Einstein Metrics of Positive Sectional Curvature on Weighted Projective Planes

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Abstract of the Dissertation

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In this dissertation, I prove the classification result that Einstein metrics of positive sectional curvature on weighted projective planes are unique (up to scale). An important intermediate theorem is that these metrics are self-dual.

To Mom and Dad

Contents

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Chapter 1

Introduction

Orbifolds are locally modeled on Euclidean Space quotient by finite group actions. In that sense, they are the simplest generalization of smooth manifolds allowing mild singularities. They arise naturally in the theory of Moduli spaces, group action and orbit spaces, as well as mathematical physics. These objects have been studied individually for a long time. However, the first formal definition has to wait till the 1950's (Satake, 1956). He (Satake, 1957) also discovered that many topological and geometrical concepts for manifolds could be generalized to the category of orbifolds, e.g., homology, cohomology, the tensor bundles and their sections, in particular, riemmannian metrics and curvature tensors.

Weighted projective planes are quotient spaces of 5-sphere ($\mathbf{S}^5$) by certain weighted circle actions. They are nice examples of 4 dimensional differentiable orbifolds. While finding smooth manifolds with certain metrics of special holonomy may be difficult, weighted projective planes have been found to admit very interesting metrics. For instance, Galicki and Lawson (1988), generalizing the symplectic quotient technique to the quaternionic spaces, demonstrated existence of self-dual Einstein metrics with positive scalar curvature on an infinite family of these spaces. On the other hand, Hitchin (1981) showed that their smooth cousin, the projective plane ($\mathbb{CP}^2$), along with $\mathbf{S}^4$ both equipped with the standard metrics, are the only examples of smooth manifolds supporting this kind of metrics. Due to their importance in riemannian geometry and mathematical physics, Galicki-Lawson metrics have been the subject of many investigations.

Apostolov and Gauduchon (2002) proved that Galicki-Lawson metrics are Einstein-Hermitian, and provided a local classification of self-dual Einstein-Hermitian metrics. Drawing on Derdziński's fundamental study (1983) of conformal geometry in dimension 4, they also set up a correspondence between the self-dual Einstein-Hermitian metrics and the self-dual Kähler metrics. Bryant (2001) in his comprehensive study of Bochner-kähler metrics proved that every weighted projective plane supports a self-dual Kähler metric which is presumably unique. Based on the above correspondence and local classification, Galicki-Lawson metrics and Bryant's metrics are locally the same. Of particular interest to us are Galicki-Lawson metrics of positive sectional curvature. The existence conditions of these metrics are provided by Dearricott (2004).

In the mean time, Gursky and LeBrun (1999) showed that the Fubini-Study metric is homothetically the unique Einstein metric of positive sectional curvature on $\mathbb{CP}^2$. In fact, their result was much stronger. They proved that the only 4 dimensional Einstein manifold with positive definite intersection form and positive sectional curvature is $\mathbb{CP}^2$. The basic idea of their proof was exploiting the Weitzenböck formulas and a lower bound on the Weyl curvature to derive inequalities between the Hirzebruch signature and the Euler characteristic of

Einstein manifolds of positive sectional curvature which are neither self-dual nor anti-self-dual. A simple calculation shows that manifolds with positive definite intersection form violate this inequality and consequently, the Einstein metrics with positive sectional curvature on such manifolds must be self-dual (with respect to the natural orientation). Quoting the above-mentioned result by Hitchin (1981), the conclusion is thus reached.

In regard of these beautiful works, we try to extend the above results on Einstein metrics of positive sectional curvature to weighted projective planes. In this direction, we have our intermediate

Theorem 1. *On The weighted projective planes, Einstein metrics of positive sectional curvature are necessarily self-dual (with respect to the canonical orientation)*

Therefore Einstein metrics of positive sectional curvature belong in the same class as the Galicki-Lawson metrics, thus must be locally isomorphic to them per the above local classification by Apostolov & Gauduchon (2002). One can furthermore infer from the recent results on self-dual Kähler geometry that the Galicki-Lawson metrics are unique in the class of self-dual Einstein metrics of positive scalar curvature, thus entails the following classification

Theorem 2. *On the weighted projective planes, Einstein metrics of positive sectional curvature are unique (up to scale).*

This is our main result.

The dissertation is organized as follows. In the next chapter, we will recall the definitions and properties of Einstein orbifolds, and give a detailed analysis of the orbifold structure of the weighted projective planes. In chapter 3, we derive geometrical and topological consequences of Einstein 4-orbifolds with positive sectional curvature, and combine the analysis in chapter 2 to prove the intermediate theorem 1. The final chapter discusses the uniqueness results of Guan (2000) and Apostolov *et. al.* (2003) and establish the main theorem.

Chapter 2

Einstein Orbifolds and Weighted Projective Planes

2.1 Four dimensional Self-dual Einstein spaces

Before discussing Einstein orbifolds, I want to set up a few definitions and point out several well-known results of 4 dimensional Einstein manifolds, which can be readily generalized to orbifolds. For more details, please refer to Besse (1987).

Definition 2.1.1. *A Riemannian manifold (M, g) is called Einstein if it has constant Ricci curvature, i.e. if its Ricci tensor r is a constant multiple of the metric, $r = \lambda g$*

Remark: Ricci curvature is the trace (with respect to the metric) on the first and the third indices of the Riemann curvature tensor. Due to the symmetry conditions, itself is a symmetric 2-tensor, hence is the same kind of tensor as the metric, making sense of the above equation. This is precisely the vacuum Einstein equation in General Relativity.

In dimension four, under the action of $SO(4)$, the curvature tensor decomposes into irreducible pieces as follows. The rank 6 bundle of 2-forms Λ^2 on an oriented Riemannian 4-manifold (M^4, g) has an invariant decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-$$

as the sum of two rank 3 vector bundles. Here $\Lambda^\pm$ are by definition the eigenspaces of the Hodge duality operator

$$* : \Lambda^2 \mapsto \Lambda^2, *^2 = 1$$

Corresponding respectively to the eigenvalue ± 1. Sections of Λ^+ are termed the self-dual 2-forms, whereas sections of Λ^- are called the anti-self-dual 2-forms. The curvature tensor of g may be thought of as a map $\Re : \Lambda^2 \mapsto \Lambda^2$, the decomposition then appears as:

$$\Re = \begin{pmatrix} s/12 + W^+ & \dot{r} \\ \dot{r} & s/12 + W^- \end{pmatrix}$$

where the self-dual and anti-self-dual Weyl curvatures $W^\pm$ are trace free as endomorphisms of $\Lambda^\pm$. The scalar curvature s acts by scalar multiplication. $\dot{r}$ represents the trace-free Ricci curvature, and vanishes iff g is Einstein. (Besse, 1987)

With the above decomposition in mind,

Definition 2.1.2. *An Einstein metric is (anti-)self-dual if $W^- = 0$ ($W^+ = 0$).*

Remark: Self-dual metrics have many nice properties. In particular, it's related to the twistor theory of Penrose. (see e.g. Atiyah, Singer & Hitchin (1978)). There has been much research done on self-dual Einstein metrics. We will discuss some of the results that are relevant to us in the last chapter.

