ANALOGUES OF THE USUAL PSEUDODIFFERENTIAL CALCULUS ON THE HEISENBERG GROUP

A DISSERTATION PRESENTED

BY

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TO

THE GRADUATE SCHOOL

IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN

MATHEMATICS

STONY BROOK UNIVERSITY

DECEMBER 2005

STONY BROOK UNIVERSITY

THE GRADUATE SCHOOL

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ABSTRACT OF THE DISSERTATION

ANALOGUES OF THE USUAL PSEUDODIFFERENTIAL CALCULUS ON THE HEISENBERG GROUP

by

Alba Susana Coré Bianchi

Doctor of Philosophy

in

Mathematics

Stony Brook University

2005

We prove the existence of an analogue of the usual pseudodifferential calculus for the Heisenberg group $\mathbb{H}^n$, with the additional objective of possible generalisations to arbitrary homogeneous groups. Taylor, Beals-Greiner, Christ-Geller-Głowacki-Polin have developed analogues of *classical* pseudodifferential operators for the Heisenberg group.The method employed in this dissertation consists in defining a class of convolution operators on $\mathbb{H}^n$, and verifying that it satisfies a list of conditions which are well known to be sufficient for the existence of a general calculus of pseudodifferential operators analogous to the usual $S^m_{1,0}$–pseudodifferential calculus on $\mathbb{R}^n$. Among these conditions, which have been identified by M. E. Taylor, is the requirement that the family of convolution operators be closed under composition. A significant part of our effort is spent in verifying this critical property.

To the memory of our parents.

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ACKNOWLEDGEMENTS

To acknowledge on this page, without omissions, all of those who, in one way or another, have contributed meaningfully is not possible. Nevertheless I would like to express my gratitude to the following notable cases:

To my advisor, Professor Daryl Neil Geller, with profound sincerity, for his guidance, patience, and specially for his support during the most difficult times.

To Professor C. Denson Hill, to whom I am greatly indebted.

To Professor Steve Preston, for dedicating his time to carefully reading my final draft, and for his valuable suggestions.

To Professor Carlos Marques, for being part of my committee.

To the many friends and wonderful people whom I met at Stony Brook, in particular to Anca Rădulescu, Harish Seshadri, Kasra Rafi, Sergei Panafidin, Szuszanna Gonye, Rukmini Dey, Abdellah Sebbar, Jennifer Slimowitz, Punita Batra, Jaeman Kim, Chayoung Sung, Carlos Cabrera Ocanas, and Martín Reires.

To Scott Wilson who had the courage to speak on my behalf.

To all the people who work at the department, in particular to Barbara Wichard, Geraldine Sciulli, and Lucille Mecci.

And to Santiago del Castillo and my brother Jorge Daniel Coré, for their special loyalty and support.

CHAPTER I

INTRODUCTION

Pseudodifferential operators are a generalisation of linear partial differential operators. which arises from the study of elliptic partial differential equations. Pseudodifferential calculus provides constructive methods for solving elliptic problems. The specific objective was to obtain a collection, or calculus of integral operators that contain all elliptic partial differential operators and their parametrices, and which be closed under composition and taking of adjoints. Having such a calculus in place, whenever one is given an elliptic partial or pseudodifferential operator, one ought to be able to write down immediately a parametrix, and obtain estimates.

Some of the basic ingredients of the theory of pseudodifferential operators, such as the study of operational calculus for noncommuting operators in quantum mechanics, appeared in the 1920s. However the main development occurred in the 1950s, in the work of Giraud, Mikhlin, Calderón and Zygmund on the theory of singular integral operators. A. P. Calderón and A. Zygmund [CalZyg57], studied integral operators of the form

$$(\mathcal{K}f)(x) = \int_{\mathbb{R}^n} K(x, x-y)\, f(y)\, dy \qquad \forall f \in C_c^\infty(\Omega), \quad \Omega \subset \mathbb{R}^n$$

i.e.

$$(\mathcal{K}f)(x) = (K_x * f)(x) \tag{I.1}$$

where $K_x(w) = K(x, w)$, and whose only singularity is at $w = 0$.

In the early 1960s Atiyah and Singer made use of the calculus of singular integral operators in their first proof of the index theorem. This and other applications stimulated interest in these operators. Then Kohn and Nirenberg [KohNir 65] switched from the representation of operators by singular integral kernels to a viewpoint based on "symbols", and introduced the terminology *pseudodifferential operator* (where apparently the prefix "pseudo" refers to the "pseudo local" character of these operators, i.e. $\mathcal{K}u$ does not have more singularities than u, and perhaps also to the fact that in general they might not be local, i.e. it might occur that $\operatorname{supp} \mathcal{K}u \not\subset \operatorname{supp} u$). They rewrote the operator $\mathcal{K}$ in the form

$$\begin{aligned}(\mathcal{K}f)(x) &= \int \widehat{\check{K}}_x(x-y)\; f(y)\; dy \\ &= \int\int e^{-2\pi i(x-y)\cdot\xi}\; a(x,\xi)\; f(y)\; d\xi\, dy \\ &= \int e^{-2\pi i x\cdot\xi}\; a(x,\xi)\; \widehat{f}(\xi)\; d\xi\end{aligned} \tag{I.2}$$

where formally the inverse Fourier transform of a in the first variable is K_x,

$$\check{a}_x(w) = K_x(w) \qquad \text{where } a_x(\xi) = a(x,\xi)$$

A fundamental idea is to pass from the calculus of pseudodifferential operators to function algebras. One establishes a bijective correspondence between certain functions called "symbols", and operators, as follows:

Definition 1. We say that a linear operator $a(x, D) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is a pseudodifferential operator with symbol $a(x, \xi)$ belonging to a certain class of distributions $\mathcal{F}$, if $a(x, D)$ can be represented by

$$\big[a(x, D)f\big](x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} a(x, \xi)\ \widehat{f}(\xi)\, d\xi \tag{I.3}$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. In this case we say that $a(x, D)$ belongs to $\mathrm{Op}\,(\mathcal{F})$.

Although this operator$\longleftrightarrow$ symbol correspondence is not a homomorphism of algebras, it preserves enough properties of the original operator algebra, and has the advantage that it is easier to manipulate symbol functions than operators. Then the crucial point is that in order to obtain collection of operators with particular features, one judiciously places demands on the class of symbols $\mathcal{F}$. In fact, if one is not careful, and in (I.3) allows general functions or distributions one obtains an enormous family of operators, that is too diverse to support an interesting theory. (E.g. any continuous linear map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ can be represented in this form with a a tempered distribution.)

The first symbol algebras were introduced by Kohn and Nirenberg, and were soon extended by Hörmander.

Definition 2. Let m, ρ, δ be real numbers, and suppose $\rho, \delta \in [0, 1]$. The *(Hörmander) symbol class of order m*, denoted by $S^m_{\rho,\delta}$ consists of those functions $a(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for any pair of multiindeces α, β, and any compact set $K \subset \mathbb{R}^n$, there exists a constant $C_{\alpha,\beta,K}$ such that

$$\left|D^\beta_x D^\alpha_\xi a(x, \xi)\right| \leqslant C_{\alpha,\beta,K} \langle\xi\rangle^{m-\rho\|\alpha\|+\delta\|\beta\|} \qquad \forall x \in K,\ \xi \in \mathbb{R}^n$$

where $\langle\xi\rangle = \big(1 + \|\xi\|^2\big)^{1/2}$.

Definition 3. The symbol $p(x,\xi)$ belongs to $S^m(\Omega)$ if $p \in S^m_{1,0}(\Omega)$ and there are smooth $p_{m-j}(x,\xi)$,homogeneous of degree $m-j$ in ξ for $|\xi| \geqslant 1$, i.e.

$$p_{m-j}(x, r\xi) = r^{m-j} p_{m-j}(x,\xi) \quad \forall |\xi| \geqslant 1,\ r \geqslant 1$$

such that

$$p(x,\xi) \sim \sum_{j\geqslant 0} p_{m-j}(x,\xi)$$

where the asymptotic condition means that

$$p(x,\xi) - \sum_{j=0}^{N} p_{m-j}(x,\xi) \in S^{m-N-1}_{1,0}(\Omega)$$

In this case we shall say that $p(x,\xi)$ is a *classical* or *polyhomogeneous* symbol. Alternative frequently used notations for this class are $\mathcal{S}^m_{\mathrm{cl}}$ and $\mathcal{S}^m_{\mathrm{ph}}$

While the presentation of Calderón and Zygmund, (I.1), and the one by Kohn and Nirenberg (I.2) appear equivalent, the second is more frequently preferred. To a great extent this is due to the fact that Fourier transforms convert convolutions to products, which are easier to manipulate. And in particular one can try to use division to invert $\mathcal{K}$.

However, as Christ, Geller, Głowacki, and Polin emphasize in [CGGP 92], when working on certain environments, such as a general Lie group, the advantages of (I.2) are largely lost, but one can still define a class of pseudodifferential operators by using (I.1), with group convolution.

In this dissertation, we shall exploit this idea, by using the Fourier transform as little as possible, and by dealing directly with convolution operators. Our goal is to define a natural class of multiplier operators on general nilpotent groups with dilations, in order to prove the existence of an analogue of the usual pseudodifferential calculus for the Heisenberg group $\mathbb{H}^n$, with the additional objective of possible generalisations to arbitrary homogeneous groups. Taylor[Taylor 84], Beals-Greiner [BeaGre 88], Christ-Geller-Głowacki-Polin [CGGP 92] have developed analogues of *classical* pseudodifferential operators for the Heisenberg group.The method we have employed consists in proving that the class of convolution operators on $\mathbb{H}^n$ which we have defined, satisfies the list of conditions which are well known to be sufficient for the existence of a general calculus of pseudodifferential operators analogous to the usual $S^m_{1,0}$–pseudodifferential calculus on $\mathbb{R}^n$. Among these conditions, which were identified by M. E. Taylor in [Taylor 84], is the requirement that the family of convolution operators be closed under composition. A significant part of our effort goes into verifying this critical property.

A brief description of the contents and main results contained in each chapter follows. The notation used is introduced in Chapter 2.

- In Chapter 1, we provide a brief introduction to the motivating ideas, and briefly delineate their historical development.

- In Chapter 2, we introduce notation, basic definitions, and results which will be needed throughout the document.

- In Chapter 3, we characterize the space of the inverse Fourier transform of multipliers, $\widetilde{\mathcal{M}}^j(G)$ with $j < 0$. Our main results are the following two theorems:

Theorem. *Let j be a real number, such that $-Q < j$, and set $k = -Q - j$.*

a) Suppose $K \in \widetilde{\mathcal{M}}^j(G)$, then for any multiindeces β, γ, there exists $C_{\beta\gamma} > 0$ such that

$$\left|\partial^\beta K(x)\right| \leqslant C_{\beta\gamma}\left|x\right|^{k-|\beta|-|\gamma|} \qquad \textit{whenever } x \neq 0.$$

Say now $j < 0$

b) *Suppose* $K \in \widetilde{\mathcal{M}}^j(G)$, *then whenever* $j + |\beta| < 0$, *the tempered distribution* $(\partial^\beta K)$ *is in fact an* L^1 *function, where the derivatives are taken in the sense of tempered distributions.*

b) *Conversely, suppose that* $K \in L^1(G)$, *that* K *is smooth away from* 0, *and that the estimate* (III.11) *holds for all* β, γ, *whenever* $x \neq 0$. *Then* $K \in \widetilde{\mathcal{M}}^j(G)$

Theorem. *Let* j *be a real number, such that* $j \leqslant -Q$, *and set* $k = -Q - j$.

a) *Suppose* $K \in \widetilde{\mathcal{M}}^j(G)$, *then for any multiindeces* β, γ, *there exists* $C_{\beta\gamma} > 0$ *such that*

$$\left|\partial^\beta K(x)\right| \leqslant C_{\beta\gamma} B_{k-|\beta|-|\gamma|}(x) \qquad \text{whenever } x \neq 0.$$

where $B_{k-|\beta|-|\gamma|}$ *is the bad power function of order* $k - |\beta| - |\gamma|$.

b) *Suppose* $K \in \widetilde{\mathcal{M}}^j(G)$, *then whenever* $j + |\beta| < 0$, *the tempered distribution* $(\partial^\beta K)$ *is in fact an* L^1 *function, where the derivatives are taken in the sense of tempered distributions.*

c) *Conversely, assume that* $n > 1$ *if* $j \in -Q\mathbb{N}$. *Suppose that* $K \in L^1$, *that* K *is smooth away from* 0, *and the estimate* (III.12) *holds for all multiindeces* β, γ *whenever* $x \neq 0$. *Then* $K \in \widetilde{\mathcal{M}}^j(G)$.

- In Chapter 4, the main result is the following theorem:

Theorem. *For* $j_1, j_2 < 0$ *the convolution of an element of* $\widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$ *with an element of* $\widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$ *belongs to* $\widetilde{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$.

- In Chapter 5, we extend this property to any $j_1, j_2 \in \mathbb{R}$.

Theorem. *Let j_1, j_2 be two real numbers, with one of them negative, and the remaining one non negative. If $K_1 \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $K_2 \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$ then their convolution $K_1 * K_2 \in \widetilde{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$.*

Theorem. *Let j_1, j_2 be two non negative real numbers. If $K_1 \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $K_2 \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$ then their convolution $K_1 * K_2 \in \widetilde{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$.*

- In Chapter 6, we discuss the existence of the calculus on the Heisenberg group, in particular we show that the family of Fréchet spaces $\{\mathcal{M}^j(\mathbb{H}^n)\}_{j\in\mathbb{R}}$ satisfies a set of hypothesis sufficient for the existence of an analogue in $\mathbb{H}^n$ of the usual pseudodifferential calculus in Euclidean space. The result is contained in this theorem

Theorem. *Suppose $\mathbb{H}^n$ is the Heisenberg group of dimension n, and $\left\{\mathcal{M}^j(\mathbb{H}^n)\right\}_{j\in\mathbb{R}}$ is the family of spaces of multipliers defined in 12. Then the following properties are satisfied*

a) $\left\{\mathcal{M}^j(\mathbb{H}^n)\right\}_{j\in\mathbb{R}}$ *is a nested family of Fréchet spaces.*

b) *If $j \geqslant 0$ then $\mathcal{M}^j(\mathbb{H}^n) \subset S^m_{\rho,\#}$ for some $\rho \in (0,1]$.*

c) *If $j < 0$ then $\mathcal{M}^j(\mathbb{H}^n) \subset S^{m\sigma}_{\rho,\#}$ for some $\sigma \in (0,1]$.*

d) $\widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n) * \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n) \subseteq \widetilde{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$ *and the product is continuous.*

e) *If $J \in \mathcal{M}^j(\mathbb{H}^n)$, and α is a multiindex, then $D^\alpha J \in \mathcal{M}^{j-\tau\|\alpha\|}(\mathbb{H}^n)$ for some $\tau \in (0,1]$.*

f) *Let $J_i \in \mathcal{M}^{j-\tau i}(\mathbb{H}^n)$, for $i = 0,1,2,\dots$, and $\tau \in (0,1]$. Then there exists a $J \in \mathcal{M}^j(\mathbb{H}^n)$ such that, for any M, if N is sufficiently large*

$$\left(J - \sum_{i=0}^{N} J_i\right) \in \mathcal{S}^{-M}_{\rho\#}.$$

g) *If $J \in \mathcal{M}^j(\mathbb{H}^n)$ then $\bar{J} \in \mathcal{M}^j(\mathbb{H}^n)$.*

And therefore on $\mathbb{H}^n$ there exists a general calculus of pseudodifferential operators analogous to the usual $S^m_{1,0}$-pseudodifferential calculus on $\mathbb{R}^n$.

In fact we prove all these conditions, except for (d), not only for the Heisenberg group, but for the case a general Lie group with dilations. This will be the object of future work.

CHAPTER II

BASIC DEFINITIONS AND BASIC RESULTS

Let $\mathfrak{g}$ be a Lie algebra, which we shall assume real and finite dimensional, and let G be the corresponding connected and simply connected Lie group. If U and V are any subsets of the algebra $\mathfrak{g}$, we shall denote by $[U, V]$ the vector subspace of $\mathfrak{g}$ generated by all the elements of the form $[\alpha, \beta]$ with $\alpha \in U$, $\beta \in V$. The *lower central series of* $\mathfrak{g}$ is defined inductively by

$$\mathfrak{g}_{(1)} = \mathfrak{g}, \qquad \mathfrak{g}_{(1)} = [\mathfrak{g}, \mathfrak{g}_{(j-1)}]$$

The $\mathfrak{g}_{(j)}$ constitute a descending chain of ideals of $\mathfrak{g}$ The Lie algebra $\mathfrak{g}$ is called *nilpotent of step* m if there exists $m \in \mathbb{N}$ for which $\mathfrak{g}_{(m+1)} = \{0\}$.

Alternatively, if U and V are subsets of the Lie group G, $[U, V]$ shall denote the subgroup of G consisting of all elements of the form $\alpha\beta\alpha^{-1}\beta^{-1}$. In this case the *lower central series of* G is defined inductively by

$$G_{(1)} = \mathfrak{g}, \qquad G_{(1)} = [G, G_{(j-1)}]$$

and these $G_{(j)}$ are a descending chain of normal subgroups of G. The Lie group $\mathfrak{g}$ is called *nilpotent of step* m if there exists $m \in \mathbb{N}$ for which $\mathfrak{g}_{(m+1)} = \{\text{identity}\}$.

Definition 4. Let V be a real vector space. A family $\{\delta_t\}_{t>0}$ of linear maps of V to itself is called a set of *dilations on* V, if there are real numbers $\lambda_j > 0$ and subspaces W_{λ_j} of V such that V is the direct sum of the W_{λ_j} and

$$\delta_t\big|_{W_{\lambda_j}} = t^{\lambda_j}\,\mathrm{Id} \qquad \forall j$$

Definition 5. A *homogeneous group* is a connected and simply connected nilpotent group G, with underlying manifold $\mathbb{R}^n$, for some n, and whose Lie algebra $\mathfrak{g}$ is endowed with a family of dilations $\{\delta_t\}_{t>0}$, which are automorphisms of $\mathfrak{g}$.

The dilations are of the form $\delta_r = \exp(A \log r)$ where A is a diagonalizable linear operator on $\mathfrak{g}$ with positive eigenvalues. The group automorphisms $\exp\circ\delta_r\circ\exp^{-1} : G \longrightarrow G$ will be called *dilations* of the group and will also be denoted by δ_r. The group G may be identified topologically with $\mathfrak{g}$ via the exponential map $\exp : \mathfrak{g} \longrightarrow G$ and with such an identification

$$\begin{array}{ccccc} \delta_r : & G & \longrightarrow & G \\ & (x_1, \dots x_n) & \longmapsto & (r^{a_1}x_1, \dots, r^{a_n}x_n) \end{array}$$

Henceforth the eigenvalues of the matrix A, listed as many times as their multiplicity, will always be denoted by $\{a_i\}_{i=1}^{n}$. Moreover, as a condition of normalisation, we shall assume without loss of generality that all the a_i are increasingly ordered and that the first is equal to 1, that is

$$1 = a_1 \leqslant \dots \leqslant a_n\,.$$

Definition 6. The *homogeneous dimension* Q of the group G is the number

$$Q = \sum_{i=1}^{n} a_i\,.$$

We shall always use the letter Q to represent this quantity as well as $A = \prod_{i=1}^{n} a_i$.

EXAMPLES OF HOMOGENEOUS GROUPS

i. **abelian groups.**

$\mathbb{R}^n$ with the usual additive structure

$$x \cdot y = x + y$$

and with dilations given by scalar multiplication

$$\delta_r(x_1, \dots, x_n) = (r^{a_1}x_1, \dots, r^{a_n}x_n)$$

Note that δ_r is an automorphism. regardless of the choice of the weights a_i.

ii. **Non-abelian, non-compact Heisenberg groups**

If $n \in \mathbb{N}$, the Heisenberg group $\mathbb{H}^n$ is the group with $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ as underlying manifold and whose multiplication law is given by

$$(x, y, t) \cdot (x', y', t') = \big(x + x', y + y', t + t' - 2(x \cdot y' - y \cdot x')\big)$$

$\mathbb{H}^n$ is a homogeneous group with dilations

$$\delta_r(x_1, \dots, x_n, y_1, \dots, y_n, t) = (rx_1, \dots, rx_n, ry_1, \dots, ry_n, r^2 t).$$

The homogeneous dimension of $\mathbb{H}^n$ is $Q = 2n + 2$.

iii. **Upper triangular groups.** Let G be the group of all $n \times n$ real matrices $\big[a_{i,j}\big]$ such that $a_{ii} = 1$ for $1 \leqslant i \leqslant n$ and $a_{ij} = 0$ when $i > j$ endowed with the usual matricial multiplication. G is a homogeneous group with dilations

$$\delta_r\Big(\big[a_{ij}\big]\Big) = \big[r^{j-i}a_{ij}\big]$$

Definition 7. Let G be a homogeneous group with dilations $\{\delta_r\}$. A *homogeneous norm* on G, relative to the given dilations, is a continuous function $|\cdot| : G \longrightarrow [0, \infty)$, smooth away from the origin satisfying

a) $|x| = 0$ if and only if x is the identity element,

b) $|x^{-1}| = |x|$ for every $x \in G$

c) $|\delta_r(x)| = r|x|$ for every $x \in G$, and $r > 0$, i.e. the norm is homogeneous of degree 1.

If $|\cdot|$ is a homogeneous norm on G then there exists a constant $C \geqslant 1$ such that

$$|xy| \leqslant C\big(|x| + |y|\big) \qquad \text{for every } x, y \in G$$

Homogeneous norms alway exist. Moreover any two homogeneous norms $|\cdot|$ and $|\cdot|'$ on G are always equivalent, i.e. there exist constants $C_1, C_2 > 0$ such that

$$C_1|x| \leqslant |x|' \leqslant C_2|x| \qquad \forall x \in G$$

In view of this, in this thesis we choose to work with the following homogeneous norm

$$|x| := \left(\sum_{i=1}^{n} x_i^{2\frac{A}{a_i}}\right)^{\frac{1}{2A}}$$

We observe that if the dilations are isotropic, i.e. all the weights a_i are equal, and satisfy our condition of normalisation $a_1 = 1$, then this homogeneous norm is simply the Euclidean norm, which will be denoted by $\|\cdot\|$.

Definition 8. If $x \in G$, $r > 0$, we define the ball of radius r about x as

$$B(x;r) = \Big\{y \in G \,:\, |x^{-1}y| < r\Big\}$$

We notice that $B(x;r) = \delta_r\big(B(0;1)\big)$.

We fix a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ such that $AX_j = a_j X_j$, i.e. a basis consisting of eigenvectors for the dilations $\{\delta_r\}$, with eigenvalues $r^{a_1}, \ldots, r^{a_n}$. Now $\mathfrak{g}$ is identified with $\mathfrak{g}_L$, the Lie algebra of left invariant vector fields on G, and under this identification the X_j are now regarded as first order left invariant differential operators on G.

For $f \in C^1(G)$ we define the action of X_j on f by the equation

$$X_j f(y) = \lim_{t \to 0} \frac{f\big(y \cdot \exp(tX_j)\big) - f(y)}{t} = \frac{d}{dt} f\big(y \cdot \exp(tX_j)\big)\Big|_{t=0}$$

A similar formula holds for the the right-invariant vector fields $\{X_j^R\}$. The differential operators $\{X_j\}$ and $\{X_j^R\}$ are homogeneous of degree a_j since they verify

$$X_j\big(f \circ \delta_r\big)(y) = r^{a_j}\big(X_j f \circ \delta_r\big)(y)$$

and the analogous relation for $\{X_j^R\}$ is also valid.

The Haar measure on G is simply the Lebesgue measure on $\mathbb{R}^n$, under the identification of G with $\mathfrak{g}_L$ and of $\mathfrak{g}_L$ with $\mathbb{R}^n$ via the basis $\{X_i\}$. We denote by $|E|$ the measure of any measurable set $E \subset G$. Then

$$\big|\delta_r(E)\big| = r^Q |E| \qquad \text{and} \qquad d\big(\delta_r(x)\big) = r^Q\, dx$$

All our integrals on G are with respect to the Haar measure. So, for any integrable function f on G

$$\int_G f\big(\delta_r(x)\big)\, dx = r^{-Q} \int_G f(x) dx$$

Proposition 9 (**p-test**). *Let $|\cdot|$ be a homogeneous norm on G. The integral*

$$\int_{|x|<1} |x|^p\, dx$$

is convergent if and only if $p > -Q$, and the integral

$$\int_{|x|>1} |x|^p \, dx$$

is convergent if and only if $p < -Q$.

Definition 10. The convolution of two functions f, g on G is defined by

$$(f * g)(x) = \int_G f(xy^{-1})g(y)\,dy = \int_G f(y)g(y^{-1}x)\,dy$$

provided that the integrals converge.

We observe that in general, if G is non-commutative, such as the case of $\mathbb{H}^n$, we have $f * g \neq g * f$.

Suppose the operator X is a left-invariant differential operator on G, and X^R a right-invariant differential operator on G, then the operators X and X^R interact with convolutions in the following way

$$X(f * g) = f * (Xg)$$
$$X^R(f * g) = (X^R f) * g$$
$$(Xf) * g = f * (X^R g)$$

$\mathcal{S}(G)$ will denote the usual Schwartz space on G, thought of as $\mathbb{R}^n$.

Definition 11. If $f \in L^1(\mathbb{R}^n)$, we define the *Fourier transform of* f, denoted by $\widehat{f}$, as

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} f(x)\,dx$$

and the *inverse Fourier transform* of f, denoted by $\check{f}$, as

$$\check{f}(x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} f(\xi)\,d\xi.$$

The following notation, definitions, and properties will be used frequently without introduction. Most of them are as they appear in [Geller]

Definition 12. Suppose $j \in \mathbb{R}$. We shall say that $r \in C^\infty(\check{G})$ is a *multiplier of order* j if for every multiindex $\alpha \in (\mathbb{Z}^+)^n$ there exists $C_\alpha > 0$ such that

$$\left|\partial^\alpha r(\xi)\right| \leqslant C_\alpha (1 + |\xi|)^{j-|\alpha|}$$

for all ξ. We shall denote the space of all multipliers of order j by $\mathcal{M}^j(G)$.

- $\mathcal{S}(G) \subseteq \mathcal{M}^j(G)$ and the inclusion is continuous.
- $\mathcal{M}^j(G) \subseteq \mathcal{S}'(G)$ and the inclusion is sequentially continuous.
- If $J \in \mathcal{M}^j(G)$ then $\check{J} \in \mathcal{E}' + \mathcal{S}$, and it is smooth away from zero.
- If α is a multiindex then $\partial^\alpha : \mathcal{M}^j(G) \longrightarrow \mathcal{M}^{j-|\alpha|}(G)$, and multiplication by ξ^α maps $\mathcal{M}^j(G)$ to $\mathcal{M}^{j+|\alpha|}(G)$.
- If $j_1 \leqslant j_2$ then $\mathcal{M}^{j_1}(G) \subseteq \mathcal{M}^{j_2}(G)$
- For $J \in \mathcal{M}^j(G)$, and $N \in \mathbb{Z}^+$, we define

$$\|J\|_{\mathcal{M}^j(G),N} = \sum_{|\alpha| \leqslant N} \left\|(1+|\xi|)^{|\alpha|-j}\, \partial^\alpha J\right\|$$

  where $\|\cdot\|$ denotes the supremun norm.
  The family $\left\{\|\cdot\|_{\mathcal{M}^j(G),N}\right\}_{n\in\mathbb{Z}^+}$ is a nondecreasing and separating sequence of seminorms on $\mathcal{M}^j(G)$.
- $\mathcal{M}^{-\infty}(G) := \mathcal{S}(G)$ by definition. The norm used for this space is

$$\|\cdot\|_{\mathcal{M}^{-\infty}(G),N} := \|\cdot\|_{\mathcal{S}(G),N}$$

- Suppose $J \in \mathcal{M}^{j}(G)$, we define m_J, the *multiplier operator with multiplier* J, as follows

$$\begin{aligned} \mathcal{S}' &\xrightarrow{\ m_J\ } \mathcal{S}' \\ f &\longmapsto \check{J} * f \end{aligned}$$

- $[\![x]\!]$ is used to denote the greatest integer less or equal to x.

Definition 13. Let t be a real number. For $x \in \mathbb{R}^n - \{0\}$, we define, B_t, the *bad power function of order* t, by

$$B_t(x) := \begin{cases} |x|^t & \text{if } t < 0, \\ \big|\log|x|\big| + 1 & \text{if } t = 0, \\ 1 & \text{if } t > 0. \end{cases} \tag{II.1}$$

Definition 14. Consider the multiindex $\beta \in (\mathbb{Z}^+)^n$ we define

$$|\beta| := \sum_{i=1}^{n} a_i \beta_i \qquad\qquad \|\beta\| := \sum_{i=1}^{n} \beta_i$$

Proposition 15. *For every multiindex* α

$$a_1\|\alpha\| \leqslant |\alpha| \leqslant a_n\|\alpha\| \tag{II.2}$$

and for all $\xi \in \mathbb{R}^n$ *there exist positive constants* c *and* C *such that*

$$c\big(1 + |\xi|\big) \leqslant \langle\xi\rangle \leqslant C\big(1 + |\xi|\big)^{a_n} \tag{II.3}$$

CHAPTER III

CHARACTERIZATION OF $\widecheck{\mathcal{M}}^j(G)$ WITH $j < 0$

Proposition 16. *Suppose j is a real number such that $-Q < j$, where, as usual, Q denotes the homogeneous dimension of G. Set $k = -Q - j$, and let m be an integer such that $j - 2mA < -Q$.*
If J is a function on $\mathbb{R}^n$, smooth away from zero, such that for every multiindex α there exists a B_α for which

$$|\partial^\alpha J(\xi)| \leqslant B_\alpha |\xi|^{j-|\alpha|}, \quad \textit{for all } \xi \neq 0, \tag{III.1}$$

then

a) J is locally integrable, $J \in S'$, and therefore $\check{J}$ is well defined.

b) Away from zero, $\check{J}$ agrees with a smooth function. Moreover, there exists a constant C, depending on n, m, and j, such that

$$|\check{J}(x)| \leqslant C\left(\sum_{|\alpha| \leqslant 2mA} B_\alpha\right)|x|^k, \quad \textit{for all } x \neq 0. \tag{III.2}$$

c) If $\varphi \in C_c^\infty(\mathbb{R}^n)$, and $\varphi = 1$ in a neighborhood of the origin, then $[(1-\varphi)J]\check{}$ is smooth away from zero, and for any $N \in \mathbb{Z}^+$ there exists $C \geqslant 0$ such that

$$|[(1-\varphi)J]\check{}\,(x)| < C|x|^{-N} \qquad \forall x \neq 0 \tag{III.3}$$

Proof.

a) The estimate (III.1) with $\alpha = (0,\ldots,0)$, and the fact that $-Q < j$ imply that J is locally integrable.

Now we select a $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\phi = 1$ in a neighborhood of 0, and we write

$$J = \phi J + (1-\phi)J$$

The first term above, (ϕJ), is a tempered distribution because $(\phi J) \in L^1(\mathbb{R}^n)$. The second term, $(1-\phi)J$, is measurable, therefore by estimate (III.1), with $|\alpha| = 0$, we have for $f \in \mathcal{S}$

$$\begin{aligned}
\Big|[(1-\phi)J](f)\Big| &= \Big|\int_{\mathbb{R}^n} [(1-\phi)J](\xi)\ f(\xi)\, d\xi\Big| \\
&\leqslant \int_{\mathbb{R}^n} \Big|[(1-\phi)J](\xi)\ f(\xi)\Big|\, d\xi \\
&\leqslant \int_{\mathbb{R}^n} C_1\, |\xi|^j\, |f(\xi)|\, d\xi \qquad \text{for some constant } C_1 \\
&= \int_{\mathbb{R}^n} C_1 \Big[\big(1+|\xi|\big)^{-(Q+1)}\Big] \Big[\big(1+|\xi|\big)^{Q+1} |\xi|^j |f(\xi)|\Big]\, d\xi \\
&\leqslant \int_{\mathbb{R}^n} \Big[\big(1+|\xi|\big)^{-(Q+1)}\Big] C_2 \|f\|_N\, d\xi \qquad \text{for some } N \in \mathbb{Z}^+ \\
&\leqslant C_3 \|\big(1+|\xi|\big)^{-(Q+1)}\|_{L^1} \|f\|_N
\end{aligned}$$

where we have used a Schwartz space norm, denoted by $\|\cdot\|_N$.

