On the topology and differential geometry of Kähler threefolds

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Abstract of the Dissertation,

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In the first part of my thesis we provide infinitely many examples of pairs of diffeomorphic, non simply connected Kähler manifolds of complex dimension 3 with different Kodaira dimensions. Also, in any allowed Kodaira dimension we find infinitely many pairs of non deformation equivalent, diffeomorphic Kähler threefolds.

In the second part we study the existence of Kähler metrics of positive total scalar curvature on 3-folds of negative Kodaira dimension. We give a positive answer for rationally connected threefolds. The proof relies on the Mori theory of minimal models, the weak factorization theorem and on a specialization technique.

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Chapter 1

Kodaira dimension of diffeomorphic threefolds

1.1 Introduction

Let M be a compact complex manifold of complex dimension n. On any such manifold the *canonical line bundle* $K_M = \wedge^{n,0}$ encodes important information about the complex structure. One can define a series of birational invariants of M,

$$P_k(M) := h^0(M, K_M^{\otimes k}),\ \ k \geq 0,$$

called the *plurigenera*. The number of independent holomorphic n-forms on M, $p_g(M) = P_1(M)$ is called the geometric genus. The *Kodaira dimension* $\mathrm{Kod}(M)$, is a birational invariant given by:

$$\mathrm{Kod}(M) = \limsup \frac{\log h^0(M, K_M^{\otimes k})}{\log k}.$$

This can be shown to coincide with the maximal complex dimension of the image of M under the pluri-canonical maps, so that $\mathrm{Kod}(M) \in \{-\infty, 0, 1, \ldots, n\}$. A compact complex n-manifold is said to be of *general type* if $\mathrm{Kod}(M) = n$.

For Riemann surfaces, the classification with respect to the Kodaira dimension, $\mathrm{Kod}(M) = -\infty, 0$ or 1 is equivalent to the one given by the *genus*, $g(M) = 0,\ 1$, and ≥ 2, respectively.

An important question in differential geometry is to understand how the complex structures on a given complex manifold are related to the diffeomorphism type of the underlying smooth manifold or further, to the topological type of the underlying topological manifold. Shedding some light on this question is S. Donaldson's result on the *failure of the h-cobordism conjecture in dimension four.* In this regard, he found a pair of *non-diffeomorphic*, *h*-cobordant, simply connected 4-manifolds. One of them was $\mathbb{CP}_2\#9\overline{\mathbb{CP}}_2$, the blow-up of $\mathbb{CP}_2$ at nine appropriate points, and the other one was a certain properly elliptic surface. For us, an important feature of these two complex surfaces is the fact that they have *different Kodaira dimensions.* Later, R. Friedman and Z. Qin [FrQi94] went further and proved that actually, for complex surfaces of Kähler type, *the Kodaira dimension is invariant under diffeomorphisms.* However, in higher dimensions, C. LeBrun and F. Catanese gave examples [CaLe97] of pairs of diffeomorphic projective manifolds of complex dimensions $2n$ with $n \geq 2$, and Kodaira dimensions $-\infty$ and $2n$.

In this thesis we address the question of the invariance of the Kodaira dimension under diffeomorphisms in complex dimension 3. We obtain the expected negative result:

Theorem A. *For any allowed pair of distinct Kodaira dimensions* (d, d'), *with the exception of* $(-\infty, 0)$ *and* $(0, 3)$, *there exist infinitely many pairs of diffeomorphic Kähler threefolds* (M, M'), *having the same Chern numbers, but* *with* $\mathrm{Kod}(M) = d$ *and* $\mathrm{Kod}(M') = d'$, *respectively.*

Corollary 1.1. *For Kähler threefolds, the Kodaira dimension is not a smooth invariant.*

Our examples also provide negative answers to questions regarding the deformation types of Kähler threefolds.

Recall that two manifolds X_1 and X_2 are called *directly deformation equivalent* if there exists a complex manifold $\mathcal{X}$, and a proper holomorphic submersion $\varpi : \mathcal{X} \to \Delta$ with $\Delta = \{|z| = 1\} \subset \mathbb{C}$, such that X_1 and X_2 occur as fibers of ϖ. The *deformation equivalence* relation is the equivalence relation generated by direct deformation equivalence.

It is known that two deformation equivalent manifolds are orientedly diffeomorphic. For complex surfaces of Kähler type there were strong indications that the converse should also be true. R. Friedman and J. Morgan proved [FrMo97] that, not only the Kodaira dimension is a smooth invariant but the plurigenera, too. However, Manetti [Man01] exhibited examples of diffeomorphic complex surfaces of general type which were *not* deformation equivalent. An easy consequence of our **Theorem** A and of the deformation invariance of plurigenera for 3-folds [KoMo92] is that in complex dimension 3 the situation is similar:

Corollary 1.2. *For Kähler threefolds the deformation type does not coincide with the diffeomorphism type.*

Actually, with a bit more work we can get:

Theorem B. *In any possible Kodaira dimension, there exist infinitely many examples of pairs of diffeomorphic, non-deformation equivalent Kähler three-folds with the same Chern numbers.*

The examples we use are Cartesian products of simply connected, $h-$cobordant complex surfaces with Riemann surfaces of positive genus. The real six-manifolds obtained will therefore be $h-$cobordant. To prove that these six-manifolds are in fact diffeomorphic, we use the $s-$Cobordism Theorem, by showing that the obstruction to the triviality of the corresponding $h-$cobordism, the *Whitehead torsion*, vanishes. Similar examples were previously used by Y. Ruan [Ruan94] to find pairs of diffeomorphic symplectic 6-manifolds which are not symplectic deformation equivalent. However, to show that his examples are diffeomorphic, Ruan uses the classification (up to diffeomorphisms) of *simply-connected*, real 6-manifolds [OkVdV95]. This restricts Ruan's construction to the case of Cartesian products by 2-spheres, a result which would also follow from Smale's h-cobordism theorem.

The examples of pairs complex structures we find are all of Kähler type with the same Chern numbers. This should be contrasted with C. LeBrun's examples [LeB99] of complex structures, mostly non-Kähler, with *different Chern numbers* on a given differentiable *real* manifold.

In our opinion, the novelty of this article is the use of the apparently forgotten *s-Cobordism Theorem.* This theorem is especially useful when combined with a theorem on the vanishing of the *Whitehead group*. For this, there exist nowadays strong results, due to F.T. Farrell and L. Jones [FaJo91].

In the next section, we will review the main tools we use to find our examples: h-cobordisms, the Whitehead group and its vanishing. In section 3 we recall few well-known generalities about complex surfaces. Sections 4 and 5 contain a number of examples and the proofs of **Theorems** A and B. In the last section we conclude with few remarks and we raise some natural questions.

1.2 The s-Cobordism Theorem

Definition 1.3. *Let M and M' be two n-dimensional closed, smooth, oriented manifolds. A cobordism between M and M' is a triplet $(W; M, M')$, where W is an $(n+1)$-dimensional compact, oriented manifold with boundary, $\partial W = \overline{\partial W}_- \bigsqcup \partial W_+$ with $\partial W_- = M$ and $\partial W_+ = M'$ (by $\overline{\partial W}_-$ we denoted the orientation-reversed version of ∂W_-).*

We say that the cobordism $(W; M, M')$ is an h-cobordism if the inclusions $i_- : M \to W$ and $i_+ : M' \to W$ are homotopy equivalences between M, M' and W.

The following well-known results [Wall62], [Wall64] allow us to easily check when two simply connected 4-manifolds are h-cobordant:

Theorem 1.4. *Two simply connected smooth manifolds of dimension 4 are h-cobordant if and only if their intersection forms are isomorphic.*

Theorem 1.5. *Any indefinite, unimodular, bilinear form is uniquely determined by its rank, signature and parity.*

In higher dimensions any h-cobordism $(W; M, M')$ is controlled by a complicated torsion invariant $\tau(W; M)$, the *Whitehead torsion,* an element of the so called *Whitehead group* which will be defined below.

Let Π be any group, and $R = \mathbb{Z}(\Pi)$ the integral unitary ring generated by Π. We denote by $GL_n(R)$ the group of all nonsingular $n \times n$ matrices over R. For all n we have a natural inclusion $GL_n(R) \subset GL_{n+1}(R)$ identifying each $A \in GL_n(R)$ with the matrix:

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(R).$$

Let $GL(R) = \bigcup_{n=1}^{\infty} GL_n(R)$. We define the following group:

$$K_1(R) = GL(R)/[GL(R), GL(R)].$$

The *Whitehead group* we are interested in is:

$$Wh(\Pi) = K_1(R)/ < \pm g \mid g \in \Pi > .$$

Theorem 1.6. *Let M be a smooth, closed manifold. For any h-cobordism W of M with $\partial_- W = M$, and with dim $W \geq 6$ there exists an element $\tau(W) \in Wh(\pi_1(M))$, called* the Whitehead torsion, *characterized by the following properties:*

- **s-Cobordism Theorem** $\tau(W) = 0$ *if and only if the h-cobordism is trivial, i.e. W is diffeomorphic to $\partial_- W \times [0, 1]$;*

- **Existence** *Given $\alpha \in Wh(\pi_1(M))$, there exists an h-cobordism W with $\tau(W) = \alpha$;*

- **Uniqueness** $\tau(W) = \tau(W')$ *if and only if there exists a diffeomorphism*

$h : W \to W'$ *such that* $h_{|M} = id_M$.

For the definition of the *Whitehead torsion* and the above theorem we refer the reader to Milnor's article [Mil66]. However, the above theorem suffices. When M is simply connected, the s-cobordism theorem is nothing but the usual h-cobordism theorem [Mil65], due to Smale.

This theorem will be a stepping stone in finding pairs of diffeomorphic manifolds in dimensions greater than 5, provided knowledge about the vanishing of the *Whitehead groups*. The vanishing theorem that we are going to use here is:

Theorem 1.7 (Farrell, Jones)**.** *Let M be a compact Riemannian manifold of non-positive sectional curvature. Then* $Wh(\pi_1(M)) = 0$.

The uniformization theorem of compact Riemann surfaces yields then the following result which, as it was kindly pointed to us by L. Jones, was also known to F. Waldhausen [Wal78], long before [FaJo91].

Corollary 1.8. *Let Σ be a compact Riemann surface. Then* $Wh(\pi_1(\Sigma)) = 0$.

