Toric Surfaces and Sasakian-Einstein 5-Manifolds

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Craig van Coevering

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Abstract of the Dissertation

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We consider toric surfaces X with an orbifold structure such that the anti-canonical line V-bundle $\mathbf{K}^{-1}$ is positive which admit a certain involution. Such a toric variety X with its orbifold structure is called a *symmetric toric Fano surface.* It is described by a convex polyhedron with integral vertices in the plane which is invariant under the antipodal map. Using the theory of multiplier ideal sheaves of A. Nadel [54, 55] we show that the appropriate Monge-Ampère equation is solvable, so X admits an orbifold Kähler Einstein metric of positive scalar curvature. By [14] the total space of a Seifert S^1-bundle on X has a Sasakian-Einstein structure. We obtain examples of smooth toric Sasakian-Einstein 5-manifolds with every

odd second Betti number. Certain divisors in the twistor space of toric anti-self-dual Einstein orbifolds $\mathcal{M}$ of positive scalar curvature (cf. [22]) are toric surfaces of the above type. The associated Sasakian-Einstein space is smooth if the 3-Sasakian orbifold associated to $\mathcal{M}$(cf. [16, 12, 13, 18]) is smooth. Thus associated to every toric 3-Sasakian manifold is a Sasakian-Einstein 5-manifold. Using the quaternionic/3-Sasakian reduction procedure as in [18] one constructs infinitely many toric 3-Sasakian manifolds providing us with a machine producing infinitely many smooth examples of toric Sasakian-Einstein 5-manifolds. All the examples constructed are diffeomorphic to $\#k(S^2 \times S^3)$ and we produce infinitely many examples for every odd $k > 1$. Furthermore, this produces Einstein metrics with infinitely many Einstein constants on each $\#k(S^2 \times S^3)$, for $1 > 0$ odd. An interesting aspect of these examples is that they are submanifolds of 3-Sasakian 7-manifolds. Furthermore, these examples are non-homogeneous Einstein manifolds of positive scalar curvature which are spin and admit real Killing spinors. See [8] for a definition of Killing spinors and [1],[53] for their relevance to physics. Toric Sasakian-Einstein manifolds have been of interest in physics very recently as examples which can be used to test the AdS/CFT correspondence [30, 51, 52]. This work expands the list of examples of toric Sasakian-Einstein manifolds to include examples of arbitrarily high second Betti number.

To my parents

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Chapter 1

Introduction

This work started with an observation in [69] concerning toric anti-self-dual Einstein orbifolds of positive scalar curvature (see [22]). These are anti-self-dual Einstein orbifolds $(\mathcal{M}, g)$ with a torus T^2 of isometries. If $\mathcal{M}$ is such an orbifold, then its twistor space $\mathcal{Z}$ is a Kähler-Einstein orbifold of positive scalar curvature with an effective divisor X_t corresponding to each nonzero $t \in \mathfrak{t} \otimes \mathbb{C}$, where $\mathfrak{t}$ is the Lie algebra of T^2. For generic t, X_t is an irreducible toric surface. Furthermore, X is Fano, that is the anti-canonical V-bundle is positive $\mathbf{K}_X^{-1} > 0$. And it is embedded as a suborbifold, meaning that X only has singularities coming from those of $\mathcal{Z}$. The analytic structure of the toric surface is described by a fan Δ in $\mathbb{Q} \times \mathbb{Q}$ which is symmetric across the origin, $-\Delta = \Delta$. This corresponds to a holomorphic involution $\beta : X \to X$ which acts on the anti-canonical cycle of curves $\cup_i D_i$ which are the compliment of the algebraic torus $T_{\mathbb{C}} = \mathbb{C}^* \times \mathbb{C}^* \subset X$. And β acts on $\cup_i D_i$ without fixed points or fixing any of the curves D_i. Such a surface is called a *symmetric toric Fano surface.*

The orbifold structure on X will be crucial. A complex orbifold has a canonical analytic structure, but the orbifold structure is much stronger. It is possible to have many different orbifold structures on the same analytic space, and the existence of certain metrics such as Kähler-Einstein metrics depends on the orbifold structure. The orbifold structure of X will be described by a convex polyhedron Δ^* in $\mathbb{Q} \times \mathbb{Q}$ with integral vertices. The fan Δ consists of the rays through the vertices of Δ^* and the 2-cones spanned by these. Again, Δ^* is symmetric under the antipodal map. There is a simple relation between the orbit structure of $\mathcal{M}$ under T^2 as described in [22] using methods as in [58] or [37] and Δ^*. In chapter 2 the relevant toric geometry is discussed, and we describe the relationships between symmetric toric Fano surfaces and anti-self-dual Einstein orbifolds.

Using the technique of multiplier ideal sheaves, as in [54, 55] and [24], in chapter 3 we show that a symmetric toric Fano surface X admits a Kähler-Einstein metric with positive scalar curvature. That is, X admits a Kähler form ω so that

$$\mathrm{Ricci}(\omega) = \lambda\omega, \text{ with } \lambda > 0$$

where $\mathrm{Ricci}(\omega)$ is the *Ricci form* of ω which in a local coordinate patch is $\mathrm{Ricci}(\omega) = \frac{i}{2\pi}\bar{\partial}\partial \log \det(\omega_{k\bar{j}})$. The basic argument is as follows.

A Kähler-Einstein metric is precisely a solution to the Monge-Ampère equation

$$(\omega_0 + \frac{i}{2\pi}\partial\bar{\partial}\phi_t)^2 = \omega_0^2 e^{-t\phi_t+f}, \quad t \in [0,1], \tag{1.1}$$

for $t = 1$ where $\omega_0 \in c_1(X)$ is a Käher form and $f \in C^\infty(X)$ is given by $\mathrm{Ricci}(\omega_0) = \omega_0 + \frac{i}{2\pi}\partial\bar{\partial}f$. The Kähler-Einstein metric is then given by $\omega = \omega_0 + \frac{i}{2\pi}\partial\bar{\partial}\phi_1$. It is well known that there exists an $\epsilon \in (0,1]$ so that (1.1) is solvable for $t \in [0, \epsilon)$. To complete the continuity argument one must show that the subset of $t \in [0, 1]$ for which (1.1) has a solution is closed. For this it is enough to have an *a priori* C^0 estimate on solutions to (1.1). (cf. [4, 71])

We have $\beta \in \mathcal{N}(T_{\mathbb{C}}) \subseteq \mathrm{Aut}(X)$, where $\mathcal{N}(T_{\mathbb{C}})$ is the normalizer of $T_{\mathbb{C}}$ in $\mathrm{Aut}(X)$. Let $G \subset \mathcal{N}(T_{\mathbb{C}})$ be a compact subgroup containing T^2 and β. Recall that $\mathcal{W}(\Delta) := \mathcal{N}(T_{\mathbb{C}})/T_{\mathbb{C}}$ is the finite group of automorphisms of the fan Δ. Let $\mathcal{W}_0 \subseteq \mathcal{W}(\Delta)$ be the subgroup preserving the polyhedron. So we may take G to be the maximal compact subgroup of $\mathcal{N}(T_{\mathbb{C}})$ which is generated by T^2 and $\mathcal{W}_0$. In the above discussion take ω_0 and $f \in C^\infty$ to be G invariant. A solution to equation (1.1) for $t < 1$ is unique, therefore must be G-invariant. Suppose that X does not admit a Kähler-Einstein metric. So equation (1.1) has no solution for $t = 1$, and the C^0 *a priori* estimate fails to hold. There exits an increasing sequence $\{t_k\} \subset (0, 1)$ and a sequence $\{\phi_k\}$ of smooth G-invariant functions such that ϕ_k is a solution to (1.1) with $t = t_k$ and with $\sup |\phi_k| \to \infty$ as $k \to \infty$. In chapter 3 we define a coherent sheaf of ideals $\mathcal{J}$ in $\mathcal{O}_X$ from the sequence $\{\phi_k\}$, the *multiplier ideal sheaf.* By construction $\mathcal{J}$ is G-invariant. And as is shown in [55] the sheaf $\mathcal{J}$ satisfies certain conditions which include:

i. $\mathcal{J}$ is proper, that is $\mathcal{J}$ is equal to neither the zero sheaf nor all of $\mathcal{O}$.

ii. For all $i > 0$, $H^i(X, \mathcal{J}) = 0$.

Denote by $V \subset X$ the subscheme, or the possibly non-reduced complex analytic subspace, determined by $\mathcal{J}$ We have $0 \leq \dim V < 2$. And a corollary to ii. is

$$H^i(V, \mathcal{O}_V) = 0 \text{ for } i > 0 \text{ and } H^0(V, \mathcal{O}_V) = \mathbb{C}. \tag{1.2}$$

From these properties it is easy to see that such a sheaf does not exist on X. Therefore (1.1) must have a solution for $t = 1$.

In chapter 4 the basics of Sasakian geometry are presented. Given a Kähler-Einstein orbifold with positive scalar curvature we construct as in [14] a Sasakian-Einstein metric on an S^1 V-bundle over X. The V-bundle is some power of the S^1 bundle associated to $\mathbf{K}_X^{-1}$. Taking the maximum root of $\mathbf{K}_X^{-1}$ gives a simply connected Sasakian-Einstein orbifold M. The condition for smoothness of M is that the local uniformizing groups of the orbifold structure on X inject into the bundle group over each uniformizing neighborhood on X. In particular, for each symmetric toric Fano surface X we have a Sasakian-Einstein orbifold M. Results of S. Smale [63] on the classification of smooth 5-manifolds imply that M is diffeomorphic to $\#k(S^2 \times S^3)$ if it is smooth and simply connected.

Applying this construction to the twistor space $\mathcal{Z}$ of the anti-self-dual Einstein orbifold $\mathcal{M}$ gives the associated 3-Sasakian space $\mathcal{S}$.(see [12] and [16]) Associated to any toric anti-self-dual Einstein orbifold $\mathcal{M}$ with $\pi_1^{orb}(\mathcal{M}) = e$, we have the following inclusions and fibre maps.

$$\begin{array}{ccc} M & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{Z} \\ & & \downarrow \\ & & \mathcal{M} \end{array} \tag{1.3}$$

The horizontal arrows are inclusions. By the adjunction formula we have $\mathbf{K}_X^{-1} = \mathbf{K}_{\mathcal{Z}}^{-\frac{1}{2}}|_X = \mathbf{L}|_X$, where $\mathbf{L}$ is the V-bundle associated with the contact structure on $\mathcal{Z}$. It follows that if $\mathcal{S}$ is smooth, then so is M. This remark and the work of C. Boyer, K. Galicki, *et al* [18] provides us with a tool for producing smooth Sasakian-Einstein 5-manifolds. One can easily produce examples of toric anti-self-dual Einstein orbifolds with positive scalar curvature by taking quaternionic Kähler quotients of $\mathbb{H}P^{m+1}$ by a torus $T^m \subset Sp(m+2)$. In [18] the condition on the $m \times m+2$ weight matrix Ω is determined for which the associated 3-Sasakian space $\mathcal{S}_\Omega$ is smooth. Examples of toric 3-Sasakian 7-manifolds are constructed with every second Betti number. We make use of a computation of the integral cohomology of these 3-Sasakian manifolds due to R. Hepworth [40]. The group $G_\Omega = H^4(\mathcal{S}_\Omega, \mathbb{Z})$ is finite, and one can make its order arbitrarily high for weight matrices satisfying the smoothness condition. Thus for each second Betti number there are infinitely many distinct 3-Sasakian 7-manifolds. We have the following:

Theorem 1.1 *Associated to every simply connected toric 3-Sasakian 7-manifold $\mathcal{S}$ is a toric Sasakian-Einstein 5-manifold M with $\pi_1(M) = e$. If $b_2(\mathcal{S}) = k$, then $b_2(M) = 2k+1$ and*

$$M \underset{diff}{\cong} \#m(S^2 \times S^3), \text{ where } m = b_2(M).$$

In particular, there exist toric Sasakian-Einstein 5-manifolds of every possible odd second Betti number.

The results of theorem (1.1) as pictured in diagram (1.3) give an invertible correspondence. That is, given either M or X, one can recover $\mathcal{M}$ and the other spaces in (1.3). This makes use of results of D. Calderbank and M. Singer [22] on the classification of toric anti-self-dual Einstein orbifolds.

The volume of a Kähler toric variety is the volume of the associated polytope. One can make use of this to determine the Einstein constants of the examples in theorem (1.1).

Theorem 1.2 *For each odd $m > 1$ there is a countably infinite number of Einstein metrics on $M = \#m(S^2 \times S^3)$ constructed in theorem (1.1). If g_i is the sequence of Einstein metrics normalized so that $\mathrm{Vol}_{g_i}(M) = 1$, then we have $\mathrm{Ric}_{g_i} = \lambda_i g_i$ with the Einstein constants $\lambda_i \to 0$ as $i \to \infty$.*

The restriction of k to be odd in the theorems is merely a limitation of the technique used. And some examples of Sasakian-Einstein metrics on $\#k(S^2 \times S^3)$ for k even can be produced by this method. Sasakian-Einstein structure are known to exist on $\#k(S^2 \times S^3)$ for all $k \geq 1$. Examples with $k = 1, \ldots, 9$ were produced by C. Boyer, K. Galicki, and M. Nakamaye [15, 19, 20]. More recently J. Kollár constructed families of Sasakian-Einstein metrics in all of the cases $k \geq 6$ [45] The Sasakian-Einstein structures produced here are distinguished from the above examples by the presence of a T^2 automorphism group.

In chapter 5 the basics on 3-Sasakian manifolds and related geometries are covered. In particular we review 3-Sasakian reduction and some results of C. Boyer and K. Galicki and others in [18]. We also cover results on anti-self-dual Einstein orbifolds and twistor spaces which are needed in completing the picture given in diagram (1.3). Then chapter 6 contains the main theorems on the new toric Sasakian-Einstein manifolds. In chapter 7 we give a more detailed description of some examples, starting with the simplest. There is an appendix giving some results on orbifolds that are needed. This includes a overview of orbifold cohomology and characteristic classes and a summary of the results of A. Haefliger and E. Salem on toric orbifolds [37].

Chapter 2

Symmetric toric Fano surfaces

We give some basic definitions in the theory of toric varieties that we will need. See [27, 56, 57] for more details. In addition we will consider the notion of a compatible orbifold structure on a toric variety and holomorphic V-bundles. We are interested in Kähler toric orbifolds, and will give a description of the Kähler structure due to V. Guillemin [34].

2.1 Toric varieties

Let $N \cong \mathbb{Z}^r$ be the free $\mathbb{Z}$-module of rank r and $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual. We denote $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$ and $M_{\mathbb{Q}} = M \otimes \mathbb{Q}$ with the natural pairing

$$\langle\ ,\ \rangle : M_{\mathbb{Q}} \times N_{\mathbb{Q}} \to \mathbb{Q}.$$

Similarly we denote $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

Let $T_{\mathbb{C}} := N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong \mathbb{C}^* \times \cdots \times \mathbb{C}^*$ be the algebraic torus. Each $m \in M$ defines a character $\chi^m : T_{\mathbb{C}} \to \mathbb{C}^*$ and each $n \in N$ defines a one-parameter subgroup $\lambda_n : \mathbb{C}^* \to T_{\mathbb{C}}$. In fact, this gives an isomorphism between M (resp.

N) and the multiplicative group $\mathrm{Hom}_{\mathrm{alg.}}(T_{\mathbb{C}}, \mathbb{C}^*)$ (resp. $\mathrm{Hom}_{\mathrm{alg.}}(\mathbb{C}^*, T_{\mathbb{C}})$).

Definition 2.1 *A subset σ of $N_{\mathbb{R}}$ is a* strongly convex rational polyhedral cone *if there are $n_1, \ldots, n_r$ so that*

$$\sigma = \mathbb{R}_{\geq 0} n_1 + \cdots + \mathbb{R}_{\geq 0} n_r,$$

and one has $\sigma \cap -\sigma = \{o\}$, where $o \in N$ is the origin.

The dimension $\dim \sigma$ is the dimension of the $\mathbb{R}$-subspace $\sigma + (-\sigma)$ of $N_{\mathbb{R}}$. The dual cone to σ is

$$\sigma^{\vee} = \{x \in M_{\mathbb{R}} : \langle x, y \rangle \geq 0 \text{ for all } y \in \sigma\},$$

which is also a convex rational polyhedral cone. A subset τ of σ is a face, $\tau < \sigma$, if

$$\tau = \sigma \cap m^{\perp} = \{y \in \sigma : \langle m, y \rangle = 0\} \text{ for } m \in \sigma^{\vee}.$$

And τ is a strongly convex rational polyhedral cone.

Definition 2.2 *A* fan *in N is a collection Δ of strongly convex rational polyhedral cones such that:*

i. For $\sigma \in \Delta$ every face of σ is contained in Δ.

ii. For any $\sigma, \tau \in \Delta$, the intersection $\sigma \cap \tau$ is a face of both σ and τ.

We will consider *complete* fans for which the support $\bigcup_{\sigma \in \Delta} \sigma$ is $N_{\mathbb{R}}$. We will denote

$$\Delta(i) := \{\sigma \in \Delta : \dim \sigma = i\}, \quad 0 \leq i \leq n.$$

Definition 2.3 *A fan in N is* nonsingular *if each $\sigma \in \Delta(r)$ is generate by r elements of N which can be completed to a $\mathbb{Z}$-basis of N. A fan in N is* simplicial *if each $\sigma \in \Delta(r)$ is generated by r elements of N which can be completed to a $\mathbb{Q}$-basis of $N_{\mathbb{Q}}$.*

If σ is a strongly convex rational polyhedral cone, $S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup. We denote by $\mathbb{C}[S_\sigma]$ the semigroup algebra. We will denote the generators of $\mathbb{C}[S_\sigma]$ by x^m for $m \in S_\sigma$. Then $U_\sigma := \operatorname{Spec}\mathbb{C}[S_\sigma]$ is a normal affine variety on which $T_{\mathbb{C}}$ acts algebraically with a (Zariski) open orbit isomorphic to $T_{\mathbb{C}}$. If σ is nonsingular, then $U_\sigma \cong \mathbb{C}^n$.

Theorem 2.4 ([27, 56, 57]) *For a fan Δ in N the affine varieties U_σ for $\sigma \in \Delta$ glue together to form an irreducible normal algebraic variety*

$$X_\Delta = \bigcup_{\sigma\in\Delta} U_\sigma.$$

Furthermore, X_Δ is non-singular if, and only if, Δ is nonsingular. And X_Δ is compact if, and only if, Δ is complete.

Proposition 2.5 *The variety X_Δ has an algebraic action of $T_{\mathbb{C}}$ with the following properties.*

i. To each $\sigma \in \Delta(i), 0 \leq i \leq n$, there corresponds a unique $(n-i)$-dimensional $T_{\mathbb{C}}$-orbit $\operatorname{Orb}(\sigma)$ *so that X_Δ decomposes into the disjoint union*

$$X_\Delta = \bigcup_{\sigma\in\Delta} \operatorname{Orb}(\sigma),$$

where $\operatorname{Orb}(o)$ *is the unique n-dimensional orbit and is isomorphic to $T_{\mathbb{C}}$.*

ii. The closure $V(\sigma)$ of $\mathrm{Orb}(\sigma)$ in X_δ is an irreducible $(n-i)$-dimensional $T_{\mathbb{C}}$-stable subvariety and

$$V(\sigma) = \bigcup_{\tau \geq \sigma} \mathrm{Orb}(\tau).$$

We will consider toric varieties with an orbifold structure.

Definition 2.6 *We will denote by Δ^* an* augmented fan *by which we mean a fan Δ with elements $n(\rho) \in N \cap \rho$ for every $\rho \in \Delta(1)$.*

Proposition 2.7 *For a complete simplicial augmented fan Δ^* we have a natural orbifold structure compatible with the action of $T_{\mathbb{C}}$ on X_Δ. We denote X_Δ with this orbifold structure by X_{Δ^*}.*

Proof. Let $\sigma \in \Delta^*(n)$ have generators $p_1, p_2, \ldots, p_n$ as in the definition. Let $N' \subseteq N$ be the sublattice $N' = \mathbb{Z}\{p_1, p_2, \ldots, p_n\}$, and σ' the equivalent cone in N'. Denote by M' the dual lattice of N' and $T'_{\mathbb{C}}$ the torus. Then $U_{\sigma'} \cong \mathbb{C}^n$. It is easy to see that

$$N/N' = \mathrm{Hom}_{\mathbb{Z}}(M'/M, \mathbb{C}^*).$$

And N/N' is the kernel of the homomorphism

$$T'_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{Z}}(M', \mathbb{C}^*) \to T_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*).$$

Let $\Gamma = N/N'$. An element $t \in \Gamma$ is a homomorphism $t : M' \to \mathbb{C}^*$ equal to 1 on M. The regular functions on $U_{\sigma'}$ consist of $\mathbb{C}$-linear combinations of x^m for $m \in \sigma'^\vee \cap M'$. And $t \cdot x^m = t(m)x^m$. Thus the invariant functions are the $\mathbb{C}$-linear combinations of x^m for $m \in \sigma^\vee \cap M$, the regular functions of U_σ. Thus $U_{\sigma'}/\Gamma = U_\sigma$. And the charts are easily seen to be compatible on intersections. □

Proposition 2.8 *Let Δ be a complete simplicial fan. Suppose for simplicity that the local uniformizing groups are abelian. Then every orbifold structure on X_Δ compatible with the action of $T_{\mathbb{C}}$ arises from an augmented fan Δ^*.*

Proof. Notice that the points with non-trivial stablizer groups are contained in $X\backslash T^n_{\mathbb{C}} = \cup_{i=1}^d D_i$. Let $\phi : \tilde{U} \to U$ be a local uniformizing chart with group Γ. We may assume that $\tilde{U} \subseteq \mathbb{C}^n$ is a neighborhood of $o \in \mathbb{C}^n$ and $\Gamma \subset GL(n, \mathbb{C})$. Let $H < \Gamma$ be the subgroup generated by $g \in \Gamma$ with $\operatorname{rank}(g - Id) = 1$. Then by results of [60], $\mathbb{C}^n/H$ is analytically isomorphic to $\mathbb{C}^n$, and we have a local uniformizing chart $\psi : \tilde{U}/H \to U$. Let $\sigma \in \Delta(n)$ be such that $\phi(o) \in U_\sigma$. Let $p_1, p_2, \ldots, p_n$ be primitive elements of N generating σ over $\mathbb{Q}$. Let $N' = \mathbb{Z}\{p_1, p_2, \ldots, p_n\}$, and let $\pi : U_{\sigma'} \cong \mathbb{C}^n \to U_\sigma$ be the uniformizing chart with group N/N' as above. By a result in [60] there is an injection $\tilde{U}/H \to U_{\sigma'}$ intertwining the group actions. Therefore, we may assume $\tilde{U}/H = U_{\sigma'}$ and $\psi = \pi$. A generator $g \in H$, i.e. $\operatorname{rank}(g - Id) = 1$, must fix a coordinate plane $\{x_i = 0\} \subset \mathbb{C}^n$, and $g^* x_i = e^{i\theta} x_i$. Thus $H = \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_n}$. Define $\tau : \mathbb{C}^n \to \mathbb{C}^n$ by $\tau(z_1, \ldots, z_n) = (z_1^{a_1}, \ldots, z_n^{a_n})$. Then we have a lifting of group actions

$$0 \to H \to \tilde{T}^n \xrightarrow{\tau_*} T^n \to 1,$$

and $\Gamma = \tau_*^{-1}(N/N')$. Thus ϕ is $\pi \circ \tau$, the latter easily seen to be canonical chart obtained by taking $\mathbb{Q}$-generators $a_1 p_1, a_2 p_2, \ldots, a_n p_n$. □

Let Δ^* be an augmented fan in N. We will assumed from now on that the fan Δ is simplicial and complete.

Definition 2.9 *A real function $h : N_{\mathbb{R}} \to \mathbb{R}$ is a Δ^** -linear support function *if for each $\sigma \in \Delta^*$ with given $\mathbb{Q}$-generators $p_1, \ldots, p_r$ in N, there is an $l_\sigma \in M_{\mathbb{Q}}$ with $h(s) = \langle l_\sigma, s\rangle$ and l_σ is $\mathbb{Z}$-valued on the sublattice $\mathbb{Z}\{p_1, \ldots, p_r\}$. And we require that $\langle l_\sigma, s\rangle = \langle l_\tau, s\rangle$ whenever $s \in \sigma \cap \tau$. The additive group of Δ^*-linear support functions will be denoted by* $\mathrm{SF}(\Delta^*)$.

Note that $h \in \mathrm{SF}(\Delta^*)$ is completely determined by the integers $h(n(\rho))$ for all $\rho \in \Delta(1)$. And conversely, an assignment of an integer to $h(n(\rho))$ for all $\rho \in \Delta(1)$ defines h. Thus

$$\mathrm{SF}(\Delta^*) \cong \mathbb{Z}^{\Delta(1)}.$$

Definition 2.10 *Let Δ^* be a complete augmented fan. For $h \in \mathrm{SF}(\Delta^*)$,*

$$\Sigma_h := \{m \in M_{\mathbb{R}} : \langle m, n\rangle \geq h(n), \text{ for all } n \in N_{\mathbb{R}}\},$$

is a, possibly empty, convex polytope in $M_{\mathbb{R}}$.

We will consider the holomorphic line V-bundles on $X = X_{\Delta*}$. All V-bundles will be *proper* in this chapter. See appendix A.1 for a definition of proper and other basics of V-bundles. The set of isomorphism classes of holomorphic line V-bundles is denoted by $\mathrm{Pic}^{\mathrm{orb}}(X)$, which is a group under the tensor product.

Definition 2.11 *A* Baily divisor *is a $\mathbb{Q}$-Weil divisor $D \in$* $\mathrm{Weil}(X) \otimes \mathbb{Q}$ *whose inverse image $D_{\tilde{U}} \in$* $\mathrm{Weil}(\tilde{U})$ *in every local uniformizing chart $\pi : \tilde{U} \to U$ is Cartier. The additive group of Baily divisors is denoted* $\mathrm{Div}^{orb}(X)$.

A Baily divisor D defines a holomorphic line V-bundle $[D] \in \mathrm{Pic}^{\mathrm{orb}}(X)$ in a way completely analogous to Cartier divisors. Given a nonzero meromorphic function $f \in \mathscr{M}$ we have the *principal divisor*

$$\mathrm{div}(f) := \sum_V \nu_V(f)V,$$

where $\nu_V(f)V$ is the order of the zero, or negative the order of the pole, of f along each irreducible subvariety of codimension one. We have the exact sequence

$$1 \to \mathbb{C}^* \to \mathscr{M}^* \to \mathrm{Div}^{\mathrm{orb}}(X) \xrightarrow{[\,]} \mathrm{Pic}^{\mathrm{orb}}(X). \tag{2.1}$$

A holomorphic line V-bundle $\pi : \mathbf{L} \to X$ is equivariant if there is an action of $T_{\mathbb{C}}$ on $\mathbf{L}$ such that π is equivariant, $\pi(tw) = t\pi(w)$ for $w \in \mathbf{L}$ and $t \in T_{\mathbb{C}}$ and the action lifts to a holomorphic action, linear on the fibers, over each uniformizing neighborhood. The group of isomorphism classes of equivariant holomorphic line V-bundles is denoted $\mathrm{Pic}^{\mathrm{orb}}{}_{T_{\mathbb{C}}}(X)$. Similarly, we have invariant Baily divisors, denoted $\mathrm{Div}^{\mathrm{orb}}{}_{T_{\mathbb{C}}}(X)$, and $[D] \in \mathrm{Pic}^{\mathrm{orb}}{}_{T_{\mathbb{C}}}(X)$ whenever $D \in \mathrm{Div}^{\mathrm{orb}}{}_{T_{\mathbb{C}}}(X)$.

Proposition 2.12 *Let $X = X_{\Delta^*}$ be compact with the standard orbifold structure, i.e. Δ^* is simplicial and complete.*

i. There is an isomorphism $\mathrm{SF}(\Delta^*) \cong \mathrm{Div}^{orb}{}_{T_{\mathbb{C}}}(X)$ *obtained by sending $h \in$* $\mathrm{SF}(\Delta^*)$ *to*

$$D_h := -\sum_{\rho\in\Delta(1)} h(n(\rho))V(\rho).$$

ii. There is a natural homomorphism $\mathrm{SF}(\Delta^*) \to \mathrm{Pic}^{orb}{}_{T_{\mathbb{C}}}(X)$ *which asso-*

ciates an equivariant line V-bundle $\mathbf{L}_h$ to each $h \in \mathrm{SF}(\Delta^*)$.

iii. Suppose $h \in \mathrm{SF}(\Delta^*)$ and $m \in M$ satisfies

$$\langle m, n \rangle \geq h(n) \text{ for all } n \in N_{\mathbb{R}},$$

then m defines a section $\psi : X \to \mathbf{L}_h$ which has the equivariance property $\psi(tx) = \chi^m(t)(t\psi(x))$.

iv. The set of sections $H^0(X, \mathcal{O}(\mathbf{L}_h))$ is the finite dimensional $\mathbb{C}$-vector space with basis $\{x^m : m \in \Sigma_h \cap M\}$.

v. Every Baily divisor is linearly equivalent to a $T_{\mathbb{C}}$-invariant Baily divisor. Thus for $D \in \mathrm{Pic}^{orb}(X)$, $[D] \cong [D_h]$ for some $h \in \mathrm{SF}(\Delta^*)$.

vi. If $\mathbf{L}$ is any holomorphic line V-bundle, then $\mathbf{L} \cong \mathbf{L}_h$ for some $h \in \mathrm{SF}(\Delta^*)$. The homomorphism in part i. induces an isomorphism $\mathrm{SF}(\Delta^*) \cong {\mathrm{Pic}^{orb}}_{T_{\mathbb{C}}}(X)$ and we have the exact sequence

$$0 \to M \to \mathrm{SF}(\Delta^*) \to \mathrm{Pic}^{orb}(X) \to 1.$$

Proof. i. For each $\sigma \in \Delta(n)$ with uniformizing neighborhood $\pi : U_{\sigma'} \to U_\sigma$ as above the map $h \to D_h$ assigns the principal divisor

$$\mathrm{div}(x^{-l_\sigma}) = - \sum_{\rho \in \Delta(1), \rho < \sigma} h(n(\rho)) V'(\rho),$$

where $V'(\rho)$ is the closure of the orbit $\mathrm{Orb}(\rho)$ in $U_{\sigma'}$. An element ${\mathrm{Div}^{\mathrm{orb}}}_{T_{\mathbb{C}}}(X)$ must be a sum of closures of codimension one orbits $V(\rho)$ in proposition (2.5), and by above remarks the map is an isomorphism.

ii. One defines $\mathbf{L}_h := [D_h]$, where $[D_h]$ is constructed as follows. Consider a uniformizing chart $\pi : U_{\sigma'} \to U_\sigma$ as in proposition (2.7). Define $\mathbf{L}_h|_{U_{\sigma'}}$ to be the invertible sheaf $\mathcal{O}_{U_{\sigma'}}(D_h)$, with D_h defined on $U_{\sigma'}$ by x^{-l_σ}. So $\mathbf{L}_h|_{U_{\sigma'}} \cong U_{\sigma'} \times \mathbb{C}$ with an action of $T'_{\mathbb{C}}$,

$$t(x,v) = (tx, \chi^{-l_\sigma}(t)v) \text{ where } t \in T'_{\mathbb{C}},\ (x,v) \in U_{\sigma'} \times \mathbb{C}.$$

Then $\mathbf{L}_h|_{U_\sigma}$ is the quotient by the subgroup $N/N' \subset T'_{\mathbb{C}}$, so it has an action of $T_{\mathbb{C}}$. And the $\mathbf{L}_h|_{U_\sigma}$ glue together equivariantly with respect to the action.

iii. For $\sigma \in \Delta$ we have $\langle m, n\rangle \geq \langle l_\sigma, n\rangle$ for all $n \in \sigma$. Then $m - l_\sigma \in M' \cap \sigma'^\vee$ and x^{m-l_σ} is a section of the invertible sheaf $\mathcal{O}_{U_{\sigma'}}(D_h)$ and is equivariant with respect to N/N' so it defines a section of $\mathbf{L}_h|_{U_\sigma}$. And these sections are compatible.

iv. We will make use of the GAGA theorems of A. Grothendieck [32, 33]. As with any holomorphic V-bundle, the sheaf of sections $\mathcal{O}(\mathbf{L}_h)$ is a coherent sheaf. It follows from GAGA that we may consider $\mathcal{O}(\mathbf{L}_h)$ as a coherent algebraic sheaf, and all global sections are algebraic. If ϕ is a global section, then $\phi \in H^0(T_{\mathbb{C}}, \mathcal{O}(\mathbf{L}_h)) \subset \mathbb{C}[M]$. And in the uniformizing chart $\pi : U_{\sigma'} \to U_\sigma$, ϕ lifts to an element of the module $\mathcal{O}_{U_{\sigma'}} \cdot x^{l_\sigma}$ which has a basis $\{x^m : m \in l_\sigma + M' \cap \sigma'^\vee\}$. So $\phi|_{U_\sigma}$ is a $\mathbb{C}$-linear combination of x^m with $m \in M$ and $\langle m, n\rangle \geq h(n)$ for all $n \in \sigma$. Thus $m \in \Sigma_h$.

v. The divisor $T_{\mathbb{C}} \cap D$ is a Cartier divisor on $T_{\mathbb{C}}$ which is also principal since $\mathbb{C}[M]$ is a unique factorization domain. Thus there is a nonzero rational function f so that $D' = D - \mathrm{div}(f)$ satisfies $D' \cap T_{\mathbb{C}} = \emptyset$. Then $D' \in \mathrm{Div^{orb}}_{T_{\mathbb{C}}}(X)$, and the result follows from i.

vi. Consider $\mathbf{L}_{U_{\sigma'}}$ on a uniformizing neighborhood $U_{\sigma'}$ as above. For each $\rho \in \Delta(1), \rho < \sigma$ the subgroup $H_\rho \subseteq N/N'$ fixing $V'(\rho)$ is cyclic and generated by $n' \in N$ where n' is the primitive element with $a_\rho n' = n(\rho)$. Now H_ρ acts linearly on the fibers of $\mathbf{L}_{U_{\sigma'}}$ over $V'(\rho)$. Suppose n' acts with weight $e^{2\pi i \frac{k}{a}}$, then let $D_\rho := kV(\rho)$. If $D' := \sum_{\rho \in \Delta(1)} D_\rho$, then $\mathbf{L}' := \mathbf{L} \otimes [-D']$ is Cartier on $X_0 := X \setminus \mathrm{Sing}(X)$, where $\mathrm{Sing}(X)$ has codimension at least two. The sheaf $\mathcal{O}(\mathbf{L}')$ is not only coherent but is a rank-1 reflexive sheaf. By GAGA $\mathcal{O}(\mathbf{L}') \cong E \otimes \mathcal{O}'$, where E is an algebraic reflexive rank-1 sheaf and $\mathcal{O}'$ is the sheaf of analytic functions. It is well known that $E = \mathcal{O}(D)$ for $D \in \mathrm{Weil}(X)$. And as a Baily divisor, we have $\mathbf{L}' \cong [D]$. So $\mathbf{L} \cong [D + D']$, and by v. we have $\mathbf{L} \cong \mathbf{L}_h$ for some $h \in \mathrm{SF}(\Delta^*)$. □

The sign convention in the proposition is adopted to make subsequent discussions involving Σ_h consistent with the existing literature, although having $D_{-m} = \mathrm{div}(x^m)$ maybe bothersome. Note also that we denote a Baily divisor by a formal $\mathbb{Z}$-linear sum the coefficient giving the multiplicity of the irreducible component in the *uniformizing chart*. This is different from its expression as a Weil divisor when irreducible components are contained in codimension-1 components of the singular set of the orbifold.