This implies that $(1-\phi)J$ is a temperate distribution, and consequently

$$J = \underbrace{\phi J}_{\in L^1 \subseteq \mathcal{S}'} + \underbrace{(1-\phi)J}_{\in \mathcal{S}'} \quad \in \mathcal{S}'(G)$$

b, c) For these parts we choose $\phi \in C_c^\infty(G)$ such that $\phi = 1$ for $|x| < 1$, and with $\operatorname{supp}\phi \subset$

$B(0;2)$, and consider the decomposition

$$J = (\phi J) + (1-\phi)J \ .$$

We define the differential operator $\mathcal{L}$ as follows

$$\mathcal{L} := D_1^{\frac{2A}{a_1}} + \cdots + D_n^{\frac{2A}{a_n}}$$

where $D_r := \dfrac{i}{2\pi}\dfrac{\partial}{\partial x_r}$ for $r = 1, \ldots, n$ and $A = \prod_{i=1}^{n} a_i$.

Our next step consists in showing that for $m' \in \mathbb{N}$ there exists a constant $M > 0$, depending on n, m', j, and G such that

$$\left|\mathcal{L}^{m'}\big[(1-\phi)J\big](x)\right| \begin{cases} = 0 & \text{if} \quad |x| < 1 \\ \leqslant M \displaystyle\sum_{|\alpha| \leqslant 2m'A} B_\alpha & \text{if} \quad 1 < |x| < 2 \\ \leqslant M \displaystyle\sum_{|\alpha| = 2m'A} B_\alpha |x|^{j-2m'A} & \text{if} \quad 2 \leqslant |x| \end{cases} \tag{III.4}$$

If $|x| < 1$, $\varphi = 1$ then $(1-\varphi)J(x) = 0$. Therefore $\mathcal{L}^{m'}\big[(1-\phi)J\big](x) = 0$. If $1 < |x| < 2$, then by Leibniz's rule we have

$$
\begin{aligned}
&\left|\mathcal{L}^{m'}\left[(1-\phi)J\right](x)\right| \\
&= \left|\sum_{\substack{1\leqslant i_l\leqslant n\\ l=1,\dots,m'}} \binom{n}{i_1,\dots,i_{m'}} \left(\frac{\partial^{\frac{2A}{a_{i_1}}+\dots+\frac{2A}{a_{i_{m'}}}}}{\partial x_{i_1}^{\frac{2A}{a_{i_1}}}\cdots\partial x_{i_{m'}}^{\frac{2A}{a_{i_{m'}}}}}(1-\phi)J\right)(x)\right| \\
&= \sum_{\substack{1\leqslant i_l\leqslant n\\ l=1,\dots,m'}} \binom{n}{i_1,\dots,i_{m'}} \left|\sum_{\substack{\alpha+\beta=\gamma\\ |\gamma|=2m'A}} C_{\alpha,\beta}\partial^{\beta}(1-\phi)(x)\partial^{\alpha}J(x)\right| \\
&= \sum_{\substack{1\leqslant i_l\leqslant n\\ l=1,\dots,m'}} \binom{n}{i_1,\dots,i_{m'}} \sum_{\substack{\alpha+\beta=\gamma\\ |\gamma|=2m'A}} C_{\alpha,\beta}C_{\beta}B_{\alpha}|x|^{j-|\alpha|} \\
&\leqslant M_1 \sum_{|\alpha|\leqslant 2m'A} B_{\alpha}|x|^{j-|\alpha|} \\
&\leqslant M_2 \sum_{|\alpha|\leqslant 2m'A} B_{\alpha}.
\end{aligned}
$$

If $|x| \geqslant 2$ then $(1-\phi)J = J$ since $\operatorname{supp}\phi \subset B(0;2)$. Therefore for $|x| \geqslant 2$

$$
\begin{aligned}
&\left|\mathcal{L}^{m'}\left[(1-\phi)J\right](x)\right| \\
&= \sum_{\substack{1\leqslant i_l\leqslant n\\ l=1,\dots,m'}} \binom{n}{i_1,\dots,i_{m'}} \left|\left(\frac{\partial^{\frac{2A}{a_{i_1}}+\dots+\frac{2A}{a_{i_{m'}}}}}{\partial x_{i_1}^{\frac{2A}{a_{i_1}}}\cdots\partial x_{i_{m'}}^{\frac{2A}{a_{i_{m'}}}}}J\right)(x)\right| \\
&\leqslant \sum_{\substack{1\leqslant i_l\leqslant n\\ l=1,\dots,m'}} \binom{n}{i_1,\dots,i_{m'}} B_{\alpha}|x|^{j-2m'A} \qquad \text{with } |\alpha| = 2m'A \\
&\leqslant M_3 \sum_{|\alpha|=2m'A} B_{\alpha}|x|^{j-2m'A}
\end{aligned}
$$

By choosing $M = \max\left\{M_2, M_3\right\}$ we can see now that what was claimed in (III.4) is valid.

Therefore for sufficiently large m' such that $j - 2m'A < -Q$, $\mathcal{L}^{m'}\left[(1-\phi)J\right]$ decreases rapidly enough at infinity so that $\mathcal{L}^{m'}\left[(1-\phi)J\right] \in L'(\mathbb{R}^n)$. Consequently by the Riemann-Lebesgue Lemma $\left[\mathcal{L}^{m'}\left[(1-\phi)J\right]\right]^{\vee}$ is continuous and bounded on $\mathbb{R}^n$.

And since

$$\begin{aligned}\left[\mathcal{L}^{m'}[(1-\phi)J]\right]^{\vee} &= \left(\sum_{l=1}^{n} x_l^{\frac{2A}{a_l}}\right)^{m'} \left[(1-\phi)J\right]^{\vee} \\ &= |x|^{2m'A}\left[(1-\phi)J\right]^{\vee}\end{aligned} \tag{III.5}$$

then $\left[(1-\phi)J\right]^{\vee}$ is continuous away from zero. Similarly, by (III.4), $x^{\alpha}\mathcal{L}^{m'}\left[(1-\phi)J\right]$ is in $L^1(\mathbb{R}^n)$, provided that $j - 2m'A + |\alpha| < -Q$. Therefore $\partial^{\alpha}\left[\mathcal{L}^{m'}[(1-\phi)J]\right]^{\vee} = (-2\pi i)^{\|\alpha\|}\left[x^{\alpha}\mathcal{L}^{m'}[(1-\phi)J]\right]^{\vee}$ exists and is continuous and bounded. Hence by (III.5) $|x|^{2m'A}\left[(1-\phi)J\right]^{\vee} \in C^{l(m')}(\mathbb{R}^n)$, where $l(m') \longrightarrow \infty$ as $m' \longrightarrow \infty$. This shows that $\left[(1-\phi)J\right]^{\vee} \in C^{l(m')}$ away from 0. Hence $\left[(1-\phi)J\right]^{\vee} \in C^{\infty}$ away from 0.

Also $\left|\left[(1-\phi)J\right]^{\vee}(x)\right| \leqslant C|x|^{-2m'A}$ for any arbitrary m', proving c).

We decompose $\check{J}$ as

$$\check{J} = (\phi J)^{\vee} + \left[(1-\phi)J\right]^{\vee}.$$

$(\phi J)^{\vee}$ is of class C^{∞}, because ϕJ is a compactly supported distribution, and $\left[(1-\phi)J\right]^{\vee}$ is C^{∞} away from 0, hence we have that $\check{J}$ is smooth away from 0, proving the first part of b).

In order to prove that there exists a constant C such that

$$\left|\check{J}(x)\right| \leqslant C\left(\sum_{|\alpha|\leqslant 2mA} B_{\alpha}\right)|x|^{k}$$

we shall consider the special case of $|x| = 1$. Then using dilations it will be extended to any $x \neq 0$.

- $\underline{|x| = 1}$:

  Since $(\phi J) \in L^1(\mathbb{R}^n)$

  $$\left|(\phi J)^{\vee}(x)\right| \leqslant \left\|\phi J\right\|_{L^1} \leqslant C_1 B_o \int_{|\xi|<2} |\xi|^{j}\, d\xi \leqslant C_o B_o \tag{III.6}$$

  We apply (III.4) with $m' = m$, and notice that by hypothesis $j - 2mA < -Q$, therefore $\mathcal{L}^m\big[(1-\phi)J\big] \in L^1(\mathbb{R}^n)$, and we have

$$
\begin{aligned}
&\Big|\big[\mathcal{L}^m(1-\phi)J\big]^{\vee}(x)\Big| \\
&\qquad \leqslant \Big\|\mathcal{L}^m\left[(1-\phi)J\right]\Big\|_{L^1} \\
&\qquad = \int\limits_{1\leqslant|\xi|\leqslant 2} \Big|\mathcal{L}^m\left[(1-\phi)J\right](\xi)\Big|\,d\xi + \int\limits_{2\leqslant|\xi|} \Big|\mathcal{L}^m\left[(1-\phi)J\right](\xi)\Big|\,d\xi \\
&\qquad \leqslant \int\limits_{1\leqslant|\xi|\leqslant 2} M \sum_{|\alpha|\leqslant 2mA} B_\alpha\, d\xi + \int\limits_{2\leqslant|\xi|} M \sum_{|\alpha|=2mA} B_\alpha\, |\xi|^{j-2mA} d\xi \\
&\qquad \leqslant C_1 \sum_{|\alpha|\leqslant 2mA} B_\alpha + M \sum_{|\alpha|=2mA} B_\alpha \int\limits_{2\leqslant|\xi|} |\xi|^{j-2mA} d\xi \\
&\qquad \leqslant C_1 \sum_{|\alpha|\leqslant 2mA} B_\alpha + C_2 \sum_{|\alpha|=2mA} B_\alpha \\
&\qquad \leqslant C_3 \sum_{|\alpha|\leqslant 2mA} B_\alpha
\end{aligned}
$$

On the other hand, $\left[\mathcal{L}^m(1-\phi)J\right]^{\vee} = \big|x\big|^{2mA}\big[(1-\phi)J\big]^{\vee} = \big[(1-\phi)J\big]^{\vee}$, because $\big|x\big|^{2mA} = 1$. Hence

$$
\Big|\left[(1-\phi)J\right]^{\vee}(x)\Big| \leqslant C_3 \sum_{|\alpha|\leqslant 2mA} B_\alpha \tag{III.7}
$$

Therefore by estimates (III.6), and (III.7)

$$
\begin{aligned}
\big|\check{J}(x)\big| &\leqslant \big|(\phi J)^{\vee}(x)\big| + \big|\left[(1-\phi)J\right]^{\vee}(x)\big| \\
&\leqslant C_o B_o + C_3\Bigg(\sum_{|\alpha|\leqslant 2mA} B_\alpha\Bigg) \\
&\leqslant C\Bigg(\sum_{|\alpha|\leqslant 2mA} B_\alpha\Bigg) \qquad \forall |x| = 1
\end{aligned}
\tag{III.8}
$$

- $\underline{x \neq 0}$:

Let $r = |x|$, and set $J_r := J \circ \delta_{\left(\frac{1}{r}\right)}$. Given that J is smooth away from zero, J_r is also smooth away from 0. Moreover for any multiindex α, and for $\xi \neq 0$, J_r satisfies

$$\begin{aligned}
\left|\partial^\alpha J_r(\xi)\right| &= \left|\partial^\alpha\big[J \circ \delta_{(\frac{1}{r})}\big](\xi)\right| \\
&= \left|r^{-|\alpha|}\big[\,(\partial^\alpha J) \circ \delta_{(\frac{1}{r})}\big](\xi)\right| \\
&= r^{-|\alpha|}\left|\big[\,(\partial^\alpha J) \circ \delta_{(\frac{1}{r})}\big](\xi)\right| \\
&\leqslant r^{-|\alpha|} B_\alpha \left|\delta_{(\frac{1}{r})}(\xi)\right|^{j-|\alpha|} \\
&= r^{-|\alpha|} B_\alpha \left(\frac{1}{r}\right)^{j-|\alpha|} |\xi|^{j-|\alpha|} \\
&= B_\alpha r^{-j} |\xi|^{j-|\alpha|}
\end{aligned}$$

$$\check{J}_r = \left(J \circ \delta_{(\frac{1}{r})}\right)^{\vee} = r^Q \left(\check{J} \circ \delta_r\right)$$

In fact, in view of inequality (III.8)

$$\left|\check{J}_r(y)\right| \leqslant C\left(\sum_{|\alpha| \leqslant 2mA} B_\alpha r^{-j}\right) \qquad \text{for } |y| = 1.$$

Therefore, if $|x| = r$ we have

$$\begin{aligned}
\left|\check{J}(x)\right| &= \left|\left(\check{J} \circ \delta_r \circ \delta_{\left(\frac{1}{r}\right)}\right)(x)\right| \\
&= r^{-Q}\left|r^{Q}\left(\check{J} \circ \delta_r\right)\left(\delta_{\left(\frac{1}{r}\right)}(x)\right)\right| \\
&= r^{-Q}\left|\check{J}_r\left(\delta_{\left(\frac{1}{r}\right)}(x)\right)\right| \\
&\leqslant r^{-Q} C \sum_{|\alpha| \leqslant 2mA} B_\alpha r^{-j} \\
&= C\left(\sum_{|\alpha| \leqslant 2mA} B_\alpha\right) r^{-j-Q} \\
&= C\left(\sum_{|\alpha| \leqslant 2mA} B_\alpha\right) r^{k} \\
&= C\left(\sum_{|\alpha| \leqslant 2mA} B_\alpha\right) |x|^{k}
\end{aligned}$$

□

The analogous result for the Fourier transform $\widehat{J}$ is also valid.

Corollary 17. *Under the same hypothesis of 16 we have*

a) *J is locally integrable, $J \in S'$, and therefore $\widehat{J}$ is well defined.*

b) *Away from zero, $\widehat{J}$ agrees with a smooth function. Moreover, there exists a constant C, depending on n,m, and $\jmath$, such that*

$$|\widehat{J}(x)| \leqslant C\left(\sum_{|\alpha|\leqslant 2mA} B_\alpha\right)|x|^k, \quad for\ all\ x \neq 0.$$

c) *If $\varphi \in C_c^\infty(\mathbb{R}^n)$, and $\varphi = 1$ in a neighborhood of zero, then $[(1-\varphi)J]^\wedge$ is smooth away from zero, and for any $N \in \mathbb{Z}^+$, there exists $C \geqslant 0$ such that*

$$\left|[(1-\varphi)J]^\wedge(x)\right| < C|x|^{-N} \qquad \forall x \neq 0 \tag{III.9}$$

Corollary 18. *If $J \in \mathcal{M}^\jmath(G)$ and $\psi \in C_c^\infty(G)$ with $\psi = 1$ in a neighborhood of zero in G, then $(1-\psi)\check{J} \in \mathcal{S}(G)$.*

Proof. We say that F is *rapidly decreasing* if for every nonnegative integer N there exists a constant $C_N \geqslant 0$ such that

$$\left|F(x)\right| \leqslant C_N|x|^{-N} \qquad \forall x \neq 0$$

Observe that we may assume that $\jmath \geqslant -Q$, since $\widetilde{\mathcal{M}}^{\jmath_1}(G) \subseteq \widetilde{\mathcal{M}}^{\jmath_2}(G)$ whenever $\jmath_1 \leqslant \jmath_2$.

Now we select $\varphi \in C_c^\infty(G)$ as in part (c) of Proposition 16 on page 17, and we write J as $J = \varphi J + (1-\varphi)J$. Therefore $\check{J} = (\varphi J)^\vee + [(1-\varphi)J]^\vee$. Consequently we can express $(1-\psi)\check{J}$ as

$$(1-\psi)\check{J} = (1-\psi)(\varphi J)^\vee + (1-\psi)\big[(1-\varphi)J\big]^\vee. \tag{III.10}$$

Since by part (c) of Proposition 16 $\big[(1-\varphi)J\big]^\vee$ is smooth away from zero and rapidly decreasing, then the second term of (III.10) $(1-\psi)\big[(1-\varphi)J\big]^{\vee}$ is smooth and rapidly decreasing. Since φJ is in $\mathcal{S}(G)$, its inverse Fourier transform is also in $\mathcal{S}(G)$. Therefore the first term of (III.10) is also in Schwartz space.

Consequently in the previous paragraph we have shown that if $J \in \mathcal{M}^j(G)$ then $(1-\psi)\check{J}$ is smooth and rapidly decreasing.

If $J \in \mathcal{M}^j(G)$, then $(\partial^\beta \check{J})^{\wedge} \in \mathcal{M}^{j+|\beta|}(G)$ for any multiindex β. Therefore $(1-\psi)\big[(\partial^\beta \check{J})^{\wedge}\big]^{\vee} = (1-\psi)\partial^\beta \check{J}$ is smooth and rapidly decreasing.

By Leibniz rule

$$\partial^\beta\big[(1-\psi)\check{J}\big] = (1-\psi)\partial^\beta \check{J} + g$$

for some $g \in C_c^\infty(G)$. And since $(1-\psi)\partial^\beta \check{J}$ is smooth and rapidly decreasing for any multiindex β, then $\partial^\beta\big[(1-\psi)\check{J}\big]$ inherits both properties. This implies that $(1-\psi)\check{J} \in \mathcal{S}(G)$, which is what we wished to prove. □

Theorem 19. *Let j be a real number, such that $-Q < j$, and set $k = -Q - j$.*

a) Suppose $K \in \widecheck{\mathcal{M}}^j(G)$, then for any multiindeces β, γ, there exists $C_{\beta\gamma} > 0$ such that

$$\left|\partial^\beta K(x)\right| \leqslant C_{\beta\gamma}\left|x\right|^{k-|\beta|-|\gamma|} \qquad \textit{whenever } x \neq 0. \tag{III.11}$$

Say now $j < 0$

b) Suppose $K \in \widecheck{\mathcal{M}}^j(G)$, then whenever $j + |\beta| < 0$, the tempered distribution $(\partial^\beta K)$ is in fact an L^1 function, where the derivatives are taken in the sense of tempered distributions.

c) Conversely, suppose that $K \in L^1(G)$, that K is smooth away from 0, and that the estimate (III.11) *holds for all β, γ, whenever $x \neq 0$. Then $K \in \widecheck{\mathcal{M}}^j(G)$*

Proof. a) Observe that $K \in \widecheck{\mathcal{M}}^j(G)$, therefore by Corollary 18, K is a smooth function away from zero, and consequently for any multiindex β it makes sense to consider $(\partial^\beta K)$.

The argument proceeds in two steps: first we do the case for the multiindex $\gamma = (0, \cdots, 0)$, and finally the general case.

- Case $|\gamma| = 0$

  We define $J := (\partial^\beta K)\hat{}$. For some constant C, we have $J = (\partial^\beta K)\hat{} = C\xi^\beta \widehat{K}$, which means that $J \in \mathcal{M}^{j+|\beta|}(G)$. Suppose $\varphi \in C_c^\infty(\mathbb{R}^n)$, with $\varphi = 1$ for $|x| < 1$, and define $J' := (1-\varphi)J$. Then $J = \varphi J + J'$. Since J and φJ are in $\mathcal{M}^{j+|\beta|}(G)$ then also J' is in $\mathcal{M}^{j+|\beta|}(G)$, and therefore J' is smooth. Given that $j > -Q$, then $j + |\beta| > -Q$. For any multiindex α there exists a constant $C_\alpha > 0$ such that for all ξ

$$\begin{aligned}\left|\partial^\alpha J'(\xi)\right| &\leqslant C_\alpha (1+|\xi|)^{j+|\beta|-|\alpha|}\\ &\leqslant B_\alpha |\xi|^{j+|\beta|-|\alpha|} \qquad \text{for some } B_\alpha.\end{aligned}$$

  We observe that the last inequality is valid for $|\xi| < 1$ because $J' = 0$ for $|\xi| < 1$. Thus by part (b) of Proposition 16 there exists a constant $C'_{\beta\gamma}$ such that for all $x \neq 0$

$$\left|(J')\check{}\,(x)\right| \leqslant C'_{\beta\gamma}|x|^{\widetilde{k}}$$

  where $\widetilde{k} = -Q - (j + |\beta|)$. Therefore for all $x \neq 0$ we have

$$\begin{aligned}\left|\partial^\beta K(x)\right| &= \left|\widecheck{\widehat{\partial^\beta K}}(x)\right|\\ &\leqslant \left|(J')\check{}\,(x)\right| + \left|(\varphi J)\check{}\,(x)\right|\\ &\leqslant C'_{\beta\gamma}|x|^{\widetilde{k}} + C''_{\beta\gamma}|x|^{\widetilde{k}}\\ &\leqslant C_{\beta\gamma}|x|^{\widetilde{k}}\end{aligned}$$

  Above, in the next to the last inequality, we have made use of the fact that since $(\varphi J) \in \mathcal{S}(G)$, its inverse Fourier transform is also in $\mathcal{S}(G)$, and since $\widetilde{k}$ is a negative integer there exists some constant $C''_{\beta\gamma} \geqslant 0$ such that $\left|(\varphi J)^{\vee}(x)\right| \leqslant C''_{\beta\gamma}|x|^{\widetilde{k}}$.

Now observing that $\widetilde{k} = (-Q-j) - |\beta| = k - |\beta|$, we conclude that for all $x \neq 0$ there exists a constant $C_{\beta\gamma}$ such that

$$\left|\partial^{\beta} K(x)\right| \leqslant C_{\beta\gamma}|x|^{k-|\beta|}$$

- <u>Case $\gamma \in (\mathbb{Z}^{+})^{n}$.</u>

  Two subcases are considered.

  ▶ <u>$0 \neq |x| \leqslant 1$</u>

  By the previous case, we know that

  $$|\partial^{\beta} K(x)| \leqslant C_{\beta}|x|^{k-|\beta|} \leqslant C_{\beta}|x|^{k-|\beta|-|\gamma|}$$

  where for the previous inequality we have used that $|x| \leqslant 1$

  ▶ <u>$1 < |x|$</u>

  $K \in \widetilde{\mathcal{M}}^{j}(G)$, therefore by Corollary 18 on page 25 if we choose $\phi \in C_c^{\infty}(G)$, with $\phi \equiv 1$ in a neighborhood of zero, and $\operatorname{supp}\phi \subset B(0,1)$, then for $1 < |x|$, $K = (1-\phi)K \in \mathcal{S}(G)$. Hence

  · if $k - |\beta| - |\gamma| \geqslant 0$, then there exists a constant $C_{\beta\gamma}$ such that

  $$\left|\partial^{\beta} K(x)\right| \leqslant C_{\beta\gamma} \leqslant C_{\beta\gamma}|x|^{k-|\beta|-|\gamma|} \qquad \text{for } 1 < |x|$$

  · if $k - |\beta| - |\gamma| < 0$, for any natural number $m \geqslant -(k - |\beta| - |\gamma|)$ there exists a constant $C_{\beta\gamma}$ such that

  $$\left|\partial^{\beta} K(x)\right| \leqslant C_{\beta\gamma}(1+|x|)^{-m} \leqslant C_{\beta\gamma}|x|^{-m} \leqslant C_{\beta\gamma}|x|^{k-|\beta|-|\gamma|} \quad \text{for } 1 < |x|$$

The consideration of these two subcases establishes the second case and therefore the proof of part (a) of the theorem is complete, i.e. we have shown that for all multiindeces β, γ there exists a constant $C_{\beta,\gamma}$ such that

$$\left|\partial^{\beta} K(x)\right| \leqslant C_{\beta,\gamma}|x|^{k-|\beta|-|\gamma|} \qquad \text{for all } x \neq 0$$

b) First we claim that in order to prove that $\partial^{\beta} K \in L^1$, it suffices to show that $K \in L^1$. This claim is valid because if we are able to establish that K is an L^1 function whenever $K \in \widetilde{\mathcal{M}}^j(G)$, then we can conclude that all its derivatives $(\partial^{\beta} K) = \left((-2\pi i)^{\|\beta\|} x^{\beta} \widehat{K}\right)^{\vee}$ are also L^1 functions whenever $(j + |\beta|) < 0$, since these derivatives clearly belong to $\widetilde{\mathcal{M}}^{j+|\beta|}$.

By part (a), $|K(x)| \leqslant C_o|x|^k$, for all $x \neq 0$. And since $k = -Q - j > -Q$, then away from zero K agrees with a locally integrable function. Also if $\phi \in C_c^{\infty}(G)$ with $\phi = 1$ in a neighborhood of zero, then by Corollary 18 $(1-\phi)K \in \mathcal{S}(G)$ therefore there exists a constant C such that $|[(1-\phi)K](x)| < C|x|^{-Q-1}$, so $(1-\phi)K \in L^1$. Consequently there exists a function F such that $F \in L^1(G)$ and which, away from zero, coincides with K.

We shall show that $F = K$ in the sense of distributions. We define $\Gamma := K - F$, and observe that it is supported at the origin, therefore, using the Dirac distribution δ, it can be written as a finite sum

$$\Gamma = K - F = \sum_{|\alpha| \leqslant N} c_{\alpha} \partial^{\alpha} \delta$$

for constants c_{α} and a certain $N \in \mathbb{Z}^+$. Since $\widehat{\Gamma} = \widehat{K} - \widehat{F}$ is a polynomial, in order to prove that $\Gamma = 0$, it suffices to show that $\widehat{\Gamma}$ approaches zero at infinity. $F \in L^1$, therefore, by the Riemann-Lebesgue lemma, $\widehat{F} \longrightarrow 0$ at infinity. On the other hand, since $\widehat{K} \in \mathcal{M}^j(G)$, there exists a constant C such that

$$|\widehat{K}(\xi)| \leqslant C(1+|\xi|)^j \longrightarrow 0 \qquad \text{as} \quad \xi \longrightarrow \infty$$

because $j < 0$. Therefore $\Gamma = 0$, and $K = F$ in the sense of distributions, hence K is an L^1 function, as desired.

c) We want to show that $\widehat{K}$ is a smooth function and that for every multiindex α there exists a constant $C_\alpha > 0$ such that

$$\left|(\partial^\alpha \widehat{K})(\xi)\right| \leqslant C_\alpha (1+|\xi|)^{j-|\alpha|} \qquad \text{for all } \xi$$

Since $K \in L^1$ and $|\partial^\beta K(x)| \leqslant C_{\beta\gamma}|x|^{k-|\beta|-|\gamma|}$ for all multiindeces β, γ and for all $x \neq 0$, it follows that

$$K \in \left(L^1_{\text{comp}} + \mathcal{S}\right) \subset \mathcal{E}' + \mathcal{S}$$

where L^1_{comp} denotes all the compactly supported elements of L^1_{loc}. Therefore $\widehat{K}$ is a smooth function.

We define $J := \left(\partial^\alpha \widehat{K}\right)^{\vee}$ and we shall show that J satisfies the hypothesis of Corollary 17. We have $J = \left(\partial^\alpha \widehat{K}\right)^{\vee} = (2\pi i)^{\|\alpha\|}\xi^\alpha K$, and since away from zero K is a smooth function, J is also smooth away from zero.

Given that $k = -Q - j > -Q$, we have that $\left(k + |\alpha|\right) > -Q$. Since the estimate (III.11) holds by hypothesis, for any β

$$\begin{aligned} \left|(\partial^\beta J)(\xi)\right| &= \left|\partial^\beta\left[(2\pi i)^{\|\alpha\|}\xi^\alpha K\right](\xi)\right| \\ &\leqslant \widetilde{C}_\beta \left|\xi\right|^{(k+|\alpha|)-|\beta|} \qquad \text{for all } \xi \neq 0, \end{aligned}$$

Then by Corollary 17, there exists a constant C such that for all $x \neq 0$

$$\left|\widehat{J}(x)\right| \leqslant C|x|^{-Q-(k+|\alpha|)}$$

Observing that $-Q-(k+|\alpha|)=(-Q-k)-|\alpha|=j-|\alpha|$, the previous inequality can be rewritten to express that for all $x \neq 0$

$$\left|\widehat{J}(x)\right| = \left|\left(\partial^{\alpha}\widehat{K}\right)(x)\right| \leqslant C|x|^{j-|\alpha|}$$

Therefore for all x

$$\left|\left(\partial^{\alpha}\widehat{K}\right)(x)\right| \leqslant C\left(1+|x|\right)^{j-|\alpha|}$$

Consequently $\widehat{K} \in \mathcal{M}^{j}(G)$, and hence $K \in \widetilde{\mathcal{M}}^{j}(G)$, and this completes the proof of part (c) of the theorem.

□

In order to get a complete characterisation of the spaces $\widetilde{\mathcal{M}}^{j}(G)$ for $j<0$, we still need to cover the case of those $\widetilde{\mathcal{M}}^{j}(G)$ with $j \leqslant -Q$. We start by establishing some auxiliary results.

We recall that $\{\delta_r : r>0\}$ denotes the family of automorphisms of the group G consisting of dilations of the type

$$\delta_r(x) = (r^{a_1}x_1, \ldots, r^{a_n}x_n)$$

and $\quad Q=\sum_{i=1}^{n} a_i \quad$ is called the homogeneous dimension of G.

Lemma 20. *Suppose c and ε are positive real numbers. Then there exists a constant C such that*

$$\int_{|x|<\varepsilon} |x|^{-Q+c}\,dx \leqslant C\varepsilon^{c}.$$

Proof. Introducing a change of variable we have

$$x = \delta_\varepsilon(x') \quad \text{so} \quad dx = \varepsilon^Q \, dx'$$

We also observe that

$$|x|^{-Q+c} = |\delta_\varepsilon(x')|^{-Q+c} = \varepsilon^{-Q+c}|x'|^{-Q+c}$$

Therefore

$$\int\limits_{|x|<\varepsilon} |x|^{-Q+c}dx = \varepsilon^{-Q+c} \int\limits_{|x'|<1} |x'|^{-Q+c}\varepsilon^Q dx' = \varepsilon^c \int\limits_{|x'|<1} |x'|^{-Q+c}dx' \leqslant C\varepsilon^c$$

where the last integral converges since $-Q+c > -Q$. □

Proposition 21. *Suppose F is a L^1 function, C^∞ away from 0, satisfying the condition*

$$|F(x)| \leqslant C|x|^{-Q+c}$$

for all $x \neq 0$, where c is a real number such that $c > a_j$. If G is an L^1 function such that $G = \partial_j F$ away from zero, then, in the distributional sense, $G = \partial_j F$.

Proof. Let $\psi \in C_c^\infty(G)$ be such that $\psi = 1$ near the origin. For $\varepsilon > 0$, we define $\psi_\varepsilon := \psi \circ \delta_{(1/\varepsilon)}$. Let $\varphi \in C_c^\infty(G)$ and observe that $\operatorname{supp}(\psi_\varepsilon\varphi) \subseteq B(0;\varepsilon)$.