An important consequence, which will be frequently used is the following:

Corollary 1.9. *Let M and M' be two simply connected, h-cobordant 4-manifolds, and Σ be a Riemann surface of positive genus. Then $M \times \Sigma$ and $M' \times \Sigma$ are diffeomorphic.*

Proof. Let W be an h-cobordism between M and M' such that $\partial_- W = M$ and $\partial_+ W = M'$ and let $\widetilde{W} = W \times \Sigma$. Then $\partial_- \widetilde{W} = M \times \Sigma$, $\partial_+ \widetilde{W} = M' \times \Sigma$, and $\widetilde{W}$ is an h-cobordism between $M \times \Sigma$ and $M' \times \Sigma$. Now, since M is simply connected $\pi_1(M \times \Sigma) = \pi_1(\Sigma)$ and so

$$Wh(\pi_1(M \times \Sigma)) = Wh(\pi_1(\Sigma)).$$

By the uniformization theorem any Riemann surface of positive genus admits a metric of non-positive curvature. Thus, by Theorem 1.7, $Wh(\pi_1(\Sigma)) = 0$, which, by Theorem 1.6, implies that $M \times \Sigma$ and $M' \times \Sigma$ are diffeomorphic. □

1.3 Generalities

To prove **Theorems** A and B we will use our Corollary 1.9, by taking for M and M' appropriate h-cobordant, simply connected, complex projective surfaces, and for Σ, Riemann surfaces of genus $g(\Sigma) \geq 1$. To find examples of h-cobordant complex surfaces, we use:

Proposition 1.10. *Let M and M' be two simply connected complex surfaces with the same geometric genus p_g, $c_1^2(M) - c_1^2(M') = m \geq 0$ and let $k > 0$ be any integer. Let X be the blowing-up of M at $k + m$ distinct points and X' be the blowing-up of M' at k distinct points. Then X and X' are h-cobordant,* $\mathrm{Kod}(X) = \mathrm{Kod}(M)$ *and* $\mathrm{Kod}(X') = \mathrm{Kod}(M')$.

Proof. By Noether's formula we immediately see that

$$b_2(M') = b_2(M) + m.$$

Since, by blowing-up we increase each time the second Betti number by one, it follows that

$$b_2(X') = b_2(X).$$

Using the birational invariance of the plurigenera, we have that

$$b_+(X') = 2p_g + 1 = b_+(X).$$

As X and X' are both non-spin, and their intersection forms have the same rank and signature, their intersection forms are isomorphic. Thus, by Theorem 1.4, X and X' are h-cobordant.

The statement about the Kodaira dimension follows immediately from the birational invariance of the plurigenera, too. □

Corollary 1.11. *Let S and S' be two simply connected, h-cobordant complex surfaces. If S_k and S'_k are the blowing-ups of the two surfaces, each at $k \geq 0$ distinct points, then S_k and S'_k are h-cobordant, too. Moreover,* $\mathrm{Kod}(S_k) = \mathrm{Kod}(S)$, *and* $\mathrm{Kod}(S'_k) = \mathrm{Kod}(S')$.

The following proposition will take care of the computation of the Kodaira dimension of our examples. Its proof is standard, and we will omit it.

Proposition 1.12. *Let V and W be two complex manifolds. Then*

$$P_m(V \times W) = P_m(V) \cdot P_m(W).$$

In particular, $\mathrm{Kod}(V \times W) = \mathrm{Kod}(V) + \mathrm{Kod}(W)$.

For the computation of the Chern numbers of the examples involved, we need:

Proposition 1.13. *Let M be a smooth complex surface with $c_1^2(M) = a$, $c_2(M) = b$, and let Σ be a smooth complex curve of genus g, and $X = M \times \Sigma$ their Cartesian product. The Chern numbers* $(\mathbf{c_1^3}, \mathbf{c_1 c_2}, \mathbf{c_3})$ *of X are*

$$((6-6g)a, (2-2g)(a+b), (2-2g)b).$$

Proof. Let $p : X \to M$, and $q : X \to \Sigma$ be the projections onto the two factors. Then the total Chern class is

$$c(X) = p^* c(M) \cdot q^* c(\Sigma),$$

which allows us to identify the Chern classes. Integrating over X, the result follows immediately. □

1.4 Diffeomorphism Type

In this section we prove **Theorem** A. All we have to do is to exhibit the appropriate examples. Thus, for each of the pairs of Kodaira dimensions stated, we provide infinitely many examples, by taking Cartesian products of appropriate $h-$cobordant Kähler surfaces with Riemann surfaces of positive genus.

Example 1: *Pairs of Kodaira dimensions* $(-\infty, 1)$ *and* $(-\infty, 2)$

Let M be the blowing-up of $\mathbb{CP}_2$ at 9 distinct points given by the intersection of two generic cubics. M is a non-spin, simply connected complex surface with $\text{Kod}(M) = -\infty$ which is also an elliptic fibration, $\pi : M \to \mathbb{CP}_1$. By taking the cubics general enough, we may assume that M has no multiple fibers, and the only singular fibers are irreducible curves with one ordinary double point.

Let M' be obtained from M by performing logarithmic transformations on two of its smooth fibers, with multiplicities p and q, where p and q are two relatively prime positive integers. M' is also an elliptic surface, $\pi' : M' \to \mathbb{CP}_1$, whose fibers can be identified to those in M except for the pair of multiple fibers F_1, and F_2. Let F be homology class of the generic fiber in M'. In homology we have $[F] = p[F_1] = q[F_2]$. By canonical bundle formula, we see that: $K_M = -F$, and

$$K_{M'} = -F + (p-1)F_1 + (q-1)F_2 = \frac{pq - p - q}{pq}F. \qquad (1.1)$$

Then $p_g(M) = p_g(M') = 0$, $c_1^2(M) = c_1^2(M') = 0$, and $\mathrm{Kod}(M') = 1$. Moreover, from [FrMo94, Theorem 2.3, page 158] M' is simply connected and non-spin.

For any $k \geq 0$, let M_k and M'_k be the blowing-ups at k distinct points of M and M', respectively, and let Σ be a Riemann surface.

- If $g(\Sigma) = 1$, according to Corollary 1.9 and Proposition 1.12, $(M_k \times \Sigma_1, M'_k \times \Sigma_1),\ k \geq 0$ will provide infinitely many pairs of diffeomorphic Kähler threefolds, whose Kodaira dimensions are $-\infty$ and 1, respectively.

- If $g(\Sigma) \geq 2$, we get infinitely many pairs of diffeomorphic Kähler threefolds of Kodaira dimensions $-\infty$, and 2, respectively.

The statement about the Chern numbers follows from Proposition 1.13.

□

Example 2: *Pairs of Kodaira dimensions* $(0, 1)$ *and* $(0, 2)$

In $\mathbb{CP}_1 \times \mathbb{CP}_2$, let M be the the generic section of line bundle

$$p_1^* \mathcal{O}_{\mathbb{CP}_1}(2) \otimes p_2^* \mathcal{O}_{\mathbb{CP}_2}(3),$$

where p_i, $i = 1, 2$ are the projections onto the two factors. Then M is a $K3$ surface, i.e. a smooth, simply connected complex surface, with trivial canonical bundle. Moreover, using the projection onto the first factor, it fibers over $\mathbb{CP}_1$ with elliptic fibers.

Kodaira [Kod70] produced infinitely many examples of properly elliptic surfaces of Kähler type, homotopically equivalent to a $K3$ surface, by performing two logarithmic transformations on two smooth fibers with relatively prime multiplicities on such elliptic $K3$. Let M' to be any such surface, and let M_k and M'_k be the blowing-ups at k distinct points of M and M', respectively.

As before, let Σ be a Riemann surface.

- If $g(\Sigma) = 1$, the Cartesian products $M_k \times \Sigma$ and $M'_k \times \Sigma$ will provide infinitely many pairs of diffeomorphic Kähler 3-folds of Kodaira dimensions 0 and 1, respectively.

- If $g(\Sigma) \geq 2$, we obtain pairs in Kodaira dimensions 1 and 2, respectively.

Again, the statement about the Chern numbers follows from Proposition 1.13. □

Example 3: *Pairs of Kodaira dimensions* $(-\infty, 2)$ *and* $(-\infty, 3)$

Arguing as before, we present a pair of simply connected, $h-$cobordant projective surfaces, one on Kodaira dimension 2, and the other one of Kodaira dimension $-\infty$.

Let M be the *Barlow surface* [Bar85]. This is a non-spin, simply connected projective surface of general type, with $p_g = 0$ and $c_1^2(M) = 1$. It is therefore h-cobordant to M', the projective plane $\mathbb{CP}_2$ blown-up at 8 points.

By taking the Cartesian product of their blowing-ups by a Riemann surface of genus 1, we obtain diffeomorphic, projective threefolds of Kodaira dimensions 3, and $-\infty$, respectively, while for a Riemann surface of bigger genus, we obtain diffeomorphic, projective threefolds of Kodaira dimensions 2, and $-\infty$, respectively. The invariance of their Chern numbers follows as usual. □

Example 4: *Pairs of Kodaira dimensions* $(0, 2)$ *and* $(1, 3)$

Following [Cat78], we will describe an example of simply connected, minimal surface of general type with $c_1^2 = p_g = 1$.

In $\mathbb{CP}_2$ we consider two generic smooth cubics F_1 and F_2, which meet transversally at 9 distinct points, $x_1, \cdots, x_9$, and let

$$\sigma : \widetilde{X} \to \mathbb{CP}_2$$

be the blowing-up of $\mathbb{CP}_2$ at $x_1, \cdots, x_9$, with exceptional divisors $\widetilde{E}_i,\ i = 1, ..., 9$. Let $\widetilde{F}_1$ and $\widetilde{F}_2$ be the strict transforms of F_1 and F_2, respectively. Then $\widetilde{F}_1$ and $\widetilde{F}_2$ are two disjoint, smooth divisors, and we can easily see that

$$\mathcal{O}_{\widetilde{X}}(\widetilde{F}_1 + \widetilde{F}_2) = \widetilde{\mathcal{L}}^{\otimes 2},$$

where

$$\widetilde{\mathcal{L}} = \sigma^* \mathcal{O}_{\mathbb{CP}_2}(3) \otimes \mathcal{O}_{\widetilde{X}}(\widetilde{E}_1 + \cdots + \widetilde{E}_9).$$

Let $\pi : \bar{X} \to \widetilde{X}$ to be the double covering of $\widetilde{X}$ branched along the smooth divisor $\widetilde{F}_1 + \widetilde{F}_2$. We denote by

$$p : \bar{X} \to \mathbb{CP}_2$$

the composition $\sigma \circ \pi$, and by $\bar{F}_1,\ \bar{F}_2$ the reduced divisors $\pi^{-1}(\widetilde{F}_1)$, and $\pi^{-1}(\widetilde{F}_2)$, respectively. Since each $\widetilde{E}_i$ intersects the branch locus at 2 distinct points, we can see that for each $i = 1, \ldots, 9,\ \bar{E}_i = \pi^{-1}(\widetilde{E}_i)$ is a smooth (-2)-curve such that

$$\pi_{|\bar{E}_i} : \bar{E}_i \to \widetilde{E}_i$$

is the double covering of $\widetilde{E}_i$ branched at the two intersection points of $\widetilde{E}_1$ with $\widetilde{F}_1 + \widetilde{F}_2$. As the $\widetilde{E}_i$'s are mutually disjoint, the $\bar{E}_i$'s will also be mutually disjoint.