For $X = X_{\Delta^*}$ there is a unique $k \in \mathrm{SF}(\Delta^*)$ such that $k(n(\rho)) = 1$ for all $\rho \in \Delta(1)$. The corresponding Baily divisor

$$D_k := -\sum_{\rho \in \Delta(1)} V(\rho) \tag{2.2}$$

is the *(orbifold) canonical divisor.* The corresponding V-bundle is $\mathbf{K}_X$, the V-bundle of holomorphic n-forms. This will in general be different from the canonical sheaf in the algebraic geometric sense.

Definition 2.13 *Consider support functions as above but which are only required to be* $\mathbb{Q}$*-valued on* $N_{\mathbb{Q}}$*, denoted* $\mathrm{SF}(\Delta, \mathbb{Q})$. h *is* strictly upper convex *if* $h(n+n') \geq h(n) + h(n')$ *for all* $n, n' \in N_{\mathbb{Q}}$ *and for any two* $\sigma, \sigma' \in \Delta(n)$*,* l_σ *and* $l_{\sigma'}$ *are different linear functions.*

Given a strictly upper convex support function h, the polytope Σ_h is the convex hull in $M_{\mathbb{R}}$ of the vertices $\{l_\sigma : \sigma \in \Delta(n)\}$. Each $\rho \in \Delta(1)$ defines a facet by

$$\langle m, n(\rho) \rangle \geq h(n(\rho)).$$

If $n(\rho) = a_\rho n'$ with $n' \in N$ primitive and $a_\rho \in \mathbb{Z}^+$ we may label the face with a_ρ to get the labeled polytope Σ_h^* which encodes the orbifold structure. Conversely, from a rational convex polytope Σ^* we associate a fan Δ^* and a support function h as follows. For an l-dimensional face $\theta \subset \Sigma^*$, define the rational n-dimensional cone $\sigma^\vee(\theta) \subset M_{\mathbb{R}}$ consisting of all vectors $\lambda(p - p')$, where $\lambda \in \mathbb{R}_{\geq 0}, p \in \Sigma$, and $p' \in \theta$. Then $\sigma(\theta) \subset N_{\mathbb{R}}$ is the $(n-l)$-dimensional cone dual to $\sigma^\vee(\theta)$. The set of all $\sigma(\theta)$ defines the complete fan Δ^*, where one assigns $n(\rho)$ to $\rho \in \Delta(1)$ if $n(\rho) = an'$ with n' primitive and a is the label on the corresponding $(n-1)$-dimensional face of Σ^*. The corresponding rational support function is then

$$h(n) = \inf\{\langle m, n \rangle : m \in \Sigma^*\} \text{ for } n \in N_{\mathbb{R}}.$$

Proposition 2.14 **([57, 27])** *There is a one-to-one correspondence between the set of pairs* (Δ^*, h) *with* $h \in \mathrm{SF}(\Delta, \mathbb{Q})$ *strictly upper convex, and rational convex marked polytopes* Σ_h^*.

We will be interested in toric orbifolds X_{Δ^*} with such a support function and polytope, Σ_h^*. More precisely will will be concerned with the following.

Definition 2.15 *Let* $X = X_{\Delta^*}$ *be a compact toric orbifold. We say that* X *is* Fano *if* $-k \in \mathrm{SF}(\Delta^*)$, *which defines the anti-canonical* V*-bundle* $\mathbf{K}_X^{-1}$, *is strictly upper convex.*

These toric variety aren't necessarily Fano in the usual sense, since $\mathbf{K}_X^{-1}$ is the *orbifold* anti-canonical class. This condition is equivalent to $\{n \in N_{\mathbb{R}} : k(n) \leq 1\} \subset N_{\mathbb{R}}$ being a convex polytope with vertices $n(\rho)$, $\rho \in \Delta(1)$. We will use Δ^* to denote both the augmented fan and this polytope in this case.

If $\mathbf{L}_h$ is a line V-bundle, then for certain $s > 0$, $\mathbf{L}_h^s \cong \mathbf{L}_{sh}$ will be a holomorphic line bundle. For example $s = \mathrm{Ord}(X)$, the least common multiple of the orders of the uniformizing groups, will do. So suppose $\mathbf{L}_h$ is a holomorphic line bundle. If the global holomorphic sections generate $\mathbf{L}_h$, by proposition (2.12) $M \cap \Sigma_h = \{m_0, m_1, \ldots, m_r\}$ and we have a holomorphic map $\psi_h : X \to \mathbb{C}P^r$ where

$$\psi_h(w) := [x^{m_0}(w) : x^{m_1}(w) : \cdots : x^{m_r}(w)]. \tag{2.3}$$

Proposition 2.16 **([57])** *Suppose* $\mathbf{L}_h$ *is a line bundle, so* $h \in \mathrm{SF}(\Delta^*)$ *is integral, and suppose* h *is strictly upper convex. Then* $\mathbf{L}_h$ *is* ample, *meaning that for large enough* $\nu > 0$

$$\psi_{\nu h} : X \to \mathbb{C}P^N,$$

is an embedding, where $M \cap \Sigma_{\nu h} = \{m_0, m_1, \ldots, m_N\}$.

Corollary 2.17 *Let* X *be a Fano toric orbifold. If* $\nu > 0$ *is sufficiently large with* $-\nu k$ *integral,* $\mathrm{K}^{-\nu}$ *will be very ample and* $\psi_{-\nu k} : X \to \mathbb{C}P^N$ *an embedding.*

2.2 Kähler structures

We review the construction of toric Kähler metrics on toric varieties. Any compact toric orbifold associated to a polytope admits a Kähler metric (see [49]). Due to T. Delzant [23] and E. Lerman and S. Tolman [49] in the orbifold case, the symplectic structure is uniquely determined up to symplectomorphism by the polytope, which is the image of the moment map. This polytope is Σ_h^* of the previous section with h generalized to be real valued. There are infinitely many Kähler structures on a toric orbifold with fixed polytope Σ_h^*, but there is a distinguished Kähler metric obtained by reduction. V. Guillemin gave an explicit formula [34, 21] for this Kähler metric. In particular, we show that every toric Fano orbifold admits a Kähler metric $\omega \in c_1(X)$.

Let Σ^* be a convex polytope in $M_{\mathbb{R}} \cong \mathbb{R}^{n*}$ defined by the inequalities

$$\langle x, u_i \rangle \geq \lambda_i, \quad i = 1, \ldots, d, \tag{2.4}$$

where $u_i \in N \subset N_{\mathbb{R}} \cong \mathbb{R}^n$ and $\lambda_i \in \mathbb{R}$. If Σ_h^* is associated to (Δ^*, h), then the u_i and λ_i are the set of pairs $n(\rho)$ and $h(n(\rho))$ for $\rho \in \Delta(1)$. We allow the λ_i to be real but require any set $u_{i_1}, \ldots, u_{i_n}$ corresponding to a vertex to form a $\mathbb{Q}$-basis of $N_{\mathbb{Q}}$.

Let $(e_1, \ldots, e_d)$ be the standard basis of $\mathbb{R}^d$ and $\beta : \mathbb{R}^d \to \mathbb{R}^n$ be the map which takes e_i to u_i. Let $\mathfrak{n}$ be the kernel of β, so we have the exact sequence

$$0 \to \mathfrak{n} \xrightarrow{\iota} \mathbb{R}^d \xrightarrow{\beta} \mathbb{R}^n \to 0, \tag{2.5}$$

and the dual exact sequence

$$0 \to \mathbb{R}^{n*} \xrightarrow{\beta^*} \mathbb{R}^{d*} \xrightarrow{\iota^*} \mathfrak{n}^* \to 0. \tag{2.6}$$

Since (2.5) induces an exact sequence of lattices, we have an exact sequence

$$1 \to N \to T^d \to T^n \to 1, \tag{2.7}$$

where the connected component of the identity of N is an $(d-n)$-dimensional torus. The standard representation of T^d on $\mathbb{C}^d$ preserves the Kähler form

$$\frac{i}{2}\sum_{k=1}^{d} dz_k \wedge d\bar{z}_k, \tag{2.8}$$

and is Hamiltonian with moment map

$$\mu(z) = \frac{1}{2}\sum_{k=1}^{d} |z_k|^2 e_k + c, \tag{2.9}$$

unique up to a constant c. We will set $c = \sum_{k=1}^{d} \lambda_k e_k$. Restricting to $\mathfrak{n}^*$ we get the moment map for the action of N on $\mathbb{C}^d$

$$\mu_N(z) = \frac{1}{2}\sum_{k=1}^{d} |z_k|^2 \alpha_k + \lambda, \tag{2.10}$$

with $\alpha_k = \iota^* e_k$ and $\lambda = \sum \lambda_k \alpha_k$. Let $Z = \mu_N^{-1}(0)$ be the zero set. By the exactness of (2.6) $z \in \mu_N^{-1}(0)$ if an only if there is a $v \in \mathbb{R}^{n*}$ with $\mu(z) = \beta^* v$. Since β^* is injective, we have a map

$$\nu : Z \to \mathbb{R}^{n*}, \tag{2.11}$$

where $\beta^* \nu(z) = \mu(z)$ for all $z \in Z$. For $z \in Z$

$$\begin{aligned} \langle \nu(z), u_i \rangle &= \langle \beta^* \nu(z), e_i \rangle \\ &= \langle \mu(z), e_i \rangle \\ &= \frac{1}{2}|z_i|^2 + \lambda_i, \end{aligned} \tag{2.12}$$

thus $\nu(z) \in \Sigma^*$. Conversely, if $v \in \Sigma^*$, then $v = \nu(z)$ for some $z \in Z$ and in fact a T^d orbit in Z. Thus Z is compact. The following is not difficult to show.

Theorem 2.18 *The action of N on Z is locally free. Thus the quotient*

$$X_{\Sigma^*} = Z/N$$

is a compact orbifold. Let

$$\pi : Z \to X$$

be the projection and

$$\iota : Z \to \mathbb{C}^d$$

the inclusion. Then X_{Σ^*} has a canonical Kähler structure with Kähler form ω

uniquely defined by

$$\pi^*\omega = \iota^*(\frac{i}{2}\sum_{k=1}^{d} dz_k \wedge d\bar{z}_k).$$

The canonical Kähler metric in the theorem is call the *Guillemin metric.*

We have an action of $T^n = T^d/N$ on X_{Σ^*} which is Hamiltonian for ω. The map ν is T^d invariant, and it descends to a map, which we also call ν,

$$\nu : X_{\Sigma^*} \to \mathbb{R}^{n*}, \tag{2.13}$$

which is the moment map for this action. The above comments show that $\mathrm{Im}(\nu) = \Sigma^*$. The action T^n extends to the complex torus $T^n_{\mathbb{C}}$ and one can show that as an analytic variety and orbifold X_{Σ^*} is the toric variety constructed from Σ^* in the previous section. See [35] for more details.

Let $\sigma : \mathbb{C}^d \to \mathbb{C}^d$ be the involution $\sigma(z) = \bar{z}$. The set Z is stable under σ, and σ descends to an involution on X. We denote the fixed point sets by Z_r and X_r. And we have the projection

$$\pi : Z_r \to X_r. \tag{2.14}$$

We equip Z_r and X_r with Riemannian metrics by restricting the Kähler metrics on $\mathbb{C}^d$ and X respectively.

Proposition 2.19 *The map (2.14) is a locally finite covering and is an isometry with respect to these metrics*

Note that Z_r is a subset of $\mathbb{R}^d$ defined by

$$\frac{1}{2}\sum_{k=1}^{d} x_k^2 \alpha_k = -\lambda. \tag{2.15}$$

Restrict to the orthant $x_k > 0$ $k = 1, \ldots, d$ of $\mathbb{R}^d$. Let Z_r' be the component of Z_r in this orthant. Under the coordinates

$$s_k = \frac{x_k^2}{2}, \quad k = 1, \ldots, d. \tag{2.16}$$

The flat metric on $\mathbb{R}^d$ becomes

$$\frac{1}{2}\sum_{k=1}^{d} \frac{(ds_k)^2}{s_k}. \tag{2.17}$$

Consider the moment map ν restricted to Z_r'. The above arguments show that ν maps Z_r' diffeomorphicly onto the interior Σ° of Σ. In particular we have

$$\langle \nu(x), u_k \rangle = \lambda_k + s_k, \ k = 1, \ldots, d, \ \text{for } x \in Z_r'. \tag{2.18}$$

Let $l_k : \mathbb{R}^{n*} \to \mathbb{R}$ be the affine function

$$l_k(x) = \langle x, u_k \rangle - \lambda_k, \quad k = 1, \ldots, d.$$

Then by equation (2.18) we have

$$l_k \circ \nu = s_k. \tag{2.19}$$

Thus the moment map ν pulls back the metric

$$\frac{1}{2}\sum_{k=1}^{d}\frac{(dl_k)^2}{l_k}, \tag{2.20}$$

on Σ° to the metric (2.17) on Z'_r. We obtain the following.

Proposition 2.20 *The moment map $\nu : X'_r \to \Sigma^\circ$ is an isometry when Σ° is given the metric (2.20).*

Let $W \subset X$ be the orbit of $T^n_{\mathbb{C}}$ isomorphic to $T^n_{\mathbb{C}}$. Then by restriction W has a T^n-invariant Kähler form ω. Identify $T^n_{\mathbb{C}} = \mathbb{C}^n/2\pi i\mathbb{Z}^n$, so there is an inclusion $\iota : \mathbb{R}^n \to T^n_{\mathbb{C}}$.

Proposition 2.21 *Let ω be a T^n-invariant Kähler form on W. Then the action of T^n is Hamiltonian if and only if ω has a T^n-invariant potential function, that is, a function $F \in C^\infty(\mathbb{R}^n)$ such that*

$$\omega = 2i\partial\bar{\partial}F.$$

Proof. Suppose the action is Hamiltonian. Any T^n-orbit is Lagrangian, so ω restricts to zero. The inclusion $T^n \subset T^n_{\mathbb{C}}$ is a homotopy equivalence. Thus ω is exact. Let γ be a T^n-invariant 1-form with $\omega = d\gamma$. Let $\gamma = \beta + \bar{\beta}$ where $\beta \in \Omega^{0,1}$. Then

$$\omega = d\gamma = \partial\beta + \bar{\partial}\bar{\beta},$$

since $\bar{\partial}\beta = \partial\bar{\beta} = 0$. Since $H^{0,k}(W)_{T^n} = 0$ for $k > 0$, there exists a T^n-invariant function f with $\beta = \bar{\partial} f$. Then

$$\omega = \partial\bar{\partial} f + \bar{\partial}\partial\bar{f} = 2i\partial\bar{\partial}\,\mathrm{Im}\, f.$$

The converse is a standard result. □

Suppose the T^n action on W is Hamiltonian with moment map $\nu : W \to \mathbb{R}^{n*}$. Denote by $x + iy$ the coordinates given by the identification $W = \mathbb{C}^n / 2\pi i \mathbb{Z}^n$.

Proposition 2.22 **([34])** *Up to a constant* ν *is the Legendre transform of* F, *i.e.*

$$\nu(x+iy) = \frac{\partial F}{\partial x} + c, \quad c \in \mathbb{R}^{n*}$$

Proof. By definition

$$d\nu_k = -\iota\left(\frac{\partial}{\partial y_k}\right)\omega.$$

But by proposition (2.21),

$$\omega = \sum_{j,k=1}^{n} \frac{\partial^2 F}{\partial x_j \partial x_k} dx_j \wedge dy_k,$$

so

$$d\nu_k = -\iota\left(\frac{\partial}{\partial y_k}\right)\omega = d\left(\frac{\partial F}{\partial x_k}\right).$$

Therefore $\nu_k = \frac{\partial F}{\partial x_k} + c_k$. □

We can eliminate c by replacing F with $F - \sum_{k=1}^{n} c_k x_k$.

Notice that the metric (2.20) on Σ° can be written

$$\sum_{j,k} \frac{\partial^2 G}{\partial y_j \partial y_k} dy_j dy_k, \tag{2.21}$$

with

$$G = \frac{1}{2} \sum_{k=1}^{d} l_k(y) \log l_k(y). \tag{2.22}$$

V. Guillemin [34] showed that the Legendre transform of G is the inverse Legendre transform of F, i.e.

$$\frac{\partial F}{\partial x} = y \text{ and } \frac{\partial G}{\partial y} = x. \tag{2.23}$$

From this it follows that

$$F(x) = \sum_{i=1}^{n} x_i y_i - G(y), \text{ where } y = \frac{\partial F}{\partial x}. \tag{2.24}$$

Define

$$l_\infty(x) = \sum_{i=1}^{d} \langle x, u_i \rangle.$$

From equations (2.22) and (2.24) it follows that F has the expression

$$F = \frac{1}{2} \nu^* \left(\sum_{k=1}^{d} \lambda_k \log l_k + l_\infty \right), \tag{2.25}$$

which gives us the following.

Theorem 2.23 ([34, 21]) *On the open $T^n_{\mathbb{C}}$ orbit of X_{Σ^*} the Kähler form ω*

is given by

$$i\partial\bar{\partial}\nu^* \left(\sum_{k=1}^{d} \lambda_k \log l_k + l_\infty \right).$$

Suppose we have an embedding as in proposition (2.16),

$$\psi_h : X_{\Sigma^*} \to \mathbb{C}P^N.$$

So Σ_h is an integral polytope and $M \cap \Sigma_h = \{m_0, m_1, \ldots, m_N\}$. Let ω_{FS} be the Fubini-Study metric on $\mathbb{C}P^N$. Note that $\psi_h^* \omega_{FS}$ is degenerate along the singular set of X, so does not define a Kähler form.

Consider the restriction of ψ_h to the open $T^n_{\mathbb{C}}$ orbit $W \subset X$. Let $\iota = \psi_h|_W$. It is induced by a representation

$$\tau : T^n_{\mathbb{C}} \to GL(N+1, \mathbb{C}), \tag{2.26}$$

with weights $m_0, m_1, \ldots, m_N$. If $z = x + iy \in \mathbb{C}^n / 2\pi i \mathbb{Z}^n = T^n_{\mathbb{C}}$, and $w = (w_0, \ldots, w_N)$, then

$$\tau(\exp z) w = (e^{\langle m_0, x+iy \rangle} w_0, \ldots, e^{\langle m_N, x+iy \rangle} w_N). \tag{2.27}$$

Recall the Fubini-Study metric is

$$\omega_{FS} = i\partial\bar{\partial} \log |w|^2. \tag{2.28}$$

Let $[w_0 : \cdots : w_N]$ be homogeneous coordinates of a point in the image of W, then

$$\iota^*\omega_{FS} = i\partial\bar{\partial}\bigg(\sum_{k=0}^{N}|w_k|^2 e^{2\langle m_k,x\rangle}\bigg). \tag{2.29}$$

From equation (2.22) we have

$$x = \frac{\partial G}{\partial y} = \frac{1}{2}\bigg(\sum_{j=1}^{d} u_j \log l_j + u\bigg),$$

where $u = \sum u_j$. Then

$$2\langle m_i, x\rangle = 2\langle m_i, \frac{\partial G}{\partial y}\rangle = \sum_{j=1}^{d}\langle m_i, u_j\rangle \log l_j + \langle m_i, u\rangle.$$

So setting $d_i = e^{\langle m_i, u\rangle}$, gives

$$e^{2\langle m_i,x\rangle} = \nu^*\bigg(d_i \prod_{j=1}^{d} l_j^{\langle m_i,u_j\rangle}\bigg).$$

But from (2.25),

$$e^{2F} = \nu^*\bigg(e^{l_\infty}\prod_{j=1}^{d} l_j^{\lambda_j}\bigg).$$

Combining these,

$$e^{2\langle m_i,x\rangle} = e^{2F}\nu^*\bigg(d_i e^{-l_\infty}\prod_{j=1}^{d} l_j^{l_j(m_i)}\bigg).$$

Let $k_i = |w_i|^2 d_i$, then summing gives

$$\sum_{i=1}^{N}|w_i|^2 e^{2\langle m_i,x\rangle} = e^{2F}\nu^*(e^{-l_\infty}Q),$$

where

$$Q = \sum_{i=1}^{N} k_i \prod_{j=1}^{d} l_j^{l_j(m_i)}.$$

Thus we have

$$\psi_h^* \omega_{FS} = \omega + i\partial\bar{\partial}\nu^*(-l_\infty + \log Q). \tag{2.30}$$

Using that Σ_h is integral, and $k_i \neq 0$ for m_i a vertex of Σ_h, it is not difficult to show that Q is a positive function on Σ_h. Thus equation (2.30) is valid on all of X.

Theorem 2.24 *Suppose $\mathbf{L}_h$ is very ample for some $h \in \mathrm{SF}(\Delta^*)$ strictly upper convex and integral, and let ω be the Guillemin metric for the polytope Σ_h. Then*

$$[\omega] = 2\pi c_1(\mathbf{L}) = [\psi_h^* \omega_{FS}].$$

Corollary 2.25 *Suppose $X = X_{\Delta^*}$ is Fano. Let ω be the Guillemin metric of the integral polytope Σ_{-k}^*. Then*

$$[\omega] = 2\pi c_1(\mathbf{K}^{-1}) = 2\pi c_1(X).$$

Thus $c_1(X) > 0$. Conversely, if $c_1(X) > 0$, then $\mathbf{K}^{-p}$ is very ample for some $p > 0$ and X is Fano as defined in definition (2.15).

Proof. For some $p \in \mathbb{Z}^+$, $-pk \in \mathrm{SF}(\Delta^*)$ is integral and $\mathbf{L}_{-pk} = \mathbf{K}^{-p}$ is very ample. Let $\tilde{\omega}$ be the Guillemin metric of the integral polytope Σ_{-pk}^*. From the theorem we have

$$[\tilde{\omega}] = 2\pi c_1(\mathbf{K}^{-p}) = 2\pi p c_1(X).$$

Let ω be the Guillemin metric for Σ_{-k}^*. Theorem (2.23) implies that $[\tilde{\omega}] = p[\omega]$

For the converse, It follows from the extension to orbifolds of the Kodaira embedding theorem of W. Baily [6] that $\mathbf{K}^{-p}$ is very ample for some $p > 0$ sufficiently large. It follows from standard results on toric varieties that $-k$ is strictly upper convex (see [57]). □

The next result will have interesting applications to the Einstein manifolds constructed later.

Proposition 2.26 *With the Guillemin metric the volume of X_{Σ^*} is $(2\pi)^n$ times the Euclidean volume of Σ.*

Proof. Let $W \subset X$ be the open $T^n_{\mathbb{C}}$ orbit. We identify W with $\mathbb{C}^n/2\pi i\mathbb{Z}^n$ with coordinates $x + iy$. The restriction of ω to W is

$$\omega|_W = \sum_{j,k=1}^{n} \frac{\partial^2 F}{\partial x_j \partial x_k} dx_j \wedge dy_k.$$

Thus

$$\frac{\omega^n}{n!} = \det\left(\frac{\partial^2 F}{\partial x_j \partial x_k}\right) dx \wedge dy.$$

Integrating over dy gives

$$\mathrm{Vol}(X, \omega) = (2\pi)^n \int_{\mathbb{R}^n} \det\left(\frac{\partial^2 F}{\partial x_j \partial x_k}\right) dx.$$

By proposition (2.22) $x \to z = \nu(x + iy) = \frac{\partial F}{\partial x}$ is a diffeomorphism from $\mathbb{R}^n$ to Σ°. By the change of variables,

$$\mathrm{Vol}(\Sigma) = \int_\Sigma dz = \int_{\mathbb{R}^n} \det\left(\frac{\partial^2 F}{\partial x_j \partial x_k}\right) dx.$$

□

Corollary 2.27 *Let $X = X_{\Delta^*}$ be a toric Fano orbifold. And let ω be any Kähler form with $\omega \in c_1(X)$. Then*

$$\mathrm{Vol}(X, \omega) = \frac{1}{n!} c_1(X)^n [X] = \mathrm{Vol}(\Sigma_{-k}).$$

Proof. Let ω_G be the Guillemin metric associated to Σ^*_{-k}, then $\frac{1}{2\pi}\omega_G \in c_1(X)$ by corollary (2.25). Then

$$\mathrm{Vol}(X, \omega) = \frac{1}{(2\pi)^n} \mathrm{Vol}(X, \omega_G) = \mathrm{Vol}(\Sigma_{-k}).$$

□

2.3 Symmetric toric orbifolds

Let X_Δ be an n-dimensional toric variety. Let $\mathcal{N}(T_{\mathbb{C}}) \subset \mathrm{Aut}(X)$ be the normalizer of $T_{\mathbb{C}}$. Then $\mathcal{W}(X) := \mathcal{N}(T_{\mathbb{C}})/T_{\mathbb{C}}$ is isomorphic to the finite group of all symmetries of Δ, i.e. the subgroup of $GL(n, \mathbb{Z})$ of all $\gamma \in GL(n, \mathbb{Z})$ with $\gamma(\Delta) = \Delta$. Then we have the exact sequence.

$$1 \to T_{\mathbb{C}} \to \mathcal{N}(T_{\mathbb{C}}) \to \mathcal{W}(X) \to 1. \tag{2.31}$$

Choosing a point $x \in X$ in the open orbit, defines an inclusion $T_{\mathbb{C}} \subset X$. This also provides a splitting of (2.31). Let $\mathcal{W}_0(X) \subseteq \mathcal{W}(X)$ be the subgroup which are also automorphisms of Δ^*; $\gamma \in \mathcal{W}_0(X)$ is an element of $\mathcal{N}(T_{\mathbb{C}}) \subset \mathrm{Aut}(X)$ which preserves the orbifold structure. Let $G \subset \mathcal{N}(T_{\mathbb{C}})$ be the compact subgroup generated by T^n, the maximal compact subgroup of $T_{\mathbb{C}}$, and $\mathcal{W}_0(X)$. Then we have the, split, exact sequence

$$1 \to T^n \to G \to \mathcal{W}_0(X) \to 1. \tag{2.32}$$

Definition 2.28 *A* symmetric Fano toric orbifold *X is a Fano toric orbifold with $\mathcal{W}_0$ acting on N with the origin as the only fixed point. Such a variety and its orbifold structure is characterized by the convex polytope Δ^* invariant under $\mathcal{W}_0$. We call a toric orbifold* special symmetric *if $\mathcal{W}_0(X)$ contains the involution $\sigma : N \to N$, where $\sigma(n) = -n$.*

Conversely, given an integral convex polytope Δ^*, inducing a simplicial fan Δ, invariant under a subgroup $\mathcal{W}_0 \subset GL(n, \mathbb{Z})$ fixing only the origin, we have a symmetric Fano toric orbifold X_{Δ^*}.

Definition 2.29 *The* index *of a Fano orbifold X is the largest positive integer m such that there is a holomorphic V-bundle $\mathbf{L}$ with $\mathbf{L}^m \cong \mathbf{K}_X^{-1}$. The index of X is denoted* $\mathrm{Ind}(X)$.

Note that $c_1(X) \in H^2_{orb}(X, \mathbb{Z})$, and $\mathrm{Ind}(X)$ is the greatest positive integer m such that $\frac{1}{m}c_1(X) \in H^2_{orb}(X, \mathbb{Z})$.

Proposition 2.30 *Let X_{Δ^*} be a special symmetric toric Fano orbifold. Then* $\mathrm{Ind}(X) = 1$ *or* 2.

Proof. We have $\mathbf{K}^{-1} \cong \mathbf{L}_{-k}$ with $-k \in \mathrm{SF}(\Delta^*)$ where $-k(n_\rho) = -1$ for all $\rho \in \Delta(1)$. Suppose we have $\mathbf{L}^m \cong \mathbf{K}^{-1}$. By proposition (2.12) there is an

$h \in \mathrm{SF}(\Delta^*)$ and $f \in M$ so that $mh = -k + f$. For some $\rho \in \Delta(1)$,

$$mh(n_\rho) = -1 + f(n_\rho)$$
$$mh(-n_\rho) = -1 - f(n_\rho).$$

Thus $m(h(n_\rho) + h(-n_\rho)) = -2$, and $m = 1$ or 2. □

We will now restrict to dimension two, symmetric toric Fano surfaces. In the smooth case every Fano surface, called a *del Pezzo surface*, is either $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2$ blown up at r points in general position $0 \leq r \leq 8$. The smooth toric Fano surfaces are $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}P^2$, the Hirzebruch surface F_1, the equivariant blow up of $\mathbb{C}P^2$ at two $T_{\mathbb{C}}$-fixed points, and the equivariant blow up of $\mathbb{C}P^2$ at three $T_{\mathbb{C}}$-fixed points. There are only three examples of smooth symmetric toric Fano surfaces, which are $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}P^2$, and the equivariant blow up of $\mathbb{C}P^2$ at three $T_{\mathbb{C}}$-fixed points. The problem of the existence of Kähler-Einstein metrics on smooth Fano surfaces is completely solved by G. Tian and S.T. Yau [67, 68]. Such a surface admits a Kähler-Einstein metric if the Lie algebra of holomorphic vector fields is reductive. These are the cases $\mathbb{C}P^1 \times \mathbb{C}P^1$, $\mathbb{C}P^2$, and $\mathbb{C}P^2$ blown up at $3 \leq r \leq 8$ points in general position. The smooth toric Fano surfaces admitting a Kähler-Einstein metric are precisely the symmetric cases. In the next chapter we will prove that all symmetric toric Fano orbifold surfaces admit Kähler-Einstein metrics. The surfaces we consider are, strictly speaking, log del Pezzo surfaces, since we require the *orbifold* anti-canonical bundle to be ample. Note that by the work of S. Bando and T. Mabuchi [7] we may suppose that the Kähler-Einstein

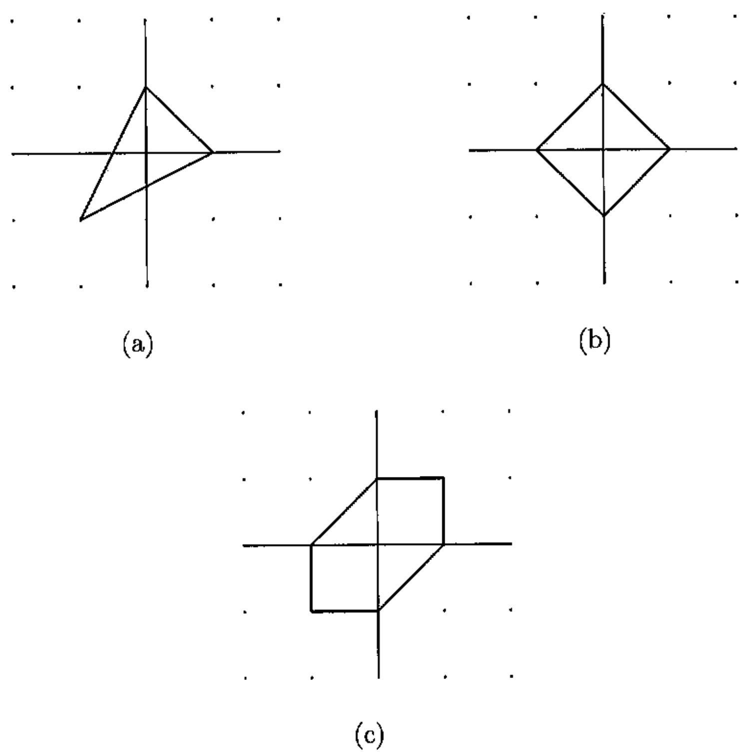


Figure 2.1: The three smooth examples

metric is $T^2 \rtimes W_0$ invariant. In this case the number of examples is infinite, including examples of every even second Betti number. This is because it is elementary to construct antipodally symmetric examples.

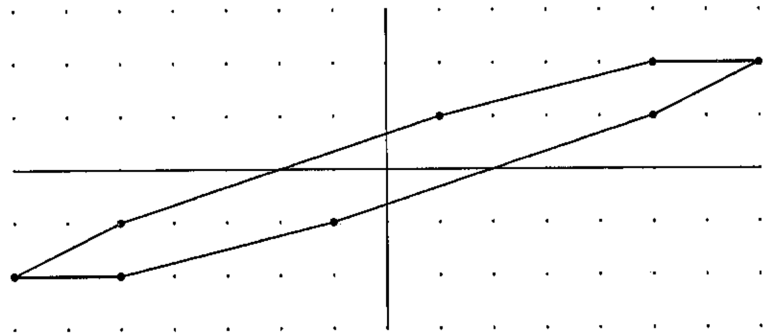


Figure 2.2: Example with 8 point singular set and $\mathcal{W}_0 = \mathbb{Z}_2$

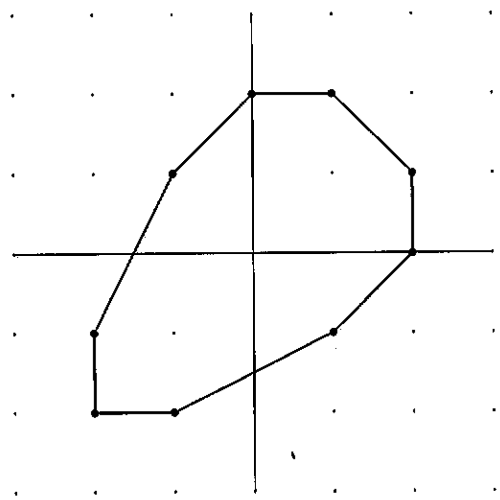


Figure 2.3: Example with $b_2 = 7$ and $\mathcal{W}_0 = D_3$

Chapter 3

Kähler-Einstein metrics

This section will present the methods and results from the analysis of Monge-Ampère equations and the theory of *multiplier ideal sheaves* that we will use to prove that symmetric toric Fano surfaces admit Kähler-Einstein metrics.

