On the Algebra and Geometry of a Manifold's Chains and Cochains

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Abstract of the Dissertation

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This dissertation consists of two parts, each of which describes new algebraic and geometric structures defined on chain complexes associated to a manifold.

In the first part we define, on the simplicial cochains of a triangulated manifold, analogues of certain objects in differential geometry. In particular, we define a cochain product and prove several results on its convergence to the wedge product of differential forms. Also, for cochains with an inner product, we define a "combinatorial Hodge star operator", and describe some applications, including a combinatorial period matrix for a triangulated

Riemann surface. There are several convergence theorems here as well; for a particularly nice cochain inner product, both of these combinatorial structures converge to their continuum analogues as the mesh of the triangulation tends to zero.

In the second part, we describe an algebraic structure on the chains of a manifold, induced by the transversal intersection of chains. We prove that, up to quasi-isomorphism, the chains form an E_∞ algebra (a generalization of a commutative algebra). This chain algebra induces the usual intersection product on homology. This result follows from a general theorem that we prove, cast in the language of operads, on partially defined algebraic structures. We also describe an application of this theorem to string topology.

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0.1 Introduction

The ideas presented in this dissertation can all be organized around one central question: What interesting algebraic or geometric structures do manifolds possess? At the heart of this is Poincaré Duality, and understanding what might be its appropriate chain-level version. In fact, my starting point for research consisted of many stimulating discussions with Dennis Sullivan on this topic, and reading the impressive thesis of Thomas Tradler [65], where several breakthroughs are made on chain-level Poincaré Duality.

It was my original intention to use the results of [65] to give a homotopy-theoretic definition of a (co)chain-level Hodge star operator. This is reasonable since the smooth Hodge star operator on forms may be expressed as the composition of Poincaré Duality and a metric. In this dissertation, a 'first order solution' is presented; what we call a *combinatorial star operator*. This is defined by combining Poincaré Duality, represented by a 'cup product' on the simplicial cochains of a triangulated manifold, with an inner product on the vector space of simplicial cochains.

With this definition at hand, we are able to prove several results on the convergence of the combinatorial star operator to the smooth Hodge star operator as the mesh of a triangulation tends to zero. All of these statements are motivated by the ideas and results presented by Dodziuk, and later Dodziuk and Patodi, in [13] and [14]. In those papers, the authors show that cochains provide a good approximation to differential forms, and that a certain discrete Laplacian converges to the smooth Laplacian.

In studying the combinatorial star operator and the convergence statements, it became clear that the cochain product deserved an analytic treatment of its own. In this dissertation, we prove several statements on the convergence of this cochain product to the wedge product of forms.

Altogether we then have a sort of "combinatorial package" $\{\delta, \cup, \bigstar\}$, which serves as a discrete model of the exterior derivative d, the wedge product $\wedge$, and the Hodge star operator $\star$, all belonging to differential geometry. This model is finite at every stage, computable, and by our convergence results, is 'accurate' to any desired level. We expect this combinatorial package will have numerous applications to computer modeling of systems involving the fundamental objects $\{d, \wedge, \star\}$ of differential geometry.

One problem suggesting the need for such a combinatorial package is the Ising Model problem, belonging to statistical mechanics. In [12], Costa-Santos and McCoy study this problem for lattices on a surface, and make several calculations supporting their conjecture that the Ising Model partition function can be written as a sum of theta-functions, evaluated at certain "discrete period matrices".

With this in mind, we show that, on a triangulated surface, the combinatorial star operator gives rise to a 'combinatorial period matrix'. We are able to prove a convergence statement here too: for a triangulated Riemann surface, and a particularly nice choice of inner product, the combinatorial period matrix convergences to the Riemann period matrix as the mesh of a triangulation tends to zero. Thus a conformal structure can be recovered from finite data, to any desired accuracy.

The second part of this dissertation is also intimately related to Poincaré Duality, as it is entirely motivated by the intersection of chains in a manifold, and therefore also the homology intersection ring of a manifold.

The algebraic structure behind intersecting chains is a priori unclear, since the intersection is only partially defined (when chains are in general position). A formalism for such partially defined algebraic structures is developed by Kriz and May in [34], and several results are proven for partially defined simplicial algebraic structures. We extend these results to complexes, giving a general theorem stating that certain partially defined algebraic structures on complexes do capture all of the important homological information. An intuitive introduction to these ideas appears in the introduction to chapter 2.

In later sections we describe applications of this theorem. First, we discuss the intersection of chains in a manifold, and show that, up to quasi-isomorphism, there is an E_∞ algebra structure on the chains.

Using ideas introduced by Chas and Sullivan in [7], we also describe applications of this theorem to string topology, the study of the algebraic structure of the free loop space of a manifold.

We expect that the general theorem (2.2.5) on partially defined algebraic structures has many more applications. For example, intersecting chains in singular spaces (i.e chain-level Intersection Homology, see [22] and [23]). Also, we expect that versions of this theorem for modules over operads, or for properads, might lead to a better understanding of chain-level Poincaré Duality.

Each chapter of this dissertation may be read independently and contains its own introduction. The references from each chapter have been combined since there is considerable overlap.

Chapter 1

Geometric Structures on the Cochains of a Manifold

1.1 Introduction

In this chapter we develop combinatorial analogues of several objects in differential and complex geometry, including the Hodge star operator and the period matrix of a Riemann surface. We define these structures on the appropriate combinatorial analogue of differential forms, namely simplicial cochains.

As we recall in section 1.3, the two essential ingredients to the smooth Hodge star operator are Poincaré Duality and a metric, or inner product. In much the same way, we'll define the combinatorial star operator using both an inner product and Poincaré Duality, the latter expressed on cochains in the form of a (graded) commutative product.

Using the inner product introduced in [13], we prove the following:

Theorem 1.1.1. *The combinatorial star operator, defined on the simplicial cochains of a triangulated Riemannian manifold, converges to the smooth Hodge* *star operator as the mesh of the triangulation tends to zero.*

We show in section 1.7 that, on a closed surface, this combinatorial star operator gives rise to a combinatorial period matrix, and prove

Theorem 1.1.2. *The combinatorial period matrix of a triangulated Riemannian 2-manifold converges to the conformal period matrix of the associated Riemann surface, as the mesh of the triangulation tends to zero.*

This suggests a link between statistical mechanics and conformal field theory, where it is known that the partition function may be expressed in terms of theta functions of the conformal period matrix [39], see also [12], [47].

The above convergence statements are made precise by using an embedding of simplicial cochains into differential forms, first introduced by Whitney [69]. This approach was used quite successfully by Dodziuk [13], and later Dodziuk and Patodi [14], to show that cochains provide a good approximation to smooth differential forms, and that the combinatorial Laplacian converges to the smooth Laplacian. This formalism will be reviewed in section 1.4.

In section 1.5 we describe the cochain product that will be used in defining the combinatorial star operator. This product is of interest in its own right, and we prove several results concerning its convergence to the wedge product on forms; see also [31],[9]. These results may be of interest in numerical analysis and the modeling of PDE's, since they give a computable discrete model which approximates the algebra of smooth differential forms. The convergence statements on the cochain product, theorems 1.5.4 through 1.5.12, are not needed for later sections.

In section 1.6 we introduce the combinatorial star operator, and show that many of the interesting relations among $\star$, d, $\wedge$, and the adjoint d^* of d, that hold in the smooth setting, also hold in the combinatorial case. Some of the relations though, are more elusive, and may only be recovered in the limit of a fine triangulation.

In section 1.7 we study the combinatorial star operator on surfaces, and prove several results on the combinatorial period matrix, as mentioned above.

In the last two sections, 1.8 and 1.9, we show how an explicit computation of the combinatorial star operator is related to "summing over weighted paths," and perform these calculations for the circle.

1.2 Background and Acknowledgments

In this section we describe previous results that are related to the contents of this chapter. My sincere apologies to anyone whose work I have left out.

The cochain product we discuss was introduced by Whitney in [69]. It was also studied by Sullivan in the context of rational homotopy theory [59], by DuPont in his study of curvature and characteristic class [16], and by Birmingham and Rakowski as a star product in lattice gauge theory [6].

In connection with our result on the convergence of this cochain product to the wedge product of forms, Kervaire has a related result for the Alexander-Whitney product $\cup$ on cochains [31]. Kervaire states that, for differential forms A, B, and the associated cochains a, b,

$$\lim_{k\to\infty} a \cup b \ (S^k c) = \int_c A \wedge B$$

for a convenient choice of subdivisions $S^k c$ of the chain c. Cheeger and Simons use this result in the context of cubical cell structures in [9]. There they construct an explicit map $E(A, B)$ satisfying

$$\int A \wedge B - a \cup b = \delta E(A, B)$$

and use it in the development of the theory of differential characters. To the best of our knowledge, our convergence theorems for the commutative cochain product in section 1.5 are the first to appear in the literature.

Several definitions of a discrete analogue of the Hodge-star operator have been made. In [12], Costa-Santos and McCoy define a discrete star operator for a particular 2-dimensional lattice and study convergence properties as it relates to the Ising Model. Mercat defines a discrete star operator for surfaces in [46], using a triangulation and its dual, and uses it to study a notion of discrete holomorphy and its relation to Ising criticality.

In [61], Tarhasaari, Kettunen and Bossavit describe how to make explicit computations in electromagnetism using Whitney forms and a star operator defined using the de Rham map from forms to cochains. Teixeira and Chew [62] have also defined Hodge operators on a lattices for the purpose of studying electromagnetic theory.

Adams [2], and also Sen, Sen, Sexton and Adams [50], define two discrete star operators using a triangulation and its dual, and present applications to lattice gauge fields and Chern-Simons theory. De Beaucé and Sen [4] define star operators in a similar way and study applications to chiral Dirac fermions; and de Beaucé and Sen [5] have generalized this to give a discretization scheme for differential geometry [5].

In the approaches using a triangulation and its dual, the star operator(s) are formulated using the duality map between the two cell decompositions. This map yields Poincaré Duality on (co)homology. We express Poincaré Duality by a commutative cup product on cochains and combine it with a non-degenerate inner product to define the star operator. Working this way, we obtain a single operator from one complex to itself.

Our convergence statements in section 1.6 are proven using the inner product introduced in [13], and to the best of our knowledge, these are the first results proving a convergence theorem for a cochain-analogue of the Hodge star operator.

In Dodziuk's paper [13], and in [14] by Dodziuk and Patodi, the authors study a combinatorial Laplacian on the cochains and proved that its eigenvalues converge to the smooth Laplacian. Such discrete notions of a Hodge structure, along with finite element method techniques, were used by Kotiuga [33], and recently by Gross and Kotiuga [24], in the study of computational electromagnetism. Jin has used related techniques in studying electrodynamics [29].

Harrison's development of 'chainlet geometry' in [26], [27], and [28] has several themes similar to those in this chapter. In her new approach to geometric measure theory, the author develops 'dual analogues' of d, $\wedge$ and $\star$ by defining them on chainlets, a Banach space defined by taking limits of polyhedral chains. Chainlets are, in a sense, dual to differential forms in that they are 'domains of integration'. The author proves several convergence results for these analogues, and it appears these constructions and results have many applications as well.

In connection with our application of the combinatorial star operator to surfaces, in particular proving the convergence of our combinatorial period matrix to the conformal period matrix, Mercat has a related result in [47]. As part of his extensive study of what he calls “discrete Riemann surfaces”, he assigns to any such object a “period matrix” of twice the expected size. He shows that there are two sub-matrices of the appropriate dimension ($g \times g$) satisfying the property that, given what he calls “a refining sequence of critical maps,” they both converge to the continuum period matrix of an associated Riemann surface. This uses his results on discrete holomorphy approximations presented in [48]. Much like the star operators described above, our approach differs in that there is no “doubling” of complexes or operators.

There is another discussion of discrete period matrices presented in [25]. There Xianfeng Gu and Shing-Tung Yau give explicit algorithms for computing a period matrix for a surface. They point out that these can be implemented on the simplicial cochains by the use of the integration map from piecewise linear forms to simplicial cochains.

1.3 Smooth Setting

We begin with a brief review of some elementary definitions. Let M be a closed oriented Riemannian n-manifold. A Riemannian metric induces an inner product on $\Omega(M) = \bigoplus_j \Omega^j = \bigoplus_j \Gamma(\bigwedge^j T^*M)$ in the following way: a Riemannian metric determines an inner product on T^*M_p for all p, and hence an inner product for each j on $\bigwedge^j T^*M_p$ (explicitly, via an orthonormal basis).

An inner product $\langle , \rangle$ on $\Omega(M)$ is then obtained by integration over M. If we denote the induced norm on $\bigwedge^j T^*M_p$ by $| \quad |_p$, then the norm $\| \quad \|$ on $\Omega(M)$ is given by

$$\|\omega\| = \Big(\int_M |\omega|_p^2 \, dV \Big)^{1/2}$$

where dV is the Riemannian volume form on M.

Let $\mathcal{L}_2\Omega(M)$ denote the completion of $\Omega(M)$ with respect to this norm. We also use $\| \quad \|$ to denote the norm on the completion. Let the exterior derivative $d : \Omega^j(M) \to \Omega^{j+1}(M)$ be defined as usual.

Definition 1.3.1. *The Poincare-Duality pairing* $(,) : \Omega^j(M) \otimes \Omega^{n-j}(M) \to \mathbb{R}$ *is defined by:*

$$(\omega, \eta) = \int_M \omega \wedge \eta.$$

The pairing $(,)$ is bilinear, (graded) skew-symmetric and non-degenerate. It induces an isomorphism $\phi : \Omega^j(M) \to (\Omega^{n-j}(M))^*$, where here $*$ denotes the linear dual. The map $\psi : \Omega^{n-j}(M) \to (\Omega^{n-j}(M))^*$ induced by $\langle , \rangle$ is also an isomorphism and one may check that the composition $\psi^{-1} \circ \phi$ equals the following operator:

Definition 1.3.2. *The Hodge star operator* $\star : \Omega^j(M) \to \Omega^{n-j}(M)$ *is defined by:*

$$\langle \star\omega, \eta \rangle = (\omega, \eta).$$

One may also define the operator $\star$ using local coordinates, see Spivak [52]. We note that this approach and the former definition give rise to the same operator $\star$ on $\mathcal{L}_2\Omega(M)$. We prefer to emphasize definition 1.3.2 since it motivates definition 1.6.1, the combinatorial star operator.

Definition 1.3.3. *The adjoint of d, denoted by d^*, is defined by $\langle d^*\omega, \eta\rangle = \langle \omega, d\eta\rangle$.*

Note that $d^* : \Omega^j(M) \to \Omega^{j-1}(M)$. The following relations hold among $\star$, d and d^*. See Spivak [52].

Theorem 1.3.4. *As maps from $\Omega^j(M)$ to their respective ranges:*

1. $\star d = (-1)^{j+1} d^* \star$
2. $\star d^* = (-1)^j d \star$
3. $\star^2 = (-1)^{j(n-j)}\,\mathrm{Id}$

Definition 1.3.5. *The Laplacian is defined to be $\Delta = d^*d + dd^*$.*

Finally, we state the Hodge decomposition theorem for $\Omega(M)$. Let $\mathcal{H}^j(M) = \{\omega \in \Omega^j(M) | \Delta\omega = d\omega = d^*\omega = 0\}$ be the space of harmonic j-forms.

Theorem 1.3.6. *There is an orthogonal direct sum decomposition*

$$\Omega^j(M) \cong d\Omega^{j-1}(M) \oplus \mathcal{H}^j(M) \oplus d^*\Omega^{j+1}(M)$$

and $\mathcal{H}^j(M) \cong H^j_{DR}(M)$, the De Rham cohomology of M in degree j.

1.4 Whitney Forms

In his book, 'Geometric Integration Theory', Whitney explores the idea of using cochains as integrands [69]. A main result is that such objects provide a reasonable integration theory that in some sense generalizes the smooth theory of integration of differential forms. This idea has been made even more precise by the work of Dodziuk [13], who used a linear map of cochains into $\mathcal{L}_2$-forms (due to Whitney [69]) to show that cochains provide a good approximation of differential forms. In this section we review some of these results. The techniques involved illustrate a tight (and analytically precise) connection between cochains and forms, and will be used later to give precise meaning to our constructions on cochains. In particular, all of our convergence statements about combinatorial and smooth objects will be cast in a similar way.

Let M be a closed smooth n-manifold and K a fixed C^∞ triangulation of M. We identify $|K|$ and M and fix an ordering of the vertices of K. Let C^j denote the simplicial cochains of degree j of K with values in $\mathbb{R}$. Given the ordering of the vertices of K, we have a coboundary operator $\delta : C^j \to C^{j+1}$. Let μ_i denote the barycentric coordinate corresponding to the i^{th} vertex p_i of K. Since M is compact, we may identify the cochains and chains of K and for $c \in C^j$ write $c = \sum_\tau c_\tau \cdot \tau$ where $c_\tau \in \mathbb{R}$ and is the sum over all j-simplices τ of K. We write $\tau = [p_0, p_1, \ldots, p_j]$ of K with the vertices in an increasing sequence with respect to the ordering of vertices in K. We now define the Whitney embedding of cochains into $\mathcal{L}_2$-forms:

Definition 1.4.1. *For τ as above, we define*

$$W\tau = j! \sum_{i=0}^{j} (-1)^i \mu_i \; d\mu_0 \wedge \cdots \wedge \widehat{d\mu_i} \wedge \cdots \wedge d\mu_j.$$

W is defined on all of C^j by extending linearly.

Note that the coordinates μ_α are not even of class C^1, but they are C^∞ on the interior of any n-simplex of K. Hence, $d\mu_\alpha$ is defined and $W\tau$ is a well defined element of $\mathcal{L}_2\Omega^j$. By the same consideration, dW is also well defined. Note both sides of the definition of W are alternating, so this map is well defined for all simplices regardless of the ordering of vertices.

Several properties of the map W are given below. See [69],[13], [14] for details.

Theorem 1.4.2. *The following hold:*

1. $W\tau = 0$ *on* $M \backslash \overline{St(\tau)}$
2. $dW = W\delta$

where St *denotes the open star and* $\overline{}$ *denotes closure.*

One also has a map $R : \Omega^j(M) \to C^j(K)$, the de Rham map, given by integration. Precisely, for any differential form ω and chain c we have:

$$R\omega(c) = \int_c \omega$$

It is a theorem of de Rham that this map is a quasi-isomorphism (it is a chain map by Stokes Theorem). RW is well defined and one can check that $RW = Id$, see [69], [13], [14].

Before stating Dodziuk and Patodi's theorem that WR is approximately equal to the identity, we first give some definitions concerning triangulations. They also appear [14].

Definition 1.4.3. *Let* K *be a triangulation of an* n*-dimensional manifold* M*.*

The mesh $\eta = \eta(K)$ *of a triangulation is:*

$$\eta = \sup r(p, q),$$

where r *means the geodesic distance in* M *and the supremum is taken over all pairs of vertices* p, q *of a* 1*-simplex in* K.

The fullness $\Theta = \Theta(K)$ *of a triangulation* K *is*

$$\Theta(K) = \inf \frac{vol(\sigma)}{\eta^n},$$

where the inf *is taken over all* n*-simplexes* σ *of* K *and* $vol(\sigma)$ *is the Riemannian volume of* σ, *as a Riemannian submanifold of* M.

A Euclidean analogue of the following lemma was proven by Whitney in [69] (IV.14).

Lemma 1.4.4. *Let* M *be a smooth Riemannian* n*-manifold.*

1. *Let* K *be a smooth triangulation of* M. *Then there is a positive constant* $\Theta_0 > 0$ *and a sequence of subdivisions* $K_1, K_2, \dots$ *of* K *such that* $\lim_{n\to\infty} \eta(K_n) = 0$ *and* $\Theta(K_n) \geq \Theta_0$ *for all* n.

2. *Let* $\Theta_0 > 0$. *There exist positive constants* C_1, C_2 *depending on* M *and* Θ_0 *such that for all smooth triangulations* K *of* M *satisfying* $\Theta(K) \geq \Theta_0$, *all* n*-simplexes of* $\sigma = [p_0, p_1, \dots, p_n]$ *and vertices* p_k *of* σ,

$$vol(\sigma) \leq C_1 \cdot \eta^n$$
$$C_2 \cdot \eta \leq r(p_k, \sigma_{p_k}),$$

where r is the Riemannian distance, $vol(\sigma)$ is the Riemannian volume, and $\sigma_{p_k} = [p_0, \ldots, p_{k-1}, p_{k+1}, \ldots, p_n]$ is the face of σ opposite to p_k.

Since any two metrics on M are commensurable, the lemma follows from Whitney's Euclidean result, see also [14].

We consider only those triangulations with fullness bounded below by some positive real constant Θ_0. By the lemma, this guarantees that the volume of a simplex is on the order of its mesh raised to the power of its dimension. Geometrically, this means that in a sequence of triangulations, the shapes do not become too thin. (In fact, Whitney's *standard subdivisions* yield only finitely many shapes, and can be used to prove the first part of the lemma.) Most of our estimates depend on Θ_0, as can be seen in the proofs. We'll not indicate this dependence in the statements.

The following theorems are proved by Dodziuk and Patodi in [14]. They show that for a fine triangulation, WR is approximately equal to the identity. In this sense, the theorems give precise meaning to the statement: for a fine triangulation, cochains provide a good approximation to differential forms.

Theorem 1.4.5. *Let ω be a smooth form on M, and σ be an n-simplex of K. There exists a constant C, independent of ω, K and σ, such that*

$$|\omega - WR\omega|_p \leq C \cdot sup\left|\frac{\partial \omega}{\partial x^i}\right| \cdot \eta$$

for all $p \in \sigma$. The supremum is taken over all $p \in \sigma$ and $i = 1, 2, \ldots n$, and the partial derivatives are taken with respect to a coordinate neighborhood containing σ.

Proof. A generalization of this theorem will be proved in this chapter; see theorem 1.5.4 and remark 1.5.5. □

By integrating the above point-wise and applying a Sobolev inequality, Dodziuk and Patodi [14] obtain the following

Corollary 1.4.6. *There exist a positive constant C and a positive integer m, independent of K, such that*

$$\|\omega - WR\omega\| \leq C \cdot \|(Id + \Delta)^m \omega\| \cdot \eta$$

for all C^∞ j-forms ω on M.