Similarly, if ℓ is a line in $\mathbb{CP}_2$ not passing through any of the intersection points of F_1 with F_2, then

$$L = p^*(\ell) = p^*\mathcal{O}_{\mathbb{CP}_2}(1)$$

is a smooth curve of genus 2, not intersecting any of the $\bar{E}_i$'s. Since

$$p^*\mathcal{O}_{\mathbb{CP}_2}(3) = \mathcal{O}_{\bar{X}}(2\bar{F}_1 + \bar{E}_1 + \cdots + \bar{E}_9),$$

we can write as before

$$\mathcal{O}_{\bar{X}}(L + \bar{E}_1 + \cdots + \bar{E}_9) = \bar{\mathcal{L}}^{\otimes 2},$$

where

$$\bar{\mathcal{L}} = p^*\mathcal{O}_{\mathbb{CP}_2}(2) \otimes \mathcal{O}_{\bar{X}}(-\bar{F}_1).$$

Let now $\phi : \bar{S} \to \bar{X}$ be the double covering of $\bar{X}$ ramified along the smooth divisor

$$L + \bar{E}_1 + \cdots + \bar{E}_9.$$

The surface $\bar{S}$ is non-minimal with exactly 9 disjoint exceptional curves of the first kind, the reduced divisors $\phi^{-1}(\bar{E}_i),\ i = 1, \ldots 9$. The surface S we were looking for is obtained from $\bar{S}$ by blowing down these 9 exceptional curves.

Lemma 1.14. *S is a simply connected, minimal surface with*

$$c_1^2(S) = p_g(S) = 1.$$

Proof. As S is obtained from $\bar{S}$ by blowing-down 9 exceptional curves,

$$c_1^2(S) = c_1^2(\bar{S}) + 9.$$

The canonical line bundle $\mathcal{K}_{\bar{S}}$ of $\bar{S}$ as a double covering of $\bar{X}$ is [BPV84, Lemma 17.1, p. 42]:

$$\mathcal{K}_{\bar{S}} = \phi^* \bar{\mathcal{L}},$$

since the canonical bundle of $\bar{X}$ is trivial. The computation of $c_1^2(\bar{S})$ follows again from [BPV84, Lemma 17.1, p. 42], and we have:

$$\begin{aligned}
c_1^2(\bar{S}) &= (\mathcal{K}_{\bar{S}} \cdot \mathcal{K}_{\bar{S}}) = (\phi^* \bar{\mathcal{L}} \cdot \phi^* \bar{\mathcal{L}}) = 2(\bar{\mathcal{L}} \cdot \bar{\mathcal{L}}) \\
&= 2(p^* \mathcal{O}_{\mathbb{CP}_2}(2) \cdot p^* \mathcal{O}_{\mathbb{CP}_2}(2)) - 4(p^* \mathcal{O}_{\mathbb{CP}_2}(2) \cdot \mathcal{O}_{\bar{X}}(\bar{F}_1)) \\
&\quad + 2(\mathcal{O}_{\bar{X}}(\bar{F}_1) \cdot \mathcal{O}_{\bar{X}}(\bar{F}_1)) \\
&= 4(\sigma^* \mathcal{O}_{\mathbb{CP}_2}(2) \cdot \sigma^* \mathcal{O}_{\mathbb{CP}_2}(2)) - 2(\pi^* \sigma^* \mathcal{O}_{\mathbb{CP}_2}(2) \cdot \pi^* \mathcal{O}_{\widetilde{X}}(\widetilde{F}_1)) \\
&\quad + \frac{1}{2}(\pi^* \mathcal{O}_{\widetilde{X}}(\widetilde{F}_1) \cdot \pi^* \mathcal{O}_{\widetilde{X}}(\widetilde{F}_1)) \\
&= 4(\mathcal{O}_{\mathbb{CP}_2}(2) \cdot \mathcal{O}_{\mathbb{CP}_2}(2)) - 4(\sigma^* \mathcal{O}_{\mathbb{CP}_2}(2) \cdot \mathcal{O}_{\widetilde{X}}(\widetilde{F}_1)) \\
&\quad + (\mathcal{O}_{\widetilde{X}}(\widetilde{F}_1) \cdot \mathcal{O}_{\widetilde{X}}(\widetilde{F}_1)) \\
&= 16 - 4(\mathcal{O}_{\mathbb{CP}_2}(2) \cdot \mathcal{O}_{\mathbb{CP}_2}(3)) \\
&= -8.
\end{aligned}$$

Thus $c_1^2(S) = 1$.

To compute $p_g(S)$ using the birational invariance of the plurigenera, it would be the same to compute

$$p_g(\bar{S}) = h^0(\bar{S}, \mathcal{K}_{\bar{S}}) = h^0(\bar{X}, \phi_* \mathcal{K}_{\bar{S}}).$$

Using the projection formula (cf. [BPV84], p. 182), we have:

$$h^0(\bar{X}, \phi_* \mathcal{K}_{\bar{S}}) = h^0(\bar{X}, \phi_* \phi^* \bar{\mathcal{L}}) = h^0(\bar{X}, \mathcal{O}_{\bar{X}}) + h^0(\bar{X}, \bar{\mathcal{L}}) = 1.$$

For the proof of the simply connectedness, we refer the interested reader to [Cat78]. □

Let S'_k be the blowing-up of a $K3$ surface at k distinct points. Let also S_k denote the blowing-up of S at $k+1$ distinct points, and let Σ be a Riemann surface.

- If $g(\Sigma) = 1$, $(S_k \times \Sigma, S'_k \times \Sigma)$ will provide infinitely many pairs of diffeomorphic Kähler threefolds of Kodaira dimensions 2 and 0, respectively;
- If $g(\Sigma) \geq 2$ we get infinitely many pairs of diffeomorphic Kähler threefolds of Kodaira dimensions 3 and 1, respectively.

The statement about the Chern classes follows as before. □

Example 5: *Pairs of Kodaira dimensions* $(1, 2)$ *and* $(2, 3)$

In $\mathbb{CP}_1 \times \mathbb{CP}_2$, let M_n be the the generic section of line bundle $p_1^* \mathcal{O}_{\mathbb{CP}_1}(n) \otimes p_2^* \mathcal{O}_{\mathbb{CP}_2}(3)$ for $n \geq 3$, where p_i, $i = 1, 2$ be the projections onto the two factors. Then M_n is a smooth, simply connected projective surface, and using the projection onto the first factor we see that M_n is a properly elliptic surface. By the adjunction formula, the canonical line bundle is:

$$K_{M_n} = p_1^* \mathcal{O}_{\mathbb{CP}_1}(n-2).$$

From this and the projection formula we can find the purigenera:

$$\begin{aligned} P_m(M_n) &= h^0(M_n, K_{M_n}^{\otimes m}) = h^0(M_n, p_1^* \mathcal{O}_{\mathbb{CP}_1}(m(n-2))) \\ &= h^0(\mathbb{CP}_1, p_{1*} p_1^* \mathcal{O}_{\mathbb{CP}_1}(m(n-2))) \\ &= h^0(\mathbb{CP}_1, \mathcal{O}_{\mathbb{CP}_1}(m(n-2))) \\ &= m(n-2)+1. \end{aligned}$$

So, $\mathrm{Kod}(M_n) = 1$, and $p_g(M_n) = n-1$. We can also see that $c_1^2(M_n) = 0$.

Let M' be any smooth sextic in $\mathbb{CP}_3$. M' is a simply connected surface of general type with $p_g(M') = 10$, and $c_1^2(M') = 24$. Let M'_k be the blowing-up of M at $24+k$ distinct points, M_k be the blowing-up of M_{11} at $k+1$ points, and let Σ be a Riemann surface. If $g(\Sigma) = 1$, $(M_k \times \Sigma, M'_k \times \Sigma)$ will provide infinitely many pairs of diffeomorphic Kähler threefolds of Kodaira dimensions 1 and 2, respectively, while if $g(\Sigma) \geq 2$ we get infinitely many pairs of diffeomorphic Kähler threefolds of Kodaira dimensions 2 and 3, respectively. The statement about the Chern classes again follows. □

1.5 Deformation Type

Similar idea can be used to prove **Theorem** B. The proof follows from the examples below.

Example 1: *Kodaira dimension* $-\infty$

Here we use again the Barlow surface M, and M', the blowing-up of $\mathbb{CP}_2$ at 8 points as two h-cobordant complex surfaces. Let S_k and S'_k denote the blowing-ups of M and M', respectively at k distinct points. Then, by the classical h-cobordism theorem, $X_k = S_k \times \mathbb{CP}_1$ and $X'_k = S'_k \times \mathbb{CP}_1$ are two diffeomorphic 3-folds with the same Kodaira dimension $-\infty$. The fact that X_k and X'_k are not deformation equivalent follows as in [Ruan94] from Kodaira's stability theorem [Kod63]. We also see immediately that they have the same Chern numbers. $\square$

Example 2: *Kodaira dimension* 2 *and* 3

We start with a *Horikawa surface*, namely a simply connected surface of general type M with $c_1^2(M) = 16$ and $p_g(M) = 10$. An example of such surface can be obtained as a ramified double cover of $Y = \mathbb{CP}_1 \times \mathbb{CP}_1$ branched at a generic curve of bi-degree $(6, 12)$. If we denote by $p : M \to Y$, its degree 2 morphism onto Y, then the canonical bundle of M is $K_M = \mathcal{O}_Y(1, 4)$, see [BPV84, page 182]. Here by $\mathcal{O}_Y(a, b)$ we denote the line bundle $p_1^*\mathcal{O}_{\mathbb{CP}_1}(a) \otimes p_2^*\mathcal{O}_{\mathbb{CP}_1}(b)$, where $p_i,\ 1 = 1, 2$ are the projections of Y onto the two factors. Notice that the formula for the canonical bundle shows that M is not spin.

Lemma 1.15. *The plurigenera of M are given by:*

$$P_n(M) = \begin{cases} 10 & n = 1 \\ 8n^2 - 8n + 11 & n \geq 2 \end{cases}$$

Proof. Cf. [BPV84] we have $p_*\mathcal{O}_M = \mathcal{O}_Y \oplus \mathcal{O}_Y(-3, -6)$. We have:

$$\begin{aligned} P_n(M) &= h^0(M, p^*\mathcal{O}_Y(n, 4n)) = h^0(Y, p_*p^*\mathcal{O}_Y(n, 4n)) \\ &= h^0(Y, \mathcal{O}_Y(n, 4n) \otimes p_*\mathcal{O}_M) \\ &= h^0(Y, \mathcal{O}_Y(n, 4n)) + h^0(Y, \mathcal{O}_Y(n-3, 4n-6)). \end{aligned}$$

Now, if $n < 3$ we get $P_n(M) = (n+1)(4n+1)$. In particular, $p_g(M) = 10$ and $P_2(M) = 27$. If $n \geq 3$, $P_n(M) = (n+1)(4n+1) + (n-2)(4n-5) = 8n^2 - 8n + 11$. $\square$

Let $M' \subset \mathbb{CP}_3$ be a smooth sextic. The adjunction formula will provide again the the canonical bundle $K_{M'} = \mathcal{O}_{M'}(2)$ and so $c_1^2(M') = 24$.