Remark: In dimension 4, thanks to the irreducible decomposition, the Chern-Gauss-Bonnet formula can be written as

$$\chi = \int_M \Omega = \frac{1}{8\pi^2} \int_M |W^+|^2 + |W^-|^2 - \frac{|\dot{r}|^2}{2} + \frac{s^2}{24} \, d\mu$$

and the Hirzebruch signature formula

$$\tau = \frac{1}{3} \int_M p_1(M) \, d\mu = \frac{1}{12\pi^2} \int_M |W^+|^2 - |W^-|^2 \, d\mu$$

To proceed, we need one more

Definition 2.1.3. *An oriented riemannian 4-manifold M is called quaternionic Kähler if M is Einstein and anti-self-dual.*

Remark: quaternionic Kähler condition can be defined for any dimension, and is a generalization of the Kähler geometry. On the other hand, most quaternionic Kähler metrics are not Kähler. In dimension 4, quaternionic Kähler condition is equivalent to requiring the metric to be Einstein and anti-self-dual. Reversing the orientation interchanges $W^{\pm}$, hence M is quaternionic Kähler iff the metric is self-dual Einstein. Besse (1987) devoted a chapter on quaternionic Khäler geometry from the point of view of Einstein metrics. Please refer to this and the references therein for more information.

2.2 Differentiable orbifolds

Orbifolds arise in many mathematical branches such as moduli spaces, algebraic varieties and mathematical physics. The concept of an orbifold was first introduced by Satake (1956) under the name of V-manifolds. It is a generalization of an orbit space of a smooth finite group action on a smooth manifold. However, we must point out that all orbifold are not global quotients spaces of finite group actions. We include parts of the definitions that are relevant to us here for the sake of completeness. For the original definitions and more details, please refer to the original papers of Satake (1956, 1957)

Definition 2.2.1. *Let X be a Hausdorff space. A C^∞ local uniformizing system (l.u.s) $\{U, G, \phi\}$ for an open set $\tilde{U}$ in X is by definition a collection of the following objects:*

U: a connected open set in R^n,

G: a finite group of C^∞-automorphisms of U, with the set of fixed points of dimension $\leq m-2$.

ϕ: a continuous map from U onto $\tilde{U}$ such that $\phi \circ \sigma = \phi$ for all $\sigma \in G$, inducing a homeomorphism from the quotient space U/G onto $\tilde{U}$.

Definition 2.2.2. *Let $\{U, G, \phi\}, \{U', G', \phi'\}$ be l.u.s. for $\tilde{U}$, $\tilde{U}'$ respectively, and let $U \subset U'$. By a C^∞ injection $\lambda : \{U, G, \phi\} \hookrightarrow \{U', G', \phi'\}$, we mean a C^∞-isomorphism λ from U onto an open subset of U' such that $\phi = \phi' \circ \lambda$. Every $\sigma \in G$ can be then considered as an injection of $\{U, G, \phi\}$ into itself.*

Also composition of injections is an injection. Conversely, we have the following lemmas.

Lemma 2.2.3. *Let λ, μ be two injections $\{U, G, \phi\} \hookrightarrow \{U', G', \phi'\}$. Then there exists a uniquely determined $\sigma' \in G'$ such that $\mu = \sigma' \circ \lambda$.*

Lemma 2.2.4. *Let λ be an injection $\{U, G, \phi\} \hookrightarrow \{U', G', \phi'\}$. If $\sigma'(\lambda(U)) \cap \lambda(U) \neq \emptyset$ with $\sigma' \in G'$, then $\sigma'(\lambda(U)) = \lambda(U)$ and σ' belongs to the image of the isomorphism $G \to G'$ defined above.*

Remark: for proofs of these lemmas, see Satake (1957).

With these preliminary results, one can give the definition of an orbifold and smooth maps between them.

Definition 2.2.5. A differentiable orbifold *is a Hausdorff topological space* $\mathbf{X}$ *with a family $\mathfrak{F}$ (called a defining family for the orbifold) of C^∞ l.u.s. for open subsets in $\mathbf{X}$ satisfying the following conditions.*

(1) Every point p of $\mathbf{X}$ is contained in at least one $\mathfrak{F}$-uniformized open set (i.e. an open set U for which there exists l.u.s. (U, G, ϕ) in $\mathfrak{F}$ such that $\phi(U) \subset U$. If p is contained in two $\mathfrak{F}$-uniformized open sets U_1, U_2, then there exists an $\mathfrak{F}$-uniformized open set U_3 such that $p \in U_3 \subset U_1 \cap U_2$.

(2) If $\{U, G, \phi\}, \{U', G', \phi'\}$ are l.u.s. in $\mathfrak{F}$ such that $\phi(U) \supset \phi'(U')$, then there exists always a C^∞ injection $\lambda : \{U, G, \phi\} \hookrightarrow \{U', G', \phi'\}$. ($\lambda$ is uniquely determined up to $\sigma' \in G'$, by Lemma 2.2.4)

Definition 2.2.6. *Let $(\mathbf{X}_1, \mathfrak{F}_1), (\mathbf{X}_2, \mathfrak{F}_2)$ be two orbifolds. We mean by* a differentiable orbifold map *h from $(\mathbf{X}_1, \mathfrak{F}_1)$ into $(\mathbf{X}_2, \mathfrak{F}_2)$ a system of mappings $h_{U_1}(\{U_1, G_1, \phi_1\} \in \mathfrak{F}_1)$ as follows:*

(1) There is a correspondence $\{U_1, G_1, \phi_1\} \mapsto \{U_2, G_2, \phi_2\}$ from $\mathfrak{F}_1$ into $\mathfrak{F}_2$ such that for any $\{U_1, G_1, \phi_1\} \in \mathfrak{F}_1$ we have a C^∞ map h_{U_1} from U_1 into U_2.

(2) Let $\{U_1, G_1, \phi_1\}, \{U_1', G_1', \phi_1'\} \in \mathfrak{F}_1, \{U_2, G_2, \phi_2\}, \{U_2', G_2', \phi_2'\} \in \mathfrak{F}_2$ *be the corresponding l.u.s. (in the sense of (1)) and let* $\phi_1(U_1) \supset \phi_1'(U_1')$*. Then for any injection* $\lambda_1 : \{U_1, G_1, \phi_1\} \hookrightarrow \{U_1', G_1', \phi_1'\}$ *there exists an injection* $\lambda_2 : \{U_2, G_2, \phi_2\} \hookrightarrow \{U_2', G_2', \phi_2'\}$ *such that* $\lambda_2 \circ h_{U_1} = h_{U_2'} \circ \lambda$

Remark: There is an equivalent definition using categorical language, see Kawasaki (1978) for details. In recent years, another definition of orbifolds surfaced using the language of groupoids. I adopt the original definitions of Satake (1957) because it's the most geometric and intuitive, and because I don't understand the other approaches that well.

