real axis and let $\delta(x) = \dim_H \Gamma_x$. It is easy to see that for every $x \in [0, 1)$ we have $\delta(x) = \max\{\dim_H E_{r_1}, \dim_H E_{r_2}\}$. So in a sense* $\dim_{\mathrm{mod}}$ *is determined by the dimension of the transverse sets. As* *the next example shows* $\dim_{\text{mod}}$ *is more related to the "thin" parts of the curve family.*

2.4.1 Cantor curtains

We consider the set Γ s.t. the intersection of Γ with every vertical line passing through $(r, 0)$ is the middle interval Cantor set E_r corresponding to $r \in (0, 1)$ of Hausdorff dimension $\log_{2/(1-r)} 2$. Unlike the previous example in this case $\delta_\Gamma(x) = \dim_H \Gamma_x$ is not constant. Given a and b in $(0, 1)$ s.t. $a < b$ we can consider the part of Γ above (a, b) and we denote the corresponding curve family by $\Gamma_{a,b}$. In this case the Hausdorff dimension decrease from $\log_{2/(1-a)} 2$ to $\log_{2/(1-b)} 2$. These type of sets were called Cantor curtains in [32].

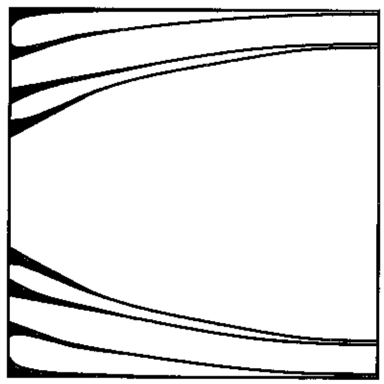


Figure 2.4: Cantor curtain.

Lemma 2.4.5 (Cantor curtains). *Suppose* $\Gamma = \Gamma_{a,b}$. *Then*

$$\begin{aligned} \dim_{\text{mod}} \Gamma &= \dim_H E_b = \inf_{[a,b]} \delta_\Gamma(x), \qquad (2.20) \\ D_{\text{mod}} \Gamma &= \dim_H E_a + 1 = \sup_{[a,b]} \delta(x) + 1. \end{aligned}$$

Proof. First let us note that the second equality follows from the first one. Indeed, by definition, for every $x \in (a, b)$ we have

$$D_{\text{mod}}\Gamma \geq \dim_{\text{mod}} \Gamma_{a,x} + 1 \overset{(2.20)}{=} \dim_H E_x + 1 \to \dim_H E_a + 1.$$

The opposite inequality holds since every nondegenerate curve family of Γ overflows $\Gamma_{x,y}$ or a subfamily of the latter with the same dimension.

We have $\Gamma = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{2^i} \Gamma_{ij}$. Since $\text{mod}\Gamma_{ij}$ is less than or equal to the area of the corresponding quadrilateral Q_{ij} we have that $\text{mod}\Gamma_{ij} \leq \left(\frac{1-a}{2}\right)^i (b-a)$. Hence we have

$$\dim_{\text{mod}} \Gamma_{ab} \leq \inf\{t : \lim_{i \to \infty} 2^i \left(\frac{1-a}{2}\right)^{it} < \infty\} \leq \dim_H E_a.$$

Now since Γ_{ab} overflows Γ_{xb} for every $x \in (a, b)$ we obtain

$$\dim_{\text{mod}} \Gamma_{ab} \leq \dim_{\text{mod}} \Gamma_{xb} \leq \dim_H E_x \to \dim_H E_b.$$

And therefore $\dim_{\text{mod}} \Gamma_{ab} \leq \dim_H E_b$. Using Lemma 2.3.14 it is easy to see that the opposite inequality also holds. $\square$

From the proof one can see that this property is true in a much more general setting.

Chapter 3

Central set of Hausdorff dimension 2

3.1 Introduction

Let D be a domain in $\mathbb{R}^2$. A ball $B = B(a, r) \subset D$ is called *maximal* in D if it is not contained in any other ball $B' \subset D$. The *central set of* D is defined as

$$C(D) = \{\, x \in D \,:\, B(x, d(x, \partial D)) \text{ } is \text{ } maximal \text{ } in \text{ } D\}$$

and the *skeleton* or *medial axis* as

$$M(D) = \{x \in D : \exists \text{ distinct } y, y' \in \partial D \text{ s.t. } d(x, y) = d(x, y') = d(x, \partial D)\},$$

where $d(A, B)$ denotes the Euclidean distance between subsets $A, B \subset \mathbb{R}^2$. Clearly $M(D) \subset C(D)$. On the other hand $C(D) \setminus M(D)$ is not always empty. For example for the domain bounded by an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$, $C(D) \setminus M(D)$ consists of the two centers of osculating circles which touch the ellipse at the points $(-a, 0)$ and $(a, 0)$ (see [21]). In the case when ∂D is piecewise C^3 the central set $C(D)$ can also be characterized as the non-differentiability locus of the function $d_D : D \ni x \mapsto d(x, \partial D)$. More precisely, in this case

$$C(D) = \{\, x \in D \,|\, g_D \,\text{is not}\; C^{1,1} \;\text{in any neighborhood of}\, x\}$$

(see [18] for the proof, actually there $C(D)$ is called the *ridge of a domain*). Another interpretation of the central set is given in [40].

In this article we are interested in the measure theoretic aspects of medial axes and central sets of planar domains (actually most of the results will be valid in higher dimensions as well). One of the earliest papers dealing with these kind of problems is [16], where Erdös proved that $M(D)$ has Hausdorff dimension 1 for planar domains. In [22] Fremlin proved, among other things, the following results about the geometry of medial axes and central sets of planar domains:

(*1*) Any medial axis is F_σ and has Hausdorff dimension at most 1;

(*2*) Any central set is G_δ and has 2-dimensional measure 0.

His paper concludes with the following questions: Can a central set have Hausdorff dimension strictly bigger than 1? Generally what can be said to narrow the gap between (*1*) and (*2*)? The goal of this paper is to prove the following result.