Lemma 1.16. *The plurigenera of M' are given by:*

$$P_n(M') = \begin{cases} \binom{2n+3}{3} & n = 1, 2 \\ 12n^2 - 12n + 11 & n \geq 3 \end{cases}$$

Proof. From the exact sequence $0 \to \mathcal{O}_{\mathbb{CP}_3}(2n-6) \to \mathcal{O}_{\mathbb{CP}_3}(2n) \to K_{M'}^{\otimes n} \to 0$, we get:

$$0 \to H^0(\mathbb{CP}_3, \mathcal{O}_{\mathbb{CP}_3}(2n-6)) \to H^0(\mathbb{CP}_3, \mathcal{O}_{\mathbb{CP}_3}(2n))$$
$$\to H^0(M', K_{M'}^{\otimes n}) \to H^1(\mathbb{CP}_3, \mathcal{O}_{\mathbb{CP}_3}(2n)) = 0.$$

So, for $n \geq 3$,

$$P_n(M') = \binom{2n+3}{3} - \binom{2n-3}{3} = 12n^2 - 12n + 11,$$

while for $n < 3$, $P_n(M') = \binom{2n+3}{3}$. In particular, $p_g(M') = 10$ and $P_2(M') = 35$. $\square$

Let M_k be the blowing-up of M at k distinct points, M'_k be the blowing-up of M' at $8+k$ distinct points, and let Σ be a Riemann surface. If $g(\Sigma) = 1$, $(M_k \times \Sigma, M'_k \times \Sigma), k \geq 0$ will provide the required examples of Kodaira dimension 2, and if $g(\Sigma) \geq 2$, will provide the required examples of Kodaira dimension 3.

To prove that they are not deformation equivalent we will use the deformation invariance of plurigenera theorem [KoMo92, page 535]. Because of the their multiplicative property cf. Proposition 1.12, it will suffice to look at the plurigenera of M and M'. But, $P_2(M) = 27$ and $P_2(M') = P_2(S) = 35$, and so $M \times \Sigma$ and $M' \times \Sigma$ are not deformation equivalent.

The statement about the Chern numbers of this examples follows immediately. $\square$

Example 3: *Kodaira dimension* 1

Here we use again the elliptic surfaces $\pi : M_{p,q} \to \mathbb{CP}_1$ obtained from the rational elliptic surface by applying logarithmic transformations on two smooth fibers, with relatively prime multiplicities p and q. From (1.1) we get $K_{M_{p,q}}^{\otimes pq} = p^*\mathcal{O}_{\mathbb{CP}_1}((pq - p - q))$. Hence $P_{pq}(M_{p,q}) = pq - p - q + 1$, while if $n \leq pq$, $P_n(M_{p,q}) = 0$, the class of F being a primitive element in $H^2(M_{p,q}, \mathbb{Z})$, cf. [Kod70]. It is easy to see now that, for example, if $(p, q) \neq (2, 3)$, $P_6(M_{p,q}) \neq P_6(M_{2,3})$. If Σ is any smooth elliptic curve, the $3-$folds $X_{p,q} = M_{p,q} \times \Sigma$ will provide infinitely many diffeomorphic Kähler threefolds of Kodaira dimension 1. Corollary 1.13 shows again that all these threefolds have the same Chern numbers. The above computation of plurigenera shows that, in general, the $X_{p,q}$'s have different plurigenera. Hence, these Kähler threefolds are not deformation equivalent. □

Example 4: *Kodaira dimension* 0

Here we are supposed to start with a simply connected minimal surface of zero Kodaira dimension. But, up to diffeomorphisms there exists only one [BPV84], the $K3$ surface. So our method fails to produce examples in this case. However, M. Gross constructed [Gro97] a pair of diffeomorphic complex threefolds with trivial canonical bundle, which are *not* deformation equivalent. For the sake of completeness we will briefly recall his examples.

Let $E_1 = \mathcal{O}_{\mathbb{CP}_1}^{\oplus 4}$ and $E_2 = \mathcal{O}_{\mathbb{CP}_1}(-1) \oplus \mathcal{O}_{\mathbb{CP}_1}^{\oplus 2}(1) \oplus \mathcal{O}_{\mathbb{CP}_1}$ be the two rank 4 vector bundles over $\mathbb{CP}_1$, and consider $X_1 = \mathbb{P}(E_1)$ and $X_2 = \mathbb{P}(E_2)$ their projectivizations. Note that E_2 deforms to E_1. Let $M_i \in |-K_{X_i}|,\ i=1,2$ general anticanonical divisors. The adjunction formula immediately shows that $K_{M_i}=0,\ i=1,2$, and so M_1 and M_2 have zero Kodaira dimension. While for M_1 is easy to see that can be chosen to be smooth, simply connected and with no torsion in cohomology, Gross shows [Gro97], [Ruan96] that the same holds for M_2. Moreover, the two 3-folds have the same topological invariants, (the second cohomology group, the Euler characteristic, the cubic form, and the first Pontrjaghin class), and so, cf. [OkVdV95], are diffeomorphic. To show that M_1 and M_2, are not deformation equivalent, note that M_2 contains a smooth rational curve with normal bundle $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$, which is stable under the deformation of the complex structure while M_1, doesn't. Obviously, M_1 and M_2 have the same Chern numbers. By blowing them up simultaneously at k distinct points, we obtain infinitely many pairs of diffeomorphic, projective threefolds of zero Kodaira dimension with the same Chern numbers. $\square$

1.6 Concluding Remarks

1. Let M and M' be any of the pairs of complex surfaces discussed in the previous two sections. A simple inspection shows that they are not spin, and so, their intersection forms will have the form $m\langle 1\rangle \oplus n\langle -1\rangle$. By a result of Wall [Wall62], if $m,n\geq 2$, the intersection form is transitive on the primitive characteristic elements of fixed square. Since, c_1 is characteristic, if it is primitive too, we can assume that the homotopy equivalence $f: M\to M'$ given by an automorphism of such intersection form will carry the first

Chern class of M' into the first Chern class of M. But this implies that the $h-$cobordism constructed between $X = M \times \Sigma$ and $X' = M' \times \Sigma$ also preserves the first Chern classes.

Following Ruan [Ruan94], we can arrange our examples such that c_1 is a primitive class. In the cases when $b_+ > 1$, which is equivalent to $p_g > 0$, it follows that there exists a diffeomorphism $F : X \to X'$ such that $F^* c_1(X') = c_1(X)$, where $F^* : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is the isomorphism induced by F. In these cases our theorems provide either examples of pairs of diffeomorphic Kähler threefolds, with the same Chern classes, but with different Kodaira dimensions, or examples of pairs of non deformation equivalent, diffeomorphic Kähler threefolds, with same Chern classes and of the same Kodaira dimension.

However, in some cases we are forced to consider surfaces with $b_+ = 1$. In these cases it is not clear whether one can arrange the $h-$cobordisms constructed between $X = M \times \Sigma$ and $X' = M' \times \Sigma$ also preserves the first Chern classes.

2. With our method it is impossible to provide examples of diffeomorphic 3-folds of Kodaira dimensions $(0, 3)$ and $(-\infty, 0)$. In the first case, our method fails for obvious reasons. In the second case, the reason is that for a projective surface of Kodaira dimension $-\infty$, the geometric genus p_g is 0, while for a simply connected projective surface of Kodaira dimension 0, $p_g \neq 0$. Thus, any two surfaces of these dimensions will have different b_+, which is preserved under blow-ups. So, no pair of projective surfaces of these Kodaira dimensions can be h-cobordant. However, this raises the following question:

Question 1.17. *Are there examples of pairs of diffeomorphic, projective* 3-

folds (M, M') *of Kodaira dimensions* $(0, 3)$ *or* $(-\infty, 0)$?

Most of the examples exhibited here have the fundamental group of a Riemann surface. Natural questions to ask would be the following:

Question 1.18. *Are there examples of diffeomorphic,* simply connected*, complex, projective* $3-$*folds of different Kodaira dimension?*

Question 1.19. *Are there examples of projective,* simply connected*, diffeomorphic, but not deformation equivalent* 3*-manifolds with the same Kodaira dimension?*

As we showed, the answer is *yes* when the Kodaira dimensions is $-\infty$ or 0, but we are not aware of such examples in the other cases.

Chapter 2

The total scalar curvature of rationally connected threefolds

2.1 Introduction

In the second part of my thesis we address the following question:

Question 2.1. *Let X be a smooth complex n-fold of Kähler type and negative Kodaira dimension. Does X admit Kähler metrics of positive total scalar curvature?*

If we denote by s_g and $d\mu_g$ the scalar curvature and the volume form of g, respectively, this is the same as asking if there is any Kähler metric g on X such that $\int_X s_g d\mu_g > 0$.

For Kähler metrics, the total scalar curvature has a simpler expression :

$$\int_X s_g d\mu_g = 2\pi n c_1(X) \cup [\omega]^{n-1} \tag{2.1}$$

where $[\omega]$ is the cohomology class of the Kähler form of g. The negativity of the Kodaira dimension is a necessary condition [Yau74] because, arguing by contradiction, if for some $m > 0$, the m^{th} power of the canonical bundle of X is either trivial or has sections one can immediately see that $c_1(X) \cup [\omega]^{n-1}$ is negative, which, by (2.1), would imply that the total scalar curvature of (X, g) was negative.

Question 2.1 has an immediate positive answer in dimension 1. The only smooth complex curve of negative Kodaira dimension is $\mathbb{P}_1$, and the Fubini-Study metric satisfies the required inequality. In complex dimension 2, a positive answer was given by Yau in [Yau74]. His proof is based on the theory of minimal models and the classification of Kähler surfaces of negative Kodaira dimension to find the required metrics on the minimal models. Then he proved that if on a given smooth surface such a metric exists, one can find Kähler metrics of positive total scalar curvature on any of its blowing-ups. Moreover, the metrics he found are Hodge metrics.

Inspired by Yau's approach, we tackle Question 2.1 in the case of projective threefolds of negative Kodaira dimension, where a satisfactory theory of minimal models exists. As in [Yau74], we look for Hodge metrics instead. In this context the natural question to ask is [1]:

Question 2.2. *Let X be a smooth projective 3-fold, with* $\mathrm{Kod}(X) = -\infty$. *Is there any ample line bundle H on X such that $K_X \cdot H^2 < 0$?*

A positive answer to this question can be connected to a deep result of S. Mori and Y. Miyaoka [MiyMo86]. Namely, Question 2.2 can be regarded as a

[1] The same question has also been raised in a different context by F. Campana, J. P. Demailly, T. Peternell and M. Schneider, [DPS96], [CaPe98].

possible effective characterization of the class of smooth, projective threefolds of negative Kodaira dimension.