Example: weighted projective spaces. Fix relatively prime positive integers $a_0, a_1, \ldots, a_n$ and define the weighted projective spaces $\mathbb{CP}^n_{(a_0,a_1,\ldots,a_n)}$ to be the quotient of $\mathbb{C}^{n+1}$ by the $\mathbb{C}^*$ action: $(z_0, z_1, \ldots, z_n) \to (\tau^{a_0} z_0, \tau^{a_1} z_1, \ldots, \tau^{a_n} z_n)$ for $\tau \in \mathbb{C}^*$. We will discuss the structure of weighted projective planes $\mathbb{CP}^2_{(a_0,a_1,a_2)}$ in detail below.

We define next the concept of orbifold bundles. Let V, B be two orbifolds with a differentiable map $\pi : V \to B$. Let further F be a smooth manifold and G be a Lie group operating on F as a C^∞ group of transformations,

Definition 2.2.7. *A pair of defining families* $(\mathfrak{F}, \mathfrak{F}^*)$*,* $\mathfrak{F}$ *being a defining family of* B *and* $\mathfrak{F}^*$ *that of* V*, is called a pair of defining families for a coordinate orbifold bundle* (V, B, π, F, G)*, if it satisfies the following conditions:*

(1) There exists a one-to-one correspondence $\{U, G, \phi\} \mapsto \{U^*, G^*, \phi^*\}$ *between* $\mathfrak{F}$ *and* $\mathfrak{F}^*$ *such that* $U^* = U \times F$ *and denoting by* π_{U^*} *the projection* $U^* \to U$*, we have* $\pi \circ \phi^* = \phi \circ \pi_U$

(2) Let $\{U, G, \phi\}, \{U^*, G^*, \phi^*\}, \{U', G', \phi'\}, \{U'^*, G'^*, \phi'^*\}$ *be two pairs of cor-*

responding l.u.s. in $(\mathfrak{F}, \mathfrak{F}^*)$ *and let* $\phi(U) \supset \phi'(U')$. *Then* $\phi^*(U^*) \supset \phi'^*U'^*$ *and there exists a one-to-one correspondence* $\lambda \to \lambda^*$ *between injections* $\lambda : \{U, G, \phi\} \hookrightarrow \{U', g', \phi'\}$ *and* $\lambda^* : \{U^*, G^*, \phi^*\} \hookrightarrow \{U'^*, G'^*, \phi'^*\}$ *such that for* $(p, q) \in U^* = U \times F$ *we have* $\lambda^*(p, q) = (\lambda(p), g_\lambda(p)q)$ *with* $g_{\lambda(p)} \in G$. *The mapping* $g_\lambda : U \to G$ *is a* C^∞ *map satisfying the relation*

$$g_{\mu\lambda}(p) = g_\mu(\lambda(p)) \circ g_\lambda(p)$$

sometimes, we call V a orbifold bundle over B for simplicity.

Let (V, B, π, F, G) be an orbifold bundle with a pair of defining families $(\mathfrak{F}, \mathfrak{F}^*)$. An orbifold map $f = f_U : (B, \mathfrak{F}) \mapsto (V, \mathfrak{F}^*)$ is called a C^∞ *cross section* of this orbifold bundle if the correspondence $\mathfrak{F} \mapsto \mathfrak{F}^*$ is given by the correspondence in the above definition and if $\pi_U \circ f_U = 1$. To give a cross section $f : (B, \mathfrak{F}) \to (V, \mathfrak{F}^*)$ is therefore to give a cross section f_U of each $U^* = U \times F$ such that for any injection $\lambda : \{U, G, \phi\} \hookrightarrow \{U', G', \phi'\}$ we have $f_U \circ \lambda = \lambda^* \circ f_U$.

Example: Tangent space and differential forms. Let $(X, \mathbb{F})$ be an orbifold. Assuming that every U is contained in $\mathbb{R}^m$, we fix a coordinate system $u^1, \ldots, u^m$ in each U once for all. Let $V = \mathbb{R}^m$ (vector space of dimension m over $\mathbb{R}$) and $G = GL(m, \mathbb{R})$ (group of all non-singular matrices of degree m) For any injection $\lambda : \{U, G, \phi\} \hookrightarrow \{U', G', \phi'\}$ put

$$G_\lambda(p) = (\frac{\partial u'^i \circ \lambda}{\partial u^i})$$

$\{u^i\}, \{u'^i\}$ being the coordinate systems in U, U' respectively. Then the system

g_λ, defines an orbifold bundle (T, X, π, F, G) with a pair of defining families $(\mathfrak{F}, \mathfrak{F}^*)$. This orbifold bundle is called the tangent bundle of X. Note that the fiber $\pi^{-1}(p)(p \in X)$ is not always a vector space, and the bundle structure is not necessarily a fiber bundle in the classical sense.

Let $p \in \phi(U), \{U, G, \phi\} \in F$ and choose $p \in U$ such that $\phi(p) = p$. Then $\pi^{-1}(p) = R^m/g_\sigma(p); \sigma \in G_p$, G_p denoting the isotropy subgroup of G at p. Now $\pi_U^{-1}(p) = p \times R^m$ can be identified with T_p (the tangent space to U at p) by the correspondence

$$P \times (x^1, \dots, x^m) \mapsto X = \sum x^i \frac{\partial}{\partial u^i}$$

Then denoting by $T_p^{G_p}$ the linear subspace of T_p formed of all G_p-invariant vectors (i.e., vectors invariant under $g_\sigma(p)(\sigma \in G_p)$), we see that $\pi^{-1}(p)$ contains a vector space $T_p = \phi^*(T_p^{G_p})$, which is independent of the choice of U and p. An element of T_p is called a tangent vector to X at p.

A cross section $\mathfrak{X}$ of the orbifold bundle T is called a (contravariant) vector field over X. In the above notations, $\mathfrak{X}_u$ being a G-invariant cross-section of $U^* = U \times F$ (i.e. a G-invariant vector field over U in the usual sense), we have $\mathfrak{X}_U(p) \in T_p^{G_p}$ and $\mathfrak{X}(p) \in T_p$. Thus $\mathfrak{X}(p)$ being a tangent vector at p for any $p \in X$, the set of all vector fields over X forms a vector space.

More generally one can construct an (r, s) tensor bundle over X by means of the system $(g_\lambda : g_\lambda(p) = \frac{\partial u'^i}{\partial u^j} \otimes \dots \frac{\partial u^j}{\partial u'^i} \otimes \dots)$ Where $\otimes$ denote the Kronecker product of matrices. (In this case, $V = R^{m(r+s)}$, and $G = GL(m, \mathbb{R})$ operating on V as an (r, s)-tensor representation.) We can also consider skew-symmetric or symmetric tensor bundles over X.