Proof. This is a special case of corollary 1.5.7. □

Now suppose the cochains $C(K)$ are equipped with a non-degenerate inner product $\langle , \rangle$ such that, for distinct i, j, $C^i(K)$ and $C^j(K)$ are orthogonal. Then one can define further structures on the cochains. In particular, we have the following

Definition 1.4.7. *The adjoint of δ, denoted by δ^*, is defined by $\langle \delta^* \sigma, \tau \rangle = \langle \sigma, \delta\tau \rangle$.*

Note that $\delta^* : C^j(K) \to C^{j-1}(K)$ is also squares to zero. One can also define

Definition 1.4.8. *The combinatorial Laplacian is defined to be $\blacktriangle = \delta^*\delta + \delta\delta^*$.*

Clearly, both δ^* and $\blacktriangle$ depend upon the choice of inner product. For any choice of non-degenerate inner product, these operators give rise to a combinatorial Hodge theory: the space of harmonic j-cochains of K is defined to be

$$\mathcal{H}C^j(K) = \{a \in C^j | \blacktriangle a = \delta a = \delta^* a = 0\}.$$

The following theorem is due to Eckmann [17]:

Theorem 1.4.9. *Let (C, δ) be a finite dimensional complex with inner product $\langle , \rangle$, and induced adjoint δ^* as above. There is an orthogonal direct sum decomposition*

$$C^j(K) \cong \delta C^{j-1}(K) \oplus \mathcal{H}C^j(K) \oplus \delta^* C^{j+1}(K)$$

and $\mathcal{H}C^j(K) \cong H^j(K)$, the cohomology of (K, δ) in degree j.

Proof. We'll write C^j for $C^j(K)$. The second statement of the theorem follows from the first.

Using the fact that δ^* is the adjoint of δ, so $\delta\delta = \delta^*\delta^* = 0$, it is easy to check that δC^{j-1}, $\mathcal{H}C^j$, and $\delta^* C^{j+1}$ are orthogonal. Thus, it suffices to show

$$\dim C^j = \dim \delta C^{j-1} \oplus \mathcal{H}C^j \oplus \delta^* C^{j+1}$$

Let δ_j^* denote δ^* restricted to C^j. By orthogonality we have

$$\dim C^j - \dim \delta^* C^j = \dim Ker(\delta_j^*) = \dim \mathcal{H}C^j + \dim \delta^* C^{j+1}.$$

The proof is complete by showing $\dim \delta^* C^j = \dim \delta C^{j-1}$. This holds because, by the adjoint property, both $\delta : \delta^* C^j \to \delta C^{j-1}$ and $\delta^* : \delta C^{j-1} \to \delta^* C^j$ are injections of finite dimensional vector spaces. $\square$

If K is a triangulation of a Riemannian manifold M, then there is a particularly nice inner product on $C(K)$, which we'll call the *Whitney inner product*. It is induced by the metric $\langle , \rangle$ on $\Omega(M)$ and the Whitney embedding of cochains into $\mathcal{L}_2$-forms. We'll use the same notation $\langle , \rangle$ for this pairing on C: $\langle \sigma, \tau \rangle = \langle W\sigma, W\tau \rangle$.

It is proven in [13] that the Whitney inner product on C is non-degenerate. Further consideration of this inner product will be given in later sections. For now, following [13] and [14], we describe how the combinatorial Hodge theory, induced by the Whitney inner product, is related to the smooth Hodge theory. Precisely, we have the following theorem due to Dodziuk and Patodi [14], which shows that the approximation $WR \approx Id$ respects the Hodge decompositions of $\Omega(M)$ and $C(K)$.

Theorem 1.4.10. *Let $\omega \in \Omega^j(M)$, $R\omega \in C^j(K)$ have Hodge decompositions*

$$\begin{aligned} \omega &= d\omega_1 + \omega_2 + d^*\omega_3 \\ R\omega &= \delta a_1 + a_2 + \delta^* a_3 \end{aligned}$$

Then,

$$\begin{aligned} \|d\omega_1 - W\delta a_1\| &\leq \lambda \cdot \|(Id + \Delta)^m \omega\| \cdot \eta \\ \|\omega_2 - W a_2\| &\leq \lambda \cdot \|(Id + \Delta)^m \omega\| \cdot \eta \\ \|d^*\omega_3 - W\delta^* a_3\| &\leq \lambda \cdot \|(Id + \Delta)^m \omega\| \cdot \eta \end{aligned}$$

where λ and m are independent of ω and K.

1.5 Cochain Product

In this section we describe a commutative, but non-associative, cochain product. It is of interest in its own right, and will be used to define the combinatorial star operator.

The product we define is induced by the Whitney embedding and the wedge product on forms, but also has a nice combinatorial description. An easy way to state this is as follows: the product of a j-simplex and k-simplex is zero unless these simplices span a common $(j+k)$-simplex, in which case the product is a rational multiple of this $(j+k)$-simplex. We will prove a convergence theorem for this product, and also show that this product's deviation from being associative converges to zero for 'sufficiently smooth' cochains.

From the point of view of homotopy theory, it is natural to consider this commutative cochain product as part of a $\mathcal{C}_\infty$-algebra. We use Sullivan's local construction of a $\mathcal{C}_\infty$-algebra [60], and show that this structure converges to the strictly commutative associative algebra given by the wedge product on forms. In particular, all of the higher homotopies of the $\mathcal{C}_\infty$-algebra converge to zero.

Only definition 1.5.1 and theorem 1.5.2 are used in later sections. We begin with the definition of a cochain product on the cochains of a fixed triangulation K.

Definition 1.5.1. *We define* $\cup : C^j(K) \otimes C^k(K) \to C^{j+k}(K)$ *by:*

$$\sigma \cup \tau = R(W\sigma \wedge W\tau)$$

Since R and W are chain maps with respect to d and δ, it follows that δ is a derivation of $\cup$, that is, $\delta(\sigma \cup \tau) = \delta\sigma \cup \tau + (-1)^{deg(\sigma)}\sigma \cup \delta\tau$. Also, since $\wedge$ is graded commutative, $\cup$ is as well: $\sigma \cup \tau = (-1)^{deg(\tau)deg(\sigma)}\tau \cup \sigma$. It follows from a theorem of Whitney [70] that the product $\cup$ induces the same map on cohomology as the usual (Alexander-Whitney) simplicial cochain product. We now give a combinatorial description of $\cup$, this also appears in [3].

Theorem 1.5.2. *Let $\sigma = [p_{\alpha_0}, p_{\alpha_1}, \ldots, p_{\alpha_j}] \in C^j(K)$ and $\tau = [p_{\beta_0}, p_{\beta_1}, \ldots, p_{\beta_k}] \in C^k(K)$. Then $\sigma \cup \tau$ is zero unless σ and τ intersect in exactly one vertex and span a $(j+k)$-simplex υ, in which case, for $\tau = [p_{\alpha_j}, p_{\alpha_{j+1}}, \ldots, p_{\alpha_{j+k}}]$, we have:*

$$\begin{aligned} \sigma \cup \tau &= [p_{\alpha_0}, p_{\alpha_1}, \ldots, p_{\alpha_j}] \cup [p_{\alpha_j}, p_{\alpha_{j+1}}, \ldots, p_{\alpha_{j+k}}] \\ &= \epsilon(\sigma,\tau)\frac{j!k!}{(j+k+1)!}[p_{\alpha_0}, p_{\alpha_1}, \ldots, p_{\alpha_{j+k}}], \end{aligned}$$

where $\epsilon(\sigma, \tau)$ is determined by:

$$orientation(\sigma) \cdot orientation(\tau) = \epsilon(\sigma,\tau) \cdot orientation(\upsilon)$$

Proof. Recall that for any simplex α, $W\alpha = 0$ on $M \backslash \overline{St(\alpha)}$. So, $\sigma \cup \tau = R(W\sigma \wedge W\tau)$ is zero if their vertices are disjoint. If σ and τ intersect in more than one vertex then $W\sigma \wedge W\tau = 0$ since it is a sum of terms containing $d\mu_{\alpha_i} \wedge d\mu_{\alpha_i}$ for some i. Thus, by possibly reordering the vertices of K, it suffices to show that for $\sigma = [p_0, p_1, \ldots, p_j]$ and $\tau = [p_j, p_{j+1}, \ldots, p_{j+k}]$, we have that $(\sigma \cup \tau)([p_0, p_1, \ldots, p_{j+k}]) = s(\sigma, \tau)\frac{j!k!}{(j+k+1)!}$. We calculate

$$\begin{aligned}
&R(W\sigma \wedge W\tau)([p_0, p_1, \ldots, p_{j+k}]) \\
&= \int_{\upsilon=[p_0,p_1,\ldots,p_{j+k}]} W([p_0, p_1, \ldots, p_j]) \wedge W([p_j, p_{j+1}, \ldots, p_{j+k}]) \\
&= j!k! \int_\upsilon \sum_{i=0}^{j+k} (-1)^i \mu_i \mu_j \; d\mu_0 \wedge \cdots \wedge \widehat{d\mu_i} \wedge \cdots \wedge d\mu_{j+k}
\end{aligned}$$

Now, $\sum_{i=0}^{j+k} \mu_i = 1$, so $d\mu_0 = -\sum_{i=0}^{j+k} d\mu_i$, and we have that the last expression

$$\begin{aligned}
&= j!k! \int_\upsilon \sum_{i=0}^{j+k} (-1)^i \mu_i \mu_j \; (-d\mu_i) \wedge d\mu_1 \wedge \cdots \wedge \widehat{d\mu_i} \wedge \cdots \wedge d\mu_{j+k} \\
&= j!k! \int_\upsilon \mu_j \sum_{i=0}^{j+k} \mu_i \; d\mu_1 \wedge \cdots \wedge d\mu_{j+k} \\
&= j!k! \int_\upsilon \mu_j \; d\mu_1 \wedge \cdots \wedge d\mu_{j+k}
\end{aligned}$$

Now, $|\int_\upsilon d\mu_1 \wedge \cdots \wedge d\mu_{j+k}|$ is the volume of a standard $(j+k)$-simplex, and thus equals $\frac{1}{(j+k)!}$. From this it is easy to show that $\int_\upsilon \mu_j \; d\mu_1 \wedge \cdots \wedge d\mu_{j+k} = \pm\frac{1}{(j+k+1)!}$, with the appropriate sign prescribed by the definition of $s(\sigma, \tau)$. $\square$

A special case of this result was derived by Ranicki and Sullivan [49] for K a triangulation of a $4k$-manifold and σ, τ of complimentary dimension. In that paper, they showed that the pairing given by $\cup$ restricted to simplices of complimentary dimension gives rise to a semi-local combinatorial formula for the signature of a $4k$-manifold.

Remark 1.5.3. *The constant 0-cochain which evaluates to 1 on all 0-simplices is the unit of the differential graded commutative (but non-associative) algebra*

$(C^*, \delta, \cup)$.

We now show that the product $\cup$ converges to $\wedge$, which perhaps is not surprising, since $\cup$ is induced by the Whitney embedding and the wedge product. Still, the statement may be of computational interest since it shows that in using cochains to approximate differential forms, the product $\cup$ is, in a analytically precise way, an appropriate analogue of the wedge product of forms.

Theorem 1.5.4. *Let $\omega_1, \omega_2 \in \Omega(M)$ and σ be an n-simplex of K. Then there exists a constant C independent of ω_1, ω_2, K and σ such that*

$$|W(R\omega_1 \cup R\omega_2)(p) - \omega_1 \wedge \omega_2(p)|_p \leq C \cdot \left(c_1 \cdot sup\left|\frac{\partial \omega_2}{\partial x^i}\right| + c_2 \cdot sup\left|\frac{\partial \omega_1}{\partial x^i}\right| \right) \cdot \eta$$

for all $p \in \sigma$, where $c_m = sup|\omega_m|_p$, the supremum is over all $i = 1, 2, \ldots n$, and the partial derivatives are taken with respect to a coordinate neighborhood containing σ.

Remark 1.5.5. *By Remark 1.5.3, Theorem 1.5.4 reduces to Theorem 1.4.5 when ω_1 is the constant function 1.*

Proof. Let $\sigma = [p_0, \ldots, p_n]$ be an n-simplex contained in a coordinate neighborhood with coordinate functions $x_1, \ldots, x_n$. Let μ_i denote the i^{th} barycentric coordinate of σ. By the triangle inequality, and a possible reordering of the coordinate functions, it suffices to consider the case

$$\omega_1 = f \; d\mu_1 \wedge \cdots \wedge d\mu_j$$

$$\omega_2 = g \; d\mu_{\alpha_1} \wedge \cdots \wedge d\mu_{\alpha_k}$$

We first compute $W(R\omega_1 \cup R\omega_2)$. We'll use the notation $[p_s, \ldots, p_{s+t}]$ to denote both the simplicial chain and the simplicial cochain taking the value one on this chain and zero elsewhere. Let

$$N = \{0, 1, 2, \ldots, n\}$$
$$J = \{1, 2, \ldots, j\}$$
$$K = \{\alpha_1, \ldots, \alpha_k\}.$$

Then

$$R\omega_1 = \sum_{\beta \in N-J} \Big(\int_{[p_\beta, p_1, \ldots, p_j]} \omega_1 \Big) \; [p_\beta, p_1, \ldots, p_j]$$
$$R\omega_2 = \sum_{\gamma \in N-K} \Big(\int_{[p_\gamma, p_{\alpha_1}, \ldots, p_{\alpha_k}]} \omega_2 \Big) \; [p_\gamma, p_{\alpha_1}, \ldots, p_{\alpha_k}].$$

Now, to compute $R\omega_1 \cup R\omega_2$, we use theorem 1.5.2. If the sets J and K intersect in two or more elements then $R\omega_1 \cup R\omega_2 = 0$ since, in this case, all products of simplices are zero.

Now suppose that J and K intersect in exactly one element. Without loss of generality, let us assume $\alpha_1 = 1$. Then the product

$$[p_\beta, p_1, \ldots, p_j] \cup [p_\gamma, p_{\alpha_1}, \ldots, p_{\alpha_k}]$$

is non-zero only if β, γ are distinct elements of the set $Q = N - (J \bigcup K)$. Using the abbreviated notation

$$\begin{aligned}
[p_s, p_J, p_K] &= [p_s, p_1, \ldots, p_j, p_{\alpha_1}, \ldots, p_{\alpha_k}] \\
\int_{[s]} \omega_1 &= \int_{[p_s, p_1, \ldots, p_j]} \omega_1 \\
\int_{[s]} \omega_2 &= \int_{[p_s, p_{\alpha_1}, \cdots, p_{\alpha_k}]} \omega_2
\end{aligned}$$

we compute

$$R\omega_1 \cup R\omega_2 = \frac{j!k!}{(j+k+1)!} \sum_{\substack{\beta,\gamma \in Q \\ \beta \neq \gamma}} \Big(\int_{[\beta]} \omega_1 \Big) \Big(\int_{[\gamma]} \omega_2 \Big) \; [p_\beta, p_\gamma, p_J, p_K].$$

If all of the coefficients (given by the integrals of ω_1 and ω_2) were equal, the above expression would vanish, since the terms would cancel in pairs (by reversing the roles of β and γ). Of course, this is not the case, but the terms are *almost* equal. We'll use some estimation techniques developed by Dodziuk and Patodi [14].

An essential estimate that we'll need for this case and the next is the following: there is a constant c, independent of ω_1, ω_2, K and σ, such that for any $p \in \sigma$, and β, γ as above,

$$\Big| j!k! \int_{[\beta]} \omega_1 \int_{[\gamma]} \omega_2 - f(p)g(p) \Big| \;\leq\; c \cdot \Big(c_1 \cdot sup \Big| \frac{\partial \omega_2}{\partial x^j} \Big| + c_2 \cdot sup \Big| \frac{\partial \omega_1}{\partial x^j} \Big| \Big) \cdot \eta^{j+k+1} \quad (1.1)$$

where $c_m = sup|\omega_m|_p$ and the supremums are taken over all $p \in \sigma$ and $i = 1, 2, \ldots n$.

To prove this, first note that by the mean value theorem, for any points $p, q \in \sigma$, $|\omega_1(q) - \omega_1(p)|_q \leq c \cdot sup|\frac{\partial \omega_1}{\partial x^j}| \cdot \eta$. (Here we're using the fact that the Riemannian metric and the flat one induced by pulling back along the coordinates x^i are commensurable.) Similarly for ω_2. Now, fix $p \in \sigma$ and let dV_β be the volume element on $[p_\beta, p_1, \ldots, p_j]$, and dV_γ be the volume element on $[p_\gamma, p_1, \ldots, p_j]$. Then

$$\begin{aligned}
&\left| j!k! \int_{[\beta]} \omega_1 \int_{[\gamma]} \omega_2 - f(p)g(p) \right| \\
&= \left| j!k! \int_{[\beta]} \omega_1 \int_{[\gamma]} \omega_2 - \frac{\int_{[\beta]} f(p) d\mu_1 \wedge \cdots \wedge d\mu_j}{\int_{[\beta]} d\mu_1 \wedge \cdots \wedge d\mu_j} \frac{\int_{[\gamma]} g(p) d\mu_{\alpha_1} \wedge \cdots \wedge d\mu_{\alpha_k}}{\int_{[\gamma]} d\mu_{\alpha_1} \wedge \cdots \wedge d\mu_{\alpha_k}} \right| \\
&= j!k! \left| \int_{[\beta]} \omega_1 \int_{[\gamma]} \omega_2 - \int_{[\beta]} \omega_1(p) \int_{[\gamma]} \omega_2(p) \right| \\
&\leq j!k! \left| \int_{[\beta]} \omega_1 \right| \left| \int_{[\gamma]} \omega_2 - \int_{[\gamma]} \omega_2(p) \right| + \left| \int_{[\gamma]} \omega_2 \right| \left| \int_{[\beta]} \omega_1 - \int_{[\beta]} \omega_1(p) \right| \\
&\leq j!k! \; c_1 \cdot \eta^j \int_{[\gamma]} |\omega_2 - \omega_2(p)|_q \; dV_\gamma + c_2 \cdot \eta^k \int_{[\beta]} |\omega_1 - \omega_1(p)|_q \; dV_\beta \\
&\leq c \cdot \left(c_1 \cdot sup \left| \frac{\partial \omega_2}{\partial x^i} \right| + c_2 \cdot sup \left| \frac{\partial \omega_1}{\partial x^i} \right| \right) \cdot \eta^{j+k+1}.
\end{aligned}$$

This implies, by the triangle inequality, for any β, γ

$$\left| \int_{[\beta]} \omega_1 \int_{[\gamma]} \omega_2 - \int_{[\gamma]} \omega_1 \int_{[\beta]} \omega_2 \right| \;\leq\; c \cdot \left(c_1 \cdot sup \left| \frac{\partial \omega_2}{\partial x^j} \right| + c_2 \cdot sup \left| \frac{\partial \omega_1}{\partial x^j} \right| \right) \cdot \eta^{j+k+1} \quad (1.2)$$

Now that we have estimated the coefficients of $W(R\omega_1 \cup R\omega_2)$, this case is completed by estimating the the product of the $d\mu_i$'s that appear in $W(R\omega_1 \cup \omega_2)$. As shown in [14],

$$|d\mu_i|_p \leq \frac{\lambda}{r(p_i, |\sigma_i|)},$$

where $\sigma_i = [p_0, \cdots, p_{j-1}, p_{j+1}, \cdots, p_N]$ is the face opposite of p_i, and r is the

Riemannian geodesic distance. So, by Lemma 1.4.4

$$|d\mu_i|_p \le \lambda' \cdot \eta^{-1}$$

for some constant λ', and therefore

$$|d\mu_{i_1} \wedge \cdots \wedge d\mu_{i_{j+k}}|_p \le |d\mu_{i_1}|_p \ldots |d\mu_{i_{j+k}}|_p \le \lambda \cdot \eta^{-(j+k)}. \tag{1.3}$$

By combining (1.2) and (1.3), we finally have, for the case that J and K intersect in exactly one element,

$$\begin{aligned} |W(R\omega_1 \cup R\omega_2)(p) - \omega_1 \wedge \omega_2(p)|_p &= |W(R\omega_1 \cup R\omega_2)(p)|_p \\ &\le C \cdot \left(c_1 \cdot sup \left| \frac{\partial \omega_2}{\partial x^i} \right| + c_2 \cdot sup \left| \frac{\partial \omega_1}{\partial x^i} \right| \right) \cdot \eta \end{aligned}$$

We now consider the case that J and K are disjoint. We first note that for any $\tau \in Q = N - (J \cup K)$, there are exactly $j + k + 1$ products

$$[p_\beta, p_1, \ldots, p_j] \cup [p_\gamma, p_{\alpha_1}, \ldots, p_{\alpha_k}]$$

which equal a nonzero multiple of $[p_\tau, p_J, p_K] = [p_\tau, p_1, \ldots, p_j, p_{\alpha_1}, \ldots, p_{\alpha_k}]$. These are given by the three mutually exclusive cases:

$$\beta = \tau, \ \gamma \in J$$

$$\gamma = \tau, \ \beta \in K$$

$$\beta = \gamma = \tau$$

Using the same notation as the previous case, we compute

$$R\omega_1 \cup R\omega_2 = \frac{j!k!}{(j+k+1)!}\Big(\sum_{\|0\|} \Big(\int_{[\beta]} \omega_1\Big)\Big(\int_{[\gamma]} \omega_2\Big)\ [p_0, p_J, p_K] \qquad (1.4)$$
$$+ \sum_{\tau \in Q-\{0\}} \sum_{\|\tau\|} \Big(\int_{[\beta]} \omega_1\Big)\Big(\int_{[\gamma]} \omega_2\Big)\ [p_\tau, p_J, p_K]\Big)$$

where the sums labeled $\sum_{\|s\|}$ are over all β, γ such that

$$[p_\beta, p_1, \ldots, p_j] \cup [p_\gamma, p_{\alpha_1}, \ldots, p_{\alpha_k}] = \frac{j!k!}{(j+k+1)!}[p_s, p_J, p_K].$$

From Lemma 1.5.6, which follows the proof of this theorem,

$$W([p_0, p_J, p_K]) = (j+k)!\ d\mu_J \wedge d\mu_K - \sum_{r \in Q-\{0\}} W([p_\tau, p_J, p_K])$$

So,

$$\begin{aligned}
&|W(R\omega_1 \cup R\omega_2) - \omega_1 \wedge \omega_2|_p \\
&\leq \frac{j!k!}{(j+k+1)} \left| \sum_{\|0\|} \Big(\int_{[\beta]} \omega_1\Big)\Big(\int_{[\gamma]} \omega_2\Big)\ d\mu_J \wedge d\mu_K - \omega_1 \wedge \omega_2 \right|_p \\
&\quad + \frac{j!k!}{(j+k+1)!} \left| \sum_{\tau \in Q-\{0\}} \Big(\sum_{\|\tau\|} \Big(\int_{[\beta]} \omega_1\Big)\Big(\int_{[\gamma]} \omega_2\Big) \right. \\
&\qquad\qquad \left. - \sum_{\|0\|} \Big(\int_{[\beta]} \omega_1\Big)\Big(\int_{[\gamma]} \omega_2\Big)\Big)\ W([p_\tau, p_J, p_K]) \right|_p
\end{aligned}$$