3.1 Kähler-Einstein metrics and the complex Monge-Ampère equation

Let X be an n-dimensional Fano orbifold and $G \subset \mathrm{Aut}(X)$ a compact group of holomorphic automorphisms. Let g be a Kähler metric on X with Kähler form $\omega \in c_1(X)$. By averaging over the compact group G we may assume that g is G invariant. In local holomorphic coordinates we have $g = \sum g_{\alpha\bar{\beta}} dz_\alpha \otimes d\bar{z}_{\bar{\beta}}$ and $\omega = \frac{i}{2\pi} \sum g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_{\bar{\beta}}$. If $\mathrm{Ric} = \sum R_{\alpha\bar{\beta}} dz_\alpha \otimes d\bar{z}_{\bar{\beta}}$ is the Ricci curvature of X, then the Ricci form $\mathrm{Ricci}(\omega)$ is the associated $(1,1)$-form and we have

$$\mathrm{Ricci}(\omega) = \frac{i}{2\pi} \bar{\partial}\partial \log \det(g_{\alpha\bar{\beta}}). \tag{3.1}$$

Definition 3.1 *A* Kähler-Einstein metric *on a Kähler orbifold X is a Kähler metric g with*

$$\text{Ricci}(\omega) = \lambda\omega,$$

where ω is the Ricci form and λ is a constant.

In our case X is Fano, so we must have $\lambda > 0$. Since ω and $\text{Ricci}(\omega)$ are in $c_1(X)$, we have

$$\text{Ricci}(\omega) = \omega + \frac{i}{2\pi}\partial\bar{\partial}f \text{ for some } f \in C^\infty(X). \tag{3.2}$$

For each $t \in [0, 1]$ consider the complex Monge-Ampère equation

$$(\omega + \frac{i}{2\pi}\partial\bar{\partial}\phi_t)^n = \omega^n e^{-t\phi_t + f}, \tag{3.3}$$

for an unknown real-valued function $\phi = \phi_t$. It is automatic that $\omega_t = \omega + \frac{i}{2\pi}\partial\bar{\partial}\phi_t > 0$. It is well known that the existence of a Kähler-Einstein metric on X is equivalent to a solution of (3.3) with $t = 1$. Take $-\frac{i}{2\pi}\partial\bar{\partial}\log$ of equation (3.3) to get

$$\begin{aligned} \text{Ricci}(\omega_t) &= \text{Ricci}(\omega) + t\frac{i}{2\pi}\partial\bar{\partial}\phi_t - \frac{i}{2\pi}\partial\bar{\partial}f \\ &= (1-t)\omega + t\omega_t \end{aligned} \tag{3.4}$$

So if $t = 1$, we have $\text{Ricci}(\omega_1) = \omega_1$. By Yau's solution to the Calabi conjecture [71], see also [5], equation (3.3) is solvable when $t = 0$. We use the continuity method to attempt to go from $t = 0$ to $t = 1$. This is not always possible as there are known obstructions to the existence of a Kähler-Einstein metric on a Fano orbifold.(cf. [50, 28])

A function $\phi \in C^2$ is *admissible* if $\omega + \frac{i}{2\pi}\partial\bar{\partial}\phi > 0$. Note that a solution to (3.3) is automatically admissible. Let Θ be the set of $C^{5+\alpha}$ admissible functions. And define a map Γ by

$$\mathbb{R} \times \Theta \ni (t, \phi) \xrightarrow{\Gamma} \log \left[\frac{(\omega + \frac{i}{2\pi}\partial\bar{\partial}\phi)^n}{\omega^n} \right] + t\phi \in C^{3+\alpha}.$$

Differentiating Γ with respect to ϕ at $t \in (0,1)$ gives

$$D_\phi \Gamma(t, \phi)\psi = -\Delta_t \psi + t\psi, \tag{3.5}$$

where Δ_t is the Laplacian with respect to the metric g_t associated to $\omega_t = \omega + \frac{i}{2\pi}\partial\bar{\partial}\phi_t$, i.e. $\Delta_t = g_t^{\alpha\bar{\beta}}\partial_\alpha\bar{\partial}_{\bar{\beta}}$. We will use the following:

Theorem 3.2 *Suppose the the Ricci curvature of the compact Kähler manifold (X, g) satisfies* $\mathrm{Ric} \geq \lambda$. *Then the first eigenvalue μ_1 of Δ satisfies $\mu_1 \geq \lambda$.*

Because of equation (3.4) we have $\mu_1 > t$. It follows from the implicit function theorem that the map $(t, \phi) \to (t, \Gamma(t, \phi))$ is a diffeomorphism of a neighborhood of $(t, \phi) \in (0,1) \times \Theta$ to a neighborhood of $(t, \Gamma(t, \phi))$, where ϕ is a solution of $(3.3)_t$. If ϕ_s is a solution to $(3.3)_t$ for $t = s \in (0,1)$, then there are solutions to $(3.3)_t$ for $t \in (s - \delta, s + \delta), \delta > 0$. Thus the set $E = \{\tau \in (0,1] : (3.3)_t \text{ has a solution with } t = \tau\}$ is open.

There is a difficulty at $t = 0$. Suppose we chose f so that $\int e^f d\mu_g = \int d\mu_g$. Then $\Gamma(0, \phi) = f$ will have a solution ϕ_0, unique up to a constant by the solution to the Calabi conjecture [5, 71]. But the map Γ is not invertible at

$(0, \phi_0)$. Therefore consider the modified map $\tilde{\Gamma} : \mathbb{R} \times \Theta \to C^{3+\alpha}$,

$$\tilde{\Gamma}(t, \phi) = \Gamma(t, \phi) + c \int \phi d\mu_g, \text{ for } c > 0. \tag{3.6}$$

Then $\tilde{\Gamma}$ is continuously differentiable with

$$D_\phi \tilde{\Gamma}(t, \phi)\psi = -\Delta_t \psi + t\psi + \int \psi d\mu_g.$$

Then $\tilde{\Gamma}$ is locally invertible at any solution to $\tilde{\Gamma}(t, \phi) = f$, for $0 \leq t < 1$. And $\tilde{\Gamma}(0, \phi) = f$ has a unique solution $\tilde{\phi}_0$ by the Calabi conjecture. Apply the implicit function theorem at $(0, \tilde{\phi}_0)$ to get for some small $\epsilon > 0$ a solution $\tilde{\Gamma}(\epsilon, \tilde{\phi}_\epsilon) = f$. Then $\phi_\epsilon = \tilde{\phi}_\epsilon + \frac{c}{\epsilon} \int \tilde{\phi}_\epsilon d\mu_g$ is a solution to $(3.3)_t$ with $t = \epsilon$. Therefore E is non-empty and open. Equation $(3.3)_t$ has a solution for $t = 1$ and X admits a Kähler-Einstein metric if E is also closed.

It is worth noting that a solution ϕ_s to $(3.3)_t$ for $t = s \in (0, 1)$ is unique. One shows that the required *a priori* estimate exits for $t \in (0, s)$ thus $(0, s] \subset E$. Then one considers solutions to 3.6 back to $t = 0$ using the implicit function theorem and the uniqueness at $t = 0$ gives the result. Thus a solution $\phi_s, s \in E, s < 1$ is invariant under G.

It to show that E is closed suffices to prove an *a priori* $C^{2+\alpha}$-estimate with $\alpha \in (0, 1)$ on the solutions $\phi_t, t \in E$ to (3.3). Given $t' \in \bar{E}$ and a sequence $\{s_i\} \subset E$ with $s_i \to t'$. By Ascoli's theorem there is a subsequence $\{s_{i_k}\}$ such that $\phi_{i_k} \to \phi_{t'} \in C^2$, where convergence is in C^2, and $\phi_{t'}$ is a solution to (3.3) with $t = t'$. It follows that Kähler-Einstein metric obtained will be G-invariant. One can improve this to merely requiring a C^0-estimate.

Lemma 3.3 ([71]) *There exist constants b and c depending only on (X, g) such that all solutions ϕ to $(3.3)_t$ satisfy*

$$0 < n + \Delta\phi \leq ce^{b(\sup\phi - \inf\phi)}.$$

Theorem 3.4 ([71]) *Let ϕ be a solution to $(3.3)_t$, then there is a C^0-estimate of the mixed derivatives $\phi_{\alpha\beta\bar{\gamma}}$ depending on*

$$\|\phi\|_{C^0}, \quad g, \quad \|f\|_{C^0}, \quad \|\nabla f\|_{C^0}, \quad \sup_{\alpha}\|\nabla_{\alpha\bar{\alpha}} f\|_{C^0}, \textit{ and } \sup_{\alpha,\beta,\gamma}\|\nabla_{\alpha\bar{\beta}\gamma} f\|_{C^0}.$$

Using the above and the ellipticity of Δ we see that if there is a constant $C > 0$ with $\|\phi\|_{C^0} \leq C$ for all solutions ϕ to $(3.3)_t, t \in E$, then there is a constant C' so that $\|\phi\|_{C^{2+\alpha}} \leq C'$ for all solutions ϕ for any $\alpha \in (0, 1)$.

For the remainder of this section we suppose $1 \notin E$. So a C^0-estimate fails to hold. There exists an increasing sequence $\{t_k\}, t_k \in (0, 1)$ and a sequence $\{\phi_k\}, \phi \in C^\infty$ such that

i. ϕ_k is a solution to equation $(3.3)_t$ for $t = t_k$, $k = 1, 2, \ldots$; In particular each ϕ_k is admissible;

ii. each $\phi_k, k = 1, 2, \ldots$ is G-invariant, and

iii. $\|\phi_k\|_{C^0} \to \infty$ as $k \to \infty$.

We will make use of the following Harnack-type inequality (cf. [64] or [66]).

Proposition 3.5 *For $\epsilon > 0$ there is a constant $C > 0$, depending on ϵ, such that*

$$\sup_X(-\phi_k) \leq (n + \epsilon)\sup_X \phi_k + C \textit{ for all } k.$$

It follows from this inequality that $\sup_M \phi_k \to \infty$ as $k \to \infty$. The following will be crucial.

Proposition 3.6 ([65, 64]) *For every* $\gamma \in (\frac{n}{n+1}, 1)$*, we have*

$$e^{\gamma \sup_M \phi_k} \int_M e^{-\gamma\phi_k} d\mu_g \to \infty \ \text{as}\ k \to \infty.$$

Proof. In the following $C > 0$ will denote an arbitrary constant, independent of k, that will change between equations. Since $\mathrm{Vol}_g = \frac{(i\pi)^n}{n!} \int \omega^n = \frac{(i\pi)^n}{n!} \int \omega_k^n = \mathrm{Vol}_{g_k}$, where $\omega_k = \omega + \frac{i}{2\pi}\partial\bar{\partial}\phi_k$ and g_k is the associated metric, we have from $(3.3)_t, t = t_k$

$$\int e^{-t_k\phi_k} d\mu_g \geq e^{-\sup f}\, \mathrm{Vol}_g\,.$$

Thus we have

$$C \int e^{-\gamma\phi_k} d\mu_g \geq \inf_X e^{(t_k-\gamma)\phi_k}, \text{ for all } k > 0. \tag{3.7}$$

Suppose the proposition does not hold, then after replacing $\{\phi_k\}$ by a subsequence we have

$$\int e^{-\gamma\phi_k} d\mu_g \leq C \inf_X e^{-\gamma\phi_k} \text{ for all } k > 0. \tag{3.8}$$

First consider the case where $t_k < \gamma$. Combining the above two inequalities we have

$$(t_k - \gamma)\sup_X \phi_k \leq -\gamma \sup_X \phi_k + C.$$

Thus $\sup_X \phi_k \leq C$. So $t_k < \gamma$ for only finitely many k. Now suppose that $t_k > \gamma$. Then we have

$$(t_k - \gamma)\inf_X \phi_k \leq -\gamma \sup_X \phi_k + C.$$

From this and proposition (3.5) we have

$$\gamma(t_k - \gamma)^{-1} \sup_X \phi_k \leq -\inf_X \phi_k + C \leq (n + \epsilon)\sup_X \phi_k + C. \tag{3.9}$$

But since $\frac{n}{n+1} < \gamma < t_k < 1$, we have $\gamma(t_k - \gamma)^{-1} > n$. Choosing $\epsilon > 0$ sufficiently small (3.9) we have $\sup_X \phi_k \leq C$ for all $k > 0$, a contradiction. □

Recall that g is invariant under the compact group $G \subset \text{Aut}(X)$ and has Kähler form $\omega \in c_1(X)$. Define

$$P_G(X, g) = \{\phi \in C^\infty : \omega + \frac{i}{2\pi}\partial\bar{\partial}\phi > 0, \text{ and } \phi \text{ is } G\text{-invariant}\}.$$

It is proved in [65] that there are positive constants α, C, depending on (X, g), such that

$$\int e^{-\alpha(\phi - \sup_x \phi)} d\mu_g \leq C \text{ for all } \phi \in P_G(X, g). \tag{3.10}$$

Definition 3.7 $\alpha_G(X) = \sup\{\alpha > 0 : \exists C > 0,$ *s.t. (3.10) holds* $\forall \phi \in P_G\}$.

It is not difficult to see that $\alpha_G(X)$ is a holomorphic invariant of a Fano orbifold, defined with respect to any metric G-invariant metric g with Kähler form in $c_1(X)$. Proposition (3.6) proves the following.

Theorem 3.8 ([65]) *Let (X, g) be a Kähler orbifold with Kähler form*

$\omega \in c_1(X)$. If $\alpha_G(X) > \frac{n}{n+1}$, then X admits a Kähler-Einstein metric.

This was used by V. Batyrev and E. Selivanova [9] to prove that any symmetric toric Fano manifold admits a Kähler-Einstein metric by proving that $\alpha_G(X) \geq 1$. We take a different approach in this work.

We will need the following proposition. See [65] for a proof.

Proposition 3.9 *After replacing $\{\phi_k\}$ by a subsequence, there is a nonempty open subset $U \subset X$ such that*

$$e^{\sup_X \phi_k} \int_U e^{-\phi_k} d\mu_g \leq O(1), \text{ as } k \to \infty.$$

Replace the each ϕ_k by $\phi_k - \sup_X \phi_k$. Then $S = \{\phi_k\}_{k=1}^{\infty}$ is a sequence of G-invariant functions which satisfy the following properties which will be crucial for the next section.

P1. There exists a Kähler-metric g with Kähler form $\omega \in c_1(X)$ such that $\omega + \frac{i}{2\pi}\partial\bar{\partial}\phi_k > 0$ for all $k > 0$.

P2. $\sup_X \phi_k = 0$ for all $k > 0$.

P3. For every $\gamma \in (\frac{n}{n+1}, 1)$ we have $\int_X e^{-\gamma\phi_k} d\mu_g \to \infty$ as $k \to \infty$.

P4. There exists a nonempty open subset $U \subset X$ such that $\int_U e^{-\phi_k} d\mu_g \leq O(1)$ as $k \to \infty$.

3.2 Multiplier ideal sheaves

Suppose S is a sequence of functions satisfying P1–P4. In this section we will show how to associate to S a coherent algebraic sheaf of ideals $\mathcal{J}(X, S)$, the *multiplier ideal sheaf.* The definition of $\mathcal{J}(X, S)$ is due to A. Nadel [54, 55]. One can work instead with coherent analytic sheaves, but the ideal sheaf one obtains is equivalent. We will need the following version of Hormander's L^2-estimate for $\bar{\partial}$ which follows from from the Weitzenböck formula for the $\bar{\partial}$-Laplacian $\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ acting on $A^{p,q}(X, \mathbf{L})$, the smooth (p, q)-forms with values in $\mathbf{L}$. Let X be any compact Kähler orbifold with Kähler form ω, $\mathbf{L}$ a hermitian holomorphic line V-bundle on X, and $f \in C^\infty$. Denote by Θ the curvature of $\mathbf{L}$.

Proposition 3.10 *Suppose that*

$$\frac{i}{2\pi}\partial\bar{\partial}f + \mathrm{Ricci}(\omega) + \frac{i}{2\pi}\Theta \geq \epsilon\omega, \tag{3.11}$$

for some $\epsilon > 0$. Let σ be a smooth $(0, p)$-form with $q > 0$ and values in $\mathbf{L}$ such that $\bar{\partial}\sigma = 0$. Then there exists an $\mathbf{L}$-valued $(0, p-1)$-form η on X such that

1. $\bar{\partial}\eta = \sigma$,

2. $\int_X |\eta|^2 e^{-f} d\mu_g \leq \frac{1}{\epsilon}\int_X |\sigma|^2 e^{-f} d\mu_g$.

Furthermore, if G acts holomorphic and isometrically on (X, g) and $\mathbf{L}$, and also f and σ are G-invariant, then we may take η to be G-invariant.

Proof. Replacing the hermitian metric h on $\mathbf{L}$ by $e^{-f}h$ reduces the proposition to the case $f = 0$. Let $G^{p,q}$ be the Green's operator of $\Delta_{\bar{\partial}}^{p,q}$ acting on $A^{p,q}(X, \mathbf{L})$. By the Weitzenböck formula for $\Delta_{\bar{\partial}}^{0,q}$ (cf. [10] p. 52) and (3.11) we have

$$(\Delta_{\bar{\partial}}^{0,q}\sigma, \sigma) \geq \epsilon\|\sigma\|^2$$

for all $\sigma \in A^{0,q}(X, \mathbf{L})$. This implies that

$$(G^{0,q}\sigma, \sigma) \leq \frac{1}{\epsilon}\|\sigma\|^2$$

for all $\sigma \in A^{0,q}(X, \mathbf{L})$. Let σ be as in the proposition and define $\eta = \bar{\partial}^* G^{0,q}\sigma$. Then we have $\bar{\partial}\eta = \bar{\partial}\bar{\partial}^* G^{0,q}\sigma = G^{0,q}\bar{\partial}\bar{\partial}^*\sigma = G^{0,q}\Delta_{\bar{\partial}}^{0,q}\sigma = \sigma$. And we have

$$\|\eta\|^2 = (\bar{\partial}^* G^{0,q}\sigma, \eta) = (G^{0,q}\sigma, \bar{\partial}\eta) = (G^{0,q}\sigma, \sigma) \leq \frac{1}{\epsilon}\|\sigma\|^2.$$

Finally, if G is as in the proposition, then $\Delta_{\bar{\partial}}$ and $G^{0,q}$ will be G-invariant. □

Definition 3.11 *Let $\{\sigma_k\}_{k=1}^{\infty} \subset C^\infty(X, \mathbf{L})$ be a sequence of sections of a smooth vector bundle. We say that $\{\sigma_k\}$ is* S-bounded *if there exists a $\gamma \in (\frac{n}{n+1}, 1)$ so that*

$$\int_X |\sigma_k| e^{-\gamma\phi_k} d\mu \leq O(1) \text{ as } k \to \infty. \tag{3.12}$$

We say that $\{\sigma_k\}$ is S-null *if there exists a $\gamma \in (\frac{n}{n+1}, 1)$ so that*

$$\int_X |\sigma_k| e^{-\gamma\phi_k} d\mu \to 0 \text{ as } k \to \infty. \tag{3.13}$$

For this definition the choice to the volume form $d\mu$ and the metric on $\mathbf{L}$ is irrelevant.

Proposition 3.12 *Suppose $\{\sigma_k\}$ and $\{\sigma'_k\}$ are sequences of smooth sections of smooth vector bundle. If the sequences $\{\sigma_k\}$ and $\{\sigma'_k\}$ are S-bounded (resp. S-null), then so is the sequence $\{\sigma_k + \sigma'_k\}$.*

Proof. Because of P2, if (3.12)(resp. (3.13)) holds for $\gamma \in (\frac{n}{n+1}, 1)$ then it holds for any smaller γ. So we may suppose both sequences satisfy (3.12)(resp. (3.13)) for the same γ. Then the proposition follows from the inequality $|\sigma_k + \sigma'_k|^2 \leq 2|\sigma_k|^2 + 2|\sigma'_k|^2$. □

Definition 3.13 *Suppose* $\mathbf{E}$ *is a holomorphic vector bundle on* X*. We define* $H^0(X, \mathbf{E})_S$ *be the set of all* $\tau \in H^0(X, \mathbf{E})$ *such that there is an* S*-bounded sequence* $\{\sigma_k\} \subset H^0(X, \mathbf{E})$ *converging uniformly to* τ*. We call* $H^0(X, \mathbf{E})_S$ *the* space of sections vanishing along S in the scheme-theoretic sense.

The following proposition is an easy consequence of proposition (3.12).

Proposition 3.14 *Suppose* $\mathbf{E}$ *and* $\mathbf{F}$ *are are holomorphic vector bundles on* X*. Then* $H^0(X, \mathbf{E})_S$ *is a complex vector subspace of* $H^0(X, \mathbf{E})$*. Also, for* $\sigma \in H^0(X, \mathbf{E})_S$ *and* $\tau \in H^0(X, \mathbf{F})$*, we have* $\sigma \otimes \tau \in H^0(X, \mathbf{E} \otimes \mathbf{F})_S$*.*

Let $\mathbf{L}$ be an ample line bundle on X and consider the homogeneous coordinate ring

$$R(X, \mathbf{L}) = \bigoplus_{\nu=0}^{\infty} H^0(X, \mathcal{O}(\mathbf{L}^\nu)) \tag{3.14}$$

of X relative to $\mathbf{L}$. We can define the homogeneous ideal of $I(X, \mathbf{L}, S)$ of $R(X, \mathbf{L})$ as

$$I(X, \mathbf{L}, S) = \bigoplus_{\nu=0}^{\infty} H^0(X, \mathcal{O}(\mathbf{L}^\nu))_S. \tag{3.15}$$

We now define the coherent sheaf of ideals $\mathcal{J}(X, \mathbf{L}, S)$ depending on $\mathbf{L}$ and S which is associated to the homogeneous ideal $I(X, \mathbf{L}, S) \subset R(X, \mathbf{L})$. Given a

Zariski open set $U \subset X$ and a regular function $f \in \Gamma(U, \mathcal{O}_X)$. We have

$$f \in \Gamma(U, \mathcal{J}(X, \mathbf{L}, S)),$$

if and only if for every $p \in U$ there exists a global section $\sigma \in H^0(X, \mathcal{O}(\mathbf{L}^\nu))$ for some $\nu > 0$ which does not vanish at p, and $f\sigma \in H^0(X, \mathcal{O}(\mathbf{L}^\nu))_S$. Since the homogeneous ring $R(X, \mathbf{L})$ is Noetherian, the ideal $I(X, \mathbf{L}, S)$ is finitely generated. Thus the sheaf $\mathcal{J}(X, \mathbf{L}, S)$ is a locally finitely generated $\mathcal{O}_X$-module, hence is coherent.

We show next that $\mathcal{J}(X, \mathbf{L}, S)$ is independent of the ample line bundle $\mathbf{L}$.

Proposition 3.15 *Suppose* $\mathbf{L}$ *and* $\mathbf{E}$ *are ample line bundles on* X. *Then* $\mathcal{J}(X, \mathbf{L}, S) = \mathcal{J}(X, \mathbf{E}, S)$.

Proof. It is sufficient to show $\mathcal{J}(X, \mathbf{L}, S) \subseteq \mathcal{J}(X, \mathbf{E}, S)$. Let $U \subset X$ be a nonempty Zariski open set, and let $f \in \Gamma(U, \mathcal{J}(X, \mathbf{L}, S))$. Given $p \in U$ there exists a section $\sigma \in H^0(X, \mathcal{O}(\mathbf{L}^\nu))$ for some $\nu > 0$ such that $\sigma(p) \neq 0$ and $f\sigma \in H^0(X, \mathcal{O}(\mathbf{L}^\nu))_S$. Now there exist an integer μ sufficiently large such that there is a section $\rho \in H^0(X, \mathcal{O}(\mathbf{L}^{-\nu} \otimes \mathbf{E}^\mu))$ such that $\rho(p) \neq 0$. This follows from the Kodaira-Baily embedding theorem [6]. Then $\sigma\rho \in H^0(X, \mathcal{O}(\mathbf{E}^\mu))$ does not vanish at p. And $f\sigma\rho \in H^0(X, \mathcal{O}(\mathbf{E}^\mu))_S$ by proposition (3.14). □

Definition 3.16 *We define* $\mathcal{J}(X, S) := \mathcal{J}(X, \mathbf{L}, S)$, *for any ample* $\mathbf{L}$. *This is independent of* $\mathbf{L}$ *by the proposition.*

In order for $\mathcal{J}(X, S)$ to be useful some further properties must be determined. For example we show that $\mathcal{J}(X, S) \subset \mathcal{O}_X$ is a proper subsheaf of ideals. This is the aim of the rest of this section.

Let $V(X,S)$ be the, possibly non-reduced, subscheme cut out by the coherent sheaf $\mathcal{J}(X,S)$.

Proposition 3.17 *The set $V(X,S)$ is nonempty.*

Proof. Suppose that $V(X,S)$ is empty. Let $p \in X$ be a arbitrary point. There exits a $\nu > 0$ and $\tau \in H^0(X, \mathcal{O}(\mathbf{L}^\nu))_S$ such that $\tau(p) \neq 0$. We may assume $\tau(p) = 1$. There exists an S-bounded sequence $\{\tau_k\} \subset H^0(X, \mathcal{O}(\mathbf{L}^\nu))$ converging uniformly to τ. There exists an open neighborhood W, in the classical topology, of p so that $\|\tau_k\| \geq \frac{1}{2}$ for large enough k. Then

$$\int_W e^{-\gamma\phi_k} d\mu_g \leq O(1) \text{ as } k \to \infty, \tag{3.16}$$

for some $\gamma \in (\frac{n}{n+1}, 1)$. By the compactness of X, there exists a finite collection $W_1, \ldots, W_m$ covering X for which (3.16) holds for $\gamma_1, \ldots, \gamma_m$. Let $\gamma = \min\{\gamma_1, \ldots, \gamma_m\}$. Then we have

$$\int_X e^{-\gamma\phi_k} d\mu_g \leq O(1) \text{ as } k \to \infty,$$

which contradicts P3. □

In order to guarantee that $\mathcal{J}(X,S)$ is nonzero we must consider what happens when we pass to a subsequence $S' \subset S$. For any subsequence S' of S we have $\mathcal{J}(X,S) \subseteq \mathcal{J}(X,S')$. We replace S by a subsequence to make $\mathcal{J}(X,S)$ as large as possible.

Proposition 3.18 *Suppose X is a projective variety and $\mathscr{C}$ is a nonempty collection of coherent sheaves of ideals partially ordered by inclusion. Then $\mathscr{C}$ has a maximal element.*

Proof. Take an embedding of X into $\mathbb{C}P^N$. Let R denote the homogeneous coordinate ring of $\mathbb{C}P^N$. Let $I \subset R$ be the homogeneous ideal of X. Each coherent sheaf $\mathcal{J}$ on X corresponds uniquely a homogeneous ideal $J \subset R$, and the correspondence $\mathcal{J} \to J$ is one-to-one (cf. [38] ex. 5.10) and order preserving. If $\mathscr{C}'$ is the collection of homogeneous ideal in R corresponding to coherent sheaves in $\mathscr{C}$, then $\mathscr{C}'$ possesses a maximal element because R is Noertherian. Thus $\mathscr{C}$ has a maximal element. □

Apply the proposition to the set of ideals $\mathcal{J}(X, S')$ where S' is any subsequence of S. We get a subsequence S' of S so that $\mathcal{J}(X, S'') = \mathcal{J}(X, S')$ for all subsequences S'' of S'. Replace S with S', then we have the further property.

P5. For every subsequence S' of S we have $\mathcal{J}(X, S') = \mathcal{J}(X, S)$

We finally have a complete definition.

Definition 3.19 *Suppose X is a Fano orbifold. A* multiplier ideal sheaf *on X is a pair $(S, \mathcal{J}(X, S))$ consisting of a sequence S satisfying P1–P5 and the coherent sheaf of ideals $\mathcal{J}(X, S)$. A* multiplier ideal subscheme *of X is a pair $(S, V(X, S))$ consisting of a sequence S satisfying P1–P5 and the subscheme $V(X, S) \subset X$.*

We are now able to prove the second half of the properness of $\mathcal{J}(X, S)$.

Proposition 3.20 *Suppose that* $\mathbf{L}$ *is an ample line bundle on* X*. Then there exists an integer* $\nu > 0$ *and a subsequence* S' *of* S *such that* $H^0(X, \mathcal{O}(\mathbf{L}^\nu))_{S'}$ *is nonzero.*

Proof. Let $U \subset X$ be a nonempty subset which satisfies P4. And let $(z_1, \ldots, z_n)$ be holomorphic coordinates centered at $p \in U$. Let $\rho \in C_c^\infty(U)$ be a compactly supported function such that $\rho = 1$ on a neighborhood $U' \ni p$. Now define for each positive integer k $\psi_k \in C^\infty(X)$ as

$$\psi_k = n\rho \log \left(|z_1|^2 + \cdots |z_n|^2 + \frac{1}{k} \right).$$

We will use the fact that $e^{-\psi_k}$ converges monotonically on U' as $k \to \infty$ to $(|z_1|^2 + \cdots + |z_n|^2)^{-n}$ which is not integrable at p. Let ω be a Kähler form as in P1. And fix a metric on $\mathbf{L}$ with positive curvature, i.e. $\frac{i}{2\pi}\Theta > 0$. There exists a constant $C_1 > 0$ so that

$$\frac{i}{2\pi}\partial\bar{\partial}\psi_k + C_1\omega \geq 0 \tag{3.17}$$

for all $k > 0$. The number C_1 exists because ψ_k is plurisubharmonic on U', and $\{\psi_k\}$ converges to $n\rho\log(|z_1|^2 + \cdots + |z_n|^2)$ in the C^∞ norm on $X \setminus U'$. By (3.17) and P1 we may choose an integer $\nu > 0$ large enough that

$$\frac{i}{2\pi}\partial\bar{\partial}(\phi_k + \psi_k) + \mathrm{Ricci}(\omega) + \nu\frac{i}{2\pi}\Theta \geq \omega, \tag{3.18}$$

for all $k > 0$. Let $\tau \in H^0(X, \mathcal{O}(\mathbf{L}^\nu))$ be nonzero at p. By proposition (3.10) and the above inequality, we obtain for each $k > 0$ a smooth section

$\tau_k \in C^\infty(X, \mathbf{L}^\nu)$ such that $\bar{\partial}\tau_k = \bar{\partial}(\rho\tau)$ and

$$\int_X |\tau_k|^2 e^{-\phi_k - \psi_k} d\mu_g \leq \int_X |\bar{\partial}(\rho\tau)|^2 e^{-\phi_k - \psi_k} d\mu_g. \tag{3.19}$$

Since $\bar{\partial}(\rho\tau)$ vanishes on U', there exist a constant $C_2 > 0$ so that

$$|\bar{\partial}(\rho\tau)|^2 e^{-\psi_k} \leq C_2 \chi_U$$

for all $k > 0$, where χ_U is the characteristic function of U. Therefore the righthand side of (3.19) is bounded by

$$C_2 \int_U e^{-\phi_k} d\mu_g,$$

which is bounded by P4 as $k \to \infty$. Thus we have

$$\int_X |\tau_k|^2 e^{-\phi_k - \psi_k} d\mu_g \leq O(1) \text{ as } k \to \infty. \tag{3.20}$$

Consider the sequence $\{\sigma_k\}$ where $\sigma_k = \tau_k - \rho\tau \in H^0(X, \mathcal{O}(\mathbf{L}^\nu))$. And we have

$$\begin{aligned} \int_X |\sigma_k|^2 e^{-\phi_k} d\mu_g &\leq 2\int_X |\tau_k|^2 e^{-\phi_k} d\mu_g + 2\int_X |\rho\tau|^2 e^{-\phi_k} d\mu_g \\ &\leq O\left(\int_X |\tau_k|^2 e^{-\phi_k - \psi_k} d\mu_g + \int_U e^{-\phi_k} d\mu_g\right) \\ &\leq O(1) \end{aligned} \tag{3.21}$$

The second inequality follows because ψ_k is bounded from above uniformly as $k \to \infty$ and $\rho\tau$ is supported in U. Since $\phi_k \leq 0$ by P2

$$\int_X |\sigma_k|^2 d\mu_g \leq O(1) \text{ as } k \to \infty. \tag{3.22}$$

Since $\{\sigma_k\}$ is a bounded sequence in the finite dimensional vector space $H^0(X, \mathcal{O}(\mathbf{L}^\nu))$, there is a subsequence $\{\sigma_{k_j}\}$ converging uniformly to $\sigma \in H^0(X, \mathcal{O}(\mathbf{L}^\nu))$. So $\sigma \in H^0(X, \mathcal{O}(\mathbf{L}^\nu))_S$, where S' is the subsequence $\{\phi_{k_j}\}$ of S. We claim that $\sigma(p) \neq 0$. Otherwise, $\rho\tau + \sigma$ is nonzero at p. Since τ_{k_j} converges uniformly to $\rho\tau + \sigma$, there is a $C_3 > 0$ and a neighborhood W of p so that

$$|\tau_{k_j}|^2 \geq C_3 \text{ on } W \tag{3.23}$$

for all large j. We have

$$\begin{aligned}\int_W e^{-\psi_{k_j}} d\mu_g &\leq O\left(\int_W e^{-\phi_{k_j} - \psi_{k_j}} d\mu_g\right) \quad \text{by P2,}\\ &\leq O\left(\int_X |\tau_{k_j}|^2 e^{-\phi_{k_j} - \psi_{k_j}} d\mu_g\right) \quad \text{by (3.23),}\\ &\leq O(1) \text{ as } j \to \infty \quad \text{by (3.20)}\end{aligned}$$

But we know that $e^{-\psi_k}$ converges monotonically to a non-integrable function.

□

Corollary 3.21 *The coherent sheaf $\mathcal{J}(X, S)$ is not identically zero, and $V(X, S)$ is not all of X.*

Proof. Let $\sigma \in H^0(X, \mathcal{O}(\mathbf{L}^\nu))_{S'}$ be not identically zero. Since $V(X, S') = V(X, S)$, it is easy to see that σ vanishes along $V(X, S)$ in the scheme theoretic sense defining nonzero sections in $\mathcal{J}(X, S)$. □

The group of symmetries $G \subset \mathrm{Aut}(X)$ of (X, g) will play a crucial role in our applications.