In particular, consider the skew-symmetric $(h, 0)$-tensor bundle over X. As in the case of the tangent vector bundle, $\pi^{-1}(p)\,(p \in X)$ contains a vector space D_p^h, which is isomorphic to the space of G_p-invariant skew-symmetric $(h, 0)$ tensors at p $(\phi(p) = p)$. D_p^1 can be regarded as a dual space of T_p. It should be noted that $\bigoplus_{h=1}^m D_p^h$ is not always an exterior algebra over D_p^1. A cross section Ω of D^h is called a differential form of degree h (h-form in short) on X. Since $\Omega(p) \in D_p^h$ for any $p \in X$, the set of all h-forms Ω on X forms a vector space. By definition, to give an h-form Ω on X is to give a (G-invariant) h-form Ω_U on each U such that it holds $\Omega_U = \Omega'_U \circ \lambda$ for any injection $\lambda : \{U, G, \phi\} \hookrightarrow \{U', G', \phi'\}$. Using these 'local expressions', we can define the operations $\wedge$ and d just as in the case of ordinary manifold. Also if $h : X \mapsto X'$ is any orbifold map, we can pull back differential forms.

The geometric concepts such as the Riemannian metric and connections and curvature forms can be defined similarly for orbifold. The expressions are similar to the case of manifolds. The Chern-Weil theory also generalizes without much modification so we can talk about the theory of characteristic classes. (The literature on orbifold is very sporadic and on-explicit. In the sequel chapters, we will have to freely use the theory of chern classes and line bundle without offering a proof or quoting a specific source. The best references for orbifold theory in general are probably the original papers of Satake (1956, 1957) and a pair of papers by his student Baily (1967)) With this understood, we can state the analog of Chern-Gauss-Bonnet formula for orbifolds. (Satake, 1957)

Theorem 3. *Let X be a compact riemannian orbifold of even dimension m.*

Then for any vector field with singularities at $p_1, \ldots, p_s$, we have

$$\chi_V(X) \equiv \sum I_{p_i}(X) \equiv \sum \frac{1}{n_i} index(p_i) = \int_M \Omega$$

Where n_i is the order of the stabilizer group of p_i. χ_V is called the orbifold Euler characteristic.

Remark: In general, χ_V does not coincide with the topological Euler characteristic $\chi(X)$ and is not necessarily an integer. By analog of Chern's argument, $\chi_V(X)$ is an invariant of the orbifold structure. In case of dimension 4, using the invariant components of the curvature tensor, the above formula can be written as

$$\chi_V(X) = \sum I_{p_i}(X) = \frac{1}{8\pi^2} \int_X |W^+|^2 + |W^-|^2 - \frac{|\dot{r}|^2}{2} + \frac{s^2}{24} \, d\mu$$

Similarly, one can define the *signature* τ_V of a 4-dimensional orbifold X as

$$\tau_V(X) = \frac{1}{3} \int_X p_1(X) \, d\mu = \frac{1}{12\pi^2} \int_X |W^+|^2 - |W^-|^2 \, d\mu$$

Remark: that $\tau_V(X)$ is an invariant of the orbifold structure was shown by Kawasaki (1978). One can also prove this using the Chern-Weil approach. Again, we use the invariant components when writing out the Pontrjagin class p_1

From these formulas, we immediately derive the analog of Hitchin-Thorpe inequality for orbifolds.

Proposition 2.2.8. *Let X be a smooth compact orbifold of dimension 4. If X* *admits an Einstein metric, then*

$$3|\tau_V(X)| \leq 2\chi_V(X)$$

Remark: This inequality places constraints on which of weighted projective planes admits Einstein metrics.

It can easily be seen by Lemma (2.2.4) that the structure of the isotropy subgroup at a point $p \in M$ does not depend on the choice of the uniformizing system. The points whose isotropy group is trivial are called the regular or manifold points. The points with non-trivial isotropy groups will be called the singular points. The collection of the singular points will be denoted ΣX. ΣX is itself an orbifold, and can be resolved as follows. Let $(1) = h_x^0, h_x^1, \ldots, h_x^{\rho_x}$ be all the conjugacy classes of elements of G_x, the uniformizing group at x. Consider the set of pairs:

$$\widetilde{\Sigma}X = \{(x, (h_x^j)|x \in \Sigma X, j = 1, 2, \ldots, \rho_x\}$$

Let $Z_{G_x}(h_x^j)$ be the centralizer of h_x^j in G_x, we see that

$$\widetilde{\Sigma}U_x \cong \{\coprod_{j=1}^{\rho_x} \widetilde{U}_x^{h_x^j} / Z_{G_x}(h_x^j)\}$$

Then we have a V-manifold structure $V_{\Sigma X}$ on $\widetilde{\Sigma}X$ defined by

$$V_{\widetilde{\Sigma}X} = \{(Z_{G_x}(h_x^j)/K_x^j, \widetilde{U}_x^{h_x^j}) \mapsto \widetilde{U}_x^{h_x^j} / Z_{G_x}(h_x^j)\}$$

Here K_x^j is the kernel of the representation $Z_{G_x}(h_x^j) \mapsto Diffeo(\widetilde{U}_x^{h_x^j})$. The number $m|K_x^j|$ is called the multiplicity of $\widetilde{\Sigma}X$ in X at $(x, (h_x^j))$.

Kawasaki (1978) also proved the corresponding signature theorem for orbifolds.

Theorem 4. *Let X be a differentiable orbifold, then the signature of X as a homology manifold is given by*

$$\tau(X) = L(X)[X] + \sum \frac{1}{m} L(X_i)[\widetilde{\Sigma}X_i]$$

Here the summand is over the components of the stratification of X described above. L is the equivariant L-class defined by Atiyah-Singer (1968).

and the following useful

Lemma 2.2.9. *The singular homology group ($\mathbb{R}$ coefficient) of a differentiable orbifold is isomorphic to its de Rham cohomology group.*

Less the singularities, an orbifold is simply a smooth manifold. The geometric definitions in the first section apply in the orbifold sense if they are *Bona Fide* objects on orbifolds and satisfy the corresponding conditions on the smooth part.

Remark: A geometric object on a manifold does not necessarily vanish at a singular point. As long as it's invariant under the group action, it's legitimate on the orbifold in question. Therefore, for a finite subgroup of the orthogonal group, in appropriate coordinates, a metric that is rotationally symmetric is allowed at the singular point. Because orbifold singularities arise from finite group actions, which regularize by unfolding via l.u.s., the singularity is not so 'wild'. We still have some control over the geometric objects on these spaces.

For instance, the tensor field never diverges at the singular points if it's not already divergent on the l.u.s, and the curvature usually doesn't blow up at the singular points.

2.3 Weighted Projective Planes

We now describe the orbifold structure of the weighted projective planes.