Also, Question 2.2 can be viewed as extracting some positivity property of the anticanonical bundle. An affirmative answer would yield in dimension three, a weak alternative to the generic semi-positivity theorem of Miyaoka asserting that, for non-uniruled manifolds, the restriction of the cotangent bundle to a general smooth complete intersection curve cut out by elements of $|mH|$ is semi-positive, for any ample divisor H and $m \gg 0$.

As an attempt to answer the original Question 2.1, in this thesis we give a partial positive answer to Question 2.2 in the case of rationally connected threefolds. Recall that cf. [KMM92], a complex projective manifold X of dimension $n \geq 2$ is called *rationally connected* if there is a rational curve passing through any two given points of X. Our result is:

Theorem A. *For every projective, rationally connected manifold X of dimension 3, there exists an ample line bundle H on X such that $K_X \cdot H^2 < 0$.*

One reason to restrict our attention to this important case comes from the observation that for rationally connected manifolds, answering affirmatively to Question 2.1 is equivalent to answering affirmatively to Question 2.2. This is automatically true for surfaces, but is a non-trivial issue in higher dimensions. This follows from their convenient cohomological properties. Namely, if X is such a manifold, then [KMM92]

$$H^i(X, \mathcal{O}_X) = 0, \text{ for } i \geq 1.$$

But in this case, from the Hodge decomposition

$$H^2(X,\mathbb{C}) = H^{0,2}(X) \oplus H^{1,1}(X) \oplus H^{2,0}(X)$$

it follows that $H^2(X,\mathbb{C}) \simeq H^{1,1}(X)$. Thus, we can see that any $(1,1)-$form with real coefficients can be approximated [2] by a $(1,1)-$form with rational coefficients. Hence, up to multiplication by positive integers, any Kähler forms can be approximated by first Chern classes of *ample line bundles.* From this the equivalence of our two questions follows easily.

However, this is not the only reason to restrict ourselves to the case of rationally connected threefolds, as it will become apparent from the proof of Theorem A.

In what follows, we outline the proof of Theorem A. To simplify the exposition, for any projective manifold X, We introduce the following definition:

Definition 2.3. *Let X be a smooth projective threefolds. We say that the property $\mathcal{P}_X$ holds true if there exists an ample line bundle H on X such that $K_X \cdot H^2 < 0$.*

Similar to the case of complex surfaces, the idea to prove this theorem is to start with an arbitrary 3-fold X of negative Kodaira dimension, and show that property $\mathcal{P}$ above holds true for its minimal model X_{min}. In the first section, we apply Mori's theory of minimal models on X to get a birational map $f : X \dashrightarrow X_{min}$ to a new three dimensional projective variety X_{min}, with at most $\mathbb{Q}-$factorial terminal singularities, which is either a Mori fiber space,

[2] Here we consider $H^2(X,\mathbb{R})$ as a finite dimensional real vector space, endowed with any metric topology

or has nef canonical bundle.

Deep results Miyaoka [Miy88], [MiPe97] exclude the possibility of X_{min} having nef canonical bundle (nef canonical bundle would imply non-negative Kodaira dimension). Hence, X_{min} has to be a Mori fiber space, i.e X_{min} is either a del Pezzo fibration, a conic bundle or a Fano variety. As in [CaPe98] and [DPS96], it is easy to see that $\mathcal{P}_{X_{min}}$ holds true.

The difficult part of this program is to show that

$$\mathcal{P}_{X_{min}} \Longrightarrow \mathcal{P}_X.$$

As a first step for a better understanding of this problem, in Section 2.3 we prove the following:

Proposition 2.4. *Let $p : X' \to X$ be a resolution of singularities of a projective $\mathbb{Q}-$factorial variety X of dimension three, with terminal singularities. Assume that p is smooth outside the singular locus* $\mathrm{Sing}(X)$. *Then*

$$\mathcal{P}_{X'} \text{ holds true} \Longleftrightarrow \mathcal{P}_X \text{ holds true.}$$

This shows that in order to answer Question 2.2 it is enough to show that $\mathcal{P}$ is a birational property of smooth projective threefolds.

Our approach is to use the weak factorization theorem [AKMW02], which says that any birational map between smooth (projective) manifolds can be decomposed into a finite sequence of blow-ups and blow-downs with nonsingular centers of (projective) manifolds. We prove the following:

Proposition 2.5. *Let $p : Y \to X$ be the blow-up of a smooth, projective 3-fold* *at a point. Then*

$$\mathcal{P}_X \text{ holds true} \Longleftrightarrow \mathcal{P}_Y \text{ holds true.}$$

For the blowing-up along curves the following result is of crucial importance:

Proposition 2.6. *Let $p : Y \to X$ be the blow-up of a smooth, projective 3-fold along a smooth curve C.*

- $\mathcal{P}_X$ holds true $\Longrightarrow \mathcal{P}_Y$ holds true.
- *If $K_X \cdot C < 0$, then*

$$\mathcal{P}_Y \text{ holds true} \Longrightarrow \mathcal{P}_X \text{ holds true.}$$

We should point out that these results do not require the rational connectedness of X.

In the last case to verify, $\mathcal{P}_Y \Longrightarrow \mathcal{P}_X$, where $Y \to X$ is the blowing-up of smooth projective threefolds along smooth curves with $K_X \cdot C \geq 0$, the methods used to prove the previous results do not work anymore. It is the last hurdle, where we use this extra assumption.

The condition $K_X \cdot C < 0$ imposed in the previous proposition can be interpreted, by the Riemann Roch theorem, as saying that the curve C "moves". Our approach is to reduce this case to the case which we have already solved. To be more precise, we are going to find a smooth curve $C' \subset X$ with $K_X \cdot C' < 0$, such that $\mathcal{P}_{Y'}$ holds true, where $Y' \to X$ is the blowing-up of X along C'. What we do in our construction is "forcing C to move", by eventually modifying it, while preserving property $\mathcal{P}$. This is done by a lengthy specialization argument, where we strongly rely on the rational connectedness hypothesis. This argument is inspired from the proof of Noether's theorem of Griffiths and Harris [GH85]. The construction presented in Section 2.4.2, on which all the computations are performed is based on the work of Graber, Harris, Starr [GHS03] and Kollár [ArKo03]. We devote to this specialization argument, the entire Section 2.4.

In Section 2.5 we prove Theorem 1.9. An appendix containing some results used intensively throughout this entire chapter is added for convenience.

Conventions: We work over the field of complex numbers and we use the standard notations and terminology of Hartshorne's Algebraic Geometry book [Har77].

2.2 Minimal models

In this section we introduce the objects which appear in Mori's theory of minimal models and we show that our problem has a positive answer for the "minimal models."

Let X be a variety with $\dim X > 1$, such that K_X is $\mathbb{Q}-$Cartier, i.e. mK_X is Cartier for some positive integer m. If $f : Y \to X$ is a proper birational morphism such that K_Y is a line bundle (e.g. Y is a resolution of X), then mK_Y is linearly equivalent to:

$$f^*(mK_X) + \sum m \cdot a(E_i) \cdot E_i,$$

where the E_i's are the exceptional divisors. Using numerical equivalence, we can divide by m and write:

$$K_Y \equiv_{\mathbb{Q}} f^*K_X + \sum a(E_i) \cdot E_i.$$

Definition 2.7. *We say that X has terminal singularities if for any resolution, and for any i, $a(E_i) > 0$.*

Definition 2.8. *We say that a variety X is $\mathbb{Q}-$factorial if for any Weil divisor D there exist a positive integer m such that mD is a Cartier divisor.*

The minimal model program (MMP) studies the structure of varieties via birational morphisms or birational maps of special types to seemingly simpler varieties. The birational morphisms which appear running the MMP are the following:

Definition 2.9 (divisorial contractions). *Let X be a projective variety with at most $\mathbb{Q}-$factorial singularities. A birational morphism $f : X \to Y$ is called a divisorial contraction if it contracts a divisor, $-K_X$ is $f-$ample and* rank $NS(X) =$ rank $NS(Y) + 1$.

The main difficulty in the higher dimensional minimal model program is the existence of non-divisorial contractions. When the variety X is $\mathbb{Q}-$factorial with only terminal singularities, one may get contractions, called contractions of flipping type $f : X \to Y$, where the exceptional locus E has codimension at least 2, but such that $-K_X$ is $f-$ample and rank $NS(X) =$ rank $NS(Y) + 1$. In this case, K_Y is no longer $\mathbb{Q}-$Cartier. The remedy in dimension 3 is the existence of special birational maps, called flips, which allow to replace X by another $\mathbb{Q}-$factorial variety X^+ with only terminal singularities, but simpler in some sense:

Definition 2.10 (flips). *Let $f : X \to Y$ be a flipping contraction as above. A variety X^+ together with a map $f^+ : X^+ \to Y$ is called a flip of f if X^+ is $\mathbb{Q}-$factorial varieties with terminal singularities and K_{X^+} is f^+-ample.*

By abuse of terminology, the birational map $X \dashrightarrow X^+$ will also be called a flip.

The minimal model program starts with an arbitrary projective $\mathbb{Q}$-factorial threefold with at most terminal singularities X on which one applies an suitable sequence of divisorial contractions and flips. To describe the outcome of the MMP, we need to introduce the following definition:

Definition 2.11 (Mori fiber spaces). *Let X and Y be two irreducible $\mathbb{Q}-$factorial varieties with terminal singularities, dim $X >$ dim Y, and $f : X \to Y$ a morphism. The triplet (X, Y, f) is called a Mori fiber space if $-K_X$ is $f-$ample, and*

$$\text{rank } NS(X) = \text{rank } NS(Y) + 1.$$

Theorem 2.12 (Mori). *Let X be a projective variety with only $\mathbb{Q}-$factorial terminal singularities, and dim $X = 3$. Then there exist a birational map $f : X \dashrightarrow X_{min}$, which is a composition of divisorial contractions and flips, such that either $K_{X_{min}}$ is nef or X_{min} has a Mori fiber space structure.*

In their approach to Question 2.2, Campana and Peternell proved some important cases. Because of the simplicity, we include their proofs.

The following easy lemma will be used frequently throughout the proofs, sometimes without referring to it.

Lemma 2.13. *Let X be a projective $\mathbb{Q}-$factorial variety of dimension 3, with (at most) terminal singularities for which there exists a nef line bundle D with $K_X \cdot D^2 < 0$. Then there exists an ample line bundle H on X such that $K_X \cdot H^2 < 0$.*

Proof. Let L be any ample on X. Then for any positive integer m, $H_m := mD + L$ is an ample line bundle with:

$$K_X \cdot H_m^2 = K_X \cdot (mD + H)^2 = mK_X \cdot D^2 + 2K_X \cdot D \cdot L + K_X \cdot L^2 < 0$$

for $m \gg 0$. □

Proposition 2.14 (Mori fiber spaces). *Let (X, Y, f) be a Mori fiber space, with dim $X = 3$. Then the property $\mathcal{P}_X$ holds true.*

Proof. Since dim $X = 3$ we have 3 cases, according to the dimension of Y :

***Case 1* (dim Y=0)** In this case X is a $\mathbb{Q}-$Fano variety with rank $NS(X) = 1$. In particular, $-mK_X$ is an ample line bundle, for some integer $m > 0$, and the property $\mathcal{P}_X$ follows immediately.