By our estimates in (1.2) and (1.3), the latter term is bounded appropriately. As for the first term, recall that the sum $\sum_{\|0\|}$ consists of j+k+1 terms. We use

(1.2) again to bound

$$\left| \sum_{\|0\|} \Big(\int_{[\beta]} \omega_1 \Big) \Big(\int_{[\gamma]} \omega_2 \Big) - (j+k+1) \Big(\int_{[0]} \omega_1 \Big) \Big(\int_{[0]} \omega_2 \Big) \right| \tag{1.5}$$

and using (1.1), for fixed $p \in \sigma$ we have a bound on

$$\left| \Big(\int_{[0]} \omega_1 \Big) \Big(\int_{[0]} \omega_2 \Big) - f(p)g(p) \right|. \tag{1.6}$$

Finally, using the triangle inequality and combining (1.5) and (1.6) with (1.3) we can conclude

$$\frac{j!k!}{(j+k+1)} \left| \sum_{\|0\|} \Big(\int_{[\beta]} \omega_1 \Big) \Big(\int_{[\gamma]} \omega_2 \Big) \, d\mu_J \wedge d\mu_K(p) - \omega_1 \wedge \omega_2(p) \right|_p$$
$$\leq C \cdot \left(c_1 \cdot sup \left| \frac{\partial \omega_2}{\partial x^j} \right| + c_2 \cdot sup \left| \frac{\partial \omega_1}{\partial x^j} \right| \right) \cdot \eta$$

□

Lemma 1.5.6. *Let* $\sigma = [p_0, p_1, \ldots, p_n]$, $N = \{1, 2, \ldots, n\}$ *and* $I = \{i_1, \ldots i_m\} \subset N$. *Then*

$$W([p_0, p_{i_1}, \ldots, p_{i_m}]) = m! \; d\mu_{i_1} \wedge \cdots \wedge d\mu_{i_m} - \sum_{r \in N - I} W([p_r, p_{i_1}, \ldots, p_{i_m}])$$

Proof. The proof is a computation. We let

$$d\mu_I = d\mu_{i_1} \wedge \cdots \wedge d\mu_{i_m}$$
$$d\mu_I^s = d\mu_{i_1} \wedge \cdots \wedge \widehat{d\mu_{i_s}} \wedge \cdots \wedge d\mu_{i_m}$$

and compute

$$
\begin{aligned}
\frac{1}{m!}W([p_0, p_{i_1}, \dots, p_{i_m}]) &= \mu_0 \ d\mu_I + \sum_{s=1}^{m} (-1)^s \mu_{i_s} \ d\mu_0 \wedge d\mu_I^s \\
&= \Big(1 - \sum_{r=1}^{n} \mu_r\Big) \ d\mu_I + \sum_{s=1}^{m} (-1)^s \mu_{i_s} \ \Big(- \sum_{r=1}^{n} d\mu_r\Big) \wedge d\mu_I^s \\
&= d\mu_I - \sum_{r=1}^{n} \mu_r \ d\mu_I - \sum_{s=1}^{m} (-1)^s \mu_{i_s} \ \Big(d\mu_{i_s} + \sum_{r \in N-I} d\mu_r\Big) \wedge d\mu_I^s \\
&= d\mu_I - \sum_{r \in N-I} \mu_r \ d\mu_I - \sum_{s=1}^{m} (-1)^s \mu_{i_s} \Big(\sum_{r \in N-I} d\mu_r\Big) \wedge d\mu_I^s \\
&= d\mu_i - \sum_{r \in N-I} \Big(\mu_r \ d\mu_I + \sum_{s=1}^{m} (-1)^s \mu_{i_s} \ d\mu_r \wedge d\mu_I^s\Big) \\
&= d\mu_I - \frac{1}{m!} \sum_{r \in N-I} W([p_r, p_{i_1}, \dots, p_{i_m}])
\end{aligned}
$$

□

Corollary 1.5.7. *There exist a constant C and positive integer m, independent of K such that*

$$\|W(R\omega_1 \cup R\omega_2) - \omega_1 \wedge \omega_2\| \leq C \cdot \lambda(\omega_1, \omega_2) \cdot \eta$$

where

$$\lambda(\omega_1, \omega_2) = \|\omega_1\|_\infty \cdot \|(Id + \Delta)^m \omega_2\| + \|\omega_2\|_\infty \cdot \|(Id + \Delta)^m \omega_1\|$$

for all smooth forms $\omega_1, \omega_2 \in \Omega(M)$, where $\| \quad \|$ is the $\mathcal{L}_2$-norm on M.

Proof. We integrate the point-wise estimate from Theorem 1.5.4, using the

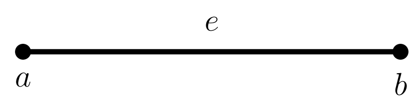

Figure 1.1: Cochain product on the unit interval

fact that M is compact and $sup|\omega_k| = \|\omega_k\|_\infty$, and the Sobolev-Inequality

$$sup\left|\frac{\partial \omega_k}{\partial x^i}\right| \leq C \cdot \|\omega_k\|_{2m} = C \cdot \|(Id + \Delta)^m \omega_k\|$$

for sufficiently large m, where $\| \quad \|_{2m}$ is the Sobolev $2m$-norm. $\square$

The convergence of $\cup$ to the associative product $\wedge$ is, a priori, a bit mysterious due to the following:

Example 1.5.8. *The product $\cup$ is not associative. For example, in Figure 1.1, $(a \cup b) \cup e = 0$, since a and b do not span a 0-simplex, but $a \cup (b \cup e) = -\frac{1}{4}e$.*

In the above example, the cochains a, b and e may be thought of as delta functions, in the sense that they evaluate to one on a single simplex and zero elsewhere. If we work with cochains which are "smoother", i.e. represented by the integral of a smooth differential form, associativity is *almost* obtained. In fact, the next theorem shows that for such cochains, the deviation from being associative is bounded by a constant times the mesh of the triangulation. Hence, associativity is recovered in the mesh goes to zero limit.

Theorem 1.5.9. *There exist a constant C and positive integer m, independent* *of K such that*

$$\|(R\omega_1 \cup R\omega_2) \cup R\omega_3 - R\omega_1(R\omega_2 \cup R\omega_3)\| \leq C \cdot \lambda(\omega_1, \omega_2, \omega_3) \cdot \eta$$

for all $\omega_1, \omega_2, \omega_3 \in \Omega(M)$, where $\| \quad \|$ is the Whitney norm and

$$\lambda(\omega_1, \omega_2, \omega_3) = \sum \|\omega_r\|_\infty \cdot \|\omega_s\|_\infty \cdot \|(Id + \Delta)^m \omega_t\|$$

where the sum is over all cyclic permutations $\{r, s, t\}$ of $\{1, 2, 3\}$.

Proof. We can prove this by first showing each of $(R\omega_1 \cup R\omega_2) \cup R\omega_3$ and $R\omega_1 \cup (R\omega_2 \cup R\omega_3)$ are close to $\omega_1 \wedge \omega_2 \wedge \omega_3$ in the point-wise norm $| \quad |_p$. The final result is then obtained by integrating and applying the Sobolev inequality to each point-wise error, then applying the triangle inequality.

Let $A \approx B$ mean

$$|A - B|_p \leq c \cdot \sum \|\omega_r\|_\infty \cdot \|\omega_s\|_\infty \cdot sup\left|\frac{\partial \omega_t}{\partial x^i}\right| \cdot \eta$$

We'll only consider the first case

$$W((R\omega_1 \cup R\omega_2) \cup R\omega_3) \approx \omega_1 \wedge \omega_2 \wedge \omega_3; \tag{1.7}$$

the second case is similar.

It suffices to consider the case

$$\omega_1 = f\ d\mu_1 \wedge \cdots \wedge d\mu_j$$

$$\omega_2 = g\ d\mu_{\alpha_1} \wedge \cdots \wedge d\mu_{\alpha_k}$$

$$\omega_3 = h\ d\mu_{\beta_1} \wedge \cdots \wedge d\mu_{\beta_l}.$$

The proof is analogous to that of Theorem 1.5.4; the only differences are that the combinatorics of two cochain products is slightly more complicated, and the estimates now involve coefficients which are triple products of integrals over simplices. Let

$$N = \{1, \ldots, n\}$$

$$J = \{1, \ldots, j\}$$

$$K = \{\alpha_1, \ldots, \alpha_k\}$$

$$L = \{\beta_1, \ldots, \beta_l\}$$

$$Q = N - (J \cup K \cup L)$$

Let us assume $J \cap K \cap L = \emptyset$; the other cases are similar. Define $A \sim B$ by

$$|A - B| \leq C \cdot \|\omega_r\|_\infty \cdot \|\omega_s\|_\infty \cdot sup\left|\frac{\partial \omega_t}{\partial x^i}\right| \cdot \eta^{j+k+l+1}$$

Using similar techniques as in the proof of theorem 1.5.4, for all $a \in N - J$,

$b \in N - K$, $c \in N - L$

$$\Big(\int_{[a]} \omega_1\Big)\Big(\int_{[b]} \omega_2\Big)\Big(\int_{[c]} \omega_3\Big) \sim \Big(\int_{[0]} \omega_1\Big)\Big(\int_{[0]} \omega_2\Big)\Big(\int_{[0]} \omega_3\Big) \quad (1.8)$$
$$j!k!l!\Big(\int_{[0]} \omega_1\Big)\Big(\int_{[0]} \omega_2\Big)\Big(\int_{[0]} \omega_3\Big) \sim f(p)g(p)h(p)$$

For any $\tau \in Q$, there are exactly

$$(j+k+1)(j+k+1) + (j+k+1)l = (j+k+1)(j+k+l+1)$$

products

$$[p_a, p_1, \ldots, p_j] \cup [p_b, p_{\alpha_1}, \ldots, p_{\alpha_k}] \cup [p_c, p_{\beta_1}, \ldots, p_{\beta_l}]$$

that equal a non-zero multiple of $[p_\tau, p_J, p_K, p_L]$. Then

$$\frac{j!k!(j+k!)l!}{(j+k+1)!(j+k+l+1)!}(j+k+1)(j+k+l+1) = \frac{j!k!l!}{(j+k+l)!}$$

so that, by applying lemma 1.5.6, and equations (1.8) and (1.3),

$$\begin{aligned} &W((R\omega_1 \cup R\omega_2) \cup R\omega_3) \\ &\approx \frac{j!k!l!}{(j+k+l)!}\bigg(\Big(\int_{[0]} \omega_1\Big)\Big(\int_{[0]} \omega_2\Big)\Big(\int_{[0]} \omega_3\Big)\; W([p_0, p_J, p_K, p_L]) \\ &\quad + \sum_{\tau \in Q - \{0\}} \Big(\int_{[0]} \omega_1\Big)\Big(\int_{[0]} \omega_2\Big)\Big(\int_{[0]} \omega_3\Big)\; W([p_\tau, p_J, p_K, p_L])\bigg) \end{aligned}$$

and this is $\approx \omega_1 \wedge \omega_2 \wedge \omega_3$ by (1.8). □

In the previous theorem, we dealt with the non-associativity of $\cup$ analytically. There is also an algebraic way to deal with this, via an algebraic generalization of commutative, associative algebras, called $\mathcal{C}_\infty$-algebras. First we'll give an abstract definition, and then unravel what it means.

Definition 1.5.10. *Let C be a graded vector space, and let $C[-1]$ denote the graded vector space C with grading shifted down by one. Let $\mathcal{L}(C) = \bigoplus_i \mathcal{L}^i(C)$ be the free Lie co-algebra on C. A $\mathcal{C}_\infty$-algebra structure on C is a degree 1 co-derivation $D : \mathcal{L}(C[-1]) \to \mathcal{L}(C[-1])$ such that $D^2 = 0$.*

A co-derivation on a free Lie co-algebra is uniquely determined by a collection of maps from $\mathcal{L}^i(C)$ to C for each $i \geq 1$. If we let m_i denote the restriction of D to $\mathcal{L}^i(C)$, then the equation $D^2 = 0$ is equivalent to a collection of equations:

$$\begin{aligned} m_1^2 &= 0 \\ m_1 \circ m_2 &= m_2 \circ m_1 \\ m_2 \circ m_2 - m_2 \circ m_2 &= m_1 \circ m_3 + m_3 \circ m_1 \\ &\vdots \end{aligned}$$

We can regard m_1 as a differential and m_2 a commutative multiplication on C. The second equation states that m_1 is a derivation of m_2. The third equation states that m_2 is associative up to the (co)-chain homotopy m_3. Note that, due to the shift of grading, m_j has degree $2 - j$.

The following theorem is due to Sullivan [60]. See also [64] for use of similar techniques.

Theorem 1.5.11. *Let (C, δ) be the simplicial cochains of a triangulated space and $\cup$ be any local commutative (possibly non-associative) cochain multiplica-*

tion on C such that δ is a derivation of $\cup$. Then there is a canonical local inductive construction which extends $(C, \delta, \cup)$ to a $\mathcal{C}_\infty$-algebra.

In this theorem, local means that the product of a j-simplex and a k-simplex is zero unless they span a $j+k$-simplex, in which case it is a multiple of this simplex. By Theorem 1.5.2, the commutative product $\cup$ defined at the beginning of this section satisfies this and the other conditions of Theorem 1.5.11.

The next theorem shows that the $\mathcal{C}_\infty$-algebra on C converges to the strict commutative and associative algebra given by the wedge product on forms in a sense analogous to the convergence statements we've made previously. In particular, all higher homotopies converge to zero as the mesh tends to zero.

Theorem 1.5.12. *Let C be the simplicial cochains of a triangulation K of M, with mesh $0 \leq \eta \leq 1$. Let $m_1 = \delta, m_2 = \cup, m_3, \ldots$ be the extension of $\delta, \cup$ to a $\mathcal{C}_\infty$- algebra on C as in theorem 1.5.11. Then there exists a constant λ independent of K such that, for all $j \geq 3$,*

$$\|W(m_j(R\omega_1, \ldots, R\omega_j))\| \leq \lambda \cdot \prod_{i=1}^{j} \|\omega_i\|_\infty \cdot \eta$$

for all $\omega_1, \ldots, \omega_k \in \Omega(M)$.

Proof. Suppose $\omega_1, \ldots, \omega_j$ are of degree $\alpha_1, \ldots, \alpha_j$, respectively. Let $\alpha = \sum \alpha_i$. We need two facts. First, for any α_i-simplex τ of K,

$$|R\omega_i(\tau)| \leq c \cdot \|\omega_i\|_\infty \cdot \eta^{\alpha_i} \tag{1.9}$$

Secondly, if p is a point in an n-simplex σ, and the r-simplices which are faces of σ are $\sigma_r^1, \ldots, \sigma_r^m$ then, by equation (1.3),

$$\left| W\left(\sum_{i=1}^{m} \sigma_r^i \right) \right|_p \leq c' \cdot \eta^{-r}. \tag{1.10}$$

Now, since m_j has degree $2-j$, $m_j(R\omega_1, \ldots, R\omega_j)$ is a linear combination of $(\alpha+2-j)$-simplices. Combining this with (1.9) and (1.10), we have for all $p \in M$ and some $\lambda \geq 0$

$$\begin{aligned} |W(m_j(R\omega_1, \ldots, R\omega_j))|_p &\leq \lambda \cdot \prod_{i=1}^{j} \|\omega_i\|_\infty \cdot \eta^\alpha \cdot \eta^{-(\alpha+2-j)} \\ &\leq \lambda \cdot \prod_{i=1}^{j} \|\omega_i\|_\infty \cdot \eta \end{aligned}$$

The result is obtained by integrating over M. □

1.6 Combinatorial Star Operator

In this section we define the combinatorial star operator ★ and prove that it provides a good approximation to the smooth Hodge-star $\star$. We also examine the relations which are expected to hold by analogy with the smooth setting. We find that some hold precisely, while others may only be recovered as the mesh goes to zero.

Definition 1.6.1. *Let K be a triangulation of a closed orientable manifold M, with simplicial cochains $C = \bigoplus_j C^j$. Let $\langle , \rangle$ be a non-degenerate positive definite inner product on C such that C^i is orthogonal to C^j for $i \neq j$. For*

$\sigma \in C^j$ we define $\bigstar\sigma \in C^{n-j}$ by:

$$\langle \bigstar\sigma, \tau \rangle = (\sigma \cup \tau)[M]$$

where $[M]$ denotes the fundamental class of M.

We emphasize that, as exemplified ed by Definition 1.3.2, the essential ingredients of a star operator are Poincaré Duality and a non-degenerate inner product. We can regard the inner product as giving some geometric structure to the space. In particular it gives lengths of edges, and angles between them. As in the smooth setting, the star operator depends on the choice of inner product (or Riemannian metric). See section 1.8 for the definition of a particularly nice class of inner products that we call *geometric inner products.*

Here are some elementary properties of $\bigstar$.

Theorem 1.6.2. *The following hold:*

1. $\bigstar\delta = (-1)^{j+1}\delta^*\bigstar$, *i.e.* $\bigstar$ *is a chain map.*

2. *For $\sigma \in C^j$ and $\tau \in C^{n-j}$, $\langle \bigstar\sigma, \tau \rangle = (-1)^{j(n-j)}\langle \sigma, \bigstar\tau \rangle$, i.e. $\bigstar$ is (graded) skew-adjoint.*

3. $\bigstar$ *induces isomorphisms $\mathcal{H}C^j(K) \to \mathcal{H}C^{n-j}(K)$ on harmonic cochains.*

Proof. The first two proofs are computational:

1. For $\sigma, \tau \in C$, we have:

$$\begin{aligned}
\langle \bigstar \delta\sigma, \tau \rangle &= (\delta\sigma \cup \tau)[M] \\
&= (-1)^{j+1}(\sigma \cup \delta\tau)[M] \\
&= (-1)^{j+1}\langle \bigstar\sigma, \delta\tau \rangle \\
&= \langle (-1)^{j+1}\delta^{*}\bigstar\sigma, \tau \rangle
\end{aligned}$$

   where we have used that fact that d is a derivation of $\cup$ and M is closed.

2. We compute:

$$\begin{aligned}
\langle \bigstar\sigma, \tau \rangle &= (\sigma \cup \tau)[M] \\
&= (-1)^{j(n-j)}(\tau \cup \sigma)[M] \\
&= (-1)^{j(n-j)}\langle \bigstar\tau, \sigma \rangle \\
&= (-1)^{j(n-j)}\langle \sigma, \bigstar\tau \rangle
\end{aligned}$$

3. Via the Hodge-decomposition of cochains, $\mathcal{H}C^{j}(K)$ may be identified with the cohomology $\mathcal{H}^{j}(K)$. Here $\bigstar$ is the composition of two isomorphisms, Poincaré Duality (since M is a manifold) and the inverse of the non-degenerate metric.

$\square$

We remark here that $\bigstar$ is in general not invertible, since the cochain product does not necessarily give rise to a non-degenerate pairing (on the cochain level!). This implies that $\bigstar$ is not an orthogonal map, and $\bigstar^2 \neq \pm\mathrm{Id}$.

For the remainder of this section, we'll fix the inner product on cochains to be the Whitney inner product, so that $\bigstar$ is the star operator induced by the Whitney inner product. This will be essential in showing that $\bigstar$ converges to the smooth Hodge star $\star$, which is defined using the Riemannian metric. First, a useful lemma. Let π denote the orthogonal projection of $\Omega^j(M)$ onto the image of $C^j(K)$ under the Whitney embedding W.

Lemma 1.6.3. $W\bigstar = \pi \star W$

Proof. Let $a \in C^j(K)$ and $b \in C^{n-j}(K)$. Note that $\star Wa$ is an $\mathcal{L}_2$-form but in general is not a Whitney form. We compute:

$$\langle W\bigstar a, Wb\rangle = \langle \bigstar a, b\rangle = \int_M Wa \wedge Wb = \langle \star Wa, Wb\rangle,$$

Thus, $W\bigstar a$ and $\star Wa$ have the same inner product with all forms in the image of W, so $W\bigstar = \pi \star W$. $\square$

Now for our convergence theorem of $\bigstar$:

Theorem 1.6.4. *Let M be a Riemannian manifold with triangulation K of mesh η. There exist a positive constant C and a positive integer m, independent of K, such that*

$$\|\star\omega - W\bigstar R\omega\| \leq C \cdot \|(Id + \Delta)^m \omega\| \cdot \eta$$

for all C^∞ differential forms ω on M.

Proof. We compute and use Theorem 1.4.5

$$\begin{aligned}
\|\star\omega - W\bigstar R\omega\| &= \|\star\omega - \perp \star W R\omega\| \\
&\leq \|\star\omega - \star W R\omega\| + \|\star W R\omega - \perp \star W R\omega\| \\
&\leq \|\star\|\|\omega - W R\omega\| + \|\star W R\omega - W R \star \omega\| \\
&\leq \|\omega - W R\omega\| + \|\star W R\omega - \star\omega\| + \|\star\omega - W R \star \omega\| \\
&\leq 2\|\omega - W R\omega\| + \|\star\omega - W R \star \omega\| \\
&\leq 3C \cdot \|(Id + \Delta)^m \omega\| \cdot \eta
\end{aligned}$$

□

The operator $\bigstar$ also respects the Hodge decompositions of $C(K)$ and $\Omega(M)$ in the following sense:

Theorem 1.6.5. *Let M be a Riemannian manifold with triangulation K of mesh η. Let $\omega \in \Omega^j(M)$ and $R\omega \in C^j(K)$ have Hodge decompositions*

$$\begin{aligned}
\omega &= d\omega_1 + \omega_2 + d^*\omega_3 \\
R\omega &= \delta a_1 + a_2 + \delta^* a_3
\end{aligned}$$

There exist a positive constant C and a positive integer m, independent of K, such that

$$\begin{aligned}
\|d \star \omega_1 - W\bigstar \delta a_1\| &\leq C \cdot \|(Id + \Delta)^m \omega\| \cdot \eta \\
\| \star \omega_2 - W\bigstar a_2\| &\leq C \cdot \|(Id + \Delta)^m \omega\| \cdot \eta \\
\| \star d^*\omega_3 - W\bigstar \delta^* a_3\| &\leq C \cdot \|(Id + \Delta)^m \omega\| \cdot \eta
\end{aligned}$$

Proof. The proof is analogous to the proof of theorem 1.6.4. □

This section ends with a discussion of convergence for compositions of the operators δ, δ^*, and $\bigstar$. We first note that δ provides a good approximation of d in the sense that $\|d\omega - W\delta R\omega\|$ is bounded by a constant times the mesh. This follows immediately from Theorem 1.4.5, using the fact that $\delta R = Rd$. In the same way, using Theorem 1.6.4, $\bigstar\delta$ provides a good approximation to $\star d$. In summary, we have:

$$\pm\delta^*\bigstar = \bigstar\delta \rightarrow \star d = \pm d^*\star$$

One would also like to know if either of $\delta\bigstar$ or $\bigstar\delta^*$ provide a good approximation to $d\star$ or $\star d^*$, respectively. Answers to these questions are seemingly harder to come by.