Theorem 3.1.1. *There is a domain $D \subset \mathbb{R}^2$ with $\dim_H(C(D)) = 2$. Moreover, given any Hausdorff measure function ϕ such that $\phi(t)/t^2 \to 0$ as $t \to 0$ there is a domain D such that $H_\phi(C(D)) > 0$.*

Here and in what follows, given a subset E of $\mathbb{R}^2$, $\dim_H(E)$ will always denote the Hausdorff dimension of E (see below for the definitions). In fact, from the proof of the theorem it would be clear that one can take D to be "almost" the unit disc. More precisely, one has the following corollary.

Corollary 3.1.2. *For any $\varepsilon > 0$ there is a domain D_ε such that $\dim_H(C(D_\varepsilon)) = 2$ and $D(0,1) \subset D_\varepsilon \subset D(0, 1+\varepsilon)$.*

Here $D(a,r)$ stands for a disc of radius r centered at a.

Now we give the definitions of Hausdorff measures and Hausdorff dimension. Given any metric space (X, d_X) and a continuous increasing function $\varphi : [0,\infty) \to [0,\infty)$ one defines Hausdorff φ-measure of X as follows

$$H_\varphi(E) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} \varphi(r_i) : E \subset \bigcup_{i=1}^{\infty} D(x_i, r_i),\ r_i < \varepsilon \right\}.$$

where $D(a,r) = \{x \in X \ : \ d_X(a,x) < r\}$. When $\varphi(t) = t^s$, for some $s > 0$, the corresponding Hausdorff measure is called s-dimensional and is denoted by H_s. The Hausdorff dimension of X is defined as

$$\dim_H(X) = \inf\{s : H_s(X) = 0\} = \sup\{s : H_s(X) = \infty\}.$$

Given a subset E of a Euclidean space it inherits the metric form the ambient space. This is the metric used to define the Hausdorff dimension of E. Since we will be interested in obtaining lower bounds for the dimension we describe here the general method of doing this.

The mass distribution principle. If $E \subset \mathbb{R}^2$ supports a positive measure μ such that

$$\mu(B(x,r)) \leq C\varphi(r),$$

for a fixed constant $C > 0$ and for all $x \in \mathbb{R}^2$ and $r > 0$ then $H_\varphi(E) \geq 0$ (see [19]).

Remark 3.1.3. *If φ can be chosen so that $\frac{\varphi(t)}{t^p} \to 0$ as $t \to 0$ for all $0 \leq p < d$ then $\dim_H(E) \geq d$. Indeed, it is easy to see that in this case one would have $H_p(E) > 0$ for every $p < d$ therefore E cannot have Hausdorff dimension less than d.*

In Section 3.2 we define a particular type of domains called "disc trees" for which the medial axes are trees. In Section 3.3 we show that in some cases the central set of a disc tree is the closure of the medial axis. In Section 3.4 we define certain trees which are medial axes of disc trees satisfying conditions of Section 3.3 and the closures of which contain 2 dimensional Cantor sets

3.2 Disc Trees

One might naively think that for a general domain the central set is the closure of its skeleton. This is not the case. Indeed, in [22] Fremlin gives an example of a domain with a skeleton which is dense in some disk. But according to *(2)* in the introduction the central set cannot have positive measure, so cannot contain a disc. In this section we define a family of domains for which the structure of the central set is easily understood. In the next section we will provide a sufficient condition for the central set to be the closure of the medial axis.

We start by considering the so called *finitely bent domains* (see [11], [12] for a detailed discussion and some pictures). These are domains which are finite unions of round discs in the plane. The reason for the terminology comes from hyperbolic geometry. Consider the plane as the boundary at infinity of the hyperbolic 3-space $\mathbb{H}^3$, modeled on the upper half space $\mathbb{R}^3_+$. Then for every closed subset $E \subset \mathbb{R}^2$ let $CH(E) \subset \mathbb{H}^3$ be the corresponding *hyperbolic convex hull* of E. This is the smallest convex subset of $\mathbb{H}^3$ which contains all the infinite hyperbolic geodesics with both endpoints in E. If E is a Jordan curve which bounds a domain Ω then the complement of the hyperbolic convex hull of E in $\mathbb{H}^3$ has two components. One of these meets $\mathbb{R}^2$ along Ω and the other one along $int\,(\Omega^c)$. We will call the boundary in $\mathbb{H}^3$ of the former one *the dome of* Ω and will denote it by S_Ω. In the case when Ω is a union of finitely many round discs, S_Ω is a piecewise smooth surface consisting of finitely many totally geodesic "faces". When two such faces meet they do so along an infinite hyperbolic geodesic. Each face is contained in a unique hyperbolic plane and the angle between two adjacent faces is defined as the angle between the two hyperbolic planes containing them. It is also equal to the angle formed by the two circles at infinity corresponding to these hyperbolic planes. This angle is called the bending angle corresponding to the common geodesic. Such a surface is called finitely bent surface. Using this terminology we can say that a domain is called finitely bent if its dome is a finitely bent surface in $\mathbb{H}^3$. For a finitely bent domain the skeleton is a finite graph and it coincides with the central set.

Let us call a domain in the plane a *crescent* if it is a region bounded by exactly two circular arcs which meet at two distinct points a, b at some (interior) angle θ. These points and the angle are called vertices and the angle of the crescent, respectively. Now for every crescent there is a one parameter family of elliptic Möbius transformations of the plane fixing the two vertices and such that the images of one of the edges foliate the crescent by circular arcs. Namely

$$T_{a,b,t}(z) = \frac{(be^{it} - a)z + ab(1 - e^{it})}{(e^{it} - 1)z + (b - ae^{it})},$$

with $0 < t < \theta$. This can be checked by conjugating this family to the family $T_{0,\infty,t}(z) = e^{it}$ by the map $w = (z - a)/(z - b)$, in which case the foliation of the crescent by the arcs corresponds to the foliation of the angular region $\{z \in \mathbb{C} : 0 < \arg z < \theta\}$ by rays starting at 0.

Start from the unit disc D_0. The skeleton M_0 in this case is the center of D_0.