***Case 2* (dim Y=1)** Take L_Y be any ample line bundle on Y. Since $-K_X$ is $f-$ample it follows that $K_X \cdot (f^*L)^2 < 0$, and as f^*L_Y is nef, from Lemma 2.13 we can see that $\mathcal{P}_X$ holds true.

***Case 3* (dim Y=2)** As before we take L_Y be any ample line bundle on Y and H_X be an ample line bundle on X. Then for any positive integer

$H_m := mf^*L + H_X$, is an ample line bundle and we have:

$$K_X \cdot H_m^2 = K_X \cdot (mf^*L + H_X)^2 = K_X \cdot H_X^2 + 2mK_X \cdot H_X \cdot f^*L_Y < 0,$$

for $m \gg 0$, again because $-K_X$ is $f-$ample.

□

Corollary 2.15. *Let X be a $\mathbb{Q}-$factorial, projective variety of dimension three with at most terminal singularities. If* $\mathrm{Kod}(X) = -\infty$ *then $\mathcal{P}_{X_{min}}$ holds true.*

Proof. From Theorem 2.12 we know that X_{min} is either a Mori fiber space, for which $\mathcal{P}_{X_{min}}$ holds true, or $K_{X_{min}}$ is nef. To exclude the second possibility, we note that from Miyaoka's abundance theorem [MiPe97, page 88], in dimension three this would imply that $\mathrm{Kod}(X_{min}) \geq 0$. However, this is impossible since the Kodaira dimension is a birational invariant. □

2.3 Various reductions

Our first reduction takes care of the singularities. Let X be a projective $\mathbb{Q}-$factorial variety X of dimension three, with terminal singularities. By Hironaka's resolution of singularities we can always find resolution $p : X' \to X$ which is an isomorphism outside the singular points of X. We begin by proving the following:

Proposition 2.16. *$\mathcal{P}_{X'}$ holds true if and only if $\mathcal{P}_X$ holds true.*

Proof. Suppose first that $\mathcal{P}_X$ holds true. Hence there exists an ample line bundle L on X such that $K_X \cdot L^2 < 0$, and let $D' = p^*L$. Then D is a nef line bundle on X' and

$$K_{X'} \cdot D'^2 = (p^* K_X + \sum_i a_i E_i) \cdot p^* L \cdot p^* L = K_X \cdot L^2 < 0.$$

Using Lemma 2.13 it follows that $\mathcal{P}_{X'}$ holds also true.

Conversely, suppose now $\mathcal{P}_{X'}$ holds true, and let H' be a an ample line on X' such that $K_{X'} \cdot H'^2 < 0$. Without loss of generality we can assume H' very ample and represented by an irreducible divisor, still denoted by H'. Let $D := p(H')$ its pushforward in X. Following [KoMo98, Lemma 3.39] we have

$$p^* D \equiv_{\mathbb{Q}} H' + \sum_i c_i E_i,$$

where the E_i 's are the exceptional divisors of the resolutions and $c_i \geq 0$.

Now, if C is any curve in X, let C' be its strict transform in X' and so $p_* C' = C$. Then, $D \cdot C = D \cdot p_* C' = p^* D \cdot C' = H' \cdot C' + \sum c_i (E_i \cdot C') > 0$, C' not being contained in any of the exceptional divisors. Thus D is a strictly nef divisor. The singularities of X being terminal, $K_{X'} \equiv_{\mathbb{Q}} p^* K_X + \sum a_i E_i$, with $a_i > 0$. Again, since the singularities of X are a finite number of isolated points, $p^* L \cdot E_i = 0$ for any cartier divisor L on X. We immediately obtain:

$$\begin{aligned} K_X \cdot D^2 &= (p^* K_X) \cdot (p^* D)^2 = (p^* K_X) \cdot (H')^2 \\ &= K_{X'} \cdot H'^2 - \sum a_i E_i \cdot H'^2 < 0. \end{aligned}$$

The proposition follows now from Lemma 2.13. ☐

Proposition 2.16 allows us to interpret the results we proved in the previous section in in the following way. From the minimal models program we obtain a birational map $f : X \dashrightarrow Y$ from a smooth projective threefold X to a singular threefold Y for which $\mathcal{P}_Y$ holds true. We can replace now Y by a *smooth* projective threefold X' for which $\mathcal{P}_{X'}$ holds true. Thus the problem we study reduces to the following:

Question 2.17. *Is $\mathcal{P}$ a birational property of the class of projective threefolds of negative Kodaira dimension?*

This is already a major simplification, because we can use now the weak factorization theorem [AKMW02] of Abramovich, Karu, Matsuki and Włodarczyk:

Theorem 2.18 (Abramovich, Karu, Matsuki, Włodarczyk). *A birational map between projective nonsingular varieties over an algebraically closed field K of characteristic zero is a composite of blowings up and blowings down with smooth centers of smooth projective varieties.*

Therefore, what is left to prove is that the property $\mathcal{P}$ is preserved under blowing-ups and blowing-downs at points and smooth curves, respectively.

2.3.1 Blowing-up at points

Proposition 2.19. *Let $p : Y \to X$ be the blow-up of a smooth, projective 3-fold at a point. Then $\mathcal{P}_X$ holds true if and only if $\mathcal{P}_Y$ holds true.*

Proof. Let E be the exceptional divisor of p. Then by [Har77, Ex. II.8.5], $\mathrm{Pic}(Y) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}[E]$ and $K_Y = p^*K_X + 2E$.

Suppose first that $\mathcal{P}_X$ holds true and let H_X be an ample line bundle on X such that $K_X \cdot H_X^2 < 0$. Then $D_Y \stackrel{\text{def}}{=} p^* H_X$ is a nef line bundle on Y such that $K_Y \cdot D_Y^2 = (p^*K_X + 2E) \cdot p^*H_X \cdot p^*H_X = p^*K_X \cdot p^*H_X \cdot p^*H_X = K_X \cdot H_X^2 < 0$. Using again Lemma 2.13 it follows that $\mathcal{P}_Y$ holds true.

Conversely, suppose that $\mathcal{P}_Y$ holds true, and let H_Y be an ample line bundle on Y such that $K_Y \cdot H_Y^2 < 0$. Then $H_Y = p^*D_X - aE$, for some line bundle $D_X \in \mathrm{Pic}(X)$, and some positive integer a. As in the proof of Proposition 2.16, we can show that D_X is nef and $K_X \cdot D_X^2 < 0$. Let C be any curve in X and let C' be its strict transform in Y. Then $p_*C' = C$, and $D_X \cdot C = D_X \cdot p_*C' = p^*D_X \cdot C' = H_X \cdot C' + aE \cdot C' > 0$, because H_X is ample and C' is not contained in E. Therefore D_X is a nef line bundle and

$$K_X \cdot D_X^2 = p^*K_X \cdot p^*D_X \cdot p^*D_X = p^*K_X \cdot H_Y \cdot H_Y = K_Y \cdot H_Y^2 - 2E \cdot H_Y^2 < 0.$$

Applying again Lemma 2.13 we can conclude the proof of the proposition. □

2.3.2 Blowing-up along curves

In the case of 1-dimensional blowing-up centers, it is easy to prove in one direction:

Proposition 2.20. *Let $p : Y \to X$ be the blow-up of a smooth, projective 3-fold along a smooth curve C. If $\mathcal{P}_X$ holds true then $\mathcal{P}_Y$ holds true.*

Proof. Let H_X be an ample line bundle on X satisfying $K_X \cdot H_X^2 < 0$ and let

$D_Y \overset{\text{def}}{=} p^*H_X$. Then D_Y is a nef line bundle on Y and we have

$$K_Y \cdot D_Y^2 = (p^*K_X + E) \cdot p^*H_X \cdot p^*H_X = p^*K_X \cdot p^*H_X \cdot p^*H_X = K_X \cdot H_X^2 < 0.$$

The conclusion follows again from Lemma 2.13. $\square$

For the converse of Proposition 2.20 the following proposition is the key step in our line of argument. Its proof is rather long, but elementary, based on Proposition C.3.

Proposition 2.21. *Let $p : Y \to X$ be the blowing-up of a smooth, projective 3-fold along a smooth curve C such that $K_X \cdot C < 0$. If $\mathcal{P}_Y$ holds true then $\mathcal{P}_X$ holds true.*

Proof. Let H_Y be an ample line bundle on X such that $\mathcal{P}_Y$ holds true. Without loss of generality, we can assume that H_X is very ample. Since $p : Y \to X$ is the blowing-up of X along $C \subset X$, the exceptional divisor $E = \mathbb{P}_C(N_{C/X}^{\vee})$ will be a ruled surface over C. Let $d = \deg_C(N_{C/X})$, and let g be the genus of C. Let f be a fiber of $p_{|_E} : E \to C$. We denote by a the intersection number $(H_Y \cdot f)$ in the Chow ring $A(Y)$. We can write:

$$H_Y = p^*L_X - aE \tag{2.2}$$

for some line bundle $L_X \in \text{Pic}(X)$. As in the proof of Proposition 2.19, we can check that $L_X \cdot C' > 0$ for any irreducible curve $C' \subset X$, different than C. Let $\tilde{C}$ be the strict transform of C'. Then:

$$L_X \cdot C' = (H_Y + aE) \cdot C' = H_Y \cdot \tilde{C} + aE \cdot \tilde{C} > 0,$$

because $E\cdot\tilde{C} \geq 0$, the curve $\tilde{C}$ is an irreducible curve, obviously not contained in E. Thus, in order to show that L_X is nef we only have to check that $L_X\cdot C \geq 0$. A straightforward application of the projection formula and of Proposition B.2 gives:

$$
\begin{aligned}
K_Y \cdot H_Y^2 &= (p^*K_X + E)\cdot(p^*L_X - aE)\cdot(p^*L_X - aE) \\
&= p^*K_X \cdot p^*L_Y \cdot p^*E - 2ap^*K_X \cdot p^*L_X \cdot E + a^2p^*K_X \cdot E^2 \\
&\quad + E\cdot p^*L_X \cdot p^*L_X - 2aE^2 \cdot p^*L_X + a^2E^3 \\
&= K_X \cdot L_X^2 - 2aE^2 \cdot p^*L_X + a^2p^*K_X \cdot E^2 + a^2E^3 \\
&= K_X \cdot L_X^2 + 2a(L_X\cdot C) - a^2(K_X\cdot C) - a^2d \\
&= K_X \cdot L_X^2 + 2a(L_X\cdot C) - a^2(2g-2).
\end{aligned}
$$

Observation 2.22. Since $K_Y \cdot H_Y^2 < 0$, to conclude the proof of the Proposition 2.21 it will suffice to prove that

$$2a(L_X\cdot C) - a^2(2g-2) \geq 0, \tag{2.3}$$

because we would obtain:

- $K_X \cdot L_X^2 < 0$,
- $L_X \cdot C \geq 0$, *if* $g \geq 1$.