Away from zero $G = \partial_j F$, therefore $(G - \partial_j F)$ is supported at 0, hence for all $\varepsilon > 0$

$$(G - \partial_j F)(\varphi) = (G - \partial_j F)\,(\psi_\varepsilon\varphi)\,.$$

We intend to apply $(\psi_\varepsilon\varphi)$ to $(G - \partial_j F)$, obtain estimates and take the limit as $\varepsilon \longrightarrow 0^+$, effectively showing that

$$\lim_{\varepsilon \to 0^+} (G - \partial_j F)\,(\psi_\varepsilon\varphi) = 0$$

which allows us to conclude that $G = \partial_j F$ in the distributional sense.

$$
\begin{aligned}
\left|\left(G - \partial_j F\right)\left(\psi_\varepsilon \varphi\right)\right| &= \left|G(\psi_\varepsilon\varphi) - \partial_j F(\psi_\varepsilon\varphi)\right| \\
&= \left|G(\psi_\varepsilon\varphi) - (-1)F\big(\partial_j(\psi_\varepsilon\varphi)\big)\right| \\
&= \left|\int_{\mathbb{R}^n} G(x)(\psi_\varepsilon\varphi)(x)\,dx + F\big(\psi_\varepsilon(\partial_j\varphi)\big) + F\big((\partial_j\psi_\varepsilon)\varphi\big)\right| \\
&\leqslant \int_{\mathbb{R}^n} \left|G(x)\right|\left|\left(\psi_\varepsilon\varphi\right)(x)\right| dx + \int_{\mathbb{R}^n} \left|F(x)\right|\left|\left(\psi_\varepsilon\partial_j\varphi\right)(x)\right| dx \\
&\quad + \int_{\mathbb{R}^n} \left|F(x)\right| \left|\left[(\partial_j\psi_\varepsilon)\varphi\right](x)\right| dx \\
&\leqslant \left[\max_{|x|\leqslant\varepsilon}(\psi_\varepsilon\varphi)\right] \int_{|x|\leqslant\varepsilon} \left|G(x)\right| dx + \left[\max_{|x|\leqslant\varepsilon}(\psi_\varepsilon\partial_j\varphi)\right] \int_{|x|\leqslant\varepsilon} \left|F(x)\right| dx \\
&\quad + \int_{\mathbb{R}^n} \left|F(x)\right| \left(\tfrac{1}{\varepsilon^{a_j}}\right) \left|\left(\partial_j\psi\right)_\varepsilon(x)\varphi(x)\right| dx \\
&\leqslant C_1 \int_{|x|\leqslant\varepsilon} \left|G(x)\right| dx + C_2 \int_{|x|\leqslant\varepsilon} \left|F(x)\right| dx \\
&\quad + \frac{1}{\varepsilon^{a_j}} \left(\max_{|x|\leqslant\varepsilon}\left[\left(\partial_j\psi\right)_\varepsilon\varphi\right]\right) \int_{|x|\leqslant\varepsilon} \left|F(x)\right| dx \\
&= C_1 \int_{|x|\leqslant\varepsilon} \left|G(x)\right| dx + C_2 \int_{|x|\leqslant\varepsilon} \left|F(x)\right| dx + C_3\left(\frac{1}{\varepsilon^{a_j}}\right) \int_{|x|\leqslant\varepsilon} \left|F(x)\right| dx \\
&\leqslant C_1 \int_{|x|\leqslant\varepsilon} \left|G(x)\right| dx + C_2 \int_{|x|\leqslant\varepsilon} \left|F(x)\right| dx + C_3\left(\frac{1}{\varepsilon^{a_j}}\right) \int_{|x|\leqslant\varepsilon} C\left|x\right|^{-Q+c} dx \\
&\leqslant C_1 \int_{|x|\leqslant\varepsilon} \left|G(x)\right| dx + C_2 \int_{|x|\leqslant\varepsilon} \left|F(x)\right| dx + C_4\left(\frac{1}{\varepsilon^{a_j}}\right) \varepsilon^{c} \qquad \text{by Lemma 20} \\
&= C_1 \int_{|x|\leqslant\varepsilon} \left|G(x)\right| dx + C_2 \int_{|x|\leqslant\varepsilon} \left|F(x)\right| dx + C_4\varepsilon^{c-a_j}
\end{aligned}
$$

Given that we have $c - a_j > 0$, the last term, as well as the other two, approach zero when $\varepsilon \longrightarrow 0^+$. Therefore

$$\lim_{\varepsilon\longrightarrow 0^+} \left(G - \partial_j F\right)\left(\psi_\varepsilon\varphi\right) = 0.$$

□

Theorem 22. *Let j be a real number, such that $j \leqslant -Q$, and set $k = -Q - j$.*

a) *Suppose $K \in \widetilde{\mathcal{M}}^j(G)$, then for any multiindeces β, γ, there exists $C_{\beta\gamma} > 0$ such that*

$$\left|\partial^\beta K(x)\right| \leqslant C_{\beta\gamma} B_{k-|\beta|-|\gamma|}(x) \qquad \textit{whenever } x \neq 0. \tag{III.12}$$

where $B_{k-|\beta|-|\gamma|}$ is the bad power function of order $k - |\beta| - |\gamma|$.

b) *Suppose $K \in \widetilde{\mathcal{M}}^j(G)$, then whenever $j + |\beta| < 0$, the tempered distribution $(\partial^\beta K)$ is in fact an L^1 function, where the derivatives are taken in the sense of tempered distributions.*

c) *Conversely, assume that $n > 1$ if $j \in -Q\mathbb{N}$. Suppose that $K \in L^1$, that K is smooth away from 0, and the estimate* (III.12) *holds for all multiindeces β, γ whenever $x \neq 0$. Then $K \in \widetilde{\mathcal{M}}^j(G)$.*

Proof.

a) First we notice that Corollary 18 guarantees that for any multiindex β, the derivatives $\partial^\beta K$ exist away from zero, because $K \in \widetilde{\mathcal{M}}^j(G)$. We start the proof by establishing estimate (III.12) for a special case, namely for a special multiindex, and then we proceed with the general case.

- Case $|\gamma| = 0$

  ► $k - |\beta| < 0$ i.e. $j + |\beta| > -Q$.

  We shall show that $\partial^\beta K$ verifies the hypothesis of part (a) of Theorem 19 on page 26, and this is all we need do for this case.

  $\widehat{K}$ belongs to $\mathcal{M}^j(G)$, therefore $(\partial^\beta K)^\wedge = (-2\pi i)^{\|\beta\|}\xi^\beta \widehat{K} \in \mathcal{M}^{j+|\beta|}(G)$, so $\partial^\beta K \in \widetilde{\mathcal{M}}^{j+|\beta|}(G)$. We have $k - |\beta| = (-Q - j) - |\beta| = (-j - |\beta|) - Q < 0$

then $j+|\beta|>-Q$, and consequently the hypothesis of part (a) of Theorem 19, are satisfied. Therefore by estimate (III.11), there exists a constant C_1 such that

$$\begin{aligned}\left|(\partial^\beta K)(x)\right| &\leqslant C_1 |x|^{-Q-(j+|\beta|)-|0|-|0|} \\ &= C_1 |x|^{k-|\beta|} \qquad \forall x \neq 0\end{aligned}$$

► $\underline{k-|\beta|=0}$ i.e. $j+|\beta|=-Q$.

First we shall estimate $[\partial_r(\partial^\beta K)]$ for $r=1,\ldots,n$. We have $[\partial_r(\partial^\beta K)]\hat{} = (-2\pi i)^{\|\beta\|+1}\xi_r\xi^\beta\widehat{K} \in \mathcal{M}^{j+|\beta|+a_r}(G)$, and this implies that $[\partial_r(\partial^\beta K)] \in \widetilde{\mathcal{M}}^{j+|\beta|+a_r}$, for $r=1,\ldots,n$. Because $j+|\beta|+a_r=-Q+a_r>-Q$ it follows that $[\partial_r(\partial^\beta K)]$ is under the hypothesis of part (a) of Theorem 19 on page 26, and consequently we can use estimate (III.11). Hence for all $x\neq 0$ there exist constants c_r, with $r=1,\ldots,n$, such that

$$\begin{aligned}\left|[\partial_r(\partial^\beta K)](x)\right| &\leqslant c_r|x|^{-Q-(j+|\beta|+a_r)} \\ &= c_r|x|^{-Q-(-Q+a_r)} \\ &= c_r|x|^{-a_r}\end{aligned}$$

Now fix $x_o=(x_{o_1},\ldots,x_{o_n})$, with $|x_o|=1$, and consider the following function

$$\begin{aligned}\Gamma_{x_o}:&(0,\infty)\longrightarrow\mathbb{C} \\ t &\longrightarrow \left[(\partial^\beta K)\circ\delta_t\right](x_o)=\left(\partial^\beta K\right)(t^{a_1}x_{o_1},\ldots,t^{a_1}x_{o_n})\end{aligned}$$

Therefore using the chain rule and the previously obtained estimate, we have

$$\begin{aligned}
\left|\Gamma'_{x_o}(t)\right| &= \left|\sum_{r=1}^{n}\left[\partial_r\left(\partial^\beta K\right)\right]\left(\delta_t(x_o)\right)\frac{d\left(\delta_t(x_o)\right)_r}{dt}\right| \\
&\leqslant \sum_{r=1}^{n}\left|\left[\partial_r\left(\partial^\beta K\right)\right]\left(\delta_t(x_o)\right)\right|\,\left|a_r t^{a_r-1}x_{o_r}\right| \\
&\leqslant \sum_{r=1}^{n} c_r\left|\delta_t(x_o)\right|^{-a_r}\,a_r\,t^{a_r-1}\,|x_{o_r}| \\
&\overset{(\text{since } |x_o|=1)}{\leqslant} \sum_{r=1}^{n} c_r\,t^{-a_r}|x_o|^{-a_r}\,a_r\,t^{a_r-1} \\
&= \sum_{r=1}^{n} c_r a_r t^{-1} \\
&= C t^{-1} \qquad \text{where} \quad C=\sum_{r=1}^{n} a_r c_r
\end{aligned}$$

On the other hand, for all $\varepsilon \in (0,\infty)$ we have

$$\begin{aligned}
\left|\Gamma_{x_o}(\varepsilon)\right| &= \left|\int_1^\varepsilon \Gamma'_{x_o}(t)\,dt + \Gamma_{x_o}(1)\right| \\
&\leqslant \left|\int_1^\varepsilon Ct^{-1}\,dt + \Gamma_{x_o}(1)\right| \\
&\leqslant \left|\int_1^\varepsilon Ct^{-1}\,dt\right| + \left|\Gamma_{x_o}(1)\right| \\
&\leqslant C\left|\log\varepsilon\right| + \left|\left(\partial^\beta K\right)(x_o)\right| \\
&\leqslant C\left|\log\varepsilon\right| + C_1 \\
&\leqslant C_2\left(\left|\log\varepsilon\right| + 1\right)
\end{aligned}$$

where $C_1 = \max\limits_{|x|=1}\left|\left(\partial^\beta K\right)(x)\right|$ and where we have used Corollary 18 to establish that K is smooth away from zero since $\partial^\beta K \in \widetilde{\mathcal{M}}^{-Q}(G)$.

For $x \neq 0$, $id = \delta_{|x||x|^{-1}} = \delta_{|x|} \circ \delta_{|x|^{-1}}$, and $\left|\delta_{|x|^{-1}}(x)\right| = |x|^{-1}|x| = 1$, hence using the previous estimate we have

$$
\begin{aligned}
\left|\left(\partial^\beta K\right)(x)\right| &= \left|\left(\partial^\beta K\right)\left(\delta_{|x|} \circ \delta_{|x|^{-1}}(x)\right)\right| \\
&= \left|\Gamma_{\left[\delta_{|x|^{-1}}(x)\right]}\left(|x|\right)\right| \\
&\leqslant C_2\left(\left|\log|x|\right| + 1\right)
\end{aligned}
$$

- $\underline{k-|\beta|>0}$ i.e. $j+|\beta|<-Q$.

  Since $\widehat{\partial^\beta K} = (-2\pi i)^{\|\beta\|}\xi^\beta \widehat{K} \in \mathcal{M}^{j+|\beta|}(G)$, therefore there exists some constant C such that $\left|\left(\partial^\beta K\right)^\wedge(\xi)\right| \leqslant C\left(1+|\xi|\right)^{j+|\beta|}$ for all ξ. Since by hypothesis $j+|\beta|<-Q$, we have $\left(\partial^\beta K\right)^\wedge$ is in $L^1(G)$, therefore its inverse Fourier transform $(\partial^\beta K)$ is a bounded function, i.e. there exists a constant C_3 such that $\left|\left(\partial^\beta K\right)(x)\right| \leqslant C_3$ for all $x \neq 0$.

Concisely, in the last few pages we have shown that for all $x \neq 0$

- if $k-|\beta|<0$ there exists a constant C_1 such that

$$
\left|\left(\partial^\beta K\right)(x)\right| \leqslant C_1|x|^{k-|\beta|}
$$

- if $k-|\beta|=0$ there exists a constant C_2 such that

$$
\left|\left(\partial^\beta K\right)(x)\right| \leqslant C_2\left(\left|\log|x|\right|+1\right)
$$

- if $k-|\beta|>0$ there exists a constant C_3 such that

$$
\left|\left(\partial^\beta K\right)(x)\right| \leqslant C_3
$$

Therefore we can conclude that there exists a constant C_β such that

$$\left|(\partial^\beta K)(x)\right| \leqslant C_\beta B_{k-|\beta|}(x)$$

for all $x \neq 0$, and where $B_{k-|\beta|}$ denotes the bad power function of order $k - |\beta|$.

- Case γ any multiindex

  We shall assume $|\gamma| \neq 0$, since $|\gamma| = 0$ has already been discussed.

  ► $k - |\beta| - |\gamma| > 0$

  This implies $k - |\beta| > |\gamma|$, therefore, applying the case already proved, there exists a constant C_β such that $\left|(\partial^\beta K)(x)\right| \leqslant C_\beta$, for all $x \neq 0$, which is what we had to prove in this case.

  ► $k - |\beta| - |\gamma| = 0$

  This implies $k - |\beta| = |\gamma|$, therefore, using the case already proved, there exists a constant C_β such that for all $x \neq 0$

$$\begin{aligned}\left|(\partial^\beta K)(x)\right| &\leqslant C_\beta \\ &\leqslant C_\beta \left(\left|\log|x|\right| + 1\right)\end{aligned}$$

  ► $k - |\beta| - |\gamma| < 0$

  The following two subcases are considered separately:

  ■ $1 \leqslant |x|$

  $K \in \widetilde{\mathcal{M}}^j$(G), therefore if we choose $\phi \in C_c^\infty(G)$ such that $\phi = 1$ in a neighborhood of zero, such that $\operatorname{supp}\phi \subset B(0;1)$, then outside of $\operatorname{supp}\phi$ we have $K = (1-\phi)K \in \mathcal{S}(G)$. Therefore given that $k - |\beta| - |\gamma| < 0$, if we choose any $m \in \mathbb{N}$ such that $-m \leqslant k - |\beta| - |\gamma|$, then there exists a positive constant

C_1' such that whenever $1 \leqslant |x|$

$$\left|\partial^\beta\big[(1-\phi)K\big](x)\right| \leqslant C_1'\big(1+|x|\big)^{-m} \leqslant C_1'|x|^{-m} \leqslant C_1'|x|^{k-|\beta|-|\gamma|}.$$

■ $\underline{0 \neq |x| < 1}$

The three possibilities for $k-|\beta|$, positive, zero, and negative, will be considered separately:

♦ $\underline{0 < k-|\beta|}$ By the case already proved there exists a constant $C_\beta > 0$ such that

$$\big|(\partial^\beta K)(x)\big| \leqslant C_\beta \leqslant C_\beta|x|^{k-|\beta|-|\gamma|}$$

where for the last inequality the facts that $k-|\beta|-|\gamma| < 0$, and that $0 \neq |x| < 1$, have been used.

♦ $\underline{k-|\beta| = 0}$ By the previous case, there exists a positive constant C_β such that

$$\big|(\partial^\beta K)(x)\big| \leqslant C_\beta\left(\big|\log|x|\big|+1\right) \leqslant \widetilde{C}_1|x|^{-|\gamma|} = \widetilde{C}_1|x|^{k-|\beta|-|\gamma|}$$

with $\widetilde{C}_1$ a positive constant, and where for establishing the second inequality above, we have used the facts that $0 \neq |x| < 1$ and $\big|\log|x|\big| \leqslant |x|^{-|\gamma|}$.

♦ $\underline{k-|\beta| < 0}$ By the case previously proved, there exists a constant $C_\beta > 0$ such that

$$\big|(\partial^\beta K)(x)\big| \leqslant C_\beta|x|^{k-|\beta|} \leqslant C_\beta|x|^{k-|\beta|-|\gamma|}$$

where for the last inequality above, we have used the facts that $0 \neq |x| < 1$ and that $k-|\beta|-|\gamma| < k-|\beta| < 0$.

Therefore for $0 \neq |x| < 1$ there exists a positive constant $C_1'' = \max\left\{C_\beta, \widetilde{C}_1\right\}$ such that

$$\big|(\partial^\beta K)(x)\big| \leqslant C_1''|x|^{k-|\beta|-|\gamma|}.$$

Consequently if $k-|\beta|-|\gamma| < 0$ there exists a positive constant $C_1 = \max\{C_1', C_1''\}$ such that

$$\left|(\partial^\beta K)(x)\right| \leqslant C_1 |x|^{k-|\beta|-|\gamma|} \quad \forall x \neq 0.$$

Summarizing, we have shown that for all $x \neq 0$

- if $k - |\beta| - |\gamma| > 0$, then there exists a constant C_β such that

$$\left|\partial^\beta K(x)\right| \leqslant C_\beta$$

- if $k - |\beta| - |\gamma| = 0$, then there exists a constant C_β such that

$$\left|\partial^\beta K(x)\right| \leqslant C_\beta \left(\left|\log|x|\right| + 1\right)$$

- if $k - |\beta| - |\gamma| < 0$, then there exists a constant C_1 such that

$$\left|\partial^\beta K(x)\right| \leqslant C_1 |x|^{k-|\beta|-|\gamma|}$$

Therefore it can be concluded that for any multiindeces β, and γ there exists a positive constant $C_{\beta\gamma}$ such that for all $x \neq 0$

$$\left|\partial^\beta K(x)\right| \leqslant C_{\beta\gamma} B_{k-|\beta|-|\gamma|}(x)\,.$$

And this concludes the proof of part (a) of the theorem.

b) The argument to prove part (b) is essentially the same as the one used to prove part (b) of Theorem 19.

In order to prove that $\partial^\beta K \in L^1(G)$ it suffices to show that $K \in L^1(G)$, because if we are able to show that K is an L^1 function whenever $K \in \widetilde{\mathcal{M}}^j(G)$, then its distributional derivatives $\partial^\beta K$ will also be L^1 functions, as long as $j + |\beta| < 0$. This because in this case $\partial^\beta K$ belongs to $\widetilde{\mathcal{M}}^{j+|\beta|}(G)$.

We know that for every $\gamma \in \left(\mathbb{Z}^+\right)^n$ there exists a positive constant $C_{o\gamma}$ such that $\left|K(x)\right| \leqslant C_{o\gamma} B_{k-|\gamma|}(x)$. Given that $j \leqslant -Q$ we have that $k = -Q - j \geqslant 0$. The following two possibilities are considered separately:

- $\underline{k = -Q - j > 0}$

  then $\left|K(x)\right| \leqslant C_{o\gamma}$ for all $x \neq 0$, therefore K agrees away from zero with a locally integrable function.

- $\underline{k = -Q - j = 0}$

  we choose any multiindex γ such that $-Q \leqslant -|\gamma|$. For example, $\gamma = (1, 0, \dots, 0)$ will do. In such case $k - |\gamma| = 0 - |\gamma| < 0$, which means

  $$\left|K(x)\right| \leqslant C_{o\gamma}|x|^{k-|\gamma|} = C_{o\gamma}|x|^{-|\gamma|}$$

  and because $Q = \sum_{i=1}^n a_i > a_1 \cdot 1 = |\gamma|$ we have that K agrees away from zero with a locally integrable function.

Given that in both cases K agrees away from zero with a locally integrable function, and that by Corollary 18 $(1-\phi)K \in L^1$, with $\phi \in C_c^\infty(G)$ and $\phi = 1$ in a neighborhood of zero, there exits a function $F \in L^1(G)$ which away from zero coincides with K. Then it can be shown that $F = K$ in the sense of distributions in a similar fashion as in part (b) of Theorem 19.

c) We need to show that $\widehat{K}$ is smooth and that for any multiindex β there exists a positive constant C_β such that the partial derivatives satisfy

$$\left|(\partial^\beta \widehat{K})(\xi)\right| \leqslant C_\beta \big(1 + |\xi|\big)^{j-|\beta|} \qquad \forall \xi .$$

Assume α is a multiindex of the form $\alpha = te_i$, where e_i is the $i-$th element of the canonical basis of $\mathbb{R}^n$, and $t \in \mathbb{N}$ such that $-Q < j + |\alpha| = j + ta_i < 0$. We shall show that $\partial^\alpha K \in \widetilde{\mathcal{M}}^{j+|\alpha|}(G) = \widetilde{\mathcal{M}}^{j+ta_i}(G)$.

We notice that such t does exist. If $n > 1$, then surely $Q > a_i$, therefore the open interval

$$\left(\frac{-Q-j}{a_i}, \frac{-j}{a_i}\right)$$

has length $\frac{Q}{a_i} > 1$, and therefore it has to contain some natural number t. Therefore $ta_i \in (-Q-j, -j)$, i.e. $-Q < j + ta_i < 0$. In the case that $n = 1$, the open interval $(-Q-j, -j)$ contains a positive multiple of $a_1 = Q$, unless $j \in -Q\mathbb{N}$.

First we observe that $\partial^\alpha K$ is smooth away from zero, because K is smooth away from zero.

Our next step is to see that $\partial^\alpha K$ satisfies estimate (III.11). Since by hypothesis we know that estimate (III.12) holds for K, then for any multiindeces β, γ there exists a constant $C_{\beta+\alpha,\gamma} > 0$ such that for all $x \neq 0$

$$\begin{aligned} \left|\partial^\beta \left(\partial^\alpha K\right)(x)\right| &\leqslant C_{\beta+\alpha,\gamma} B_{k-|\alpha|-|\beta|-|\gamma|}(x) \\ &= C_{\beta+\alpha,\gamma} |x|^{k-|\alpha|-|\beta|-|\gamma|}. \end{aligned}$$

In the previous equation we have made use of the fact that $-Q < j + |\alpha|$ which implies that $k - |\alpha| - |\beta| - |\gamma| = -Q - j - |\alpha| - |\beta| - |\gamma| < 0$. Then estimate (III.11) holds for $\partial^\alpha K$.

Our next and final step will consist in using induction to prove that the distributional derivative $\partial^\alpha K = \partial^{te_i} K$ is in $L^1(G)$. In fact we show by induction $\partial^{se_i} K \in L^1(G)$, for $1 \leqslant s \leqslant t,\ s \in \mathbb{N}$. Assume $s = 1$. By hypothesis K is in $L^1(G)$ and C^∞ away from 0. Since (III.12) holds for K, then for any multiindeces β, γ, there exists $C_{\beta\gamma}$ such that

$$\left|\partial^\beta K(x)\right| \leqslant C_{\beta\gamma} B_{k-|\beta|-|\gamma|}(x)$$

where with ∂^β is the derivative in the usual sense. Then since $k = -Q - j \geqslant 0$

$$\left|K(x)\right| \leqslant C_o B_k(x) = \begin{cases} C_o & \text{if } k > 0 \\ C_o\left(\left|\log|x|\right| + 1\right) & \text{if } k = 0 \end{cases}$$

A more succinct estimate is

$$\left|K(x)\right| \leqslant C_1 |x|^{-Q+c} \text{ with } c > a_i$$

since in the case that $k > 0$ we can take $c = Q$, and in the case that $k = 0$ we can take $c = Q - \varepsilon$ for any $\varepsilon > 0$ such that $Q - \varepsilon > a_i$.

Given that for any multiindex γ

$$\left|\partial_i K(x)\right| \leqslant C_\gamma B_{k-|\alpha|-|\gamma|}(x) = C_\gamma B_{k-a_i-|\gamma|}(x)\,,$$

choosing γ such that $k - a_i - |\gamma| < -Q$ we see that $\partial_i K$ is integrable near infinity. And if $|\gamma| = 0$, we have $k - a_i - |\gamma| = k - a_i$ which is greater than $-Q$, then $\partial_i K$ is integrable near zero. Consequently the usual derivative $\partial_i K$ is an L^1 function away from zero. Therefore by Proposition 21 the derivative $\partial_i K$ in the sense of distributions agrees with the usual derivatives $\partial_i K$ a.e. Then the distributional derivative $\partial_i K$ is in $L^1(G)$.

Now assume the thesis to be valid for $(s-1)$, that is $\partial_i^{s-1} K$ is in $L^1(G)$. By hypothesis $\partial_i^{s-1} K$ is C^∞ away from zero and $\left|\partial_i^{s-1} K(x)\right| \leqslant C B_{k-(s-1)a_i}(x)$. Since $k - sa_i \geqslant k - ta_i > -Q$ then $k - (s-1)a_i \geqslant k - (t-1)a_i > -Q + a_i$. Hence there exists $c > a_i$ such that

$$\left|\partial_i^{s-1} K(x)\right| \leqslant C B_{k-(s-1)a_i}(x) \leqslant C_1 |x|^{-Q+c}.$$

Recalling that

$$\left|\partial_i^s K(x)\right| \leqslant C_\gamma B_{k-sa_i-|\gamma|}(x)\,,$$

and arguing in a similar fashion as for $s = 1$, we complete the inductive step. So we can conclude that $\partial^\alpha K$ is a L^1 function away from zero. Then by Proposition 21 the distributional derivative $\partial^\alpha K$ agrees with the usual derivative $\partial^\alpha K$ a.e. Therefore the distributional derivative $\partial^\alpha K$ is in $L^1(G)$.

We have verified that $\partial^\alpha K$ satisfies all the hypothesis of part (c) of Theorem 19 and consequently $(\partial^\alpha K) \in \widetilde{\mathcal{M}}^{j+|\alpha|}(G) = \widetilde{\mathcal{M}}^{j+ta_i}(G)$. Hence $(\partial^\alpha K)^\wedge = (-2\pi i)^{\|\alpha\|}\xi^\alpha \widehat{K} = (-2\pi i)^{\|\alpha\|}\xi_i^t \widehat{K} \in \mathcal{M}^{j+ta_i}(G)$. Which means that $\xi_i^t \widehat{K} \in \mathcal{M}^{j+ta_i}(G)$.

By choosing a $\varphi \in C_c^\infty(G)$, equal to 1 in a neighborhood of zero, one can write K as follows

$$K = \underbrace{(\varphi K)}_{\in L^1_{\text{comp}} \subset \mathcal{E}'} + \underbrace{(1-\varphi)K}_{\in \mathcal{S}}$$

Hence by (III.12), and the fact that $K \in L^1(G)$ we have shown that $K \in (L^1_{\text{comp}}) + \mathcal{S} \subset \mathcal{E}' + \mathcal{S}$, and therefore $\widehat{K} \in C^\infty(G)$.

Let S be the *homogeneous unit sphere*, i.e. $S = \{x \in G \,:\, |x| = 1\}$, where $|\cdot|$ denotes the homogeneous norm of G. Given that S is clearly compact, we can find a positive number c, such that S be covered by the family of sets

$$U_s = \left\{\, x \in S \,:\, |x_s| > c \,\right\} \qquad \text{for } s = 1, \ldots, n\,.$$

Now we construct a covering of $\mathbb{R}^n - \{0\}$ consisting of those cones obtained by including for each point $x \in U_s$, the "radial ray" passing through that point,

$$\Gamma_s := \left\{\, \delta_r(x) \,:\, x \in U_s, \text{ and } r > 0 \right\} = \bigcup_{r>0} \delta_r(U_s) \qquad \text{for } s = 1, \ldots, n.$$

Since every point in $\mathbb{R}^n - \{0\}$ is of the form $\delta_r(x)$ for some $r > 0$ and $x \in S$, the family of cones $\{\Gamma_s\}_{s=1}^n$ covers $\mathbb{R}^n - \{0\}$. Therefore if $\xi \in \mathbb{R}^n - \{0\}$, for some i we have that $\xi \in \Gamma_i$, i.e. for a certain $x \in U_i \subset S$

$$\xi_i = \left[\delta_{|\xi|}(x)\right]_i = |\xi|^{a_i} x_i, \qquad \text{thus} \qquad |\xi_i| = |\xi|^{a_i} |x_i| > c|\xi|^{a_i}. \tag{III.13}$$

Next we show that for any multiindex β there exists a positive constant C_β such that

$$\left|\partial^\beta \widehat{K}(\xi)\right| \leqslant C_\beta \left(1 + |\xi|\right)^{j-|\beta|} \qquad \forall \xi \tag{III.14}$$

If $|\xi| \geqslant 1$ we have

$$\begin{aligned}
\left|\left(\partial^\beta \widehat{K}\right)(\xi)\right| &= \left|\partial^\beta \left[\frac{1}{\xi_i^t}\left(\xi_i^t \widehat{K}\right)\right]\right| \\
&\overset{\text{(Leibniz rule)}}{=} \left|\sum_{le_i+\tau=\beta} \binom{\beta}{le_i} \left[\partial^{le_i}\left(\xi_i^{-t}\right) \partial^\tau \left(\xi_i^t \widehat{K}\right)\right](\xi)\right| \\
&\leqslant C_1 \sum_{le_i+\tau=\beta} \left|\partial_i^l \left(\xi_i^{-t}\right)(\xi) \;\; \partial^\tau \left(\xi_i^t \widehat{K}\right)(\xi)\right| \\
&\overset{(\xi_i^t \widehat{K} \in \mathcal{M}^{j+ta_i})}{\leqslant} C_2 \sum_{le_i+\tau=\beta} |\xi_i|^{-t-l} \left(1 + |\xi|\right)^{j+ta_i-|\tau|} \\
&= C_2 \sum_{le_i+\tau=\beta} \frac{\left(1+|\xi|\right)^{ta_i}}{|\xi_i|^{t+l}} \;\; \left(1+|\xi|\right)^{j-|\tau|} \\
&\overset{(\xi \in \Gamma_i, \text{ then } |\xi_i| > c|\xi|^{a_i})}{\leqslant} C_2 \sum_{le_i+\tau=\beta} \frac{\left(1+|\xi|\right)^{ta_i}}{\tilde{c}|\xi|^{(t+l)a_i}} \;\; \left(1+|\xi|\right)^{j-|\tau|} \\
&= C_2 \sum_{le_i+\tau=\beta} \frac{\left(1+|\xi|\right)^{ta_i}}{\tilde{c}|\xi|^{ta_i}} \;\; \frac{\left(1+|\xi|\right)^{j-|\tau|}}{|\xi|^{la_i}} \\
&\leqslant C_3 \sum_{le_i+\tau=\beta} \left(1+|\xi|\right)^{j-la_i-|\tau|} \\
&\overset{(|\beta|=|le_i|+|\tau|=la_i+|\tau|)}{\leqslant} C_4 \left(1+|\xi|\right)^{j-|\beta|}
\end{aligned}$$

If $|\xi| \leqslant 1$, by the continuity of $\left(1+|\xi|\right)^{j-|\beta|}$, there exists a positive constant m such that

$$0 < m \leqslant \left(1+|\xi|\right)^{j-|\beta|}.$$

Hence for all $|\xi| \leqslant 1$

$$\left|\partial^{\beta}\widehat{K}(\xi)\right| \leqslant C \leqslant \frac{C}{m}\left(1+|\xi|\right)^{j-|\beta|}$$

Choosing $C_\beta = \max\left\{\frac{C}{m}, C_4\right\}$ we obtain (III.14). And therefore $\widehat{K} \in \mathcal{M}^j(G)$, which is what we had to prove.

$\square$

CHAPTER IV

CONVOLUTION IN $\widecheck{\mathcal{M}}^j(\mathbb{H}^n)$ WITH $j < 0$

IV.1 BASIC DEFINITIONS AND RESULTS

Definition 23. The *Heisenberg group* $\mathbb{H}^n$ is a non-commutative homogeneous group with underlying manifold $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, and the group law

$$(x, y, t) \cdot (x', y', t') = \left(x + x', y + y', t + t' - 2 \sum_{j=1}^{n} \left(x_j y'_j - x'_j y_j \right) \right)$$

The dilations in $\mathbb{H}^n$ are given by

$$\delta_r(x, y, t) = (rx, ry, r^2 t) \qquad \text{for } r > 0.$$

The homogeneous dimension of $\mathbb{H}^n$ is $Q = 2n + 2$, which is always greater or equal to 4.

The homogeneous norm in $\mathbb{H}^n$ is given by

$$\left|(x,y,t)\right| = \left(|x|^4 + |y|^4 + t^2\right)^{1/4}$$

Consider $\mathfrak{g}_L$, the Lie algebra of left-invariant vector fields on $\mathbb{H}^n$. We fix a basis

$$\left\{X_1, \dots, X_n, Y_1, \dots, Y_n, T\right\}$$

of $\mathfrak{g}_L$, by choosing X_i, Y_i, and T as the left-invariant vector fields which agree with $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial y_i}$, and $\frac{\partial}{\partial t}$ at the origin. Similarly we have

$$\left\{X_1^R, \dots, X_n^R, Y_1^R, \dots, Y_n^R, T^R\right\}$$

the corresponding basis of right invariant vector fields on $\mathbb{H}^n$ chosen by requiring that X_i^R, Y_i^R, and T^R be the unique right-invariant vector fields which agree with $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial y_i}$, and $\frac{\partial}{\partial t}$ at the origin. It is known that

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \qquad && j = 1, \dots n \\ Y_j &= \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \qquad && j = 1, \dots n \\ T &= \frac{\partial}{\partial t} \end{aligned}$$

$$\begin{aligned} X_j^R &= \frac{\partial}{\partial x_j} - 2y_j \frac{\partial}{\partial t} \qquad && j = 1, \dots n \\ Y_j^R &= \frac{\partial}{\partial y_j} + 2x_j \frac{\partial}{\partial t} \qquad && j = 1, \dots n \\ T^R &= \frac{\partial}{\partial t} \end{aligned}$$

The differential operators X_j, Y_j, X_j^R, Y_j^R are homogeneous of degree 1, and T, T^R are homogeneous of degree 2.

We have the commutation relations

$$[X_j, Y_j] = -4T$$

and all other commutators of $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, T\}$ vanish.