Proposition 3.22 *Suppose every every element in the sequence S is G-invariant. Then the coherent sheaf $\mathcal{J}(X, S)$ is G-invariant. Thus both $\mathcal{J}(X, S)$ and $V(X, S)$ are $G_{\mathbb{C}}$-invariant, where $G_{\mathbb{C}}$ is the complexification of G.*

Proof. Take $\mathbf{L} = \mathbf{K}_X^{-m}$, where m is taken to be a multiple of $\mathrm{Ord}(X)$, so $\mathbf{L}$ is an ample line bundle. And G lifts to an action on $\mathbf{L}$. It follows that G acts on $H^0(X, \mathcal{O}(\mathbf{L}^\nu))$ for each $\nu > 0$ and the subspace $H^0(X, \mathcal{O}(\mathbf{L}^\nu))_S$ is invariant. So the sheaf of ideals $\mathcal{J}(X, S) = \mathcal{J}(X, \mathbf{L}, S)$ is G-invariant. □

So as to simplify notation, in what follows we denote $\mathcal{J} = \mathcal{J}(X, S)$ and $V = V(X, S)$. We have the vanishing theorem of A. Nadel [55].

Theorem 3.23 $H^k(X, \mathcal{J}) = 0$ *for* $k \geq 1$.

The proof for the case of a Fano orbifold X goes through verbatim as the smooth case in [55]. The proof makes use of the fact that X is covered by affine Zariski open sets $U_1, \ldots, U_r$ and $H^k(U_i, \mathcal{J}) = 0$, for $k > 0$, because $\mathcal{J}$ is coherent. One then uses Leray's theorem and the Weitzenböck formula, proposition (3.10).

Corollary 3.24 $H^k(V, \mathcal{O}_V) = 0$ *for* $k \geq 1$, *and* $H^0(V, \mathcal{O}_V) = \mathbb{C}$.

Proof. The proof follows from the exact sequence of coherent sheaves

$$0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{O}_V \to 0.$$

By Kodaira vanishing we have $H^k(X, \mathcal{O}_X) = 0$ for $k > 0$. Also, note that $H^0(X, \mathcal{J}) = 0$. Now take the long exact cohomology sequence. □

3.3 Kähler-Einstein metrics on symmetric toric Fano surfaces

For our applications X is 2-dimensional, so the irreducible components of the subscheme $V(X,S)$ can be 0 or 1-dimensional. Also from corollary (3.24) it is clear that $V(X,S)$ is connected. We consider the case in which $V = V(X,S)$ is a 1-dimensional subscheme. Let V_{red} be the reduced scheme associated to V.

Theorem 3.25 *Suppose V is a 1-dimensional multiplier ideal subscheme of X. Then every irreducible component of V_{red} is isomorphic to $\mathbb{C}P^1$, any two irreducible components meet at at most one point, and V_{red} does not contain a cycle.*

In other words, V_{red} is a tree of $\mathbb{C}P^1$'s. We first prove a series of lemmas.

Lemma 3.26 *Suppose V is a 1-dimensional projective scheme with $H^1(V, \mathcal{O}) = 0$, then $H^1(V_{red}, \mathcal{O}_{V_{red}}) = 0$.*

Proof. We have the exact sequence of coherent sheaves on V:

$$0 \to \mathcal{R} \to \mathcal{O}_V \to \mathcal{O}_{V_{red}} \to 0,$$

where $\mathcal{R}$ is the sheaf of nil-radicals of $\mathcal{O}_V$. The lemma follows from the cohomology exact sequence:

$$H^1(V, \mathcal{O}_V) \to H^1(V_{red}, \mathcal{O}_{V_{red}}) \to H^2(V, \mathcal{R}) = 0,$$

where the last group is zero because $\dim V = 1$. □

In the following W will denote a reduced complex scheme of pure dimension one. Consider a morphism $\pi : \tilde{W} \to W$ of reduced schemes of pure dimension one which is finite, surjective, and the cokernel of $\mathcal{O}_W \to \pi_* \mathcal{O}_{\tilde{W}}$ has zero-dimensional support. We call such a map a *partial normalization.*

Lemma 3.27 *Suppose* $H^1(W, \mathcal{O}_W) = 0$ *and* $\pi : \tilde{W} \to W$ *is a partial normalization. Then* $H^1(\tilde{W}, \mathcal{O}_{\tilde{W}}) = 0$.

Proof. We have the exact sequence of coherent sheaves on W

$$0 \to \mathcal{O}_W \to \pi_* \mathcal{O}_{\tilde{W}} \to \mathcal{L} \to 0,$$

where the sheaf $\mathcal{L}$ has zero dimensional support, i.e. is a skyscraper sheaf. The long exact cohomology sequence gives $H^1(\tilde{W}, \mathcal{O}_{\tilde{W}}) = 0$. □

The following part of the cohomology sequence will be useful:

$$0 \to H^0(W, \mathcal{O}_W) \to H^0(\tilde{W}, \mathcal{O}_{\tilde{W}}) \to H^0(W, \mathcal{L}) \to 0.$$

Thus

$$\#\{\text{connected components of } \tilde{W}\} = \#\{\text{connected components of } W\} + h^0(W, \mathcal{L}). \tag{3.24}$$

Lemma 3.28 *If W is irreducible and $H^1(W, \mathcal{O}_W) = 0$, then $W \cong \mathbb{C}P^1$.*

Proof. Let $\pi : \tilde{W} \to W$ be the normalization of W. Then (3.24) gives $1 = 1 + H^0(W, \mathcal{L})$. So $\mathcal{L} = 0$ and π is an isomorphism. Thus W is nonsingular and $H^1(W, \mathcal{O}_W) = 0$ by the last lemma. □

Lemma 3.29 *Suppose $H^1(W, \mathcal{O}_W) = 0$ and $W' \subset W$ is a nonempty collection of irreducible components of W. Then we have $H^1(W', \mathcal{O}_{W'}) = 0$.*

Proof. Let W'' be the irreducible components not contained in W'. So $W = W' \cup W''$. And let $\pi : W' \sqcup W'' \to W$ be the partial normalization, where $W' \sqcup W''$ is the disjoint union. Then the lemma follows from lemma (3.27). □

Lemma 3.30 *Suppose $H^1(W, \mathcal{O}_W) = 0$. The number of singular points of W is strictly less that the number of irreducible components of W. So W cannot be a cycle of $\mathbb{C}P^1$'s.*

Proof. Let $\pi : \tilde{W} \to W$ be a desingularization. Then we have from (3.24)

$$h^0(W, \mathcal{L}) < \#\{\text{connected components of } \tilde{W}\}$$
$$= \#\{\text{irreducible components of } W\}$$

We also have $\#\{\text{singular points of } W\} \leq h^0(W, \mathcal{L})$. □

Proof.(of theorem) Since $H^1(V, \mathcal{O}_V) = 0$, we have $H^1(V_{red}, \mathcal{O}_{V_{red}}) = 0$. Every component of V_{red} is isomorphic to $\mathbb{C}P^1$ by lemma (3.28). Every two irreducible components meet at at most one point and V_{red} does not contain a cycle by lemmas (3.29) and (3.30). □

For remainder of this section X will be a symmetric toric Fano surface. So X is represented by a complete fan Δ in $\mathbb{Z} \times \mathbb{Z}$ and a convex polygon Δ^*. Let $G \subset \mathcal{N}(T_{\mathbb{C}})$ be the compact subgroup generated by T^2 and $\mathcal{W}_0(X)$ as in chapter (2). And we may consider $\mathcal{W}_0(X)$ to be the subgroup of $GL(2, \mathbb{Z})$ which preserves Δ^*. By hypothesis the only fixed point of the action of $\mathcal{W}_0(X)$ on $\mathbb{Z} \times \mathbb{Z}$ is the origin.

Theorem 3.31 *Let X be a symmetric toric Fano surface. Then X admits a Kähler-Einstein metric invariant under G.*

Proof. Suppose that X does not admit a Kähler-Einstein metric. Then there is a multiplier ideal sheaf $\mathcal{J}(X, S)$ with multiplier ideal subscheme $V = V(X, S)$, both invariant under $G_{\mathbb{C}}$, the complexification of G. Note that $T^2_{\mathbb{C}} \subset G_{\mathbb{C}}$. If V is zero dimensional, then V_{red} consists of a single point.

But there are no fixed points of $G_{\mathbb{C}}$. Suppose V is one dimensional. Then V_{red} consists of a non-cyclic connected chain of anti-canonical curves. If V_{red} consists of an odd number of curves then $G_{\mathbb{C}}$ fixes a curve. And if it consists of an even number then $G_{\mathbb{C}}$ fixes a point. Either case is a contradiction. $\square$

Chapter 4

Sasakian-Einstein manifolds

We define the notion of Sasakian and in particular Sasakian-Einstein manifolds. Under relatively weak assumptions a Sasakian manifold is a Seifert S^1-bundle with additional structure over a Hodge Kähler orbifold. We use an inversion construction which goes back to Kobayashi [43] and Hatakeyama [39] to construct 5-dimensional Sasakian-Einstein orbifolds and manifolds from the Kähler-Einstein orbifold surfaces of chapter 3.

4.1 Fundamentals of Sasakian geometry

Definition 4.1 *Let (M, g) be a Riemannian manifold of dimension $n = 2m+1$, ∇ the Levi-Civita connection, and $R(X,Y) \in End(TM)$ The Riemannian curvature. Then (M, g) is Sasakian if either of the following equivalent conditions hold:*

(i) There exists a unit length Killing vector field ξ on M so that the $(1,1)$

tensor $\Phi(X) = \nabla_X \xi$ *satisfies the condition*

$$(\nabla_X \Phi)(Y) = g(\xi, Y)X - g(X, Y)\xi$$

for vector fields X *and* Y *on* M.

(ii) There exists a unit length Killing vector field ξ *on* M *so that the Riemann curvature tensor satisfies*

$$R(X, \xi)Y = g(\xi, Y)X - g(X, Y)\xi$$

for vector fields X *and* Y *on* M.

We say that the triple $\{g, \xi, \Phi\}$ defines a Sasakian structure on M. Define η to be the one form dual to ξ, i.e. $\eta(X) = g(X, \xi)$. Let $N_\Phi(X, Y) = \Phi[X, \Phi Y] + \Phi[\Phi X, Y] - [\Phi X, \Phi Y] - \Phi^2[X, Y]$ be the Nijenhuis tensor of Φ. The following are easy consequences of definition (4.1).

Proposition 4.2 *The elements of a Sasakian structure satisfy the equations*

i. $\Phi(\xi) = 0, \quad \eta(\Phi(x)) = 0,$

ii. $\Phi^2(X) = -X + \eta(X)\xi,$

iii. $g(\Phi X, Y) + g(X, \Phi Y) = 0, \quad g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y),$

iv. $d\eta(X, Y) = g(\Phi X, Y), \quad N_\Phi(X, Y) = 2d\eta(X, Y) \otimes \xi.$

Easy calculation shows that η is a contact form with Reeb vector field ξ, and Φ defines a CR-structure on the orthogonal compliment to the subbundle of

TM defined by ξ. Sometimes one denotes a Sasakian manifold by (M, g, ξ, Φ) to be more explicit.

Proposition 4.3 *Suppose* (M, g) *has a unit length Killing vector field* ξ, *so that the* $(1,1)$ *tensor* $\Phi(X) = \nabla_X \xi$ *satisfies* $\Phi^2(X) = -X + \eta(X)\xi$. *If the CR-structure satisfies* $N_\Phi(X,Y) = 2d\eta(X,Y) \otimes \xi$, *then* $\{g, \xi, \Phi\}$ *is a Sasakian structure.*

Proof. First note that by the hypothesis all the properties in proposition (4.2) are satisfied. Easy computation shows that

$$N_\Phi(X,Y) = -(\nabla_{\Phi X}\Phi)(Y) + (\nabla_{\Phi Y}\Phi)(X) + \Phi \circ \nabla_X \Phi(Y) - \Phi \circ \nabla_Y \Phi(X). \quad (4.1)$$

Differentiating the equation $\Phi^2(X) = -X + \eta(X)\xi$ gives

$$\Phi \circ \nabla_X \Phi(Y) = -\nabla_X \Phi \circ (Y) + g(\Phi X, Y)\xi + \eta(Y)\Phi(X). \quad (4.2)$$

Applying this to the last two terms of (4.1) gives

$$\begin{aligned} g(N_\Phi(X,Y), Z) = {} & [-g(\nabla_X \Phi(\Phi Y), Z) + g(\nabla_{\Phi Y}\Phi(X), Z)] + g(\Phi X, Y)\eta(Z) \\ & + g(\Phi X, Z)\eta(Y)\, [-g(\nabla_{\Phi X}\Phi(Y), Z) + g(\nabla_Y \Phi(\Phi X), Z)] \\ & - g(\Phi Y, X)\eta(Z) - g(\Phi Y, Z)\eta(X). \quad (4.3) \end{aligned}$$

Since $d\eta$ is closed we have

$$g(\nabla_U \Phi(V), W) + g(\nabla_V \Phi(W), U) + g(\nabla_W \Phi(U), V) = 0.$$

Thus

$$g(N_\Phi(X,Y),Z) = g(\nabla_Z\Phi(X),\Phi Y) + g(\nabla_Z\Phi(\Phi X),Y)$$
$$+ g(\Phi X,Y)\eta(Z) + g(\Phi X,Z)\eta(Y) - g(\Phi Y,X)\eta(Z) - g(\Phi Y,Z)\eta(X) \quad (4.4)$$

Applying equation (4.2) to the second term on the right of the equality gives

$$g(N_\Phi(X,Y),Z) = 2g(\nabla_Z\Phi(X),\Phi Y) - 2g(\Phi Y,Z)\eta(X) + 2g(\Phi X,Y)\eta(Z) \quad (4.5)$$

By assumption, we have

$$g(N_\Phi(X,Y),Z) = 2g(\Phi X,Y)\eta(Z).$$

Thus

$$g(\nabla_Z\Phi(X),\Phi Y) = g(\Phi Y,Z)\eta(X). \quad (4.6)$$

Using $\Phi^2(X) = -X + \eta(X)\xi$, we have

$$\begin{aligned} g(\nabla_Z\Phi(X),Y) &= g(\nabla_Z\Phi(X),\xi)\eta(Y) + g(Y,Z)\eta(X) - \eta(Y)\eta(X)\eta(Z) \\ &= -g(\Phi X,\Phi Z)\eta(Y) + g(Y,Z)\eta(X) - \eta(Y)\eta(X)\eta(Z) \\ &= -g(X,Z)\eta(Y) + g(Y,Z)\eta(X) \end{aligned}$$

which gives the formula in part *i.* of definition (4.1). □

The following can be taken as a definition of a Sasakian manifold. It shows that Sasakian manifolds are the odd dimensional versions of Kähler manifolds in the sense that contact manifolds are the odd dimensional versions of symplectic manifolds. The metric cone of a Riemannian manifold (M, g) is the manifold $C(M) = \mathbb{R}_+ \times M$ with the cone metric $\bar{g} = dr^2 + r^2 g$.

Proposition 4.4 *Let (M, g) be a Riemannian manifold of dimension $n = 2m+1$. Then (M, g) is Sasakian if, and only if, the holonomy of the metric cone $(C(M), \bar{g})$ is a subgroup of $U(m+1)$. In other words, $(C(M), \bar{g})$ is Kähler.*

We see from definition (4.1) that $\mathrm{Ric}(X, \xi) = 2m\eta(X)$. Thus if (M, g) is Einstein, then the scalar curvature of g is $s = 2m(2m+1)$.

Definition 4.5 *A Sasakian manifold (orbifold) (M, g) is* Sasakian-Einstein *if the Riemannian metric g is Einstein, in which case, the scalar curvature is $s = 2m(2m+1)$.*

Direct calculation shows the following.

Proposition 4.6 *A Sasakian manifold (orbifold) (M, g) is Sasakian-Einstein if, and only if, $(C(M), \bar{g})$ is Ricci flat. That is, $(C(M), \bar{g})$ is Calabi-Yau which is equivalent to the restricted holonomy $Hol_0(\bar{g}) \subseteq SU(m+1)$.*

By Meyer's theorem if (M, g) is Sasakian-Einstein then $\pi_1(M)$ is finite and $\mathrm{diam}(M) \leq \pi$. Furthermore, we have

Proposition 4.7 *If (M, g) is a simply connected Sasakian-Einstein manifold then M is spin.*

Proof. The frame bundle of $(C(M), \bar{g})$ reduces to $SU(m+1)$. It follows that the frame bundle of M reduces to $SU(m) \times 1$. $\square$

Let $\mathcal{F}$ be the 1-dimensional foliation defined by ξ. The foliation $\mathcal{F}$ is *quasi-regular* is every $p \in M$ has cubical neighbor such that every leaf $\mathcal{L}$ intersects a transversal slice through p at most a finite number of times $N(p)$. This is equivalent to all the leafs of $\mathcal{F}$ being compact. We call M *regular* if $N(p) = 1$ for all $p \in M$. In this case M is an S^1 fibre bundle. In the quasi-regular case ξ generates a locally free circle action, and the space of leaves $\mathcal{Z}$ is an orbifold. The projection $\pi : M \to \mathcal{Z}$ is a Seifert fibration.

Theorem 4.8 *Let (M, g) be a compact quasi-regular Sasakian manifold of dimension $2m+1$, and let $\mathcal{Z}$ be the space of leaves of the foliation $\mathcal{F}$ then*

i. The leaf space $\mathcal{Z}$ is a compact complex orbifold with Kähler metric h and Kähler form ω such that $\pi : (M, g) \to (\mathcal{Z}, h)$ is an orbifold Riemannian submersion, and a multiple of $[\omega]$ is in $H^2_{orb}(\mathcal{Z}, \mathbb{Z})$.

ii. $\mathcal{Z}$ is a normal $\mathbb{Q}$-factorial algebraic variety.

iii. $(\mathcal{Z}, h)$ has positive Ricci curvature if and only if $\mathrm{Ric}_g > -2$. In this case $\pi_1(\mathcal{Z}) = e$, and $\mathcal{Z}$ is uniruled with Kodaira dimension $\kappa(\mathcal{Z}) = -\infty$.

iv. (M, g) is Sasakian-Einstein if and only if $(\mathcal{Z}, h)$ is Kähler-Einstein with scalar curvature $4m(m+1)$.

See [13, 14] for more details. We want to invert theorem (4.8). First a couple definitions.

Definition 4.9 *A Kähler orbifold $(\mathcal{Z}, \omega)$ is* Hodge *if $[\omega] \in H^2_{orb}(\mathcal{Z}, \mathbb{Z})$.*

Recall the definition (2.29) in chapter 2 of $\mathrm{Ind}(\mathcal{Z})$ and that it is the largest integer d such that $\frac{c_1(\mathbf{K}_{\mathcal{Z}}^{-1})}{d}$ is an element of $H^2_{orb}(\mathcal{Z}, \mathbb{Z})$. If $d = \mathrm{Ind}(\mathcal{Z})$, there exists a holomorphic line V-bundle $\mathbf{F}$ with $\mathbf{F}^d = \mathbf{K}_{\mathcal{Z}}^{-1}$. The following is promised inversion result.

Theorem 4.10 *Let $(\mathcal{Z}, h)$ be a Hodge Kähler orbifold. Then there is an S^1 V-bundle $\pi : M \to \mathcal{Z}$ with first Chern class $[\omega]$. Let θ be the connection 1-form with $\frac{1}{2\pi} d\theta = \omega$, then the 1-form $\eta = \frac{1}{2\pi}\theta$ and the metric $g = \eta \otimes \eta + \pi^* h$ define a Sasakian structure on M.*

Proof. Let ξ be the vector field generated by the S^1 action on M. Then clearly ξ is a unit length Killing vector field for g. Since we have $d\eta = \pi^*\omega$, $\Phi(X) = \nabla_X \xi$ defines a $(1,1)$ tensor which is the lift of the complex structure J on $\mathcal{Z}$. That is, $d\eta(X,Y) = \pi^*\omega(X,Y) = g(\Phi(X), Y)$. And we have

$$\Phi^2(X) = -X + \eta(X)\xi, \quad \Phi(\xi) = 0, \quad \eta(\Phi(X)) = 0.$$

Let X, Y be vector fields π-related to $\tilde{X}, \tilde{Y}$ on $\mathcal{Z}$. Since $\pi_*\Phi(X) = J\pi_*(X) = J\tilde{X}$,

$$\pi_* N_\Phi(X,Y) = J[\tilde{X}, J\tilde{Y}] + J[J\tilde{X}, \tilde{Y}] - [J\tilde{X}, J\tilde{Y}] - J^2[\tilde{X}, \tilde{Y}] = 0.$$

Thus $N_\phi(X,Y)$ is vertical. And

$$g(N_\Phi(X,Y), \xi) = -\eta([\Phi X, \Phi Y]) = 2d\eta(\Phi X, \Phi Y) = 2d\eta(X,Y).$$

Thus it follows from proposition (4.3) that $\{g, \xi, \Phi\}$ is a Sasakian structure.

□

Note that we have a 1-parameter family of Sasakian structures $\{g_a, \xi_a, \Phi_a\}$ for $a \in \mathbb{R}^+$, where $g_a = a^2\eta \otimes \eta + a\pi^*h$ and $\xi_a = \frac{1}{a}\xi$.

In general M in the above theorem is an orbifold rather than a smooth manifold. We are interested in construction smooth manifolds. Suppose $\pi : M \to Z$ is an S^1 V-bundle over an orbifold Z. For $z \in Z$, let $\{\phi, U, \Gamma\}$ be a local uniformizing neighborhood centered at $z \in Z$. Thus $U \subset Z$ is open and covered by $\phi : \tilde{U} \to U$, where the finite group Γ fixes $\tilde{z}$, $\phi(\tilde{z}) = z$, and acts on $\tilde{U}$ with $\tilde{U}/\Gamma \cong U$. Then Γ acts on $\tilde{U} \times S^1$ by $(\tilde{x}, u) \to (\gamma^{-1}\tilde{x}, uh_z(\gamma))$ for $\gamma \in \Gamma$, where $h_z : \Gamma \to S^1$ is a homomorphism. And $\pi^{-1}(U)$ is isomorphic to $\tilde{U} \times S^1/\Gamma$. We see that M is a manifold if, and only if, h_z is an injection for each z. We come to the inversion result of main interest.

Corollary 4.11 *Let $(\mathcal{Z}, h)$ be a compact Fano orbifold with $\pi_1^{orb}(\mathcal{Z}) = e$. Let $\pi : M \to \mathcal{Z}$ be the S^1 V-bundle with first Chern class $\frac{1}{d}c_1(\mathcal{Z})$, where $d =$ $\mathrm{Ind}(\mathcal{Z})$. Suppose that the local uniformizing groups inject into S^1. Then M is a simply connected manifold and has a Sasakian structure $\{g, \xi, \Phi\}$ with $\mathrm{Ric}_g > 0$. If $(\mathcal{Z}, h)$ is Kähler-Einstein, then (M, g, ξ, Φ) is Sasakian-Einstein.*

Proof. By theorem (4.10) there is a family of Sasakian structures $\{g_a, \xi_a, \Phi_a\}$ on M, which is a smooth compact manifold, with $g_a = a^2\eta \otimes \eta + a\pi^*h$, for $a > 0$. By the solution to the Calabi conjecture [71, 5] we may assume that h is a Kähler metric of positive Ricci curvature. The O'Niell tensors T and N vanish, and for A we have $A_XY = -g(\Phi_aX, Y)\xi$ and $A_X\xi = \Phi_aX$ for horizontal vectors X and Y.(cf. [10]) And we have

$$\mathrm{Ric}_{g_a}(X,Y) = \mathrm{Ric}_h(X,Y) - 2g_a(X,Y). \tag{4.7}$$

Since $\mathrm{Ric}_{g_a}(X,\xi) = 2m\eta(X)$ and M is compact, we see that for a sufficiently small $\mathrm{Ric}_{g_a} > 0$. If $(\mathcal{Z},h)$ is Einstein then (4.7) shows that there is a unique $a > 0$ with $\mathrm{Ric}_{g_a} = 2mg_a$.

Suppose M is not simply connected, then M has at most a finite cover $\tilde{M}$ by Meyer's theorem. Since $\pi_1^{orb}(\mathcal{Z}) = e$, $\tilde{\pi} : \tilde{M} \to \mathcal{Z}$ is an S^1 V-bundle covering $\pi : M \to \mathcal{Z}$ which contradicts that the Chern class of $\pi : M \to \mathcal{Z}$ is not divisible in $H^2_{orb}(\mathcal{Z},\mathbb{Z})$. □

The following will have interesting applications.

Proposition 4.12 *Let (M,g,ξ,Φ) be a simply connected quasi-regular Sasakian-Einstein manifold(orbifold) of dimension $n = 2m+1$. Then*

$$\mathrm{Vol}(M,g) = \frac{\pi d}{m+1}\mathrm{Vol}(\mathcal{Z},h) = \frac{d}{m!}\Big(\frac{\pi}{m+1}\Big)^{m+1} c_1(\mathcal{Z})^m[\mathcal{Z}],$$

where h is the Kähler-Einstein metric on $\mathcal{Z}$ such that $\pi : M \to \mathcal{Z}$ is a Riemannian submersion and $d = \mathrm{Ind}(\mathcal{Z})$.

Proof. Since $\pi_1^{orb}(\mathcal{Z}) = e$ from an exact sequence in appendix A.2, M is the total space of an S^1 V-bundle L with $c_1(L) = \frac{1}{d}c_1(\mathcal{Z})$. Recall that $\mathrm{Ric}_h = 2(m+1)h$. Let $\eta = \frac{1}{2\pi}\theta$ where θ is the connection induced on L with curvature $(\frac{2(m+1)}{d})\omega$, where ω is the Kähler form of h. Then it follows from the above arguments that

$$g = \big(\tfrac{\pi d}{m+1}\big)^2\eta\otimes\eta + \pi^* h$$

is the Sasakian-Einstein metric. Integrating over the fibre gives the result. □

4.2 Construction of toric Sasakian-Einstein 5-manifolds

The results in chapter 3 and corollary (4.11) give a method for constructing toric Sasakian-Einstein 5-manifolds. We then calculate their homology using know results on Seifert S^1-bundles. We then make use of the classification of simply connected spin 5-manifolds of S. Smale to determine the diffeomorphism type of the smooth examples M that our method constructs when $\pi_1(M) = e$.

Theorem 4.13 *Let X be a symmetric toric Fano surface. Let $\pi : M \to X$ be the S^1 V-bundle with Chern class $\frac{1}{d}c_1(X)$, with $d = \mathrm{Ind}(X)$. Then M has a Sasakian-Einstein structure. If $\pi_1^{orb}(X) = e$, then M is simply connected.*

Proof. This follows from theorem (3.31) and corollary (4.11) □

In most of our examples X will be special symmetric, in which case $\mathrm{Ind}(X) = 1$ or 2 from proposition (2.30). Then the M will be the S^1 V-bundle associated to $\mathbf{K}_X^{-1}$ or $\mathbf{K}_X^{-\frac{1}{2}}$.

Proposition 4.14 *Let (M, g, ξ, Φ) be a simply connected quasi-regular Sasakian-Einstein 5-manifold(orbifold) with leaf space X a toric orbifold surface. Then*

$$\mathrm{Vol}(M, g) = d\left(\frac{\pi}{3}\right)^3 \mathrm{Vol}(\Sigma_{-k}),$$

where Σ_{-k} is the polytope associated to X and its anti-canonical support function $-k$.

Proof. This follows from proposition (4.12) and corollary (2.27). □

This will be useful for the families of examples we will construct. for which $d = 1$ or 2.

In general M constructed in this theorem is an orbifold. See the remarks before corollary (4.11) for the condition necessary for the smoothness of M. Producing smooth examples is not difficult. In chapter (5) we will develop a technique for producing infinitely many smooth examples via theorem 4.13. This will give infinitely many examples of Sasakian-Einstein manifolds (M, g, ξ, Φ) with $b_2(M) = m$ for every odd $m \geq 3$.

For any orbifold X with local uniformizing systems $\{\tilde{U}_i, \Gamma_i, \phi_i\}$, the *order* of X is the least common multiple of the orders of the uniformizing groups Γ_i and is denoted $\text{Ord}(X)$. In the case we are considering X is a projective surface with an orbifold structure. Let $S_X \subset X$ be the *orbifold singular set* of X Let $D_j, j = 1, \dots, n$ be the irreducible curves contained in S_X. The order of the stablizer group Γ_x for $x \in D_j$ is constant on an open dense subset. Denote it by m_j. Note that the singular set of X as a complex variety has codimension two, i.e. is discreet in this case, and is contained in S_X.

Since the Seifert S^1-bundle $M \to X$ is an S^1 V-bundle, it has a Chern class $c_1(M/X) \in H^2(X, \mathbb{Q})$. Taking its $\text{Ord}(X)$ power we get an S^1 fiber bundle. Thus we can take $\text{Ord}(X)c_1(M/X) \in H^2(X, \mathbb{Z})$. Let d be the largest integer such that $\text{Ord}(X)c_1(M/X) \in H^2(X, \mathbb{Z})$ is divisible by d. The following is due to J. Kollár.

Theorem 4.15 **([44, 46])** *Let $M \to X$ be an S^1 Seifert bundle with smooth total space over a projective orbifold surface. Suppose that $H_1(M,\mathbb{Q}) = 0$ and $H_1^{orb}(X,\mathbb{Z}) = 0$. If $r = rank H^2(X,\mathbb{Q})$, then the cohomology of M is*

p	0	1	2	3	4	5
H^p	$\mathbb{Z}$	0	$\mathbb{Z}^{r-1} + \mathbb{Z}/d$	$\mathbb{Z}^{r-1} + \sum_i (\mathbb{Z}/m_i)^{2g(D_i)}$	$\mathbb{Z}/d$	$\mathbb{Z}$

where $g(D_i) = \dim H^1(D_i, \mathcal{O}_{D_i})$ is the genus of D_i.

Note that if X toric, then the curves $D_j, j = 1, \ldots, n$ are a subset of the anti-canonical divisor, and each has genus zero.

We make use of the classification, due to S. Smale, of smooth compact simply connected 5-manifolds which in addition are spin. Recall that M is spin when $w_2(M) = 0$, where $w_2(M)$ is the second Stiefel-Whitney class of M.

Theorem 4.16 **([63])** *There is a one-to-one correspondence between compact smooth simply connected spin 5-manifolds M and finitely generated abelian groups. The correspondence is given as follows.*

i. For any such M, $H_2(M,\mathbb{Z}) \cong \mathbb{Z}^m \oplus T \oplus T$, where T is torsion.

ii. For any finite abelian group T and $m \geq 0$ there is a unique M with $H_2(M,\mathbb{Z}) \cong \mathbb{Z}^m \oplus T \oplus T$.

In particular, if $H_2(M,\mathbb{Z}) \cong \mathbb{Z}^m$ then $M \underset{\text{diff}}{\cong} \#m(S^2 \times S^3)$. We have the following restriction on M.

Corollary 4.17 *Suppose the Sasakian-Einstein space in theorem (4.13) is smooth and simply connected. Then*

$$M \underset{\textit{diff}}{\cong} \#m(S^2 \times S^3),$$

where $m = b_2(X) - 1$.

4.3 Some classification results

We consider the problem of which smooth 5-manifolds M admit toric Sasakian-Einstein structures. Of course, we have the well known properties discussed above. We must have $\pi_1(M)$ finite, and the universal cover $\tilde{M}$ is spin. We have the following partial converse to the result of the previous section.

Theorem 4.18 *Let M be a simply connected toric Sasakian-Einstein 5-manifold which is quasi-regular. Then the leaf space $\mathcal{Z}$ of $\mathcal{F}$ is a toric variety and $M \underset{diff}{\cong} \#m(S^2 \times S^3)$, for some m. Furthermore, the Sasakian-Einstein structure (M, g, ξ, Φ) is non-deformable as a Sasakian-Einstein manifold fixing the foliation $\mathcal{F}$ determined by ξ.*

The condition that $\mathcal{F}$ is preserved in the last statement can probably be removed. It is sufficient to assume that quasi-regularity is preserved.

Proof. We have T^2 acting on M a Sasakian isometries. By theorem (4.8) $(\mathcal{Z}, h)$ is a normal orbifold surface with a positive scalar curvature Kähler-Einstein metric h. Thus $\mathbf{K}_{\mathcal{Z}}^{-1} > 0$. And for some $n > 0$, $\mathbf{K}_{\mathcal{Z}}^{-n}$ is very ample. Thus $\iota_{|\mathbf{K}_{\mathcal{Z}}^{-n}|} : \mathcal{Z} \to \mathbb{P}(W^*)$, where $W = H^0(\mathcal{Z}, \mathcal{O}(\mathbf{K}_{\mathcal{Z}}^{-n}))$, is an embedding. Since T^2 acts holomophically on $\mathcal{Z}$, it complexifies to an action of $T^2_{\mathbb{C}} = \mathbb{C}^* \times \mathbb{C}^*$. Furthermore, $T^2_{\mathbb{C}}$ acts on W^*. Thus the action of $T^2_{\mathbb{C}}$ is the restriction of that on $\mathbb{P}(W^*)$ to $\mathcal{Z} \subset \mathbb{P}(W^*)$. The action of the torus on W^* is completely reducible. Let M be the group of characters of $T^2_{\mathbb{C}}$. Then $W^* = \oplus_{m \in M} W_m$, where $W_m = \{w \in W^* : tw = m(t)w \text{ for all } t \in T^2_{\mathbb{C}}\}$. Thus $T^2_{\mathbb{C}}$ acts algebraically on $\mathcal{Z}$.

There exists a 2-dimensional orbit $P \subset \mathcal{Z}$ which must be dense. It is easy to see that the stablizer of $z \in P$ is a finite subgroup of $T^2 \subset T^2_{\mathbb{C}}$. Thus $P \cong T^2_{\mathbb{C}}$. And it follows that $\mathcal{Z}$ is a toric variety. (See [56])

We have the exact sequence [37]

$$\cdots \to \pi_1(S^1) \to \pi_1(M) \to \pi_1^{orb}(\mathcal{Z}) \to e.$$

Thus $\pi_1^{orb}(\mathcal{Z}) = e$. And corollary (4.17) determines the diffeotype.