Let $\mathcal{G}_1 = \{D_{1,i}\}_{i=1}^{n_1}$ (first generation) be a collection of finitely many discs with centers in D_0 such that the corresponding crescents $C_{1,i} = D_{1,i} \setminus \overline{D_0}$ are mutually disjoint. Let $D_1 = D_0 \bigcup_{i=1}^{n_1} D_{1,i}$. The skeleton M_1 of D_1 is obtained from M_0 by adding (radial) segments connecting M_0 to the centers of the discs $D_{1,i}$. Indeed, for every i the corresponding family $T_{b_i^+,b_i^-,t}$, defines a one parameter family of disks interpolating between D_0 and $D_{1,i}$ and passing through the *bending points* $\{b_i^+, b_i^-\} = \partial D_0 \cap \partial D_{1,i}$. Each of these disks is contained in $D_0 \cup D_{1,i}$ and therefore is maximal in $D_0 \cup D_{1,i}$. Their centers sweep out the interval between the centers of D_0 and $D_{1,i}$. Now, since the crescents $C_{1,i}$ are mutually disjoint the maximal circles of $D_0 \cup D_{1,i}$ are still maximal in D_1. Hence M_1 is a tree with one vertex of degree n_1 and n_1 vertices of degree one. Let

$$B_1 = \{x \in D_1 |\, dist(x, \partial D_1) = dist(x, \partial D_0)\}.$$

Then $D_1 \setminus B_1$ can be written as a disjoint union of circular *sectors* $S_{1,i}$ of $D_{1,i}$ corresponding to the crescents $C_{1,i}$ (see white regions in figure 1). $B_1 \setminus M_1$ consists of two type of points: those for which the closest points on ∂D_1 are bending points (the dark grey triangular regions in fig. 1) and the rest (light grey circular sectors, which could possibly degenerate to a line segment if two successive sectors have a common bending point, and white sectors).

Let $\mathcal{G}_2 = \{D_{2,i}\}_{i=1}^{n_2}$ (second generation) be a collection of discs with centers in $\cup_{i=1}^{n_1} S_{1,i}$. Denote by $\tilde{D}_{2,j} \in \mathcal{G}_1$ the (first generation) disc which contains the center of $D_{2,j}$. We will assume that these second generation discs satisfy the following:

(i) $C_{2,i} := D_{2,i} \setminus D_1 = D_{2,i} \setminus \tilde{D}_{2,j}$;

(ii) $C_{2,i} \cap C_{2,j} = \emptyset$ whenever $i \neq j$.

Let $D_2 = D_1 \cup_{i=1}^{n_2} D_{2,i}$. The skeleton M_2 of D_2 is obtained from M_1 by adding edges connecting the centers of second generation discs to the corresponding degree one vertices of M_1, i.e. two centers are connected if one of them is contained in the disc corresponding to the other. Let

$$B_2 = int\{x \in D_2 |\, dist(x, \partial D_2) = dist(x, \partial D_1)\}.$$

Again $D_2 \setminus B_2 = \cup_{i=1}^{n_2} S_{2,i}$, ($S_{2,i}$ is a sector of $D_{2,j}$ with the arc $\partial D_{2,i} \setminus D_1$) where these sectors are mutually disjoint and $B_2 \setminus M_2$ can be written as a union of triangles corresponding to bending points and circular sectors.

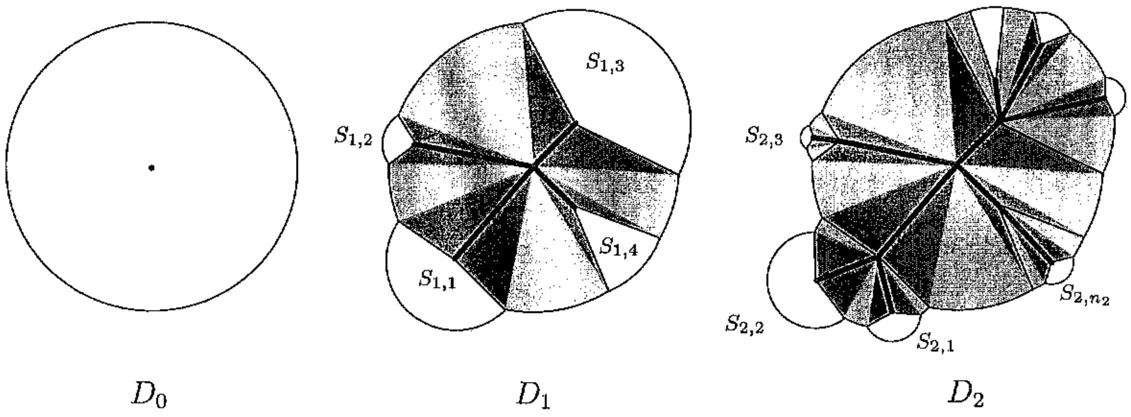


Figure 3.1: Construction of the domain D. Shown are D_0, D_1 and D_2. Thick point and the graphs represent M_0,M_1 and M_2. The union of the grey regions in D_1 is B_1. Dark grey triangles in D_1 correspond to the first generation bending points. In the third picture the dark triangles correspond to the first and second generation bending points. B_2 is the union of the grey parts in D_2.

In general suppose D_k has been constructed from D_{k-1} by adding discs $D_{k,i}$, $i = 1, \ldots, n_k$ or, which is the same, circular sectors $S_{k,i}$. Let $\mathcal{G}_{k+1} = \{D_{k+1,i}\}_{i=1}^{n_{k+1}}$ be a collection of discs with centers in $\cup_{i=1}^{n_k} S_{k,i}$. Denote by $\tilde{D}_{k+1,j} \in \mathcal{G}_k$ the (k-th generation) disc which contains the center of $D_{k+1,j}$ (the "parent" of $D_{k+1,j}$). Assume that $k+1$ generation discs satisfy the following conditions:

(i) $C_{k+1,i} := D_{k+1,i} \setminus D_k = D_{k+1,i} \setminus \tilde{D}_{k+1,i}$;

(ii) $C_{k+1,i} \cap C_{k+1,j} = \emptyset$ whenever $i \neq j$.