Similarly

$$[X_j^R, Y_j^R] = 4T$$

and all other commutators of $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, T\}$ vanish. Since $T = \frac{\partial}{\partial t}$ then

$$T(f * g) = \big(Tf\big) * g = f * \big(Tg\big).$$

Definition 24. For $j = 1, \ldots, 2n+1$ we define the *multiplication map*

$$\begin{aligned} M_j : \mathcal{S}(\mathbb{H}^n) &\longrightarrow \mathcal{S}(\mathbb{H}^n) \\ f \ &\longmapsto M_j(f) \end{aligned}$$

where

$$(M_j f)(x, y, t) := \begin{cases} x_j f(x, y, t) & \text{if } j = 1, \ldots, n \\ y_j f(x, y, t) & \text{if } j = n+1, \ldots, 2n \\ t f(x, y, t) & \text{if } j = 2n+1. \end{cases} \tag{IV.1}$$

We observe that if f belongs to the Schwartz space $\mathcal{S}(\mathbb{H}^n)$, then clearly $(M_j f)$ is in $\mathcal{S}(\mathbb{H}^n)$ for all $j = 1, \ldots, 2n+1$. Moreover if $f, g \in \mathcal{S}(\mathbb{H}^n)$, then $f * g$ is also in $\mathcal{S}(\mathbb{H}^n)$ and therefore $M_j(f * g)$ makes sense.

Substitution of $M_j(f)$ by $x_j f$, or by $y_j f$, or by tf, are abuses of notation which will be made whenever the risk of confusion be minimal.

By duality the multiplication maps can be extended to S'. More precisely the maps

$M_j : \mathcal{S}(\mathbb{H}^n) \longrightarrow \mathcal{S}(\mathbb{H}^n)$ are extended to $M_j : \mathcal{S}'(\mathbb{H}^n) \longrightarrow \mathcal{S}'(\mathbb{H}^n)$ by the formula

$$[M_j(\Lambda)](f) = \Lambda[M_j(f)] \qquad \forall \Lambda \in \mathcal{S}'(\mathbb{H}^n), \quad f \in \mathcal{S}$$

Definition 25. For $f, g \in L^1(\mathbb{H}^n)$ we define $f * g$, the *convolution of f with g*, as follows $u = (x, y, t)$, $v = (x', y', t')$

$$\begin{aligned}
&(f*g)(u) \\
&\quad := \int_{\mathbb{H}^n} f(uv^{-1}) g(v)\, dv \\
&\quad = \int_{\mathbb{H}^n} f\Big(x - x',\ y - y',\ t - t' + 2\sum_{j=1}^{n}(x_j y_j' - x_j' y_j)\Big)\, g(x', y', t')\ dx'\, dy'\, dt'
\end{aligned}$$

Here $v^{-1} = (-x', -y', -t')$.

Proposition 26.

a) Say $f, g \in \mathcal{S}(\mathbb{H}^n)$ then

$$x_j(f * g) = (x_j f) * g + f * (x_j g) \qquad j = 1, \ldots, n \tag{IV.2}$$

$$y_j(f * g) = (y_j f) * g + f * (y_j g) \qquad j = 1, \ldots, n \tag{IV.3}$$

$$t(f * g) = (tf) * g + f * (tg) - 2\sum_{j=1}^{n}\big[(x_j f) * (y_j g) - (y_j f) * (x_j g)\big] \tag{IV.4}$$

b) The relations (IV.2), (IV.3), *and* (IV.4) *remain valid for each of the following cases:*

i. $f \in \mathcal{E}'(\mathbb{H}^n)$ *and* $g \in \mathcal{S}(\mathbb{H}^n)$.

ii. $g \in \mathcal{S}(\mathbb{H}^n)$ *and* $f \in \mathcal{E}'(\mathbb{H}^n)$.

iii. $f, g \in \mathcal{E}'(\mathbb{H}^n)$.

Proof. We start with the first equation, (IV.2). For $j = 1, \ldots, n$

$$
\begin{aligned}
\big[M_j(f*g)\big](u) &= \big[x_j(f*g)\big](u) \\
&= \big((x_j - x'_j) + x'_j\big)\int_{\mathbb{H}^n} f(uv^{-1})g(v)\,dv \\
&= (x_j - x'_j)\int_{\mathbb{H}^n} f(uv^{-1})g(v)\,dv + x'_j\int_{\mathbb{H}^n} f(uv^{-1})g(v)\,dv \\
&= \int_{\mathbb{H}^n} (M_j f)(uv^{-1})g(v)\,dv + \int_{\mathbb{H}^n} f(uv^{-1})(M_j g)(v)\,dv \\
&= \big[(M_j f)*g\big](u) + \big[f*(M_j g)\big](u)
\end{aligned}
$$

The proof of the second identity, (IV.3), is completely analogous and therefore it will be omitted. The proof of the last identity, (IV.4), follows.

$$
\begin{aligned}
&\Big[M_{2n+1}(f*g)\Big](u) \\
&\qquad = \big[t(f*g)\big](u) \\
&\qquad = \left[\left(t - t' + 2\sum_{j=1}^{n}\big(x_j y'_j - x'_j y_j\big) + t' - 2\sum_{j=1}^{n}\big(x_j y'_j - x'_j y_j\big)\right)(f*g)\right](u) \\
&\qquad = \left(t - t' + 2\sum_{j=1}^{n}\big(x_j y'_j - x'_j y_j\big)\right)\int_{\mathbb{H}^n} f(uv^{-1})g(v)\,dv \\
&\qquad\quad + t'\int_{\mathbb{H}^n} f(uv^{-1})g(v)\,dv - 2\sum_{j=1}^{n}\big(x_j y'_j - x'_j y_j\big)\int_{\mathbb{H}^n} f(uv^{-1})g(v)\,dv \\
&\qquad = \int_{\mathbb{H}^n}\big(M_{2n+1}f\big)(uv^{-1})\,g(v)\,dv + \int_{\mathbb{H}^n} f(uv^{-1})\big(M_{2n+1}g\big)(v)\,dv \\
&\qquad\quad - 2\sum_{j=1}^{n}\big[(x_j - x'_j)y'_j - x'_j(y_j - y'_j)\big]\int_{\mathbb{H}^n} f(uv^{-1})\,g(v)\,dv \\
&\qquad = \big[\big(M_{2n+1}f\big)*g\big](u) + \big[f*\big(M_{2n+1}g\big)\big](u) \\
&\qquad\quad - 2\sum_{j=1}^{n}\int_{\mathbb{H}^n}\big[\big(M_j f\big)(uv^{-1})\big(M_{j+n}g\big)(v) - \big(M_{n+j}f\big)(uv^{-1})\big(M_j g\big)(v)\big]\,dv
\end{aligned}
$$

$$= \left[(M_{2n+1} f) * g \right] (u) + \left[f * (M_{2n+1} g) \right] (u)$$

$$- 2 \sum_{j=1}^{n} \left[(M_j f) * (M_{n+j} g) - (M_{n+j} f) * (M_j g) \right] (u)$$

Part (a) has been proved. Part (b) follows immediately from part (a), and the fact that $C_c^\infty(\mathbb{H}^n)$ is dense in $\mathcal{E}'(\mathbb{H}^n)$. □

By the use of induction the following corollary follows from Proposition 26.

Corollary 27.

a) Say $f, g \in \mathcal{S}(\mathbb{H}^n)$ *and* $k \in \mathbb{N}$ *then*

$$x_j^k (f * g) = \sum_{p+q=k} \binom{k}{p} (x_j^p f * x_j^q g) \qquad j = 1, \ldots, n \quad \text{(IV.5)}$$

$$y_j^k (f * g) = \sum_{p+q=k} \binom{k}{p} (y_j^p f * y_j^q g) \qquad j = 1, \ldots, n \quad \text{(IV.6)}$$

$$t^k (f * g) = \left\{ \begin{array}{c} \textit{linear combination of terms of the form } (Rf) * (Pg) \\ \textit{with } R \textit{ and } P \textit{ homogeneous polynomials in x,y,t} \\ \textit{such that} \quad \operatorname{hom\,deg} R + \operatorname{hom\,deg} P = 2k \end{array} \right\} \quad \text{(IV.7)}$$

b) Relations (IV.5), (IV.6), *and* (IV.7) *remain valid in each of the following cases*

i. $f \in \mathcal{E}'(\mathbb{H}^n)$ *and* $g \in \mathcal{S}(\mathbb{H}^n)$.

ii. $g \in \mathcal{S}(\mathbb{H}^n)$ *and* $f \in \mathcal{E}'(\mathbb{H}^n)$.

iii. $f, g \in \mathcal{E}'(\mathbb{H}^n)$.

Proposition 28. *If* $K_i \in \widetilde{\mathcal{M}}^{j_i}(\mathbb{H}^n)$, $i = 1, 2$, *then*

a)

$$\frac{\partial}{\partial x_j}(K_1 * K_2) = \left(X_j^R K_1\right) * K_2 + 2\,(y_j\, T K_1) * K_2$$
$$+ \tfrac{1}{2}\,(X_j\, K_1) * \left(Y_j^R y_j\, K_2\right) - \tfrac{1}{2}\,(Y_j\, K_1) * \left(X_j^R y_j\, K_2\right).$$

An alternative expression is

$$\frac{\partial}{\partial x_j}(K_1 * K_2) = K_1 * (X_j K_2) - 2K_1 * (y_j\, T K_2)$$
$$+ \tfrac{1}{2}\,(Y_j y_j K_1) * \left(X_j^R\, K_2\right) - \tfrac{1}{2}\,(X_j y_j K_1) * \left(Y_j^R\, K_2\right).$$

b)

$$\frac{\partial}{\partial y_j}(K_1 * K_2) = \left(Y_j^R\, K_1\right) * K_2 + 2\,(x_j\, T K_1) * K_2$$
$$+ \tfrac{1}{2}\,(Y_j\, K_1) * \left(X_j^R x_j\, K_2\right) - \tfrac{1}{2}\,(X_j\, K_1) * \left(Y_j^R x_j\, K_2\right).$$

An alternative expression is

$$\frac{\partial}{\partial y_j}(K_1 * K_2) = K_1 * (Y_j K_2) - 2K_1 * (x_j\, T K_2)$$
$$+ \tfrac{1}{2}\,(X_j x_j K_1) * \left(Y_j^R\, K_2\right) - \tfrac{1}{2}\,(Y_j x_j K_1) * \left(X_j^R\, K_2\right).$$

c)

$$\frac{\partial}{\partial t}(K_1 * K_2) = \tfrac{1}{4}\,(X_1\, K_1) * \left(Y_1^R K_2\right) - \tfrac{1}{4}\,(Y_1\, K_1) * \left(X_1^R K_2\right).$$

d) Alternative expressions for the previous are

$$\frac{\partial}{\partial t}(K_1 * K_2) = (T K_1) * K_2$$

$$\frac{\partial}{\partial t}(K_1 * K_2) = K_1 * (T K_2).$$

Proof. We use Proposition 26 noting that by Corollary 18, $K_1, K_2 \in \mathcal{E}'(\mathbb{H}^n) + \mathcal{S}(\mathbb{H}^n)$.

a)
$$\begin{aligned}
\frac{\partial}{\partial x_j}(K_1 * K_2) &= \left(X_j^R + 2y_jT\right)(K_1 * K_2) \\
&= X_j^R(K_1 * K_2) + 2y_jT(K_1 * K_2) \\
&= \left(X_j^R K_1\right) * K_2 + 2Ty_j(K_1 * K_2) \\
&= \left(X_j^R K_1\right) * K_2 + 2T\left((y_jK_1) * K_2 + K_1 * (y_j * K_2)\right) \\
&= (X_j^R K_1) * K_2 + 2(y_jTK_1) * K_2 + K_1 * (2Ty_jK_2) \\
&= (X_j^R K_1) * K_2 + 2(y_jTK_1) * K_2 \\
&\quad + K_1 * \left(\tfrac{1}{2}[X_j^R, Y_j^R]y_jK_2\right) \\
&= (X_j^R K_1) * K_2 + 2(y_jTK_1) * K_2 \\
&\quad + K_1 * \left(\tfrac{1}{2}(X_j^R Y_j^R - Y_j^R X_j^R)y_jK_2\right) \\
&= (X_j^R K_1) * K_2 + 2(y_jTK_1) * K_2 \\
&\quad + \tfrac{1}{2}(X_jK_1) * (Y_j^R y_jK_2) - \tfrac{1}{2}(Y_jK_1) * (X_j^R y_jK_2).
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial x_j}(K_1 * K_2) &= (X_j - 2y_jT)\,(K_1 * K_2)\\
&= X_j(K_1 * K_2) - 2y_jT(K_1 * K_2)\\
&= K_1 * (X_jK_2) - 2Ty_j(K_1 * K_2)\\
&= K_1 * (X_jK_2) - 2T\left((y_jK_1) * K_2 + K_1 * (y_jK_2)\right)\\
&= K_1 * (X_jK_2) - 2K_1 * (y_jTK_2) - (2Ty_jK_1) * K_2\\
&= K_1 * (X_jK_2) - 2K_1 * (y_jTK_2)\\
&\quad + \left(\tfrac{1}{2}[X_j, Y_j]y_jK_1\right) * K_2\\
&= K_1 * (X_jK_2) - 2K_1 * (y_jTK_2)\\
&\quad + \tfrac{1}{2}\left((X_jY_j - Y_jX_j)y_jK_1\right) * K_2\\
&= K_1 * (X_jK_2) - 2K_1 * (y_j\,TK_2)\\
&\quad + \tfrac{1}{2}\,(Y_jy_jK_1) * \left(X_j^R\,K_2\right) - \tfrac{1}{2}\,(X_jy_jK_1) * \left(Y_j^R\,K_2\right).
\end{aligned}
$$

b) The proof of part (b) is analogous to that of part (a).

c)
$$
\begin{aligned}
\frac{\partial}{\partial t}(K_1 * K_2) &= T(K_1 * K_2)\\
&= (TK_1) * K_2\\
&= \left(-\tfrac{1}{4}[X_1, Y_1]K_1\right) * K_2\\
&= \left(-\tfrac{1}{4}(X_1Y_1 - Y_1X_1)K_1\right) * K_2\\
&= \tfrac{1}{4}\,(Y_1X_1K_1) * K_2 - \tfrac{1}{4}\,(X_1Y_1K_1) * K_2\\
&= \tfrac{1}{4}\,(X_1\,K_1) * \left(Y_1^RK_2\right) - \tfrac{1}{4}\,(Y_1\,K_1) * \left(X_1^RK_2\right).
\end{aligned}
$$

d) There is nothing to prove in this case since $T = \frac{\partial}{\partial t}$ and

$$\left(\frac{\partial}{\partial t}K_1\right) * K_2 = K_1 * \left(\frac{\partial}{\partial t}K_2\right).$$

$\square$

Remark 29. Given that the differential operators X_i, Y_i, X_i^R, Y_i^R are homogeneous of degree 1, and T is homogeneous of degree 2, it follows from the previous proposition that for $K_i \in \widetilde{\mathcal{M}}^{j_i}$ it is possible to write

$$\frac{\partial}{\partial x_j}(K_1 * K_2) = \left\{ \begin{array}{c} \text{Sum of four terms of the form } \widetilde{K}_1 * \widetilde{K}_2, \\ \text{with } \widetilde{K}_1 \in \widetilde{\mathcal{M}}^{j_1+1}(\mathbb{H}^n) \text{ and } \widetilde{K}_2 \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n) \end{array} \right\}$$

or alternatively is also possible to express this

$$\frac{\partial}{\partial x_j}(K_1 * K_2) = \left\{ \begin{array}{c} \text{Sum of four terms of the form } \widetilde{K}_1 * \widetilde{K}_2, \\ \text{with } \widetilde{K}_1 \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n) \text{ and } \widetilde{K}_2 \in \widetilde{\mathcal{M}}^{j_2+1}(\mathbb{H}^n) \end{array} \right\}$$

The analogous statements are valid for $\frac{\partial}{\partial y_j}$.

For $\frac{\partial}{\partial t}(K_1 * K_2)$ we have the expression

$$\frac{\partial}{\partial t}(K_1 * K_2) = \left\{ \begin{array}{c} \text{Sum of two terms of the form } \widetilde{K}_1 * \widetilde{K}_2, \\ \text{with } \widetilde{K}_1 \in \widetilde{\mathcal{M}}^{j_1+1}(\mathbb{H}^n) \text{ and } \widetilde{K}_2 \in \widetilde{\mathcal{M}}^{j_2+1}(\mathbb{H}^n) \end{array} \right\}$$

Corollary 30. *Assume $K_1, K_2 \in \widetilde{\mathcal{M}}^{j_i}(\mathbb{H}^n)$, and $m \in \mathbb{Z}^+$.*

a) For each $i = 1, \ldots, n$, there exist $K^i_{1,r} \in \widetilde{\mathcal{M}}^{j_1+m}(\mathbb{H}^n)$, and $K^i_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$ such that

$$\frac{\partial^m}{\partial x_i^m}\Big(K_1 * K_2\Big) = \sum_{r=1}^{4^m}\Big(K^i_{1,r} * K^i_{2,r}\Big)$$

Alternatively for each $i = 1, \ldots, n$, there exist $K^i_{1,r} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $K^i_{2,r} \in \widetilde{\mathcal{M}}^{j_2+m}(\mathbb{H}^n)$ such that the previous expression is valid.

b) For each $i = 1, \ldots, n$, there exist $K^i_{1,r} \in \widetilde{\mathcal{M}}^{j_1+m}(\mathbb{H}^n)$, and $K^i_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$ such that

$$\frac{\partial^m}{\partial y_i^m}\Big(K_1 * K_2\Big) = \sum_{r=1}^{4^m}\Big(K^i_{1,r} * K^i_{2,r}\Big)$$

Alternatively for each $i = 1, \ldots, n$, there exist $K^i_{1,r} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $K^i_{2,r} \in \widetilde{\mathcal{M}}^{j_2+m}(\mathbb{H}^n)$ such that for the previous expression is valid.

Proof.

a) Induction over the order of differentiation, m, will be used. By the previous remark we know that

$$\frac{\partial}{\partial x_i}\Big(K_1 * K_2\Big) = \sum_{r=1}^{4}\Big(K^i_{1,r} * K^i_{2,r}\Big)$$

with $K^i_{1,r} \in \widetilde{\mathcal{M}}^{j_1+1}(\mathbb{H}^n)$, and $K^i_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$. Now we assume the thesis to be valid for $m-1$, that is

$$\frac{\partial^{m-1}}{\partial x_i^{m-1}}\Big(K_1 * K_2\Big) = \sum_{r=1}^{4^{m-1}}\Big(K^i_{1,r} * K^i_{2,r}\Big)$$

with $K^i_{2,r} \in \widetilde{\mathcal{M}}^{j_1+m-1}(\mathbb{H}^n)$, and $K^i_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$.

Therefore

$$\begin{aligned}
\frac{\partial^m}{\partial x_i^m}\left(K_1 * K_2\right) &= \frac{\partial}{\partial x_i}\frac{\partial^{m-1}}{\partial x_i^{m-1}}\left(K_1 * K_2\right) \\
&= \frac{\partial}{\partial x_i}\sum_{r=1}^{4^{m-1}}\left(K_{1,r}^i * K_{2,r}^i\right) \\
&= \sum_{r=1}^{4^{m-1}}\frac{\partial}{\partial x_i}\left(K_{1,r}^i * K_{2,r}^i\right) \\
&= \sum_{r=1}^{4^{m-1}}\sum_{s=1}^{4}\left(K_{1,s}^i * K_{2,s}^i\right) \\
&= \sum_{r=1}^{4^m}\left(K_{1,r}^i * K_{2,r}^i\right).
\end{aligned}$$

with $K_{1,r}^i \in \widetilde{\mathcal{M}}^{j_1+m-1+1}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_1+m}(\mathbb{H}^n)$, and $K_{2,r}^i \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$, which is what we wanted to prove.

The proof of the other parts is completely analogous.

□

Say $(x, y, t) \in \mathbb{H}^n$, and $\beta = (\beta_1, \ldots, \beta_{2n+1}) \in (\mathbb{Z}^+)^{2n+1}$ a multiindex. We shall designate by β_x, β_y, and β_t those multiindeces in $(\mathbb{Z}^+)^{2n+1}$ having the same components as β, but replacing with zeroes those components which do **not** affect the x_i, the y_i, and t, respectively. Therefore if, as usual for $\mathbb{H}^n$, we denote by $|\beta| := \sum_i^{2n} \beta_i + 2\beta_{2n+1}$, then the following two equations hold

$$\beta = \beta_x + \beta_y + \beta_t, \qquad \text{and} \qquad |\beta| = |\beta_x| + |\beta_y| + |\beta_t|.$$

We shall refer to the three multiindeces β_x, β_y, and β_t as the *coördinatewise subordinated decomposition of* the multiindex β, although when confusion appear improbable, only *subordinated decomposition* will be used.

Proposition 31. *If* $K_i \in \widetilde{\mathcal{M}}^{j_i}(\mathbb{H}^n)$, $i = 1,2$, $\beta \in (\mathbb{Z}^+)^{2n+1}$ *with* β_x, β_y, *and* β_t *its subordinated decomposition. Then*

a) There exist $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\beta_x|}(\mathbb{H}^n)$, *and* $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$, *such that*

$$\partial^{\beta_x}(K_1 * K_2) = \sum_{r=1}^{4^{|\beta_x|}} (K_{1,r} * K_{2,r})$$

Alternatively there exist $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, *and* $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2+|\beta_x|}(\mathbb{H}^n)$, *such that the previous expression is valid.*

b) There exist $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\beta_y|}(\mathbb{H}^n)$, *and* $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$.

$$\partial^{\beta_y}(K_1 * K_2) = \sum_{r=1}^{4^{|\beta_y|}} (K_{1,r} * K_{2,r})$$

Alternatively there exist $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, *and* $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2+|\beta_y|}(\mathbb{H}^n)$, *such that the previous expression is valid.*

c)

$$\partial^{\beta_t}(K_1 * K_2) = \left(T^{(|\beta_t|/2)} K_1\right) * K_2$$

where $T^{(|\beta_t|/2)} = \underbrace{T \circ T \circ \ldots \circ T}_{\frac{1}{2}|\beta_t| \text{ times}}$, *and with* $T^{(|\beta_t|/2)} K_1 \in \widetilde{\mathcal{M}}^{j_1+|\beta_t|}(\mathbb{H}^n)$.

Alternatively

$$\partial^{\beta_t}(K_1 * K_2) = K_1 * \left(T^{(|\beta_t|/2)} K_2\right)$$

with $T^{(|\beta_t|/2)} K_2 \in \widetilde{\mathcal{M}}^{j_2+|\beta_t|}(\mathbb{H}^n)$.

Proof. We shall prove part (a) by induction on $k = |\beta_x|$. If $k = 1$, then $\beta_x = e_i$, where e_i is the $i-$th element of he canonical base of $\mathbb{R}^{2n+1}$, $1 \leqslant i \leqslant n$. Therefore by Remark 29 we have that

$$\partial^{\beta_x}(K_1 * K_2) = \frac{\partial}{\partial x_i}(K_1 * K_2) = \sum_{r=1}^{4} (K_{1,r} * K_{2,r})$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+1}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_1+|\beta_x|}(\mathbb{H}^n)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$. We assume that for $k = |\beta_x| - 1$ the thesis is valid. Consider some $\beta_i \neq 0$, with $1 \leqslant i \leqslant n$, then we can express β_x as $\beta_x = \sigma + e_i = (\beta_1, \ldots, \beta_i - 1, \ldots, \beta_n, 0, \ldots, 0) + e_i$. Note $|\sigma| = |\beta_x| - 1$. Therefore by the inductive hypothesis we have

$$\begin{aligned}
\partial^{\beta_x}(K_1 * K_2) &= \partial^{e_i}\partial^{\sigma}(K_1 * K_2) \\
&= \partial^{e_i} \sum_{r=1}^{4^{|\beta_x|-1}} (K_{1,r} * K_{2,r}) \\
&= \sum_{r=1}^{4^{|\beta_x|-1}} \partial^{e_i}(K_{1,r} * K_{2,r}) \\
&= \sum_{r=1}^{4^{|\beta_x|-1}} \sum_{s=1}^{4} (K_{1,r_s} * K_{2,r_s}) \\
&= \sum_{r=1}^{4^{|\beta_x|}} (K_{1,r} * K_{2,r})
\end{aligned}$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j+|\beta_x|}(\mathbb{H}^n)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$, as desired.

The alternative case, as well as part (b), and its alternative, can be proved with an analogous argument.

Part (c) is proved by induction on $k = \beta_{2n+1}$. In case that $k = \beta_{2n+1} = 1$, we have $\beta_t = e_{2n+1}$, and hence, by part (d) of Proposition 28,

$$\partial^{\beta_t}(K_1 * K_2) = \frac{\partial}{\partial t}(K_1 * K_2) = (TK_1 * K_2)$$

with $TK_1 \in \widetilde{\mathcal{M}}^{j_1+2}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_1+|\beta_t|}(\mathbb{H}^n)$.

Now we assume that the thesis is valid for $k - 1 = \beta_{2n+1} - 1$ and we shall prove that it is valid for β_{2n+1}. Set $\beta_t = e_{2n+1} + \sigma = e_{2n+1} + (0, \ldots, 0, \beta_{2n+1} - 1)$. Therefore by the inductive hypothesis

$$\begin{aligned}
\partial^{\beta_t}(K_1 * K_2) &= \partial^{\sigma}\partial^{e_{2n+1}}(K_1 * K_2) \\
&= \partial^{\sigma}\left(\left(TK_1\right) * K_2\right) \\
&= \left(T^{\frac{|\sigma|}{2}}\left(TK_1\right)\right) * K_2 \\
&= \left(T^{\frac{|\sigma+e_{2n+1}|}{2}}K_1\right) * K_2 \\
&= \left(T^{\frac{|\beta_t|}{2}}K_1\right) * K_2\,,
\end{aligned}$$

with $T^{\frac{|\beta_t|}{2}}K_1 \in \widetilde{\mathcal{M}}^{j_1+|\sigma|+|e_{2n+1}|}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j+|\beta_t|}(\mathbb{H}^n)$, as desired.

The proof of the alternative statement can be obtained by an analogous argument. □

As an immediate consequence we can state the following:

Corollary 32. *If $K_1, K_2 \in \widetilde{\mathcal{M}}^{j_i}(\mathbb{H}^n)$, and $\beta \in (\mathbb{Z}^+)^{2n+1}$ with β_x, β_y, and β_t its subordinated decomposition, then there exist $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\beta|}(\mathbb{H}^n)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$, such that*

$$\partial^{\beta}(K_1 * K_2) = \sum_{r=1}^{4^{|\beta_x+\beta_y|}} \left(K_{1,r} * K_{2,r}\right) \tag{IV.8}$$

Alternatively there exist $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2+|\beta|}(\mathbb{H}^n)$ such that the previous expression is valid.

Proof. Observe that $\beta = \beta_x + \beta_y + \beta_t$, therefore if we apply the previous proposition to

β_t, β_y, and β_x respectively, we have

$$
\begin{aligned}
\partial^\beta \big(K_1 * K_2\big) &= \partial^{\beta_x} \partial^{\beta_y} \partial^{\beta_t} \big(K_1 * K_2\big) \\
&= \partial^{\beta_x} \partial^{\beta_y} \left(\left(T^{|\beta_t|/2} K_1 \right) * K_2 \right) \\
&= \partial^{\beta_x} \sum_{r=1}^{4^{|\beta_y|}} (K_{1,r} * K_{2,r}) \\
&= \sum_{r=1}^{4^{|\beta_y|}} \partial^{\beta_x} (K_{1,r} * K_{2,r}) \\
&= \sum_{r=1}^{4^{|\beta_y|}} \sum_{s=1}^{4^{|\beta_x|}} (K_{1,r_s} * K_{2,r_s}) \\
&= \sum_{r=1}^{4^{|\beta_x + \beta_y|}} (K_{1,r} * K_{2,r})
\end{aligned}
$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1 + |\beta_t| + |\beta_y| + |\beta_x|}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_1 + |\beta|}(\mathbb{H}^n)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$.

The proof of the alternative is analogous. $\square$

Proposition 33. *Let β be a multiindex in $(\mathbb{Z}^+)^{2n+1}$, with subordinated decomposition β_x, β_y, β_t, and p, q, nonnegative integers such that $p + q = |\beta_x| + |\beta_y| + |\beta_t| = |\beta|$. Then any linear combination of terms of the form $\partial^{\beta_x} \partial^{\beta_y} \partial^{\beta_t} (K_1 * K_2) = \partial^\beta \big(K_1 * K_2\big)$ with $K_1 \in \widetilde{\mathcal{M}}^{j_1 - p}(\mathbb{H}^n)$, and $K_2 \in \widetilde{\mathcal{M}}^{j_2 - q}(\mathbb{H}^n)$, is equal to a linear combination of terms of the form $(\widetilde{K}_1 * \widetilde{K}_2)$, with $\widetilde{K}_1 \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$ and $\widetilde{K}_2 \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$.*

Proof. By induction on the value of $k = p + q$. If $p + q = 0$, then $p = q = 0$, and $|\beta_x| + |\beta_y| + |\beta_t| = 0$. Therefore

$$
\sum_{i=1}^{m} c_i \, \partial^{\beta_x} \partial^{\beta_y} \partial^{\beta_t} \big(K_{1,i} * K_{2,i}\big) = \sum_{i=1}^{m} c_i \big(K_{1,i} * K_{2,i}\big)
$$

with $K_{1,i} \in \widetilde{\mathcal{M}}^{j_1 + 0}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $K_{2,i} \in \widetilde{\mathcal{M}}^{j_2 + 0}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$.

We assume now that the thesis is valid for all $k \leqslant (p+q)-1$. We shall consider two possibilities for β_x:

- $|\beta_x| \neq 0$:

  If $|\beta_x| \neq 0$, then for some $1 \leqslant l \leqslant n$, we must have some $\frac{\partial}{\partial x_l}$ in ∂^{β_x}. Therefore β_x can be expressed by $\beta_x = \widetilde{\beta_x} + e_l$, where e_l is the $l-$th element of the canonical basis of $\mathbb{R}^{2n+1}$. Using Corollary 30 we are able to introduce $\frac{\partial}{\partial x_l}$ into the first factor of each $K_{1,i} * K_{2,i}$ in the case that $p > 0$, or into the second factor, in the case $p = 0$. Then we use the inductive hypothesis to conclude the proof.

  ► $p > 0$:

  By Corollary 30 part (a), we have

$$\sum_{i=1}^{m} c_i \partial^{\beta_x} \partial^{\beta_y} \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) = \sum_{i=1}^{m} c_i \partial^{\widetilde{\beta_x}} \partial^{\beta_y} \partial^{\beta_t} \left(\sum_{s=1}^{4} \left(K_{1,i_s} * K_{2,i_s}\right) \right)$$
$$= \sum_{r=1}^{4m} c_r \partial^{\widetilde{\beta_x}} \partial^{\beta_y} \partial^{\beta_t} \left(K_{1,r} * K_{2,r}\right)$$

  with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1 - p + 1}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_1 - (p-1)}(\mathbb{H}^n)$, $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2 - q}(\mathbb{H}^n)$, and $|\widetilde{\beta_x}| + |\beta_y| + |\beta_t| = (p-1) + q = (p+q) - 1$. Then by the inductive hypothesis, we have that

$$\sum_{i=1}^{m} c_i \partial^{\beta_x} \partial^{\beta_y} \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) = \sum_{s} \widetilde{c}_s \left(\widetilde{K}_{1,s} * \widetilde{K}_{2,s} \right)$$

  with $\widetilde{K}_{1,s} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $\widetilde{K}_{2,s} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$, which is what we wanted to prove.

  ► $p = 0$:

  If $p = 0$, then $q \neq 0$. By Corollary 30, part (a), alternative version, on page 57, we have that

$$\sum_{i=1}^{m} c_i \partial^{\beta_x} \partial^{\beta_y} \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) = \sum_{r=1}^{4m} c_r \partial^{\widetilde{\beta_x}} \partial^{\beta_y} \partial^{\beta_t} \left(K_{1,r} * K_{2,r}\right)$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1-p}(\mathbb{H}^n)$, $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2-q+1}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_2-(q-1)}(\mathbb{H}^n)$, and $|\widetilde{\beta_x}| + |\beta_y| + |\beta_t| = p + (q-1) = (p+q) - 1$. Then by the inductive hypothesis, we have

$$\sum_{i=1}^{m} c_i \partial^{\beta_x} \partial^{\beta_y} \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) = \sum_{s} \widetilde{c}_s \left(\widetilde{K}_{1,s} * \widetilde{K}_{2,s}\right)$$

with $\widetilde{K}_{1,s} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $\widetilde{K}_{2,s} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$.

- $|\beta_x| = 0$:

  - $|\beta_y| > 0$:

    If $|\beta_y| > 0$, then for some $1 \leqslant l \leqslant n$, we must have some $\frac{\partial}{\partial y_l}$ in ∂^{β_y}. Therefore β_y can be expressed as $\beta_y = \widetilde{\beta}_y + e_l$. Using part (b) of Corollary 30 we are able to introduce $\frac{\partial}{\partial y_l}$ into the first factor of each $K_{1,i} * K_{2,i}$, in the case that $p > 0$, or into the second factor in the case that $p = 0$. Then we use inductive hypothesis to conclude the proof.