Definition 2.3.1. *For $(x, y, z) \in \mathbb{C}^3$ such that $|x|^2 + |y|^2 + |z|^2 = 1$, let $\tau \in \mathbb{S}^1$ acts by $(\tau^p x, \tau^q y, \tau^r z)$ where p, q, r are positive integers and $gcd(p, q, r) = 1$.*

The quotient space of $\mathbf{S}^5$ by the above action of $\mathbf{S}^1$ is called the a weighted projective plane *of weight (p, q, r), denoted as $\mathbb{CP}^2_{(p,q,r)}$.*

Remark: when $p = q = r = 1$, this is the standard definition of the projective plane $\mathbb{CP}^2$, hence the notation. Otherwise, we have an infinite family of differentiable orbifolds. Let $U_0 \subset \mathbb{CP}^2_{(p,q,r)}$ be the open neighborhood given in homogeneous coordinates by $U_0 = \{[x, y, z], x \neq 0\}$, it's easy to see that the map $\phi_0 : \mathbb{C}^2 \mapsto U_0$ is a l.u.s for U_0 with finite group $G_0 = \mathbb{Z}_p$. Similarly, one can define U_1 and U_2, together they make up the l.u.s. for weighted projective planes.

Remark: This definition is easily generalized to the higher dimensions to weighted projective spaces $\mathbb{CP}^m_{(a_0,a_1,\ldots,a_m)}$

When $z \equiv 0$, the action restricts to the (x, y) plane as $(\tau^p x, \tau^q y)$, geometrically it corresponds to the action on the toroidal decomposition of $\mathbf{S}^3$ where $\mathbf{S}^1$ acts on the tori by p^{th} power in one homological direction and q^{th} power in the other. The action on each torus is free and the quotient is a circle. At each of the center circles, $\mathbf{S}^1$ acts by p^{th} power and q^{th} power respectively, and the quotients are two points with stabilizers $\mathbb{Z}_p$ and $\mathbb{Z}_q$. Thus, the quotient space of $\mathbf{S}^3$ by this action is a 2-sphere of two singular points with these stabilizer groups. When $d = (p, q) \neq 1$, the action on each torus is not free. The quotient is still a circle all of whose points are singular with a stabilizer group $\mathbb{Z}_d$. The action on the center circles remains the same. The quotient space is again a sphere with 2 special points with stabilizers $\mathbb{Z}_p$ and $\mathbb{Z}_q$. However, the other points also has non-trivial stabilizer group $\mathbb{Z}_d$.

Now let $|x|^2 + |y|^2 \neq 0$, this certainly contains the 2 sphere described above. This is topologically a complex line bundle quotient by the finite group action $\mathbb{Z}_r$ where $\mathbb{Z}_r$ acts in the z direction (fiber direction) (See Galicki & Lawson (1988) for the explicit coordinates). They also showed that Chern class of this line bundle is r. Adding in the infinite point, we obtain topologically the Thom space of a topological line bundle with Chern class r, with base a singular sphere. It's clear that this picture is symmetric with respect to the cyclic order of (p, q, r). We therefore find three Thom spaces on a weighted projective plane, depending on which point is chosen to the ∞.

Another way to look at these spaces is topologically we have 3 singular spheres (corresponding to the 3 coordinate planes of $\mathbb{CP}^2$) attached on $\mathbb{C}^* \times \mathbb{C}^*$. For this reason, weighted projective planes are the so called *Toric Varieties.* From this picture, it's not hard to see the cell structure of the weighted projective planes consists of a 0-cell, 2-cell and a 4-cell. And the singular homology groups are the same as those of $\mathbb{CP}^2$.

2.4 Symplectic Quotients and Galicki-Lawson metrics

Symplectic Quotient construction was developed by Marsden and Weinstein (1974). The initial intent was to reduce the degree of freedom of a symplectic manifold acted on by a Lie group as a symplectomorphism. Modding out the manifold by the group action and the orbit space turns out to have a natural symplectic structure. In the following years, this construction turns out to be extremely useful in constructing new metrics of special holonomy. Most notably, Hitching *et. al.* (1987) employed this technique to construct new hyperkähler metrics.

Galicki and Lawson (1988) generalized further this technique to quaternionic Kähler metrics. They consider the action of $\mathbf{S}^1$ on the quaternionic projective spaces $\mathbb{HP}^n$. Many of the quotient space they obtained are however, riemannian orbifolds. In particular, they included the following examples when $n = 2$.

Theorem 5. *Each of the weighted projective planes $\mathbb{CP}^2_{(2a,a+b,a+b)}$ for $a+b$ odd carries a self-dual Einstein orbifold metric with positive scalar curvature. This is also true of the weighted projective planes $\mathbb{CP}^2_{(a,\frac{a+b}{2},\frac{a+b}{2})}$ when $a+b$ is even.*

Remark: Choosing other weights for the $\mathbf{S}^1$-action on $\mathbf{HP}^2$ gives similar metrics on $\mathbb{CP}^2_{(p,q,r)}$ for other values of the weights. For many combinations of (p,q,r) (determined by the Hitchin-Thorpe inequality, there exists a quaternionic Kähler metric on the corresponding weighted projective planes. For integers $0 \le p \le q \le r$ satisfying $r < p + q$, the scalar curvature of the metric is positive.

Dearricott (2001) claims that infinitely many of these metrics in fact, have positive sectional curvature, hence established the existence of positively curved Einstein metrics on an infinite family of these spaces. Understanding Bryant's method (2001) for Bochner Kähler metrics, he gave furthermore the following classification (2004)

Theorem 6. *The Galicki-Lawson metrics on $\mathbb{CP}^2_{p,q,r}, p < q < r$ have positive sectional curvature if and only if .*

$$\sigma_3(p+q+r, -3p+q+r, p-3q+r, p+q-3r) > 4(p+q+r)^3$$

where σ_3 is the third symmetric polynomial.

2.5 Orbifold Euler characteristic and Signature of Weighted Projective Planes

In this section, we compute the orbifold Euler characteristic χ_V and the signature τ_V for weighted projective planes. In the next chapter, we will use these numbers to prove our main results. In these directions, we have the following

Theorem 7. *For the weighted projective plane $\mathbb{CP}^2_{p,q,r}$, the orbifold Euler characteristic χ_V and the signature τ_V are given by*

$$\chi_V = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$

$$\tau_V = \frac{1}{3}\frac{p^2+q^2+r^2}{pqr}$$

Proof. Choosing a vector field that vanishes at the three singular points. In fact, any vector field on a weighted projective plane satisfies this condition by definition. Less the singularities, we have a smooth manifold with Euler characteristic 0, hence the sum of the index of the vector field over the smooth part is 0. As a matter of fact, we can construct such a vector field explicitly using the cell structure of weighted projective planes. Take the constant radial vector field on the 4-cell, attaching the 2-cell with the radial vector field, it's clear that the vector fields match up. Add in the 0-cell. We have a global vector field vanishing at the three singular points with index +1 and nowhere else. hence the orbifold Euler characteristic follows from the formula of Satake.

To prove the second formula, we need to compute the first Pontrjagin class of weighted projective planes. To this end, look at the following orbifold map $\phi : \mathbb{CP}^2 \mapsto \mathbb{CP}^2_{(p,q,r)}$ that sends $[x, y, z]$ to $[x^p, y^q, z^r]$. It's easy to see this map is well defined and is an orbifold map in the above sense. It's also easy to see that the map is a pqr-fold covering map.