Let $D_{k+1} = D_k \cup_{i=1}^{n_{k+1}} D_{k+1,i}$. The skeleton M_{k+1} of D_{k+1} is obtained from M_k by adding edges connecting the centers of discs of $\mathcal{G}_{k+1}$ to the corresponding degree one vertices of M_k. Let

$$B_{k+1} = int\{x \in D_{k+1} | \, dist(x, \partial D_{k+1}) = dist(x, \partial D_k)\}.$$

Just as before $D_{k+1} \setminus B_{k+1} = \cup_{i=1}^{n_{k+1}} S_{k+1,i}$, these sectors are mutually disjoint and $B_k \setminus M_{k+1}$ is a union of "bending" triangles and circular sectors. So for every edge e of M_k there are two triangles T^+ and T^- which have e as a common edge and the corresponding bending points b^+ and b^- as vertices, respectively.

Finally we obtain an increasing sequence of finitely bent domains $D_0 \subset D_1 \subset \ldots \subset D_k \subset \ldots$. Let $D = \bigcup_{i=1}^{\infty} D_i$. We will call a domain *a disc tree* if it can be constructed as above. Let us call the vertices of the k-th generation crescents k-th generation *bending points.*

3.3 Decomposition of disc trees

Let D be a disc tree and let $B = \bigcup_{k=1}^{\infty} B_i$ and $L = D \setminus B$. Then for every point $x \in L$ there is a positive integer $k(x)$ and an infinite sequence of (open) sectors $S_k(x) := S_{k,i_k}, k \geq k(x)$, each of which contains x. By taking $k(x)$ to be the smallest integer with this property we obtain the *maximal* sequence of sectors containing x. This means that every such sequence will be a subsequence of the maximal one. Let $\alpha_k(x)$ be the central angle of $S_k(x)$ and let $e_k(x)$ be the edge connecting the centers of $S_k(x)$ and $S_{k+1}(x)$. To proceed we need the following definition: for a domain $W \subset \mathbb{R}^2$ let f_W denote the set valued function which associates to a point $x \in W$ the subset of closest points on the boundary,

$$f_W(x) = \{y \in \partial W : |y - x| = dist(x, \partial W)\}. \tag{3.1}$$

By definition $M(D) = \{x \in D \,|\, card\{f(x)\} > 1\}$. It is clear from the construction that $f_D|_{B_k} = f_{D_{k+m}}|_{B_k}$ for any $k, m \in \mathbb{N}$.

Lemma 3.3.1. *If D is as above and $x \in \overline{M} \setminus M \cap D$ then $x \in L$.*

Proof. Suppose $x \in D$ and $x \notin L$. Then $x \in B$ and there is a $k \in \mathbb{N}$ s.t. $x \in B_k$. By the description of B_k given above it follows then that either $x \in M_k \subset M$ or there is a neighborhood $V \subset B_k$ of x such that $\#\{f_D|_V(x)\} = \#\{f_{D_k}|_V(x)\} = 1$ and therefore $V \cap M = \emptyset$. In any case $x \notin \overline{M} \setminus M$. □

M has a tree structure and each vertex x of M is a vertex of some sector $S_{i,j}$. So x has a parent $\tilde{x}$ and a number of children $C(x) = \{c_1(x), \ldots, c_{n(x)}(x)\}$. If x, y are vertices we will write $y \prec x$, and say that y is a descendant of x, if there are vertices $x = z_1, \ldots, z_n = y$ such that $z_{i+1} \in C(z_i)$. We will call a sequence of vertices $\{x_i\}$ *admissible* if $x_{i+1} \prec x_i$. Let $\widetilde{M}$ be the collection of limit points of admissible sequences of M. This set in the plane is a realization of the space of ends of M.

Theorem 3.3.2. *If D is a disc tree such that every sector has a central angle less than π then*

$$\widetilde{M}(D) \cap D \subset C(D).$$

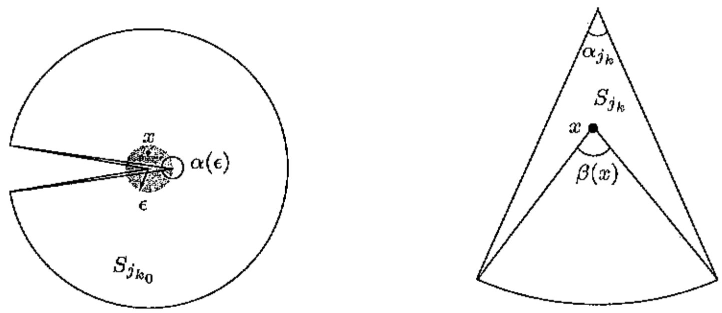
Proof. To prove the theorem we will need the following statements.

Lemma 3.3.3. *If $x \in (\widetilde{M} \setminus M) \cap D$ then $\alpha_k(x) \to 0$.*

Lemma 3.3.4. *If* $x \in (\widetilde{M} \setminus C) \cap D$ *then* $\liminf_{k\to\infty} \alpha_k(x) \geq \pi$.

The theorem will follow since the assumption about central angles at the vertices of the sectors implies that there are no points in D which satisfy the condition of Lemma 3.3.4. Equivalently $\widetilde{M} \setminus M \subset C(D)$. $\square$

Proof of Lemma 3.3.3. Since $x \notin M$ there is a unique closest point b to x in ∂D. Also, since $\widetilde{M} \subset \overline{M}$, by lemma 3.3.1 $x \in L$ and therefore there is a unique maximal sequence $S_j(x), j \geq j_0$ such that $x \in \bigcap_{j \geq j_0} S_j$. Denote by b_j^-, b_j^+ and c_j the corresponding (distinct) bending points and the vertex of S_j, respectively, and let $l_j^\pm = [c_j, b_j^\pm)$. We have that x is a limit point of c_j, i.e. $x = \lim_{k\to\infty} c_{j_k}$. Consider the triangular regions $T_{j_k}^-, T_{j_k}^+$ adjacent to the edges e_{j_k}. We know that $f_D(T_{j_k}^\pm) = b_{j_k}^\pm$. Clearly there is a sequence of points $x_{j_k}^\pm \in T_{j_k}^\pm$ such that $x_{j_k}^\pm \to x$ (just pick two points in, say $1/k$, neighborhoods of e_{j_k} in $T_{j_k}^- \cup e_{j_k} \cup T_{j_k}^+$). Therefore $f_D(x_{j_k}^\pm) = b_{j_k}^\pm \to b$ since $f|_{D\setminus M}$ is a continuous *function* (see [21], 3.3). It follows that $\alpha_{j_k} \to 0$ or 2π, since $x \in D$ and the radii of the sectors S_{j_k} are bounded from below and tend to $|x - b| > 0$. From our construction it follows that $\alpha_{j_k} \nrightarrow 2\pi$. Indeed, since $c_{j_k} \to x$, for every ϵ there is an integer k_0 such that $|c_{j_{k_0}} - c_{j_{k_0+m}}| < \epsilon$ for all $m > 0$. By construction $l_{j_{k_0}}^\pm \cap l_{j_{k_0+m}}^\pm = \emptyset$ for $m > 0$. Now, it is easy to see that there is an upper bound $\alpha(\epsilon) < 2\pi$ such that every circular sector with a center in the ϵ neighborhood of $c_{j_{k_0}}$ and vertices outside of $S_{j_{k_0}}$ will have a central angle $\leq \alpha_j(\epsilon) < 2\pi$ (see fig. 2 (a)). Therefore $\alpha_{j_k} \to 0$. A similar argument implies that the complete sequence $\alpha_i \to 0$. Indeed, if a sector with a center in S_{j_k} contains x and has vertices outside of S_{j_k} then there is an upper bound on its