Suppose $\{g_t, \eta_t, \Phi_t\}$ is a 1-parameter family of Sasakian-Einstein structure. Then we have a family $(\mathcal{Z}_t, h_t)$ of Kähler-Einstein structures, i.e. each $\mathcal{Z}_t$ carries a complex structure an Kähler-Einstein metric on the same underlying orbifold. Let $D = \sum_i D_i$ be the anti-canonical divisor of $\mathcal{Z}$. Let $\Omega^1_{\mathcal{Z}}$ and $\Omega^1(\log D)$ denote respectively the *orbifold* sheaves of differential forms and differential forms with logarithmic poles along $D = \sum D_i$. That is, they are the quotients of the corresponding sheaves on uniformizing neighborhoods. Note that they are coherent analytic sheaves. Then

$$0 \to \Omega^1_{\mathcal{Z}} \longrightarrow \Omega^1_{\mathcal{Z}}(\log D) \longrightarrow \bigoplus_{i=1}^{d} \mathcal{O}_{D_i} \to 0 \tag{4.8}$$

and

$$\Omega^1_{\mathcal{Z}}(\log D) \cong \mathcal{O}_{\mathcal{Z}} \oplus \mathcal{O}_{\mathcal{Z}}. \tag{4.9}$$

See [27, 57] for a proof. By Serre duality $H^1(\mathcal{Z}, \Theta_{\mathcal{Z}}) \cong H^1(\mathcal{Z}, \Omega^1(\mathbf{K}_{\mathcal{Z}}))$. Tensor (4.8) with $\mathcal{O}(\mathbf{K}_{\mathcal{Z}})$ and take the long exact cohomology sequence

$$\cdots \to \bigoplus_i H^0(D_i, \mathcal{O}_{D_i}(\mathbf{K}_{\mathcal{Z}})) \to H^1(\mathcal{Z}, \Omega^1(\mathbf{K}_{\mathcal{Z}})) \to H^1(\mathcal{Z}, \mathcal{O}(\mathbf{K}_{\mathcal{Z}}))^{\oplus 2} \to \cdots.$$

Since $\mathbf{K}_{\mathcal{Z}} < 0$, Kodaira vanishing shows that $H^0(D_i, \mathcal{O}_{D_i}(\mathbf{K}_{\mathcal{Z}})) = H^1(\mathcal{Z}, \mathcal{O}(\mathbf{K}_{\mathcal{Z}})) = 0$, thus $H^1(\mathcal{Z}, \Omega^1(\mathbf{K}_{\mathcal{Z}})) = 0$. Also $H^2(\mathcal{Z}, \Theta_{\mathcal{Z}}) \cong H^0(\mathcal{Z}, \Omega^1(\mathbf{K}_{\mathcal{Z}})) = 0$. Thus for small t the $\mathcal{Z}_t$ are biholomorphic to $\mathcal{Z}_0$.

By the work of S. Bando and T. Mabuchi [7] the Kähler-Einstein metric h_t is unique up to the action of $\mathrm{Aut}(\mathcal{Z}_t)_0$, the connected component of the identity in $\mathrm{Aut}(\mathcal{Z}_t)$. Thus there exists a family $\phi_t \in \mathrm{Aut}(\mathcal{Z}_t)_0 \cong \mathrm{Aut}(\mathcal{Z}_0)_0$ with $\phi_t^* h_t = h_0$. □

Proposition 4.19 *Let (M, g, ξ, Φ) be a quasi-regular toric Sasakian-Einstein 5-manifold with $\pi_1(M) = e$ and $H_2(M, \mathbb{Q}) = 0$. Then $M \underset{diff}{\cong} S^5$ and $\{g, \xi, \Phi\}$ is the standard round Sasakian structure.*

Proof. Since $w_2(M) = 0$ and M is simply connected, by corollary (4.17) we have $M \underset{\text{diff}}{\cong} S^5$. Also the space of leaves $\mathcal{Z}$ of the foliation $\mathcal{F}$ has $\pi_1^{orb}(\mathcal{Z}) = e$ and $b_2(\mathcal{Z}) = 1$. And the toric orbifold $\mathcal{Z}$ is characterized by three vectors $\{\sigma_1, \sigma_2, \sigma_3\} \subset \mathbb{Z} \times \mathbb{Z}$, and has cyclic orbifold uniformizing groups. Furthermore, from theorem A.8 we have

$$\cdots \to \pi_2^{orb}(W) \to \mathbb{Z}^2/\mathbb{Z}\{\sigma_1, \sigma_2, \sigma_3\} \to \pi_1^{orb}(\mathcal{Z}) \to \pi_1^{orb}(W) \to e, \qquad (4.10)$$

where $W \cong \mathcal{Z}/T^2$ is an orbifold with boundary. In our case W is homeomorphic to a disk, so $\mathbb{Z}\{\sigma_1, \sigma_2, \sigma_3\} = \mathbb{Z} \times \mathbb{Z}$. Consider the sublattice $\mathbb{Z}\{\sigma_1, \sigma_2\} \subset \mathbb{Z} \times \mathbb{Z}$. We have $\mathbb{Z} \times \mathbb{Z}/\mathbb{Z}\{\sigma_1, \sigma_2\} \cong \mathbb{Z}_p$. Thus $p\sigma_3 = -a_1\sigma_1 - a_2\sigma_2$ with a_1, a_2 positive. If $\tau_3 = \mathbb{R}_{\geq 0}\sigma_1 + \mathbb{R}_{\geq 0}\sigma_2$, then let $U_{\tau_3} \subset \mathcal{Z}$ be the the corresponding affine neighborhood. We have the uniformizing system

$\mathbb{C}^2 \to U_{\tau_3}$ with group $\mathbb{Z}_p$ acting by $(z_1, z_2) \to (e^{2\pi i \frac{a_1}{p} k} z_1, e^{2\pi i \frac{a_2}{p} k} z_2)$, for $k = 0, 1, \ldots, p-1$. Set $a_3 = p$. Then repeating the same argument gives affine neighborhoods U_{τ_1}, U_{τ_2} with uniformizing groups $\mathbb{Z}_{a_1}$ and $\mathbb{Z}_{a_2}$, where U_{τ_1} is the quotient of $\mathbb{C}^2$ by the action $(z_1, Z_2) \to (e^{2\pi i \frac{a_2}{a_1} k} z_1, e^{2\pi i \frac{a_3}{a_1} k} z_2)$, for $k = 0, 1, \ldots, a_1$, and *mutatis mutandis* for U_{τ_2}. We see that $\mathcal{Z} \cong \mathbb{C}P^2_{a_1, a_2, a_3}$, the weighted projective plane with the standard orbifold structure. Then the proposition follows from the next proposition. □

We are considering the weighted projective space $\mathbb{C}P^n_{a_0, \ldots, a_n}$ with the following orbifold structure. For $i \in \{0, 1, \ldots, n\}$, consider the map $\phi_i : \mathbb{C}^n \to \mathbb{C}P^n_{a_0, \ldots, a_n}$ given by $\phi_i(z_1, \ldots, z_n) = [z_1 : \cdots : z_i : 1 : z_{i+1} : \cdots : z_n]$. Then the $\{\phi_i, \Gamma_i = \mathbb{Z}_{a_i}\}$ are a system of uniformizing neighborhoods. Note that as analytic spaces or toric varieties we have $\mathbb{C}P^n_{a_0, \ldots, a_n} \cong \mathbb{C}P^n / \mathbb{Z}_{a_0} \times \cdots \times \mathbb{Z}_{a_n}$. But they are not isometric orbifolds. For one thing $\pi_1^{orb}(\mathbb{C}P^n_{a_0, \ldots, a_n}) = e$ and $\pi_1^{orb}(\mathbb{C}P^n / \mathbb{Z}_{a_0} \times \cdots \times \mathbb{Z}_{a_n}) = \mathbb{Z}_{a_0} \times \cdots \times \mathbb{Z}_{a_n}$. The existence of metrics with certain properties depends strongly on the orbifold structure.

Proposition 4.20 *The only weighted projective plane admitting a Kähler-Einstein metric is $\mathbb{C}P^2_{1,1,1} = \mathbb{C}P^2$, and in this case it is up to isometry and homothety the Fubini-Study metric.*

Proof. Consider the canonical V-bundle $\mathbf{K}$ of $\mathbb{C}P^m_{a_0, \ldots, a_m}$. Denote by γ the action of $\mathbb{C}^*$ on $\mathbb{C}^{m+1}$, $\gamma(w)(z_0, \ldots, z_m) = (w^{a_0} z_0, \ldots, w^{a_m} z_m)$. The meromorphic $m+1$-form $\Omega = \frac{dz_0}{z_0} \wedge \cdots \wedge \frac{dz_m}{z_m}$ is invariant under this action. Let Y be the vector field generated by γ, $Y = \gamma_*(\frac{\partial}{\partial z})$. Then $\omega = Y \lrcorner \Omega$ is a meromorphic section of $\mathbf{K}$. We have $Y = \Sigma_{k=0}^m a_k z_k \frac{\partial}{\partial z_k}$, and

$$\omega = \sum_{k=0} m(-1)^k a_k \frac{dz_0}{z_0} \wedge \cdots \wedge \frac{\hat{dz_k}}{z_k} \wedge \cdots \wedge \frac{dz_m}{z_m}. \tag{4.11}$$

Note that we are considering the V-manifold canonical bundle. As an algebraic variety the canonical sheaf and divisors on $\mathbb{C}P^m_{a_0,\ldots,a_m}$ are in general different. Consider the V-manifold map $\psi : \mathbb{C}P^m \longrightarrow \mathbb{C}P^m_{a_0,\ldots,a_m}$ with $\psi([z_0,\ldots,z_m]) = [z_0^{a_0},\ldots,z_m^{a_m}]$. Then $\psi^* K$ is the line bundle on $\mathbb{C}P^m$ associated to the divisor $-a_0 H_0 - a_1 H_1 - \cdots - a_m H_m$, where H_k is the hyperplane $z_k = 0$, which is linearly equivalent to $-(a_0 + \cdots + a_m)H$.

Suppose $m = 2$, and $X = \mathbb{C}P^2_{a_0,a_1,a_2}$. We have

$$\psi^* c_1(X)^2[\mathbb{C}P^2] = \psi^* c_1(K)^2[\mathbb{C}P^2] = (a_0 + a_1 + a_2)^2.$$

Thus we have $c_1(X)^2[X] = \frac{(a_0+a_1+a_2)^2}{a_0 a_1 a_2}$. Note that here, and in the following, we are using V-bundle characteristic classes which can be computed via Chern-Weil forms. (cf. [36])

For any compact smooth orbifold X we define the *orbifold Euler characteristic* $\chi_{orb}(X) = e(X)[X]$, where $e(X)$ is the Euler class of X. And define the *orbifold signature* to be $\tau_{orb}(X) = \frac{1}{3}p_1(X)[X]$. On a Kähler orbifold we have (see Besse [10])

$$\chi_{orb}(X) = \frac{1}{8\pi^2}\int_X \left[|B_0|^2 + \frac{s^2}{12} - |\rho_0|^2\right] d\mu, \tag{4.12}$$

where $B_0 = W^-$ is the Bochner curvature and ρ_0 is the trace-free Ricci form. Likewise, we have

$$\tau_{orb}(X) = \frac{1}{12\pi^2} \int_X \left[\frac{s^2}{24} - |B_0|^2 \right] d\mu. \tag{4.13}$$

Combining (4.12) and (4.13) we get

$$\chi_{orb}(X) - 3\tau_{orb}(X) = \frac{1}{8\pi^2} \int_X \left[3|B_0|^2 - |\rho_0|^2 \right] d\mu. \tag{4.14}$$

Thus if $\rho_0 = 0$ we get the orbifold Miyoaka-Yau inequality $\chi_{orb} \geq 3\tau_{orb}$. Suppose $M = \mathbb{C}P^2_{a_0,a_1,a_2}$ with $a_0 \leq a_1 \leq a_2$ and $(a_0, a_1, a_2) = 1$. Then $\chi_{orb}(X) = \frac{1}{a_0} + \frac{1}{a_1} + \frac{1}{a_2}$. (see [62]) We have

$$\tau_{orb}(X) = \frac{1}{3}(c_1^2 - 2c_2)[X] = \frac{1}{3}(c_1^2[X] - 2\chi_V) = \frac{1}{3}\frac{a_0^2 + a_1^2 + a_2^2}{a_0 a_1 a_2}, \tag{4.15}$$

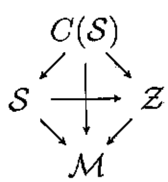
and we have $\chi_V = \frac{a_1a_2 + a_0a_2 + a_0a_1}{a_0a_1a_2}$. But it is easy to see that $a_1a_2 + a_0a_2 + a_0a_1 \leq a_0^2 + a_1^2 + a_2^2$ with equality only if $a_0 = a_1 = a_2$. So we have a contradiction unless $a_0 = a_1 = a_2 = 1$. In this case from (4.14) we have $B_0 = 0$, and the metric has constant holomorphic sectional curvature. □

In other words, any toric Sasakian-Einstein rational homology 5-sphere is just a quotient of the round metric on S^5. There are infinitely many quasi-regular toric Sasakian structures on S^5 with positive Ricci curvature. The associated Kähler orbifolds $\mathcal{Z}$ are the weighted projective planes which admit positive Ricci curvature Kähler metrics by the solution to the Calabi conjecture. (cf. [71] or [5])

Chapter 5

3-Sasakian manifolds

In this chapter we define 3-Sasakian manifolds, the closely related quaternionic-Kähler spaces, and their twistor spaces. These are sister geometries where one is able to pass from one to the other two by considering the appropriate orbifold fibration. Given a 3-Sasakian manifold $\mathcal{S}$ there is the associated twistor space $\mathcal{Z}$, quaternionic-Kähler orbifold $\mathcal{M}$, and hyperkähler cone $C(\mathcal{S})$. This is characterized by the *diamond*:



The equivalent 3-Sasakian and quaternionic-Kähler reduction procedures provide an elementary method for constructing 3-Sasakian and quaternionic-Kähler orbifolds. This method is effective in producing smooth 3-Sasakian manifolds, though the quaternionic-Kähler spaces obtained are rarely smooth. In particular, we are interested in toric 3-Sasakian 7-manifolds $\mathcal{S}$ and their associated four dimensional quaternionic-Kähler orbifolds $\mathcal{M}$. Here toric means that the structure is preserved by an action of the real two torus T^2. In four dimensions quaternionic-Kähler means that $\mathcal{M}$ is Einstein and anti-self-dual, i.e. the self-dual half of the Weyl curvature vanishes $W_+ \equiv 0$. These examples are well known and they are all obtained by reduction. (cf. [18] and [22]) In this case we will associate two more Einstein spaces to the four Einstein spaces in the diamond. To each diamond of a toric 3-Sasakian manifold we have a special symmetric toric Fano surface X and a Sasakian-Einstein manifold M and the following diagram where the horizontal arrows are inclusions.

$$\begin{array}{ccc} M & \longrightarrow & \mathcal{S} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{Z} \\ & & \downarrow \\ & & \mathcal{M} \end{array} \tag{5.1}$$

The motivation is twofold. First, it adds two more Einstein spaces to the examples on the right considered by C. Boyer, K. Galicki, and others in [18, 13] and also by D. Calderbank and M. Singer [22]. Second, M is smooth when the 3-Sasakian space $\mathcal{S}$ is. And the smoothness of $\mathcal{S}$ is ensured by a relatively mild condition on the moment map. Thus we get infinitely many Sasakian-Einstein manifolds with arbitrarily high second Betti numbers paralleling the 3-Sasakian manifolds constructed in [18].

5.1 Definitions and basic properties

Definition 5.1 *Let $(\mathcal{S}, g)$ be a Riemannian manifold of dimension $n = 4m+3$. Then $\mathcal{S}$ is 3-Sasakian if it admits three Killing vector fields $\{\xi^1, \xi^2, \xi^3\}$ each satisfying definition (4.1) such that $g(\xi^i, \xi^k) = \delta_{ij}$ and $[\xi^i, \xi^j] = 2\epsilon_{ijk}\xi^k$.*

We have a triple of Sasakian structures on $\mathcal{S}$. For $i = 1, 2, 3$ we have $\eta^i(X) = g(\xi^i, X)$ and $\Phi^i(X) = \nabla_X \xi^i$. We say that $\{g, \xi^i, \eta^i, \Phi^i : i = 1, 2, 3\}$ defines a *3-Sasakian structure* on $\mathcal{S}$. Each of the three Sasakian structures satisfies the properties of proposition (4.2), and the quaternionic nature of a 3-Sasakian structure is reflected in the following.

Proposition 5.2 *The tensors $\Phi^i, i = 1, 2, 3$ satisfy the following identities.*

i. $\Phi^i(\xi^j) = -\epsilon_{ijk}\xi^k$,

ii. $\Phi^i \circ \Phi^j = -\epsilon_{ijk}\Phi^k + \xi^i \otimes \eta^j - \delta_{ij} Id$

Notice that if $\alpha = (a_1, a_2, a_3) \in S^2 \subset \mathbb{R}^3$ then $\xi(\alpha) = a_1\xi^1 + a_2\xi^2 + a_3\xi^3$ is a Sasakian structure. Thus a 3-Sasakian manifold come equipped with an S^2 of complex structure.

Similar to proposition (4.4), the following proposition can be taken as a definition of a 3-Sasakian manifold.

Proposition 5.3 *Let $(\mathcal{S}, g)$ be a Riemannian manifold of dimension $n = 4m+3$. Then $(\mathcal{S}, g)$ is 3-Sasakian if, and only if, the holonomy of the metric cone $(C(M), \bar{g})$ is a subgroup of $Sp(m+1)$. In other words, $(C(M), \bar{g})$ is hyperkähler.*

Proof.(sketch) Let $\phi = r\frac{\partial}{\partial r}$ be the Euler vector field on $C(M)$. Then define almost complex structures $I_i, i = 1, 2, 3$ by

$$I_i X = \Phi^i(X) - \eta^i(X)\phi, \text{ and } I_i\phi = \xi^i.$$

It is straight forward to verify that they satisfy $I_i \circ I_j = \epsilon_{ijk} I_k - \delta_{ij} Id$. And from the integrability condition on each Φ^i in definition (4.1) each $I_i, i = 1, 2, 3$ is parallel. □

Since a hyperkähler manifold is Ricci flat, the remarks following proposition (4.4) imply the following.

Corollary 5.4 *A 3-Sasakian manifold $(\mathcal{S}, g)$ of dimension $n = 4m + 3$ is Einstein with positive scalar curvature $s = 2(2m + 1)(4m + 3)$. Furthermore, if $(\mathcal{S}, g)$ is complete, then it is compact with finite fundamental group.*

The structure group of a 3-Sasakian manifold reduces to $Sp(m) \times \mathbb{I}_3$ where $\mathbb{I}_3$ is the 3×3 identity matrix. Thus we have

Corollary 5.5 *A 3-Sasakian manifold (M, g) is spin.*

Suppose $(\mathcal{S}, g)$ is compact. This will be the case in all examples considered here. Then the vector fields $\{\xi^1, \xi^2, \xi^3\}$ are complete and define a locally free action of $Sp(1)$ on $(\mathcal{S}, g)$. This defines a foliation $\mathcal{F}_3$, the *3-Sasakian foliation*. The generic leaf is either $SO(3)$ or $Sp(1)$, and all the leaves are compact. So $\mathcal{F}_3$ is quasi-regular, and the space of leaves is a compact orbifold, denoted $\mathcal{M}$. The projection $\varpi : \mathcal{S} \to \mathcal{M}$ exhibits $\mathcal{S}$ as an $SO(3)$ or $Sp(1)$ V-bundle over $\mathcal{M}$. The leaves of $\mathcal{F}$ are constant curvature 3-Sasakian 3-manifolds which must be homogeneous spherical space forms. Thus a leaf is $\Gamma \backslash S^3$ with $\Gamma \subset Sp(1)$. We say that $(\mathcal{S}, g)$ is *regular* if the foliation $\mathcal{F}_3$ is regular.

For $\beta \in S^2$ we also have the characteristic vector field $\xi(\beta)$ with the associated 1-dimensional foliation $\mathcal{F}_\beta \subset \mathcal{F}_3$. In this case $\mathcal{F}_\beta$ is automatically quasi-regular. Denote the leaf space of $\mathcal{F}_\beta$ as $\mathcal{Z}_\beta$ or just $\mathcal{Z}$. Then the natural projection $\pi : \mathcal{S} \to \mathcal{Z}$ is an S^1 Seifert fibration. And $\mathcal{Z}$ has all the properties of theorem (4.8).

Fix a Sasakian structure $\{\xi^1, \Phi^1, \eta^1\}$ on $\mathcal{S}$. The horizontal subbundle $\mathcal{H} = \ker \eta^1$ to the foliation $\mathcal{F}$ of ξ^1 with the almost complex structure $I = -\Phi^1|_{\mathcal{H}}$ define a CR structure on $\mathcal{S}$. The form $\eta = \eta^2 + i\eta^3$ is of type $(1,0)$ with respect to I. And $d\eta|_{\mathcal{H}\cap\ker(\eta)} \in \Omega^{2,0}(\mathcal{H} \cap \ker(\eta))$ is nondegenerate as a complex 2-form on $\mathcal{H} \cap \ker(\eta)$. Consider the complex 1-dimensional subspace $P \subset \Lambda^{1,0}\mathcal{H}$ spanned by η. Letting $\exp(it\xi^1)$ denote an element of the circle subgroup $U(1) \subset Sp(1)$ generated by ξ^1 one see that $\exp(it\xi^1)$ acts on P with character e^{-2it}. Then $\mathbf{L} \cong \mathcal{S} \times_{U(1)} P$ defines a holomorphic line V-bundle over $\mathcal{Z}$. And we have a holomorphic section θ of $\Lambda^{1,0}(\mathcal{Z}) \otimes \mathbf{L}$ such that

$$\theta(X) = \eta(\tilde{X}),$$

where $\tilde{X}$ is the horizontal lift of a vector field X on $\mathcal{Z}$. Let $D = \ker(\theta)$ be the complex distribution defined by θ. Then $d\theta|_D \in \Gamma(\Lambda^2 D \otimes \mathbf{L})$ is nondegenerate. Thus $D = \ker(\theta)$ is complex contact structure on $\mathcal{Z}$, that is, a maximally non-integrable holomorphic subbundle of $T^{1,0}\mathcal{Z}$. Also, $\theta \wedge (d\theta)^m$ is a nowhere zero section of $\mathbf{K}_{\mathcal{Z}} \otimes \mathbf{L}^{m+1}$. Thus $\mathbf{L} \cong \mathbf{K}_{\mathcal{Z}}^{-\frac{1}{m+1}}$ as holomorphic line V-bundles. We have the following strengthened version of (4.8) for 3-Sasakian manifolds.

Theorem 5.6 *Let $(\mathcal{S}, g)$ be a compact 3-Sasakian manifold of dimension $n = 4m+3$, and let $\mathcal{Z}_\beta$ be the leaf space of the foliation $\mathcal{F}_\beta$ for $\beta \in S^2$. Then $\mathcal{Z}_\beta$ is a compact $\mathbb{Q}$-factorial contact Fano variety with a Kähler-Einstein metric h with scalar curvature $s = 8(2m+1)(m+1)$. The projection $\pi : \mathcal{S} \to \mathcal{Z}$ is* *an orbifold Riemannian submersion with respect to the metrics g on $\mathcal{S}$ and h on $\mathcal{Z}$.*

The space $\mathcal{Z} = \mathcal{Z}_\beta$ is, up to isomorphism of all structures, independent of $\beta \in S^2$. We call $\mathcal{Z}$ the *twistor space* of $\mathcal{S}$. Consider again the natural projection $\varpi : \mathcal{S} \to \mathcal{M}$ coming from the foliation $\mathcal{F}_3$. This factors into $\pi : \mathcal{S} \to \mathcal{Z}$ and $\rho : \mathcal{Z} \to \mathcal{M}$. The generic fibers of ρ is a $\mathbb{C}P^1$ and there are possible singular fibers $\Gamma\backslash\mathbb{C}P^1$ which are simply connected and for which $\Gamma \subset U(1)$ is a finite group. And restricting to a fiber $\mathbf{L}|_{\mathbb{C}P^1} = \mathcal{O}(2)$, which is an V-bundle on singular fibers. Consider $g = \exp(\frac{\pi}{2}\xi^2) \in Sp(1)$ which gives an isometry of $\mathcal{S}$ $\varsigma_g : \mathcal{S} \to \mathcal{S}$ for which $\varsigma_g(\xi^1) = -\xi^1$. And ς_g descends to an anti-holomorphic isometry $\sigma : \mathcal{Z} \to \mathcal{Z}$ preserving the fibers.

We now consider the orbifold $\mathcal{M}$ more closely. Let $(\mathcal{M}, g)$ be any $4m$ dimensional Riemannian orbifold. An *almost quaternionic* structure on $\mathcal{M}$ is a rank 3 V-subbundle $\mathcal{Q} \subset End(T\mathcal{M})$ which is locally spanned by almost complex structures $\{J_i\}_{i=1,2,3}$ satisfying the quaternionic identities $J_i^2 = -Id$ and $J_1J_2 = -J_2J_1 = J_3$. We say that $\mathcal{Q}$ is compatible with g if $J_i^*g = g$ for $i = 1, 2, 3$. Equivalently, each $J_i, i = 1, 2, 3$ is skew symmetric.

Definition 5.7 *A Riemannian orbifold $(\mathcal{M}, g)$ of dimension $4m, m > 1$ is* quaternionic Kähler *if there is an almost quaternionic structure $\mathcal{Q}$ compatible with g which is preserved by the Levi-Civita connection.*

This definition is equivalent to the holonomy of $(\mathcal{M}, g)$ being contained in $Sp(1)Sp(m)$. For orbifolds this is the holonomy on $\mathcal{M} \setminus S_{\mathcal{M}}$ where $S_{\mathcal{M}}$ is the singular locus of $\mathcal{M}$. Notice that this definition always holds on an oriented Riemannian 4-manifold ($m = 1$). This case requires a different definition.

Consider the *curvature operator* $\mathcal{R} : \Lambda^2 \to \Lambda^2$ of an oriented Riemannian 4-manifold. With respect to the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$, we have

$$\mathcal{R} = \begin{pmatrix} W_+ + \frac{s}{12} & \overset{\circ}{r} \\ \overset{\circ}{r} & W_- + \frac{s}{12} \end{pmatrix}, \tag{5.2}$$

where W_+ and W_- are the selfdual and anti-self-dual pieces of the Weyl curvature and $\overset{\circ}{r} = \mathrm{Ric} - \frac{s}{4}g$ is the trace-free Ricci curvature. An oriented 4 dimensional Riemannian orbifold $(\mathcal{M}, g)$ is quaternionic Kähler if it is Einstein and anti-self-dual, meaning that $\overset{\circ}{r} = 0$ and $W_+ = 0$.

Theorem 5.8 *Let $(\mathcal{S}, g)$ be a compact 3-Sasakian manifold of dimension $n = 4m+3$. Then there is a natural quaternionic Kähler structure on the leaf space of $\mathcal{F}_3$, $(\mathcal{M}, \breve{g})$, such that the V-bundle map $\varpi : \mathcal{S} \to \mathcal{M}$ is a Riemannian submersion. Furthermore, $(\mathcal{M}, \breve{g})$ is Einstein with scalar curvature $16m(m+2)$).*

Proof. For any $x \in \mathcal{M}$ choose $z \in \varpi^{-1}(x)$ and a slice $W \subset \mathcal{S}$ through z invariant under $\Gamma = \mathrm{Stab}(z) \subset Sp(1)$. This gives a local uniformizing chart for $\mathcal{M}$, $\phi : \tilde{U} = W \to U$, where $W/\Gamma = U \subset \mathcal{M}$. And $W \times Sp(1) \to W \times_\Gamma Sp(1)$ uniformizes the V-bundle $\varpi : \mathcal{S} \to \mathcal{M}$. Let $\mathcal{H}$ be the horizontal distribution to the foliation $\mathcal{F}_3$. Given $X \in T\mathcal{S}$ denote its horizontal projection by hX. The bundle $\mathcal{Q}$ of almost complex structures is given over $\tilde{U}$ by $\tilde{J}_i = -\Phi^i|_{\mathcal{H}}$ with local basis of section $J_i, i = 1, 2, 3$ over W given by $(J_i)_w = (\tilde{J}_i)_w$. Here we are identifying the tangent space of W with $\mathcal{H}$ thereby giving W the submersion metric. Then proposition (5.2) ii) shows that $J_i, i = 1, 2, 3$ satisfy the quaternionic identities. To compute the covariant derivative of the $J_i, i = 1, 2, 3$ notice that they are the sections of the associated bundle $W \times Sp(1) \times_{Sp(1)} \mathfrak{sp}(1)$ corresponding to W and $\{\xi^1, \xi^2, \xi^3\} \subset \mathfrak{sp}(1)$. Let $X \in T_w W$ with $X = hX + V$ with vertical component induced by $\zeta \in \mathfrak{sp}(1)$. Let Y be a basic vector field on $W \times Sp(1)$, meaning horizontal and projectable. Then if $\check{\nabla}$ denotes the Levi-Civita connection on W, we have

$$\begin{aligned}\check{\nabla}_X J_i(Y) &= -h\nabla_X \Phi^i(Y) - \nabla_Y[\zeta, \xi^i] \\ &= -h[g(\xi^i, Y)X - g(X, Y)\xi^i] - \nabla_Y[\zeta, \xi^i] \\ &= -\nabla_Y[\zeta, \xi^i].\end{aligned} \tag{5.3}$$

And since $-\nabla[\zeta, \xi^i]$ is a local section of $\mathcal{Q}$, we have a quaternionic Kähler structure. Any quaternionic Kähler space is Einstein (cf. [61]). The scalar curvature is a simple consequence of the O'Niell tensors.

We will settle the $m = 1$ case. Let X, Y, Z, W be basic vector fields on $\mathcal{S}$. For simplicity they will be identified with their projections on $\mathcal{M}$. The O'Niell tensors are

$$A_X Y = \sum_i g(X, \Phi^i Y)\xi^i, \quad A_X \xi^i = \Phi^i(X), \quad T \equiv 0. \tag{5.4}$$

See Besse [10] for more details on Riemannian submersions. Straight forward calculation gives the relation between the Riemannian curvature tensors R of

$(\mathcal{S}, g)$ and $\check{R}$ of $(\mathcal{M}, \check{g})$:

$$R(X,Y)Z = \check{R}(X,Y)Z - 2\sum_i g(X, \Phi^i Y)\Phi^i(Z) + \sum_i g(Y, \Phi^i Z)\Phi^i(X) - \sum_i g(X, \Phi^i Z)\Phi^i(Y) \quad (5.5)$$

Let J be the almost complex structure on $\tilde{U}$ associated to ξ_β for $\beta \in S^2$ and $\Phi = -\nabla\xi_\beta$. If R_{XY} denotes the curvature of $\mathrm{End}(T\mathcal{S})$, then from (4.1)

$$R_{XY}\Phi(Z) = g(\Phi X, Z)Y - g(\Phi Y, Z)X + g(X, Z)\Phi(Y) - g(Y, Z)\Phi(X). \quad (5.6)$$

Using the curvature formulae (5.5) and (5.6) and the identities in proposition (5.2) we have

$$\begin{aligned}
\check{R}_{XY}J(Z) &= \check{R}(X,Y)J(Z) - J\check{R}(X,Y)Z \\
&= -R(X,Y)\Phi(Z) + \Phi(R(X,Y)Z) \\
&= -R_{XY}\Phi(Z) - g(Y,Z)\Phi(X) + g(Y, \Phi Z)X + g(X,Z)\Phi(Y) - g(X, \Phi Z)Y \\
&\quad - 2\sum_i g(X, \Phi^i Y)[\Phi^i \circ \Phi(Z) - \phi \circ \Phi^i(Z)] \\
&= 2\sum_i \check{g}(X, J_i Y)[J_i \circ JZ - J \circ J_i Z].
\end{aligned} \quad (5.7)$$

Thus the curvature $\check{R}$ of $\mathrm{End}(T\mathcal{M})$ restricts to $\mathrm{Skewsym}(\mathcal{Q}) \subset \mathrm{Skewsym}(T\mathcal{M}) \cong \Lambda^2$. Let $\Lambda^2_+ \subset \Lambda^2$ denote the subspace corresponding to

$\mathcal{Q}$ under this identification. So

$$\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_\perp, \tag{5.8}$$

where $\Lambda^2_\perp$ is the orthogonal compliment to Λ^2_+. Furthermore, denote by $\Lambda^2_- \subset \Lambda^2_\perp$ the subspace commuting with Λ^2_+. So $\Lambda^2_- \cong \mathfrak{sp}(m)$. And using the natural identification $\mathrm{Skewsym}(\mathcal{Q}) \cong \mathcal{Q}$, $J \to \frac{1}{2}[J, -] \in \mathrm{Skewsym}(\mathcal{Q})$, if $\pi_+ : \Lambda^2 \to \Lambda^2_+$ is the projection,

$$\check{R} = c\pi_+, \text{ for } c > 0. \tag{5.9}$$

Let $\mathcal{R} : \Lambda^2 \to \Lambda^2$ be the curvature operator of $(\mathcal{M}, \check{g})$. If $\alpha \in \Lambda^2_\perp$ and $\gamma \in \Lambda^2_+$, then

$$0 = \check{R}(\alpha)\gamma = [\mathcal{R}(\alpha), \gamma].$$

So $\mathcal{R}(\alpha) \in \Lambda^2_-$. Also, by the symmetry of the curvature operator,

$$\langle \mathcal{R}(\gamma), \alpha \rangle = \langle \mathcal{R}(\alpha), \gamma \rangle = 0.$$

So for any $\gamma \in \Lambda^2_+$, $\mathcal{R}(\gamma) \in \Lambda^2_+$. And we see that $\mathcal{R}(\gamma) = c\gamma$. Thus the curvature operator is

$$\mathcal{R} = c\pi_+ + \psi, \tag{5.10}$$

where $\psi(\beta) \in \Lambda^2_-$ for all $\beta \in \Lambda^2$ and vanishes on $(\Lambda^2_-)^\perp \subset \Lambda^2$. When $m = 1$, and $\mathcal{M}$ is 4 dimensional, Λ^2_+ is the usual bundle of selfdual forms, and $\Lambda^2_- = \Lambda^2_\perp$ is the bundle of anti-self-dual forms. In view of the decomposition (5.2) of $\mathcal{R}$ we have precisely that $(\mathcal{M}, \check{g})$ is anti-self-dual and Einstein. □

Both theorems (5.6) and (5.8) can be inverted. See [12] for the inverse of theorem (5.6). The inverse of theorem (5.8) was first proved by Konishi [47] in the regular case. We state it below for orbifolds as we will make use of it later. Alternatively, starting with a positive scalar curvature quaternionic Kähler orbifold $\mathcal{M}$ one constructs its twistor space $\mathcal{Z}$ as the bundle of unit vectors in $\mathcal{Q}$. Then as in [61] $\mathcal{Z}$ is positive scalar curvature Einstein with a complex contact structure and anti-holomorphic involution preserving the fibers, which are generically $\mathbb{C}P^1$. Then as in corollary (4.11) one constructs $\mathcal{S}$, and the complex contact form lifts to define the 3-Sasakian structure on $\mathcal{S}$.(cf. [12])

Theorem 5.9 *Let $(\mathcal{M}, \breve{g})$ a quaternionic Kähler orbifold of dimension $4n$ with positive scalar curvature normalized to $16n(n+2)$. Then there is a principle $SO(3)$ V-bundle $\varpi : \mathcal{S} \to \mathcal{M}$, for which the total space $\mathcal{S}$ admits a 3-Sasakian structure making ϖ a Riemannian submersion.*

In many cases the $SO(3)$ bundle lifts to an $Sp(1)$ bundle. The obstruction to this lifting is the Marchiafava-Romani class. An almost quaternionic structure is a reduction of the frame bundle to an $Sp(1)Sp(m)$ bundle. Let $\mathcal{G}$ be the sheaf of germs of smooth maps to $Sp(1)Sp(m)$. An almost quaternionic structure is an element $s \in H^1_{orb}(\mathcal{M}, \mathcal{G})$. Consider the exact sequence

$$0 \to \mathbb{Z}_2 \to Sp(1) \times Sp(m) \to Sp(1)Sp(m) \to 1. \qquad (5.11)$$

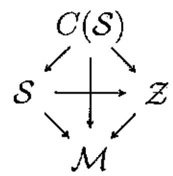
Definition 5.10 *The* Marchiafava-Romani class *is $\varepsilon = \delta(s)$, where*

$$\delta : H^1_{orb}(\mathcal{M}, \mathcal{G}) \to H^2_{orb}(\mathcal{M}, \mathbb{Z}_2)$$

is the connecting homomorphism.