On $U_0 = \{\lambda = x/z, \mu = y/z, z \neq 0\} \subset X = \mathbb{CP}^2_{(p,q,r)}$, consider the meromorphic two form in the inhomogeneous coordinates,

$$\omega = r\frac{d\lambda}{\lambda} \wedge \frac{d\mu}{\mu}$$

when we change coordinates to U_1, U_2, this form has similar presentations with coefficients p and q. (The coefficients can be most easily seen by observing that U_0 is the quotient of $\mathbb{C}^2$ by $\mathbb{Z}_r$, hence needs a multiple r to compensate for the fact that the volume is $\frac{1}{r}$ of the l.u.s.) Hence this form is global and defines a section of the canonical line bundle K on X. It's clear that this form has divisor $-H'_0-H'_1-H'_2$, where H'_i are the three singular spheres. By the orbifold map ϕ, the divisor is pulled back to the divisor $-pH_0 - qH_1 - rH_2$ on $\mathbb{CP}^2$, which is linearly equivalent to $-(p+q+r)H$. That this is true can be seen as follows. Under ϕ, the inverse images of the singular spheres are precisely the coordinate planes H_i in $\mathbb{CP}^2$. In fact, H_i is a pq (qr, rp resp.) covering H'_i, with multiplicity r (p, q resp.) (Take, e.g., $H_0 = [0, y, z]$, it's easy to see that ϕ is a qr fold covering. Now fix y, z, and consider ϕ in a neighborhood of $x = 0$, the map is locally of the form x^p, hence H_0 has multiplicity p). Moreover, it's easy to check that $\phi^*\omega$ is a section of the canonical bundle of $\mathbb{CP}^2$. The above assertion also follow from the Riemann-Hurwitz formula, because ϕ is a branched covering with branch points the three singular spheres with multiplicities p, q and r. Hence we have

$$\phi^* c_1^2(X)[\mathbb{CP}^2] = \phi^* c_1^2(K)[\mathbb{CP}^2] = c_1^2(\phi^* K) = (p+q+r)^2$$

since ϕ is a pqr-fold covering, and the covering map is compatible with the orbifold structure

$$c_1^2(X)[X] = \frac{(p+q+r)^2}{pqr}$$

therefore

$$\tau_V(X) = \frac{1}{3}(c_1^2(X) - 2c_2(X)) = \frac{1}{3}(c_1^2(X) - 2\chi_V(X)) = \frac{1}{3}\frac{p^2+q^2+r^2}{pqr}$$

□

Van Coevering (2004) obtained the same numbers using slightly different methods. He also established the orbifold Miyaoka-Yau inequality and its consequences in his Ph.D thesis.

Remark: Using these formulas, it's easy to see that the constraints on weighted projective planes admitting Galicki-Lawson metrics are in agreement with the Hitchin-Thorpe inequality above.

Remark: The orbifold Euler Characteristic formulas generalize easily to the Weighted Projective Spaces so that

$$\chi_V(\mathbb{CP}^m_{(a_0,a_1,\ldots,a_m)}) = \frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_m}$$

The computation of c_1 also gives similar results. I don't know how to compute the higher Chern numbers in this case. This could be a very interesting research project.

Chapter 3

Main estimates and the proof of the main theorem

In this chapter, we derive estimates involving various invariant components of the curvature tensor, and as a consequence, inequalities involving the orbifold Euler characteristics and the signature for Einstein metrics of positive sectional curvature on weighted projective planes. Most of the results were worked out in Gursky & LeBrun (1999) for smooth manifolds, here I show that they generalize to the category of orbifolds. I give detailed proofs only in the case where they differ from the original source. Combining with the computation on the characteristic numbers in the previous chapter, we obtain the intermediate theorem I. Note that for local results, we choose to present them on the smooth part of the orbifolds, i.e., a non-compact manifold. The orbifold singularities are irrelevant. Some of the results actually are true on the singular parts as well, but have no bearing on the final results at all. For the global results involving integrals over the whole space, since the singular points are of measure zero, and since on a local uniformization system, all the concepts are invariant under the group action. Choosing the invariant measure, one sees that the integrations go through just as for smooth manifolds.

We begin with a few preliminary results.

Lemma 3.0.1. *Let (X, g) be an oriented 4 dimensional Einstein orbifolds. If the sectional curvature of g is non-negative, then away from the singular points,*

$$\frac{s}{\sqrt{6}} \geq |W^+| + |W^-|$$

Proof. Under the Hodge decomposition, every 2-form ϕ on X can be uniquely written as $\phi = \phi^+ + \phi^-$, where $\phi^\pm \in \Lambda^\pm$. Now a 2-form is decomposable iff $\phi \wedge \phi = 0$. But this condition can be rewritten as $|\phi^+|^2 - |\phi^-|^2 = 0$, since the decomposition is orthogonal. Thus the sectional curvature of g is non-negative iff the curvature operator $\Re : \Lambda^2 \mapsto \Lambda^2$ satisfies

$$\langle \phi^+ + \phi^-, \Re(\phi^+ + \phi^-)\rangle \geq 0$$

for all unit-length self-dual 2-form ϕ^+ and all unit-length anti-self-dual 2-forms ϕ^-. But for an Einstein manifold, the invariant decomposition tells us that this can be rewritten as

$$\frac{s}{6} + \lambda_+ + \lambda_- \geq 0 \tag{3.1}$$

where, for each $x \in M$, $\lambda_\pm(x) \leq 0$ is by definition the smallest eigenvalue of the trace- free endomorphism $W_x^\pm : \Lambda_x^\pm \mapsto \Lambda_x^\pm$.

The claim will thus follow immediately from the above inequality (3.1) if we can show that

$$|\lambda_\pm| \geq \frac{1}{\sqrt{6}}|W^\pm|$$

To see this, let $\lambda_+ \leq \mu_+ \leq \nu_+$ be the eigenvalues of W^+. Thus

$$|W^+|^2 = \lambda_+^2 + \mu_+^2 + \nu_+^2$$

But since W^+ is trace-free, $\lambda_+ + \mu_+ + \nu_+ = 0$, hence

$$|W^+|^2 = \lambda_+^2 + \mu_+^2 + \nu_+^2 + (\lambda_+ - \mu_+ - \nu_+)(\lambda_+ + \mu_+ + \nu_+) = 2[\lambda_+^2 - \mu_+\nu_+]$$

If $\mu_\pm \geq 0$, this last expression is less than $2|\lambda_+|^2$. Otherwise, $\lambda_+ \leq \mu_\pm \leq 0$, $0 < \nu_+ \leq 2|\lambda_+|$, and hence $|W^+|^2 \leq 6|\lambda_+|^2$. We have $|\lambda_-| \geq \frac{1}{\sqrt{6}}|W^-|$ by the same argument, and the lemma is proved. □