(a) (b)

Figure 3.2:

central angle which goes to 0 if $\alpha_{j_k} \to 0$ (see fig.3.3 (b)). □

Proof of Lemma 3.3.4. Since $x \notin C(D)$ there is a unique point y closest to x on ∂D and a point $x' \in D$ such that $|x' - y| > |x - y|$ and the disc $D' = B(x', |x' - y|)$ contains $B(x, |x - y|)$ and is contained in D. Note that $x' \in L$ since otherwise we would have had $y \in f(B)$ and hence $x \in B$ contradicting Lemma 3.3.1. Take a point $y' = (1 - t)x + ty$ with $t \in (0, 1)$. For k_0 large enough $[y', x'] \subset \bigcap_{k \geq k_0} S_k$. Since $c_k \to x$ it follows that $\alpha_k \geq \pi$ for k large enough. □

Remark 3.3.5. *It is not true that $\overline{M} \cap D \subset C(D)$ even for domains as nice as we described. But it is not hard to give conditions such that $\overline{M} = M \cup \widetilde{M}$. Next we give such a condition.*

Let l_k be the length of the longest k-th generation edge of M.

Lemma 3.3.6. *If $l_k \to 0$ then $\overline{M} = \widetilde{M} \cup M$.*

Proof. We only need to show that $\overline{M} \setminus M \subset \widetilde{M}$. Suppose $x = \lim_{i \to \infty} x_i$, where $x_i \in M$ and $x \notin M$. By lemma 3.3.1 $x \in L$ and $x \in \bigcap S_k(x)$. Then for every

$i \in \mathbb{N}$ large enough, there is a k_i such that $x_i \in S_{k_i}(x) \setminus S_{k_i+1}(x)$ and $k_i \to \infty$ as $i \to \infty$. This means that $x_i \in M_{k_i+1} \setminus M_{k_i}$ and hence $|c_{k_i}(x) - x_i| \leq l_{k_i}$ (here $c_k(x)$ is the center of $S_k(x)$). So,

$$|c_{k_i}(x) - x| \leq |c_{k_i}(x) - x_i| + |x_i - x| \to 0$$

since $l_{k_i} \to 0$. □

3.4 Proof of theorem 3.1.1

We first construct a particular type of trees which can be realized as medial axes of domains constructed as above and satisfy the condition of Lemma 3.3.6. Then we show that the closure of such a tree can have positive measure with respect to the Hausdorff measure H_φ for the needed type of φ. The result then follows from Lemma 3.3.6 and Theorem 3.3.2.

The construction of the trees is by induction. Fix two sequences $\mathbf{p} = \{p_i\}$ and $\mathbf{n} = \{n_i\}$ of integers such that n_i is divisible by p_i. Let $q_i = p_i + 1$ and

$$N_k = \prod_{i=1}^{k} n_i, \quad P_k = \prod_{i=1}^{k} p_i, \quad Q_k = \prod_{i=1}^{k} q_i.$$

Step 1. Let 0 be the root of the tree. Divide the plane into n_1 congruent infinite sectors by n_1 rays emanating from 0 such that angles between adjacent rays are all equal to $\alpha_1 := 2\pi/n_1$. On the bisectors of the angular regions lay segments of lengths

$$\frac{1}{Q_1}, \frac{2}{Q_1}, \dots, \frac{p_1}{Q_1};$$
$$\frac{1}{Q_1}, \frac{2}{Q_1}, \dots, \frac{p_1}{Q_1}; \dots$$

say in the counterclockwise direction. These are the first generation edges. The new endpoints of the intervals are the first generation vertices (we will denote the collection of these by V_1). Note that V_1 can be thought of as consisting of p_1 "rows", the vertices which are equidistant from 0, each of which containing n_1/p_1 vertices.

Step 2. (part I). For each first generation vertex $v \in V_1$ construct its n_2 children in a similar fashion, this time having p_2 rows with n_2/p_2 vertices in every row. Namely, consider the ray $r(v) = \{(1-t)\tilde{v} + tv : t > 1\}$, where $\tilde{v}$ denotes the parent of v (in this case 0), and let $r^+(v)$ and $r^-(v)$ be the rays emanating from v for which the angle between each of these and $r(v)$ is $\alpha_1/2$. In particular, $r^{\pm}(v)$ are parallel to the rays starting at $\tilde{v}$ and separating v from its "siblings". Denote by $\mathcal{C}(v)$ the cone with vertex v, sides $r^{\pm}(v)$ and angle at v equal to α_1. Now repeat the step 1 with $\mathcal{C}(v)$ instead of the plane.

Step 2 (part II). In other words, divide $\mathcal{C}(v)$ by rays into n_2 congruent cones of angle $\alpha_2 = \alpha_1/n_2 = 2\pi/N_2$. Lay edges of lengths

$$\frac{1}{Q_2}, \frac{2}{Q_2}, \dots, \frac{p_2}{Q_2};$$
$$\frac{1}{Q_2}, \frac{2}{Q_2}, \dots, \frac{p_2}{Q_2}; \dots$$

(in the same direction) along the bisectors of these cones. The new endpoints are the second generation vertices V_2.