    - $p > 0$:

      By part (b) of Corollary 30, we have

$$\begin{aligned}
&\sum_{i=1}^{m} c_i \, \partial^{\beta_x} \, \partial^{\beta_y} \, \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) \\
&\qquad = \sum_{i=1}^{m} c_i \, \partial^{\widetilde{\beta}_y} \, \partial^{\beta_t} \Big(\sum_{s=1}^{4} \left(K_{1,i_s} * K_{2,i_s}\right)\Big) \\
&\qquad = \sum_{r=1}^{4m} c_r \, \partial^{\widetilde{\beta}_y} \, \partial^{\beta_t} \left(K_{1,r} * K_{2,r}\right)
\end{aligned}$$

      with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1-p+1}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_1-(p-1)}(\mathbb{H}^n)$, $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2-q}(\mathbb{H}^n)$, and $|\widetilde{\beta}_y| + |\beta_t| = (p-1) + q = p + q - 1$. Therefore applying the inductive hypothesis we have

$$\sum_{i=1}^{m} c_i\, \partial^{\beta_x} \partial^{\beta_y} \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) = \sum_{s} \widetilde{c}_s \left(\widetilde{K}_{1,s} * \widetilde{K}_{2,s}\right)$$
with $\widetilde{K}_{1,s} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $\widetilde{K}_{2,s} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$.

- $\underline{p = 0}$:

  If $p = 0$, then $q \neq 0$. By part (b) of Corollary 30, in its alternative version, we have that

$$\sum_{i=1}^{m} c_i\, \partial^{\beta_x} \partial^{\beta_y} \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) = \sum_{r=1}^{4m} c_r\, \partial^{\widetilde{\beta}_y} \partial^{\beta_t} \left(K_{1,r} * K_{2,r}\right)$$
  with $\widetilde{K}_{1,r} \in \widetilde{\mathcal{M}}^{j_1-p}(\mathbb{H}^n)$, $\widetilde{K}_{2,r} \in \widetilde{\mathcal{M}}^{j_2-q+1}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_2-(q-1)}(\mathbb{H}^n)$, and $|\widetilde{\beta}_y| + |\beta_t| = p + (q-1) = p + q - 1$. Therefore by the inductive hypothesis applied to the previous line

$$\sum_{i=1}^{m} c_i\, \partial^{\beta_x} \partial^{\beta_y} \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) = \sum_{s} \widetilde{c}_s \left(\widetilde{K}_{1,s} * \widetilde{K}_{2,s}\right)$$
  with $\widetilde{K}_{1,s} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $\widetilde{K}_{2,s} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$, as desired.

- $\underline{|\beta_y| = 0}$:

  If $|\beta_x| = |\beta_y| = 0$, then $p + q = |\beta_t|$ which is greater or equal than 2. We distinguish the following two possibilities for p:

  - $\underline{p \geqslant 2}$:

    Since $|\beta_t| \neq 0$, then there exists some $\frac{\partial}{\partial t}$ in ∂^{β_t}. Therefore β_t can be expressed as $\beta_t = \widetilde{\beta}_t + e_{2n+1}$. Using then by part (d) of Proposition 28 we are able to introduce $\frac{\partial}{\partial t}$ into the first factor of each $K_{1,i} * K_{2,i}$ to obtain

$$\begin{aligned}\sum_{i=1}^{m} c_i\, \partial^{\beta_x} \partial^{\beta_y} \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) &= \sum_{i=1}^{m} c_i\, \partial^{\widetilde{\beta}_t} \frac{\partial}{\partial t}\left(K_{1,i} * K_{2,i}\right) \\ &= \sum_{i=1}^{m} c_i\, \partial^{\widetilde{\beta}_t} \left(\widetilde{K}_{1,i} * K_{2,i}\right)\end{aligned}$$
    with $\widetilde{K}_{1,i} \in \widetilde{\mathcal{M}}^{j_1-p+2}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_1-(p-2)}(\mathbb{H}^n)$, $K_{2,i} \in \widetilde{\mathcal{M}}^{j_2-q}(\mathbb{H}^n)$, and

$|\widetilde{\beta}_t| = (p-2) + q = p + q - 2$. Therefore by the inductive hypothesis

$$\sum_{i=1}^{m} c_i\, \partial^{\beta_x}\, \partial^{\beta_y}\, \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) = \sum_{s} \widetilde{c}_s \big(\widetilde{\widetilde{K}}_{1,s} * \widetilde{\widetilde{K}}_{2,s}\big)$$

with $\widetilde{\widetilde{K}}_{1,s} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $\widetilde{\widetilde{K}}_{2,s} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$, as desired.

- $\underline{p < 2}$:

  We discriminate the following two cases

  - $\underline{q \geqslant 2}$: Since β_t can be expressed as $\beta_t = \widetilde{\beta}_t + e_{2n+1}$ then by part (d) of Proposition 28, we are able to introduce $\frac{\partial}{\partial t}$ into the second factor of each $K_{1,i} * K_{2,i}$ and obtain

    $$\sum_{i=1}^{m} c_i\, \partial^{\beta_x}\, \partial^{\beta_y}\, \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) = \sum_{i=1}^{m} c_i\, \partial^{\widetilde{\beta}_y} \big(\widetilde{\widetilde{K}}_{1,i} * \widetilde{\widetilde{K}}_{2,i}\big)$$

    with $K_{1,i} \in \widetilde{\mathcal{M}}^{j_1 - p}(\mathbb{H}^n)$, $\widetilde{K}_{2,i} \in \widetilde{\mathcal{M}}^{j_2 - q + 2}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_2 - (q-2)}(\mathbb{H}^n)$, and $|\widetilde{\beta}_t| = p + q - 2$. Therefore by the inductive hypothesis

    $$\sum_{i=1}^{m} c_i\, \partial^{\beta_x}\, \partial^{\beta_y}\, \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) = \sum_{s} \widetilde{c}_s \big(\widetilde{\widetilde{K}}_{1,s} * \widetilde{\widetilde{K}}_{2,s}\big)$$

    with $\widetilde{\widetilde{K}}_{1,s} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $\widetilde{\widetilde{K}}_{2,s} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$, as desired.

  - $\underline{q < 2}$: Given that $p + q = |\beta_t| \geqslant 2$, and p and q are less than 2, therefore $p = q = 1$. Since $\beta_t = \widetilde{\beta}_t + e_{2n+1}$ then by Remark 29 we are able to introduce the $\frac{\partial}{\partial t}$ into both factors of each $K_{1,i} * K_{2,i}$

    $$\begin{aligned}
    \sum_{i=1}^{m} c_i\, \partial^{\beta_x}\, \partial^{\beta_y}\, \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) &= \sum_{i=1}^{m} c_i\, \partial^{\widetilde{\beta}_t} \frac{\partial}{\partial t} \left(K_{1,i} * K_{2,i}\right) \\
    &= \sum_{i=1}^{m} c_i\, \partial^{\widetilde{\beta}_t} \sum_{s=1}^{2} \left(K_{1,i_s} * K_{2,i_s}\right) \\
    &= \sum_{r=1}^{2m} c_r\, \partial^{\widetilde{\beta}_t} \big(\widetilde{K}_{1,r} * K_{2,r}\big)
    \end{aligned}$$

    with $\widetilde{K}_{1,r} \in \widetilde{\mathcal{M}}^{j_1 - p + 1}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_1 - (p-1)}(\mathbb{H}^n)$,

$K_{2,r} \in \widetilde{\mathcal{M}}^{j_2-q+1}(\mathbb{H}^n) = \widetilde{\mathcal{M}}^{j_2-(q-1)}(\mathbb{H}^n)$,

and $|\widetilde{\beta}_t| = (p-1) + (q-1) = p+q-2$. Therefore by the inductive hypothesis

$$\sum_{i=1}^{m} c_i\, \partial^{\beta_x}\, \partial^{\beta_y}\, \partial^{\beta_t} \left(K_{1,i} * K_{2,i}\right) = \sum_{s} \widetilde{c}_s \left(\widetilde{K}_{1,s} * \widetilde{K}_{2,s}\right)$$

with $\widetilde{K}_{1,s} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $\widetilde{K}_{2,s} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$, as desired.

This concludes the proof of the proposition. □

Definition 34. We define the following spaces

$$\mathcal{P}^k(G) := \left\{ p \,:\, p \text{ homogeneous polynomial of degree } k \text{ on } G \right\}$$

A homogeneous distribution on G is said to be *regular* if it is smooth away from zero.

$$\mathfrak{Rhom}_k(G) := \left\{ K \,:\, K \text{ regular homogeneous distribution of degree } k \text{ on } G \right\}$$

$$\mathcal{K}^k(G) := \begin{cases} \mathfrak{Rhom}_k(G) & \text{if } k \notin \mathbb{Z}^+ \\ \left\{ K + p(x)\log|x| \,:\, K \in \mathfrak{Rhom}_k(G), \text{ and } p \in \mathcal{P}^k \right\} & \text{if } k \in \mathbb{Z}^+ \end{cases}$$

$$\mathcal{S}_o(G) := \left\{ f \in \mathcal{S}(G) \,:\, \int_G x^\alpha f(x)\, dx = 0 \text{ for all monomials } x^\alpha \right\}$$

The following three propositions are from [CGGP 92] and their proofs will be omitted.

Proposition 35. *If $K \in \mathcal{K}^k(G)$ then there exists an operator*

$$\begin{array}{rccc} \mathcal{O}_o(K) : & \mathcal{S}_o(G) & \longrightarrow & \mathcal{S}_o(G) \\ & f & \longmapsto & f * K \end{array}$$

Proposition 36. *If $K_i \in \mathcal{K}^{k_i}(G)$ then*

$$\mathcal{O}_o(K_1)\mathcal{O}_o(K_2) = \mathcal{O}_o(K)$$

for some $K \in \mathcal{K}^{k_1+k_2+Q}$. This K is uniquely determined modulo $\mathcal{P}^{k_1+k_2+Q}(G)$, and it will be denoted by $K_1 \underline{*} K_2$.

Proposition 37 (Christ-Geller-Glowacki-Polin). *Say $\varphi_i \in C_c^\infty(G)$, with $\varphi_i = 1$ near zero, for $i = 1, 2$. Then we may convolve the compactly supported distributions $\varphi_1 K_1$ and $\varphi_2 K_2$, and we have*

$$K_1 \underline{*} K_2 - \varphi_1 K_1 * \varphi_2 K_2 \text{ is smooth on } G.$$

Lemma 38. *Assume that $K \in \mathcal{K}^k(G)$*

a) If $-Q < k \leqslant 0$, then there exists a constant $C > 0$ such that

$$\left|K(x)\right| \leqslant CB_k(x)\,.$$

b) If $k > 0$, and $\varphi \in C_c^\infty(G)$ such that $\varphi = 1$ in a neighborhood of 0, then there exists a constant $C > 0$ such that

$$\left|(\varphi K)(x)\right| \leqslant CB_k(x)\,.$$

Proof.

a) $\underline{-Q < k < 0}$

If $-Q < k < 0$, then $K \in \mathfrak{Rhom}_k(G)$. Since $K \in C^\infty(G - \{0\})$, on the homogeneous unit sphere we have $\left|K(x)\right| \leqslant C$, with $C = \max_{|x|=1} \{|K(x)|\}$. K is homogeneous of degree k, then for all $x \neq 0$

$$\left|K(x)\right| = \left|K\left(|x|\frac{x}{|x|}\right)\right| = |x|^k \left|K\left(\frac{x}{|x|}\right)\right| \leqslant C|x|^k = CB_k(x).$$

$\underline{k = 0}$

$K \in \mathcal{K}^o(G)$ then $K(x) = K_1(x) + p(x)\log|x|$, with $K_1 \in \mathfrak{Rhom}_o(G)$ and $p \in \mathcal{P}^o(G)$. Hence there exist constants C_1 and c such that $|K_1(x)| \leqslant C_1$, and $p(x) = c$. Therefore for all $x \neq 0$

$$\begin{aligned}|K(x)| &= \big|K_1(x) + c\log|x|\big| \\ &\leqslant C_1 + |c|\big|\log|x|\big| \\ &\leqslant C\big(1 + \big|\log|x|\big|\big) \\ &= CB_o(x)\end{aligned}$$

b) $\underline{0 < k \notin \mathbb{Z}^+}$

If $0 < k \notin \mathbb{Z}^+$ then $K \in \mathfrak{Rhom}_k(G)$. Thus there exists a constant C_1, such that $|K(x)| \leqslant C_1|x|^k$. Therefore for all $x \in \operatorname{supp}(\varphi K)$

$$\big|(\varphi K)(x)\big| \leqslant C_2C_1|x|^k \leqslant C = CB_k(x).$$

where the last inequality is valid since we are working on a compact set.

$\underline{0 < k \in \mathbb{Z}^+}$

If $0 < k \in \mathbb{Z}^+$ then $K = K_1(x) + p(x)\log|x|$, with $K_1 \in \mathfrak{Rhom}_k(G)$ and $p \in \mathcal{P}^k(G)$. Therefore there exist constants C_1, C_2 and C_3 such that $|K_1(x)| \leqslant C_1|x|^k$, $|p(x)| \leqslant C_2|x|^k$, and $|\log|x|| \leqslant C_3|x|^{-k/2}$ on $B(0;1)$. Then for $x \in \operatorname{supp}(\varphi K)$

$$\begin{aligned}\big|(\varphi K)(x)\big| &\leqslant \big|(\varphi K_1)(x)\big| + \big|(\varphi p)(x)\log|x|\big| \\ &\leqslant C_4|x|^k + C_5|x|^{k-k/2} \\ &\leqslant C \\ &= CB_k(x)\end{aligned}$$

$\square$

Corollary 39. *Suppose $-Q < k_1, k_2$ and $\varphi_1, \varphi_2 \in C_c^\infty(G)$, with $\varphi_i = 1$ near the origin, then there exists a constant $C > 0$ such that*

$$\left|\varphi_1 B_{k_1}(u) * \varphi_2 B_{k_2}(u)\right| \leqslant C B_{k_1 + k_2 + Q}(u)$$

Proof. Two cases will be considered separately.

- $-Q < k_1, k_2 < 0$:

  Since $k_1, k_2 < 0$ then $B_{k_1}(u) = |u|^{k_1}$, and $B_{k_2}(u) = |u|^{k_2}$. Therefore $B_{k_1} \in \mathcal{K}^{k_1}(G)$, and $B_{k_2} \in \mathcal{K}^{k_2}(G)$. By Proposition 37 there exists $\psi \in C^\infty(G)$ such that $\varphi_1 B_{k_1} * \varphi_2 B_{k_2} = K_1 \underline{*} K_2 + \psi$. Since $K_1 \underline{*} K_2 \in \mathcal{K}^{k_1+k_2+Q}(G)$, and since the convolution of the compactly supported distributions $\varphi_i B_{k_i}$ is also a compactly supported distribution, then by Lemma 38 there exists a constant C_1 such that for all $u \in \operatorname{supp}\left(\varphi_1 B_{k_1} * \varphi_2 B_{k_2}\right)$

  $$\left|(K_1 \underline{*} K_2)(u)\right| \leqslant C_1 B_{k_1+k_2+Q}(u)$$

  Therefore

  $$\begin{aligned} \left|(\varphi_1 B_{k_1})(u) * (\varphi_2 B_{k_2})(u)\right| &= \left|(K_1 \underline{*} K_2)(u) + \psi(u)\right| \\ &\leqslant C_1 B_{k_1+k_2+Q}(u) + C_2 \\ &\leqslant C B_{k_1+k_2+Q}(u) \end{aligned}$$

  In the last inequality we have made use of the fact that we are working on a compact set.

- At least one $k_i \geqslant 0$:

  Assume that $k_1 \geqslant 0$. Then since $k_2 > -Q$, $k_1 + k_2 + Q > 0$, and we only need show that $\varphi_1 B_{k_1} * \varphi_2 B_{k_2}$ is bounded. All the possibilities to be considered will be handled using the following strategy: $\varphi_1 B_{k_1} * \varphi_2 B_{k_2}$ is a convolution in $L^p(G) * L^q(G)$, with p, q *conjugate*, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, therefore it is in $L^\infty(G)$, and hence bounded.

► $\underline{k_1 > 0,\ -Q < k_2}$:

$k_1 > 0$ then $\varphi_1 B_{k_1} = C_1\varphi_1 \in L^\infty(G)$.

$$\varphi_2 B_{k_2}(u) = \begin{cases} \varphi_2 |u|^{k_2} & \text{if } -Q < k_2 < 0 \\ \varphi_2\big(\big|\log|u|\big| + 1\big) & \text{if } k_2 = 0 \\ C_2\varphi_2 & \text{if } 0 < k_2 \end{cases}$$

All of the possibilities above are in $L^1(G)$. For the second one, one only needs observe that near zero $\big(\big|\log|u|\big|+1\big) \leqslant C|u|^{-1/2}$, which is in $L^1(G)$ when multiplied by φ_2. Therefore $\varphi_1 B_{k_1} * \varphi_2 B_{k_2} \in L^\infty(G) * L^1(G) \subset L^\infty(G)$.

► $\underline{k_1 = k_2 = 0}$:

$\varphi_i B_{k_i} = \varphi_i\big(\big|\log|u|\big| + 1\big)$. We choose any $\varepsilon > 0$ such that $Q - \varepsilon > 0$. Then $\big(\big|\log|u|\big| + 1\big) \leqslant C|u|^{-\left(\frac{Q-\varepsilon}{2}\right)}$ on $B(0;1)$. Thus $\big(\big|\log|u|\big| + 1\big)^2 \leqslant C^2|u|^{-(Q-\varepsilon)}$, near zero, and hence the product with φ_2 belongs to $L^1(G)$, and so $\varphi_i B_{k_i} \in L^2(G)$. Therefore $\varphi_1 B_{k_1} * \varphi_2 B_{k_2} \in L^2(G) * L^2(G) \subset L^\infty(G)$.

► $\underline{k_1 = 0,\ -Q < k_2 < 0}$:

$\varphi_2 B_{k_2} = \varphi_2|u|^{k_2}$. We choose any $\varepsilon > 0$ such that $q = \frac{Q}{-k_2} - \varepsilon > 1$, then $qk_2 > -Q$. Consequently $\varphi_2 B_{k_2} \in L^q(G)$. Since $k_1 = 0$ then $\varphi_1 B_{k_1} = \varphi_1\big(\big|\log|u|\big| + 1\big)$. Let p be the conjugate of q, thus $\big(\big|\log|u|\big| + 1\big) \leqslant C|u|^{-\frac{1}{2p}}$ on $B(0;1)$, and so $\varphi_1 B_{k_1} \in L^p(G)$. Therefore $\varphi_1 B_{k_1} * \varphi_2 B_{k_2} \in L^p(G) * L^q(G) \subset L^\infty(G)$.

□

Remark 40. The following statements are special cases of the previous corollary

I. $\underline{k_1, k_2 \text{ as well as } k_1 + k_2 + Q, \text{ negative}}$:

If $\varphi_1, \varphi_2 \in C_c^\infty(G)$ such that $\varphi_i = 1$ near the origin, and $-Q < k_1, k_2 < 0$, with

$k_1 + k_2 + Q < 0$, then there exist some constant C such that

$$\left|\varphi_1 |u|^{k_1} * \varphi_2 |u|^{k_2}\right| \leqslant C |u|^{k_1+k_2+Q}$$

II. k_1, k_2 negative and $k_1 + k_2 + Q = 0$:

If $\varphi_1, \varphi_2 \in C_c^\infty(G)$ such that $\varphi_i = 1$ near the origin, and $-Q < k_1, k_2 < 0$, with $k_1 + k_2 + Q = 0$, then there exist some constant C such that

$$\left|\varphi_1 |u|^{k_1} * \varphi_2 |u|^{k_2}\right| \leqslant C \left(\left|\log |u|\right| + 1\right)$$

III. k_1, k_2 negative and $k_1 + k_2 + Q$ positive:

If $\varphi_1, \varphi_2 \in C_c^\infty(G)$ such that $\varphi_i = 1$ near the origin, and $-Q < k_1, k_2 < 0$, with $k_1 + k_2 + Q > 0$, then there exist some constant C such that

$$\left|\varphi_1 |u|^{k_1} * \varphi_2 |u|^{k_2}\right| \leqslant C$$

IV. k_o negative, k_1 positive:

If $\varphi_1, \varphi_2 \in C_c^\infty(G)$ such that $\varphi_i = 1$ near the origin, $-Q < k_o < 0$, and $k_1 > 0$, then there exists a constant C such that

$$\left|\varphi_1 \left(\left|\log |u|\right| + 1\right) * \varphi_2 |u|^{k_o}\right| \leqslant C$$

IV.2 MAIN THEOREM

Theorem 41. *For $j_1, j_2 < 0$ the convolution of an element of $\widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$ with an element of $\widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$ belongs to $\widetilde{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$.*

Proof. Given that the tempered distributions K_i are in $\widetilde{\mathcal{M}}^{j_i}(\mathbb{H}^n)$, then by part (b) of Theorems 19, and 22 each K_i is in $L^1(\mathbb{H}^n)$, and therefore $K_1 * K_2$ is in $L^1(\mathbb{H}^n)$.

In view of part (c) of Theorems 19 and 22, it suffices to show that for every multiindex β, there exists a positive constant C_β such that for all $u \neq 0$.

$$\left|\partial^\beta (K_1 * K_2)(u)\right| \leqslant C_\beta B_{k_1+k_2+Q-|\beta|}(u) \tag{IV.1}$$

where $B_{k_1+k_2+Q-|\beta|}$ is the bad power function of order $k_1 + k_2 + Q - |\beta|$. Recall that the k_i were defined as $k_i = -Q - j_i$.

First we observe that it suffices to verify (IV.1) for the case of K_1 and K_2 compactly supported. Choosing $\varphi_i \in C_c^\infty(\mathbb{H}^n)$ such that $\varphi_i = 1$ in a neighborhood of zero, we can write

$$K_i = \varphi_i K_i + (1 - \varphi_i) K_i \qquad i = 1, 2.$$

Therefore

$$\begin{aligned} K_1 * K_2 = \varphi_1 K_1 * \varphi_2 K_2 + [(1 - \varphi_1) K_1] * \varphi_2 K_2 \\ + \varphi_1 K_1 * [(1 - \varphi_2) K_2] + [(1 - \varphi_1) K_1] * [(1 - \varphi_2) K_2] \end{aligned} \tag{IV.2}$$

In view of the fact that the convolution of elements of $\mathcal{S}$ is again in $\mathcal{S}$ (i.e. $\mathcal{S} * \mathcal{S} \subseteq \mathcal{S}$), it can be claimed that $[(1 - \varphi_1) K_1] * [(1 - \varphi_2) K_2]$ is in $\mathcal{S}$. Also, in view of the fact that the convolution of a compactly supported distribution with elements of $\mathcal{S}$ belongs to $\mathcal{S}$, $(\mathcal{S} * \mathcal{E}' \subseteq \mathcal{S})$, it can be claimed that $\varphi_1 K_1 * [(1 - \varphi_2) K_2]$ and $[(1 - \varphi_1) K_1] * \varphi_2 K_2$ are in $\mathcal{S}$. Then it is clear that the last three terms above belong to $\mathcal{S}$, and hence satisfy estimate (IV.1). Hence we have shown that the verification of (IV.1)can be reduced to the case of compactly supported K_i.

Henceforth we can assume without loss of generality that K_1 and K_2 have compact support.

Notice that since we may choose φ_1 and φ_2 to have compact support in an arbitrarily small neighborhood of the origin, and that the last three terms of (IV.2)belong to $\mathcal{S}(\mathbb{H}^n)$, hence $K_1 * K_2$ is smooth away from zero.

We consider the following possibilities for k_1, and k_2.

- CASE I — $k_1, k_2 < 0,\ k_1 + k_2 + Q < 0$: Let β be multiindex with β_x, β_y, and β_t its subordinated decomposition. We observe that for some positive constant C

$$|u|^{|\beta|} = |(x,y,t)|^{|\beta|} \leqslant C\left[\sum_{i=1}^{n}|x_i|^{|\beta|} + \sum_{i=1}^{n}|y_i|^{|\beta|} + |t|^{|\beta|/2}\right] \tag{IV.3}$$

In what follows we shall work on obtaining estimates for

$|x_i|^{|\beta|}\left|\partial^\beta(K_1 * K_2)(u)\right|$, $|y_i|^{|\beta|}\left|\partial^\beta(K_1 * K_2)(u)\right|$, and $|t|^{|\beta|/2}\left|\partial^\beta(K_1 * K_2)(u)\right|$

discriminating cases according to the parity of $|\beta|$.

▶ $|\beta|$ even: Denoting by e_i the i–th element of the canonical base of $\mathbb{R}^{2n+1}$, and using the dual Leibniz rule, we have:

$$\begin{aligned}
|x_i|^{|\beta|}\left|\partial^\beta(K_1 * K_2)\right| &= \left|x_i^{|\beta|}\partial^{\beta_x}\partial^{\beta_y}\partial^{\beta_t}(K_1 * K_2)\right| \\
&= \left|\partial^{\beta_y}\partial^{\beta_t}\sum_{\substack{\gamma+ae_i=\beta_x\\ a\in\mathbb{Z}^+}}(-1)^{|ae_i|}\binom{\beta_x}{ae_i}\partial^\gamma\left[\left(\partial^{ae_i}x_i^{|\beta|}\right)(K_1 * K_2)\right]\right| \\
&= \left|\partial^{\beta_y}\partial^{\beta_t}\sum_{\substack{\gamma+ae_i=\beta_x\\ a\in\mathbb{Z}^+}}(-1)^{a}\binom{(\beta_x)_i}{a}c_a\,\partial^\gamma\left[x_i^{|\beta|-a}(K_1 * K_2)\right]\right| \\
&= \left|\partial^{\beta_y}\partial^{\beta_t}\sum_{\substack{\gamma+ae_i=\beta_x\\ a\in\mathbb{Z}^+}}(-1)^{a}\tbinom{(\beta_x)_i}{a}c_a\,\partial^\gamma\left[\sum_{p+q=|\beta|-a}\tbinom{|\beta|-a}{p}\left(x_i^pK_1 * x_i^qK_2\right)\right]\right| \\
&\leqslant C_i'\sum_{\substack{\gamma+ae_i=\beta_x\\ a\in\mathbb{Z}^+}}\ \sum_{p+q=|\gamma|+|\beta_y|+|\beta_t|}\left|\partial^\gamma\partial^{\beta_y}\partial^{\beta_t}\left(x_i^pK_1 * x_i^qK_2\right)\right|
\end{aligned}$$

where

- $|\beta_x| = |\gamma + ae_i| = |\gamma| + a$,
- $p + q = |\beta| - a = (|\beta_x| + |\beta_y| + |\beta_t|) - a = |\gamma| + |\beta_y| + |\beta_t|$,

$\cdot \; x_i^p K_1 \in \widetilde{\mathcal{M}}^{j_1-p}(\mathbb{H}^n), \quad \text{and} \quad x_i^q K_2 \in \widetilde{\mathcal{M}}^{j_2-q}(\mathbb{H}^n).$

By Proposition 33, from the previous inequality we are able to conclude that for some $K_{i,s} \in \widetilde{\mathcal{M}}^{j_i}(\mathbb{H}^n)$

$$|x_i|^{|\beta|}\left|\partial^\beta(K_1 * K_2)(u)\right| \leqslant \sum_s c_s\left|\left(K_{1,s} * K_{2,s}\right)(u)\right|$$

$$\leqslant C_{1_i}|u|^{k_1+k_2+Q} \quad \forall u \neq 0$$

Remark 40.(I) has been used in the last inequality.

Now setting $C_1 = \max\limits_{i=1,\dots,n}\{C_{1_i}\}$ we have that for any $i = 1,\dots,n$

$$|x_i|^{|\beta|}\left|\partial^\beta\big(K_1 * K_2\big)(u)\right| \leqslant C_1|u|^{k_1+k_2+Q} \quad \forall u \neq 0, \tag{IV.4}$$

The analogous estimate involving y_i can be established with identical proof and will be omitted. Namely setting $C_2 = \max\limits_{i=1,\dots,n}\{C_{2_i}\}$ we have that for any $i = 1,\dots,n$

$$|y_i|^{|\beta|}\left|\partial^\beta\big(K_1 * K_2\big)(u)\right| \leqslant C_2|u|^{k_1+k_2+Q}. \quad \forall u \neq 0,\ i = 1,\dots,n \tag{IV.5}$$

Now we prove the third estimate.

$$
\begin{aligned}
|t|^{|\beta|/2}\left|\partial^{\beta}\left(K_1 * K_2\right)\right| &= \left|t^{|\beta|/2}\,\partial^{\beta_x}\,\partial^{\beta_y}\,\partial^{\beta_t}\left(K_1 * K_2\right)\right| \\
&= \left|\partial^{\beta_x}\,\partial^{\beta_y}\sum_{\substack{\gamma+ae_{2n+1}=\beta_t\\ a\in\mathbb{Z}^+}}(-1)^{|ae_{2n+1}|}\binom{\beta_t}{a\,e_{2n+1}}\partial^{\gamma}\left[\partial^{ae_{2n+1}}\left(t^{|\beta|/2}\right)\left(K_1 * K_2\right)\right]\right| \\
&= \left|\partial^{\beta_x}\,\partial^{\beta_y}\sum_{\substack{\gamma+ae_{2n+1}=\beta_t\\ a\in\mathbb{Z}^+}}(-1)^{a}\binom{\beta_{2n+1}}{a}c_a\,\partial^{\gamma}\left[t^{\frac{|\beta|}{2}-a}\left(K_1 * K_2\right)\right]\right| \\
&\leqslant C'\sum_{\substack{\gamma+ae_{2n+1}=\beta_t\\ a\in\mathbb{Z}^+}}\left|\partial^{\beta_x}\,\partial^{\beta_y}\,\partial^{\gamma}\left[t^{\frac{|\beta|}{2}-a}\left(K_1 * K_2\right)\right]\right| \\
&= C'\sum_{\substack{\gamma+ae_{2n+1}=\beta_t\\ a\in\mathbb{Z}^+}}\left|\partial^{\beta_x}\,\partial^{\beta_y}\,\partial^{\gamma}\sum_{m}c_m\left[\left(P_mK_1\right) * \left(Q_mK_2\right)\right]\right|
\end{aligned}
$$

where

$P_mK_1 \in \widetilde{\mathcal{M}}^{j_1-p}(\mathbb{H}^n)$, and $Q_mK_2 \in \widetilde{\mathcal{M}}^{j_2-q}(\mathbb{H}^n)$, with

P_m and Q_m homogeneous polynomials in x, y, and t, such that the sum of their homogeneous degrees equals

$$2\left(\tfrac{|\beta|}{2}-a\right) = |\beta_x| + |\beta_y| + |\gamma| = p + q.$$

By Proposition 33 for some $K_{1,s} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $K_{2,s} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$, we have

$$
\begin{aligned}
|t|^{|\beta|/2}\left|\partial^{\beta}\left(K_1 * K_2\right)(u)\right| &\leqslant \sum_{c_s} c_s\left|\left(K_{1,s} * K_{2,s}\right)(u)\right| \\
&\leqslant C_3|u|^{k_1+k_2+Q} \qquad u \neq 0
\end{aligned}
$$

Remark 40 (I) was used in the last inequality.