Applying Lemma 3.0.2 gives us the following estimates

Lemma 3.0.2. *Let (X, g) be a compact 4-dimensional Einstein orbifold of non-negative sectional curvature. If g is not flat, then*

$$\chi_V(X) \leq \frac{5}{8\pi^2} \int_X \frac{s_g^2}{24}\, d\mu_g \tag{3.2}$$

Proof. Because g is not flat, and the sectional curvature is non-negative, our Einstein metric g must have positive scalar curvature, and hence positive ricci Curvature. Lemma (3.0.2) now tells us that

$$|W_g^+|^2 + |W_g^-|^2 \leq (|W_g^+| + |W_g^-|)^2 \leq \frac{s_g^2}{6}$$

so that

$$\chi_V(X) = \frac{1}{8\pi^2} \int_X [|W_g^+|^2 + |W_g^-|^2 + \frac{s_g^2}{24}]\, d\mu_g \leq \frac{5}{8\pi^2} \int_X \frac{s_g^2}{24}\, d\mu_g$$

□

Remark: in the case of smooth manifold, using the Chern-Gauss-Bonnet formulas, Gursky & LeBrun 1999) showed that the inequality must be strict.

This has an interesting consequence

Lemma 3.0.3. *Let (X, g) be a compact 4-dimensional Einstein orbifold of non-negative sectional curvature. Then $\chi_V(M) \leq 10$*

Proof. We may assume that g has positive scalar curvature, since otherwise the orbifold Euler characteristic would vanish. By rescaling, we can thus arrange for our Einstein metric to have Ricci tensor $r = 3g$. A generalization of the Bishop's inequality to orbifolds then asserts that the total volume of (M, g) is less than or equal to that of the 4-sphere with its standard metric g_s. Since both g and g_s have $s = 12$, Lemma (3.0.3) now asserts that

$$\chi_V(X) \leq \frac{5}{8\pi^2} \int_X \frac{s_g^2}{24} \, d\mu_g \leq \frac{5}{8\pi^2} \int_{S^4} \frac{s_g^2}{24} \, d\mu_{g_s} = 5\chi(S^4) = 10$$

□

Remark: because for smooth manifolds the inequalities in the previous Lemma is strict, it's possible to claim the stronger result that $\chi(M) \leq 9$ for (M, g) a smooth Einstein 4-manifold.

The above estimates were basically point-wise in nature, we now turn to some global results, starting with

Lemma 3.0.4. *Suppose (X, g) is a compact oriented Einstein 4-Orbifold of positive scalar curvature. Then either $W^+ \equiv 0$, or else there is a smooth,* *conformally related metric $\hat{g} = u^2 g$ such that*

$$\int_X [s_{\hat{g}} - 2\sqrt{6}|W^+_{\hat{g}}|_{\hat{g}}]\, d\mu_{\hat{g}} \leq 0$$

moreover, one can either arrange for the inequality to be strict, or for the metric $\hat{g}$ to be locally Kähler.

Proof. The proof in Gursky & LeBrun (1999) goes through without any modification. □

Remark: the proof shows that Einstein metrics of positive sectional curvature on weighted projective planes have degenerate W^+, meaning that two of the three eigenvalues of W^+ coincide, hence are locally hermitian with respect to some complex structure according to a Riemannian version of the Goldberg-Sachs theorem (see, e.g., Apostlov & Gauduchon, 2001).

This implies a surprising 'gap theorem' for W^+

Theorem 8. *Let (X, g) be a compact oriented 4-dimensional Einstein orbifold with $s > 0$ and $W^+ \neq 0$. Then*

$$\int_X |W^+_g|^2_g \geq \int_X \frac{s_g^2}{24} d\mu_g,$$

with equality iff $\nabla W^+ = 0$.

Proof. By the Chern-Gauss-Bonnet type formula

$$\chi_V(X) = c \int_X [|W^+|^2 + |W^-|^2 + \frac{s_g^2}{24} - \frac{|\dot{r}|^2}{2}]\, d\mu_g$$

for g Einstein, $\dot{r}_g$ vanishes. Now consider a conformal rescaling of our Einstein metric $\hat{g} = u^2 g$, we have $\dot{r}_{\hat{g}} \neq 0$ hence makes a negative contribution to the above integral. While the left hand side, being an orbifold invariant remains constant, hence

$$\frac{\int_X s_g d\mu_g}{\sqrt{\int_X d\mu_g}} \leq \frac{\int_X s_{\hat{g}} d\mu_{\hat{g}}}{\sqrt{\int_X d\mu_{\hat{g}}}}$$

However, assuming that $W^+ \neq 0$, Lemma 3.0.5 tells us that u can be chosen so that

$$\int_X s_{\hat{g}}\, d\mu_{\hat{g}} \leq 2\sqrt{6} \int |W_{\hat{g}}^+|\, d\mu_{\hat{g}} \leq (24 \int |W_{\hat{g}}^+|^2\, d\mu_{\hat{g}})^{\frac{1}{2}} (\int \mu_{\hat{g}})^{\frac{1}{2}}$$

Since s_g is constant, and because the L^2 norm of W^+ is conformally invariant, it therefore follows that

$$(\int_X s_g^2\, d\mu_g)^{1/2} = \frac{\int_X s_g\, d\mu_g}{\sqrt{\int_X d\mu_g}} \leq \frac{\int_X s_{\hat{g}} d\mu_{\hat{g}}}{\sqrt{\int_X d\mu_{\hat{g}}}} \leq (24 \int |W_{\hat{g}}^+|^2 d\mu_{\hat{g}})^{\frac{1}{2}} = (24 \int |W_g^+|^2 d\mu_g)^{\frac{1}{2}}$$

Moreover, equality can occur only if $\hat{g}$ is both locally Kähler and isometric to a constant times g. The latter, of course, happen iff g is itself locally Kähler. But since $s \neq 0$ is constant and $W^+ \neq 0$, the latter is equivalent to requiring that $\nabla W^+ \equiv 0$

□

Remark: in the case of smooth manifolds, the first inequality is a consequence of Obata's result that any Einstein metric is a unique Yamabe minimizer. Van Coevering (2005) has shown that this result generalizes to the orbifold category, thus arrived at the same inequality from another approach.

Reversing the orientation, we have

Corollary 3.0.5. *Let (X, g) be an oriented compact 4-dimensional Einstein orbifold with $s > 0$ and $W^- \neq 0$. Then*

$$\int_X |W_g^+|_g^2 \geq \int_X \frac{s_g^2}{24} d\mu_g, \tag{3.3}$$

$$\frac{2\chi_V - 3\tau_V}{3} \geq \frac{1}{4\pi^2} \int_X \frac{s_g^2}{24} d\mu_g. \tag{3.4}$$

Moreover, both these inequalities are strict unless $\nabla W^- = 0$.