One has that ε is the Stiefel-Whitney class $w_2(\mathcal{Q})$. Also, ε is the obstruction to the existence of a square root $\mathbf{L}^{\frac{1}{2}}$ of L. In the four-dimensional case $n = 1$, $\varepsilon = w_2(\Lambda^2_+) = w_2(T\mathcal{M})$. When $\varepsilon = 0$ for the 3-Sasakian space $\mathcal{S}$ associated to $(\mathcal{M}, \breve{g})$ we will always mean the one with $Sp(1)$ generic fibres.

The above is all encapsulated in the following "diamond" where the maps are orbifold fibrations. See [13] for more details.



(5.12)

On a quaternionic Kähler orbifold $(\mathcal{M}, g)$ one has the *moment map* $\mu \in \Gamma(\mathcal{M}, \mathfrak{g}^* \otimes \mathcal{Q})$, where $\mathfrak{g} \subseteq \mathfrak{Isom}(g)$ is a Lie subalgebra of the Lie algebra of Killing vector fields. If $X \in \mathfrak{g}$, then ∇X is in the subspace of $\mathrm{End}(T\mathcal{M})$ determined by the holonomy algebra $\mathfrak{sp}(1) \oplus \mathfrak{sp}(m)$ by a result of B. Kostant [48]. Since $\nabla_X - \nabla X = \mathcal{L}_X$, X preserves the quaternionic structure. We define

$$\langle \mu_{\mathcal{M}}, X \rangle = \pi_+(\nabla X). \tag{5.13}$$

We will make use of the *quaternionic Kähler quotient* to construct new examples. Let $G \subseteq \mathrm{Isom}(g)$ be a compact subgroup with moment map $\mu \in \Gamma(\mathcal{M}, \mathfrak{g}^* \otimes \mathcal{Q})$. Then G acts on $\mu^{-1}(0)$. If the action is locally free, then

$$\mathcal{M} /\!\!/ G := \mu^{-1}(0)/G \tag{5.14}$$

naturally has the structure of a quaternionic Kähler orbifold. Each of the four spaces in the diamond (5.12) has a moment map and reduction procedure which results in another diamond. For our purposes it will be more convenient to use the equivalent notion of 3-Sasakian reduction discussed in the next section.

5.2 3-Sasakian reduction

We will need 3-Sasakian reduction to construct examples of 3-Sasakian 7-manifolds. In particular, we are interested in toric 3-Sasakian 7-manifolds which have T^2 preserving the the 3-Sasakian structure. Up to coverings they are all obtainable by taking 3-Sasakian quotients of S^{4n-1} by a torus T^k, $k = n - 2$. See [18, 13] for more details.

Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold. And let $I(\mathcal{S}, g)$ be the subgroup in the isometry group $\mathrm{Isom}(\mathcal{S}, g)$ of 3-Sasakian automorphisms. Of course $\mathrm{Isom}(\mathcal{S}, g)$ contains the group generated by $\{\xi^1, \xi^2, \xi^3\}$. We mention the following.

Proposition 5.11 ([**13**]) *Let $(\mathcal{S}, g)$ be a complete 3-Sasakian manifold which is not of constant curvature. Then* $\mathrm{Isom}(\mathcal{S}, g) = I(\mathcal{S}, g) \times Sp(1)$ *or* $\mathrm{Isom}(\mathcal{S}, g) = I(\mathcal{S}, g) \times SO(3)$. *If $(\mathcal{S}, g)$ does have constant curvature, then* $\mathrm{Isom}(\mathcal{S})$ *strictly contains* $\mathrm{Isom}(\mathcal{S}, g) = I(\mathcal{S}, g) \times Sp(1)$ *or* $\mathrm{Isom}(\mathcal{S}, g) = I(\mathcal{S}, g) \times SO(3)$ *with* $\mathrm{Isom}(\mathcal{S}, g)$ *the centralizer of $Sp(1)$ or $SO(3)$.*

Let $G \subseteq I(\mathcal{S}, g)$ be compact. The group $I(\mathcal{S}, g)$ extends to the group $I(C(\mathcal{S}), \bar{g})$ of hyperkähler isometries of $(C(\mathcal{S}), \bar{g})$ preserving the factor $\mathbb{R}_+$. See proposition (5.3). Due to [41] there is a moment map $\mu : C(\mathcal{S}) \to \mathfrak{g}^* \otimes \mathbb{R}^3$, where $\mathfrak{g}$ is the Lie algebra of G. One can define the *3-Sasakian moment map*

$$\mu_{\mathcal{S}} : \mathcal{S} \to \mathfrak{g}^* \otimes \mathbb{R}^3 \tag{5.15}$$

by restriction $\mu_{\mathcal{S}} = \mu|_{\mathcal{S}}$. For 3-Sasakian reduction to work we must require the level set $\mu_{\mathcal{S}}^{-1}(0)$ to be invariant under the $Sp(1)$ action generated by the 3-Sasakian vector fields $\xi^i, i = 1, 2, 3$. From this requirement the 3-Sasakian moment map must have the following form. (cf. [13])

Let $\tilde{X}$ be the vector field on $\mathcal{S}$ induced by $X \in \mathfrak{g}$. The moment map is given by

$$\langle \mu_{\mathcal{S}}^a, X \rangle = \frac{1}{2}\eta^a(\tilde{X}), \quad a = 1, 2, 3 \text{ for } X \in \mathfrak{g}, \tag{5.16}$$

We have the following version of reduction.

Proposition 5.12 *Let $(\mathcal{S}, g)$ be a 3-Sasakian manifold and $G \subset I(\mathcal{S}, g)$ a connected compact subgroup. Assume that G act freely (locally freely) on $\mu_{\mathcal{S}}^{-1}(0)$. Then $\mathcal{S} /\!\!/ G = \mu_{\mathcal{S}}^{-1}(0)/G$ has the structure of a 3-Sasakian manifold (orbifold). Let $\iota : \mu_{\mathcal{S}}^{-1}(0) \to \mathcal{S}$ and $\pi : \mu_{\mathcal{S}}^{-1}(0) \to \mu_{\mathcal{S}}^{-1}(0)/G$ be the corresponding embedding and submersion. Then the metric $\breve{g}$ and 3-Sasakian vector fields are defined by $\pi^*\breve{g} = \iota^* g$ and $\pi_*\xi^i|_{\mu_{\mathcal{S}}^{-1}(0)} = \breve{\xi}^i$.*

Note that for $x \in \mathcal{S}$, $\operatorname{Im} d(\mu_{\mathcal{S}})_x = \mathfrak{g}_x^\perp \otimes \mathbb{R}^3$ where $\mathfrak{g}_x^\perp \subseteq \mathfrak{g}^*$ is the annihilator of the Lie subalgebra of the stablizer of $x \in \mathcal{S}$. So G acts locally freely on $\mu_{\mathcal{S}}^{-1}(0)$ if, and only if, 0 is a regular value of $\mu_{\mathcal{S}}$.

Consider the unit sphere $S^{4n-1} \subset \mathbb{H}^n$ with the metric g obtained by restricting the flat metric on $\mathbb{H}^n$. Give S^{4n-1} the standard 3-Sasakian structure induced by the right action of $Sp(1)$. Then $I(S^{4n-1}, g) = Sp(n)$ acting by the standard linear representation on the left. We have the maximal torus $T^n \subset Sp(n)$ and every representation of a subtorus T^k is conjugate to an inclusion $\iota_\Omega : T^k \to T^n$ which is represented by a matrix

$$\iota_\Omega(\tau_1, \ldots, \tau_k) = \begin{pmatrix} \prod_{i=1}^k \tau_1^{a_1^i} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \prod_{i=1}^k \tau_n^{a_n^i} \end{pmatrix}, \tag{5.17}$$

where $(\tau_1, \ldots, \tau_k) \in T^k$. Every such representation is defined by the $k \times n$ integral *weight matrix*

$$\Omega = \begin{pmatrix} a_1^1 & \cdots & a_k^1 & \cdots & a_n^1 \\ a_1^2 & \cdots & a_k^2 & \cdots & a_n^2 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_1^k & \cdots & a_k^n & \cdots & a_n^k \end{pmatrix} \tag{5.18}$$

Let $\{e_i\}, i = 1, \ldots, k$ be a basis for the dual of the Lie algebra of T^k, $\mathfrak{t}_k^* \cong \mathbb{R}^k$. Then the moment map $\mu_\Omega : S^{4n-1} \to \mathfrak{t}_k^* \otimes \mathbb{R}^3$ can be written as $\mu_\Omega = \sum_j \mu_\Omega^j e_j$ where

$$\mu_\Omega^j(\mathbf{u}) = \sum_l \bar{u}_l i a_l^j u_l. \tag{5.19}$$

In terms of complex coordinates $u_l = z_l + w_l j$ on $\mathbb{H}^n$ we have

$$\mu_\Omega^j(\mathbf{z}, \mathbf{w}) = i \sum_l a_l^j (|z_l|^2 - |w_l|^2) + 2k \sum_l a_l^j \bar{w}_l z_l. \tag{5.20}$$

Assume $\text{rank}(\Omega) = k$ otherwise we just have an action of a subtorus of T^k.

Denote by

$$\Delta_{\alpha_1,\ldots,\alpha_k} = \det \begin{pmatrix} a^1_{\alpha_1} & \cdots & a^1_{\alpha_k} \\ \vdots & & \vdots \\ a^k_{\alpha_1} & \cdots & a^k_{\alpha_k} \end{pmatrix} \tag{5.21}$$

the $\binom{n}{k}$ $k \times k$ minor determinants of Ω.

Definition 5.13 *Let $\Omega \in \mathscr{M}_{k,n}(\mathbb{Z})$ be a weight matrix.*

(i) Ω is non-degenerate *if $\Delta_{\alpha_1,\ldots,\alpha_k} \neq 0$, for all $1 \le \alpha_1 < \cdots < \alpha_k \le n$.*

Let Ω be non-degenerate, and let d be the gcd *of all the $\Delta_{\alpha_1,\ldots,\alpha_k}$, the kth determinantal divisor. Then Ω is* admissible

(ii) if $\gcd(\Delta_{\alpha_2,\ldots,\alpha_{k+1}}, \ldots, \Delta_{\alpha_1,\ldots,\hat{\alpha}_t,\ldots,\alpha_{k+1}}, \ldots, \Delta_{\alpha_1,\ldots,\alpha_k}) = d$ *for all length $k+1$ sequences $1 \le \alpha_1 < \cdots < \alpha_t < \cdots < \alpha_{k+1} \le n+1$.*

The quotient obtained in proposition (5.12) $\mathcal{S}_\Omega = S^{4n-1} /\!\!/ T^k(\Omega)$ will depend on Ω only up to a certain equivalence. Choosing a different basis of $\mathfrak{t}_k$ results in an action on Ω by an element in $Gl(k,\mathbb{Z})$. We also have the normalizer of T^n in $Sp(n)$, the Weyl group $\mathscr{W}(Sp(n)) = \Sigma_n \times \mathbb{Z}_2^n$ where Σ_n is the permutation group. $\mathscr{W}(Sp(n))$ acts on S^{4n-1} preserving the 3-Sasakian structure, and it acts on weight matrices by permutations and sign changes of columns. The group $Gl(k,\mathbb{Z}) \times \mathscr{W}(Sp(n))$ acts on $\mathscr{M}_{k,n}(\mathbb{Z})$.

The gcd d_j of the jth row of Ω divides d. We may assume that the gcd of each row of Ω is 1 by merely reparametrizing the coordinates τ_j on T^k. We say that Ω is in reduced form if $d = 1$.

Lemma 5.14 *Every non-degenerate weight matrix Ω is equivalent to a reduced matrix.*

So $\mathcal{S}_\Omega$ will only depend on the reduced form of Ω and equivalence up to the action of $Gl(k,\mathbb{Z}) \times \mathscr{W}(Sp(n))$.

Theorem 5.15 **([13, 18])** *Let $\Omega \in \mathscr{M}_{k,n}(\mathbb{Z})$ be reduced.*

(i) If Ω is non-degenerate, then $\mathcal{S}_\Omega$ is an orbifold.

(ii) Supposing Ω is non-degenerate, $\mathcal{S}_\Omega$ is smooth if and only if Ω is admissible.

Notice that the automorphism group of $\mathcal{S}_\Omega$ contains $T^{n-k} \cong T^n/\iota_\Omega(T^k)$.

We are primarily interested in 7-dimensional toric quotients. In this case there are infinite families of distinct quotients. We may take matrices of the form

$$\Omega = \begin{pmatrix} 1 & 0 & \cdots & 0 & a_1 & b_1 \\ 0 & 1 & \cdots & 0 & a_2 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_k & b_k \end{pmatrix}. \tag{5.22}$$

Proposition 5.16 **([18])** *Let $\Omega \in \mathscr{M}_{k,k+2}(\mathbb{Z})$ be as above. Then Ω is admissible if and only if $a_i, b_j, i, j = 1, \ldots, k$ are all nonzero, $\gcd(a_i, b_i) = 1$ for $i = 1, \ldots, k$, and we do not have $a_i = a_j$ and $b_i = b_j$, or $a_i = -a_j$ and $b_i = -b_j$ for some $i \neq j$.*

Proposition (5.16) shows that for $n = k + 2$ there are infinitely many reduced admissible weight matrices. One can, for example, choose $a_i, b_j, i, j = 1, \ldots k$ be all pairwise relatively prime. We will make use of the cohomology computation of R. Hepworth [40] to show that we have infinitely many smooth 3-Sasakian 7-manifolds of each second Betti number $b_2 \geq 1$. Let $\Delta_{p,q}$ denote the $k \times k$ minor determinant of Ω obtained by deleting the p^{th} and q^{th} columns.

Theorem 5.17 ([40][17, 18]) *Let $\Omega \in \mathscr{M}_{k,k+2}(\mathbb{Z})$ be a reduced admissible weight matrix. Then $\pi_1(\mathcal{S}_\Omega) = e$. And the cohomology of $\mathcal{S}_\Omega$ is*

p	0	1	2	3	4	5	6	7
H^p	$\mathbb{Z}$	0	$\mathbb{Z}^k$	0	G_Ω	$\mathbb{Z}^k$	0	$\mathbb{Z}$

where G_Ω is a torsion group of order

$$\sum |\Delta_{s_1,t_1}| \cdots |\Delta_{s_{k+1},t_{k+1}}|$$

with the summand with index $s_1, t_1, \ldots, s_{k+1}, t_{k+1}$ included if and only if the graph on the vertices $\{1, \ldots, k+2\}$ with edges $\{s_i, t_i\}$ is a tree.

If we consider weight matrices as in proposition (5.16) then the order of G_Ω is greater than $|a_1 \cdots a_k| + |b_1 \cdots b_k|$. We have the following.

Corollary 5.18 ([40][18]) *There are smooth toric 3-Sasakian 7-manifolds with second Betti number $b_2 = k$ for all $k \geq 0$. Furthermore, there are infinitely many possible homotopy types of examples $\mathcal{S}_\Omega$ for each $k > 0$.*

Let $\Omega \in \mathscr{M}_{k,k+2}(\mathbb{Z})$ be a reduced admissible weight matrix, so $\mathcal{S}_\Omega$ is a smooth 3-Sasakian 7-manifold. Recall that we have a right action of $Sp(1)$ on $\mathcal{S}_\Omega$ and $\mathcal{S}_\Omega/Sp(1) = \mathcal{M}_\Omega$ is the associated quaternionic kähler orbifold. We will denote $N(\Omega) = \mu_\Omega^{-1}(0)$ which has an action of $T^{k+2} \times Sp(1)$. Since Ω is reduced, the exact sequence

$$e \to T^k \xrightarrow{\iota_\Omega} T^{k+2} \to T^2 \to e, \tag{5.23}$$

induces an exact sequence

$$0 \to \mathbb{Z}^k \to \mathbb{Z}^{k+2} \to \mathbb{Z}^2 \to 0. \tag{5.24}$$

Denote $B(\Omega) := N(\Omega)/Sp(1)$ and $Q(\Omega) := T^{k+2}\backslash B(\Omega)$. Then we have the following commutative diagram

$$\begin{array}{ccc} N(\Omega) & & \\ \downarrow & & \\ B(\Omega) & & \\ \downarrow & \searrow & \\ \mathcal{M}_\Omega & \to & Q(\Omega) \end{array} \tag{5.25}$$

where the upper left arrow is a principal $Sp(1)$ fibration, lower left is a locally free T^k fibration, the diagonal has generic T^{k+2} fibers, and the bottom map has generic T^2 fibers. We will define a stratification of $B(\Omega)$, and likewise $N(\Omega)$. From the non-degeneracy of Ω it follows that at most one coordinate $u_\alpha, \alpha = 1, \ldots, k+2$ can vanish on either set.

$$\begin{aligned} N_0(\Omega) =& \{\mathbf{u} \in N(\Omega) : u_\alpha = 0, \text{ for some } \alpha = 1, \ldots, k+2\} \\ N_1(\Omega) =& \{\mathbf{u} \in N(\Omega) : u_\alpha \neq 0, \text{ for all } \alpha = 1, \ldots, k+2 \text{ and there exists a pair} \\ & (u_\alpha, u_\beta) \text{ lying on the same } \mathbb{C}\text{-line in } \mathbb{H}\} \\ N_2(\Omega) =& \{\mathbf{u} \in N(\Omega) : u_\alpha \neq 0, \text{ for all } \alpha = 1, \ldots, k+2 \text{ and there is no pair} \\ & (u_\alpha, u_\beta) \text{ lying on the same } \mathbb{C}\text{-line in } \mathbb{H}\} \end{aligned} \tag{5.26}$$

Let $\tilde{T}^{k+2}$ be the quotient of T^{k+2} by the diagonal element $(-1, \ldots, -1)$ which acts on $B(\Omega)$.

Proposition 5.19 *Let $\Omega \in \mathscr{M}_{k,k+2}(\mathbb{Z})$ be reduced and non-degenerate. Then* *(i)* $\tilde{T}^{k+2}$ *acts freely on* $B_2(\Omega)$.

(ii) $B_1(\Omega)$ *consists of* $k+2$ *components* $\bigsqcup_{\alpha=1}^{k+2} B_1(\alpha, \Omega)$ *where the stablizer of each point of* $B_1(\alpha, \Omega)$ *is a subgroup* $G_\alpha \subset \tilde{T}^{k+2}$ *with* $G_\alpha \cong S^1$.

(iii) $B_0(\Omega)$ *consists of* $k+2$ *orbits of* $\tilde{T}^{k+2}$ *with stabilizer groups* $F_\alpha \subset \tilde{T}^{k+2}$ *with* $F_\alpha \cong T^2$ *generated by* G_α *and* $G_{\alpha+1}$.

(iv) $Q(\Omega)$ *is a polygon with edges and vertices corresponding to components of* $B_1(\Omega)$ *and* $B_0(\Omega)$ *respectively.*

Proof. The action of $T^k \times Sp(1)$ on $N(\Omega)$ in terms of quaternionic coordinates u_α is given by $u_\alpha \to e^{i\theta_\alpha} u_\alpha q$ for $\alpha = 1, \ldots, k+2$ and $q \in Sp(1)$. We will consider the u_α as homogeneous coordinates on $B(\Omega) \subset \mathbb{H}P^{k+1}$. Consider complex coordinates $u_\alpha = z_\alpha + w_\alpha j$. We may act by an element of $Sp(1)$ so that $w_\beta = 0$ for any $\beta \in \{1, \ldots, k+2\}$. Suppose $[\mathbf{u}] \in B_1(\Omega)$, so each $u_\alpha \neq 0$. After acting by $Sp(1)$ we have $u_\alpha = z_\alpha$ and $u_\beta = z_\beta$, and it follows from equation (5.20) and non-degeneracy that $z_\alpha w_\alpha = 0$, for all $\alpha = 1, \ldots, k+2$. Acting by j if necessary, we may assume that w_{k+2}=0. The stablizer of $[\mathbf{u}]$ is the projection of the stablizer of $\mathbf{u}$ which is contain in $T^{k+2} \cdot S^1$ onto $\tilde{T}^{k+2}$. The action of $T^{k+2} \cdot S^1$ is

$$z_\alpha \to e^{i(\theta_\alpha + \phi)} z_\alpha \text{ and } w_\alpha \to e^{i(\theta_\alpha - \phi)} w_\alpha, \text{ for } \alpha = 1, \ldots, k+2, \tag{5.27}$$

where $e^{i\phi}$ is the coordinate of the S^1 factor. We have $e^{i\theta_{k+2}} = e^{-i\phi}$ and $e^{i\theta_\alpha} = e^{\pm i\phi}$ for $i = 1, \ldots, k+1$ depending on whether $z_\alpha = 0$ or $w_\alpha = 0$. So the stablizer group of $\mathbf{u} \in B_1(\Omega)$ is the S^1 subgroup $(e^{\pm i\theta}, \ldots, e^{\pm i\theta}, e^{i\theta}) \in \tilde{T}^{k+2}$, where there is a $+$ if $w_\alpha = 0$ and a $-$ if $z_\alpha = 0$.

Suppose $[\mathbf{u}] \in B_2(\Omega)$. Then as before we may suppose $w_{k+2} = 0$. There exists a β with $z_\beta w_\beta \neq 0$. Again the stablizer of $[\mathbf{u}]$ is the projection onto $\tilde{T}^{k+2}$ of the stablizer of $\mathbf{u}$ in $T^{k+2} \cdot S^1$. From equations (5.27) we have either $e^{i\theta_\beta} = e^{i\phi} = 1$ or $e^{i\theta_\beta} = e^{i\phi} = -1$. But since either $z_\alpha \neq 0$ or $w_\alpha \neq 0$ for all α, we have either $e^{i\theta_\alpha} = 1$ for $\alpha = 1, \ldots, k+2$ or $e^{i\theta_\alpha} = -1$ for $\alpha = 1, \ldots, k+2$. Thus $\tilde{T}^{k+2}$ act freely on $B_2(\Omega)$.

By a transformation by an element of $Gl(k, \mathbb{Q})$ we may normalize Ω to get

$$\Omega' = \begin{pmatrix} 1 & 0 & \cdots & 0 & f_1 & g_1 \\ 0 & 1 & \cdots & 0 & f_2 & g_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & f_k & g_k \end{pmatrix}, \tag{5.28}$$

for which we have $\mu_{\Omega'}^{-1}(0) = \mu_\Omega^{-1}(0)$. Then the $f_i, i = 1, \ldots, k$ and $g_i, i = 1, \ldots, k$ are nonzero. And we may assume that $g_1/f_1 < \cdots < g_i/f_i < \cdots < g_k/f_k$ after making a further transformation. Then the equations for $N(\Omega)$ in complex coordinates $u_\alpha = z_\alpha + w_\alpha j$ become

$$\begin{gathered} |z_l|^2 - |w_l|^2 + f_l(|z_{k+1}|^2 - |w_{k+1}|^2) + g_l(|z_{k+2}|^2 - |w_{k+2}|^2) = 0, \text{ and} \\ \bar{w}_l z_l + f_l \bar{w}_{k+1} z_{k+1} + g_l \bar{w}_{k+2} z_{k+2} = 0 \quad l = 1, \ldots, k. \end{gathered} \tag{5.29}$$

Suppose $[\mathbf{u}] \in B_1(\Omega)$, with $w_{k+2} = 0$ and $z_\alpha w_\alpha = 0$ for $\alpha = 1, \ldots, k+2$. Assume that $0 < g_1/f_1 < \cdots < g_i/f_i < \cdots < g_k/f_k$, for simplicity. The general case is only slightly more complicated. To $[\mathbf{u}]$ we associate a vector, which denotes the stablizer group, with a $+$ in the α^{th} component if $z_\alpha \neq 0$ and $w_\alpha = 0$ and a $-$ otherwise. We have the following: $v_i = (+, \ldots, +, -, \ldots, -, +)$ for $i = 1, \ldots, k$, which has i plus signs, $v_{k+1} = (-, \ldots, -, +, +)$, and $v_{k+2} =$

$(-,\ldots,-,-,+)$. Denote the corresponding stablizer group to v_i by G_i.

Let $B_0(\beta,\Omega)$ be the subset of $B_0(\Omega)$ with $u_\beta = 0$. Suppose $[\mathbf{u}] \in B_0(\beta,\Omega)$. As before, we may fix $u_{k+2} = z_{k+2}$, where we choose another coordinate if $\beta = k+2$. Then we have either $z_\alpha = 0$ or $w_\alpha = 0$ for $\alpha = 1,\ldots,k+2$. The stablizer of $[\mathbf{u}]$ is the S^1 subgroup generated by $e^{i\theta_\beta}$ along with the S^1 subgroup in the $N_1(\Omega)$ case. Using equations (5.29) and $\|\mathbf{u}\| = 1$, one can show that $B_0(\beta,\Omega)$ consists of a single orbit of $\tilde{T}^{k+2}$.

As discussed in [58] or [37] the quotient $Q(\Omega)$ of $B(\Omega)$ by $\tilde{T}^{k+2}$ is a closed polygon with $k+2$ edges, that we may label with the v_i, and $k+2$ vertices. □

Recall the orbifold $\mathcal{M}_\Omega$ has an action of $T^2 \cong T^{k+2}/\iota_\Omega(T^k)$, and can be characterized as in [58] and [37] by its orbit space and stablizer groups. See Appendix A.3 for more details. The orbit space is $\mathcal{M}_\Omega/T^2 = Q_\Omega$. We also use $G_i, i = 1,\ldots,k+2$ to denote the image of the G_i in T^2, which are S^1 subgroups by the non-degeneracy of Ω. Then Q_Ω is a polygon with $k+2$ edges $C_1, C_2, \ldots, C_{k+2}$, labeled in cyclic order, with the interior of C_i are orbits with stablizer G_i. Choose an explicit surjective homomorphism $\Phi : \mathbb{Z}^{k+2} \longrightarrow \mathbb{Z}^2$ annihilating the rows of Ω. So

$$\Phi = \begin{pmatrix} b_1 & b_2 & \cdots & b_{k+2} \\ c_1 & c_2 & \cdots & c_{k+2} \end{pmatrix} \tag{5.30}$$

It will be helpful to normalize Φ. After acting on the columns of Φ by $\mathscr{W}(Sp(k+2))$ and on the right by $Gl(2,\mathbb{Z})$ we may assume that $b_i > 0$ for $i = 1,\ldots,k+2$ and $c_1/b_1 < \cdots < c_i/b_i < \cdots < c_{k+2}/b_{k+2}$. Now in the above proof one has, up to a cyclic permutation, $v_1 = (+ - \cdots -), \ldots, v_i = (+ \cdots + - \cdots -), \ldots, v_{k+2} = (+ \cdots +)$. Then the stablizer groups $G_i \subset T^2$ are characterized by $(m_i, n_i) \in \mathbb{Z}^2$ where

$$(m_i, n_i) = \sum_{l=1}^{i} (b_l, c_l) - \sum_{l=i+1}^{k+2} (b_l, c_l), \quad i = 1, \cdots k+2. \tag{5.31}$$

It is convenient to take $(m_0, n_0) = -(m_{k+2}, n_{k+2})$.

5.3 Anti-self-dual Einstein orbifolds

We will consider toric anti-self-dual Einstein orbifolds in greater detail. Such an orbifold $\mathcal{M}$ is toric if it admits an effective action of T^2. By the previous section quaternionic Kähler reduction gives us infinitely many examples. By reducing $\mathbb{H}P^{k+1}$ by a subtorus $T^k \subset Sp(k+2)$ defined by an admissible matrix Ω we get a toric anti-self-dual orbifold $\mathcal{M}_\Omega$ with $b_2(\mathcal{M}) = k$. The orbifold $\mathcal{M}$ is characterized by a polygon $Q_\Omega = \mathcal{M}/T^2$ with $k+2$ edges labeled in cyclic order with $(m_0, n_0), (m_1, n_1), \ldots, (m_{k+2}, n_{k+2})$ in $\mathbb{Z}^2$ with $(m_0, n_0) = -(m_{k+2}, n_{k+2})$. These vectors satisfy the following:

a. The sequence m_i, $i = 0, \ldots k+2$ is strictly increasing.

b. The sequence $(n_i - n_{i-1})/(m_i - m_{i-1})$, $i = 1, \ldots k+2$ is strictly increasing.

We will make use of the following classification result of D. Calderbank and M. Singer.

Theorem 5.20 *Let $\mathcal{M}$ be a compact toric 4-orbifold with $\pi_1^{orb}(\mathcal{M}) = e$ and $k = b_2(\mathcal{M})$. Then the following are equivalent.*

i. One can arrange that the isotropy data of $\mathcal{M}$ satisfy a. and b. above by cyclic permutations, changing signs, and acting by $Gl(2,\mathbb{Z})$.

ii. $\mathcal{M}$ admits a toric anti-self-dual Einstein metric unique up to homothety and equivariant diffeomorphism. Furthermore, $(\mathcal{M}, g)$ is isometric to the quaternionic Kähler reduction of $\mathbb{H}P^{k+1}$ by a torus $T^k \subset Sp(k+2)$.

It is well known that the only possible smooth compact anti-self-dual Einstein spaces with positive scalar curvature are S^4 and $\overline{\mathbb{C}P}^2$, which are both toric. Note that the stablizer vectors $v_0 = (m_0, n_0), v_1 = (m_1, n_1), \ldots, v_{k+2} = (m_{k+2}, n_{k+2})$ form half a convex polygon with edges of increasing slope.

Theorem 5.21 *There is a one to one correspondence between compact toric anti-self-dual Einstein orbifolds $\mathcal{M}$ with $\pi_1^{orb}(\mathcal{M}) = e$ and special symmetric toric Fano orbifold surfaces X with $\pi_1^{orb}(X) = e$. By theorem (3.31) X has a Kähler-Einstein metric of positive scalar curvature. Under the correspondence if $b_2(\mathcal{M}) = k$, then $b_2(X) = 2k+2$.*

Proof. Suppose $\mathcal{M}$ has isotropy data $v_0, v_1, \ldots, v_{k+2}$. Then it is immediate that $v_0, v_1, \ldots, v_{k+2}, -v_1, -v_2, \ldots, -v_{k+1}$ are the vertices of a convex polygon in $N_{\mathbb{R}} = \mathbb{R}^2$, which defines an augmented fan Δ^* defining X. The symmetry of X is clear.

Suppose X is a special symmetric toric Fano surface. Then X is characterized by a convex polygon Δ^* with vertices $v_0, v_1, \ldots, v_{2k+4}$ with $v_{2k+4} = v_0$. Choose a primitive $p = (u, w) \in \mathbb{Z} \times \mathbb{Z}, w > 0$ which is not proportional to any $v_i - v_{i-1}, i = 1, \ldots, k+2$. Choose $s, t \in \mathbb{Z}$ with $su + tw = 1$. Then let $v_i', i = 0, \ldots, 2k+4$ be the images of the v_i under $\begin{bmatrix} w & -u \\ s & t \end{bmatrix}$ There is a $v'_j = (m'_j, n'_j)$ with m'_j smallest. And $v'_j, v'_{j+1}, \ldots, v'_{j+k+2}$, where the subscripts are mod $2k+4$, satisfy a. and b. Such a toric orbifold is simply connected if and only if the isotropy data span $\mathbb{Z} \times \mathbb{Z}$. One can show that the correspondence does not depend on the particular isotropy data. $\square$

In the next section we will prove a more useful geometric correspondence between toric anti-self-dual Einstein orbifolds and symmetric toric Kähler-Einstein surfaces.

Example. Consider the admissible weight matrix

$$\Omega = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}.$$

Then the 3-Sasakian space $\mathcal{S}_\Omega$ is smooth and $b_2(\mathcal{S}_\Omega) = b_2(\mathcal{M}_\Omega) = 2$. And the anti-self-dual orbifold $\mathcal{M}_\Omega$ has isotropy data

$$v_0 = (-7,-2), (-5,-2), (-1,-1), (5,1), (7,2) = v_4.$$

The singular set of $\mathcal{M}$ consists of two points with stablizer group $\mathbb{Z}_3$ and two with $\mathbb{Z}_4$. The associated toric Kähler-Einstein surface is that in figure (2.3). $\diamondsuit$

Proposition 5.22 *Let X be the symmetric toric Fano surface associated to the anti-self-dual Einstein orbifold $\mathcal{M}$. Then* $\mathrm{Ind}(X) = 2$ *if and only if* $w_2(\mathcal{M}) = 0$. *In other words,* $\mathbf{K}_X^{-1}$ *has a square root if and only if the contact line bundle on $\mathcal{Z}$,* $\mathbf{L}$, *does.*

Recall that $w_2(\mathcal{M})$ is equal to the Marchiafava-Romani class ε. Thus the vanishing of $w_2(\mathcal{M})$ is equivalent to the existence of a square root $\mathbf{L}^{\frac{1}{2}}$ of the contact line bundle $\mathbf{L}$ on $\mathcal{Z}$.