As a precursor, we point out that there is not a complete answer as to whether or not δ^* converges to d^*. In [51], Smits does prove convergence for the case of 1-cochains on a surface. To the author's mind, and as can be seen in the work of [51], one difficulty (with the general case) is that the operator δ^* is not local, since it involves the inverse of the cochain inner product.[1] A first attempt to understand this inverse is described in Section 1.8.

The issue becomes further complicated when considering the operator $\bigstar\delta^*$. We have no convergence statements about this operator. On the other hand, the operator $\delta\bigstar$, which incidentally does not equal $\pm\bigstar\delta^*$, is a bit less myste-

[1] If the cochain inner product is written as a matrix M with respect to the basis given by the simplices, then $\delta^* = M^{-1}\partial M$ where ∂ is the usual boundary operator on chains.

rious, and we have weak convergence in the sense that

$$\langle W\delta \bigstar R\omega_1 - d \star \omega_1, \omega_2 \rangle$$

is bounded by a constant λ (depending on ω_1 and ω_2) times the mesh.

Finally, one might ask if $\bigstar^2$ approaches $\pm$Id for a fine triangulation. While we have no analytic result to state, our calculations for the circle in section 1.9 suggest this is the case. One can show that a graded symmetric operator squares to $\pm$Id if and only if it is orthogonal. Hence one might view $\bigstar^2 \neq \mathrm{Id}$ as the failure of orthogonality, which at least for applications to surfaces in section 1.7, presents no difficulty.

1.7 Applications to Surfaces

In this section we study applications of the combinatorial star operator on a triangulated closed surface. As motivation, let us recall some facts from the analytic setting.

Let M be a Riemann surface. The Hodge-star operator on the complex valued 1-forms of M may be defined in local coordinates by $\star dx = dy$ and $\star dy = -dx$ and extended over $\mathbb{C}$ linearly. One can check that this is well defined using the Cauchy-Riemann equations for the coordinate interchanges. The Hodge-star operator restricts to an orthogonal automorphism of complex valued 1-forms that squares to $-$Id. Furthermore, the harmonic 1-forms split into an orthogonal sum of holomorphic and anti-holomorphic 1-forms corresponding to the $-i$ and $+i$ eigenspaces of the Hodge-star operator.

Riemann studied how the periods, the integrals of holomorphic and anti-holomorphic 1-forms, are related to the underlying complex structure. He showed that for any fixed homology basis these periods satisfy the so-called *bi-linear relations*. Furthermore, choosing a particular basis for the holomorphic 1-forms gives rise to a *period matrix*, which, by Torelli's theorem, determines the conformal structure of the Riemann surface. These period matrices lie in what is called the Siegel upper half space. (Two references for this material are [53] and [18].) An unsolved problem, called the Schottky problem, is to determine which points in the Siegel upper half space represent the period matrix of a Riemann surface.

In this section, we'll show that the combinatorial Hodge-star operator on a triangulated surface induces similar structures. In particular, given any hermitian inner product on the complex valued simplicial 1-cochains, the harmonic cochains split as holomorphic and anti-holomorphic 1-cochains. We'll prove analogues of the bilinear relations of Riemann, and show how one obtains a combinatorial period matrix. This construction yields its own combinatorial Schottky problem, but we won't discuss that here.

After describing our combinatorial construction, we'll show that if the complex valued simplicial cochains of a triangulated closed orientable Riemannian 2-manifold are equipped with the inner product induced by the Whitney embedding, then all of these structures provide a good approximation to the their continuum analogues. In particular, the holomorphic and anti-holomorphic 1-cochains converge to the holomorphic and anti-holomorphic 1-forms, and the combinatorial period matrix converges to the conformal period matrix of the associated Riemann surface, as the mesh of the triangulation tends to zero.

Hence, every conformal period matrix is a limit point of a sequence of combinatorial period matrices.

These statements may be interpreted as saying that a triangulation of a surface, endowed with an inner product on the associated cochains, determines a conformal structure. Furthermore, for triangulations of a Riemannian 2-manifold, a conformal structure is recovered (in the limit) from algebraic and combinatorial data. Statements like this have been expressed by physicists for some time in various field theories and in statistical mechanics, see [12].

We now describe the construction of combinatorial period matrices. First, we need to extend some of our definitions from previous sections to the case of complex valued cochains. Let $\langle , \rangle$ be any non-degenerate positive definite hermitian inner product on the complex valued simplicial 1-cochains of a triangulated topological surface K. We define the associated combinatorial star operator $\bigstar$ by:

$$\langle \bigstar a, b \rangle = (a \cup \overline{b})[M],$$

where the bar denotes complex conjugation and $\cup$ is as in Section 1.5, extended over $\mathbb{C}$ linearly. Just as with real coefficients, we have a Hodge decomposition

$$C^1(K) \cong \delta C^0(K) \oplus H^1(K) \oplus \delta^* C^2(K)$$

where H^1 is the space of complex valued harmonic 1-cochains. Since $\delta^* \bigstar = \bigstar \delta$, by Theorem 1.6.2, $\bigstar$ induces an isomorphism of H^1.

Definition 1.7.1. *Let K, $\langle , \rangle$, and $\bigstar$ be as above. We define the subspace of* *holomorphic* 1*-cochains by*

$$\mathcal{H}^{1,0}(K) = \{\sigma \in H^1(K) | \bigstar\sigma = -i\lambda\sigma \quad \textit{for some} \quad \lambda \geq 0\}$$

and the subspace of anti-holomorphic 1*-cochains by*

$$\mathcal{H}^{0,1}(K) = \{\sigma \in H^1(K) | \bigstar\sigma = i\lambda\sigma \quad \textit{for some} \quad \lambda \geq 0\}$$

Since $\bigstar$ is not an orthogonal map, λ may not equal one. The following theorem shows that the space of harmonic 1-cochains splits into the subspaces of holomorphic and anti-holomorphic cochains.

Theorem 1.7.2. *Let K be a triangulation of a surface M of genus g. A hermitian inner product on the simplicial 1-cochains of K gives an orthogonal direct sum decomposition*

$$H^1(K) \cong \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1}$$

where $\mathcal{H}^{1,0}$ and $\mathcal{H}^{0,1}$ are defined as a above, using the induced operator $\bigstar$. Each summand on the right has complex dimension g and complex conjugation maps $\mathcal{H}^{1,0}$ to $\mathcal{H}^{0,1}$ and vice versa.

Proof. The last assertion follows since $\bigstar$ is $\mathbb{C}$-linear. To prove the decomposition, we first note that the induced map $\bigstar$ on $\mathcal{H}$ has pure imaginary eigenvalues since it is skew-adjoint: $\langle \bigstar\sigma, \tau\rangle = -\langle \sigma, \bigstar\tau\rangle$. If $\sigma_1 \in \mathcal{H}^{1,0}$ and

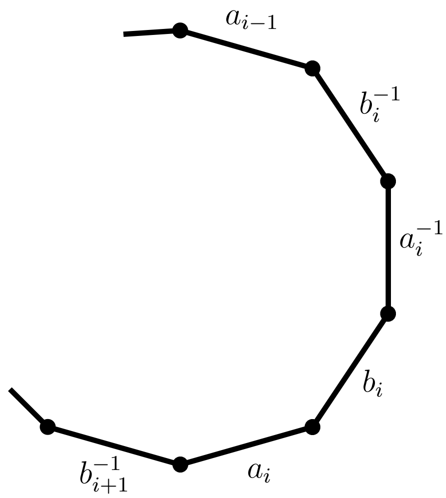

Figure 1.2: Fundamental domain of a surface

$\sigma_2 \in \mathcal{H}^{0,1}$ then for some $\lambda_1, \lambda_2 > 0$

$$-i\lambda_1\langle\sigma_1, \sigma_2\rangle = \langle\bigstar\sigma_1, \sigma_2\rangle = -\langle\sigma_1, \bigstar\sigma_2\rangle = i\lambda_2\langle\sigma_1, \sigma_2\rangle$$

so $\mathcal{H}^{1,0}$ and $\mathcal{H}^{0,1}$ are orthogonal. Finally, $\dim \mathcal{H}^{1,0} = \dim \mathcal{H}^{1,0} = g$ since $\dim \mathcal{H} = 2g$ and the eigenvalues of $\bigstar$ are all non-zero and occur in conjugate pairs. $\square$

We'll now study further properties of holomorphic and anti-holomorphic 1-cochains. As in the smooth case, there is much to be gained by analyzing the periods of these cochains. We begin with a brief description of the homology basis we'll use to evaluate these cochains on.

Without loss of generality, we assume that M is obtained by identifying the sides of a $4g$-gon, as in Figure 1.2. The basis $\{a_1, a_2, \ldots a_g, b_1, b_2, \ldots b_g\}$ for the first homology is classically referred to as the canonical basis [18], [53], since it

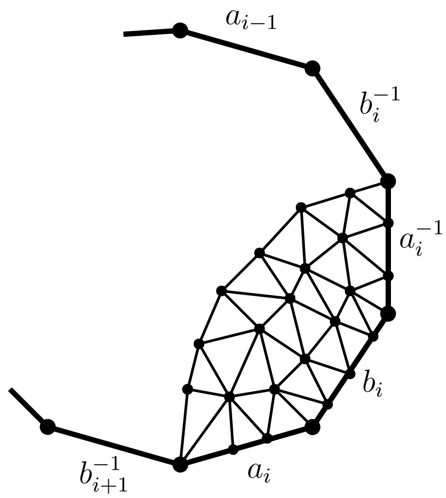

Figure 1.3: Triangulated surface

satisfies the following nice property: the intersection of any two basis elements is non-zero only for a_i and b_i, in which case it equals one. Of course, this basis is not truly canonical; nevertheless, we'll work with it. (Note that, the discussion below is basis-independent up to an action of the modular symplectic group; we omit details here.) We assume our triangulation K is a subdivision of the cellular decomposition given by the canonical homology basis. For any such subdivision, each element of the canonical homology basis is represented as a sum of the edges into which it is subdivided, as in Figure 1.3. Evaluating a cochain of K on an element of the canonical homology basis, means evaluating it on this subdivided representative.

Definition 1.7.3. *For $h \in \mathcal{H}^{1,0}$, the A-periods and B-periods of h are the following complex numbers:*

$$A_i = h(a_i) \quad B_i = h(b_i) \qquad \textit{for} \quad 1 \leq i \leq g$$

Theorem 1.7.4. $[\mathrm{Riemann'sBi-linearrelations}]$ *If $\sigma, \sigma' \in \mathcal{H}^{1,0}$, then the A-periods and B-periods satisfy:*

$$-i\lambda\langle\sigma, \overline{\sigma'}\rangle = \sum_{i=1}^{g}(A_iB_i' - B_iA_i') = 0$$

where λ is such that $\bigstar\sigma = -i\lambda\sigma$.

Proof. Since $\sigma' \in \mathcal{H}^{1,0}$, $\overline{\sigma'} \in \mathcal{H}^{0,1}$ it follows that $\langle\sigma, \overline{\sigma'}\rangle = 0$. To show the bi-linear relation we compute:

$$-i\lambda\langle\sigma, \overline{\sigma'}\rangle = \langle\bigstar\sigma, \overline{\sigma'}\rangle = (\sigma \cup \sigma')[M]$$

where the fundamental class $[M]$ of M may be represented by the sum of the 2-cells of K appropriately oriented. Now let $p : U \to M$ be the universal cover, with U triangulated so that p is locally a linear isomorphism onto the triangulation K of M. Let S denote a fundamental domain in the triangulation of U so that the induced map p_* maps the 2-simplices of S isomorphically onto the 2-simplices of K. Then $p_*(S) = [M]$, so the last expression equals

$$(\sigma \cup \sigma')([M]) = (p^*\sigma \cup p^*\sigma')(S)$$

where p^* denotes the pull back on cohomology. Since σ is holomorphic, it is closed, as is $p^*\sigma$. Since $\overline{S}$ is contractible to a point, the restriction of $p^*\sigma$ to $\overline{S}$ may be written as $p^*\sigma = \delta f$ for some 0-cochain f. Thus, since $\delta\sigma' = 0$ we

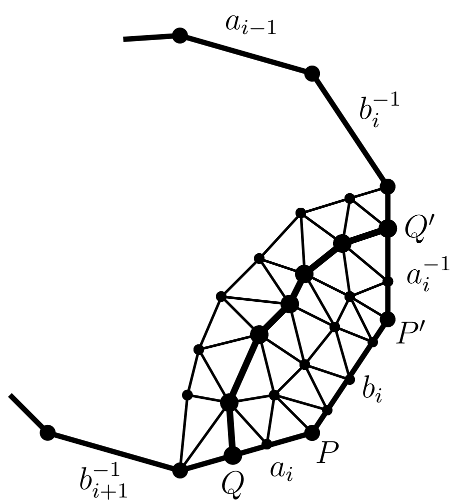

Figure 1.4: A 1-cycle α from Q to Q'

have:

$$\begin{aligned} -i\lambda\langle\sigma,\overline{\sigma'}\rangle &= (\delta f \cup p^*\sigma')(S) \\ &= (f \cup p^*\sigma')(\partial S) \\ &= \sum_{i=1}^{g}(f \cup p^*\sigma')(a_i + a_i^{-1} + b_i + b_i^{-1}) \end{aligned}$$

It remains to show that this last expression equals $\sum_{i=1}^{g}(A_iB_i' - B_iA_i')$. To do this, we first derive a simple relation for the values of f on the 0-simplices contained in the cycles of the canonical homology basis. Consider Figure 1.4. The chain α from Q to Q' is a cycle. Since α is homologous to the cycle made up of chains from Q to P, P to P' and P' to Q', and since the first and third project to the same chains on K, we have that

$$f(Q) - f(Q') = f(\partial\alpha) = \delta f(\alpha) = p^*\sigma(\alpha) = p^*\sigma(b_i) = B_i$$

which means that for any 1-cochain τ

$$(f \cup \tau)(a_i^{-1}) = -((f + B_i) \cup \tau)(a_i) = -(f \cup \tau)(a_i) - B_i\tau(a_i).$$

Similarly,

$$(f \cup \tau)(b_i^{-1}) = -((f - A_i) \cup \tau)(b_i) = -(f \cup \tau)(b_i) + A_i\tau(a_i)$$

So, we finally have that

$$\begin{aligned}
-i\lambda\langle\sigma, \overline{\sigma'}\rangle &= \sum_{i=1}^{g}(f \cup p^*\sigma')(a_i + a_i^{-1} + b_i + b_i^{-1}) \\
&= \sum_{i=1}^{g} -B_i p^*\sigma'(a_i) + A_i p^*\sigma'(b_i) \\
&= \sum_{i=1}^{g}(A_i B_i' - B_i A_i')
\end{aligned}$$

$\square$

Replacing $\overline{\sigma'}$ with σ' in the previous proof shows if $\sigma, \sigma' \in \mathcal{H}^{1,0}$ then

$$-i\lambda\langle\sigma, \sigma'\rangle = \sum_{i=1}^{g}(A_i\overline{B_i'} - B_i\overline{A_i'})$$

where $\bigstar\sigma = -i\lambda\sigma$. If we apply this to $\sigma' = \sigma$ we obtain the following expression for the norm of a holomorphic 1-cochain in terms of its periods.

Corollary 1.7.5. *If $\sigma, \in \mathcal{H}^{1,0}$ satisfies $\bigstar\sigma = -i\lambda\sigma$ with periods A_i and B_i then*

$$\|\sigma\|^2 = \langle\sigma, \sigma\rangle = \frac{i}{\lambda}\sum_{i=1}^{g}(A_i\overline{B_i} - B_i\overline{A_i}) \geq 0$$

$$
\begin{array}{c|cccccccc}
 & a_1 & a_2 & \cdots & a_g & b_1 & b_2 & \cdots & b_g \\
\hline
\sigma_1 & 1 & 0 & \cdots & 0 & \sigma_1(b_1) & \sigma_1(b_2) & \cdots & \sigma_1(b_g) \\
\sigma_2 & 0 & 1 & \cdots & 0 & \sigma_2(b_1) & \sigma_2(b_2) & \cdots & \sigma_2(b_g) \\
\vdots & & & \ddots & & & & & \\
\sigma_g & 0 & 0 & \cdots & 1 & \sigma_g(b_1) & \sigma_g(b_2) & \cdots & \sigma_g(b_g)
\end{array}
$$

Figure 1.5: The period matrix

Proof. □

This corollary immediately yields:

Corollary 1.7.6. *Let σ be a holomorphic 1-cochain.*

1. *If the A-periods or B-periods of σ vanish then $\sigma = 0$.*

2. *If the A-periods and B-periods of σ are real then $\sigma = 0$.*

Proof. □

Now let $\{\tau_1, \tau_2, \ldots, \tau_g\}$ be a basis for the space of holomorphic cochains. As just proved, if all the A-periods of a linear combination of these basis elements vanish, then this linear combination is identically zero. This implies we can solve uniquely for coefficients $c_{i,j}$ such that:

$$\sum_{i=1}^{g} c_{i,j}\ \tau_i(a_k) = \delta_{j,k}.$$

We put $\sigma_j = \sum_{i=1}^{g} c_{i,j}\ \tau_i$ and we call the basis $\{\sigma_1, \sigma_2, \ldots, \sigma_g\}$ *the canonical basis for the space of holomorphic 1-cochains.* This gives a matrix of periods as in Figure 1.5.

Definition 1.7.7. *Let* $\{\sigma_1, \sigma_2, \ldots, \sigma_g\}$ *be the canonical basis for the space of holomorphic* 1*-cochains and* $\{a_1, a_2, \ldots a_g, b_1, b_2, \ldots b_g\}$ *the canonical homology basis, so* $\sigma_i(a_j) = \delta_{i,j}$*. We define the period matrix* $\Pi = (\pi_{i.j})$ *to be the* $g \times g$ *matrix of* B*-periods:*

$$\pi_{i,j} = \sigma_i(b_j)$$

When we wish to emphasize the dependence of Π *on* K *or* $\langle , \rangle$ *we'll write* Π_K *or* $\Pi_{K,\langle,\rangle}$*.*

Remark 1.7.8. *Let* K *be fixed. If two inner products on* $C^1(K)$ *differ by a constant multiple then the associated period matrices are equal. Hence, the combinatorial period matrix is a "conformal invariant".*

Theorem 1.7.9. *Let* K *be a triangulated closed surface with a simplicial cochain inner product. The associated period matrix* Π *is symmetric and* $\mathrm{Im}(\Pi)$ *is positive definite.*

Proof. It suffices to show $\sigma_i(b_j) = \sigma_j(b_i)$ for $1 \le i, j \le g$, where σ_i and σ_j are canonical holomorphic cochain basis elements. We apply Theorem 1.7.4 and compute:

$$\begin{aligned} 0 &= -i\lambda_i \langle \sigma_i, \overline{\sigma_j} \rangle \\ &= \sum_{k=1}^{g} \sigma_i(a_k)\sigma_j(b_k) - \sigma_i(b_k)\sigma_j(a_k) \\ &= \sum_{k=1}^{g} \delta_{i,k}\sigma_j(b_k) - \delta_{j,k}\sigma_i(b_k) \\ &= \sigma_j(b_i) - \sigma_i(b_j) \end{aligned}$$

To prove the second statement, let $\sigma = \sum_{i=1}^{g} c_i \sigma_i$ be a nontrivial $\mathbb{R}$-linear combination of elements of the canonical basis of holomorphic cochains. Then $\sigma(a_i) = c_i$. We show

$$\sigma \cdot \mathrm{Im}(\Pi) \cdot \sigma > 0$$

by using Corollary 1.7.5 and computing:

$$\begin{aligned}
0 &< \frac{i}{\lambda} \sum_{k=1}^{g} \sigma(a_k)\overline{\sigma(b_k)} - \sigma(b_k)\overline{\sigma(a_k)} \\
&= \frac{i}{\lambda} \sum_{k=1}^{g} c_k \Big(\sum_{i=1}^{g} c_i \overline{\sigma_i(b_k)} \Big) - c_k \Big(\sum_{i=1}^{g} c_i \sigma_i(b_k) \Big) \\
&= \frac{i}{\lambda} \sum_{i=1}^{g} \sum_{k=1}^{g} c_k c_i \overline{\sigma_i(b_k)} - c_k c_i \sigma_i(b_k) \\
&= \frac{2}{\lambda} \sum_{i=1}^{g} \sum_{k=1}^{g} c_k c_i \; Im(\sigma_i(b_k)) \\
&= \frac{2}{\lambda} \; (\sigma \cdot Im(\Pi) \cdot \sigma)
\end{aligned}$$

□

Up to this point, we have assumed that K is a triangulated closed topological surface and $\langle , \rangle$ is a non-degenerate inner product on the simplicial cochains of K. As remarked in the beginning of this section, the structures we have uncovered (splitting of harmonics, bilinear relations, period matrix etc.) also appear for 1-forms on a Riemann surface. In fact, all of the statements proven above hold for forms as well [53], except one should set $\lambda = 1$, since in this case the Hodge star operator $\star$ is an orthogonal transformation.

Now let M be an orientable closed Riemannian 2-manifold. The Riemannian metric induces an operator $\star$ which squares to $-\mathrm{Id}$, and (identifying tangent and cotangent space via the metric) this operator $\star$ gives an almost complex structure. Gauss proved that M admits a unique complex structure, i.e. a Riemann surface structure, that is compatible with this almost complex structure. This theorem is, a priori, non-trivial, and involves a transcendental construction of holomorphic coordinate charts. By Torelli's theorem the resulting complex structure is determined uniquely by the period matrix of the associated Riemann surface M. We now describe how this conformal period matrix of M is related to a combinatorial period matrix in the case that M is triangulated.

Let K be a triangulation of a Riemannain 2-manifold M. The usual $\mathcal{L}_2$ inner product on the vector space of real valued 1-forms may be extended to a hermitian inner product on the space of complexified 1-forms canonically, by declaring

$$\langle \omega_1 \otimes z_1, \omega_2 \otimes z_2 \rangle = z_1 \overline{z_2} \langle \omega_1, \omega_2 \rangle \tag{1.11}$$

for $\omega_i \otimes z_i \in T^*M \bigotimes \mathbb{C}$. Let $\| \ \|$ denote the induced norm on $T^*M \bigotimes \mathbb{C}$.

The Whitney embedding of complex valued 1-cochains into $T^*M \bigotimes \mathbb{C}$ induces an inner product on complex valued 1-cochains. For the remainder of this section, we work only with this inner product. We remark here that while the approximation theorems from Section 1.4 and 1.6 (using the Whitney inner product) involved real-valued forms and cochains, the proofs hold verbatim for complex coefficients as well.