Step $k+1$ (part I). Suppose the generation k edges and vertices V_k have been constructed. For each $v \in V_k$ let $r(v)$ be the ray starting at $\tilde{v}$ and passing through v, as before. Also let $r^{\pm}(v)$ make an angle $\alpha_k/2$ with $r(v)$, where $\alpha_k = \alpha_{k-1}/n_k = 2\pi/N_k$. Let $\mathcal{C}(v)$ be the cone with sides $r^{\pm}(v)$ and angle α_k at v.

Step $k+1$ (part II). Divide $\mathcal{C}(v)$ into n_{k+1} congruent cones of opening $\alpha_{k+1} = 2\pi/N_{k+1}$. On the bisectors of these cones lay the edges of generation $k+1$ of lengths

$$\frac{1}{Q_{k+1}}, \frac{2}{Q_{k+1}}, \ldots, \frac{p_{k+1}}{Q_{k+1}};$$
$$\frac{1}{Q_{k+1}}, \frac{2}{Q_{k+1}}, \ldots, \frac{p_{k+1}}{Q_{k+1}}; \ldots$$

counterclockwise. Thus the new endpoints, children of v, can be written as a union of p_{k+1} "rows" of cardinality n_{k+1}/p_{k+1}. The collection of the children of *all* vertices of generation k gives V_{k+1}. Continue by induction.

Denote the resulting tree by $\Gamma = \Gamma(\mathbf{p}, \mathbf{n})$.

Remark 3.4.1. *We will see that Γ has diameter no more than 2 (see 3.4.2 for the proof) and therefore to construct a domain for which it is the medial axis one needs to start with a disc D_0 of radius larger than 1. For each first generation vertex $v \in V_1$ consider the disc $D(v)$ centered at v such that $\partial D \cap \partial D_0 = (r^+(v) \cup r^-(v)) \cap \partial D_0$. Define $D_1 = D_0 \bigcup_{v \in V_1} D(v)$. For a $v \in V_2$ denote by $D(v)$ the disc centered at v such that $\partial D(v) \cap \partial D_2 = (r^+(v) \cup r^-(v)) \cap \partial D_2$. Then $D_2 = D_1 \cup_{v \in V_2} D(v)$. Continuing by induction we get a sequence of domains $D_0 \subset D_1 \subset \ldots \subset D_k \subset \ldots$ and get $D = \bigcup_{k=0}^{\infty} D_k$. According to the* *previous section* Γ *is the medial axis of* D *and* $C(D) = \overline{\Gamma}$.

To estimate the dimension of $\overline{\Gamma}$ we consider a special covering of $\widetilde{\Gamma}$ by circular sectors. To do that first note that if v is a vertex of generation k then all its descendants are contained in a subsector of the cone $\mathcal{C}(v)$. We are interested in the smallest such subsector. To find the radius of the latter we note that the children of v are at most p_{k+1}/Q_{k+1} away from v, the grand children are at most $\frac{p_{k+1}}{Q_{k+1}} + \frac{p_{k+2}}{Q_{k+2}}$ away and so on. Hence, we see that any descendant of v is at most $\sum_{i=k+1}^{\infty} p_i/Q_i$ away. Let us denote by $C(v)$ the circular sector centered at v of angle α_k and radius $\sum_{i=k+1}^{\infty} p_i/Q_i$. Then

$$\widetilde{\Gamma} = \bigcap_{i=1}^{\infty} \bigcup_{v \in V_i} C(v).$$

Indeed, if V_i is the collection of i-th generation vertices then $\widetilde{\Gamma} \subset \bigcup_{v \in V_i} C(v)$ for all i. Furthermore, if v' is a descendant of v then $C(v') \subset C(v)$. Therefore $\widetilde{\Gamma} \subset \bigcap_{i=1}^{\infty} \bigcup_{v \in V_j} C(v)$. On the other hand $\bigcap_{i=1}^{\infty} \bigcup_{v \in V_i} C(v) \subset \overline{\Gamma} \setminus \Gamma = \widetilde{\Gamma}$ by construction.

Before we continue let us calculate the radius of $C(v)$.

Lemma 3.4.2. *Let* $v \in V_k$ *and* l_k *denote the diameter (radius) of* $C(v)$. *Then*

$$l_k = \frac{1}{Q_k}. \tag{3.2}$$

Proof. First, note that

$$l_k = \lim_{n \to \infty} \sum_{i=k+1}^{n} \frac{p_i}{Q_i} = \frac{L_{k+1}}{Q_k}, \tag{3.3}$$

where $L_{k+1} := \lim_{n\to\infty}\left[\frac{p_{k+1}}{q_{k+1}} + \ldots + \frac{p_n}{q_{k+1}\cdots q_n}\right]$. We claim $L_k = 1$, for every k. Indeed, the general term of the sequence can be rewritten using the fact that $q_i = p_i + 1$ as follows

$$
\begin{aligned}
&\frac{p_k}{q_k} + \frac{1}{q_k}\frac{p_{k+1}}{q_{k+1}} + \ldots + \left(\frac{1}{q_k}\frac{1}{q_{k+1}}\cdots\frac{1}{q_{n-1}}\right)\frac{p_n}{q_n} \\
= \; &\frac{p_k}{q_k} + \left(1 - \frac{p_k}{q_k}\right)\frac{p_{k+1}}{q_{k+1}} + \ldots + \prod_{i=k}^{n-1}\left(1 - \frac{p_i}{q_i}\right)\frac{p_n}{q_n}.
\end{aligned}
\tag{3.4}
$$

Now, given a sequence of numbers $c_i < 1$, the following can be seen from geometric considerations

$$
\begin{aligned}
&c_k + (1 - c_k)c_{k+1} + \ldots + \prod_{i=k}^{n-1}(1 - c_i)c_n \\
= &1 - \prod_{i=1}^{n}(1 - c_i).
\end{aligned}
\tag{3.5}
$$