Proof. Suppose $\mathrm{Ind}(X) = 2$ which is equivalent to $w_2(X) = 0$, where W_2 denotes the orbifold Seifel-Whitney class. Recall that the orbit space of $\mathcal{M}$ is a $k+2$-gon W with labeled edges $C_1, \ldots, C_{k+2}$. Since $\pi_1^{orb}(\mathcal{M}) = e$, there exists an edge C_i for which the orbifold uniformizing group Γ has odd order. Let U be a tubular neighborhood of an orbit in C_i. So $U \cong S^1 \times I \times D/\Gamma$, where I is an open interval and D is a 2-disk. And let V be a neighborhood homotopically equivalent to $\mathcal{M} \setminus U$ with $U \cup V = \mathcal{M}$. Consider the exact homology sequence in $\mathbb{Z}_2$-coefficients,

$$\cdots \to H_2(BU) \oplus H_2(BV) \to H_2(B\mathcal{M}) \to H_1(B(U \cap V)) \\ \to H_1(BU) \oplus H_1(BV) \to 0. \quad (5.32)$$

We have $BU \cong S^1 \times I \times EO(4)/\Gamma$. Since $EO(4)$ is contractible, $H_*(EO(4)/\Gamma, A) = H_*(\Gamma, A)$ for any abelian group A. In particular, $H^n(\Gamma, \mathbb{Z}_2) = 0$ for all $n > 0$, since $|\Gamma|$ is odd. Thus $H_2(BU, \mathbb{Z}_2) = 0$ and $H_1(BU, \mathbb{Z}_2) = \mathbb{Z}_2$. Similarly, it not hard to show that $H_1(B(U \cap V, \mathbb{Z}_2) = \mathbb{Z}_2$. From the exact sequence (5.32) the inclusion $j : V \to \mathcal{M}$ induces a surjection $j_* : H_2(BV, \mathbb{Z}_2) \to H_2(B\mathcal{M}, \mathbb{Z}_2)$. Considering the orbit spaces one sees that there is a smooth embedding $\iota : V \to X$. The tangent V-bundle $T\mathcal{M}$ lifts to a genuine vector bundle on $B\mathcal{M}$ which will also be denoted T. See appendix A.2. Then

$$w_2(\mathcal{M}) = w_2(T\mathcal{M}) \in H^2(B\mathcal{M}, \mathbb{Z}_2) = \mathrm{Hom}(H_2(B\mathcal{M}, \mathbb{Z}_2), \mathbb{Z}_2).$$

Let $\alpha \in H_2(B\mathcal{M}, \mathbb{Z}_2)$. Then there exists a $\beta \in H_2(BV, \mathbb{Z}_2)$ with $j_*\beta = \alpha$. Then

$$w_2(T\mathcal{M})(\alpha) = w_2(TV)(\beta) = w_2(TX)(\iota_*\beta) = 0.$$

Thus $w_2(\mathcal{M}) = 0$.

The converse statement will follow from the main result of the next section. □

5.4 Twistor space and divisors

We will consider the twistor space $\mathcal{Z}$ introduced in theorem (5.6) more closely for the case when $\mathcal{M}$ is an anti-self-dual Einstein orbifold. For now suppose $(\mathcal{M}, [g])$ is an anti-self-dual, i.e. $W_+ \equiv 0$, conformal orbifold. There exists a complex three dimensional orbifold $\mathcal{Z}$ with the following properties:

a. There is a smooth V-bundle fibration $\varpi : \mathcal{Z} \to \mathcal{M}$.

b. The general fiber of $P_x = \varpi^{-1}(x), x \in \mathcal{Z}$ is a projective line $\mathbb{C}P^1$ with normal bundle $N \cong \mathcal{O}(1) \oplus \mathcal{O}(1)$, which holds over singular fibers with N a V-bundle.

c. There exists an anti-holomorphic involution σ of $\mathcal{Z}$ leaving the fibers P_x invariant.

Let T be an oriented real 4-dimensional vector space with inner product g. Let $C(T)$ be set of orthogonal complex structures inducing the orientation, i.e. if $r, s \in T$ is a complex basis then r, Jr, s, Js defines the orientation. One has $C(T) = S^2 \subset \Lambda^2_+(T)$, where S^2 is the sphere of radius $\sqrt{2}$. Now take T to be $\mathbb{H}$. Recall that $Sp(1)$ is the group of unit quaternions. Let

$$Sp(1)_+ \times Sp(1)_- \tag{5.33}$$

act on $\mathbb{H}$ by

$$w \to gwg'^{-1}, \text{ for } w \in \mathbb{H} \text{ and } (g, g') \in Sp(1)_+ \times Sp(1)_-. \tag{5.34}$$

Then we have

$$Sp(1)_+ \times_{\mathbb{Z}_2} Sp(1)_- \cong SO(4), \tag{5.35}$$

where $\mathbb{Z}_2$ is generated by $(-1, -1)$. Let

$$\begin{aligned} C &= \{ai + bj + ck : a^2 + b^2 + c^2 = 1, a, b, c \in \mathbb{R}\} \\ &= \{g \in Sp(1)_+ : g^2 = -1\} \cong S^2. \end{aligned} \tag{5.36}$$

Then $g \in C$ defines an orthogonal complex structure by

$$w \to gw, \text{ for } w \in \mathbb{H},$$

giving an identification $C = C(\mathbb{H})$. Let $V_+ = \mathbb{H}$ considered as a representation of $Sp(1)_+$ and a right $\mathbb{C}$-vector space. Define $\pi : V_+ \setminus \{0\} \to C$ by $\pi(h) = -hih^{-1}$. Then the fiber of π over hih^{-1} is $h\mathbb{C}$. Then π is equivariant if $Sp(1)_+$ acts on C by $q \to gqg^{-1}, g \in Sp(1)_+$. We have a the identification

$$C = V_+ \setminus \{0\}/\mathbb{C}^* = \mathbb{P}(V_+). \tag{5.37}$$

Fix a Riemannian metric g in $[g]$. Let $\phi : \tilde{U} \to U \subset \mathcal{M}$ be a local uniformizing chart with group Γ. Let $F_{\tilde{U}}$ be the bundle of orthonormal frames on $\tilde{U}$. Then

$$F_{\tilde{U}} \times_{SO(4)} \mathbb{P}(V_+) = F_{\tilde{U}} \times_{SO(4)} C \tag{5.38}$$

defines a local uniformizing chart for $\mathcal{Z}$ mapping to

$$F_{\tilde{U}} \times_{SO(4)} \mathbb{P}(V_+)/\Gamma = F_{\tilde{U}}/\Gamma \times_{SO(4)} \mathbb{P}(V_+).$$

Right multiplication by j on $V_+ = \mathbb{H}$ defines the anti-holomorphic involution σ which is fixed point free on (5.38). We will denote a neighborhood as in (5.38) by $\tilde{U}_{\mathcal{Z}}$.

An almost complex structure is defined as follows. At a point $z \in \tilde{U}_{\mathcal{Z}}$ the Levi-Civita connection defines a horizontal subspace H_z of the real tangent space T_z and we have a splitting

$$T_z = H_z \oplus T_z P_x = T_x \oplus T_z P_x, \tag{5.39}$$

where $\varpi(z) = x$ and T_x is the real tangent space of $\tilde{U}$. Let J_z be the complex structure on T_x given by $z \in P_x = C(T_x)$, and let J'_z be complex structure on $T_x \oplus T_z P_x$ arising from the natural complex structure on P_x. Then the almost complex structure on T_z is the direct sum of J_z and J'_z. This defines a natural almost complex structure on $Z_{\tilde{U}}$ which is invariant under Γ. We get an almost complex structure on $\mathcal{Z}$ which is integrable precisely when $W_+ \equiv 0$.

Assume that $\mathcal{M}$ anti-self-dual Einstein with non-zero scalar curvature. Then $\mathcal{Z}$ has a complex contact structure $D \subset T^{1,0}\mathcal{Z}$ with holomorphic contact form $\theta \in \Gamma(\Lambda^{1,0}\mathcal{Z} \otimes \mathbf{L})$ where $\mathbf{L} = T^{1,0}\mathcal{Z}/D$.

The group of isometries $\mathrm{Isom}(\mathcal{M})$ lifts to an action on $\mathcal{Z}$ by real holomorphic transformations. Real means commuting with σ. This extends to a holomorphic action of the complexification $\mathrm{Isom}(\mathcal{M})_{\mathbb{C}}$. For $X \in \mathfrak{Isom}(\mathcal{M}) \otimes \mathbb{C}$, the Lie algebra of $\mathrm{Isom}(\mathcal{M})_{\mathbb{C}}$, we will also denote by X the holomorphic vector field induced on $\mathcal{Z}$. Then $\theta(X) \in H^0(\mathcal{Z}, \mathcal{O}(\mathbf{L}))$. By a well known twistor correspondence the map $X \to \theta(X)$ defines an isomorphism

$$\mathfrak{Isom}(\mathcal{M}) \otimes \mathbb{C} \cong H^0(\mathcal{Z}, \mathcal{O}(\mathbf{L})), \tag{5.40}$$

which maps real vector fields to real sections of $\mathbf{L}$.

Suppose for now on that $\mathcal{M}$ is a toric anti-self-dual Einstein orbifold with twistor space $\mathcal{Z}$. We will assume that $\pi_1^{orb}(\mathcal{M}) = e$ which can alway be arranged by taking the orbifold cover. Then as above T^2 acts on $\mathcal{Z}$ by holomorphic transformations. And the action extends to $T^2_{\mathbb{C}} = \mathbb{C}^* \times \mathbb{C}^*$, which in this case is an algebraic action. Let $\mathfrak{t}$ be the Lie algebra of T^2 with $\mathfrak{t}_{\mathbb{C}}$ the Lie algebra of $T^2_{\mathbb{C}}$. Then we have from (5.40) the pencil

$$P = \mathbb{P}(\mathfrak{t}_{\mathbb{C}}) \subseteq |L|, \tag{5.41}$$

where for $t \in P$ we denote $X_t = (\theta(t))$ the divisor of the section $\theta(t) \in H^0(\mathcal{Z}, \mathcal{O}(\mathbf{L}))$. Note that P has an equator of real divisors. Also, since $T^2_{\mathbb{C}}$ is abelian, every $X_t, t \in P$ is $T^2_{\mathbb{C}}$ invariant.

Consider again the T^2-action on $\mathcal{M}$. Let K_x denote the stablizer of $x \in \mathcal{M}$. Recall the set with non-trivial stabilizers of the T^2-action on $\mathcal{M}$ is $B = \bigcup_{i=1}^{k+2} B_i$ where B_i is topologically a 2-sphere. Denote $x_i = B_i \cap B_{i+1}$, $B_i' = B_i \setminus \{x_i, x_{i-1}\}$ and $B' = \bigcup_{i=1}^{k+2} B_i'$. And denote the stabilizer of $B_i' = B_i \setminus \{x_i, x_{i-1}\}$ by $K_i = S^1(m_i, n_i)$. The stablizer of x_i is $K = T^2$. We will first determine the singular set $\Sigma \subset \mathcal{Z}$ for the T^2-action on $\mathcal{Z}$.

Lemma 5.23 *For $x \in B$ there exists on P_x precisely two fixed points z^+, z^- for the action of K_x which are σ conjugate. For $x \in B'$, the stablizer group in T^2 of any other $z \in P_x$ is trivial.*

Proof. Let $\phi : \tilde{U} \to U$ be a uniformizing chart centered at x with group γ. We may assume that $\tilde{K}_x$ acts on $\tilde{U}$ with $\gamma \subset \tilde{K}_x$ and $\tilde{K}_x/\gamma = K_x$. Then the uniformized tangent space splits

$$T_{\tilde{x}} = T_1 \oplus T_2. \tag{5.42}$$

When $x \in B'$ we take T_1 to be the space on which $\tilde{K}_i$ acts trivially and T_2 on which $\tilde{K}_i$ act faithfully. When $x = x_i$, $\tilde{K}_x = \tilde{K}_i \oplus \tilde{K}_{i+1}$ assume $\tilde{K}_i$ acts faithfully on T_1 and trivially on T_2, and $\tilde{K}_{i+1}$ trivially on T_1 and faithfully on T_2.

We determine the action of $\tilde{K}_x$ on $z \in \tilde{P}_x$. Identify (5.42) with $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$, considered as a right $\mathbb{C}$-vector space. The action of $\tilde{K}_x = S^1(t)$ in the first case is

$$(x, y) \to (x, ty),$$

and the action of $\tilde{K}_x = S^1(s) \times S^1(t)$ in the second is

$$(x, y) \to (sx, ty).$$

If $(u, v) \in S^1 \times S^1 \subset Sp(1)_+ \times Sp(1)_-$, then the action of (u, v) on T_x is

$$(x, y) \to (uv^{-1}x, (uv)^{-1}y).$$

In the first case the action of $\tilde{K}_i$ is realized by the subgroup $\{(u, u)\}$ with $t = u^{-1}$. Considering the representation of $Sp(1)_+$ on $V_+ = \mathbb{H}$, u acts by

$$(w, z) \to (uw, u^{-1}z).$$

One sees that the only fixed points on $\tilde{P}_x = \mathbb{P}(V_+)$ are $[1 : 0]$ and $[0 : 1]$. It is easy to see that $\tilde{K}_i$ acts freely on every other point of $\tilde{P}_x$. This also proves the statement for $x = x_i$. □

Denote the two K_x fixed points on P_x for $x = x_i$ by $z_i^{\pm}$. We will denote $P_i := P_{x_i}, i = 1, \ldots, k+2$. The next result is an easy consequence of the last lemma.

Lemma 5.24 *There exist two irreducible rational curves $C_i^{\pm}, i = 1, \ldots, k+2$ mapped diffeomorphically to B_i by ϖ. Furthermore, $\sigma(C_i^{\pm}) = C_i^{\mp}$.*

The singular set for the T^2-action on $\mathcal{Z}$ is the union of rational curves

$$\Sigma = \left(\cup_{i=1}^{k+2} P_i\right) \bigcup \left(\cup_{i+1}^{k+2} C_i^+ \cup C_i^-\right). \tag{5.43}$$

The fixed points for T^2 are $z_i^\pm, i = 1, \ldots, k+2$. And the stablizer group of $C_i'^\pm = C_i^\pm \setminus \{z_i^\pm, z_{i-1}^\pm\}$ is K_i. If $S_\mathcal{Z}$ is the orbifold singular set, then $S_\mathcal{Z} \subset \Sigma$. In this case $S_\mathcal{Z} = \mathrm{Sing}(\mathcal{Z})$, the singular set of $\mathcal{Z}$ as an analytic variety.

We will denote the union of the curves $C_i^\pm$ by

$$C = \bigcup_{i=1}^{k+2} (C_i^+ \cup C_i^-).$$

Then either C is a connected cycle, or it consists of two σ-conjugate cycles. It will turn out that C is always connected. Thus it may be more convenient to denote its components by $C_i, i = 1, \ldots, 2n$, where $n = k+2$, and the points $z_i^\pm$ by z_i and z_{i+n} such that

$$z_i = C_i \cap C_{i+1}, i = 1, \ldots, 2n,$$

where we take the index to be mod $2n$.

We now consider the action of $T^2_\mathbb{C}$ on $\mathcal{Z}$. The stablizer group of $z \in \mathcal{Z}$ in $T^2_\mathbb{C}$ will be denoted G_z. Let $G_i \subset T^2_\mathbb{C}$ be the complexification of K_i.

Lemma 5.25 *For $z \in C_i', i = 1, \ldots, 2n$, the stablizer group G_z coincides with G_i.*

Proof. We have $G_i \subset G_z$ with $\dim G_i = 1$. Suppose $G_i \neq G_z$ then G_z/G_i is a discrete subgroup of $T^2_\mathbb{C}/G_i \cong \mathbb{C}^*$. It is easy to see that G_z/G_i is an infinite cyclic subgroup of $T^2_{\mathbb{C}}/G_i$. Then the orbit of z, $C'_i \cong T^2_{\mathbb{C}}/G_z$ must be a one dimensional complex torus, which is a contradiction. $\square$

Recall that a parametrization of a stablizer group $K_i = S^1(m_i, n_i)$, of B_i, $i = 1, \ldots, n$, is only fixed up to sign. This amounts to a choice of orientation of B_i. In view of proposition (5.27) for the stablizer group G_i of C'_i, $i = 1, \ldots, 2n$, there is a fixed parametrization $\rho_i : \mathbb{C}^* \to T^2_{\mathbb{C}}$. One picks one of two possibilities by the rule: For z in a sufficiently small neighborhood of a point of C'_i one has

$$\lim_{t\to 0} \rho_i(t) z \in C_i.$$

Lemma 5.26 *We have $\rho_i = -\rho_{i+n}$ for $i = 1, \ldots, n$, where we consider the ρ_i to be elements of the $\mathbb{Z}^2$ lattice of one parameter subgroups of $T^2_{\mathbb{C}}$.*

Proof. Let $x \in B'_i$. And consider the action of G_i on the twistor line P_i as described in the proof of lemma (5.23). If $z \in P_i$, then $\lim_{t\to 0} \rho_i(z) = z_+ \in C_i$ implies $\lim_{t\to 0} \rho_i^{-1}(z) = z_+ \in C_{i+n}$. $\square$

We now consider the isotropy representations of G_z. The proof of the following is straight forward.

Proposition 5.27 *Let $z \in C$ with $\varpi(z) = x$. And let $\phi : \tilde{U} \to U$ be a $\tilde{K}_x$-invariant local uniformizing chart with group $\gamma \subset \tilde{K}_x$. Also $\tilde{G}_i$ denotes the complexification of $\tilde{K}_x$.*

i. Let $z \in C'_i, i = 1, \ldots, 2n$. Then there are $\mathbb{C}$-linear coordinates (u, v, w) on

$T_{\tilde{z}}\tilde{U}_Z$ *and an identification* $\tilde{G}_z \cong \mathbb{C}^*(t)$ *so that* $\tilde{G}_z$ *acts by*

$$(u, v, w) \to (u, tv, tw).$$

And the subspace $v = w = 0$ *maps to the tangent space of* C_i' *at* z.

ii. Let $z = z_i$ *for* $i = 1, \dots, 2n$. *Then there are* $\mathbb{C}$*-linear coordinates* (u, v, w) *on* $T_{\tilde{z}}\tilde{U}_Z$ *and an identification* $\tilde{G}_z \cong \mathbb{C}^*(s) \times \mathbb{C}^*(t)$ *so that* $\tilde{G}_z$ *acts by*

$$(u, v, w) \to (stu, sv, tw).$$

And the uniformized tangent space of P_i *(resp.* C_i*, and* C_{i+1}*) at* z *is the subspace* $v = w = 0$ *(resp.* $u = v = 0$ *and* $u = w = 0$*).*

We will determine the $T^2_{\mathbb{C}}$-action in a neighborhood of C. Let $z \in C$, and let $\tilde{U}_Z$ be a $\tilde{K}_z$-invariant uniformizing neighborhood as above with local group $\gamma \subset \tilde{K}_z$. Then there is

i. a $\tilde{K}_z$-invariant neighborhood W of the origin in $T_{\tilde{z}}\tilde{U}_Z$,

ii. a $\tilde{K}_z$-invariant neighborhood V of $\tilde{z}$ in $\tilde{U}_Z$, and

iii. a $\tilde{K}_z$-invariant biholomorphism $\varphi : W \to V$, i.e.

$$\varphi(gx) = g\varphi(x), \text{ for } x \in W, g \in \tilde{K}_z. \tag{5.44}$$

This is a well known; see for example [11].

This linear action extend locally to $\tilde{G}_z$, where $\tilde{G}_z$ is the complexification of $\tilde{K}_z$. Let $W_0 \subset W$ be a connected relatively compact neighborhood of the origin. And define the open set $A = \{(g, w) \in \tilde{G}_z \times W_0 : gw \in W\}$, and let $A_0 \subset A$ be the connected component containing $\tilde{K}_z \times W_0$. Then for any $(g, w) \in A_0$, we have $g\varphi(w) \in V$ and (5.44).

We now describe the local action of $T^2_{\mathbb{C}}$ around a point $z \in C$. There are two cases, i. and ii., distinguished as in proposition (5.27). In case i. $z \in C'_i$ for some $i = 1, \ldots, 2n$. And in case ii. $z = z_i$ for some $i = 1, \ldots, 2n$. we will use proposition (5.27) and the above remarks to produce a neighborhood U of z as follows.

Case i. Suppose $z \in C'_i$. There exists an equivariant uniformizing neighborhood $\phi : \tilde{U} \to U$ centered at z with group $\gamma \subset \tilde{K}_i$. One can lift the corresponding one parameter group $\tilde{\rho}_i : \mathbb{C}^*(t) \to \tilde{T}_{\mathbb{C}}$ with image $\tilde{G}_i$. Let $\tilde{G}' = \mathbb{C}^*(s)$ be a compliment to $\tilde{G}_i$ in $\tilde{T}_{\mathbb{C}}$. There exists coordinates (u, v, w) in $\tilde{U}$ so that

$$\tilde{U} = \{(u, v, w) : |u - 1| < \epsilon, |v| < 1, |w| < 1\}, \epsilon > 0, \tilde{z} = (1, 0, 0). \tag{5.45}$$

And $v = w = 0$ is the subset mapped to C and $\tilde{G}_i$ acts by

$$(u, v, w) \to (u, tv, tw), \text{ for } |t| \leq 1. \tag{5.46}$$

The action of $\tilde{G}'$ is given by $(u, v, w) \to (su, v, w)$ for $|su - 1| < \epsilon$.

Case ii. Suppose $z = z_i$ for some $i = 1, \ldots, 2n$. There exists an equivariant uniformizing neighborhood $\phi : \tilde{U} \to U$ centered at z with group $\gamma \subset \tilde{K}_z = \tilde{T}^2$. And one can lift the one parameter groups to $\tilde{\rho}_i$ and $\tilde{\rho}_{i+1}\tilde{\rho}_{i+1}$ to give an isomorphism

$$\tilde{\rho}_i \times \tilde{\rho}_{i+1} : \mathbb{C}^*(s) \times \mathbb{C}^*(t) \to \tilde{T}^2_{\mathbb{C}},$$

where $\tilde{T}^2_{\mathbb{C}}$ is the complexification of $\tilde{T}^2$. There exists coordinates (u, v, w) in $\tilde{U}$ so that

$$\tilde{U} = \{(u, v, w) : |u| < 1, |v| < 1, |w| < 1\}, \tilde{z} = (0, 0, 0), \qquad (5.47)$$

where the equations $u = v = 0$, $u = w = 0$, and $v = w = 0$ are the equations defining the subsets mapped to C_i, C_{i+1}, and P_i respectively. And the action of $(s, t) \in \mathbb{C}^*(s) \times \mathbb{C}^*(t)$ is given by

$$(u, v, w) \to (stu, sv, tw), \text{ for } |s| \leq 1, |t| \leq 1. \qquad (5.48)$$

We will call such a neighborhood U of a point of C an *admissible neighborhood*, and $\phi : \tilde{U} \to U$ with group γ an *admissible uniformizing system*. Let U be an admissible neighborhood. We set

$$U' := U \setminus \Sigma.$$

Denote by $\tilde{U}'$ the preimage of U' in $\tilde{U}$. We will define subsets $\tilde{U}'_{ab}, \tilde{U}'_{01}$, and $\tilde{U}''_{01}$ of $\tilde{U}'$.

Case i. For $(a, b) \neq 0$,

$$\tilde{U}'_{ab} := \{(u, v, w) \in \tilde{U}' : av = bw\}.$$

Case ii. For (a, b) with $a \neq 0$,

$$\tilde{U}'_{ab} := \{(u, v, w) \in \tilde{U}' : au = bvw\},$$

and

$$\tilde{U}'_{01} := \{(u, v, w) \in \tilde{U}' : v = 0\}, \text{ and } \tilde{U}''_{01} := \{(u, v, w) \in \tilde{U}' : w = 0\}.$$

Lemma 5.28 *The subsets defined above are connected closed submanifolds of $\tilde{U}'$ and each consists of a single local $\tilde{T}^2_{\mathbb{C}}$-orbit with these being all the orbits. And the closure of each orbit is an analytic submanifold of $\tilde{U}$.*

This follows from the above description of the $\tilde{T}^2_{\mathbb{C}}$-action. Note that γ preserves the orbits so this gives a description of the local orbits of $T^2_{\mathbb{C}}$ in U. We will denote by U'_{ab}, U'_{01}, and U''_{01} the corresponding local orbits in U.

We have the local leaf structure of the orbits in an admissible neighborhood. In most cases this gives the global leaf structure.

Lemma 5.29 *Let U be an admissible neighborhood. Let $E, F \subset U'$ be separate local leaves not both being of type U'_{01} or U''_{01}. Then E and F are not contained in the same $T^2_{\mathbb{C}}$-orbit.*

Proof. After acting by an element of $T^2_{\mathbb{C}}$ we may assume U is an admissible neighborhood as in case i. with coordinates (u, v, w) and $v = w = 0$ defining $C_i \cap U$. Let $z \in E$ and $z' \in F$ both have $u = 1$. There is a $g \in T^2_{\mathbb{C}}$ with $gz = z'$. Let $z_0 = \lim_{t \to 0} \rho_i(t) z = \lim_{t \to 0} \rho_i(t) z'$. Then

$$gz_0 = g\left(\lim_{t \to 0} \rho_i(t) z\right) = \lim_{t \to 0} \rho_i(t) gz = \lim_{t \to 0} \rho_i(t) z' = z_0.$$

So $g \in G_i$, and $g = \rho_i(t_0)$. If $|t_0| \leq 1$, then g preserves the local leaves. If $|t_0| > 0$, the equation $z = g^{-1}z'$ gives a contradiction. $\square$

Lemma 5.30 *For any $z \in U'$, an admissible neighborhood, the stablizer group G_z is the identity.*

Proof. If $g \in G_z$, then g fixes the entire $T^2_{\mathbb{C}}$-orbit of z. Therefore g fixes the entire set U'_{ab} containing z. But the closure of U'_{ab} intersects either C_i or C_{i+1}. So g is contained in either G_i or G_{i+1}. But from the above description of the action on U', we see that $g = e$. $\square$

Lemma 5.31 *Let z be any point of $P'_i = P_i \setminus \{z_i, z_{i+n}\}$. And let U be an admissible neighborhood of z_i or z_{i+n}. Then there exists a neighborhood V of z and $g \in T^2_{\mathbb{C}}$ so that $g(V) \subset U$.*

Proof. The stablizer group of P'_i is the image of the one parameter group ${\rho_i\rho_{i+1}}^{-1} : C^*(s) \to T^2_{\mathbb{C}}$. Then the orbit of z by G_i for example is P'_i. So a suitable element $g \in G_i$ will work. $\square$

By lemmas (5.30) and (5.31) there is a small neighborhood W of $\Sigma \subset \mathcal{Z}$, so that if we set $W' := W \setminus \Sigma$, the stablizer of every point or W' in $T^2_{\mathbb{C}}$ is the identity.

Our goal is to determine the structure of the divisors in the pencil P. As before we will consider the one parameter groups $\rho_i \in N = \mathbb{Z} \times \mathbb{Z}$, where N

Define elements $t_i \in P$ by $t_i = \rho_{i+1} - \rho_i, i = 1, \ldots, n$. Recall that the stabilizer of P_i is $\rho_i \rho_{i+1}^{-1} : \mathbb{C}^* \to T^2_{\mathbb{C}}$. If $t \in P \setminus \{t_1, t_2, \ldots, t_n\}$, then a vector field induced by t is tangent to, and non-vanishing on, $P_i, i = 1, \ldots, n$. Since the contact structure $D = \ker \theta$ is transverse to the twistor lines, $P_i \cap X_t = \{z^+, z^-\}$. Let $z \in Z_t$ be in an admissible neighborhood of C. Then the $T^2_{\mathbb{C}}$-orbit O of z satisfies $\bar{O} \setminus O \subset C$. The intersection of O with any admissible neighborhood is a leaf U'_{ab} which has analytic closure. Let $Y = \bar{O}$, then Y is an analytic subvariety.

Suppose C consists of two disjoint cycles with $Y \cap C = \bigcup_{i=1}^{n} C_i$. Then Y is a degree one divisor, i.e. intersecting a generic twistor line at one point. If $\bar{Y} = \sigma(Y)$, then $Y \cap \bar{Y} = \bigcup_{i=1}^{m} P_{x_i}$, a disjoint union of twistor lines with $x_i \notin B, i = 1, \ldots, m$. Since $Y \cap \bar{Y}$ is $T^2_{\mathbb{C}}$-invariant, we must have $Y \cap \bar{Y} = \emptyset$. Thus Y intersects each twistor line at one point. This is impossible. Y defines a, positively oriented, almost complex structure J on $\mathcal{M}$. Then if $c_1 = c_1(\mathcal{M}, J)$, $c_1^2 = 2\chi_{orb} + 3\tau_{orb}$ where χ_{orb} and τ_{orb} are defined as in the proof of proposition (4.20). We have

$$2\chi_{orb} + 3\tau_{orb} = \frac{1}{4\pi^2} \int_{\mathcal{M}} \frac{s^2}{24} d\mu > 0.$$

But a familiar Bochner argument shows the intersection form is negative definite. Therefore $C \subset Y$, Y is a degree two divisor, and $X_t = Y$. From the description of the admissible uniformizing systems and the local leaves, we see that X_t is a suborbifold. Since X_t is the closure of an orbit isomorphic to $T^2_{\mathbb{C}}$ it is a toric variety and has the anti-canonical cycle C and stabilizers ρ_i defining Δ^*.

The adjunction for $X = X_t$ formula gives $\mathbf{K}_X \cong \mathbf{K}_{\mathcal{Z}} \otimes [X]|_X = \mathbf{K}_{\mathcal{Z}}^{-\frac{1}{2}}|_X = \mathbf{K}_{\mathcal{Z}}^{\frac{1}{2}}|_{\mathcal{Z}}$. Thus $\mathbf{K}_X^{-1} > 0$. Now corollary (2.25) implies that X is Fano and Δ^* is a convex polytope. It follows that $t_1, \ldots, t_n \subset P$ form a cycle of distinct points.

Suppose $t = t_i, i = 1, \ldots, n$. Then $X_t \cap \Sigma = C \cup P_i$. Let $z \in X_t$ be in an admissible neighborhood of type i. with orbit O. Let $D = \bar{O}$. Then $D \setminus O \subset C \cup P_i$. And $P_i \subset D$, for otherwise we would have $D = X_t$ as in the last paragraph. For an admissible neighborhood U of z_i or z_{i+n} O must intersect U in a leaf U'_{01} or U''_{01}. This can be seen from lemma (5.29). We must have either $D \cap \Sigma = C \cup P_i$ or a cycle of the form $C_1, \ldots, C_i, P_i, C_{i+n+1}, \ldots, C_{2n}$. In the first case $D = X_t$ is irreducible. Since X_t is a real divisor, arguments as in [59] show that X_t must be a suborbifold, i.e. smooth on a uniformizing neighborhood. But X_t has a crossing singularity along P_i, a contradiction. Therefore, $D \cap \Sigma = C_1, \ldots, C_i, P_i, C_{i+n+1}, \ldots, C_{2n}$, and D is an analytic subvariety, and a suborbifold. Since $D = \bar{O}$ it is a toric variety. Since X_t is real, $\bar{D} \subset X_t$. And $D \cup \bar{D} = X_t$ as both are degree two. □

Note that if the isotropy data of $\mathcal{M}$ is normalized to satisfy conditions a. and b. before (5.20), then we have the identification

$$\rho_1 = (m_1, n_1), \ldots, \rho_{k+2} = (m_{k+2}, n_{k+2}), \rho_{k+3} = -(m_1, n_1), \ldots$$
$$\ldots, \rho_{2k+4} = -(m_{k+2}, n_{k+2}) = (m_0, n_0). \quad (5.49)$$

Here, as above, we identify ρ_i with a lattice point in $N = \mathbb{Z} \times \mathbb{Z}$.

5.5 Sasakian submanifolds

Associated to each compact toric anti-self-dual Einstein orbifold $\mathcal{M}$ with $\pi_1^{orb}(\mathcal{M}) = e$ is the twistor space $\mathcal{Z}$ and a family of embeddings $X_t \subset \mathcal{Z}$ where $t \in P \setminus \{t_1, t_2, \ldots, t_{k+2}\}$ and $X = X_t$ is the symmetric toric Fano surface canonically associated to $\mathcal{M}$. We denote the family of embeddings by

$$\iota_t : X \to \mathcal{Z}. \tag{5.50}$$

Let M be the total space of the S^1-Seifert bundle associated to $\mathbf{K}_X^{-1}$ or $\mathbf{K}_X^{-\frac{1}{2}}$, depending on whether $\mathrm{Ind}(X) = 1$ or 2.