First we prove the following

Lemma 1.7.10. *Let M be a Riemannian 2-manifold with triangulation K of mesh η, and let $\mathfrak{h}$ be a complex valued holomorphic 1-form on M, so $\star\mathfrak{h} = -i\mathfrak{h}$.*

By the Hodge decomposition of cochains and Theorem 1.7.2 we may write

$$R\mathfrak{h} = \delta g + h_1 + h_2 + \delta^* k$$

uniquely for $h_1 \in \mathcal{H}^{1,0}$ *and* $h_2 \in \mathcal{H}^{0,1}$. *Then there exists a positive constant* C, *independent of* K, *such that*

$$\|Wh_1 - \mathfrak{h}\| \leq C \cdot \eta$$

Proof. By Theorems 1.4.10 and 1.6.5, there is a positive constant C, independent of K, such that

$$\begin{aligned} C \cdot \eta &\geq \|W\bigstar(h_1 + h_2) - \star\mathfrak{h}\| + \|\mathfrak{h} - W(h_1 + h_2)\| \\ &= \|W\bigstar(h_1 + h_2) - \star\mathfrak{h}\| + \| \star \mathfrak{h} + iW(h_1 + h_2)\| \\ &\geq \|W\bigstar h_1 + W\bigstar h_2 + iW(h_1 + h_2)\| \\ &= \|\bigstar h_1 + \bigstar h_2 + i(h_1 + h_2)\| \end{aligned}$$

Since $h_1 \in \mathcal{H}^{1,0}$ and $h_2 \in \mathcal{H}^{0,1}$ we may write $\bigstar h_1 = -i\lambda_1 h_1$ and $\bigstar h_2 = i\lambda_2 h_2$ for some $\lambda_1, \lambda_2 > 0$. Using the fact that $\mathcal{H}^{1,0} \perp \mathcal{H}^{0,1}$ we then have

$$\begin{aligned} C^2 \cdot \eta^2 &\geq \| - i\lambda_1 h_1 + i\lambda_2 h_2 + ih_1 + ih_2\|^2 \\ &= \|(1 - \lambda_1)h_1 + (1 + \lambda_2)h_2\|^2 \\ &= \langle(1 - \lambda_1)h_1, (1 - \lambda_1)h_1\rangle + \langle(1 + \lambda_2)h_2, (1 + \lambda_2)h_2\rangle \\ &= |1 - \lambda_1|^2\|h_1\|^2 + |1 + \lambda_2|^2\|h_2\|^2 \end{aligned}$$

So we conclude

$$\|h_2\| \le \frac{C \cdot \eta}{|1+\lambda_2|} \le C \cdot \eta$$

and finally,

$$\|Wh_1 - \mathfrak{h}\| \le \|W(h_1 + h_2) - \mathfrak{h}\| + \|h_2\| \le 2C \cdot \eta.$$

$\square$

Remark 1.7.11. *A closer examination of the proof shows that, for a fine enough triangulation, $1-\lambda_1$ is bounded by a constant times the mesh. There is an analogous statement for anti-holomorphic 1-forms $\mathfrak{h}$ and the anti-holomorphic part of the cochain $R\mathfrak{h}$.*

One can check that the hermitian inner product on 1-forms of M, defined in (1.11), agrees with the usual inner product on the 1-forms of the Riemann surface associated to M, given by

$$\langle \omega, \eta \rangle = \int_M \omega \wedge \star\overline{\eta}.$$

It is a peculiarity of working in the middle dimension (here 1) that this inner product, and the Hodge star operator, depend only on the conformal class of the Riemannian metric. This implies that the period matrix of the Riemann surface associated to M can be computed by using the inner product in (1.11) in the following way: split off the harmonic 1-forms and evaluate the appropriate basis of the $-i$ eigenspace of $\star$ on the canonical homology basis. Note that this involves a transcendental procedure in the Hodge decomposition of forms. The point of the following theorem is that the period matrix, and therefore the complex structure is computable, to any desired accuracy, from algebraic and combinatorial data.

Theorem 1.7.12. *Let M be a closed orientable Riemannian 2-manifold and let Π be the period matrix of the Riemann surface associated to M. Let K_n be a sequence of triangulations of M with mesh converging to zero. Then, for each n, the induced Whitney inner product on the simplicial 1-cochains of K_n gives rise to a combinatorial period matrix Π_{K_n}, and*

$$\lim_{n\to\infty} \Pi_{K_n} = \Pi.$$

Proof. Let $\mathfrak{h}_g, \cdots, \mathfrak{h}_g$ be the canonical basis of holomorphic 1-forms with periods

$$\begin{aligned} \mathfrak{h}_i(a_j) = \int_{a_j} \mathfrak{h}_i \;&=\; \delta_{i,j} \\ \mathfrak{h}_i(b_j) = \int_{b_j} \mathfrak{h}_i \;&=\; \pi_{i,j} \end{aligned}$$

for $1 \leq i, j \leq g$, and $\pi_{i,j}$ the (i, j) entry of Π.

For each n, let $\varphi_1^n, \cdots, \varphi_g^n$ be a basis for the holomorphic cochains on K_n. Then the periods are

$$\varphi_i^n(a_j) = \delta_{i,j}$$

$$\varphi_i^n(b_j) = \pi_{i,j}^n$$

for $1 \leq i, j \leq g$, and $\pi_{i,j}^n$ the (i, j) entry of Π_{K_n}. Our goal is to show, for all

$1 \leq i, j \leq g$,

$$\lim_{n\to\infty} \varphi_i^n(b_j) = \mathfrak{h}_i(b_j).$$

Let R_n denote the integration map taking 1-forms to cochains on K_n. We denote by h_i^n the holomorphic part of the cochain $R_n \mathfrak{h}_i$. By the previous lemma, $h_i^n \to \mathfrak{h}_i$ as $n \to \infty$. Hence,

$$\lim_{n\to\infty} h_i^n(a_j) = \mathfrak{h}_i(a_j) = \delta_{i,j} \tag{1.12}$$

For each n and $1 \leq i \leq g$ we may write

$$h_i^n = \sum_{k=1}^{g} c_{i,k}^n \varphi_k^n$$

and by evaluating on the cycle a_j we see that

$$c_{i,j}^n = \sum_{k=1}^{g} c_{i,k}^n \varphi_k^n(a_j) = h_i^n(a_j)$$

Combining this with equation (1.12), we have

$$\lim_{n\to\infty} c_{i,j}^n = \delta_{i,j}$$

which implies

$$\lim_{n\to\infty} \|\varphi_i^n - h_i^n\| = 0$$

By the lemma, $\|h_i^n\| \to \|\mathfrak{h}_i\|$, so the sequences $\|h_i^n\|$ and $\|\varphi_i^n\|$ are bounded. Finally, we have

$$\lim_{n\to\infty} \varphi_i^n(b_j) = \lim_{n\to\infty} h_i^n(b_j) = \mathfrak{h}_i(b_j)$$

□

Corollary 1.7.13. *Let M be a closed Riemann surface with period matrix Π. Let K_n be a sequence of triangulations of M with mesh converging to zero, and combinatorial period matrices Π_{K_n} induced by the Whitney metric. Then,*

$$\lim_{n\to\infty} \Pi_{K_n} = \Pi.$$

Proof. While there isn't a notion of geodesic length on a Riemann surface, a distance converging to zero is well defined since it depends only on a conformal class of metrics. So the statement of the corollary makes sense. Then one can choose any Riemannian metric on M in the conformal class of metrics determined by M, and apply the above theorem. □

Corollary 1.7.14. *Every conformal period matrix is the limit of a sequence of combinatorial period matrices.*

1.8 Inner Products and Their Inverses

In this section we study inner products on cochains, as well as the induced "inverse inner product". Smits also studied the inverse of inner products in [51], where he proved results on the convergence of the divergence operator d^* on a surface.

Definition 1.8.1. *A* geometric inner product *on the real vector space of simplicial cochains $C = \bigoplus_j C^j$ of a triangulated space K is a non-degenerate positive definite inner product $\langle , \rangle$ on C satisfying:*

1. $C^i \perp C^j$ *for* $i \neq j$

2. locality: $\langle a, b\rangle \neq 0$ *only if* $St(a) \cap St(b)$ *is non-empty.*

Remark 1.8.2. *A geometric inner product restricted to 1-cochains gives a notion of lengths of edges and the angles between them. It may be interesting to study the consequences of an inner product with signature other than the one considered here.*

We assume in this section that all cochain inner products are geometric in the above sense. Note that the Whitney inner product is geometric.

An inner product on C^* induces an isomorphism from C^* to the linear dual of C^*, which we denote by C_* and refer to as the simplicial chains (to be more precise, this is the double dual of chains, but we'll confuse the two since we're assuming K is compact). The "inverse of the inner product" is that induced by the inverse of the isomorphism $C^* \to C_*$, which is an isomorphism $C_* \to C^*$. This gives an inner product on the (simplicial) chains C_* and will be denoted by $\langle , \rangle^{-1}$.

If one represents a geometric cochain inner product as a matrix, using the standard basis given by the simplices, then the locality property roughly states that this matrix is "near diagonal". Of course, the inverse of a diagonal matrix is diagonal, but the inverse of a near diagonal matrix is *not* near diagonal (e.g. see Section 1.9). Rather, it can have all entries non-zero; i.e. the inverse inner product on chains is *not* geometric.[2]

[2]It is true is that the matrix entries decrease in absolute value as they move from the diagonal, so that the inner product of two chains decays rapidly as a function of "geometric distance".

In the rest of this section, we describe the inverse inner product $\langle , \rangle^{-1}$ on chains in a geometric way by showing it can be computed as a weighted sum of paths in a collection of graphs associated to K. This will be useful in the next section for making explicit computations of the combinatorial star operator. We begin with some definitions:

Definition 1.8.3. *A graph Γ (without loops) consists of a set S, called vertices, and a collection of two-element subsets of S, called* edges *Two edges of Γ are said to be incident if their intersection (as subsets of S) is nonempty. A weighted graph is a graph with an assignment of a real number $w(e)$ to each edge e. A path γ in a graph is a sequences of edges $\{e_i\}_{i\in I}$ such that e_i and e_{i+1} are incident for each i. The weight $w(\gamma)$ of a path γ in a weighted graph is the product of the weights of the edges in γ. By convention, we say there is a unique path of length zero between any vertex and itself, and the weight of this path is one.*

Definition 1.8.4. *Let K be the simplicial cochain complex of a triangulated n-manifold M. We define the* graph associated to the j-simplicies of K, *denoted $\Gamma(K, j)$, to be the following graph: The vertices of $\Gamma(K, j)$ are the set $\{\sigma_\alpha\}$ of j-simplices of K; two distinct vertices σ_1, σ_2 of $\Gamma(K, j)$ are joined by an edge if and only if they are faces of a common n-simplex of K (i.e $St(\sigma_1) \cap St(\sigma_2)$ is non-empty).*

Corollary 1.8.5. *Let K be the simplicial cochain complex of a triangulated n-manifold M.*

1. *Paths in $\Gamma(K, j)$ correspond to sequences $\{s_i\}_{i\in I}$ of j-simplices in K such that, for each i, s_i and s_{i+1} are faces of a common n-simplex.*

2. $\Gamma(K,0)$ *is isomorphic to* K_1*, the* 1*-skeleton of* K *(the union of its vertices and edges).*

Now suppose the cochains C^* of K are endowed with a geometric inner product $\langle,\rangle$. (Our motivating example is the Whitney metric on C^*, but other examples arise when considering interactions on simplicial lattices.) In this case we associate to $(C^*, \langle,\rangle)$ the following collection of weighted graphs.

Definition 1.8.6. *Let* C^* *be the cochains of a finite triangulation* K *of a manifold, with geometric cochain inner product* $\langle,\rangle$*. We define the* weighted j-cochain graph of K $\Gamma_w(K,j)$ *to have the underlying graph of* $\Gamma(K,j)$ *with the edge* $e = \{\sigma_1, \sigma_2\}$ *weighted by*

$$w(e) = \frac{\langle \sigma_1, \sigma_2 \rangle}{\|\sigma_1\| \cdot \|\sigma_2\|}$$

where $\|\sigma\| = \sqrt{\langle \sigma, \sigma \rangle}$

Remark 1.8.7. *The appropriate analogue of corollary 1.8.5 for weighted graphs holds as well.*

The following describes how the metric $\langle,\rangle^{-1}$ on C_j can be computed by counting weighted paths in the weighted j-cochain graph $\Gamma_w(K,j)$.

Theorem 1.8.8. *For* $\sigma_1, \sigma_2 \in C_j$

$$\langle \sigma_1, \sigma_2 \rangle^{-1} = \frac{1}{\|\sigma_1\| \cdot \|\sigma_2\|} \sum_{i \geq 0} (-1)^i \sum_{\gamma_i \in \Gamma_w(K,j)} w(\gamma_i)$$

where γ_i *is a path in* $\Gamma_w(K,j)$ *of length* i*, starting at* σ_1 *and ending at* σ_2*.*

Proof. Let M be the matrix for $\langle , \rangle$ with respect to a fixed ordering of the basis given by the simplices of K. Let D be the diagonal matrix, with respect to the same ordered basis, whose diagonal entries are the norm of a simplex. Let $M_D = D^{-1}MD^{-1}$. Note that the entries of M_D are normalized since the entries of D^{-1} are of the form $\frac{1}{\|\sigma\|}$. In particular the diagonal entries of M_D equal 1, so we may write

$$M^{-1} = D^{-1}(M_D)^{-1}D^{-1} = D^{-1}(I+A)^{-1}D^{-1}$$

It is easy to check that A is precisely the weighted adjacency matrix for the weighted graph $\Gamma_w(K,j)$. Recall that the i^{th} power of a weighted adjacency matrix counts the sum of the weights of all paths of length i. By the Cauchy-Schwartz inequality, all of the entries a of A satisfy $0 \leq a < 1$, so the formula

$$(I+A)^{-1} = \sum_{i \geq 0} (-1)^i A^i$$

may be applied above, and we conclude that

$$\langle \sigma_1, \sigma_2 \rangle^{-1} = \sigma_1 M^{-1} \sigma_2 = \frac{1}{\|\sigma_1\| \cdot \|\sigma_2\|} \sum_{i \geq 0} (-1)^i \sum_{\gamma_i \in \Gamma_w(K,j)} w(\gamma_i)$$

$\square$

Remark 1.8.9. *1. The above theorem in the case $j = 0$, in light of Remark 1.8.7, shows that for vertices p and q of K, $\langle p, q \rangle^{-1}$ may be expressed as a weighted sum over all paths in the* 1*-skeleton $K_1 \subset K$.*

2. These expressions for $\langle , \rangle^{-1}$ not only provide a nice geometric interpre-

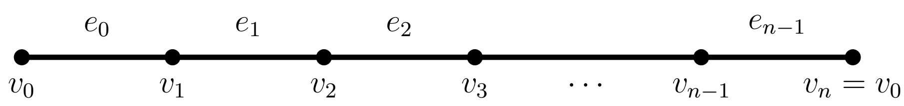

Figure 1.6: Triangulation of S^1

tation, but are also useful for computations, as we will see in Section 1.9 where we compute $\bigstar$ *for the circle.*

1.9 Computation for S^1

In this section we compute the operator $\bigstar$ explicitly for the circle S^1. We take S^1 to be the unit interval $[0, 1]$ with 0 and 1 identified. We consider a sequence of subdivisions, the n^{th} triangulation being given by vertices at the points $v_i = \frac{i}{n}$ for $0 \leq i \leq n$. We denote the edge from v_i to v_{i+1} by e_i for $0 \leq i \leq n$ and orient this edge from v_i to v_{i+1}. See Figure 1.6.

All operators will be written as matrices with respect to the ordered basis $\{v_0, \ldots, v_{n-1}, e_0, \ldots, e_{n-1}\}$.

Recall that the operator $\bigstar$ is defined by $\langle \bigstar\sigma, \tau\rangle = (\sigma \cup \tau)[S^1]$ where here $[S^1]$ is the sum of all the edges with their chosen orientations. We'll use the cochain inner product $\langle , \rangle$ induced by the Whitney embedding and the standard metric on S^1 (i.e. $\langle \mathrm{d}t, \mathrm{d}t\rangle = 1$). Let M denote the matrix for the cochain inner product and let C denote the matrix for the pairing given by $(\sigma, \tau) \mapsto (\sigma \cup \tau)[S^1]$. Then $\bigstar = M^{-1}C$. (We suppress the dependence of these operators on the level of subdivision; the n^{th} level M and C are size $2n \times 2n$.)

By the definition of $\cup$ and our chosen orientations we have that

$$C = \left(\begin{array}{c|c} 0 & A \\ \hline A^t & 0 \end{array} \right)$$

where

$$A = \begin{pmatrix} 1/2 & 0 & 0 & \dots & 0 & 1/2 \\ 1/2 & 1/2 & 0 & \dots & \dots & 0 \\ 0 & 1/2 & 1/2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1/2 & 1/2 \end{pmatrix}$$

and t denotes transpose.

One can compute explicitly:

$$\langle \sigma, \tau \rangle = \begin{cases} \frac{2}{3n} & \sigma = \tau \text{ is a vertex} \\ \frac{1}{6n} & \sigma,\ \tau \text{ are vertices in the boundary of a common edge} \\ n & \sigma = \tau \text{ is an edge} \\ 0 & \text{otherwise} \end{cases}$$

So, in our chosen basis, the matrix for the inner product is given by:

$$M = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & nI \end{array} \right)$$

where I denotes the $n \times n$ identity matrix and

$$B = \begin{pmatrix} 2/3n & 1/6n & 0 & \dots & 0 & 1/6n \\ 1/6n & 2/3n & 1/6n & \dots & \dots & 0 \\ 0 & 1/6n & 2/3n & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & \ddots & 1/6n \\ 1/6n & 0 & \dots & 0 & 1/6n & 2/3n \end{pmatrix}.$$

We now compute B^{-1}. Note that one can write $B = \frac{2}{3n}(\frac{1}{4}D + I)$ where

$$D = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & \dots & \dots & 0 \\ 0 & 1 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}$$

then

$$\begin{aligned} B^{-1} &= \frac{3n}{2}\left(\frac{1}{4}D + I\right)^{-1} \\ &= \frac{3n}{2}(I - \frac{1}{4}D + \frac{1}{4^2}D^2 - \frac{1}{4^3}D^3 \pm \cdots) \\ &= \frac{3n}{2}\sum_{k \geq 0}(-1/4)^k D^k \end{aligned}$$

Note that D is the adjacency matrix for the graph corresponding to the original triangulation K, or rather, $\frac{1}{4}D$ is the weighted adjacency matrix for

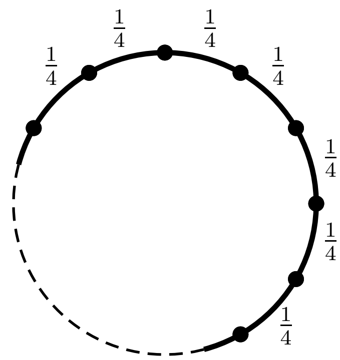

Figure 1.7: The weighted graph corresponding to $\frac{1}{4}D$

the weighted graph in Figure 1.7.

As shown in Section 1.8, the matrices $\frac{1}{4^k}D^k$ have a geometric interpretation: the (i, j) entry equals the total weight of all paths from v_i to v_j of length k. Since in this case all weights are $\frac{1}{4}$, we'll simply compute the the (i, j) entry of D^k, i.e the total number of paths from v_i to v_j of length k.

We first note that for the real line with integer vertices, the number of paths of length r between two vertices distance s apart is the binomial coefficient $\binom{r}{\frac{r+s}{2}}$. By considering the standard covering of the circle with n vertices by the line we have

$$d_{i,j}^k = \sum_{t\in\mathbb{Z}} \binom{k}{\frac{k+|i-j|+nt}{2}}$$

where the above binomial coefficient is zero unless $\frac{k+|i-j|+nt}{2}$ is a non-negative integer less than or equal to k. Hence,

$$M^{-1} = \left(\begin{array}{c|c} \left[\frac{3n}{2}\sum_{k\geq 0}\left(\frac{-1}{4}\right)^k d_{i,j}^k\right] & 0 \\ \hline 0 & \frac{1}{n}I \end{array}\right).$$

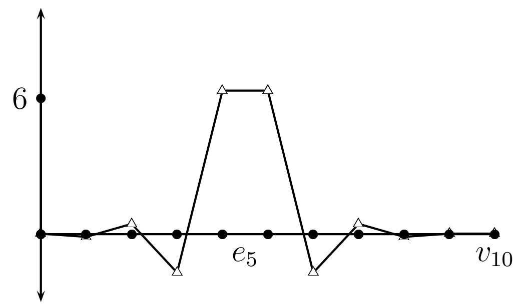

Figure 1.8: Coefficients of v_i appearing in $\bigstar e_5$, for $n = 10$

We conclude that:

$$\begin{aligned}
\bigstar v_i &= \frac{1}{2n}(e_{i-1} + e_i) \\
\bigstar e_i &= \frac{3n}{4} \sum_{0 \le j \le n-1} \left(\sum_{k \ge 0} \left(\frac{-1}{4} \right)^k \sum_{t \in \mathbb{Z}} \binom{k}{\frac{k+|i-j|+nt}{2}} + \binom{k}{\frac{k+|i-(j+1)|+nt}{2}} \right) v_j
\end{aligned}$$

In the Figures 1.8, 1.9 and 1.10, we plot $\bigstar e_{n/2}$ for $n = 10, 20, 50$. In each figure, the x-axis denotes the circle, triangulated with black dots as vertices. For fixed n, and each $0 \le i \le n$, we plot the coefficient of v_i appearing in $\bigstar e_{n/2}$. We've used a triangle to denote this value. To suggest that the plots are roughly a "delta function" supported in a small neighborhood, we have connected consecutive plot points with a line.

The matrices and plots we have encountered are reminiscent of those that appear in the study of discrete differential operators. We emphasize here that this phenomenon results from the inner product or metric, in particular its inverse. From our computation of $\bigstar$ one can easily compute $\bigstar^2$, and it is clear that this operator approximates a delta-type function.