Applying this in our case we get

$$
L_k = \lim_{n\to\infty}\left[1 - \frac{1}{q_k\ldots q_n}\right] = 1, \tag{3.6}
$$

since $q_i = p_i + 1 > 2, \forall i$. $\square$

Now we are ready to calculate the Hausdorff dimension of $\widetilde{\Gamma}$. We will use the mass distribution principle. Define the probability measure μ on $\widetilde{\Gamma}$ by distributing it evenly among all the sectors of the same generation:

$$
\mu(C(v)) = \frac{1}{N_i}, \quad \forall v \in V_i.
$$

To have an estimate on μ it will be important to have an estimate on the

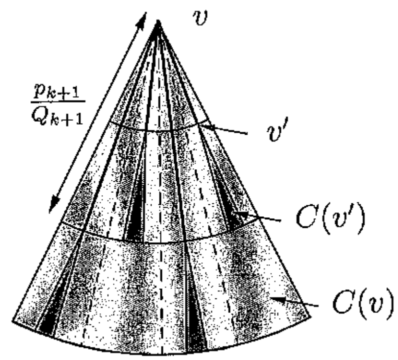


Figure 3.3: The covering of $\widetilde{\Gamma}$ by circular sectors.

number of sectors intersecting a ball B. For that reason we use the following notation. For a ball $B \subset \mathbb{R}^2$ and $i \in \mathbb{N}$ let $\nu_i(B)$ be the number of i-th generation sectors which have positive μ-mass when intersected with B, or

$$\nu_i(B) = \#\{C(v) \,:\, \mu(C(v) \cap B) > 0, v \in V_i\}. \tag{3.7}$$

Lemma 3.4.3. *With the notations as above*

$$\nu_m(B) \leq 2\frac{n_m}{p_m}\nu_{m-1}(B), \qquad \textit{if} \qquad |B| < l_m \tag{3.8}$$

$$\nu_m(B) < \frac{n_m}{p_m}\nu_{m-1}(B), \qquad \textit{if} \qquad |B| \leq l_{m+1}\cos(\alpha_m/2) \tag{3.9}$$

$$\nu_m(B) \leq 2\frac{n_m}{p_m}\frac{|B|}{r_{m-1}/q_m}, \qquad \textit{if} \qquad r_m \leq |B| < r_{m-1}, \tag{3.10}$$

where $|B|$ denotes the diameter of B and $r_m = l_m/N_m$ (i.e. up to a constant it's the length of the "base" of the sector $C(v)$ for $v \in V_m$).

Let us first prove our main theorem using this lemma.

Theorem 3.4.4. *If φ is a measure function s.t. $\phi(t) = \frac{\varphi(t)}{t^2} \xrightarrow{t\to 0} \infty$ then there are sequences $\boldsymbol{p} = \{p_i\}$ and $\boldsymbol{n} = \{n_i\}$ such that $H_\varphi(\overline{\Gamma(\boldsymbol{p}, \boldsymbol{n})}) > 0$.*

Proof. Let $h(t) = \min\left\{\frac{1}{t^\alpha}, \sqrt[3]{\phi\left(t^{\frac{1}{1+\alpha}}\right)}\right\}$ where $\alpha \in (0,1)$. Pick $p_1 \in \mathbb{N}$ and define for $k > 1$

$$q_{k+1} = \left[\frac{P_k}{Q_k} \cdot h\left(\frac{1}{Q_k}\right)\right] \tag{3.11}$$

where $[\cdot]$ denotes the integer part. Note that from (3.11) it follows that $q_{k+1} \leq Q_k^\alpha$ and therefore

$$Q_{k+1} \leq {Q_k}^{1+\alpha}.$$

Now, choose k and m so that

$$\begin{aligned} r_{k+1} \leq \ & |B| & < r_k, \\ l_{m+2} \leq \ & |B| & < l_{m+1}. \end{aligned}$$

Then by applying (3.8) once and $m - k - 3$ times (3.9) we obtain

$$\nu_m(B) \quad \leq 2 \prod_{i=k+2}^{m} \frac{n_i}{p_i} \cdot \nu_{k+1}(B) = 2\frac{N_m/N_{k+1}}{P_m/P_{k+1}}\nu_{k+1}(B).$$

Now, by (3.10)

$$\nu_m(B) \leq 4\frac{N_m/N_{k+1}}{P_m/P_{k+1}} \cdot \frac{|B|}{r_k/q_{k+1}}\frac{n_{k+1}}{p_{k+1}} = 4\frac{N_m/N_k}{P_m/P_k} \cdot \frac{|B|}{r_k}q_{k+1}.$$

Therefore

$$\begin{aligned} \mu(B) \leq \frac{\nu_m(B)}{N_m} &\lesssim \frac{1}{N_m}\frac{N_m/N_k}{P_m/P_k}\frac{|B|}{r_k}q_{k+1} \\ &= |B|\frac{P_k}{N_k P_m}Q_k N_k q_{k+1} \end{aligned}$$

(here and below $A \lesssim B$ stands for $A \leq C \cdot B$ where $C < \infty$ is an absolute constant).

Multiplying and dividing the right hand side by $|B|$ we get

$$\mu(B) \lesssim |B|^2 \frac{P_k Q_k Q_{m+2}}{P_m} q_{k+1} = |B|^2 P_k Q_k q_{k+1} q_{m+2} \frac{Q_{m+1}}{P_m},$$

since $|B|^{-1} \leq l_{m+2}^{-1} = Q_{m+2}$. Hence from (3.11) we obtain

$$\mu(B) \lesssim |B|^2 Q_k^2 q_{k+1} h\left(\frac{1}{Q_{m+1}}\right) h\left(\frac{1}{Q_m}\right),$$

and since h is decreasing

$$\mu(B) \lesssim |B|^2 Q_k^2 q_{k+1} h^2\left(\frac{1}{Q_{m+1}}\right).$$

By the remark in the beginning $Q_{m+1} \leq Q_m^{1+\alpha}$. Hence we obtain the following

$$\mu(B) \lesssim |B|^2 Q_k^2 Q_{k+1} h^2\left(\frac{1}{Q_m^{1+\alpha}}\right) \leq |B|^2 Q_k^2 Q_{k+1} (h(|B|^{1+\alpha}))^2$$