Proof. Reversing the orientation of X interchanges W^+ and W^-. Applying this observation to theorem 8 immediately yields the first inequality. But this and the Gauss-Bonnet type formula then tells us that

$$(2\chi_V - 3\tau_V)(X) = \frac{1}{4\pi^2} \int_X [2|W_g^-|^2 + \frac{s^2}{24}]\, d\mu_g \geq \frac{3}{4\pi^2} \int_X \frac{s_g^2}{24}\, d\mu_g$$

Thus proving the corollary □

Now we can prove our main estimates,

Theorem 9. *Let (X, g) be a smooth compact oriented 4-dimensional Einstein orbifold with non-negative sectional curvature. Assume moreover that g is neither self-dual or anti-self-dual. Then the orbifold Euler characteristic and the signature of X satisfy*

$$10 \geq \chi_V \geq \frac{15}{4} |\tau_V|$$

Proof. Combining (3.2) and (3.4), we have

$$\frac{2}{3}\chi_V - \tau_V \geq \frac{1}{4\pi^2}\int \frac{s^2}{24}d\mu > \frac{2}{5}\chi_V$$

or in other words $\chi_V \geq \frac{15}{4}\tau_V$. Reversing the orientation of X, we also have $\chi_V \geq -\frac{15}{4}\tau_V$. Since Lemma 3.0.3 tells us that $\chi_V \leq 10$, we are therefore done. □

The main result now follows from the Gauss-Bonnet theorem and the signature theorem for orbifolds

Theorem 10. *On the weighted projective planes, any orbifold quaternionic Kähler metrics of non-negative sectional curvature must be self dual.*

Proof. From the formulas for χ_V and τ_V for weighet projective planes respectively, we see that $\chi_V \leq 3\tau_V$, since $p^2+q^2+r^2 > pq+qr+rp$ hence the above inequality is violated. Choosing the correct orientation, we see that Einstein metrics of positive sectional curvature on the weighted projective planes must be self-dual. □

Chapter 4

Uniqueness and relation to Kähler geometry

4.1 Relations to Kähler geometry

Derdziński (1983) in his groundbreaking investigation on 4-dimensional conformal geometry showed the following facts for a Riemannian Einstein 4-manifold with degenerate self-dual Weyl curvature (i.e., at any point at least two of the three eigenvalues of W^+ coincide)

Lemma 4.1.1. *Let (M, g) be a connected oriented Einstein 4-manifold such that W^+ has at most 2 eigenvalues at each point, then either $W^+ \equiv 0$, or else W^+ has exactly 2 distinct eigenvalues at each point. In the latter case, moreover, the conformally related metric $\hat{g} = 2\sqrt[3]{3}|W^+|^{2/3}g$ is locally Kähler, and is locally compatible with exactly one pair $\pm J$ of oriented complex structures. The scalar curvature s of $\hat{g}$ is then nowhere zero, and $g = s^{-2}\hat{g}$*

This result conformally relates Einstein metrics with degenerate self-dual Weyl curvature to Kähler metrics.

Bryant (2001) in his comprehensive study of Bochner-Kähler geometry, proved that

Theorem 11. *Every weighted projective plane supports a self-dual Kähler metric*

Remark: He also states that these metrics 'are presumably unique, although all details have not been checked.' Incidentally, when $r < p + q$, these metrics have positive scalar curvature.

The Galicki-Lawson's metrics are Einstein Hermitian with respect to some complex structure because W^+ of these metrics are degenerate. This conclusion follows from a Riemannian version of the Goldberg-Sachs theorem (Apostolov and Gauduchon, 2000). Building on Derdziński's result, they also proved the following bijection between self-dual Einstein-Hermitian metrics and self-dual Kähler metrics

Lemma 4.1.2. *Non-locally-symmetric self-dual Einstein-Hermitian metrics are in one-to-one correspondence with self-dual Kähler metrics of nowhere vanishing and non-constant scalar curvature (in the same conformal class)*

Remark: this result is local and does not require compactness of the manifold, it holds on the open dense subset of a manifold such that the above notions make sense, therefore, obviously true on orbifolds. According to this Lemma, the uniqueness of the Bryant's metrics and the uniqueness of the Galicki-Lawson metrics are equivalent.

Moreover, by solving a system of over determined partial differential equations, they gave a local classification of the self-dual Einstein Hermitian structures, and concluded that the Galicki-Lawson's metrics and the Bryant's metrics are locally the same.

4.2 Uniqueness

The above correspondence between Einstein-Hermitian and Kähler metrics provide strong evidence for the uniqueness of self-dual Einstein metrics of positive scalar curvature, as the local solutions are completely known. However, it remains the possibility that there exists globally different such metrics on weighted projective planes. Here we present a different approach to the uniqueness of the Galicki-Lawson's metrics using the existing results on extremal metrics on toric varieties. It can easily be shown that self-dual Kähler metrics are extremal, so the existence of extremal metrics follows from Bryant's theorem.

Guan (2000) using a modified version of the Mabuchi functional, proved the following

Theorem 12. *For any two Kähler metrics in a Kähler class which are invariant under a maximal connected subgroup of the automorphism group on a toric variety there is a geodesic curve connecting them. In particular, there is at most one extremal metric in a Kähler class on a toric variety.*

Proof. (Sketch) Fix a riemannian toric variety X, One may define the modified Mabuchi functional and show that the extremal metrics in a Kähler class. Now the moduli space of the modified Mabuchi functional is geodesically convex. Moreover, on any geodesy that connects two metrics, the second derivative of the functional with respect to the geodesic length is positive, hence the uniqueness of extremal metrics. □

Remark: he states his theorem only for smooth toric varieties, the proof generalizes verbatim to the toric varieties that are also orbifolds. From this we conclude that on each weighted projective plane, there is a unique self-dual Kähler metric, since the de Rham cohomology group is one dimensional, thus there is only one Kähler class (this follows from Kawasaki's lemma that de Rham cohomology is isomorphic to the $\mathbb{R}$ coefficient singular homology and the cell structure of weighted projective planes. It also follows from the Smith-Gysin sequence on the orbit space of the defining S^1 action for weighted projective planes.) By the above correspondence, this says that the Galicki-Lawson metrics are the unique self-dual Einstein metrics with positive scalar curvature on weighted projective planes. Hence, we arrive at our main

Theorem 13. *The only weighted projective planes admitting Einstein metrics of positive sectional curvatures are the ones that also admits the Galicki-Lawson metrics of positive sectional curvature. That is, the ones satisfy the Dearricott's criterion. These metrics are unique up to scale.*

Remark: I thank Professor Ziller and Professor Calderbank for pointing out this approach. The uniqueness of Galicki-Lawson metrics is a brief remark in (Apostolov et al. 2004), which they brought to my attention.

Remark: some of the above results can readily be generalized to weighted projective spaces. It's possible to state a stronger result of the type of Gursky & LeBrun (1999) in this direction, that all self-dual Einstein-Hermitian orbifolds of positive scalar curvature are the weighted projective spaces with the Galicki-Lawson metrics (Ziller 2004). See also Calderbank & Singer (2004) for the 4-dimensional results from another perspective.

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