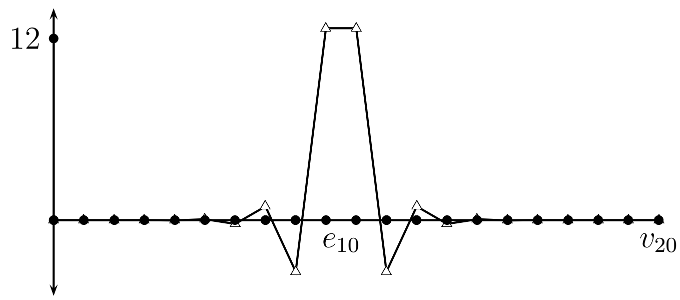

Figure 1.9: Coefficients of v_i appearing in $\bigstar e_{10}$, for $n = 20$

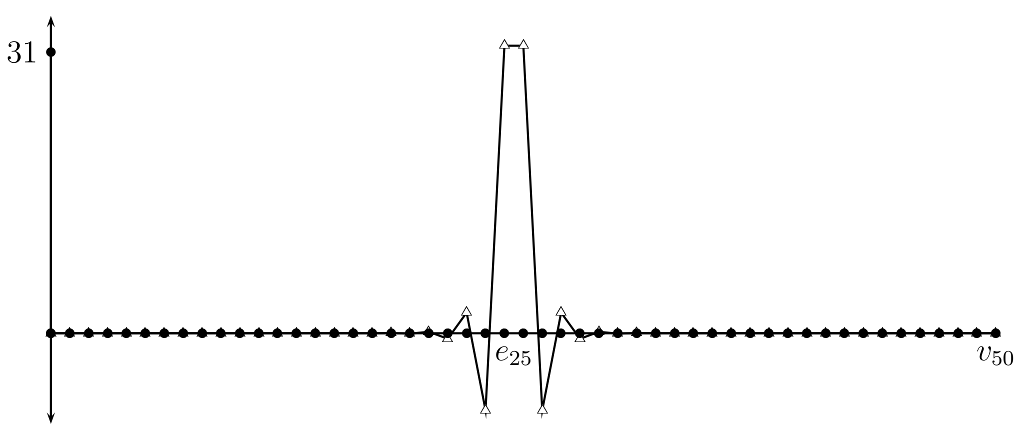

Figure 1.10: Coefficients of v_i appearing in $\bigstar e_{25}$, for $n = 50$

Chapter 2

Partial Algebras and Applications

2.1 Introduction

It has long been known to algebraic topologists that the intersection of chains in a manifold is not fully defined. As a perhaps worst-case example, a chain is never transverse to itself, and therefore its naive self-intersection does not even give the correct homological dimension. In this chapter we prove a result which allows one to deal with situations such as this, where an algebraic structure on a complex is only partially defined. An important consequence is that, although having fully defined algebraic structures on a complex is a psychological luxury, certain partial structures are sufficient for capturing all of the important homological information.

Let us first describe how the properties of chains in general position lead naturally to a precise algebraic theory. Imagine one has a collection of j chains of a smooth manifold in general position. Of course, for $j = 1$, this condition is vacuous. Note that any subset of this collection also consists of chains in general position. Finally, it is almost a classical result that homology classes may be represented by cycles in general position. Abstracting these properties, one is led naturally to the definition of a *domain* in a complex: A domain in a complex C, is a collection of subcomplexes C_j of $C^{\otimes j}$ such that $C_1{=}C$, C_j is a subcomplex of $C^{\otimes j_1} \otimes \cdots \otimes C^{\otimes j_k}$ for all ordered partitions $j_1, \cdots, j_k$ of the integer j, and finally all inclusions of subcomplexes induce isomorphisms on homology (i.e. are quasi-isomorphisms) [58], [34].

Continuing to be led by the example of chains, one can also ask how the operation of intersection behaves with respect to the collection of chains in general position. One important property can be described as follows. Suppose we have a collection of j chains and partition them into a collection of k subsets. Then for each subset we intersect the chains to obtain a new chain. The result is a collection of k chains *which are in general position.* Of course, if we then intersect this collection of k chains the resulting chain is that same as we if had simply intersected the original collection of j chains. Abstracting this property we are led naturally to the notion of a partially defined algebraic structure, [34].

Operads provide a convenient way to abstractly describe the operations and relations of an algebraic structure. An algebra over an operad is roughly then the imposition of these abstract operations on a particular set, complex, space, etc. In [34], Kriz and May define partial algebras over operads and proved several results concerning them. As a continuation of this work, we show in Section 2.2 that partial algebras over operads of complexes do capture all of the important homological information. In particular, they are quasi-isomorphic to genuine algebras.

In sections that follow we describe an application of this theorem to the chains of a manifold, showing in particular that the intersection of chains induces the structure of an E_∞ algebra on a complex quasi-isomorphic to the chains. Related results and announcements appear in McClure's paper [42], see also Section 2.3 below.

In Section 2.7 we use the results of Chas and Sullivan [7] to describe an application to string topology, showing that 'free resolutions of the Lie operad' act on a complex quasi-isomorphic to any complex of chains of the free loopspace.

In Appendix A we describe ways of constructing geometrically defined chain complexes suitable for intersection theory.

2.2 Algebraic Result

In this section we prove the algebraic result that will be used in later sections. Our main result concerns partial algebras over an operad, and here are the necessary definitions.

Definition 2.2.1. *An operad of complexes over a ring R is a collection of complexes $\mathcal{O}(j)$ over R, $j \geq 0$, together with a unit map $\eta : R \to \mathcal{O}(1)$, an action of the symmetric group Σ_j on $\mathcal{O}(j)$ for each j, and chain maps*

$$\Theta : \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \to \mathcal{O}(j_1 + \cdots j_k)$$

for all $k \geq 1$ and $j_i \geq 0$. The maps Θ are required to be associative, equivariant with repsect to the Σ actions, and unital with respect to the unit η. See [34].

One should think that the component $\mathcal{O}(j)$ encodes operations with j

inputs and one output. The maps Θ tell one how to compose operations. Morphisms of operads are defined naturally.

Definition 2.2.2. *A domain in a complex C is a collection $\{C_j\}$ of subcomplexes of $C^{\otimes j}$ satisfying the following:*

1. $C_1 = C$.

2. *For all $j = j_1 + \ldots j_k$, C_j is a Σ_j-invariant subcomplex of $C^{\otimes j_1} \otimes \cdots \otimes C^{\otimes j_k}$.*

3. *The inclusion map $C_j \hookrightarrow C^{\otimes j}$ is a quasi-isomorphism.*

One should think of a domain in a complex as describing a domain on which certain operations are defined. These operations, from j inputs to one output, are encoded by the component $\mathcal{O}(j)$ in the following way.

Definition 2.2.3. *Let $\mathcal{O}$ be an operad. A partial algebra over the operad $\mathcal{O}$ is a domain $\{C_j\}$ in a complex C and a collection of Σ_j-equivariant maps*

$$\Theta_j : \mathcal{O}(j) \otimes C_j \to C$$

satisfying

1. *For all $j = j_1 + \cdots + j_k$, the maps*

   $$\Theta_j : \mathcal{O}(j_1) \otimes \ldots \mathcal{O}(j_k) \otimes C_j \to C$$

   factor through C_k.

2. *The maps Θ_j form an action with respect to the operad composition.*

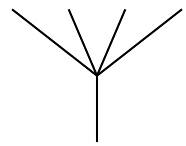

Figure 2.1: A tree as an operation

Figure 2.2: The unit element

Remark 2.2.4. *An algebra over an operad is a partial algebra where the domain* $\{C_j\}$ *is the trivial one satisfying* $C_j = C^{\otimes j}$*.*

For simplicity, we work over the ring R of integers or rationals, though our results apply to any Dedekind ring. Let $\mathcal{O} = \bigoplus_{k \geq 0} \mathcal{O}(k)$ be an operad of complexes over R such that each $\mathcal{O}(k)$ is a projective $R[\Sigma_k]$-module and let A be a flat complex over R. These assumptions are crucial and imply that inclusions and quasi-isomorphisms are preserved under tensor products.

We represent elements of $\mathcal{O}(k)$ by trees with k inputs, as in figure 2.1, and the unit in $\mathcal{O}(1)$ as in Figure 2.2.

We represent an element of $\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k)$ by a collection of k trees, as in Figure 2.3, where we have left spaces between trees to indicate this is a tensor product of elements of $\mathcal{O}$.

Any operad $\mathcal{O}$ corresponds to a monad in the category of complexes, also denoted by $\mathcal{O}$, defined by $\mathcal{O}(X) = \bigoplus_{k \geq 0} \mathcal{O}(k) \otimes X^k$ where X^k is the k-fold tensor product of X. The monad structure is given by the operad composition and unit. Algebras over $\mathcal{O}$ correspond to algebras over the monad $\mathcal{O}$. We

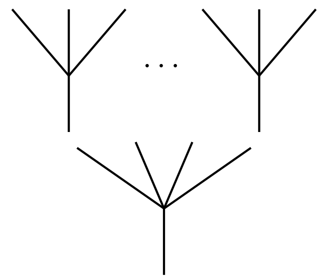

Figure 2.3: An element of $\mathcal{O}(k) \otimes \mathcal{O}(j_1) \cdots \mathcal{O}(j_k)$

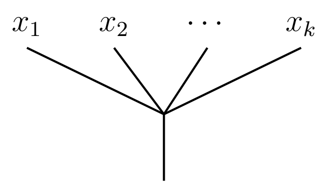

Figure 2.4: An element of $\mathcal{O}(k) \otimes X^k$

represent an element of $\mathcal{O}(k) \otimes X^k$ by a diagram consisting of a tree labeled by elements of X, as in Figure 2.4, where $x_1 \otimes \cdots \otimes x_k \in X^k$.

An operad $\mathcal{O}$ also gives rise to a monad in the category of domains of complexes, which we denote by $\mathcal{O}_*$. If $X_* = \{X_j\}$ is a domain in a complex X then we define $\mathcal{O}_*(X_*) = \bigoplus_{k \geq 0} \mathcal{O}(k) \otimes X_k$ and a domain in this complex is given by elements in $(\bigoplus_{k \geq 0} \mathcal{O}(k) \otimes X_k)^j$ which may be represented a diagram of j trees labeled by elements of X that satisfy $x_1 \otimes \cdots \otimes x_i \in X_i$, as in Figure 2.5. Similarly, partial algebras over $\mathcal{O}$ correspond to algebras over the monad $\mathcal{O}_*$ in the category of domains of complexes.

As shown in [34], these monadic interpretations allow one to make use of the two-sided bar construction $B_*(-,-,-)$.

Theorem 2.2.5. *Let A be a flat complex and $\mathcal{O} = \bigoplus_{k \geq 0} \mathcal{O}(k)$ be an operad*

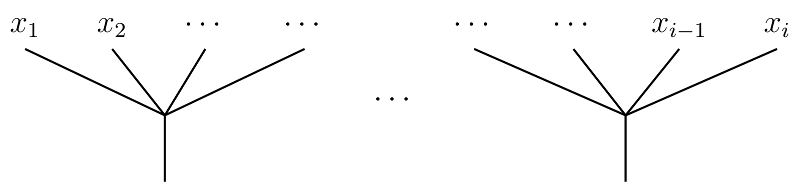

Figure 2.5: An element in the domain X_i

of complexes such that each $\mathcal{O}(k)$ is a projective $R[\Sigma_k]$-module. There is a functor W that assigns to any partial $\mathcal{O}$-algebra A_* an $\mathcal{O}$-algebra WA_* such that A_* and WA_* are quasi-isomorphic as partial $\mathcal{O}$-algebras.

The rest of this section is devoted to the proof of this theorem; many the techniques used appear in [34].

Using the notation in [34], from a domain $A_* = \{A_j\}_{j\geq 1}$ we obtain the following diagram of simplicial domains of complexes, each of which is a simplicial partial $\mathcal{O}$-algebra:

$$\underline{A_*} \underset{\varphi}{\overset{\eta}{\rightleftarrows}} B_* \xrightarrow{\delta} W_* \tag{2.1}$$

where $\underline{A_*}$ is the constant simplicial object, $B_* = B_*(\mathcal{O}_*, \mathcal{O}_*, A_*)$ and $W_* = (R\mathcal{O}L, \mathcal{O}_*, A_*)$. Here R and L are the obvious functors from complexes to domains of complexes, and vice versa, respectively. We now give a diagrammatic description of B_*, W_* and the maps above.

By definition $B_* = \{B_q\}_{q\geq 0}$ is a simplicial domain of complexes where the q-simplicies are $B_q = \mathcal{O}_*^{q+1}(A_*)$. By our above diagram conventions, an element of $B_{q,j} = \mathcal{O}_*^{q+1}(A_j)$ is represented by a collection of j objects, each a stacking of trees $q + 1$ high, labeled on top by elements of A such that $a_1 \otimes \cdots \otimes a_i \in A_i$. See Figure 2.6.

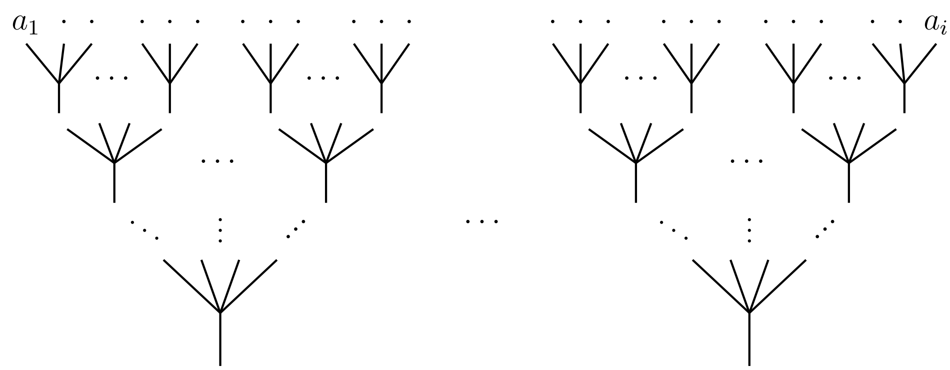

Figure 2.6: An element of $B_{q,j}$

Note that this does in fact define a domain in $B_q = \mathcal{O}_*^{q+1}(A)$. Let us refer to top row of trees at the 1^{st}, the next below the 2^{nd}, etc. The 0^{th} face operator of this simplicial object is given by evaluating the elements of A on the 1^{st} row of trees using the partial algebra structure of A_* over $\mathcal{O}$. For $1 \leq i \leq q$, the i^{th} face operator is given by composing the i^{th} and $(i+1)^{st}$ rows of this diagram using the operad structure. The 0^{th} degeneracy operator of this simplicial object is given by inserting a row of units of $\mathcal{O}$ between the elements of A and the first row of trees. For $1 \leq i \leq q$, the i^{th} degeneracy operator is given by inserting a row of units of $\mathcal{O}$ between the i^{th} and $(i+1)^{st}$ rows of this diagram. B_* is a simplicial partial algebra over $\mathcal{O}$ in the following way: given an element of $\mathcal{O}(j)$ (diagrammatically a tree with j inputs), and an object of $B_{q,j} = \mathcal{O}_*^{q+1}(A_j)$, as in the diagram above, we compose at the bottom using the operad composition.

The diagrammatic description of W_* is similar. Let $W_{q,j} = R\mathcal{O}L\mathcal{O}_*^q(A_j)$. An element of $W_{q,j}$ is represented by a collection of j objects, each a stacking of trees $q+1$ high, labeled on top by elements of A satisfying a certain property

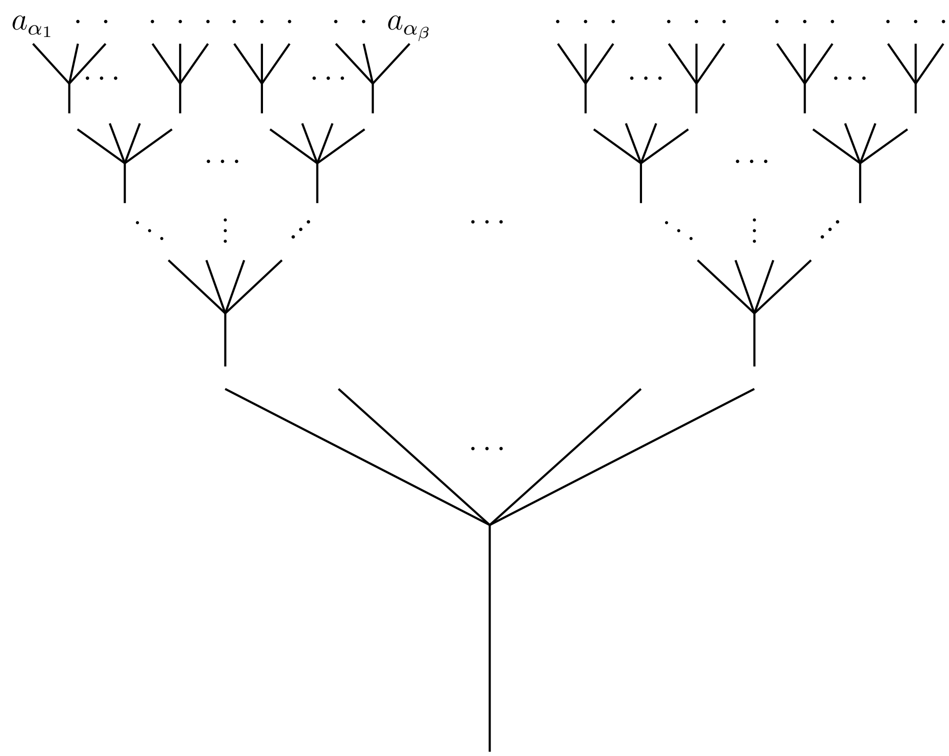

Figure 2.7: An element of $W_{q,j}$

common to each of these j objects. To illustrate this property, consider any one of these j objects, as in Figure 2.7. In terms of this diagram we may express the property as follows: for each tree in the q^{th} (top) row, the elements $a_{\alpha_1}, \cdots, a_{\alpha_\beta}$ of A "lying above" this tree satisfy $a_{\alpha_1} \otimes \cdots \otimes a_{\alpha_\beta} \in A_\beta$. In Figure 2.7 we have show this only for the first tree in the q^{th} row (of one of the j objects).

The face and degeneracy operators of W_* are diagrammatically the same as for B_*. Note that for each q, $W_{q,j} = (W_{q,1})^{\otimes j}$ and, as before, W_* is a simplicial $\mathcal{O}_*$-algebra by composition at the bottom, and hence can be thought of as a simplicial $\mathcal{O}$-algebra.

The maps η, φ, δ have simple descriptions in terms of these diagrams. The map δ is induced by the given domain A_* of A. In terms of diagrams this means the identity map on trees and the inclusion on elements of A, which is a quasi-isomorphism by the assumption that A_* is a domain. The map φ is described diagrammatically by fully evaluating an object of $B_{q,j}$ using the partial algebra structure, and considering the output element of A_j as an element of $A_{q,j}$ in the constant simplicial object $\underline{A_*}$. It is easy to see that φ is a map of partial algebras. Lastly, η is given by including an element of $A_{q,j}$ into $B_{q,j}$ by stacking units $q+1$ high under each element of A.

It is easy to check that $\varphi \circ \eta = \mathrm{Id}$. Moreover, as proven in [45], there is a simplicial homotopy from $\eta \circ \varphi$ to Id. Hence φ is a quasi-isomorphism of simplicial partial $\mathcal{O}$-algebras.

We wish to apply a normalization map $C_\#$ to this diagram to obtain a quasi-isomorphism of partial $\mathcal{O}$-algebras. As is pointed out in [34], the obvious way to do this does not necessarily take simplicial domains of complexes to domains in complexes. We verify directly that for the diagram of simplicial domains in equation (2.1), this is indeed the case. Let us first define $C_\#$.

If $X_{q,p}$ is a simplicial complex, with q denoting the simplicial grading, p the complex grading, we define $C_\#(X)_n = \sum_{p+q=n} X_{q,p}/D$ where D denotes the set of degenerate simplicies, i.e. the sum of the images of the degeneracy maps. This forms a complex with differential equal to the sum of the simplicial differential $\sum(-1)^i\partial_i$ plus $(-1)^q$ times the complex differential. $C_\#$ is defined on morphisms by adding maps along fixed degree $n = q + p$. One can show that $C_\#$ takes simplicial maps to chain maps, simplicial homotopies to chain homotopies and simplicial quasi-isomorphisms to quasi-isomorphisms [34].

For $X_{q,r}, Y_{p,s}$ simplicial complexes the shuffle map

$$g : C_\#(X)_{q+r} \otimes C_\#(Y)_{p+s} \to C_\#(X \otimes Y)_{q+p+r+s}$$

is defined by

$$g(a \otimes b) = \sum_{(u,v)} \pm (s_{\nu_q} \cdots s_{\nu_1} a \otimes s_{\mu_p} \cdots s_{\mu_1} b)$$

where s_* are the degeneracy operators, the sum is over all shuffles $\nu_1 < \cdots < \nu_q$ and $\mu_1 < \cdots < \mu_p$ of $\{0, 1, \cdots, p+q+1\}$, and the sign is determined by the signature of the corresponding permutation of $\{0, 1, \cdots, p+q+1\}$. See [45]. According to [34], g is commutative, associative and unital. We also denote iterates of the shuffle map by g.

We define the Alexander-Whitney map $f : C_\#(X \otimes Y) \to C_\#(X) \otimes C_\#(Y)$ by:

$$f(a \otimes b) = \sum_{i=0}^{n} \tilde{\partial}^{n-i} a \otimes \partial_0^i b$$

where ∂_* are the face operators and $\tilde{\partial}$ denotes the last face operator. One can show that $f \circ g = \mathrm{Id}$ (note we're working on the normalized level).

We now describe a way to define $C_\#$ on a simplicial domain of a complex $(X_{q,p})_* = \{X_{q,p,j}\}_{j \geq 1}$ where $X_{q,p,1} = X_{q,p}$. For each j, $X_{q,p,j}$ is a simplicial complex and there is a quasi-isomorphism $X_{q,p,j} \hookrightarrow (X_{q,p,1})^{\otimes j}$. Here $\otimes$ means the tensor product as simplicial complexes. We define a domain in $C_\#(X_{q,p,1})$ by:

$$C_\#(X_{q,p,1})_j = f\Big(g\big(C_\#(X_{q,p,1})^{\otimes j}\big) \cap C_\#(X_{q,p,j})\Big), \tag{2.2}$$

i.e. map in via the shuffle map, intersect with the given simplicial domain,

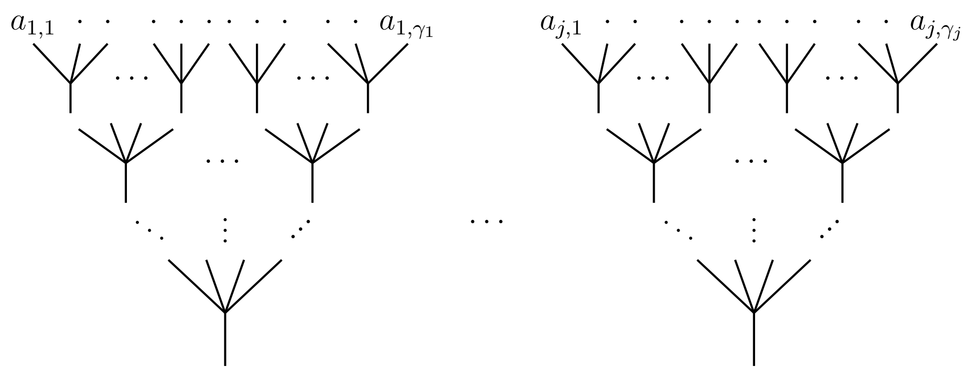

Figure 2.8: An element of $(C_\# B_1)^{\otimes j}$

and then map out via the Alexander-Whitney map. Let $f \circ \cap \circ g$ denote this map. As pointed out in [34], this not necessarily yield a quasi-isomorphism $C_\#(X_{q,p,1})_j \hookrightarrow C_\#(X_{q,p,1})^{\otimes j}$. We now verify directly that this is indeed the case when applied to $\underline{A_*}$, B_*, and W_*.