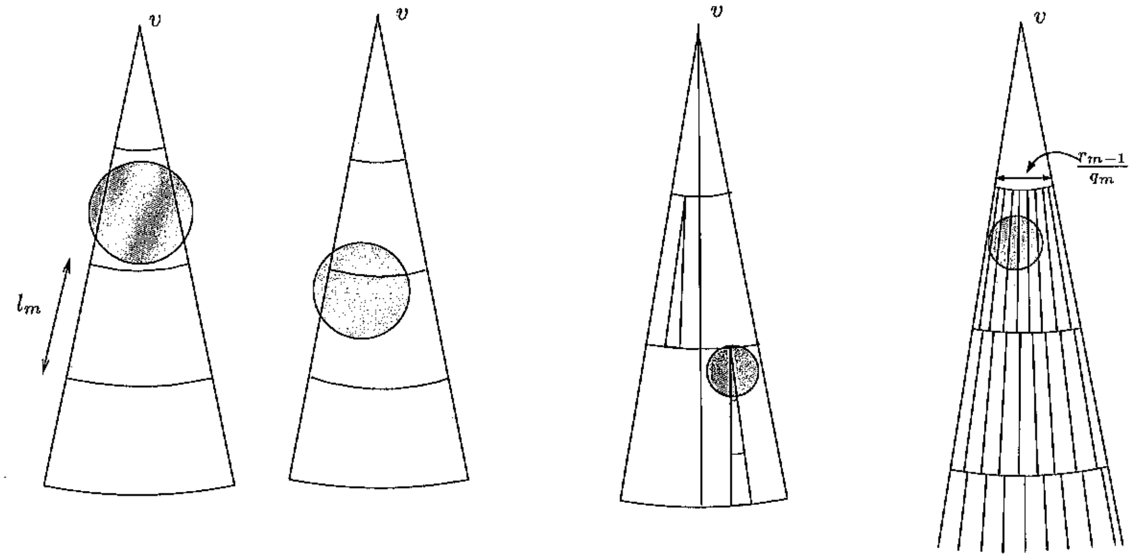
since $|B| < Q_m^{-1}$. Now choose N_k so that $Q_k^2 Q_{k+1} \leq h\left(\frac{1}{Q_k N_k}\right) \leq h(|B|)$. Then

$$\mu(B) \lesssim |B|^2 h(|B|) h^2(|B|^{1+\alpha}) \leq |B|^2 h^3(|B|^{1+\alpha}) \leq |B|^2 \phi(|B|),$$

according to our choice of h. This means there is a constant $C < \infty$ s.t. $\mu(B) \leq C\varphi(|B|)$ for every ball $B \subset \mathbb{R}^2$. By mass distribution principle $H_\varphi(\overline{\Gamma}) > 0$. □

Proof of Lemma 3.4.3. Suppose $v \in V_{m-1}$. First, recall that $C(v)$ can be decomposed into "strips" of the form $C(v) \cap \{x : \frac{i}{Q_m} \leq |x-v| \leq \frac{i+1}{Q_m}\}$, $(0 \leq i \leq p_{k+1})$, and thickness l_m, each of which contains exactly n_m/p_m sectors of generation m.

Proof of (3.8). There are two possibilities (see fig. 3.4(a)) depending on



(a) (b) (c)

Figure 3.4: Estimating the dimension.

wether there is an $i \in \{1, \ldots, p_m\}$ for which

$$B \cap \{x : |x-v| = i \cdot l_m\}$$

is empty or not (B is completely contained in some strip). Since $|B| < l_m$ it can intersect at most two such "strips". This means

$$\nu_{m+1}(B) \leq 2\frac{n_{m+1}}{p_{m+1}}\nu_m(B).$$

Proof of (3.9). Again there are two cases. If $B \cap \{x : |x - v| = i \cdot l_m\} = \emptyset$ for every $i \in \{1, \ldots, p_{m+1}\}$ then the inequality is clear.

Now, suppose $\exists i$ such that $B \cap \{x : |x - v| = il_m\} \neq \emptyset$. Then B doesn't intersect any $m + 1$ generation sector which is contained in $C(v) \setminus B(v, il_m)$. Indeed, for every such sector C' we have by construction

$$dist(C', \{x : |x - v| = i \cdot l_m\}) > l_{m+1} \cos \alpha_m > |B|,$$

(see fig.3.4(b)). In particular $\mu(B \cap C') = 0$. Hence B can get positive measure only from the sectors in the i-th "row". This means that for every $v \in V_{m-1}$ there are at most n_m/p_m subsectors of $C(v)$ of generation m which have positive μ measure when intersected with B. Therefore

$$\nu_m(B) \leq \frac{n_m}{p_m} \nu_{m-1}(B).$$

Proof of (3.10) Start from $v \in V_{m-1}$. For m large we may assume without loss of generality $r_{m-1} << l_{m+1}$. First consider B such that

$$\frac{r_{m-1}}{q_m} \leq |B| \leq r_{m-1} < l_{m+1} \cos(\alpha_m/2).$$

Then by (3.9)

$$\nu_m(B) \leq \frac{n_m}{p_m} \nu_{m-1}(B) \leq \frac{n_m}{p_m} \frac{|B|}{r_{m-1}/q_m} \nu_{m-1}(B).$$

On the other hand, if $r_m \le |B| \le r_{m-1} l_m$ then

$$\nu_m(B) \le \frac{n_m}{p_m} \frac{2|B|}{r_{m-1}/q_m} \nu_{m-1}(B)$$

since distance between any two adjacent children of $C(v)$ which are in a strip $\{x : il_m \le |x - v| \le (i+1)l_m\}$ is the same and there are at most $\frac{n_m}{p_m}$ of them (see fig.3.4(c), where the lines crossing the grey ball are in fact generation m sectors with very small angle α_m). It follows easily from our construction that $\nu_{m-1}(B) \le 1$ for $|B| < r_{m-1}$. Hence we obtain the desired inequality. □

Remark 3.4.5. *It is not hard to see that the constructed domain has a Lipschitz boundary provided the bending angles are decaying fast enough. So it is natural to ask the following question.*

Question 1. Is there a domain with smooth (C^1, C^k, or C^∞) boundary and central set of Hausdorff dimension 2?

For the disc trees we considered $C(D) \setminus M(D)$ is totally disconnected.

Question 2. Is there a domain $D \subset \mathbb{C}$ such that $C(D) \setminus M(D)$ is connected and $\dim_H(C(D)) = 2$?

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