Therefore we have shown that for some positive constant C_3

$$|t|^{|\beta|/2}\left|\partial^{\beta}\left(K_1 * K_2\right)(u)\right| \leqslant C_3|u|^{k_1+k_2+Q}. \tag{IV.6}$$

Using the three estimates which we have just proved, namely (IV.4), (IV.5) and (IV.6), we have

$$
\begin{aligned}
&|u|^{|\beta|}\left|\partial^\beta\big(K_1 * K_2\big)(u)\right| \\
&\quad\leqslant C\left(\sum_{i=1}^{n}|x_i|^{|\beta|}+\sum_{i=1}^{n}|y_i|^{|\beta|}+|t|^{|\beta|/2}\right)\left|\partial^\beta\big(K_1 * K_2\big)(u)\right| \\
&\quad\leqslant C\left(nC_1|u|^{k_1+k_2+Q}+nC_2|u|^{k_1+k_2+Q}+C_3|u|^{k_1+k_2+Q}\right) \\
&\quad\leqslant C_\beta|u|^{k_1+k_2+Q} \qquad \forall u\neq 0
\end{aligned}
\tag{IV.7}
$$

Therefore we can conclude that for every multiindex β, with $|\beta|$ even, there exists a positive constant C_β such that for all $u \neq 0$

$$
\left|\partial^\beta\big(K_1 * K_2\big)(u)\right| \leqslant C_\beta|u|^{k_1+k_2+Q-|\beta|}
$$

- $|\beta|$ odd: We shall work with $|\beta| \pm 1$ and then we shall proceed as in the case when $|\beta|$ was an even number.

$$
\begin{aligned}
&|x_i|^{|\beta|\pm 1}\left|\partial^\beta\big(K_1 * K_2\big)\right| \\
&\quad= \left|x_i^{|\beta|\pm 1}\,\partial^\beta(K_1 * K_2)\right| \\
&\quad= \left|\partial^{\beta_y}\,\partial^{\beta_t}\sum_{\substack{\gamma+ae_i=\beta_x\\ a\in\mathbb{Z}^+}}(-1)^a\binom{(\beta_x)_i}{a}c_a\,\partial^\gamma\left(\sum_{\substack{p+q=\\ |\beta|\pm 1-a}}\binom{|\beta|\pm 1-a}{p}\big(x_i^pK_1 * x_i^qK_2\big)\right)\right| \\
&\quad\leqslant C_i'\sum_{\substack{\gamma+ae_i=\beta_x\\ a\in\mathbb{Z}^+}}\ \sum_{\substack{p+q=\\ |\gamma|+|\beta_y|+|\beta_t|\pm 1}}\left|\partial^\gamma\,\partial^{\beta_y}\,\partial^{\beta_t}\left(\widetilde{K}_{1,p} * \widetilde{K}_{2,q}\right)\right|
\end{aligned}
$$

where $\widetilde{K}_{1,p} := x_i^pK_1$ and $\widetilde{K}_{2,q} := x_i^qK_2$.

Hence we may regard

$$\widetilde{K}_{1,p} \in \widetilde{\mathcal{M}}^{(j_1 \mp 1)-(p \mp 1)}(\mathbb{H}^n), \quad \text{and} \quad \widetilde{K}_{2,q} \in \widetilde{\mathcal{M}}^{j_2 - q}(\mathbb{H}^n),$$

or alternatively

$$\widetilde{K}_{1,p} \in \widetilde{\mathcal{M}}^{j_1 - p}(\mathbb{H}^n) \quad \text{and} \quad \widetilde{K}_{2,q} \in \widetilde{\mathcal{M}}^{(j_2 \mp 1)-(q \mp 1)}(\mathbb{H}^n)$$

with

$$(p \mp 1) + q = p + (q \mp 1) = |\gamma| + |\beta_y| + |\beta_t|$$

By Proposition 33, from the last inequality we can conclude that for some

$$K_{1,s} \in \widetilde{\mathcal{M}}^{j_1 \mp 1}(\mathbb{H}^n) \ \text{ and } \ K_{2,s} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n),$$

or alternatively for some

$$K_{1,s} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n) \ \text{ and } \ K_{2,s} \in \widetilde{\mathcal{M}}^{j_2 \mp 1}(\mathbb{H}^n),$$

we have

$$\begin{aligned} |x_i|^{|\beta| \pm 1} \left|\partial^\beta (K_1 * K_2)(u)\right| &\leqslant \sum_s c_s \left|\left(K_{1,s} * K_{2,s}\right)(u)\right| \\ &\leqslant C_{1_i} |u|^{k_1 + k_2 + Q \pm 1} \qquad \forall u \neq 0 \end{aligned}$$

where Remark 40.(I) was used to establish the last inequality.

Therefore we have shown that for some positive constant $C_1 = \max\limits_{i=1,\dots,n} \{C_{1_i}\}$

$$|x_i|^{|\beta| \pm 1} \left|\partial^\beta (K_1 * K_2)(u)\right| \leqslant C_1 |u|^{k_1 + k_2 + Q \pm 1} \tag{IV.8}$$

for all $u \neq 0$, and $i = 1, \dots, n$. Similarly it follows that for some positive constant $C_2 = \max\limits_{i=1,\dots,n} \{C_{2_i}\}$

$$|y_i|^{|\beta| \pm 1} \left|\partial^\beta (K_1 * K_2)(u)\right| \leqslant C_2 |u|^{k_1 + k_2 + Q \pm 1} \tag{IV.9}$$

for all $u \neq 0$, and $i = 1, \ldots, n$.

Now we compute the third estimate.

$$
\begin{aligned}
&|t|^{(|\beta|\pm 1)/2} \left|\partial^\beta \big(K_1 * K_2\big)\right| \\
&\qquad = \left| t^{\frac{|\beta|\pm 1}{2}}\, \partial^{\beta_x}\, \partial^{\beta_y}\, \partial^{\beta_t} \big(K_1 * K_2\big) \right| \\
&\qquad \leqslant C' \sum_{\substack{\gamma + a e_{2n+1} = \beta_t \\ a \in \mathbb{Z}^+}} \left| \partial^{\beta_x}\, \partial^{\beta_y}\, \partial^{\gamma} \left[t^{\left[\frac{|\beta|\pm 1}{2} - a\right]} \big(K_1 * K_2\big) \right] \right| \\
&\qquad = C' \sum_{\substack{\gamma + a e_{2n+1} = \beta_t \\ a \in \mathbb{Z}^+}} \left| \partial^{\beta_x}\, \partial^{\beta_y}\, \partial^{\gamma} \sum_m c_m \left[\big(P_m K_1\big) * \big(Q_m K_2\big) \right] \right|
\end{aligned}
$$

where P_m, Q_m are homogeneous polynomials in x, y and t such that the sum of their homogeneous degrees equals $2\left[\frac{|\beta|\pm 1}{2} - a\right] = |\beta_x| + |\beta_y| + |\gamma| \pm 1$. Hence we may regard

- either

  $P_m K_1 \in \widetilde{\mathcal{M}}^{(j_1 \mp 1) - (p \mp 1)}(\mathbb{H}^n)$ and $Q_m K_2 \in \widetilde{\mathcal{M}}^{j_2 - q}(\mathbb{H}^n)$,

- or alternatively

  $P_m K_1 \in \widetilde{\mathcal{M}}^{j_1 - p}(\mathbb{H}^n)$ and $Q_m K_2 \in \widetilde{\mathcal{M}}^{(j_2 \mp 1) - (q \mp 1)}(\mathbb{H}^n)$

with $(p \mp 1) + q = p + (q \mp 1) = |\beta_x| + |\beta_y| + |\gamma| = |\beta| - 2a$

By Proposition 33 on page 62, on page 62, for some

$$K_{1,s} \in \widetilde{\mathcal{M}}^{j_1 \mp 1}(\mathbb{H}^n) \text{ and } K_{2,s} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$$

or for some

$$K_{1,s} \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n) \text{ and } K_{2,s} \in \widetilde{\mathcal{M}}^{j_2 \mp 1}(\mathbb{H}^n)$$

we have

$$
\begin{aligned}
\left|\partial^\beta \big(K_1 * K_2\big)(u)\right| &\leqslant \sum_s c_s \left| \big(K_{1,s} * K_{2,s}(u)\right| \\
&\leqslant C_3 |u|^{k_1 + k_2 + Q \pm 1} \qquad \forall u \neq 0
\end{aligned}
$$

Remark 40.(I) was used in the last inequality.

Therefore we have shown that for some positive constant C_3

$$\left|t\right|^{(|\beta|\pm1)/2}\left|\partial^{\beta}\left(K_1 * K_2\right)(u)\right| \leqslant C_3\left|u\right|^{k_1+k_2+Q\pm1} \quad \forall u \neq 0 \qquad \text{(IV.10)}$$

Using the estimates which we have just proved, (IV.8), (IV.9), and (IV.10), we have

$$\begin{aligned}
&\left|u\right|^{|\beta|\pm1}\left|\partial^{\beta}\left(K_1 * K_2\right)(u)\right| \\
&\leqslant C\left(\sum_{i=1}^{n}|x_i|^{(|\beta|\pm1)}+\sum_{i=1}^{n}|y_i|^{(|\beta|\pm1)}+|t|^{\frac{(|\beta|\pm1)}{2}}\right)\left|\partial^{\beta}\left(K_1 * K_2\right)(u)\right| \\
&\leqslant C\left(nC_1\left|u\right|^{k_1+k_2+Q\pm1}+nC_2\left|u\right|^{k_1+k_2+Q\pm1}+nC_3\left|u\right|^{k_1+k_2+Q\pm1}\right) \\
&\leqslant C_{\beta}\left|u\right|^{k_1+k_2+Q\pm1}
\end{aligned}$$

Hence we can conclude that for all $u \neq 0$

$$\left|u\right|^{|\beta|}\left|\partial^{\beta}(K_1 * K_2)(u)\right| \leqslant C_{\beta}\left|u\right|^{k_1+k_2+Q}$$

Note:

We observe that in the proof of the case $|\beta|$ odd, our use of Remark 40.(I) is valid. When working with $|\beta| \pm 1$ the hypothesis of that remark are satisfied when

- either $j_1 \mp 1 < 0,\ j_2 < 0$, and $[(j_1 \mp 1) + j_2] > -Q$,
- or when $j_1 < 0,\ (j_2 \mp 1) < 0$, and $[j_1 + (j_2 \mp 1)] > -Q$.

Since j_1 and j_2 are negative then $j_i < 1,\ i = 1, 2$. On the other hand $k_1 + k_2 + Q$ is negative so $k_1 + k_2 + Q = (-Q - j_1) + (-Q - j_2) + Q < 0$, then $j_1 + j_2 > -Q$, which implies $j_1 + j_2 > -Q - 1$. Consequently in order for the hypothesis of part I of Remark 40 to hold we need

- $j_1 < -1$ or $j_2 < -1$, when working with $|\beta| - 1$,
- $j_1 + j_2 > -Q + 1$, when working with $|\beta| + 1$.

So, the hypothesis of the remark fail to hold

- if $j_1 \geqslant -1$ and $j_2 \geqslant -1$, when working with $|\beta| - 1$
- if $j_1 + j_2 \leqslant -Q + 1$, when working with $|\beta| + 1$.

However the last three inequalities cannot hold, simultaneously, because if this were the case then

$$-2 = -1 - 1 \leqslant j_1 + j_2 \leqslant -Q + 1,$$

hence $Q \leqslant 3$, which is impossible because the smallest homogeneous dimension for the Heisenberg group is 4.

It follows that $|\beta| - 1$ can be used when $|\beta| + 1$ makes the hypothesis of part I of Remark 40 fail; alternatively, if $|\beta| - 1$ makes the hypothesis of the remark fail, $|\beta| - 1$ will be used.

Therefore we have shown that if $k_i < 0,\ i = 1, 2$ and $k_1 + k_2 + Q < 0$, then for each multiindex β there exists some positive constant C_β such that

$$\left|\partial^\beta (K_1 * K_2)(u)\right| \leqslant C_\beta |u|^{k_1+k_2+Q-|\beta|} \qquad \forall u \neq 0.$$

- <u>CASE II — $k_1, k_2 < 0,\ |\beta| = 0$</u>: We have $k_i = -j_i - Q > -Q$ because the j_i are negative, hence the hypothesis of Corollary 39are satisfied, and therefore we are allowed to claim that there exists a constant C_o such that

$$\left|(K_1 * K_2)(u)\right| \leqslant C_o\, B_{k_1+k_2+Q}(u) \qquad \forall u \neq 0.$$

- <u>CASE III — $k_1, k_2 < 0,\ k_1 + k_2 + Q \geqslant 0$</u>

  Our argument here consists in reducing this case to one of the previous ones, depending on the multiindex β.

We start by working on the first convolution factor of $\partial^\beta(K_1 * K_2)$. We discriminate the following two possible cases:

► $|\beta| < -[\![j_1]\!] - 1$: By Corollary 32 we obtain

$$\begin{aligned}\partial^\beta(K_1 * K_2) &= \partial^{\beta_x}\,\partial^{\beta_y}\,\partial^{\beta_t}\,(K_1 * K_2)\\ &= \sum_{r=1}^{4^{|\beta_x+\beta_y|}} (K_{1,r} * K_{2,r})\end{aligned}$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\beta|}(\mathbb{H}^n)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$.

By hypothesis, $|\beta| < -[\![j_1]\!] - 1 = -[\![j_1+1]\!]$. Therefore $j_1 + |\beta| < j_1 - [\![j_1+1]\!] < 0$. The number $k_1 = -Q - j_1$ is negative by hypothesis, hence $-Q < j_1 < j_1 + |\beta| < 0$, which implies that $-Q - (j_1 + |\beta|) < 0$. Consequently Case II can be applied to each $K_{1,r} * K_{2,r}$.

Then, for all $u \neq 0$

$$\begin{aligned}\left|\partial^\beta(K_1 * K_2)(u)\right| &= \left|\sum_{r=1}^{4^{|\beta_x+\beta_y|}} (K_{1,r} * K_{2,r})(u)\right|\\ &\leqslant \sum_{r=1}^{4^{|\beta_x+\beta_y|}} \left|(K_{1,r} * K_{2,r})(u)\right|\\ &\leqslant \sum_{r=1}^{4^{|\beta_x+\beta_y|}} C_r B_{[-Q-(j_1+|\beta|)]+(-Q-j_2)+Q}(u)\\ &\leqslant C_\beta B_{k_1+k_2+Q-|\beta|}(u)\end{aligned}$$

with $C_\beta = \max\limits_{r=1,\ldots,4^{|\beta_x+\beta_y|}} \left\{C_r\right\}$.

► $|\beta| \geqslant -[\![j_1]\!] - 1$: Given that $|\beta| \geqslant -[\![j_1]\!] - 1$, the multiindex β can be written as $\beta = \gamma + \alpha$, with $|\alpha| = -[\![j_1]\!] - 1$, or $|\alpha| = -[\![j_1]\!] - 2$.

The derivatives corresponding to the multiindex α will be applied on the first factor

$\partial^\beta(K_1 * K_2)$, and the derivatives corresponding to the multiindex γ will be applied on the second factor of the convolution of each element of the linear combination which results from the derivatives corresponding to the multiindex α.

By Corollary 32 we obtain

$$\begin{aligned}
\partial^\beta(K_1 * K_2) &= \partial^\gamma\, \partial^\alpha(K_1 * K_2)\\
&= \partial^\gamma\, \partial^{\alpha_x}\, \partial^{\alpha_y}\, \partial^{\alpha_t}(K_1 * K_2)\\
&= \sum_{r=1}^{4^{|\alpha_x+\alpha_y|}} \partial^\gamma\,(K_{1,r} * K_{2,r})\,.
\end{aligned}$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\alpha|}(\mathbb{H}^n)$ and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$.

In what follows the following two possibilities for γ are considered separately:

- $|\gamma| < -[\![j_2]\!] - 1$:

  By Corollary 32

$$\begin{aligned}
\partial^\beta(K_1 * K_2) &= \sum_{r=1}^{4^{|\alpha_x+\alpha_y|}} \partial^\gamma(K_{1,r} * K_{2,r})\\
&= \sum_{r=1}^{4^{|\alpha_x+\alpha_y|}} \partial^{\gamma_x}\, \partial^{\gamma_y}\, \partial^{\gamma_t}\,(K_{1,r} * K_{2,r})\\
&= \sum_{r=1}^{4^{|\alpha_x+\alpha_y|}} \left(\sum_{s=1}^{4^{|\gamma_x+\gamma_y|}} (K_{1,r_s} * K_{2,r_s})\right)\\
&= \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\gamma_x+\gamma_y|}} (K_{1,r} * K_{2,r})
\end{aligned} \tag{IV.11}$$

  with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\alpha|}(\mathbb{H}^n)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2+|\gamma|}(\mathbb{H}^n)$.

  The fact that $|\gamma| < -[\![j_2]\!] - 1$ implies that $j_2 + |\gamma|$ is negative. Since by hypothesis $k_2 = -Q - j_2 < 0$, then $-Q < j_2 \leqslant j_2 + |\gamma| < 0$. Hence the number $-Q - (j_2 + |\gamma|)$ is negative. By hypothesis, $|\alpha| = -[\![j_1]\!] - 1$, or $|\alpha| = -[\![j_1]\!] - 2$, hence $-2 \leqslant j_1 + (-[\![j_1]\!] - 2) \leqslant j_1 + |\alpha| \leqslant j_1 + (-[\![j_1]\!] - 1) < 0$;

that is $-2 \leqslant j_1 + |\alpha| < 0$. Since in the case of the Heisenberg group the smallest possible value for the homogeneous dimension is 4, then $-Q - (j_1 + |\alpha|)$ is negative. Consequently we can use Case II for each $K_{1,r} * K_{2,r}$ of (IV.11) and conclude for all $u \neq 0$

$$\begin{aligned}
\left|\partial^\beta (K_1 * K_2)(u)\right| &= \left|\sum_{r=1}^{4^{|\alpha_x+\alpha_y+\gamma_x+\gamma_y|}} (K_{1,r} * K_{2,r})(u)\right| \\
&\leqslant \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\gamma_x+\gamma_y|}} \left|(K_{1,r} * K_{2,r})(u)\right| \\
&= \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\gamma_x+\gamma_y|}} C_r B_{[-Q-(j_1+|\alpha|)]+[-Q-(j_2+|\gamma|)]+Q}\,(u) \\
&= C_\beta B_{k_1+k_2+Q-(|\alpha|+|\gamma|)}\,(u) \\
&= C_\beta B_{k_1+k_2+Q-|\beta|}\,(u).
\end{aligned}$$

with $C_\beta = \max\limits_{r=1,\ldots,4^{|\alpha_x+\alpha_y+\gamma_x+\gamma_y|}} \left\{C_r\right\}$.

- $|\gamma| \geqslant -[\![j_2]\!] - 1$:

Since $|\gamma| \geqslant -[\![j_2]\!]-1$, then we can write γ as $\gamma = \tau+\sigma$, with $|\sigma| = -[\![j_2]\!]-1$, or $|\sigma| = -[\![j_2]\!] - 2$.

By Corollary 32

$$
\begin{aligned}
\partial^{\beta}(K_1 * K_2) &= \sum_{r=1}^{4^{|\alpha_x+\alpha_y|}} \partial^{\gamma}(K_{1,r} * K_{2,r}) \\
&= \sum_{r=1}^{4^{|\alpha_x+\alpha_y|}} \partial^{\tau}\, \partial^{\sigma_x}\, \partial^{\sigma_y}\, \partial^{\sigma_t}(K_{1,r} * K_{2,r}) \\
&= \sum_{r=1}^{4^{|\alpha_x+\alpha_y|}} \partial^{\tau} \left(\sum_{s=1}^{4^{|\sigma_x+\sigma_y|}} (K_{1,r_s} * K_{2,r_s}) \right) \\
&= \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \partial^{\tau}(K_{1,r} * K_{2,r})
\end{aligned}
\tag{IV.12}
$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\alpha|}(\mathbb{H}^n)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2+|\sigma|}(\mathbb{H}^n)$.

Next we intend to apply Case I to each $\partial^{\tau}(K_{1,r} * K_{2,r})$. Therefore we have to do some formal verification that the hypothesis of Case I are indeed satisfied. Thus we need to verify that: $j_1 + |\alpha|$, $j_2 + |\sigma|$, $-Q - (j_1 + |\alpha|)$, $-Q - (j_2 + |\sigma|)$, and $[-Q - (j_1 + |\alpha|)] + [-Q - (j_2 + |\sigma|)] + Q$ are negative numbers. By hypothesis, $|\alpha| = -[\![j_1]\!] - 1$, or $|\alpha| = -[\![j_1]\!] - 2$, hence $-2 \leqslant j_1 + (-[\![j_1]\!] - 2) \leqslant j_1 + |\alpha| \leqslant j_1 + (-[\![j_1]\!] - 1) < 0$; that is $-2 \leqslant j_1 + |\alpha| < 0$. Since in the case of the Heisenberg group the smallest possible value of the homogeneous dimension, is 4, then $-Q - (j_1 + |\alpha|)$ is negative. Similarly, by hypothesis $|\sigma| = -[\![j_2]\!] - 1$, or $|\sigma| = -[\![j_2]\!] - 2$. Therefore $-2 \leqslant j_2 + |\sigma| < 0$, which implies that $-Q - (j_2 + |\sigma|)$ is negative.

It only remains to see that $[-Q - (j_1 + |\alpha|)] + [-Q - (j_2 + |\sigma|)] + Q < 0$. Given that $-2 \leqslant j_1 + |\alpha| < 0$, and $-2 \leqslant j_2 + |\sigma| < 0$, then $[-Q - (j_1 + |\alpha|)] + [-Q - (j_2 + |\sigma|)] + Q \leqslant -Q + 4$. Because the homogeneous dimension Q is always greater than 4 when $n > 3$, all the hypothesis of Case I are satisfied, unless $n = 3$.

Since $Q = 4$ when $n = 3$, then $[-Q - (j_1 + |\alpha|)] + [-Q - (j_2 + |\sigma|)] + Q = -(j_1 + |\alpha|) - (j_2 + |\sigma|) - 4 < 0$, unless $j_1 + |\alpha| = -2$, and $j_2 + |\sigma| = -2$.

Consequently, all the hypothesis of Case I are satisfied unless $n = 3$, and $j_1 + |\alpha| = j_2 + |\sigma| = -2$.

Consider the situation when all hypothesis of Case I are satisfied. Using Case I in identity (IV.12) yields that, for all $u \neq 0$

$$
\begin{aligned}
\left|\partial^\beta (K_1 * K_2)(u)\right| &= \left| \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \partial^\tau (K_{1,r} * K_{2,r})(u) \right| \\
&\leqslant \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \left|\partial^\tau (K_{1,r} * K_{2,r})(u)\right| \\
&\leqslant \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} C_{\tau_r} |u|^{[-Q-(j_1+|\alpha|)]+[-Q-(j_2+|\sigma|)]+Q-|\tau|} \\
&= \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} C_{\tau_r} |u|^{(-Q-j_1)+(-Q-j_2)+Q-(|\alpha|+|\sigma|+|\tau|)} \\
&= \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} C_{\tau_r} |u|^{k_1+k_2+Q-|\beta|} \\
&\leqslant C_\beta \, |u|^{k_1+k_2+Q-|\beta|}
\end{aligned}
$$

with $C_\beta = \max\limits_{r=1,\dots,4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \left\{C_{\tau_r}\right\}$,

and with $|\beta| = |\alpha| + |\gamma| = |\alpha| + |\sigma| + |\tau|$.

Now, since $k_1 + k_2 + Q - |\beta| < 0$, the last inequality above implies that, for all $u \neq 0$

$$\left|\partial^\beta (K_1 * K_2)(u)\right| = C_\beta \, B_{k_1+k_2+Q-|\beta|}(u).$$

Let us discuss now the situation when $n = 3$, and $j_1 + |\alpha| = j_2 + |\sigma| = -2$. We shall discriminate two possible cases, namely:

- $\underline{|\tau| = 0}$

  If $|\tau| = 0$ then (IV.12) becomes

$$\partial^\beta(K_1 * K_2) = \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} (K_{1,r} * K_{2,r})$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\alpha|}(\mathbb{H}^1) = \widetilde{\mathcal{M}}^{-2}(\mathbb{H}^1)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2+|\sigma|}(\mathbb{H}^1) = \widetilde{\mathcal{M}}^{-2}(\mathbb{H}^1)$.

Since each $K_{1,r} * K_{2,r}$ is under the hypothesis of Case II then we can conclude that for all $u \neq 0$

$$\begin{aligned}
\left|\partial^\beta(K_1 * K_2)(u)\right| &= \left|\sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} (K_{1,r} * K_{2,r})(u)\right| \\
&\leqslant \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \left|(K_{1,r} * K_{2,r})(u)\right| \\
&\leqslant \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} C_r B_{[-Q-(j_1+|\alpha|)]+[-Q-(j_2+|\sigma|)]+Q}(u) \\
&= \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} C_r B_{k_1+k_2+Q-(|\alpha|+|\sigma|)}(u) \\
&\leqslant C_\beta\, B_{k_1+k_2+Q-|\beta|}(u)
\end{aligned}$$

with $C_\beta = \max_{r=1,\ldots,4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \left\{C_r\right\}$.

♦ $\underline{|\tau| \neq 0}$

If $|\tau| \neq 0$ then in ∂^τ we must have some $\frac{\partial}{\partial x}$, or $\frac{\partial}{\partial y}$, or $\frac{\partial}{\partial t}$. Next we shall introduce one of these derivatives on each $K_{1,r} * K_{2,r}$ of (IV.12) using Remark 29, and then we shall end up in Case I.

For example, suppose we have $\frac{\partial}{\partial x}$ in ∂^τ, and we wish to introduce into the first factor of $K_{1,r} * K_{2,r}$ in (IV.12). We know that

$$\partial^\beta(K_1 * K_2) = \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \partial^\tau(K_{1,r} * K_{2,r})$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\alpha|}(\mathbb{H}^1) = \widetilde{\mathcal{M}}^{-2}(\mathbb{H}^1)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2+|\sigma|}(\mathbb{H}^1) =$

$\widetilde{\mathcal{M}}^{-2}(\mathbb{H}^1)$.

There exists some $\frac{\partial}{\partial x}$ in ∂^τ, therefore $\tau = \tilde{\tau} + e_1$ where e_1 is the first element of the canonical base of $\mathbb{R}^3$. Therefore applying remark 29 we obtain

$$\begin{aligned}
\partial^\beta(K_1 * K_2) &= \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \partial^{\tilde{\tau}} \frac{\partial}{\partial x}(K_{1,r} * K_{2,r}) \\
&= \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \partial^{\tilde{\tau}} \left(\sum_{s=1}^{4} (K_{1,r_s} * K_{2,r_s}) \right) \\
&= \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|+1}} \partial^{\tilde{\tau}}(K_{1,r} * K_{2,r})
\end{aligned}$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\alpha|+1}(\mathbb{H}^1) = \widetilde{\mathcal{M}}^{-1}(\mathbb{H}^1)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2+|\sigma|}(\mathbb{H}^1) = \widetilde{\mathcal{M}}^{-2}(\mathbb{H}^1)$.

We also note that $|\beta| = |\alpha| + |\sigma| + |\tau| = |\alpha| + |\sigma| + |\tilde{\tau}| + 1$. Since for each $\partial^{\tilde{\tau}}(K_{1,r} * K_{2,r})$ we are under the hypothesis of Case I then for all $u \neq 0$

$$\begin{aligned}
\left|\partial^\beta(K_1 * K_2)(u)\right| &\leqslant \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|+1}} C_{\tilde{\tau}_r}\, B_{[-Q-(j_1+|\alpha|+1)]+[-Q-(j_2+|\sigma|)]+Q-|\tilde{\tau}|}(u) \\
&\leqslant C_\beta\, B_{k_1+k_2+Q-|\beta|}(u)
\end{aligned}$$

with $C_\beta = \max\limits_{r=1,\ldots,4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|+1}} \left\{ C_{\tilde{\tau}_r} \right\}$.

If we have some $\frac{\partial}{\partial y}$ in ∂^τ and we want to use it, we proceed in an analogous fashion as our previous case. In this situation we have that $\tau = \tilde{\tau} + e_2$, where e_2 is the second element of the canonical base of $\mathbb{R}^3$, and we shall use Remark 29.

Finally if there is some $\frac{\partial}{\partial t}$ in ∂^τ and we want to use it, we imitate our previous work. Here $\tau = \tilde{\tau} + e_3$ and we apply Remark 29. Therefore

$$\partial^\beta(K_1 * K_2) = \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \partial^{\tilde{\tau}} \frac{\partial}{\partial t}(K_{1,r} * K_{2,r})$$
$$= \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \partial^{\tilde{\tau}} \left(\sum_{s=1}^{2}(K_{1,r_s} * K_{2,r_s})\right)$$
$$= \sum_{r=1}^{2\cdot 4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \partial^{\tilde{\tau}}(K_{1,r} * K_{2,r})$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\alpha|+1}(\mathbb{H}^1) = \widetilde{\mathcal{M}}^{-1}(\mathbb{H}^1)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2+|\sigma|+1}(\mathbb{H}^1) = \widetilde{\mathcal{M}}^{-1}(\mathbb{H}^1)$.

Then using Case I, we obtain for $u \neq 0$

$$\left|\partial^\beta(K_1 * K_2)(u)\right| \leqslant \sum_{r=1}^{2\cdot 4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} C_{\tilde{\tau}_r}\, B_{[-Q-(j_1+|\alpha|+1)]+[-Q-(j_2+|\sigma|+1)]+Q-|\tilde{\tau}|}(u)$$
$$\leqslant C_\beta\, B_{k_1+k_2+Q-|\beta|}(u)$$

with $C_\beta = \max\limits_{r=1,\ldots,2\cdot 4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \left\{C_{\tilde{\tau}_r}\right\}$.

Observe that in this case we cannot use part (d) of Proposition 28, because it will result in

$$\partial^\beta(K_1 * K_2) = \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \partial^{\tilde{\tau}}\Big((TK_{1,r}) * K_{2,r}\Big)$$

where $TK_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\alpha|+2}(\mathbb{H}^1) = \widetilde{\mathcal{M}}^0(\mathbb{H}^1)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{-2}(\mathbb{H}^1)$, and still we do not know how to handle the situation of $\widetilde{\mathcal{M}}^0(\mathbb{H}^n) * \widetilde{\mathcal{M}}^j(\mathbb{H}^n)$.

- CASE IV — SOME $k_i \geqslant 0$

Suppose without loss of generality that $k_1 \geqslant 0$. Since $k_2 > -Q$, and $k_1 \geqslant 0$, then $k_1 + k_2 + Q > 0$.

Two subcases will be distinguished:

► $|\beta| = 0$

By Corollary 39 it can be concluded that then there exists a positive constant C_o such that for all $u \neq 0$

$$\left|(K_1 * K_2)(u)\right| \leqslant C_o B_{k_1+k_2+Q}(u) .$$

► $|\beta| \neq 0$

■ $|\beta| < [\![k_1]\!] + 1$

If $|\beta| < [\![k_1]\!] + 1$ Corollary 32 will be applied to $\partial^\beta(K_1 * K_2)$. Then for $u \neq 0$

$$\begin{aligned}
\left|\partial^\beta(K_1 * K_2)(u)\right| &= \left|\partial^{\beta_x}\,\partial^{\beta_y}\,\partial^{\beta_t}(K_1 * K_2)(u)\right| \\
&= \left|\sum_{r=1}^{4^{|\beta_x+\beta_y|}} (K_{1,r} * K_{2,r})(u)\right| \\
&\leqslant \sum_{r=1}^{4^{|\beta_x+\beta_y|}} \left|(K_{1,r} * K_{2,r})(u)\right|
\end{aligned}$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\beta|}(\mathbb{H}^n)$ and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$.

Now, if $-Q-(j_1+|\beta|) < 0$, and $k_2 = -Q - j_2 < 0$, we apply Case II to each $K_{1,r} * K_{2,r}$. And if $-Q-(j_1+|\beta|) \geqslant 0$, or $k_2 = -Q - j_2 \geqslant 0$ we apply Case IV with multiindex $(0,\ldots,0)$. Then for all $u \neq 0$

$$\begin{aligned}
\left|\partial^\beta(K_1 * K_2)(u)\right| &\leqslant \sum_{r=1}^{4^{|\beta_x+\beta_y|}} C_r\, B_{[-Q-(j_1+|\beta|)]+k_2+Q}\,(u) \\
&\leqslant C_\beta\, B_{k_1+k_2+Q-|\beta|}\,(u)
\end{aligned}$$

with $C_\beta = \max\limits_{r=1,\ldots,4^{|\beta_x+\beta_y|}} \{C_r\}$.

■ $|\beta| \geqslant [\![k_1]\!] + 1$

Since $|\beta| \geqslant [\![k_1]\!] + 1$, then we can write β as $\beta = \gamma + \alpha$, with $|\alpha| = [\![k_1]\!] + 1$, or $|\alpha| = [\![k_1]\!] + 2$. By Corollary 32 applied to $\partial^\beta(K_1 * K_2)$ we obtain

$$\begin{aligned}\left|\partial^\beta (K_1 * K_2)(u)\right| &= \left|\partial^\gamma \partial^{\alpha_x} \partial^{\alpha_y} \partial^{\alpha_t} (K_1 * K_2)(u)\right| \\ &= \left|\partial^\gamma \sum_{r=1}^{4^{|\alpha_x+\alpha_y|}} (K_{1,r} * K_{2,r})(u)\right| \\ &\leqslant \sum_{r=1}^{4^{|\alpha_x+\alpha_y|}} \left|\partial^\gamma (K_{1,r} * K_{2,r})(u)\right|\end{aligned} \tag{IV.13}$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\alpha|}(\mathbb{H}^n)$ and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$.