Recall that $\underline{A_*}$ is the constant simplicial object of domains in the complex A. Then $C_\#(\underline{A_*}) = C(\underline{A_*})_1 = A_*$, since we are working on the normalized level and all of the degeneracy maps are the identity.

Claim 1. *Applying $C_\#$ to B_* and defining $(C_\# B_*)_j$ as in (2.2) yields a domain in $C_\# B_*$.*

Proof. An element Z of $(C_\# B_1)^{\otimes j}$ is represented as a collection of j objects, as in Figure 2.8, where for $1 \leq k \leq j$, the k^{th} object is a stacking of trees $q_k + 1$ high, and the elements $a_{k,1}, \cdots, a_{k,\gamma_k}$ of A labeling the k^{th} object satisfy $a_{k,1} \otimes \cdots \otimes a_{k,\gamma_k} \in A_{\gamma_k}$.

We now compute $(C_\# B)_j$ explicitly and show that $(C_\# B)_j \hookrightarrow (C_\# B_1)^{\otimes j}$ is a quasi-isomorphism.

Let S_j denote the subcomplex of $(C_\# B_1)^{\otimes j}$ consisting of those elements Z'

which satisfy the further restriction

$$a_{1,1} \otimes \cdots \otimes a_{j,\gamma_j} \in A_\gamma \tag{2.3}$$

where $\gamma = \sum_{i=1}^{j} \gamma_i$. We claim $(C_\# B)_j = S_j$ and first show

$$(C_\# B)_j = f\Big(g\big((C_\# B_1)^{\otimes_j}\big) \cap C_\#(B_j)\Big) \subset f\big(C_\#(B_j)\big) \subset S_j.$$

The first containment follows set theoretically. For the second we note that if $Y \in C_\#(B_j)$ then, when represented diagrammatically, its labeling A-elements satisfy (2.3). Recall that the map f is defined by applying first and last face operators, which here are induced by the partial algebra structure on A_* and the operad composition in $\mathcal{O}$. The last face operators are no concern; it is simply operad composition at the bottom. The first face operators may be regarded as an iteration of partial actions on one of the j objects and the identity on the others. By the definition of a partial algebra, under the partial action the output consists of elements of A *which lie in the domain*. Then $f(Y)$ also satisfies (2.3), so $f(Y) \in S_j$.

We now argue that we have the opposite inclusion:

$$S_j \subset (C_\# B)_j = f\Big(g\big((C_\# B_1)^{\otimes_j}\big) \cap C_\#(B_j)\Big).$$

Recall that g is defined by applying degeneracy operators, which here are given by the operad unit. If $Y \in S_j \subset C_\#(B_1)^{\otimes_j}$ then $g(Y) \in C_\#(B_j)$ since the elements of A satisfying (2.3) are unaffected by the insertion of units of

$\mathcal{O}$. Then

$$f\Big(g(Y) \cap C_{\#}(B_j)\Big) = f\big(g(Y)\big) = Y$$

so $Y \in (C_{\#}B)_j$.

Thus we have shown that $S_j = (C_{\#}B)_j$, i.e that the map $f \circ \cap \circ g$ is simply the restriction to those elements Z whose labeling A-elements satisfy (2.3). Then the inclusion $i : (C_{\#}B)_j \hookrightarrow (C_{\#}B_1)^{\otimes_j}$ can be seen as being induced by the quasi-isomorphism given in the domain A_* of A. By our assumptions that A is flat and $\mathcal{O}$ is projective, this inclusion i is also a quasi-isomorphism. Lastly, there is a factoring

$$(C_{\#}B)_j \hookrightarrow (C_{\#}B)_{j_1} \otimes \cdots \otimes (C_{\#}B)_{j_k}$$

for all ordered partitions $\{j_1, \cdots j_k\}$ of the integer j, induced by that of the domain A_*, so $(C_{\#}B)_j$ is a domain in $C_{\#}B$. $\square$

Claim 2. *Applying $C_{\#}$ to W_* and defining $(C_{\#}W_*)_j$ as in (2.2) yields a domain in $C_{\#}W_*$ such that $(C_{\#}W_*)_j = (C_{\#}W_1)^{\otimes_j}$.*

Proof. It suffices to show that $(C_{\#}W_*)_j = (C_{\#}W_1)^{\otimes_j}$. Recall that each element of $(C_{\#}W_1)^{\otimes_j}$ is represented by a collection of j objects, the k^{th} object a stacking of trees $q_k + 1$ high labeled on top by elements of A such that for each tree in the q_k^{th} row, the elements $a_{\alpha_1}, \cdots, a_{\alpha_\beta}$ of A "lying above" this tree satisfy $a_{\alpha_1} \otimes \cdots \otimes a_{\alpha_\beta} \in A_\beta$. See Figure 2.7.

Recall that the map g introduces units of $\mathcal{O}$ as before. Now, $g((C_{\#}W_1)^{\otimes_j}) \subset C_{\#}(W_j)$ since the insertion of rows of units (between any two rows) has no effect on the condition that elements of A lying above the ${q_k}^{th}$ level trees are in the domain. Then,

$$C_{\#}(W_*)_j = f\Big(g(C_{\#}W_1)^{\otimes j} \cap C_{\#}(W_j)\Big) = f\Big(g(C_{\#}W_1)^{\otimes j}\Big) = C_{\#}(W_1)^{\otimes j}$$

since, on the normalized level $f \circ g = \mathrm{Id}$. Thus the claim is proved. $\square$

We are now ready to complete the proof of the theorem. We have shown that $C_{\#}$ sends the diagram:

$$\begin{tikzcd} \underline{A_*} \arrow[r, shift left, "\eta"] & B_* \arrow[l, shift left, "\varphi"] \arrow[r, "\delta"] & W_* \end{tikzcd}$$

of domains of simplicial complexes to domains of complexes. Then each of $C_{\#}(\underline{A_*})$, $C_{\#}(B_*)$, and $C_{\#}(W_*)$ are partial $\mathcal{O}$-algebras via precomposition with the shuffle map g. Also, $C_{\#}(W_*)$ is an $\mathcal{O}$-algebra.

Also, both $C_{\#}\delta$ and $C_{\#}\varphi$ are maps of partial $\mathcal{O}$-algebras. By our previous remarks that $C_{\#}$ takes simplicial maps to chain maps, simplicial homotopies to chain homotopies and simplicial quasi-isomorphisms to quasi-isomorphisms, both $C_{\#}\delta$ and $C_{\#}\varphi$ are quasi-isomorphisms, so the theorem is proved.

Remark 2.2.6. *In [66], Vallette generalizes operads to what he calls 'properads'. These encode operations with $k \geq 1$ outputs, and algebras over properds are defined naturally. Examples are Lie bialgebras, Hopf algebras etc. With the appropriate modification of definition 2.2.3, one can define partial algebras over properads and their morphisms. We expect these can be functorially replaced by genuine algebras over properads. A complete proof is work in progress, and may be much the same as above, in this case using the bar construction for properads, which is described in [66].*

2.3 Applications to Intersection of Chains

We now wish to apply Theorem 2.2.5 to chains of a manifold, giving full meaning to the partially defined operation of transversal intersection. Much of the hard work in setting up this application has very recently been done by McClure [42] for piecewise linear chains in a PL manifold. In that paper he defines a new and useful intersection pairing for PL chains in general position, and proves that they form a “partial Leinster algebra”. These same constructions also show the existence of a partial algebra, in the above operadic sense, over the commutative operad (see Definition 2.5.1). Pulling back this partial action to the action of an E_∞ algebra, Theorem 2.2.5 can be applied, giving an E_∞ algebra on a complex quasi-isomorphic to the PL chains of a PL manifold (i.e. a PL-chain example of Theorem 2.6.2). This result was announced by McClure in [42] and is expected to appear in [43]. (It remains to give a construction that assigns to any partial Leinster algebra an E_∞ algebra, and it will be interesting to see how this construction relates to the general theorem 2.2.5 above.)

The story does not end here. In seems useful to have a theory of intersecting chains that are represented by maps into a manifold, rather than as subsets of a manifold. Such chains have important applications to mapping spaces: the free loopspace in String Topology and Sullivan’s interpretation of the Gromov-Witten Theory in algebraic topology, [57]. Furthermore, it is instructive to work out all of the details of intersection of chains in the (piecewise) smooth category, including signs, since they may also be used to describe the “homology intersection ring of a manifold” and the “umkehr map”. We’ll accomplish these tasks in the next two sections, while also laying the ground work for applications of Theorem 2.2.5.

2.4 Transversal Intersection of Chains of a Manifold

In this section, we describe the transversal intersection of chains in a manifold. This will be a first step in describing a partial action of the commutative operad on the chains. The basic intersection constructions in this section also appear in [7].

Let M be a closed oriented n-manifold. Let C be a flat chain complex constructed by mapping oriented stratified objects into M. In Appendix A we suggest a way one might construct such a complex C, though fully understanding all such complexes is work in progress. [1]

The properties this complex must satisfy for our later constructions are:

1. The transversal intersection of any two chains is a chain in C.

2. Cycles, and relative cycles, of C can represented by chains in general position. See definition 2.5.4.

3. The identity map $\mathrm{Id} : M \to M$ is an element of C.[2]

[1] To sketch those ideas here, we could take M to be smooth, the stratified objects to be compact, connected and modeled on smooth manifolds, and the maps, restricted to each stratum, to be smooth. The algebraic boundary operator is induced by the geometric boundary operator, and the restriction of the map.

[2] This is only necessary if one desires a unit on the chain level.

Remark 2.4.1. *None of our constructions require that the complex C give the usual homology of M. We'll refer to any such complex C as* a complex of chains of M *(as oppposed to "the chains" or "the complex of chains").*

Let us denote a chain in C by a triple (c_1, o_1, f_1), where c_1 is the underlying space of the chain, o_1 is an orientation, and f_1 is a map of c_1 into M, see Appendix A. When referring to orientations below, it is helpful imagine we're in the smooth category.

We now describe how to define the intersection of two transversal chains of M (giving the first property in 2.4). Let (c_1, o_1, f_1) and (c_2, o_2, f_2) be chains in C such that the map $f_1 \times f_2 : c_1 \times c_2 \to M \times M$ restricted to each stratum of $c_1 \times c_2$, is transverse to the diagonal D of $M \times M$. Then the preimage of the diagonal is a stratified subset $c_1 \cdot c_2$ of $c_1 \times c_2$ [10], and the restriction f' of the map $f_1 \times f_2$ to this locus may be regarded as a map into M. We may assume this locus is connected as otherwise it may be written uniquely as the algebraic sum of its connected components. We orient $c_1 \cdot c_2$ in the following way: the normal bundle N_D of D in $M \times M$ is oriented so that

$$o_{N_D} \oplus o_D = o_{M \times M}.$$

We pull back o_N to obtain an orientation of the normal bundle $N_{c_1 \cdot c_2}$ of $c_1 \cdot c_2$ in $c_1 \times c_2$, and orient $c_1 \cdot c_2$ so that

$$o_{N_{c_1 \cdot c_2}} \oplus o_{c_1 \cdot c_2} = o_{c_1} \oplus o_{c_2},$$

We define the intersection of the chains (c_1, o_1, f_1) and (c_2, o_2, f_2) by

$$(c_1, o_1, f_1) \pitchfork (c_2, o_2, f_2) = (-1)^{n|c_1|} \; (c_1 \cdot c_2, o_{c_1 \cdot c_2}, f')$$

where $|c_1|$ is the degree of c_1. We extend this map bi-linearly over the transversal generators to a map defined on a subcomplex C_2 of $C \otimes C$, and denote it by $\pitchfork \colon C_2 \to C$. For brevity, we may write c_1 for (c_1, o_1, f_1).

Remark 2.4.2. *The map $\pitchfork$ has degree $-n$. If we shift the complex C down by $n = \dim M$, then the map $\pitchfork$ has degree 0, and the introduction of the sign $(-1)^{n|c_1|}$ makes $\pitchfork$ graded commutative, and ∂ a graded derivation of $\pitchfork$, as the following lemma shows.*

Lemma 2.4.3. *The map $\pitchfork \colon C_2 \to C$ satisfies:*

1. $[M] \pitchfork c_1 = c_1 \pitchfork [M] = c_1$, *where $[M]$ is the identity map* $\mathrm{Id} : M \to M$*, i.e. $[M] \in C$ is a unit with respect to $\pitchfork$.*
2. $c_1 \pitchfork c_2 = (-1)^{(|c_1|+n)(|c_2|+n)} \; c_2 \pitchfork c_1$
3. $\partial(c_1 \pitchfork c_2) = \partial c_1 \pitchfork c_2 + (-1)^{|c_1|+n} \; c_1 \pitchfork \partial c_2$

Proof. Up to sign, the first statement is clear[3]. We first check that the orientations of $[M] \pitchfork c_1$ and c_1 agree,

$$\begin{aligned} o_M \oplus o_{N_{M \pitchfork c_1}} \oplus o_{M \pitchfork c_1} &= (-1)^{n^2} \; o_{M \times M} \oplus o_{c_1} \\ &= (-1)^{n^2} \; o_{N_D} \oplus o_D \oplus o_{c_1} \\ &= o_D \oplus o_{N_D} \oplus o_{c_1}. \end{aligned}$$

[3]Considering signs is like changing one's clothes; very few do it in public, and when one does, it's often a turn-off.

Since $o_M = o_D$ and $o_{N_D} = N_{M \pitchfork c_1}$, $[M]$ is a left unit. Similarly, the orientations of $c_1 \pitchfork [M]$ and c_1 agree

$$\begin{aligned} o_{c_1} \oplus o_{N_D} \oplus o_D &= o_{c_1} \oplus o_{M \times M} \\ &= (-1)^{nc_1} \ o_{N_{c_1 \pitchfork M}} \oplus o_{c_1 \pitchfork M} \oplus o_M \\ &= o_{c_1 \pitchfork M} \oplus o_{N_{c_1 \pitchfork M}} \oplus o_M. \end{aligned}$$

For the second statement, it suffices to show

$$c_1 \cdot c_2 = (-1)^{|c_1| \cdot |c_2| + n^2} \ c_2 \cdot c_1$$

since then

$$\begin{aligned} c_1 \pitchfork c_2 &= (-1)^{n|c_1|} \ c_1 \cdot c_2 \\ &= (-1)^{n|c_1|} (-1)^{|c_1| \cdot |c_2| + n^2} \ c_2 \cdot c_1 \\ &= (-1)^{n|c_1| + |c_1| \cdot |c_2| + n^2 + n|c_2|} c_2 \pitchfork c_1 \\ &= (-1)^{(|c_1|+n)(|c_2|+n)} \ c_2 \pitchfork c_1 \end{aligned}$$

Recall that $c_i \cdot c_j$ is oriented so that

$$o_{N_{c_i \cdot c_j}} \oplus o_{c_i \cdot c_j} = o_{c_i} \oplus o_{c_j}. \tag{2.4}$$

Consider the following commutative diagram

$$\begin{tikzcd}
c_1 \times c_2 \arrow[r, "f_1 \times f_2"] \arrow[d, "\sigma"'] & M \times M \arrow[d, "\sigma"] \\
c_2 \times c_1 \arrow[r, "f_2 \times f_1"] & M \times M
\end{tikzcd}$$

where σ is the map that interchanges coordinates. The left vertical map has degree $(-1)^{|c_1|\cdot|c_2|}$, so interchanging the roles of i and j in (2.4), the right hand side of (2.4) changes by a sign of $(-1)^{|c_1|\cdot|c_2|}$. As for the left hand side of (2.4), consider the induced commutative diagram of isomorphisms of n-dimensional normal bundles

$$\begin{tikzcd}
N_{c_1\cdot c_2} \arrow[r, "f_1 \times f_2"] \arrow[d, "\sigma"'] & N_D \arrow[d, "\sigma"] \\
N_{c_2\cdot c_1} \arrow[r, "f_2 \times f_1"] & N_D
\end{tikzcd}$$

The horizontal maps are orientation preserving by definition. Therefore the sign of the left vertical map equals the sign of the right vertical map, which is $(-1)^{n^2}$, since the degree of $\sigma : M \times M \to M \times M$ is $(-1)^{n^2}$, N_D is oriented by

$$o_{N_D} \oplus o_D = o_{M\times M},$$

and $o_D = o_M$ canonically. Therefore, interchanging the roles of i and j in (2.4), the right hand side of (2.4) changes by a sign of $(-1)^{n^2}$. So, the total sign change is $(-1)^{|c_1|\cdot|c_2|+n^2}$.

For the last statement, note that any point in $\partial c'$ is either in $\partial c_1 \pitchfork c_2$ or $c_1 \pitchfork \partial c_2$; again we check the signs. Here we must pay particular attention to the orientation induced on the boundary; if X is a stratum with outward point normal vector μ_X, then ∂X is given the orientation $o_{\partial X}$ satisfying

$$o_X = \mu_x \oplus o_{\partial X}$$

Let $c' = c_1 \pitchfork c_2$. For a point in $\partial c_1 \pitchfork c_2$, the vector $\mu_{c'}$ pointing outward of c' equals the vector μ_{c_1} pointing outwards of c_1. So,

$$\begin{aligned} o_{c_1} \oplus o_{c_2} &= (-1)^{n|c_1|}\ o_{N_{c'}} \oplus o_{c'} \\ &= (-1)^{n|c_1|}\ o_{N_{c'}} \oplus \mu_{c'} \oplus o_{\partial c'} \\ &= (-1)^{n|c_1|}\ o_{N_{c'}} \oplus \mu_{c_1} \oplus o_{\partial c'} \\ &= (-1)^{n|c_1|+n}\ \mu_{c_1} \oplus o_{N_{c'}} \oplus o_{\partial c'} \end{aligned} \tag{2.5}$$

Computing another way we see that

$$\begin{aligned} o_{c_1} \oplus o_{c_2} &= \mu_{c_1} \oplus o_{\partial c_1} \oplus o_{c_2} \\ &= \mu_{c_1} \oplus o_{\partial c_1 \times c_2} \\ &= (-1)^{n(|c_1|-1)}\ \mu_{c_1} \oplus o_{N_{\partial c_1 \pitchfork c_2}} \oplus o_{\partial c_1 \pitchfork c_2} \end{aligned} \tag{2.6}$$

Comparing (2.5) and (2.6), and noting that $o_{N_{c'}} = o_{N_{\partial c_1 \pitchfork c_2}}$, we see the signs of $o_{\partial c'}$ and $o_{\partial c_1 \pitchfork c_2}$ agree.

We make a similar computation for a point in $c_1 \pitchfork \partial c_2$, where in this case,

$\mu_{c'} = \mu_{c_2}$, so

$$\begin{aligned}
(-1)^{n|c_1|+n}\ \mu_{c_2} \oplus o_{N_{c'}} \oplus o_{\partial c'} &= o_{c_1} \oplus o_{c_2} \\
&= o_{c_1} \oplus \mu_{c_2} \oplus o_{\partial c_2} \\
&= (-1)^{|c_1|}\ \mu_{c_2} \oplus o_{c_1} \oplus o_{\partial c_2} \\
&= (-1)^{|c_1|}\ \mu_{c_2} \oplus o_{c_1 \times \partial c_2} \\
&= (-1)^{|c_1|+n|c_1|}\ \mu_{c_2} \oplus o_{N_{c_1 \pitchfork \partial c_2}} \oplus o_{c_1 \pitchfork \partial c_2} \quad (2.7)
\end{aligned}$$

From (2.7) we use the fact that $o_{N_{c'}} = o_{N_{c_1 \pitchfork \partial c_2}}$ to conclude

$$o_{\partial c'} = (-1)^{|c_1|+n}\ o_{c_1 \pitchfork \partial c_2}.$$

□

2.5 Partial Commutative Algebra

In this section we show that the operations ∂ and $\pitchfork$ defined previously induce the structure of a partial algebra over the commutative operad. For generality, we'll assume that all coefficents are $\mathbb{Z}$. First some definitions.

Definition 2.5.1. *The unital commutative associative operad $\mathscr{C}$ is defined by the following: for $j \geq 0$, $\mathscr{C}(j)$ is isomorphic to the complex with $\mathbb{Z}$ in degree zero, and 0 in all other degrees; the Σ_j-actions are trivial and the operad composition is given by multiplication.*

Remark 2.5.2. *A complex C with the structure of an algebra over the operad*

$\mathscr{C}$ is a unital differential graded commutative associative algebra with multiplication of degree 0.

Remark 2.5.3. *The operad $\mathscr{C}$ can also be described as the being generated over $\mathbb{Z}$ by degree zero elements $u \in \mathscr{C}(0)$, $\mathrm{id} \in \mathscr{C}(1)$ and $m \in \mathscr{C}(2)$, with trivial Σ_j-actions and the appropriate unital and associativity relations. In particular,*

$$m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m).$$

Our aim is to show that C, shifted down by the dimension of M, forms a partial algebra over the operad $\mathscr{C}$, see Definition 2.2.3. First, we must define the *domain* for this partial algebra (see Definition 2.2.2).

Definition 2.5.4. *1. Let $M^j = M \times \cdots \times M$. A collection of $j \geq 2$ maps $f_i : N_i \to M$, from smooth manifolds N_i into the smooth manifold M are in* general position *if the map*

$$f_1 \times \cdots \times f_j : N_i \times \cdots \times N_j \to M^j$$

is transverse to the the diagonal

$$D_j = \{(x, \dots, x) \mid x \in M\} \subset M^j.$$

2. A collection of chains $c_1, \dots, c_j$, with maps $f_1, \dots, f_j$ into M, is in general position if the restriction of $f_1 \times \cdots \times f_j$ to each stratum of $c_1 \times \cdots \times c_j$ is transverse to D.