If $g = 0$, we still have to check that $L_X \cdot C \geq 0$.

For a better understanding of (2.3) the following considerations are necessary.

On $E = \mathbb{P}_C(N^\vee_{C/X})$, let C_0 be the section of minimal self-intersection $C_0^2 = -e$. We use $\{C_0, f\}$ as a basis for $\mathrm{Num}_{\mathbb{Z}}(E)$. With respect to this basis:

$$H_{X|_E} \equiv aC_0 + bf, \text{ for some } b \in \mathbb{Z} ;$$
$$E_{|_E} \equiv xC_0 + yf,$$

where x and y can be determined as follows:

$$-1 = E \cdot f = E_{|_E} \underset{E}{\cdot} f = (xC_0 + yf) \underset{E}{\cdot} f = x;$$
$$-d = E^3 = E_{|_E} \underset{E}{\cdot} E_{|_E} = (-C_0 + yf)^2 = -e - 2y,$$

so $y = \frac{d-e}{2}$. Here we denoted by " $\underset{E}{\cdot}$ " the intersection product on the exceptional smooth divisor E.

Remark 2.23. Note that the two invariants, d and e, of $E = \mathbb{P}_C(N^\vee_{C/X})$, have the same parity.

Lemma 2.24. *In the above notations, we have:*

$$2a(L_X \cdot C) - a^2(2g-2) = 2ab - a^2 e - a^2(K_X \cdot C). \tag{2.4}$$

Proof. Computing $H_Y \cdot E \cdot p^* L_X$ in two ways, we obtain:

$$H_Y \cdot E \cdot p^* L_X = (L_X \cdot C) L_X \cdot f = a(L_X \cdot C);$$
$$H_Y \cdot E \cdot p^* L_X = H_Y \cdot E \cdot (H_Y + aE) = H_Y^2 \cdot E + aH_Y \cdot E^2.$$

Thus

$$2a(L_X \cdot C) - a^2(2g-2) = 2(H_Y^2 \cdot E + aH_Y \cdot E^2) - a^2(2g-2)$$
$$= 2(H_Y^2 \cdot E + aH_Y \cdot E^2) - a^2 d - a^2(K_X \cdot C).$$

Furthermore:

$$2(H_Y^2 \cdot E + aH_Y \cdot E^2) = 2[(H_{X|_E} \underset{E}{\cdot} H_{X|_E}) + a(H_{X|_E} \underset{E}{\cdot} E_{|_E})]$$
$$= 2(aC_0 + bf)^2 + a(aC_0 + bf) \underset{E}{\cdot} [-2C_0 + (d-e)f]$$
$$= 2a^2C_0^2 + 4ab - 2a^2C_0^2 - 2ab + a^2(d-e)$$
$$= 2ab + a^2(d-e).$$

Therefore $2a(L_X \cdot C) - a^2(2g-2) = 2ab - a^2 e - a^2(K_X \cdot C)$. ☐

We can finish now the proof of Proposition 2.21:

- If $g \geq 0$, by Proposition C.3, we have two subcases:

i) ***Case*** $\mathbf{e \geq 0}$: Since H_Y is ample, $H_{Y|_E}$ is ample, and so, by Proposition C.3, $a > 0$ and $b > ae$. Remembering that $K_X \cdot C < 0$, from (2.4) we can see that:

$$2a(L_X \cdot C) - a^2(2g-2) = 2ab - a^2 e - a^2(K_X \cdot C)$$
$$> a^2 e - a^2(K_X \cdot C) > 0.$$

By the crucial Observation 2.22 and by Proposition 2.13 we are done.

ii) ***Case*** $\mathbf{e<0}$: Similarly, since H_Y is ample, $H_{Y|_E}$ is ample, too. Thus, by Proposition C.3, $a>0$ and $b>\frac{1}{2}ae$. Then:

$$\begin{aligned}2a(L_X\cdot C)-a^2(2g-2)=&2ab-a^2e-a^2(K_X\cdot C)\\ &>-a^2(K_X\cdot C)>0,\end{aligned}$$

and we are again done.

- If $g=0$ then $e\geq 0$, and in this case it suffices to show that $2ab+a^2(d-e)\geq 0$. Since $H_{Y|_E}$ is ample, $a>0$ and $b>ae$. We have:

$$2ab+a^2(d-e)>a^2e+a^2d=a^2(e-2-K_X\cdot C).$$

So, if $K_X\cdot C\leq -2$ it follows immediately that $L_X\cdot C>0$, and with the help of Observation 2.22 and Lemma 2.13 we are done again. If $K_X\cdot C=-1$, then $d=-1$ and since d and e have the same parity, $e\geq 1$ and we obtain again $L_X\cdot C>0$, and we can conclude as above.

With this Proposition 2.21 is completely proved. □

Remark 2.25. The proof of Proposition 2.21 also works when $K_X\cdot C=0$ and $g>0$. However, when C is a rational curve $d=\deg N_{C/X}=-2$, and the above arguments show that a possible exception occurs only when $e=0$, and $0<b<a$. In this case, $N_{C/X}\simeq\mathcal{O}_{\mathbb{P}_1}(-1)\oplus\mathcal{O}_{\mathbb{P}_1}(-1)$, and what fails is only the nefness of L_X.

When $K_X\cdot C>0$, nothing can be said with the above approach.

This remark inspires the following conjecture:

Conjecture 2.26. *Let Y be the blowing up of of a smooth, projective threefold X along a curve $C \simeq \mathbb{P}_1$ with $N_{C/X} \cong \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$. If the contraction of the exceptional divisor of Y along the "other direction" is projective, then:*

$$\mathcal{P}_Y \text{ holds true} \Longrightarrow \mathcal{P}_X \text{ holds true.}$$

In the next section we will give a proof of a special case of this conjecture as a part of our argument.

2.4 Specialization argument

From what we proved so far, to show that $\mathcal{P}$ is a birational property in the class of smooth, projective threefolds it would be enough to answer affirmatively to the following question:

Question 2.27. *Let $p : X_C \to X$ be the blowing up of smooth projective threefold X along a smooth curve $C \subset X$ with $K_X \cdot C \geq 0$. Suppose that $\mathcal{P}_{X_C}$ holds true. Does $\mathcal{P}_X$ also hold true?*

Proposition 2.21 is inspirational, suggesting that a positive answer is possible if we can replace the blowing-up $p : X_C \to X$ of X along the curve C by the blowing-up $p' : X_{C'} \to X$ of X along a smooth curve $C' \subset X$, but such that $K_X \cdot C' < 0$, as long as we are able to show that $\mathcal{P}_{X_{C'}}$ also holds true. We will show that such an approach works in the case of *rationally connected* projective threefolds.

A more precise description of our strategy to answer Question 2.27, and the outline of the structure of this section is the following. In the next subsection we introduce the results from the theory of rationally connected manifolds we need. Then using the outcome of Theorem 2.31, in subsection 2.4.2, we construct a smooth family over the unit disk $\mathcal{X} \to \Delta$, whose general fiber X_{C_t} is the blowing-up of X along a smooth curve C_t with $K_X \cdot C_t < 0$. The central fiber of this family will be a normal crossing divisor whose irreducible components are smooth rationally connected threefolds. In subsection 2.4.3 we show that any line bundle on the central fiber of $\mathcal{X} \to \Delta$ extends to $\mathcal{X}$. Moreover, if the line bundle on the central fiber is chosen to be ample, its extension restricted to X_{C_t} will also be ample, by eventually shrinking Δ. In subsection 2.4.5, we apply the results obtained in the previous subsection, to show how to construct ample line bundles on one of the components of the central fiber of $\mathcal{X} \to \Delta$. In the next subsection, we show how to use the result of the previous subsection to construct an ample line bundles on the whole central fiber of $\mathcal{X} \to \Delta$. Finally, in the last subsection, we set up the intersection theory of the central fiber, and show that on the central fiber of $\mathcal{X} \to \Delta$ satisfies property $\mathcal{P}$ holds true, which will imply that $\mathcal{P}_{X_{C_t}}$ holds true, too.

2.4.1 Rationally connected manifolds

In this section we collect the necessary information from the theory of rationally connected manifolds. For the definitions and the main results presented we refer the interested reader to [KMM92], [Kol96] and especially to [ArKo03].

Let X denote a complex projective manifold with $\dim X \geq 2$.

Definition 2.28. *A nonsingular, complex, projective variety X will be called* rationally connected *if any pair of points in X can be connected by a rational curve.*

The main properties and characterizations of rationally connected manifolds are summarized in the following:

Theorem 2.29. 1) *Rationally connectedness is a birational property and is invariant under smooth deformations.*

2) *Rationally connected manifolds are simply connected and satisfy*

$$H^0(X, \Omega_X^{\otimes m}) = 0 \text{ for } m > 0 \text{ and } H^i(X, \mathcal{O}_X) = 0 \text{ for } i > 0.$$

3) *X is rationally connected if and only if for any point $x \in X$ there exists a smooth rational curve $L \subset X$ passing through x, with arbitrarily prescribed tangent direction and such that its normal bundle $N_{L|X}$ is ample.*

We should point out that the statement in 3) is not one of the usual characterizations of rationally connectedness. However, it easily follows from [Deb01, page 110].

Definition 2.30. *A* comb *with* n teeth *is a projective curve with $n+1$ irreducible components $C, L_1, \ldots, L_n$ such that:*

- *The curves $L_1, \cdots, L_n$ are mutually disjoint, smooth rational curves.*
- *Each L_i, $i \neq 0$ meets C transversely in a single smooth point of C.*

The curve C is called the handle *of the comb, and $L_1, \ldots, L_n$ are called the* teeth.

The key result we use is the following theorem of Graber, Harris and Starr [GHS03], which we present in the shape given by J. Kollár [ArKo03]:

Theorem 2.31. *Let X be a smooth, complex, projective variety of dimension at least* 3*. Let $C \subset X$ be a smooth irreducible curve. Let $L \subset X$ be a rational curve with ample normal bundle intersecting C and let $\mathcal{L}$ be a family of rational curves on X parametrized by a neighborhood of $[L]$ in* $\mathrm{Hilb}(X)$*. Then there are curves $L_1, \ldots, L_n \in \mathcal{C}$ such that $C_0 = C \cup L_1 \cup \cdots \cup L_n$ is a comb and satisfies the following conditions:*

1) *The sheaf $N_{C_0/X}$ is generated by the global sections.*

2) $H^1(C_0, N_{C_0/X}) = 0$.

Obviously the hypotheses are fulfilled in the case of rationally connected manifolds.

For a better understanding of this theorem the following corollary [ArKo03] is very useful. Since we consider that its proof gives some useful information about our construction, we include for convenience Kollár's proof.