Two subcases will be distinguished, namely

- $\underline{|\gamma| < [\![k_2]\!] + 1}$

  Applying Corollary 32 to each $\partial^\gamma (K_{1,r} * K_{2,r})$ of equation (IV.13) we obtain

$$\begin{aligned}\left|\partial^\beta (K_1 * K_2)(u)\right| &\leqslant \sum_{r=1}^{4^{|\alpha_x+\alpha_y|}} \left|\partial^{\gamma_x} \partial^{\gamma_y} \partial^{\gamma_t} (K_{1,r} * K_{2,r})(u)\right| \\ &= \sum_{r=1}^{4^{|\alpha_x+\alpha_y|}} \left|\sum_{s=1}^{4^{|\gamma_x+\gamma_y|}} (K_{1,r_s} * K_{2,r_s})(u)\right| \\ &\leqslant \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\gamma_x+\gamma_y|}} \left|(K_{1,r} * K_{2,r})(u)\right|\end{aligned}$$

  with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\alpha|}(\mathbb{H}^n)$ and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2+|\gamma|}(\mathbb{H}^n)$.

  Observe that $-Q-(j_1+|\alpha|) = k_1 - |\alpha|$. Since $|\alpha| = [\![k_1]\!]+1$, or $|\alpha| = [\![k_1]\!]+2$, then $-2 \leqslant -Q-(j_1+|\alpha|) < 0$. Therefore if $-Q-(j_2+|\gamma|) < 0$, we are under the hypothesis of Case II, and if $-Q-(j_2+|\gamma|) \geqslant 0$, we are under the hypothesis of Case IV with multi-index $(0,\ldots,0)$.

  Therefore for all $u \neq 0$

$$\left|\partial^\beta (K_1 * K_2)(u)\right| \leqslant \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\gamma_x+\gamma_y|}} C_r\, B_{[-Q-(j_1+|\alpha|)]+[-Q-(j_2+|\gamma|)]+Q}(u)$$

$$\leqslant C_\beta\, B_{k_1+k_2+Q-|\alpha+\gamma|}(u)$$

$$\leqslant C_\beta\, B_{k_1+k_2+Q-|\beta|}(u)$$

with $C_\beta = \max_{r=1,\ldots,4^{|\alpha_x+\alpha_y+\gamma_x+\gamma_y|}} \Big\{C_r\Big\}.$

♦ $\underline{|\gamma| \geqslant [\![k_2]\!]+1}$

Since $|\gamma| \geqslant [\![k_1]\!]+2$, then one can write $\gamma = \tau+\sigma$ with $|\sigma| = [\![k_2]\!]+1$, or $|\sigma| = [\![k_2]\!]+2$. By Corollary 32 applied to (IV.13)

we have for all $u \neq 0$

$$\left|\partial^\beta (K_1 * K_2)(u)\right| \leqslant \sum_{r=1}^{4^{|\alpha_x+\alpha_y|}} \left|\partial^\tau\, \partial^{\sigma_x}\, \partial^{\sigma_y}\, \partial^{\sigma_t}(K_{1,r} * K_{2,r})(u)\right|$$

$$= \sum_{r=1}^{4^{|\alpha_x+\alpha_y|}} \left|\partial^\tau \sum_{s=1}^{4^{|\sigma_x+\sigma_y|}} (K_{1,r_s} * K_{2,r_s})(u)\right|$$

$$\leqslant \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \left|\partial^\tau (K_{1,r} * K_{2,r})(u)\right|$$

with $K_{1,r} \in \widetilde{\mathcal{M}}^{j_1+|\alpha|}(\mathbb{H}^n)$, and $K_{2,r} \in \widetilde{\mathcal{M}}^{j_2+|\sigma|}(\mathbb{H}^n)$.

Since $|\alpha| = [\![k_1]\!]+1$, or $|\alpha| = [\![k_1]\!]+2$, then $-2 \leqslant -Q-(j_1+|\alpha|) = k_1 - |\alpha|$. Similarly, since $|\sigma| = [\![k_2]\!]+1$, or $|\sigma| = [\![k_2]\!]+2$, then $-2 \leqslant -Q-(j_2+|\sigma|) = k_2-|\sigma|$. And given that the homogeneous dimension Q in the case of the Heisenberg group is greater or equal to 4, then $[-Q-(j_1+|\alpha|)]+[-Q-(j_2+|\sigma|)]+Q \geqslant 0$. Therefore we are under the hypothesis of Case III.

Then for all $u \neq 0$, we have

$$
\begin{aligned}
\left|\partial^{\beta}(K_1 * K_2)(u)\right| \quad &\leqslant \quad \sum_{r=1}^{4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} C_{\tau_r}\, B_{[-Q-(j_1+|\alpha|)]+[-Q-(j_2+|\sigma|)]+Q-|\tau|}(u) \\
&\leqslant \quad C_{\beta}\, B_{k_1+k_2+Q-(|\tau|+|\sigma|+|\alpha|)}\,(u) \\
&\overset{(\gamma=\tau+\sigma)}{=} C_{\beta}\, B_{k_1+k_2+Q-|\beta|}\,(u)
\end{aligned}
$$

with $C_{\beta} = \max_{r=1,\ldots,4^{|\alpha_x+\alpha_y+\sigma_x+\sigma_y|}} \left\{C_{\tau_r}\right\}.$

This concludes the proof of Case IV, as well as the proof of the theorem.

$\square$

CHAPTER V

$\widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n) * \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$ WITH SOME $j_i \geqslant 0$

V.1 Basic definitions and results

Let G be a homogeneous group with dilations $\{\delta_r\}_{r>0}$ defined by

$$\delta_r(x_1, \ldots, x_n) = (r^{a_1}x_1, \ldots, r^{a_n}x_n)$$

If necessary coördinates are assumed to be relabeled, so that the exponents a_i form a increasing sequence, $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$. Recall the definition of the homogeneous norm in G as

$$\left|(x_1, \ldots, x_n)\right| = \left(\sum_{i=1}^{n} x_i^{\frac{2A}{a_r}}\right)^{\frac{1}{2A}}$$

with $A = \prod_{i=1}^{n} a_i$.

Proposition 42. *Let G be a homogeneous group, and suppose that $\{X_j\}_{i=1}^n$ are the left invariant vector fields which agree with $\frac{\partial}{\partial x_j}$ at the origin. Then*

a) Any left invariant vector field X_j can be written as

$$X_j = \frac{\partial}{\partial x_j} + \sum_{k=j+1}^{n} p_{j,k}(x)\frac{\partial}{\partial x_k} \qquad j = 1, \ldots, n$$

where $p_{j,k}(x) = p_{j,k}(x_1, \ldots, x_{j-1})$ are homogeneous polynomials, with respect to the group dilations, of degree $a_k - a_j$.

b) Any $\frac{\partial}{\partial x_j}$ can be written as

$$\frac{\partial}{\partial x_j} = X_j + \sum_{k=j+1}^{n} q_{j,k}(x)X_k \qquad j = 1, \ldots, n$$

where $q_{j,k}$ are homogeneous polynomials, with respect to the group dilations, of degree $a_k - a_j$.

Entirely analogous formulae express X_j^R in terms of $\left\{\frac{\partial}{\partial x_{j+k}}\right\}_{k=0}^n$, as well as $\frac{\partial}{\partial x_j}$ in terms of $\left\{X_{j+k}^R\right\}_{k=0}^n$.

Proof. See [FolSte 82]. □

Proposition 43. *Say $j \in \mathbb{R}$. If $K \in \widetilde{\mathcal{M}}^j(G)$, then there exist $K_r \in \widetilde{\mathcal{M}}^{j-a_r}(G)$, $r = 1, \ldots, n$, and possibly an error term $K_o \in \widetilde{\mathcal{M}}^{j-2A}(G)$, such that*

$$K = \sum_{r=1}^{n} X_r K_r + K_o$$

Proof. Set

$$F_r = \left[\left(\frac{i}{2\pi}\right) \frac{\xi_r^{\left(\frac{2A}{a_r}-1\right)}}{|\xi|^{2A}+1}\, \widehat{K} \right]^{\vee} \qquad r = 1, \ldots, n$$

and

$$F_o = \left(\frac{1}{|\xi|^{2A}+1}\, \widehat{K} \right)^{\vee} . \tag{V.1}$$

Since $K \in \widetilde{\mathcal{M}}^j(G)$, then $\xi_r^{\left(\frac{2A}{a_r}-1\right)} \widehat{K} \in \mathcal{M}^{j+a_r\left(\frac{2A}{a_r}-1\right)} = \mathcal{M}^{j+2A-a_r}(G)$. Therefore $\widehat{F}_r \in \mathcal{M}^{j+2A-a_r-2A}(G) = \mathcal{M}^{j-a_r}(G)$ for all $r = 1, \ldots, n$. And $\widehat{F}_o \in \mathcal{M}^{j-2A}(G)$. Consequently $F_r \in \widetilde{\mathcal{M}}^{j-a_r}(G)$, $r = 1, \ldots, n$, and $F_o \in \widetilde{\mathcal{M}}^{j-2A}(G)$.

We have

$$\begin{aligned}
\widehat{K} &= \frac{|\xi|^{2A}+1}{|\xi|^{2A}+1}\, \widehat{K} \\
&= \frac{\sum_{r=1}^{n} \xi_r^{\frac{2A}{a_r}} + 1}{|\xi|^{2A}+1}\, \widehat{K} \\
&= \sum_{r=1}^{n} \frac{\xi_r^{\frac{2A}{a_r}} + 1}{|\xi|^{2A}+1}\, \widehat{K} + \frac{1}{|\xi|^{2A}+1}\, \widehat{K} \\
&= \sum_{r=1}^{n} (-2\pi i \xi_r) \frac{\left(\frac{i}{2\pi}\right) \xi_r^{\frac{2A}{a_r}-1}}{|\xi|^{2A}+1}\, \widehat{K} + \frac{1}{|\xi|^{2A}+1}\, \widehat{K} \\
&= \sum_{r=1}^{n} (-2\pi i \xi_r) \widehat{F}_r + \widehat{F}_o \\
&= \sum_{r=1}^{n} \widehat{\frac{\partial}{\partial x_r} F_r} + \widehat{F}_o \\
&= \left(\sum_{r=1}^{n} \frac{\partial}{\partial x_r} F_r + F_o \right)^{\wedge}
\end{aligned}$$

Therefore

$$K = \sum_{r=1}^{n} \frac{\partial}{\partial x_r} F_r + F_o \tag{V.2}$$

where $F_r \in \widetilde{\mathcal{M}}^{j-a_r}(G),\ r = 1, \ldots, n$, and $F_o \in \widetilde{\mathcal{M}}^{j-2A}(G)$.

By Proposition 42 we know that we can express $\frac{\partial}{\partial x_i}$ in terms of the left invariant vector fields $X_i's$ as

$$\frac{\partial}{\partial x_k} = X_k + \sum_{s=k+1}^{n} q_{k,s}(x) X_s \qquad k = 1, \ldots, n$$

where $q_{k,s}$ are homogeneous polynomials of degree $a_s - a_k$.

Then

$$\begin{aligned}
K &= \frac{\partial}{\partial x_1} F_1 + \frac{\partial}{\partial x_2} F_2 + \cdots + \frac{\partial}{\partial x_n} F_n + F_o \\
&= \left(X_1 + \sum_{k=2}^{n} q_{2,k}(x) X_k \right) F_1 + \left(X_2 + \sum_{k=3}^{n} q_{3,k}(x) X_k \right) F_2 + \cdots + X_n F_n + F_o \\
&= X_1 F_1 + X_2 (q_{1,2}(x) F_1) + \cdots + X_n \left(\sum_{r=1}^{n} q_{r,n}(x) F_n \right) + F_o
\end{aligned}$$

Therefore if we define

$$K_r = \sum_{k=1}^{r} q_{k,r}(x) F_k \qquad r = 1, \ldots, n \qquad \text{and} \quad K_o = F_o \tag{V.3}$$

with $q_{r,r}(x) := 1$. Then we can write K as

$$K = \sum_{r=1}^{n} X_r K_r + K_o.$$

Since $F_k \in \widetilde{\mathcal{M}}^{j-a_k}(G),\ r = 1, \ldots, n$, and $q_{k,r}$ are homogeneous polynomials of degree $a_r - a_k$, hence $q_{k,r} F_k \in \widetilde{\mathcal{M}}^{j-a_k-(a_r-a_k)}(G) = \widetilde{\mathcal{M}}^{j-a_r}(G),\ k = 1, \ldots, r$. Therefore $K_r = \sum_{k=1}^{r} q_{k,r} F_k \in \widetilde{\mathcal{M}}^{j-a_r}(G)$, for $r = 1, \ldots, n$ and $K_o = F_o \in \widetilde{\mathcal{M}}^{j-2A}(G)$. $\square$

Corollary 44. *If $K \in \widetilde{\mathcal{M}}^j(G)$, with $j \in \mathbb{R}$, then there exist $K_r \in \widetilde{\mathcal{M}}^{j-a_r}(G)$, $r = 1, \dots, n$, and possibly an error term K_o in $\widetilde{\mathcal{M}}^{j-2A}(G)$, such that*

$$K = \sum_{r=1}^{n} X_r^R K_r + K_o$$

Proof. The proof is analogous to the proof of Proposition 43 □

Proposition 45. *Let $j \in \mathbb{R}$. If $K \in \widetilde{\mathcal{M}}^j(G)$, then for any $s \in \mathbb{N}$ there exist $K_r \in \widetilde{\mathcal{M}}^{j-a_r}(G)$, $r = 1, \dots, n$, and $K_o \in \widetilde{\mathcal{M}}^{j-2As}(G)$ such that K can be written as*

$$K = \sum_{r=1}^{n} X_r K_r + K_o \tag{V.4}$$

Similarly for any $s \in \mathbb{N}$ there exist $K_r \in \widetilde{\mathcal{M}}^{j-a_r}(G)$, $r = 1, \dots, n$, and $K_o \in \widetilde{\mathcal{M}}^{j-2As}(G)$ such that K can be written as

$$K = \sum_{r=1}^{n} X_r^R K_r + K_o \tag{V.5}$$

Proof. We proceed by induction on s. By Proposition 43on page 95,K can be written as

$$K = \sum_{r=1}^{n} X_r K_r + K_o \qquad \text{with} \quad K_r \in \widetilde{\mathcal{M}}^{j-a_r}(G) \quad \text{and} \quad K_o \in \widetilde{\mathcal{M}}^{j-2A}(G).$$

For $s-1$ assume that we have $K = \sum_{r=1}^{n} X_r K_r + K_o$ with $K_r \in \widetilde{\mathcal{M}}^{j-a_r}(G)$ and $K_o \in \widetilde{\mathcal{M}}^{j-2A(s-1)}(G)$. Applying Proposition 43 to K_o we obtain

$$\begin{aligned} K &= \sum_{r=1}^{n} X_r K_r + \sum_{k=1}^{n} X_k K_{o,k} + K_{o,o} \\ &= \sum_{r=1}^{n} X_r \left(K_r + K_{o,r}\right) + K_{o,o} \end{aligned}$$

where $K_r \in \widetilde{\mathcal{M}}^{j-a_r}(G)$, $K_{o,r} \in \widetilde{\mathcal{M}}^{j-2A-a_r}(G)$, and $K_{o,o} \in \widetilde{\mathcal{M}}^{j-2A(s-1)-2A}(G) = \widetilde{\mathcal{M}}^{j-2As}(G)$.

Since $\widetilde{\mathcal{M}}^{j-2A-a_r}(G) \subset \widetilde{\mathcal{M}}^{j-a_r}(G)$, then $(K_r + K_{o,r}) \in \widetilde{\mathcal{M}}^{j-a_r}(G)$. Therefore

$$K = \sum_{r=1}^{n} X_r \widetilde{K}_r + \widetilde{K}_o$$

where $\widetilde{K}_r = K_r + K_{o,r} \in \widetilde{\mathcal{M}}^{j-a_r}(G)$, and $\widetilde{K}_o \in \widetilde{\mathcal{M}}^{j-2As}(G)$, as desired.

The proof of the right invariant case is completely analogous.

□

Proposition 46. *Assume $j \in \mathbb{R}$. If $K \in \widetilde{\mathcal{M}}^{j}(G)$, then for any $k, s \in \mathbb{N}$ there exist $K_{r_1,r_2,\dots,r_k} \in \widetilde{\mathcal{M}}^{j-a_{r_1}-\dots-a_{r_k}}(G)$, and $K_o \in \widetilde{\mathcal{M}}^{j-2As}(G)$ such that K can be written as*

$$K = \sum_{1 \leqslant r_1 \dots, r_k \leqslant n} X_{r_1} \cdots X_{r_k} K_{r_1,\dots,r_k} + K_o \tag{V.6}$$

Analogous formulae are valid for the right invariant vector fields $\{X_i^R\}_i$

Proof. We proceed by induction on k. By Proposition V.4 it is possible to write K as $K = \sum_{r=1}^{n} X_r K_r + K_o$ with $K_r \in \widetilde{\mathcal{M}}^{j-a_r}(G)$, and $K_o \in \widetilde{\mathcal{M}}^{j-2As}(G)$. We assume that for $k-1$ the thesis is valid, i.e. we assume that we can express

$$K = \sum_{1 \leqslant r_1 \dots, r_{k-1} \leqslant n} X_{r_1} \cdots X_{r_{k-1}} K_{r_1,\dots,r_{k-1}} + \widetilde{K}_o \tag{V.7}$$

with $K_{r_1,\dots,r_{k-1}} \in \widetilde{\mathcal{M}}^{j-a_{r_1}-\dots-a_{r_{k-1}}}(G)$, and $\widetilde{K}_o \in \widetilde{\mathcal{M}}^{j-2As}(G)$. Now, applying Proposition 45 to each $K_{r_1,\dots,r_{k-1}}$ and substituting in (V.7) we obtain

$$\begin{aligned} K &= \sum_{1 \leqslant r_1 \dots, r_{k-1} \leqslant n} X_{r_1} \cdots X_{r_{k-1}} \left(\sum_{r_k=1}^{n} X_{r_k} K_{r_1,\dots,r_k} + K_{o,r_1,\dots,r_{k-1}} \right) + \widetilde{K}_o \\ &= \sum_{1 \leqslant r_1 \dots, r_k \leqslant n} X_{r_1} \cdots X_{r_k} K_{r_1,\dots,r_k} + \sum_{1 \leqslant r_1 \dots, r_{k-1} \leqslant n} X_{r_1} \cdots X_{r_{k-1}} K_{o,r_1,\dots,r_{k-1}} + \widetilde{K}_o \end{aligned}$$

with $K_{r_1,\dots,r_k} \in \widetilde{\mathcal{M}}^{j-a_{r_1}-\dots-a_{r_k}}(G)$, and with $X_{r_1} \cdots X_{r_{k-1}} K_{o,r_1,\dots,r_{k-1}} \in \widetilde{\mathcal{M}}^{j-2As}(G)$.

Therefore

$$K = \sum_{1\leqslant r_1 \dots, r_k \leqslant n} X_{r_1} \cdots X_{r_{k-1}} K_{r_1,\dots,r_k} + K_o$$

with $K_o = \left(\sum_{1\leqslant r_1 \dots, r_{k-1} \leqslant n} X_{r_1} \cdots X_{r_{k-1}} K_{o,r_1,\dots,r_{k-1}} + \widetilde{K}_o \right) \in \widetilde{\mathcal{M}}^{j-2As}(G)$, and with $K_{r_1,\dots,r_k} \in \widetilde{\mathcal{M}}^{j-a_{r_1}-\dots-a_{r_k}}(G)$.

A completely analogous argument, which is being omitted, proves the right invariant case. $\square$

V.2 MAIN THEOREMS

Theorem 47. *Let j_1, j_2 be two real numbers, with one of them negative, and the remaining one non negative. If $K_1 \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $K_2 \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$ then their convolution $K_1 * K_2 \in \widetilde{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$.*

Proof.

- $j_1 < 0$ and $j_2 \geqslant 0$:

  Consider any natural numbers k and s, such that $j_2 - ka_1 < 0$, and $j_2 - 2As < 0$. By Proposition 46 on the previous page

$$K_2 = \sum_{1\leqslant r_1 \dots, r_k \leqslant n} X_{r_1} \cdots X_{r_k} K_{r_1,\dots,r_k} + K_o$$

  with $K_{r_1,\dots,r_k} \in \widetilde{\mathcal{M}}^{j_2-a_{r_1}-\dots-a_{r_k}}(\mathbb{H}^n)$, and $K_o \in \widetilde{\mathcal{M}}^{j_2-2As}(\mathbb{H}^n)$.

Therefore

$$K_1 * K_2 = K_1 * \left(\sum_{1 \leqslant r_1 \ldots, r_k \leqslant n} X_{r_1} \cdots X_{r_k} K_{r_1,\ldots,r_k} \right) + K_o$$
$$= K_1 * \left(\sum_{1 \leqslant r_1 \ldots, r_k \leqslant n} X_{r_1} \cdots X_{r_k} K_{r_1,\ldots,r_k} \right) + K_1 * K_o$$
$$= \sum_{1 \leqslant r_1 \ldots, r_k \leqslant n} X_{r_1} \cdots X_{r_k} \left(K_1 * K_{r_1,\ldots,r_k} \right) + K_1 * K_o$$

Notice that $K_1 \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, with $j_1 < 0$, and $K_{r_1,\ldots,r_k} \in \widetilde{\mathcal{M}}^{j_2 - a_{r_1} - \ldots - a_{r_k}}(\mathbb{H}^n)$. Since $j_2 - a_{r_1} - \ldots - a_{r_k} \leqslant j_2 - ka_1 < 0$ by our hypothesis on k, then $K_{r_1,\ldots,r_k} \in \widetilde{\mathcal{M}}^{j_2 - a_{r_1} - \ldots - a_{r_k}}(\mathbb{H}^n)$, with $j_2 - a_{r_1} - \ldots - a_{r_k} < 0$. Therefore, by Theorem 41, each $K_1 * K_{r_1,\ldots,r_k} \in \widetilde{\mathcal{M}}^{j_1 + j_2 - a_{r_1} - \ldots - a_{r_k}}(\mathbb{H}^n)$. Hence $X_{r_1} \cdots X_{r_k} \left(K_1 * K_{r_1,\ldots,r_k} \right) \in \widetilde{\mathcal{M}}^{j_1 + j_2}(\mathbb{H}^n)$. And therefore $\sum_{1 \leqslant r_1 \ldots, r_k \leqslant n} X_{r_1} \cdots X_{r_k} \left(K_1 * K_{r_1,\ldots,r_k} \right) \in \widetilde{\mathcal{M}}^{j_1 + j_2}(\mathbb{H}^n)$.

Observe that $K_o \in \widetilde{\mathcal{M}}^{j_2 - 2As}(\mathbb{H}^n)$, and that $j_2 - 2As < 0$. Then by Theorem 41 we know $K_1 * K_o \in \widetilde{\mathcal{M}}^{j_1 + j_2 - 2As}(\mathbb{H}^n)$. Since $\widetilde{\mathcal{M}}^{j_1 + j_2 - 2As}(\mathbb{H}^n) \subset \widetilde{\mathcal{M}}^{j_1 + j_2}(\mathbb{H}^n)$, we have that $K_1 * K_o \in \widetilde{\mathcal{M}}^{j_1 + j_2}(\mathbb{H}^n)$.

Consequently

$$K_1 * K_2 \in \widetilde{\mathcal{M}}^{j_1 + j_2}(\mathbb{H}^n).$$

- $j_1 \geqslant 0$ and $j_2 < 0$:

Consider any natural numbers k and s, such that $j_1 - ka_1 < 0$ and $j_1 - 2As < 0$. By Proposition 46

$$K_1 = \sum_{1 \leqslant r_1 \ldots, r_k \leqslant n} X^R_{r_1} \cdots X^R_{r_k} K_{r_1,\ldots,r_k} + K_o$$

with $K_{r_1,\ldots,r_k} \in \widetilde{\mathcal{M}}^{j_1 - a_{r_1} - \ldots - a_{r_k}}(\mathbb{H}^n)$, and $K_o \in \widetilde{\mathcal{M}}^{j_1 - 2As}(\mathbb{H}^n)$. With an identical argument as in the previous case we can establish that $K_1 * K_2 \in \widetilde{\mathcal{M}}^{j_1 + j_2}(\mathbb{H}^n)$.

$\square$

Theorem 48. *Let j_1, j_2 be two non negative real numbers. If $K_1 \in \widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, and $K_2 \in \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n)$ then their convolution $K_1 * K_2 \in \widetilde{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$.*

Proof. Consider arbitrary natural numbers k, s such that $j_2 - ka_1 < 0,$ *and* $j_2 - 2sA < 0$. Then by Proposition 46, K_2 has the following expression

$$K_2 = \sum_{1 \leqslant r_1 \dots, r_k \leqslant n} X_{r_1} \cdots X_{r_k} K_{1, r_1, \dots, r_k} + K_o$$

with $K_{1,r_1,\dots,r_k} \in \widetilde{\mathcal{M}}^{j_2 - a_{r_1} - \dots - a_{r_k}}(\mathbb{H}^n)$, and $K_o \in \widetilde{\mathcal{M}}^{j_2 - 2sA}(\mathbb{H}^n)$.

Then we have

$$\begin{aligned} K_1 * K_2 &= K_1 * \left(\sum_{1 \leqslant r_1 \dots, r_k \leqslant n} X_{r_1} \cdots X_{r_k} K_{r_1, \dots, r_k} + K_o \right) \\ &= \sum_{1 \leqslant r_1 \dots, r_k \leqslant n} X_{r_1} \cdots X_{r_k} \left(K_1 * K_{r_1, \dots, r_k} \right) + K_1 * K_o \,. \end{aligned}$$

K_1 is in $\widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n)$, with $j_1 \geqslant 0$. Since $j_2 - a_{r_1} - \dots - a_{r_k} \leqslant j_2 - ka_1 < 0$, by our hypothesis over k, then $K_{r_1,\dots,r_k}$ is in $\widetilde{\mathcal{M}}^{j_2 - a_{r_1} - \dots - a_{r_k}}(\mathbb{H}^n)$, with $j_2 - a_{r_1} - \dots - a_{r_k} < 0$. Therefore by Theorem 47, case II, each $K_1 * K_{r_1,\dots,r_k} \in \widetilde{\mathcal{M}}^{j_1 + j_2 - a_{r_1} - \dots - a_{r_k}}(\mathbb{H}^n)$. So, $X_{r_1}, \dots, X_{r_k}$ $(K_1 * K_{r_1,\dots,r_k}) \in \widetilde{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$. And therefore $\sum_{1 \leqslant r_1 \dots, r_k \leqslant n} X_{r_1} \cdots X_{r_k} (K_1 * K_{r_1,\dots,r_k})$ is in $\widetilde{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$.

$K_o \in \widetilde{\mathcal{M}}^{j_2 - 2sA}(\mathbb{H}^n)$, with $j_2 - 2sA < 0$. Then by Theorem 47, case II, $K_1 * K_o \in \widetilde{\mathcal{M}}^{j_1 + j_2 - 2sA}(\mathbb{H}^n)$. Given that $\widetilde{\mathcal{M}}^{j_1 + j_2 - 2sA}(\mathbb{H}^n) \subset \widetilde{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$, then $K_1 * K_o \in \widetilde{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$.

Consequently, $K_1 * K_2$ is in $\widetilde{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$, as desired. $\square$

CHAPTER VI

EXISTENCE OF PSEUDODIFFERENTIAL CALCULI

We begin by recalling some basic facts on pseudodifferential operators of Hörmander type $(\rho, 0)$, and how the elements of $\operatorname{Op}\left(\mathcal{S}^m_{\rho,0}(\mathbb{R}^n)\right)$ can be characterised.

Definition 49 (Hörmander). Let m, ρ, δ be real numbers, and suppose $\rho, \delta \in [0,1]$. The *symbol class of order* m, denoted by $S^m_{\rho,\delta}$ consists of those functions $a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for any pair of multiindeces α, β, and any compact set $K \subset \mathbb{R}^n$, there exists a constant $C_{\alpha,\beta,K}$ such that

$$\left|D^\beta_x D^\alpha_\xi a(x,\xi)\right| \leqslant C_{\alpha,\beta,K}\langle\xi\rangle^{m-\rho\|\alpha\|+\delta\|\beta\|} \qquad \forall x \in K,\ \xi \in \mathbb{R}^n$$

where $\langle\xi\rangle = \left(1 + \|\xi\|^2\right)^{1/2}$.

Definition 50. We say that a linear operator $a(x, D) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is a pseudodifferential operator with symbol $a(x, \xi)$ of class $S^m_{\rho,\delta}$, if $a(x, D)$ can be represented by

$$\big[a(x, D)f\big](x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} a(x, \xi)\ \widehat{f}(\xi)\, d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. In this case we say that $a(x, D)$ belongs to $\mathrm{Op}\left(S^m_{\rho,\delta}\right)$.

If $a(x, \xi) \in S^m_{\rho,0}$ then the operator $a(x, D) : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ can be extended to $a(x, D) : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ by

$$\big[a(x, D)J\big] = \widehat{J}(a_f)$$

where

$$a_f(\xi) := \int e^{-2\pi x\cdot\xi} a(x, \xi)\, f(x)\, dx \qquad \forall f \in \mathcal{S}(\mathbb{R}^n).$$

Definition 51. We say that an operator $A(D)$ in $\mathcal{S}(\mathbb{R}^n)$ is a *Fourier multiplier operator* if

$$\big[A(D)f\big](x) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} a(\xi)\ \widehat{f}(\xi)\, d\xi$$

for $f \in \mathcal{S}(\mathbb{R}^n)$. The reason for this name is that these operators multiply the Fourier transform of f by a function $a(\xi)$, called the multiplier.

Notice that Fourier multiplier operators can be written as convolution operators

$$\big[A(D)f\big](x) = \big(a\widehat{f}\big)^{\check{}}\,(x) = \big(\check{a} * f\big)(x)$$

$S^m_{\rho,\#}$ will be used to denote the space of those Fourier multipliers which are in $S^m_{\rho,0}$. With the family of seminorms defined by

$$\|p\|_{\alpha,m,\rho} = \sup_{\xi} \left\{ \langle\xi\rangle^{-m+\rho\|\alpha\|} \left| D_{\xi}^{\alpha} p(\xi) \right| \right\}$$

$S^{m}_{\rho,\#}$ is a Fréchet space.

Suppose $a(x,\xi) \in S^{m}_{\rho,0}$. For any $x \in \mathbb{R}^n$ we define $a_x(\xi) = a(x,\xi)$. If $A(y)$ is the operator of Fourier multiplication by $a_y(\xi)$, then for any $f \in \mathcal{S}(\mathbb{R}^n)$

$$\begin{aligned} \big[a(x,D)f\big](x) &= \int e^{-2\pi i x\cdot\xi}\, a(x,\xi)\, \widehat{f}(\xi)\, d\xi \\ &= \int e^{-2\pi i x\cdot\xi}\, a_x(\xi)\, \widehat{f}(\xi)\, d\xi \qquad \text{(VI.1)} \\ &= \big[A(x)f\big](x) \end{aligned}$$

Therefore one can say that locally, a pseudodifferential operator is a multiplier operator, where the multiplier depends smoothly upon the point at which we are.

Since $[A(x)f](x) = (\check{a}_x * f)(x)$ then

$$[a(x,D)f](x) = (\check{a}_x * f)(x).$$

Therefore, locally one can always represent a pseudodifferential operator as a family of convolution operators, where one convolves with an element $\check{a}_x \in \left(S^{m}_{\rho,\#}\right)^{\check{}}$. This point of view is very useful when working with pseudodifferential operators on homogeneous groups G, where the group Fourier transform is cumbersome to use. Consequently pseudodifferential operators on $\mathbb{R}^n$ with symbols in $S^{m}_{\rho,0}$ can be obtained from families of convolution operators on $\mathbb{R}^n$.

M. E. Taylor shows similarly that classes of pseudodifferential operators on a Lie G group can be constructed from smooth families of operators on G. If G is a Lie group, instead of Fourier multiplier operators, we consider convolution operators on G , defined by

$$[Af](x) = \int_G a(y)\, f(y^{-1}x)\, dy \qquad \forall f \in C_c^\infty$$

where dy stands for the Haar measure on the group. And the Fourier transform of a, $\widehat{a}$ belongs to $\mathfrak{X}$, a Fréchet space contained in $C^\infty(\mathfrak{g}')$, where $\mathfrak{g}'$ stands for the linear dual of the Lie algebra of G.

We assume that $\mathfrak{X} \subseteq S^m_{\rho,\#}(\mathbb{R}^n)$ for some $m \in \mathbb{R}$ and $\rho \in (0,1]$, and we say that $A \in \operatorname{Op}(\mathfrak{X})$.