Remark 2.5.5. *If a collection of chains $c_1, \dots, c_j$ is in general position, then* *so is any sub-collection, since for $j' \leq j$, the projection map $\pi : M^j \to M^{j'}$ is a submersion that restricts to a diffeomorphism from D_j to $D_{j'}$. Note that by definition, for any $1 \leq k \leq j$, $c_1, \ldots, \partial c_k, \ldots, c_j$ are also in general position since their product is a codimension-one stratum of $c_1 \times \cdots \times c_j$.*

Definition 2.5.6. *Let $C(M)$ be a complex of chains of M. Then $C_j(M)$ is defined to be the linear span of all elements $c_1 \otimes \cdots \otimes c_j \in C(M)^{\otimes j}$, where $c_1, \ldots, c_j$ are chains in M which are in general position.*

Lemma 2.5.7. *The collection of complexes $\{C_j(M)\}$ is a domain in the complex $C(M)$.*

Proof. We verify the conditions of Definition 2.2.2. First, $C_1 = C$ by definition. By Remark 2.5.5, for all j and all $j_1 + \ldots j_k = j$, C_j is a subcomplex $C^{\otimes j_1} \otimes \cdots \otimes C^{\otimes j_k}$. Also, C_j is Σ_j invariant since the condition of general position is order independent. It remains to show that the inclusion map $i : C_j \hookrightarrow C^{\otimes j}$ is a quasi-isomorphism[4]. Let us consider the case $j = 2$. We use the Kunneth formula, which states that

$$H(C \otimes C) \approx H(C) \otimes H(C) \oplus Tor\big(H(C), H(C)\big),$$

see MacLane [37]. This theorem, and its proof, show that $H(C \otimes C)$ is spanned by two types of cycles. The first are those of the form $c_1 \otimes c_2$ where c_1 and c_2 are cycles of C, and the second are those built out of chains c of the form

$$c = c_1 \otimes c_2 + (-1)^{|c_1|+1}\, c_3 \otimes c_4$$

[4]A entirely different proof of the analogous result for PL chains appears in [42].

where c_1 and c_4 are cycles, and for some integer m, $\partial c_2 = mc_4$ and $\partial c_3 = mc_1$. Furthermore, the homology class of such a c is determined by the cycles c_1 and c_4 and the integer m, and the map

$$\operatorname{Tor}\big(H(C), H(C)\big) \to \frac{H(C \otimes C)}{p\big(H(C) \otimes H(C)\big)}$$

is an isomorphism, where p is the homology product. Following MacLane's notation, we'll write the class in Tor as (c_1, m, c_4).

To show that $C_2 \hookrightarrow C^{\otimes_2}$ is surjective on homology, it suffices to show that that it is surjective onto each summand in the Kunneth decomposition. Let c_1 and c_2 be cycles of C. Our standing assumption on C is that cycles, and relative cycles, can be put into general position. We'll show how to do this in the piecewise smooth category, using an argument of Abraham [1] and Morse [44], see also [21]. We choose smooth manifolds N_i and submersions $\phi : c_i \times N_i \to M$. Then the map $\phi_1 \times \phi_2$, evaluated at $(n_1, n_2) \in N_1 \times N_2$, is transverse to the diagonal D precisely when the projection map

$$\pi_1 \times \pi_2 : c_1 \times N_1 \times c_2 \times N_2 \to N_1 \times N_2$$

restricted to $(\phi_1 \times \phi_2)^{-1}(D)$ has (n_1, n_2) as a regular value. By Sard's theorem we can choose such a regular value; this gives a perturbation of c_1 and c_2 to transverse cycles, so the inclusion induces a map on homology that is surjective onto the first summand.

Now suppose $c = c_1 \otimes c_2 + (-1)^{|c_1|+1}\, c_3 \otimes c_4$ is an element of $\operatorname{Tor}\big(H(C), H(C)\big)$, as above. Since this class depends only on the class of the cycles c_1 and c_4, we can choose homologous cycles a and b that are transverse. Let $\partial x = a - c_1$ and $\partial y = b - c_2$. Then put $A = x + c_3$ and $B = y + c_2$ so that $\partial A = a$ and $\partial B = b$. Since a and b are transverse, the Cartesian map from $A \times B$ is transverse to the diagonal in a neighborhood of $a \times b$. Using a relative version of the perturbation argument described above, we can can perturb the map on $a \times B$, keeping the map on a fixed, to a chain B' is transverse to a. Similarly, we can make perturb A to A' that is transverse to b, giving a representative

$$a \otimes B' + (-1)^{|c_1|+1}\, A' \otimes b$$

for the class $c \in \mathrm{Tor}$ that is in C_2. Hence, the inclusion induces a surjective map on homology.

The proof of injectivity is similar. Suppose a cycle in $H(C_2)$ maps to zero in $H(C \otimes C)$. We'll consider each summand in the Kunneth formula. For the first summand, if two cycles c_1 and c_2 are transverse, and c_1 is a boundary, then using a relative perturbation, one can obtain a cycle homologous to c_1 that is transverse to c_2, so such a cycle is in fact zero in $H(C_2)$.

Now suppose $c = (c_1, m, c_4) \in \mathrm{Tor}$ is a boundary (this occurs only if $1/m$ is in the ground ring). We can write $c = c_1 \otimes c_2 + (-1)^{|c_1|+1}\, c_3 \otimes c_4$ and since c_1 and c_2 are transverse, and $\partial c_3 = mc_1$, c_2 is transverse to c_3 in a neighborhood of c_1. Then we can perturb c_2 to a chain c_2' that is transverse to c_3, keeping c_4 fixed so that $\partial c_2' = c_4$. Then $c_3 \otimes c_2'$ is in the image of i_*, and its boundary is a representative for c. Hence the inclusion induces a map on homology that is injective. So $C_j \hookrightarrow C^{\otimes j}$ is a quasi-isomorphism.

The proof for $j \geq 3$ uses the same transversality arguments applied to the

Kunneth formula for $H(C^{\otimes_j})$. $\square$

Now that we've show that the collection $\{C_j\}$ form a domain, our aim is to show that the this forms a partial algebra over $\mathscr{C}$, see Definition 2.2.3. First, we need two lemmas.

Lemma 2.5.8. *If $c_1, \ldots, c_j$ are in general position, then so are $(c_1 \pitchfork c_2), c_3, \ldots, c_j$.*

Proof. Suppose $c_1, \ldots, c_j$ are in general position. Let $p \in M$ be a point of the naive intersection. Let d_i be the dimension of the push forward of the tangent space of c_i to this point, and d be the dimension of the intersection of all of these vector spaces. Then from the definition of general position we have

$$\sum_{i=1}^{j} d_i + n - d = kn$$

By Remark 2.5.5, c_1 and c_2 are in general position, so the dimension of the push forward of the tangent space of $c_1 \pitchfork c_2$ to any intersection point is $d_1 + d_2 - n$. We want to show that the push forward of the tangent of spaces of $c_1 \pitchfork c_2$, $c_3, \ldots, c_j$ and D_{k-1} together span $D \times M^{k-2}$. We can compute that the dimension of this span is

$$(d_1 + d_2 - n) + \sum_{i=3}^{j} d_i + n - d$$

which equals (k-1)n, the dimension of $D \times M^{k-2}$, so we are done. $\square$

Lemma 2.5.9. *The maps $\pitchfork (\pitchfork \otimes id)$ and $\pitchfork (id \otimes \pitchfork) : C_3 \to C$ are equal.*

Proof. Let c_1, c_2, c_3 be chains of C in general position. By Remark 2.5.5 and Lemma 2.5, both $\pitchfork (\pitchfork \otimes id)$ and $\pitchfork (id \otimes \pitchfork)$ are well defined. Up to sign, the result it clear. We compute the orientations $o_{(c_1 \pitchfork c_2) \pitchfork c_3}$ and $o_{c_1 \pitchfork (c_2 \pitchfork c_3)}$. First,

$$
\begin{aligned}
o_{c_1} \oplus o_{c_2} \oplus o_{c_3} &= (-1)^{n|c_1|} \ o_{N_{c_1 \pitchfork c_2}} \oplus o_{c_1 \pitchfork c_2} \oplus o_{c_3} \\
&= (-1)^{n|c_1|}(-1)^{n(|c_1|+|c_2|-n)} \ o_{N_{c_1 \pitchfork c_2}} \oplus o_{N_{(c_1 \pitchfork c_2) \pitchfork c_3}} \oplus o_{(c_1 \pitchfork c_2) \pitchfork c_3} \\
&= (-1)^{n|c_2|+n} \ o_{N_{c_1 \pitchfork c_2}} \oplus o_{N_{(c_1 \pitchfork c_2) \pitchfork c_3}} \oplus o_{(c_1 \pitchfork c_2) \pitchfork c_3}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
o_{c_1} \oplus o_{c_2} \oplus o_{c_3} &= (-1)^{n|c_2|} \ o_{c_1} \oplus o_{N_{c_2 \pitchfork c_3}} \oplus o_{c_2 \pitchfork c_3} \\
&= (-1)^{n|c_1|}(-1)^{n|c_2|} \ o_{N_{c_2 \pitchfork c_3}} \oplus o_{c_1} \oplus o_{c_2 \pitchfork c_3} \\
&= (-1)^{n|c_2|} \ o_{N_{c_2 \pitchfork c_3}} \oplus o_{N_{c_1 \pitchfork (c_2 \pitchfork c_3)}} \oplus o_{c_1 \pitchfork (c_2 \pitchfork c_3)}.
\end{aligned}
$$

The proof is complete by showing that

$$
(-1)^n o_{N_{c_1 \pitchfork c_2}} \oplus o_{N_{(c_1 \pitchfork c_2) \pitchfork c_3}} = o_{N_{c_2 \pitchfork c_3}} \oplus o_{N_{c_1 \pitchfork (c_2 \pitchfork c_3)}}
$$

This follows by considering the orientations of normal bundles in the triple product of M. Consider the product $M_1 \times M_2 \times M_3$ with diagonal D_{123} and normal bundle N_{123}. Let $M_{ij} = M_i \times M_j$, D_{ij} be the diagonal in M_{ij} and N_{ij}

the normal bundle of D_{ij} in M_{ij}. We compute

$$\begin{aligned} o_{N_{12}} \oplus o_{N_{123}} \oplus o_{D_{123}} &= o_{N_{12}} \oplus o_{D_{12}} \oplus o_{M_3} \\ &= o_{M_1 \times M_2 \times M_3} \\ &= o_{M_1} \oplus o_{N_{23}} \oplus o_{D_{23}} \\ &= (-1)^n \; o_{N_{23}} \oplus o_{M_1} \oplus o_{D_{23}} \\ &= (-1)^n \; o_{N_{23}} \oplus o_{N_{123}} \oplus o_{D_{123}} \end{aligned}$$

□

We are now ready to show that the domain in C prescribed by general position is a partial algebra over the operad $\mathscr{C}$, with the action induced by transversal intersection. In what follows, we have (implicitly) shifted the complex C down by the dimension of M, see Remark 2.4.2.

Theorem 2.5.10. *Let C be chains of M, with domain C_j as in Definition 2.5.6. Let $\Theta_0 : \mathscr{C}(0) \otimes C_0 \to C$ be the unit map, $\Theta_1 : \mathscr{C}(1) \otimes C_1 \to C$ be the identity and $\Theta_2 : \mathscr{C}(2) \otimes C_2 \to C$ be given by $\pitchfork$. For $j \geq 3$, define*

$$\Theta_j : \mathscr{C}(j) \otimes C_j \to C,$$

by iterates of $\pitchfork$. These maps give the structure of a partial algebra over the operad $\mathscr{C}$.

Proof. By Remark 2.5.5 and Lemma 2.5, any choice of iterates of $\pitchfork$ is defined on C_j for $j \geq 3$, and this action factors appropriately through the domain, see Definition 2.2.3. By Lemma 2.5.9 any two choices of iterates are equal, so Θ_j, for $j \geq 3$ is well-defined. The maps Θ_j respect the symmetric group action by Lemma 2.4.3, part (b), and are chain maps by Lemma 2.4.3, part (c). Lemmas 2.5.9 and 2.4.3, part (a), imply that the maps Θ_j define an action with respect to the operad composition. $\square$

2.6 Induced E_∞ Algebra

We'd now like to apply Theorem 2.2.5 to show that the partial algebra $\{C_j\}$ over $\mathscr{C}$ is quasi-isomorphic to a genuine algebra structure. A technical assumption of Theorem 2.2.5, is that the operad must satisfy the property that, for each $j > 0$, the j^{th} component of the operad is a free (or even projective) $\mathbb{Z}[\Sigma_j]$-module. Over $\mathbb{Z}$, $\mathscr{C}$ does not satisfy this property (since the Σ_j actions are trivial). Following Kriz and May in [34], we'll pull back this partial $\mathscr{C}$ algebra to an operad that is a $\mathbb{Z}[\Sigma]$-free resolution of $\mathscr{C}$, i.e. an E_∞ operad.

Definition 2.6.1. *An E_∞ operad is a unital operad $\mathcal{O}$, i.e. $\mathcal{O}(0) \approx \mathbb{Z}$, such that the maps*

$$\mathcal{O}(j) \otimes \mathcal{O}(0)^j \to \mathcal{O}(0) \approx \mathbb{Z}$$

are quasi-isomorphisms, and each $\mathcal{O}(j)$ is a free $\mathbb{Z}[\Sigma_j]$-module.

There are several explicit constructions of E_∞ operads. See [34].

Using the given quasi-isomorphisms

$$\mathcal{O}(j) \approx \mathbb{Z} = \mathscr{C}(j),$$

we obtain a partial algebra over $\mathcal{O}$, and from Theorem 2.2.5 we obtain the following

Theorem 2.6.2. *Let C be a complex of chains of a closed oriented manifold M, see Remark 2.4.1. There is a complex quasi-isomorphic to C that is an algebra over any E_∞ operad.*

Remark 2.6.3. *From any integral E_∞ algebra one obtains mod-p homology operations, for all primes p. See [34].*

Remark 2.6.4. *In characteristic zero, an E_∞ algebra on a complex X can be transfered to any complex chain equivalent to X. This follows from a general theorem of Markl [40]. For example, if the complex C in Theorem 2.6.2 is chain equivalent to the singular chain complex of M, or to the complex of currents, then these inherit the structure of an E_∞ algebra.*

2.7 Application to String Topology

In this section we describe an application of Theorem 2.2.5 to String Topology. Almost all of the ideas required for this are described by Sullivan and Chas in [7].

Let C be a complex of chains of the free loopspace of M, as in Section 2.4. By the adjoint property of mappings, we can regard an element (c_1, o_1, f_1) of this group as map $f : c_1 \times S^1 \to M$.

One basic idea of Chas and Sullivan is that operations on chains of the free loopspace can be defined by transversally intersecting chains and then composing loops. For example, the diagram in Figure 2.9 describes the $*$-product operation in the following way: Given chains (c_1, o_1, f_1) and (c_2, o_2, f_2),

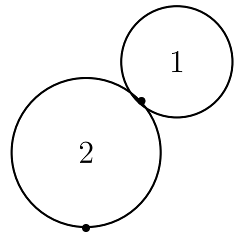

Figure 2.9: The $*$ product

intersect the map $f_1(-,0)$ with the map f_2. (Of course, transversality is required.) This gives a subset of $c_1 \times c_2 \times S^1$ on which one can define a map from S^1 by composing loops (first along the loop of c_2 to the intersection point, then along the loop of c_1, then back along the rest of the loop of c_2). Explicit formulas are given in [7]. The signs can be computed as in Section 2.4 by regarding $c_1 \times S^1$ as having orientation $o_{c_1} \oplus o_{S^1}$. Note that this generically produces a chain of degree $|c_1| + |c_2| + 1$.

Remark 2.7.1. *The usual composition of loops is not associative. In order for the desired relations below to hold precisely for transversal chains, one can compose loops as suggested by Moore. There may also be an interesting way to work in the A_∞ structure of naive loop composition, as described by Stasheff [54], [55].*

In [7], the authors then define the loop bracket operation $\{,\}$ as the graded symmetrization of the $*$ product, and prove the following:

Theorem 2.7.2. *The operations ∂ and $\{,\}$ define a differential graded Lie algebra (transversally) on the complex C shifted down by $dim(M)+1$. In other words, they satisfy the graded derivation property, graded skew-symmetry and the graded Jacobi identity.*

One can define a subcomplex C_j of $C^{\otimes j}$ that is spanned tensor products of chains that are appropriately transversal, i.e. any choice of iterated brackets $\{,\}$ is defined for any permutation of the tensor factors (and their boundaries). A proof that these subcomplexes form a domain, as in Definition 2.2.2, would be much the same as that of Lemma 2.5.7 above.

Now, the Lie operad $\mathscr{L}$ can be described by via generators and relations with one binary generator in $\mathscr{L}(2)$ and the relations of skew-symmetry and Jacobi. The transversally defined Lie algebra structure of Theorem 2.7.2 may then be described as a partial algebra over the operad $\mathscr{L}$ with domain $\{C_j\}$.

Theorem 2.7.3. *Let C be a complex of chains of the free loopspace of an oriented manfiold M, shifted down by $\dim M + 1$, and C_j a domain for the partial algebra over the operad $\mathscr{L}$ induced by the loop bracket $\{,\}$. Let $\mathscr{O}$ be any $\mathbb{Z}[\Sigma]$-free operad of complexes admitting a quasi-isomorphism $\mathscr{O} \to \mathscr{L}$. Then there is an $\mathscr{O}$-algebra on a complex quasi-isomorphic to C.*

Proof. Pull back along the quasi-isomorphism to obtain a partial $\mathscr{O}$ algebra, then apply Theorem 2.2.5. $\square$

An explicit construction of such an operad $\mathscr{O}$ can be made by inductively taking $\mathbb{Z}[\Sigma_j]$-free resolutions of $\mathscr{L}$ starting at $\mathscr{L}(2)$.

We expect there are many more applications of Theorem 2.2.5 to string topology; this is work in progress.

Appendix A

On Chains

In this section we present some ideas on constructing chain complexes which are suitable for intersection theory.

First, we point out that Eilenberg's singular chain functor may not be suitable for this purpose, since the transversal intersection of two maps from standard simplicies is not a map from a standard simplex. Instead, we enlarge the set of objects we consider. To do so, we construct a complex using ideas appearing in the work of Sullivan [57]. The idea is to consider maps from "abstract chains" into a space. A similar approach was used by Lefschetz where he considered maps of abstract cells into a space [36].

What we now describe is in the smooth category. One can make similar constructions in the real analytic or piece-wise linear categories. Kontseivich and Sobelman have a related discussion in the piecewise algebraic category in [32].

Let X be a smooth manifold.

Definition A.0.4. *An* oriented k-prechain *of* X *is a triple* (c, o, f) *consisting of a compact connected* k*-dimensional Whitney stratified subset* c *of some Eu-*

clidean space, with a choice o of orientation for each k-dimensional stratum of c, and a map $f : c \to X$ which restricts to a smooth map on each stratum of c.

We refer the reader to [21] for definitions of stratified objects, compare [63].

For any space X, the collection of oriented k-prechains is a proper class. We now consider equivalence classes of oriented k-prechains, where (c_1, o, f_1) is equivalent to (c_2, o, f_2) if there is a strata preserving diffeomorphism ϕ such that the following diagram commutes:

$$\begin{tikzcd}
c_1 \arrow[dd, "\phi"'] \arrow[dr, "f_1"] & \\
& X \\
c_2 \arrow[ur, "f_2"'] &
\end{tikzcd}$$

We'll use the same notation (c, o, f) for a prechain and it's equivalence class.

Let $F_k(X)$ be the free abelian group on this set of equivalence classes. The group $F_k(X)$ is isomorphic to a direct sum of copies of $\mathbb{Z}$, indexed by the diffeomorphism classes of stratified sets. There is a map $\partial : F_k(X) \to F_{k-1}(X)$ defined on (equivalence classes of) generators (c, o, f) in the following way: ∂c is the algebraic sum, counted with multiplicity, of the connected components of the $(k-1)$-dimensional strata of c, each with orientation $o_{\partial c}$ induced by $o = o_c$, so that if u_X is an outward pointing normal vector then

$$u_X \oplus o_{\partial c} = o_c.$$

The map $\partial f : \partial c \to X$ is given by the restriction of f to the $(k-1)$-dimensional strata of c.

Let $Q_k(X)$ be the quotient of $F_k(X)$ by the additive relation that the equivalence class of $(c, -o, f)$ equals negative one times the equivalence class of (c, o, f), where $-o$ is the opposite orientation of o. This relation implies $\partial^2 = 0$.

Up to this point, we have only recast Lefschetz's definition of chains, in which used oriented cells, into the more general context of stratified sets modeled on manifolds (in some category). Lefschetz introduced these definitions in [35]. He mistakenly assumed that this construction yields a free abelian group. It was pointed out by Cêch that if a prechain admits an orientation reversing diffeomorphism such that the following diagram commutes

$$\begin{tikzcd} c \arrow[dr, "f_1"] \arrow[dd, "\phi"'] & \\ & X \\ c \arrow[ur, "f_2"'] & \end{tikzcd}$$

then its image under the quotient represents an element of order 2.

As Lefschetz points out in [36], Cêch suggests that these torsion elements may be considered "degenerate". (In [36] Lefschetz works modulo what he calls "degenerate chains", and shows the homology of the resulting complex equals that obtained from a triangulation of the space.) According to Steenrod [56], it was later pointed out by Tucker that these torsion elements are boundaries.[1]

Of course, if one is content to work over the rationals, then there are no torsion issues and the resulting complex C is flat.

[1]Eilenberg later avoided all of these issues with the introduction of his "singular chain complex", which uses ordered simplicies instead of oriented cells.

Over the integers we suggest introducing a further relation on Q, that we call *finite additivity*, and declares that a chain is equivalent to any of it's subdivisions (of the type in the category to which it belongs). More precisely, any prechain is equivalent to the algebraic sum of the closure of the the top dimensional strata in any substratification of c (with the obvious restriction orientations and maps). From a geometric stand-point, this relation seems natural since the stratification of a cycle isn't important.[2] The boundary operator respects this relation. Let C denote the resulting complex.

We expect that in several categories of stratified objects, finite additivity has the following consequence: If c is an element admitting an orientation reversing automorphism ϕ as above, then c can be stratified into two pieces, say a and b, such that ϕ restricts to an orientation reversing automorphism from a to b. Then by finite additivity, $c = a + b$; but via the map ϕ, $b = -a$, so in fact $c = 0$. In this case, the 2-torsion elements are in fact zero. We conjecture that in such cases C is in fact free abelian.

Finally, we remark that there is a chain map from the singular chain complex into any of the chain complexes constructed above, because an ordered simplex can be regarded as an oriented stratified object modeled on smooth, real analytic, PL etc. manifolds. We expect this map is a quasi-isomorphism since such stratified objects can be triangulated [21],[20],[30],[67].

[2]One now has to be careful to define transversality to mean for *some* stratification.

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