Theorem 5.34 *Let $\mathcal{M}$ be a compact toric anti-self-dual Einstein orbifold with $\pi_1^{orb}(\mathcal{M}) = e$. There exists a Sasakian structure $\{\bar{g}, \xi, \Phi\}$ on M. So that if $(X, \bar{h})$ is the Kähler structure making $\pi : M \to X$ a Riemannian submersion, then we have the following diagram where the horizontal maps are isometric embeddings.*

$$\begin{array}{ccc} M & \xrightarrow{\bar{\iota}_t} & \mathcal{S} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\iota_t} & \mathcal{Z} \\ & & \downarrow \\ & & \mathcal{M} \end{array} \tag{5.51}$$

If the 3-Sasakian space $\mathcal{S}$ is smooth, then so is M. If M is smooth, then

$$M \underset{diff}{\cong} \#k(S^2 \times S^3), \text{ where } k = 2b_2(\mathcal{S}) + 1.$$

Proof. The adjunction formula gives

$$\mathbf{K}_X \cong \mathbf{K}_{\mathcal{Z}} \otimes [X]|_X = \mathbf{K}_{\mathcal{Z}} \otimes \mathbf{K}_{\mathcal{Z}}^{-\frac{1}{2}}|_X = \mathbf{K}_{\mathcal{Z}}^{\frac{1}{2}}|_X.$$

Thus

$$\mathbf{K}_X^{-1} \cong \mathbf{K}_{\mathcal{Z}}^{-\frac{1}{2}}|_X. \tag{5.52}$$

Let h be the Kähler-Einstein metric on $\mathcal{Z}$ related to the the 3-Sasakian metric g on $\mathcal{S}$ by Riemannian submersion. So $\mathrm{Ric}_h = 8h$. Recall that $\mathcal{S}$ is the total space of the S^1-Seifert bundle associated to $\mathbf{L}$, or $\mathbf{L}^{\frac{1}{2}}$ if $w_2(\mathcal{M}) = 0$. Also M is the total space of the S^1-Seifert bundle associated to either $\mathbf{K}_X^{-1}$ or $\mathbf{K}_X^{-\frac{1}{2}}$. Using the isomorphism in (5.52) we lift ι_t to $\bar{\iota}_t$. Then we have

$$g = \eta \otimes \eta + \pi^* h,$$

and $\eta = \frac{d}{8}\theta$ with θ a connection on $\mathbf{L}$ or $\mathbf{L}^{\frac{1}{2}}$ and where $d = \mathrm{Ind}(\mathcal{Z}) = 2$ or 4 respectively. Then it is not difficult to see that by pulling the connection back by $\iota_t^*\mathbf{L} \cong \mathbf{K}_X^{-1}$ we can pull η back to $\bar{\eta}$ on M. And define $\bar{h} = \iota_t^*(h)$. Then

$$\bar{g} = \bar{\eta} \otimes \bar{\eta} + \pi^* \bar{h}$$

is a Sasakian metric on M.

If $\mathcal{S}$ is smooth, then locally the orbifold groups act on $\mathbf{L}$ (or $\mathbf{L}^{\frac{1}{2}}$) without non-trivial stabilizers. By (5.52) this holds for the bundle $\mathbf{K}_X^{-1}$ (or $\mathbf{K}_X^{-\frac{1}{2}}$) on X.

We have $\pi_1^{orb}(X) = e$ from theorem A.8 of appendix A.3. If M is smooth, then theorem (4.13) and corollary (4.17) imply that $\pi_1(M) = e$ and give the diffeomorphism. □

We are more interested in M with the Sasakian-Einstein metric that exists by theorem (4.13). In this case the horizontal maps are not isometries.

Chapter 6

Main Theorems

In this chapter we present the new infinite families of Sasakian-Einstein manifolds. This gives us the diagram (5.1).

Theorem 6.1 *Let $(\mathcal{S}, g)$ be a toric 3-Sasakian 7-manifold with $\pi_1(\mathcal{S}) = e$. Canonically associated to $(\mathcal{S}, g)$ are a special symmetric toric Fano surface X and a toric Sasakian-Einstein 5-manifold M which fit in the commutative diagram (5.51). We have $\pi_1^{orb}(X) = e$ and $\pi_1(M) = e$. And*

$$M \underset{diff}{\cong} \#k(S^2 \times S^3), \text{ where } k = 2b_2(\mathcal{S}) + 1$$

Furthermore $(\mathcal{S}, g)$ can be recovered from either X or M with their torus actions.

Proof. The homotopy sequence

$$\cdots \to \pi_1(G) \to \pi_1(\mathcal{S}) \to \pi_1^{orb}(\mathcal{M}) \to e,$$

where $G = SO(3)$ or $Sp(1)$, shows that $\pi_1^{orb}(\mathcal{M}) = e$. The surface X is uniquely determined by theorem 5.33. It follows from the exact sequence in theorem A.8 involving the fundamental group of a toric 4-orbifold that $\pi_1^{orb}(X) = e$. It follows from theorem (5.34) that M is smooth and the diffeomorphism holds. And an application of theorem (4.13) give the Sasakian-Einstein structure on M and shows it is simply connected. Given X or M with its Sasakian structure we can recover the orbifold $\mathcal{M}$, which has a unique toric anti-self-dual Einstein metric by theorem (5.20). This uniquely determines the 3-Sasakian manifold by results of chapter 5. □

Theorem 6.2 *For each odd $k \geq 3$ there is a countably infinite number of toric Sasakian-Einstein structures on $\#k(S^2 \times S^3)$.*

Proof. Recall from corollary (5.18) there are infinitely homotopically distinct smooth simply connected 3-Sasakian manifolds $\mathcal{S}$ with $b_2(\mathcal{S}) = k$ for $k > 0$. From theorem (6.1) associated to each $\mathcal{S}$ is a distinct Sasakian-Einstein manifold diffeomorphic to $\#m(S^2 \times S^3)$, where $m = 2k + 1$. □

Our construction only produces the homogeneous Sasakian-Einstein structure on $S^2 \times S^3$. The restriction of k to be odd is merely a limitation on the techniques used. In the next chapter there is an example of toric Sasakian-Einstein structure on $\#6(S^2 \times S^3)$.

If a simply connected manifold has two Sasakian-Einstein structures for the same metric g then it is S^5.

Corollary 6.3 *For each odd $k \geq 3$ there is a countably infinite number of cohomogeneity 2 Einstein metrics on $\#k(S^2 \times S^3)$. In particular, the identity component of the isometry group is T^3.*

These metrics have the following curious property.

Proposition 6.4 *For $M = \#k(S^2 \times S^3)$ with $k > 1$ odd, let g_i be the sequence of Einstein metrics in the theorem normalized so that $\mathrm{Vol}_{g_i}(M) = 1$. Then we have $\mathrm{Ric}_{g_i} = \lambda_i g_i$ with the Einstein constants $\lambda_i \to 0$ as $i \to \infty$.*

Proof. From proposition 4.14 we have

$$\mathrm{Vol}(M, g) = d\left(\frac{\pi}{3}\right)^3 \mathrm{Vol}(\Sigma_{-k}),$$

for the volume of a Sasakian-Einstein manifold with toric leaf space X the anti-canonical polytope Σ_{-k}. We have $d = 1$ or 2. The above Sasakian-Einstein manifolds have leaf spaces X_i, where $X_i = X_{\Delta_i^*}$. Observe that the polygons Δ_i^* get arbitrarily large, and the anti-canonical polytopes $(\Sigma_{-k})_i$ satisfy

$$\mathrm{Vol}((\Sigma_{-k})_i) \to 0, \text{ as } i \to \infty.$$

□

This implies the following.

Theorem 6.5 *The moduli space of Einstein structures on each of the manifolds $\#k(S^2 \times S^3)$ for $k \geq 1$ odd has infinitely many connected components.*

The case $k = 1$ is covered by homogeneous examples by M. Wang and W. Ziller [70].

There are a couple of consequence of these examples following from some finiteness results. There is a result of M. Gromov [31] that says that a manifold which admits a metric of nonnegative sectional curvature satisfies a bound on the total Betti number depending only on the dimension. Further, he proved that if the diameter is bounded, then as the total Betti number goes to infinity the infimum of the sectional curvatures goes to $-\infty$. For any $\kappa \leq 0$ there exists k_0 so that, for $k > k_0$, $\#k(S^2 \times S^3)$ does not admit a metric with sectional curvature $K \geq \kappa$. We have the following.

Theorem 6.6 *For any $\kappa \leq 0$ there are infinitely many simply connected Einstein 5-manifolds which do not admit metrics with sectional $K \geq \kappa$.*

One can also consider these examples in relation to a compactness result of M. Anderson [3]. He showed that the space of Riemannian n-manifolds (M, g), $\mathscr{M}(\lambda, c, d)$ with $\mathrm{Ric}_g = \lambda g$, $\mathrm{inj}(g) \geq c > 0$, and $\mathrm{diam} \leq d$ is compact in the C^∞ topology. For fixed $k > 1$ odd in theorem (6.2) the Sasakian-Einstein metrics g_i on $M = \#k(S^2 \times S^3)$ have $\lambda = 4$. We have $\mathrm{Vol}_{g_i}(M) \to 0$ as $i \to \infty$, so no subsequence converges. We have the following.

Theorem 6.7 *For the the sequence of Einstein manifolds (M, g_i) we have $\mathrm{inj}(g_i) \to 0$ as $i \to \infty$. Also, take any sequence $k_i > 1$ of odd integers and examples from theorem (6.2) $(\#k_i(S^2 \times S^3), g_i)$, then we have $\mathrm{inj}(g_i) \to 0$ as $i \to \infty$.*

Examples of Einstein 7-manifolds with properties as in theorem (6.6) and in the second statement of theorem (6.7) have been given in [18]. These are the toric 3-Sasakian 7-manifolds $\mathcal{S}_\Omega$ considered here.

Chapter 7

Examples

We consider some of the examples obtained starting with the simplest. In particular we can determine some of the spaces in diagram (5.1) associated to a smooth toric 3-Sasakian 7-manifold more explicitly in some cases.

7.1 Smooth examples

It is well known that there exists only two complete examples of positive scalar curvature anti-self-dual Einstein manifolds [42] [26], S^4 and $\mathbb{C}P^2$ with the round and Fubini-Study metrics respectively. Note that we are considering $\mathbb{C}P^2$ with the opposite of the usual orientation.

$\mathcal{M} = S^4$

Considering the spaces in diagram (5.1) we have: $\mathcal{M} = S^4$ with the round metric; its twistor space $\mathcal{Z} = \mathbb{C}P^3$ with the Fubini-study metric; the quadratic divisor $X \subset \mathcal{Z}$ is $\mathbb{C}P^1 \times \mathbb{C}P^1$ with the homogeneous Kähler-Einstein metric; $M = S^2 \times S^3$ with the homogeneous Sasakian-Einstein structure; and $\mathcal{S} = S^7$

has the round metric. In this case diagram (5.1) becomes the following.

$$\begin{array}{ccc} S^2 \times S^3 & \longrightarrow & S^7 \\ \downarrow & & \downarrow \\ \mathbb{C}P^1 \times \mathbb{C}P^1 & \longrightarrow & \mathbb{C}P^3 \\ & & \downarrow \\ & & S^4 \end{array} \tag{7.1}$$

This is the only example, I am aware of, for which the horizontal maps are isometric immersions when the toric surface and Sasakian space are equipped with the Einstein metrics.

$\mathcal{M} = \mathbb{C}P^2$

In this case $\mathcal{M} = \mathbb{C}P^2$ with the Fubini-Study metric; its twistor space is $\mathcal{Z} = F_{1,2}$, the manifold of flags $V \subset W \subset \mathbb{C}^3$ with $\dim V = 1$ and $\dim W = 2$, with the homogeneous Kähler-Einstein metric. The projection $\pi : F_{1,2} \to \mathbb{C}P^2$ is as follows. If $(p, l) \in F_{1,2}$ so l is a line in $\mathbb{C}P^2$ and $p \in l$, then $\pi(p, l) = p^{\perp} \cap l$, where $p^{\perp}$ is the orthogonal compliment with respect to the standard hermitian inner product. We can define $F_{1,2} \subset \mathbb{C}P^2 \times (\mathbb{C}P^2)^*$ by

$$F_{1,2} = \{([p_0 : p_1 : p_2], [q^0 : q^1 : q^2]) \in \mathbb{C}P^2 \times (\mathbb{C}P^2)^* : \sum p_i q^i = 0\}.$$

And the complex contact structure is given by $\theta = q^i dp_i - p_i dq^i$. Fix the action of T^2 on $\mathbb{C}P^2$ by

$$(e^{i\theta}, e^{i\phi})[z_0 : z_1 : z_2] = [z_0 : e^{i\theta} z_1 : e^{i\phi} z_2].$$

Then this induces the action on $F_{1,2}$

$$(e^{i\theta}, e^{i\phi})([p_0 : p_1 : p_2], [q^0 : q^1 : q^2]) = ([p_0 : e^{i\theta}p_1 : e^{i\phi}p_2], [q^0 : e^{-i\theta}q^1 : e^{-i\phi}q^2]).$$

Given $[a, b] \in \mathbb{C}P^1$ the one parameter group $(e^{ia\tau}, e^{ib\tau})$ induces the holomorphic vector field $W_\tau \in \Gamma(T^{1,0}F_{1,2})$ and the quadratic divisor $X_\tau = (\theta(W_\tau))$ given by

$$X_\tau = (ap_1q^1 + bp_2q^2 = 0, \quad p_iq^i = 0).$$

One can check directly that X_τ is smooth for $\tau \in \mathbb{C}P^1 \setminus \{[1,0],[0,1],[1,1]\}$ and $X_\tau = \mathbb{C}P^2_{(3)}$, the equivariant blow-up of $\mathbb{C}P^2$ at 3 points. For $\tau \in \{[1,0],[0,1],[1,1]\}$, $X_\tau = D_\tau + \bar{D}_\tau$ where both $D_\tau, \bar{D}_\tau$ are isomorphic to the Hirzebruch surface $F_1 = \mathbb{P}(\mathcal{O}_{\mathbb{C}P^1} \oplus \mathcal{O}_{\mathbb{C}P^1}(1))$

The Sasakian-Einstein space is $M = \#3(S^2 \times S^3)$. And we have $\mathcal{S} = \mathcal{S}(1,1,1) = SU(3)/U(1)$ with the homogeneous 3-Sasakian structure. This case has the following diagram.

$$\begin{array}{ccc} \#3(S^2 \times S^3) & \hookrightarrow & SU(3)/U(1) \\ \downarrow & & \downarrow \\ \mathbb{C}P^2_{(3)} & \to & F_{1,2} \\ & & \downarrow \\ & & \mathbb{C}P^2 \end{array} \tag{7.2}$$

7.2 Galicki-Lawson quotients

The simplest examples of quaternionic-Kähler quotients are the Galicki-Lawson examples first appearing in [29] and further considered in [16]. These are circle quotients of $\mathbb{H}P^2$. In this case the weight matrices are of the form

$\Omega = \mathbf{p} = (p_1, p_2, p_3)$ with the admissible set

$$\{\mathcal{A}_{1,3}(Z) = \{\mathbf{p} \in \mathbb{Z}^3 | p_i \neq 0 \text{ for } i = 1,2,3 \text{ and } \gcd(p_i, p_j) = 1 \text{ for } i \neq j\}$$

We may take $p_i > 0$ for $i = 1, 2, 3$. The zero locus of the 3-Sasakian moment map $N(\mathbf{p}) \subset S^{11}$ is diffeomorphic to the Stiefel manifold $V^{\mathbb{C}}_{2,3}$ of complex 2-frames in $\mathbb{C}^3$ which can be identified as $V^{\mathbb{C}}_{2,3} \cong U(3)/U(1) \cong SU(3)$. Let $f_{\mathbf{p}} : U(1) \to U(3)$ be

$$f_{\mathbf{p}}(\tau) = \begin{bmatrix} \tau^{p_1} & 0 & 0 \\ 0 & \tau^{p_2} & 0 \\ 0 & 0 & \tau^{p_3} \end{bmatrix}.$$

Then the 3-Sasakian space $\mathcal{S}(\mathbf{p})$ is diffeomorphic to the quotient of $SU(3)$ by the action of $U(1)$

$$\tau \cdot W = f_{\mathbf{p}}(\tau) W f_{(0,0,-p_1-p_2-p_3)}(\tau) \text{ where } \tau \in U(1) \text{ and } W \in SU(3).$$

Thus $\mathcal{S}(\mathbf{p}) \cong SU(3)/U(1)$ is a biquotient similar to the examples considered by Eschenburg in [25].

The action of the group $SU(2)$ generated by $\{\xi^1, \xi^2, \xi^3\}$ on $N(\mathbf{p}) \cong SU(3)$ commutes with the action of $U(1)$. We have $N(\mathbf{p})/SU(2) \cong SU(3)/SU(2) \cong S^5$ with $U(1)$ acting by

$$\tau \cdot v = f_{(-p_2-p_3,-p_1-p_3,-p_1-p_2)} v \text{ for } v \in S^5 \subset \mathbb{C}^3.$$

We see that $\mathcal{M}_\Omega \cong \mathbb{C}P^2_{a_1,a_2,a_3}$ where $a_1 = p_2 + p_3, a_2 = p_1 + p_3, a_3 = p_1 + p_2$ and the quotient metric is anti-self-dual with the reverse of usual orientation.

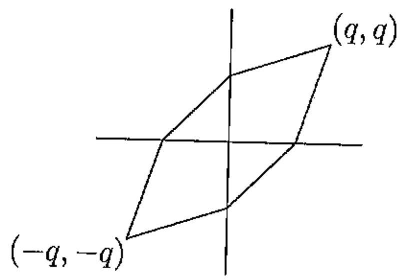


Figure 7.1: infinite Fano orbifold structures on $\mathbb{C}P^2_{(3)}$

If p_1, p_2, p_3 are all odd then the generic leaf of the 3-Sasakian foliation $\mathcal{F}_3$ is $SO(3)$. If exactly one is even, then the generic leaf is $Sp(1)$. Denote by X_{p_1,p_2,p_3} the toric Fano divisor, which can be considered as a generalization of $\mathbb{C}P^2_{(3)}$. We have the following spaces and embeddings.

$$\begin{array}{ccc} \#3(S^2 \times S^3) & \rightarrowtail & \mathcal{S}(p_1,p_2,p_3) \\ \downarrow & & \downarrow \\ X_{p_1,p_2,p_3} & \longrightarrow & \mathcal{Z}(p_1,p_2,p_3) \\ & & \downarrow \\ & & \mathbb{C}P^2_{a_1,a_2,a_3} \end{array} \tag{7.3}$$

A simple series of examples can be obtained by taking $\mathbf{p} = (2q-1, 1, 1)$ for any $q \geq 1$. Then the anti-self-dual Einstein space is $\mathcal{M} = \mathbb{C}P^2_{1,q,q}$ which is homeomorphic to $\mathbb{C}P^2$, but its metric is ramified along a $\mathbb{C}P^1$ to order q. For the toric divisor $X \subset \mathcal{Z}$ we have $X = \mathbb{C}P^2_{(3)}$ with the metric ramified along two $\mathbb{C}P^1$'s to order q. We get an sequence of distinct Sasakian-Einstein structures on $M \cong \#3(S^2 \times S^3)$.

It is possible to construct new examples of toric Sasakian-Einstein manifolds M with b_2 even. The surface in figure (2.3) is a "generalized blow-up" of $\mathbb{C}P^2$ at 6 points. One can check that the total space of $\mathbf{K}_X^{-1}$ minus the zero section is smooth. Thus we obtain a toric Sasakian-Einstein structure on $M \cong \#6(S^2 \times S^3)$.

Appendix A

Orbifolds

A.1 Definitions

The notion of an orbifold, also called a V-manifold, was developed by I. Satake in 1956. Subsequently he and W.L. Baily developed the theory of real and complex orbifolds generalizing familiar results for manifolds. For example, we make use of the Gause-Bonnet theorem for V-manfolds [62] and Baily's extention of the Kodiara embedding theorem to V-manifolds. Many familiar results for manifolds can be proved almost verbatim for V-manifolds. This is true for de Rham's theorem, the Hodge decomposition theorem, and Kodaira-Nakano vanishing as proved in [2].

Definition A.1 *A smooth(holomorphic)* orbifold *is a second countable Hausdorff space X together with an open covering $\{U_i\}_{i \in A}$ satisfying:*

i. $\{U_i\}_{i \in A}$ is an open cover of X such that if $x \in U_i \cap U_j$ there is a $k \in A$ so that $x \in U_k \subset U_i \cap U_j$.

ii. For each $i \in A$ there is a triple $\{\tilde{U}_i, \Gamma_i, \phi_i\}$ where $\tilde{U}_i$ is a connected open *subset of* $\mathbb{R}^n(\mathbb{C}^n)$ *containing the origin,* Γ_i *is a finite group of diffeomorphisms (biholomorphisms) acting effectively and properly on* $\tilde{U}_i$*, and* $\phi_i : \tilde{U}_i \to U_i$ *is continuous with* $\phi_i \circ \gamma = \phi_i$*, for each* $\gamma \in \Gamma_i$*, and the induced map* $\tilde{U}_i/\Gamma_i \to U_i$ *is a homeomorphism.*

iii. If $U_i \subset U_j$*, there is a diffeomorphism(biholomorphism)* $g_{ji} : \tilde{U}_i \to \tilde{U}_j$ *onto an open subset such that* $\phi_i = \phi_j \circ g_{ji}$*. This implies that for any* $\gamma_i \in \Gamma_i$ *there is a unique* $\gamma_j \in \Gamma_j$ *so that* $\gamma_j \circ g_{ji} = g_{ji} \circ \gamma_i$*.*

One can always take the uniformizing groups $\Gamma_i \subset SO(n)$ acting linearly in the smooth case. In the complex case one can take holomorphic uniformizing systems so that $\Gamma_i \subset GL(n, \mathbb{C})$ (see [11]). For $x \in X$ let $\tilde{x} \in \phi^{-1}(x)$ for any uniformizing neighborhood system $\{U, \Gamma, \phi\}$ containing x. Then, up to conjugation, the isotropy group of $\tilde{x}$, only depends on x. We denote this group Γ_x. The open dense subset of X of points x with $\Gamma_x = \{e\}$ are the *regular points.* The points with nontrivial isotopy groups are the *orbifold singular points.* The set of orbifold singular points is denoted S_X. The least common multiple of the orders of all the $\Gamma_x, x \in X$ is the *order* of the orbifold X and is denoted $\mathrm{Ord}(X)$.

We make much use of the orbifold analogue of a fiber bundle.

Definition A.2 *A* V-bundle E *over* X *consists of a bundle* $B_{\tilde{U}_i}$ *over each* $\tilde{U}_i$ *for each local uniformizing system* $\{\tilde{U}_i, \Gamma_i, \phi_i\}$ *with group* G *and fiber* F *together with a finite group* Γ_i^* *and a homomorphism* $h_{\tilde{U}_i} : \Gamma_i^* \to \mathrm{Aut}(B_{\tilde{U}_i})$ *satisfying:*

i. The group homomorphism $\pi_* : \Gamma_i^* \to \Gamma_i$ *induced by* $\pi : B_{\tilde{U}_i} \to \tilde{U}_i$ *is surjective.*

ii. If $g_{ji} : \tilde{U}_i \to \tilde{U}_j$ is a diffeomorphism onto an open subset, there is a bundle homomorphism $g^*_{ji} : B_{\tilde{U}_i} \to B_{\tilde{U}_i}|_{g_{ji}(\tilde{U}_i)}$, such that if $\gamma^*_i \in \Gamma^*_i$ there is a unique $\gamma^*_j \in \Gamma^*_j$ so that $h_{\tilde{U}_j}(\gamma^*_j) \circ g^*_{ji} = g^*_{ij} \circ h_{\tilde{U}_i}(\gamma^*_i)$. And if $g_{kj} : \tilde{U}_j \to \tilde{U}_k$ is another diffeomorphism, then $(g_{kj} \circ g_{ji})^* = g^*_{kj} \circ g^*_{ji}$.

If F is a vector space with the group G acting linearly and each $\mathrm{Im}(h_{\tilde{U}_i})$ consisting of vector bundle automorphisms, then E is a vector V-bundle. Similarly, if F is a Lie group and each $\mathrm{Im}(h_{\tilde{U}_i})$ consists of principle bundle automorphism, then E is a principle V-bundle.

We have the notion of a holomorphic V-bundle by making the obvious changes to the definition. The total space of E is an orbifold. In most cases the homomorphism π_* in i. will be injective and $\Gamma^*_i = \Gamma_i$ for every $i \in A$, then E is a *proper V-bundle.* If E is proper then for each $x \in X$ there is a homomorphism $h_x : \Gamma_x \to \mathrm{Aut}(F)$. If h_x is trivial for all $x \in X$, then E is an *absolute V-bundle* which is a fiber in the usual sense.

An important case is that of holomorphic line V-bundles.

Definition A.3 *Let X be a complex orbifold. Then $\mathrm{Pic}^{orb}(X)$ is the abelian group of equivalence classes of proper holomorphic line V-bundles on X.*

The usual Picard group $\mathrm{Pic}(X)$ is a subgroup, and the inclusion $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}^{\mathrm{orb}}(X)$ induces an isomorphism

$$\mathrm{Pic}(X) \otimes \mathbb{Q} \cong \mathrm{Pic}^{\mathrm{orb}}(X) \otimes \mathbb{Q}. \tag{A.1}$$

A.2 Classifying space and invariants

In this section we give a review of A. Haefliger's construction of the classifying space of an orbifold. For more details see [36]. This will be used to define invariants including characteristic classes of V-bundles. Let M be an n dimensional orbifold with uniformizing system $\{\tilde{U}_i, \Gamma_i, \phi_i\}$. Suppose M is given any Riemannian metric. Let $L_{\tilde{U}_i}$ be the bundle of orthogonal frames on $\tilde{U}_i$. The $L_{\tilde{U}_i}$ glue together to form the priciple V-bundle L_M of orthogonal frames on M. Consider the Stiefel manifold $V_{n,n+k} = O(n+k)/O(k)$ of orthogonal n-frames in $\mathbb{R}^{n+k}$. $V_{n,n+k}$ has an action of $O(n)$ and is k-universal as a principal $O(n)$-bundle. The sequence $\cdots \subset V_{n,n+k} \subset V_{n,n+k+1} \subset \cdots$ gives rise to the direct limit $EO(n)$, which is a universal $O(n)$-bundle $EO(n) \to BO(n)$. Define $BM = L_M \times_{O(n)} EO(n)$. Then there is a natural projection $p : BM \to M$ whose generic fiber is the contractible space $EO(n)$. Then the orbifold (co)homology and homotopy groups are

$$H^i_{orb}(M, \mathbb{Z}) = H^i(BM, \mathbb{Z}),\ H_i^{orb}(M, \mathbb{Z}) = H_i(BM, \mathbb{Z}),\ \pi_i^{orb}(M) = \pi_i(BM). \tag{A.2}$$

An application of the Leray spectral sequence to the map $p : BM \to M$ gives the following.

Proposition A.4 ([36]) *The map $p : BM \to M$ induces an isomorphism $H^i_{orb}(M, A) \cong H^i(M, A)$ for $A = \mathbb{Q}, \mathbb{R}, \mathbb{C},$ or $\mathbb{Z}_p$ where p is relatively prime to* $\mathrm{Ord}(M)$.

We will make use of the following generalization of the long exact homotopy sequence. Let G be a compact group acting locally freely on an orbifold N

with quotient M. Then G acts on BN with quotient BN/G can be seen to be a classifying space for M. So we have

$$\cdots \to \pi_i(G) \to \pi_i^{orb}(N) \to \pi_i^{orb}(M) \to \pi_{i-1}(G) \to \cdots . \tag{A.3}$$

The classifying space BM of M gives a convenient description of V-bundles on M. See [14].

Proposition A.5 *There is a one-to-one correspondence between isomorphism classes of proper V-bundles on M with group G and generic fiber F and isomorphism classes of bundles on BM with group G and generic fiber F.*

There are two ways of defining characteristic classes of a V-bundle E. First, one takes the usual characteristic classes of the corresponding bundle on BM giving an element in $H^*_{orb}(M, A)$ for $A = \mathbb{Z}, \mathbb{Z}_2$, or $\mathbb{R}$. Second, when using real coeficients it is more convenient to use Chern-Weil theory giving an element of de Rham cohomology $H^*(M, \mathbb{R})$. Both definitions are equivalent as we will explain. Let G be a compact group, the structure group of E, $S(\mathfrak{g}^*)$ the invariant polynomials on the Lie algebra $\mathfrak{g}$, and BG the classifying space of G-bundles. Then we have the commutative diagram:

$$\begin{array}{ccc} S(\mathfrak{g}^*) & \xrightarrow{\text{C-W}} & H^*(M,\mathbb{R}) \\ \downarrow & & \downarrow \\ H^*(BG,\mathbb{R}) & \longrightarrow & H^*(BM,\mathbb{R}) \end{array} \tag{A.4}$$

The top map is the Chern-Weil homomorphism taking $f \in S(\mathfrak{g}^*)$ to $[f(\Theta)]$, where Θ is any connection on E. The left vertical map is an isomorphism, the universal Chern-Weil map. The bottom map is induced by the classifying map of E, and the right vertical map is the isomorphism in proposition (A.4).

A.3 Torus actions

We summarize the description of compact orbifolds with a torus action due to A. Haefliger and É. Salem [37]. Let M be a smooth compact oriented n dimensional orbifold with a smooth effective action of an m dimensional torus $G = T^m = \mathfrak{g}/\Lambda$, $m \leq n$. Let $\{\hat{U}, \gamma, \phi\}$, $\phi : \hat{U} \to U$ be a uniformizing system with γ centered at $x \in U$, and let $H < G$ be the stabilizer of x. Let U be a G-invariant tubular neighborhood of $G \cdot x$. Let $\Gamma = \pi_1^{orb}(U)$. Since $\pi_2(U) = e$, the universal orbifold cover $\pi : \tilde{U} \to U$ is smooth [36]. Let $\tilde{G}$ be the group of diffeomorphisms of $\tilde{U}$ projecting to the group of diffeomorphism G of U. So $\Gamma \lhd \tilde{G}$, normal subgroup, and $\tilde{G}/\Gamma = G$. Let $\tilde{H}$ be the stabilizer of a point $\tilde{x} \in \pi^{-1}(x)$. We will apply the differentiable slice theorem. There exists a ball $B \subset \mathbb{R}^{n-m+k}$ and a representation $\tilde{H} \to SO(n-m+k)$. The tubular neighborhood U can be chosen so that there is a G-equivariant diffeomorphism

$$U \cong (\tilde{G} \times_{\tilde{H}} B)/\Gamma.$$

Furthermore, $\Gamma \cap \tilde{H} = \gamma$, and $(\tilde{G} \times_{\tilde{H}} B)/\Gamma = (\tilde{G}/\Gamma) \times_{\tilde{H}/\gamma} (B/\gamma) = G \times_H (B/\gamma)$.

Since $\tilde{G}/\tilde{H}$ is the universal cover of G/H, we have $\tilde{G}/\tilde{G}_0 = \tilde{H}/\tilde{H}_0 = D$, a finite group, where $\tilde{G}_0 = \mathfrak{g}/\Lambda_0$ and $\tilde{H}_0 = \mathfrak{h}/\Lambda_0$ are the identity components. Here $\Lambda_0 \subseteq \Lambda$ is a sublattice. We have the following classification of G-invariant tubular neighborhoods.

Proposition A.6 **([37])** *Let $G = \mathfrak{g}/\Lambda$ be an m-torus acting effectively on an* *oriented n-orbifold X. And let $G \cdot x$ be an orbit with k-dimensional stabilizer H. Then we have*

i. a rank k sublattice $\Lambda_0 \subseteq \Lambda$,

ii. a finite group D and a central extension

$$0 \to \Lambda/\Lambda_0 \to \Gamma \to D \to 1,$$

iii. and a faithful representation $\tilde{H} \to SO(n - m + k)$, where $\tilde{H}$ is the maximal compact subgroup of $\tilde{G} = \Gamma \times_{\Lambda/\Lambda_0} \mathfrak{g}/\Lambda_0$.

A G-invariant tubular neighborhood U of $G \cdot x$ is G-equivariantly diffeomorphic to $(\tilde{G} \times_{\tilde{H}} B)/\Gamma$. This data classifies such a tubular neighborhood up to equivariant diffeomorphism.

Suppose $k = n - m$ or $k = n - m - 1$, then $\tilde{H}_0$ is a maximal torus in $SO(2k)$ or $SO(2k + 1)$. Also, $\tilde{H} = \tilde{H}_0$, since $\tilde{H}$ is in the centralizer of $\tilde{H}_0$. Thus $\Gamma = \Lambda/\Lambda_0$ and $D = 1$, and the tubular neighborhood U is determined by $\Lambda_0 \subset \Lambda$ with $\tilde{H} = \mathfrak{h}/\Lambda_0$. The tubular neighborhoods are as follows:

a. $k = n - m$, U/G is homeomorphic to $[0, 1)^{n-m}$, $\Lambda_0 = \bigoplus_{i=1}^{n-m} \Lambda_i$, where the Λ_i are rank one linearly independent sublattices of Λ and $\Lambda_i \otimes \mathbb{R}/\Lambda_i$ is the stabilizer of orbits over the i^{th} face of U/G.

b. $k = n - m - 1$, U/G is homeomorphic to $[0, 1)^{n-m-1} \times (-1, 1)$, and $\Lambda_0 = \bigoplus_{i=1}^{n-m-1} \Lambda_i$.

We now pass from the local classification to the global. Let $W = X/G$.

Proposition A.7 ([37]) *Let $W = \bigcup_i W_i$ be a union of open sets, and let $\{(X_i, W_i, \pi_i)\}$ be a set of G-orbifolds with orbit maps $\pi_i : X_i \to W_i$. Then there is a G-orbifold X with orbit map $\pi : X \to W$ and with $\pi^{-1}(W_i)$ G-equivariantly diffeomorphic to X_i if and only if a Čech cohomology class in $H^3(W, \Lambda)$ associated to $\{(X_i, W_i, \pi_i)\}$ vanishes. When this is the case, the set of such G-orbifolds one to one with elements of $H^2_{orb}(W, \Lambda)$.*

Consider the cohomogeneity two case, $n - m = 2$ and $\dim W = 2$. Let X_0 be the open dense subset of m-dimensional orbits. Then $W_0 = X_0/G$ is a 2-orbifold. The only other possible orbits are of dimensions $m - 1$ and $m - 2$. That is, with stabilizers of dimensions $k = n - m - 1$ and $k = n - m$, respectively cases b) and a) above.

Theorem A.8 ([37]) *Let X be a compact connected oriented n-orbifold with a smooth effective action of an $m = (n-2)$-torus $G = \mathfrak{g}/\Lambda$.*

i. Then $W = X/G$ is a compact connected oriented 2-orbifold with edges and corners with each edge labeled with a Λ_i, where Λ_i is a rank 1 sublattice of Λ such that the two sublattices at a corner are linearly independent.

ii. For any data as in i. there is a G-orbifold. And if $H^2_{orb}(W, \Lambda) = 0$, then X is unique up to G-equivariant diffeomorphism.

iii. We have the exact sequence

$$\pi_2^{orb}(W_0) \to \Lambda/\textstyle\sum_i \Lambda_i \to \pi_1^{orb}(X) \to \pi_1^{orb}(W_0) \to e.$$

Thus $\pi_1^{orb}(X) = e$ if and only if W_0 is a smooth disk and $\sum_i \Lambda_i = \Lambda$, or $W_0 = W$ is a simply connected orbifold two sphere and we have $\pi_1^{orb}(W) = \mathbb{Z}$

and $\pi_1^{orb}(W) \to \Lambda$ *is an isomorphism,* $m = 1$.

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