If $\widehat{a}(y,\xi) = \int e^{2\pi i x\cdot\xi}\, a(y,x)\, dx$ is a smooth function of y, with values in $\mathfrak{X}$, for y in a neighborhood of the identity element $e \in G$, then $A(y)$ defined using the modular function Δ by

$$\big[A(y)f\big](x) = \int_G a(y, xz^{-1})\, f(z)\, \Delta(z^{-1})\, dz \qquad \forall f \in C_c^\infty(G)$$

is a smooth function of y, taking values in the Fréchet space $\operatorname{Op}(\mathfrak{X})$. $\Delta = 1$ in a homogeneous group. Then we associate the operator

$$\big[\mathfrak{A}f\big](x) = \big[A(x)f\big](x)$$

and we say that $\mathfrak{A} \in \operatorname{Op}\left(\widetilde{\mathfrak{X}}\right)$. Notice the analogy with (VI.1). If $a(y,z)$ has compact support in y, we say $\mathfrak{A} \in \operatorname{Op_c}\left(\widetilde{\mathfrak{X}}\right)$

In the following theorem we have collected a list of remarks and results from M.E. Taylor.

Theorem 52. *Suppose G is a Lie group, and $\{\mathfrak{X}^m\}_{m\in\mathbb{R}}$ is a nested family of Fréchet spaces satisfying the following properties*

a) *If $m \geqslant 0$ then $\mathfrak{X}^m \subset S^m_{\rho,\#}$ for some $\rho \in (0,1]$.*

b) *If $m < 0$ then $\mathfrak{X}^m \subset S^{m\sigma}_{\rho,\#}$ for some $\sigma \in (0,1]$.*

c) If $A \in \operatorname{Op}(\mathfrak{X}^{m_1})$, and $B \in \operatorname{Op}(\mathfrak{X}^{m_2})$, then $AB \in \operatorname{Op}(\mathfrak{X}^{m_1+m_2})$. Moreover the composition of convolution operators is continuous.

d) If $p(\xi) \in \mathfrak{X}^m$, then $D_\xi^\alpha p(\xi) \in \mathfrak{X}^{m-\tau\|\alpha\|}$ for some $\tau \in (0,1]$.

e) If $K_j \in \mathfrak{X}^{m-\tau j}$, then there exists $K \in \mathfrak{X}^m$ such that, for any M, if N is sufficiently large,

$$K - (K_o + \cdots + K_N) \in S_{\rho\#}^{-M}$$

f) If $p(\xi) \in \mathfrak{X}^m$ then $\bar{p}(\xi) \in \mathfrak{X}^m$.

If $\mathfrak{A} \in \operatorname{Op}\left(\widetilde{\mathfrak{X}}^{m_1}\right)$ and $\mathfrak{B} \in \operatorname{Op}_c\left(\widetilde{\mathfrak{X}}^{m_2}\right)$ then $\mathfrak{A}\mathfrak{B} \in \operatorname{Op}\left(\widetilde{\mathfrak{X}}^{m_1+m_2}\right)$, and it has an asymptotic expansion

$$[\mathfrak{A}\mathfrak{B}f](x) \sim \sum_{\gamma \geqslant 0} A^{[\gamma]}(x) B_{[\gamma]}(x)\, f(x)$$

If $\mathfrak{A} \in \operatorname{Op}\left(\widetilde{\mathfrak{X}}^{m}\right)$ then the adjoint $\mathfrak{A}^* \in \operatorname{Op}\left(\widetilde{\mathfrak{X}}^{m}\right)$ and it has the following asymptotic expansion

$$[\mathfrak{A}^* f](x) \sim \sum_{\gamma \geqslant 0} A^{[\gamma]}(x) f(x)$$

The proof can be found in [Taylor 84]

Corollary 53. *In order to prove the existence on a Lie group G of an analogue of the usual pseudodifferential calculus on Euclidean space, it suffices to construct a family $\{\mathfrak{X}^m\}_{m\in\mathbb{R}}$ satisfying the hypothesis of the previous theorem.*

We consider the family $\left\{\mathcal{M}^j(G)\right\}_{j\in\mathbb{R}}$, where the spaces of multipliers $\mathcal{M}^j(G)$ are as in definition 12, and we shall show that this family verifies the hypothesis of Theorem 52.

Proposition 54. $\left\{\mathcal{M}^j(G)\right\}_{j\in\mathbb{R}}$ *is a nested family of Fréchet spaces.*

Proof. Each $\mathcal{M}^j(G)$ is easily seen to be a Fréchet space. Suppose $j_1 \leqslant j_2$ and $J \in \mathcal{M}^{j_1}(G)$. If α is a multiindex, then there exists a positive constant C_α such that for all ξ

$$\begin{aligned} \left|\partial^\alpha J(\xi)\right| &\leqslant C_\alpha(1+|\xi|)^{j_1-|\alpha|} \\ &\leqslant C_\alpha(1+|\xi|)^{j_2-|\alpha|} \qquad \text{since } j_1 \leqslant j_2 \end{aligned}$$

Therefore $J \in \mathcal{M}^{j_2}(G)$. We also notice that the inclusion map is continuous. □

Proposition 55. *If $j \geqslant 0$ then $\mathcal{M}^j(G) \subseteq S^j_{\rho\#}$, for some $\rho \in (0,1]$.*

Proof. If $J \in \mathcal{M}^j(G)$, and $\alpha \in (\mathbb{Z}^+)^n$ then there exists a positive constant C_α such that for all ξ

$$\begin{aligned} \left|\partial^\alpha J(\xi)\right| &\leqslant C_\alpha(1+|\xi|)^{j-|\alpha|} \\ &= C_\alpha(1+|\xi|)^{j}(1+|\xi|)^{-|\alpha|} \\ &\leqslant C_\alpha\frac{(1+|\xi|)^j}{(1+|\xi|)^{\|\alpha\|a_1}} \qquad \text{since } \|\alpha\|a_1 \leqslant |\alpha| \\ &\leqslant C'_\alpha\frac{(1+|\xi|)^j}{\langle\xi\rangle^{\frac{a_1}{a_n}\|\alpha\|}} \\ &\leqslant C''_\alpha\frac{\langle\xi\rangle^j}{\langle\xi\rangle^{\frac{a_1}{a_n}\|\alpha\|}} \\ &= C''_\alpha\langle\xi\rangle^{j-\frac{a_1}{a_n}\|\alpha\|} \end{aligned}$$

Then $J \in S^j_{\left(\frac{a_1}{a_n}\right)\#}$. Therefore $\mathcal{M}^j(G) \subseteq S^j_{\rho\#}$ with $\rho = \frac{a_1}{a_n} \in (0,1]$. □

Proposition 56. *If $j < 0$, then $\mathcal{M}^j(G) \subseteq \mathcal{S}^{j\sigma}_{\rho\#}$, for some $\sigma \in (0,1]$.*

Proof. If $J \in \mathcal{M}^j(G)$, and $\alpha \in (\mathbb{Z}^+)^n$, then there exists a positive constant C_α such that for all ξ

$$\begin{aligned} \left|\partial^\alpha J(\xi)\right| &\leqslant C_\alpha (1+|\xi|)^{j-|\alpha|} \\ &\leqslant C'_\alpha \frac{(1+|\xi|)^j}{\langle\xi\rangle^{\|\alpha\| a_1}} \\ &\leqslant C''_\alpha \frac{\langle\xi\rangle^{\frac{1}{a_n}j}}{\langle\xi\rangle^{\|\alpha\|\frac{a_1}{a_n}}} \\ &= C''_\alpha \langle\xi\rangle^{\frac{1}{a_n}j - \frac{a_1}{a_n}\|\alpha\|} \end{aligned}$$

Then $J \in \mathcal{S}^{j\frac{1}{a_n}}_{\left(\frac{a_1}{a_n}\right)\#}$. Therefore $\mathcal{M}^j(G) \subseteq \mathcal{S}^{j\sigma}_{\rho\#}$, with $\sigma = \frac{1}{a_n}$, and $\rho = \frac{a_1}{a_n}$. $\square$

Proposition 57. $\widecheck{\mathcal{M}}^{j_1}(\mathbb{H}^n) * \widecheck{\mathcal{M}}^{j_2}(\mathbb{H}^n) \subseteq \widecheck{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$ *and the product is continuous.*

Proof. In chapters III and IV we have established that $\widecheck{\mathcal{M}}^{j_1}(\mathbb{H}^n) * \widecheck{\mathcal{M}}^{j_2}(\mathbb{H}^n) \subseteq \widecheck{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$ for all $j_1, j_2 \in \mathbb{R}$.

In order to prove the continuity of the product, for a fixed pair of $j_1, j_2 \in \mathbb{R}$ we define the following bilinear map

$$T : \mathcal{M}^{j_1}(\mathbb{H}^n) \times \mathcal{M}^{j_2}(\mathbb{H}^n) \longrightarrow \mathcal{M}^{j_1+j_2}(\mathbb{H}^n)$$

$$(J_1, J_2) \longmapsto (\check{J}_1 * \check{J}_2)^\wedge$$

and show that it is continuous.

We consider the following mappings, where "double" arrows are used to denote mapping which are sequentially continuous

$$\begin{tikzcd}
 & & \mathcal{S}'(\mathbb{R}^{2n+1}) \\
\mathcal{M}^{j_1}(\mathbb{H}^n) \times \mathcal{M}^{j_2}(\mathbb{H}^n) \arrow[urr, Rightarrow, "\widetilde{T}"] \arrow[rr, "T"'] & & \mathcal{M}^{j_1+j_2}(\mathbb{H}^n) \arrow[u, Rightarrow, "i"']
\end{tikzcd}$$

where i denotes the inclusion. $\widetilde{T}$ is the same map as T but it is thought as a map from $\mathcal{M}^{j_1}(\mathbb{H}^n) \times \mathcal{M}^{j_2}(\mathbb{H}^n)$ to $\mathcal{S}'(\mathbb{R}^{2n+1})$. Notice that the spaces, $\mathcal{M}^{j_1+j_2}(\mathbb{H}^n)$, and $\mathcal{M}^{j_1}(\mathbb{H}^n) \times \mathcal{M}^{j_2}(\mathbb{H}^n)$, are Fréchet spaces.

In order to prove the continuity of T it suffices to prove that i is sequentially continuous and that $\widetilde{T}$ is separately sequentially continuous and then by the Closed Graph Theorem, T will result continuos.

Suppose $\{J_k\}_k$ is a sequence in $\mathcal{M}^{j_1+j_2}(\mathbb{H}^n)$, $J \in \mathcal{M}^{j_1+j_2}(\mathbb{H}^n)$, and $J_k \longrightarrow J$ as $k \longrightarrow \infty$. Therefore

$$\left\| J_k - J \right\|_{\mathcal{M}^{j_1+j_2},N} = \sum_{|\alpha| \leqslant N} \left\| \left(1 + |\xi|\right)^{|\alpha|-(j_1+j_2)} \partial^\alpha \left(J_k - J\right) \right\| \xrightarrow[k \longrightarrow \infty]{} 0$$

So, for $N = 0$

$$\left\| J_k - J \right\|_{\mathcal{M}^{j_1+j_2},0} = \left\| \left(1 + |\xi|\right)^{-(j_1+j_2)} \left(J_k - J\right) \right\| \xrightarrow[k \longrightarrow \infty]{} 0$$

For $f \in \mathcal{S}(\mathbb{R}^{2n+1})$ we have

$$\begin{aligned}
&\left|(J_k - J)f\right| \\
&\quad = \left|\int_{\mathbb{R}^{2n+1}} (J_k - J)(\xi)\, f(\xi)\, d\xi\right| \\
&\quad \leqslant \int_{\mathbb{R}^{2n+1}} \left(1+|\xi|\right)^{-Q-1}\left|\left(1+|\xi|\right)^{-(j_1+j_2)}(J_k - J)(\xi)\right|\,\left|\left(1+|\xi|\right)^{j_1+j_2+Q+1} f(\xi)\right|\, d\xi \\
&\quad \leqslant \left(\sup_\xi \left|\left(1+|\xi|\right)^{-(j_1+j_2)}(J_k - J)(\xi)\right|\right)\left(\sup_\xi \left|\left(1+|\xi|\right)^{j_1+j_2+Q+1} f(\xi)\right|\right)\int_{\mathbb{R}^{2n+1}} \left(1+|\xi|\right)^{-Q-1} d\xi \\
&\quad \leqslant C\|f\|_M \|J_k - J\|_{\mathcal{M}^{j_1+j_2},0} \xrightarrow[k \to \infty]{} 0
\end{aligned}$$

for some $M \in \mathbb{Z}^+$, and where $\|\cdot\|_M$ denotes a Schwarz space seminorm. Consequently $i : \mathcal{M}^{j_1+j_2}(\mathbb{H}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^{2n+1})$ is sequentially continuous.

Now we want to prove that $\widetilde{T}$ is separately sequentially continuous, which means that we want to show that for each fixed $J_2 \in \mathcal{M}^{j_2}(\mathbb{H}^n)$ and $J_1 \in \mathcal{M}^{j_1}(\mathbb{H}^n)$ the operators

$$\begin{array}{ccc}
\mathcal{M}^{j_1}(\mathbb{H}^n) & \xrightarrow{T_{J_2}} & \mathcal{S}'(\mathbb{R}^{2n+1}) \\
J_1 & \longmapsto & \left(\check{J}_1 * \check{J}_2\right)^{\wedge}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{M}^{j_2}(\mathbb{H}^n) & \xrightarrow{T_{J_1}} & \mathcal{S}'(\mathbb{R}^{2n+1}) \\
J_2 & \longmapsto & \left(\check{J}_1 * \check{J}_2\right)^{\wedge}
\end{array}$$

are sequentially continuous.

In order to show that the mapping T_{J_2} is sequentially continuous, we shall see that it can be written as the composition of sequentially continuous mappings F, G defined as follows, and the Fourier transform.

$$\begin{array}{ccccccc}
\mathcal{M}^{j_1}(\mathbb{H}^n) & \xrightarrow{F} & \mathcal{E}' \oplus \mathcal{S} & \xrightarrow{G} & \mathcal{S}'(\mathbb{R}^{2n+1}) & \xrightarrow{\wedge} & \mathcal{S}'(\mathbb{R}^{2n+1}) \\
J_1 & \longmapsto & \left(\varphi\check{J}_1, (1-\varphi)\check{J}_1\right) & \longmapsto & \left(\varphi\check{J}_1\right) * \check{J}_2 + (1-\varphi)\check{J}_1 * \check{J}_2 & \longmapsto & \left[\left(\varphi\check{J}_1\right) * \check{J}_2 + (1-\varphi)\check{J}_1 * \check{J}_2\right]^{\wedge}
\end{array}$$

where $\varphi \in C_c^\infty(\mathbb{R}^{2n+1})$, $\varphi = 1$ in a neighborhood of zero.

We say that $h_n \longrightarrow h$ in $\mathcal{E}' \oplus \mathcal{S}$ if $h_n = (f_n, g_n)$, $h = (f, g)$ with $f_n, f \in \mathcal{E}'$, $g_n, g \in \mathcal{S}$ $f_n \longrightarrow f$ in $\mathcal{E}'$ and $g_n \longrightarrow g$ in $\mathcal{S}$. Since $\mathcal{E}' * \mathcal{S}' \subset \mathcal{S}'$, we have $\mathcal{S} * \mathcal{S}' \subseteq \mathcal{S}$, and where the inclusions are sequentially continuous, therefore G is sequentially continuous.

Since $\widehat{\ }$ is the Fourier transform from $\mathcal{S}'$ to $\mathcal{S}'$, it is continuous.

Finally in order to show that F is continuous, for a fixed $\varphi \in C_c^\infty(\mathbb{R}^{2n+1})$, we shall show that the mappings

$$\begin{array}{ccccccc} \mathcal{M}^{j_1}(\mathbb{H}^n) & \xrightarrow{R_1} & \mathcal{E}' & \qquad & \mathcal{M}^{j_1}(\mathbb{H}^n) & \xrightarrow{R_2} & \mathcal{S} \\ J_1 & \longmapsto & \varphi \check{J}_1 & \qquad & J_1 & \longmapsto & (1-\varphi)\check{J}_1 \end{array}$$

are continuous, and hence $F = R_1 + R_2$ will be shown to be continuous.

Consider the following mappings

$$\begin{tikzcd} & \mathcal{S}'(\mathbb{R}^{2n+1}) \\ \mathcal{M}^{j_1}(\mathbb{H}^n) \arrow[ur, Rightarrow, "\widetilde{R}_2"] \arrow[r, "R_2"'] & \mathcal{S}(\mathbb{R}^{2n+1}) \arrow[u, Rightarrow, "i"'] \end{tikzcd}$$

The sequential continuity of $\widetilde{R}_2$ follows from the following observations. First, the fact that the inclusion $i : \mathcal{M}^{j_1}(\mathbb{H}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^{2n+1})$ is sequentially continuous. Second, the continuity of the inverse Fourier transform in $\mathcal{S}'(\mathbb{R}^{2n+1})$. And finally the continuity in $\mathcal{S}'$ of multiplication by $(1-\varphi)$. The continuity of R_2 follows from the the Closed Graph Theorem, since we also know that the inclusion $i : \mathcal{S}(\mathbb{R}^{2n+1}) \hookrightarrow \mathcal{S}'(\mathbb{R}^{2n+1})$ is sequentially continuous and that the spaces $\mathcal{M}^{j_1}(\mathbb{H}^n)$ and $\mathcal{S}(\mathbb{R}^{2n+1})$ are of Fréchet type.

To see that R_1 is continuous we fix $f_o \in C^\infty$ and observe that the function

$$\begin{array}{ccc} \mathcal{M}^{j_1}(\mathbb{H}^n) & \xrightarrow{\widetilde{R}_1} & \mathbb{C} \\ J_1 & \longmapsto & (\varphi\check{J}_1)(f_o) \end{array}$$

is continuous since $(\varphi\check{J}_1)(f_o) = \check{J}_1(\varphi f_o) = J_1[(\varphi f_o)\widehat{\ }]$, so $\widetilde{R}(J_1) = J_1[(\varphi f_o)\widehat{\ }]$, and $(\varphi f_o)\widehat{\ } \in \mathcal{S}$, and convergence in $\mathcal{M}^{j_1}$ implies convergence in $\mathcal{S}'$.

Therefore $F = R_1 + R_2$ is a continuous map. Therefore T_{J_2} is sequentially continuous.

The sequential continuity of T_{J_1} follows in a completely analogous fashion. Hence $\widetilde{T}$ is separately sequentially continuous, and therefore T is continuous. $\square$

Proposition 58. *Let $J_i \in \mathcal{M}^{j-\tau i}(G)$, for $i = 0, 1, 2, \ldots$, and $\tau \in (0, 1]$. Then there exists a $J \in \mathcal{M}^{j}(G)$ such that, for any M, if N is sufficiently large $\left(J - \sum_{i=0}^{N} J_i\right) \in \mathcal{S}^{-M}_{\rho\#}$.*

Proof. Let $\psi : \mathbb{R}^n \longrightarrow [0, 1]$ be a C^{∞} function such that $\psi = 0$ for $|\xi| \leqslant 1$, and $\psi = 1$ for $|\xi| \geqslant 2$. Let $\{t_i\}_i$ be a positive decreasing sequence such that $0 < \ldots < t_1 < t_o \leqslant 1$ with $\lim\limits_{i \to 0^+} t_i = 0$. These t_i will be specified later. Define

$$J(\xi) := \sum_{i=0}^{\infty} \psi\big(\delta_{t_i}(\xi)\big) J_i(\xi)$$

Since $t_i \longrightarrow 0$ as $i \longrightarrow 0^+$, then for any fixed ξ, $\psi(\delta_{t_i}(\xi)) = 0$ for all, except for a finite number of i; so there are only finitely many non zero terms in the previous sum. Consequently this sum is well defined, and it follows that $J \in C^{\infty}(\mathbb{R}^n)$.

Our intention is to show that $J - \sum_{i=0}^{N} J_i \in \mathcal{M}^{j-\tau(N+1)}(G)$ for any $N \in \mathbb{Z}^+$. Then by Propositions 55 and 56 we shall have

$$J - \sum_{i=0}^{N} J_i \in \mathcal{M}^{j-\tau(N+1)}(G) \subseteq \mathcal{S}^{[j-\tau(N+1)]\sigma}_{\left(\frac{a_1}{a_n}\right)\#} \qquad \text{for some } \sigma \in (0, 1]$$

Hence if we choose N sufficiently large such that $j - \tau(N + 1) < -M$, then we have $[j - \tau(N + 1)]\sigma < -M$, which implies that

$$J - \sum_{i=0}^{N} J_i \in \mathcal{S}^{-M}_{\left(\frac{a_1}{a_n}\right)\#}$$

as desired.

Since $\psi = 0$ for $|\xi| \leqslant 1$, and $\psi = 1$ for $|\xi| \geqslant 2$, then for $|\beta| \neq 0$, $\left(\partial^{\beta}\psi\right)\left(\delta_t(\xi)\right) = 0$, when $|\delta_t(\xi)| = t|\xi| \leqslant 1$, or when $|\delta_t(\xi)| = t|\xi| \geqslant 2$. Then for $|\beta| \neq 0$, $\left(\partial^{\beta}\psi\right)\left(\delta_t(\xi)\right) \neq 0$

implies that $1 < |\delta_t(\xi)| = t|\xi| < 2$, i.e. $t^{-1} < |\xi| < 2t^{-1}$. If $0 < t \leqslant 1$ then $1 + |\xi|$ is also comparable with t^{-1}, which implies that there exists some positive constant C' such that $t < C'(1 + |\xi|)^{-1}$. Therefore if $0 < t \leqslant 1$, we have

$$
\begin{aligned}
\left|\partial^\beta[\psi\left(\delta_t(\xi)\right)]\right| &= \left|t^{|\beta|}\left[(\partial^\beta\psi)(\delta_t(\xi))\right]\right| \\
&= t^{|\beta|}\left|(\partial^\beta\psi)(\delta_t(\xi))\right| \\
&\leqslant C''t^{|\beta|} \\
&\leqslant C'C''(1 + |\xi|)^{-|\beta|} \\
&= C_\beta(1 + |\xi|)^{-|\beta|} .
\end{aligned}
$$

Therefore $\left\{\psi \circ \delta_t\right\}_{0<t\leqslant 1} \subset \mathcal{M}^0(G)$.

For any multiindex α, and $i \in \mathbb{Z}^+$, from Leibniz's rule, and the fact that $J_i \in \mathcal{M}^{j-\tau i}(G)$, we have

$$
\begin{aligned}
\left|\partial^\alpha\left[\psi(\delta_t\xi)J_i(\xi)\right]\right| &= \left|\sum_{\beta+\gamma=\alpha} C_{\beta,\gamma}\partial^\beta(\psi(\delta_t\xi))\partial^\gamma J_i(\xi)\right| \\
&\leqslant \sum_{\beta+\gamma=\alpha} C_{\beta,\gamma}\left|\partial^\beta(\psi(\delta_t\xi))\right|\left|\partial^\gamma J_i(\xi)\right| \\
&\leqslant \sum_{\beta+\gamma=\alpha} C_{\beta,\gamma}C_\beta(1 + |\xi|)^{-|\beta|}C_{i,\gamma}(1 + |\xi|)^{(j-\tau i)-|\gamma|} \\
&\leqslant C_{i,\gamma}(1 + |\xi|)^{(j-\tau i)-|\beta+\gamma|} \\
&\leqslant C_{i,\gamma}(1 + |\xi|)^{(j-\tau i)-|\alpha|} .
\end{aligned}
$$

Therefore $\left\{\left(\psi \circ \delta_t\right) J_i\right\}_{0<t\leqslant 1} \subset \mathcal{M}^{j-\tau i}(G)$.

We define $C_i := \max\left\{C_{i,\alpha} \,:\, |\alpha| \leqslant i\right\}$. Since $\psi(\xi) = 0$ if $|\xi| \leqslant 1$ then $\psi\big(\delta_t(\xi)\big) \neq 0$, implies $|\delta_t(\xi)| = t|\xi| > 1$, i.e. $|\xi| > t^{-1}$. So, we select $t_i > 0$ such that $t_i < t_{i-1}$, $|\xi| > t_i^{-1}$, and $C_i t_i^\tau \leqslant 2^{-i}$. Then $\psi\big(\delta_{t_i}(\xi)\big) = 0$ if $(1 + |\xi|)^{-\tau} \geqslant t_i^\tau$. Therefore for any multiindex α such that $|\alpha| \leqslant i$ we have

$$\begin{aligned}\left|\partial^\alpha\big[\psi(\delta_{t_i}\xi)J_i\big]\right| &\leqslant C_i(1+|\xi|)^{j-\tau i-|\alpha|}\\ &= C_i\big(1+|\xi|\big)^{-\tau}\big(1+|\xi|\big)^{j-\tau i+\tau-|\alpha|}\\ &< C_i t_i^\tau(1+|\xi|)^{j-\tau i+\tau-|\alpha|}\\ &\leqslant 2^{-i}(1+|\xi|)^{j-\tau i+\tau-|\alpha|}\end{aligned} \tag{VI.2}$$

For any multiindex β we choose i_o such that $i_o \geqslant |\beta|$, and we express J as

$$J = \sum_{i=0}^{i_o}\big(\psi\circ\delta_{t_i}\big)J_i + \sum_{i=i_o+1}^{\infty}\big(\psi\circ\delta_{t_i}\big)J_i$$

Since $\sum_{i=0}^{i_o}\big(\psi\circ\delta_{t_i}\big)J_i$ is a finite sum and $\big(\psi\circ\delta_{t_i}\big)J_i \in \mathcal{M}^{j-\tau i}(G) \subseteq \mathcal{M}^{j}(G)$; then $\sum_{i=0}^{i_o}\big(\psi\circ\delta_{t_i}\big)J_i \in \mathcal{M}^{j}(G)$. Therefore there exists a positive constant C such that for all ξ

$$\left|\partial^\beta\sum_{i=0}^{i_o}\psi\big(\delta_{t_i}(\xi)\big)J_i(\xi)\right| \leqslant C(1+|\xi|)^{j-|\beta|}.$$

By (VI.2)

$$\begin{aligned}\left|\partial^\beta\sum_{i=i_o+1}^{\infty}\psi\big(\delta_{t_i}(\xi)\big)J_i(\xi)\right| &\leqslant \sum_{i=i_o+1}^{\infty}2^{-i}\big(1+|\xi|\big)^{j-\tau i+\tau-|\beta|}\\ &\leqslant \left(\sum_{i=i_o+1}^{\infty}2^{-i}\right)\big(1+|\xi|\big)^{j-|\beta|}\\ &\leqslant \big(1+|\xi|\big)^{j-|\beta|}.\end{aligned}$$

Consequently

$$\begin{aligned}\left|\partial^\beta J(\xi)\right| &\leqslant \left|\partial^\beta \sum_{i=0}^{i_o} \psi\big(\delta_{t_i}(\xi)\big) J_i(\xi)\right| + \left|\partial^\beta \sum_{i=i_o+1}^{\infty} \psi\big(\delta_{t_i}(\xi)\big) J_i(\xi)\right| \\ &\leqslant C\big(1+|\xi|\big)^{j-|\beta|} + \big(1+|\xi|\big)^{j-|\beta|} \\ &= (C+1)\big(1+|\xi|\big)^{j-|\beta|}\end{aligned}$$

Since this estimate holds for any multiindex β, then $J \in \mathcal{M}^j(G)$.

Moreover, for $N \in \mathbb{Z}^+$, writing

$$J - \sum_{i=0}^{N} J_i = \sum_{i=0}^{N} \big[\big(\psi \circ \delta_{t_i}\big) - 1\big] J_i + \sum_{i=N+1}^{\infty} \big(\psi \circ \delta_{t_i}\big) J_i$$

and working in the same fashion as before, we obtain $\sum\limits_{i=N+1}^{\infty} (\psi \circ \delta_{t_i}) J_i \in \mathcal{M}^{j-\tau(N+1)}(G)$. On the other hand, since $\psi(\xi) = 1$ for $|\xi| \geqslant 2$, $\psi(\delta_{t_i}(\xi)) - 1 = 0$ for $|\delta_{t_i}(\xi)| = t_i|\xi| \geqslant 2$. So, $\psi(\delta_{t_i}(\xi))) - 1 = 0$ for $|\xi| \geqslant 2t_N^{-1}$. Then $\sum_{i=0}^{N} \big[(\psi \circ \delta_{t_i}) - 1\big] J_i \in \mathcal{M}^{-\infty}(G) = \mathcal{S}(G)$. Consequently for any N

$$\left(J - \sum_{i=0}^{N} J_i\right) \in \mathcal{M}^{j-\tau(N+1)}(G)$$

i.e. $J \sim \sum_{i=0}^{\infty} J_i$. □

Proposition 59. *If $J \in \mathcal{M}^j(G)$ then for any multiindex α, $D^\alpha J \in \mathcal{M}^{j-\tau\|\alpha\|}(G)$ for some $\tau \in (0,1]$.*

Proof. Suppose β is any multiindex. Since $J \in \mathcal{M}^j(G)$, then for all ξ

$$\begin{aligned}\left|\partial^\beta(\partial^\alpha J)(\xi)\right| &= \left|\partial^{\beta+\alpha} J(\xi)\right| \\ &\leqslant C_{\alpha+\beta}\big(1+|\xi|\big)^{j-|\alpha+\beta|} \\ &= C_{\alpha+\beta}\big(1+|\xi|\big)^{(j-|\alpha|)-|\beta|}\end{aligned}$$

Therefore $\partial^\alpha J \in \mathcal{M}^{j-|\alpha|}(G)$, so $D^\alpha J \in \mathcal{M}^{j-|\alpha|}(G) \subseteq \mathcal{M}^{j-\tau\|\alpha\|}$ with τ any number in the interval $(0,1]$. $\square$

Proposition 60. *If $J \in \mathcal{M}^j(G)$ then $\bar{J} \in \mathcal{M}^j(G)$*

Proof. Let α be a multiindex, since $J \in \mathcal{M}^j(G)$ then there exists a positive constant C_α such that for all ξ

$$\left|\partial^\alpha J(\xi)\right| \leqslant C_\alpha \left(1 + |\xi|\right)^{j-|\alpha|}.$$

Since $\left|\partial^\alpha \bar{J}(\xi)\right| = \left|\partial^\alpha J(\xi)\right|$, then for all ξ

$$\left|\partial^\alpha \bar{J}(\xi)\right| \leqslant \left(1 + |\xi|\right)^{j-|\alpha|}$$

Therefore $\bar{J} \in \mathcal{M}^j(G)$. $\square$

The following theorem contains a summary of the results from the previous seven propositions, and establishes the existence on the Heisenberg group of the general calculus analogous to the usual $S^m_{1,0}$–pseudodifferential calculus on $\mathbb{R}^n$.

Theorem 61 (Existence of a usual pseudodifferential calculus for $\mathbb{H}^n$). *Suppose $\mathbb{H}^n$ is the Heisenberg group of dimension n, and $\left\{\mathcal{M}^j(\mathbb{H}^n)\right\}_{j\in\mathbb{R}}$ is the family of spaces of multipliers defined in 12. Then the following properties are satisfied*

a) $\left\{\mathcal{M}^j(\mathbb{H}^n)\right\}_{j\in\mathbb{R}}$ *is a nested family of Fréchet spaces.*

b) *If $j \geqslant 0$ then $\mathcal{M}^j(\mathbb{H}^n) \subset S^m_{\rho,\#}$ for some $\rho \in (0,1]$.*

c) *If $j < 0$ then $\mathcal{M}^j(\mathbb{H}^n) \subset S^{m\sigma}_{\rho,\#}$ for some $\sigma \in (0,1]$.*

d) $\widetilde{\mathcal{M}}^{j_1}(\mathbb{H}^n) * \widetilde{\mathcal{M}}^{j_2}(\mathbb{H}^n) \subseteq \widetilde{\mathcal{M}}^{j_1+j_2}(\mathbb{H}^n)$ *and the product is continuous.*

e) *If $J \in \mathcal{M}^j(\mathbb{H}^n)$, and α is a multiindex, then $D^\alpha J \in \mathcal{M}^{j-\tau\|\alpha\|}(\mathbb{H}^n)$ for some $\tau \in (0,1]$.*

f) *Let $J_i \in \mathcal{M}^{j-\tau i}(\mathbb{H}^n)$, for $i = 0,1,2,\ldots$, and $\tau \in (0,1]$. Then there exists a $J \in \mathcal{M}^j(\mathbb{H}^n)$ such that, for any M, if N is sufficiently large*

$$\left(J - \sum_{i=0}^{N} J_i\right) \in S^{-M}_{\rho\#}.$$

g) *If $J \in \mathcal{M}^j(\mathbb{H}^n)$ then $\bar{J} \in \mathcal{M}^j(\mathbb{H}^n)$.*

And therefore on $\mathbb{H}^n$ there exists a general calculus of pseudodifferential operators analogous to the usual $S^m_{1,0}$–pseudodifferential calculus on $\mathbb{R}^n$.

We note that for the case of a general homogeneous group we have proved all the conditions in 52, except for part (d). We believe this generalisation can be proved adapting the techniques used in this work.

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