Corollary 2.32. $\mathrm{Hilb}(X)$ *has a unique irreducible component containing $[C_0]$. This component is smooth at $[C_0]$ and a non-empty subset of it parametrizes smooth, irreducible curves in X.*

Proof. Since the curve C_0 is locally complete intersection, its normal sheaf $N_{C_0/X}$ is locally free. We have an exact sequence

$$0 \to N_{C/X} \to N_{C_0/X}{}_{|C} \to Q \to 0$$

where Q is a torsion sheaf supported at the points $P_i = C \cap L_i$, for $i = 1, \ldots n$. Since $N_{C_0/X}$ is globally generated, we can find a global section $s \in H^0(C_0, N_{C_0/X})$ such that, for each i, the restriction of s to a neighborhood of P_i is not in the image of $N_{C/X}$. This means that s corresponds to a first-order deformation of C_0 that smoothes the nodes P_i of C_0. From the vanishing of $H^1(C_0, N_{C_0/X})$ we see that there are no obstructions finding a global deformation of C_0 that smoothes its nodes P_i.

To be more explicit, we choose local holomorphic coordinates, so that near one of its nodes P, C_0 is given by:

$$z_1 z_2 = z_3 = \cdots = z_n = 0.$$

Consider now a general $1-$parameter deformation corresponding to a section of $N_{C_0/X}$ which does not belong to the subspace of $N_{C_0/X,P}$ generated by $z_3, \cdots, z_n$. This deformation will be given by the equations:

$$z_1 z_2 + tf(t, \mathbf{z}) = z_3 + tf_3(t, \mathbf{z}) = \cdots = z_n + tf_n(t, \mathbf{z}),$$

and $f(t, \mathbf{z}) \neq 0$, by assumption. We can change new coordinates $z_1{}' := z_1$, $z_2{}' := z_2$ and $z_i{}' := z_i + tf_i(t, \mathbf{z})$ for $i = 3, \ldots n$, to get new, simpler equations:

$$z_1{}' z_2{}' + t(a + F(t, \mathbf{z})) = z_3{}' = \cdots = z_n{}' = 0, \tag{2.5}$$

where $a \neq 0$ and $F(0, 0) = 0$. The singular points are given by the equations:

$$z_1{}' + t\frac{\partial F}{\partial z_2{}'} = z_2{}' + t\frac{\partial F}{\partial z_1{}'} = \cdots = z_n{}' = 0.$$

Substituting back these equations into ${z_1}'{z_2}' + t(a + F(t, \mathbf{z}) = 0$ we get a new equation for the supposed singular point:

$$ta = -tF(t, \mathbf{z}) - t^2 \frac{\partial F}{\partial {z_1}'} \frac{\partial F}{\partial {z_2}'}$$

The latter has no solution for $t \neq 0$ and $t, {z_1}', {z_2}', \ldots {z_n}'$ small since $a \neq 0$ and $F(0, 0) = 0$. □

Remark 2.33. Using the implicit function theorem we can change one more time the coordinates in (2.5) such that near the node $P_i,\ C_0$ is given by:

$$z_1 z_2 + t = z_3 = \cdots = z_n = 0. \tag{2.6}$$

This change of coordinates is given by $z_i := {z_i}'$ for $i = 1, \ldots, n$, and $t := t(a + F(t, \mathbf{z}))$.

2.4.2 Construction of the specialization

We start with our blowing-up $p : X_C \to X$ of a projective, rationally connected threefold X along a smooth curve $C \subset X$. Let E be the exceptional divisor. We will construct a degeneration having an appropriate blowing-up of X_C as one of the components of the central fiber.

Since X is rationally connected, we can always attach [ArKo03] to the curve $C \subset X$ a finite number of disjoint, smooth rational curves $L_1, \ldots L_n \subset X$, with ample normal bundle, meeting C at transversely at exactly one point $P_i = C \cap L_i,\ i = 1, \ldots, n$. Using Theorem 2.31 and Corollary 2.32, the comb $C_0 = C \cup L_1 \cup \cdots \cup L_n$ is smoothable for $n \gg 0$. As in the proof of Corollary

2.32 this means that we can find a small deformation of C_0 parametrized by a one dimensional disk $\Delta \subset \text{Hilb}(X)$ centered in $[C_0]$. That is there exists a smooth submanifold $\mathcal{C} \subset X \times \Delta$, such that its projection $\pi : \mathcal{C} \to \Delta$ is flat, and

$$\pi^{-1}(t) = \begin{cases} C_0, & \text{if } t = 0 \\ C_t, & \text{if } t \neq 0, \end{cases}$$

where C_t is a smooth irreducible curve. From Corollary 2.32 and Remark 2.33, in local coordinates chosen w.r.t a neighborhood of the node P_i, π is the projection

$$(z_1, z_2, z_3, t) \mapsto t$$

and $\mathcal{C}$ is given by $z_1 z_2 + t = z_3 = 0$. In these local coordinates, $C \subset \mathcal{C}$ is given by $z_1 = z_3 = t = 0$, and L_i by $z_2 = z_3 = t = 0$.

Let $\varpi : \mathcal{X}_{\mathcal{C}} \to X \times \Delta$ be the blow-up of $X \times \Delta$ along $\mathcal{C}$, and let

$$\Pi : \mathcal{X}_{\mathcal{C}} \to \Delta$$

be the projection onto Δ.

Lemma 2.34. (Structure of $\Pi : \mathcal{X}_{\mathcal{C}} \to \Delta$)

i) *$\mathcal{X}_{\mathcal{C}}$ is a smooth variety, and $\Pi : \mathcal{X}_{\mathcal{C}} \to \Delta$ is a flat, proper family of projective varieties.*

ii) *For $t \neq 0$, $X_{\mathcal{C},t} = \Pi^{-1}(t)$ is the blowing-up of X along C_t, while $X_{\mathcal{C},0} = \Pi^{-1}(0)$ is the blowing-up of X along the ideal sheaf of $C_0 \subset X$.*

Proof. i) This are standard facts about blowing-up, see sections II. 7 and II. 8 of [Har77].

ii) For the proof we can either quote the universal property of the blowing-up, Corollary II.7.15 of [Har77] or use local equations as in the proof of Corollary 2.32. We adopt the latter. The results we want to prove here are of local nature. In a neighborhood of a node of $C_0 \subset U \subset X \times \Delta$, $\mathcal{C}$ is given by the equations:

$$z_1 z_2 + t = z_3 = 0.$$

$\mathcal{X}_{\mathcal{C}|_U} \subset U \times \mathbb{P}_1$ will therefore given by the equations:

$$(z_1 z_2 + t)v = z_3 u, \tag{2.7}$$

where $[u : v]$ are the homogeneous coordinates on $\mathbb{P}_1$, and the conclusion follows now immediately. □

Let $\Pi_0 : X_{\mathcal{C},0} \to X$ denote the blowing-up map of X along the ideal sheaf of C_0.

Lemma 2.35. (Structure of the central fiber $X_{\mathcal{C},0}$)

i) *$X_{\mathcal{C},0}$ has exactly n distinct ordinary double points as singularities.*

ii) *The exceptional divisor of Π_0, denoted by E^* is a union of smooth Weil divisors $E^*_C,\ E^*_1, \ldots, E^*_n$.*

Proof. i) From the arguments used in the above Lemma we can see that the singular points of $X_{\mathcal{C},0}$ can occur only over the singular points of C_0. The type of this singularities can be seen from (2.7) for $t = 0$. It follows that for node of C_0, in the above coordinates, there is exactly one singular point of $X_{\mathcal{C},0}$, which appears in the chart where $v \neq 0$ and is given by the local equation

$$z_1 z_2 = z_3 u', \tag{2.8}$$

where $u' = \frac{u}{v}$.

ii) Since the center of the blowing-up has exactly $n+1$ components, it follows the exceptional divisor of $X_{\mathcal{C},0}$ has $n+1$ components too, one over each of the components of C_0. Using (2.8) the other claims easily follow. □

Let $Q_i \in \mathcal{X}_{\mathcal{C}},\ 1 = 1, \ldots, n$ denote the singular points of $X_{\mathcal{C},0}$.

Remark 2.36. It can be seen that $X_{\mathcal{C},0}$ is a Gorenstein non $\mathbb{Q}-$factorial variety. Hence push-forward arguments, as the ones we used in the previous section cannot be applied.

In order to perform the computation to follow, we need a better understanding of the components E_i^*'s of E^*.

Proposition 2.37. (The components $E_i^*,\ i = 1, \ldots n$)

i) $E_i^* = \mathbb{P}_{L_i}(N^{\vee}_{C_0/X_{|L_i}})$.

ii) The conormal bundle of E_i^* is given by the extension:

$$0 \longrightarrow \mathcal{O}_{E_i^*} \longrightarrow N^{\vee}_{E_i^*/\mathcal{X}_{\mathcal{C}}} \longrightarrow \mathcal{J}_{Q_i} \otimes \mathcal{O}_{E_i^*}(1) \otimes \mathcal{O}_{E_i^*}(f) \longrightarrow 0,$$

where, by $\mathcal{J}_{Q_i}$ we denoted the ideal sheaf of Q_i on E_i^*, $\mathcal{O}_{E_i^*}(1)$ is the dual of the tautological bundle of the ruled surface E_i^*, and f is its fiber.

Proof. i) This is a well-known fact. We include a short proof for convenience. The general theory of blowing-ups tells us that, since $L_i \subset \mathcal{C}$, $E_i^* = \mathbb{P}_{L_i}(N^{\vee}_{\mathcal{C}/X\times\Delta_{|L_i}})$. To compute $N^{\vee}_{\mathcal{C}/X\times\Delta_{|L_i}}$ we use the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{L_i} & = & \mathcal{O}_{L_i} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^{\vee}_{\mathcal{C}/X\times\Delta_{|L_i}} & \longrightarrow & N^{\vee}_{L_{i/X}} \oplus \mathcal{O}_{L_i} & \longrightarrow & \mathcal{O}_{L_i}(P_i) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^{\vee}_{C_0/X_{|L_i}} & \longrightarrow & N^{\vee}_{L_i/X} & \longrightarrow & \mathcal{O}_{P_i}(P_i) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
\tag{2.9}
$$

The first row is given by the exact sequence of conormal bundles of the inclusions $L_i \subset \mathcal{C} \subset X \times \Delta$. We have the obvious isomorphism $N^{\vee}_{L_{i/X\times\Delta}} \simeq N^{\vee}_{L_{i/X}} \oplus \mathcal{O}_{L_i}$.

On the smooth surface $\mathcal{C}$, since the L_i's are mutually disjoint *rational curves* and meet C transversally at exactly one point, we have:

$$0 = L_i \cdot C_t = L_i \cdot C_0 = L_i \cdot (C + L_1 + \cdots L_n) = 1 + L_i^2.$$

Therefore, the L_i's are actually $(-1)-$curves and $N^{\vee}_{L_{i|\mathcal{C}}} \simeq \mathcal{O}_{L_i}(P_i)$.

The second row is the exact sequence of Andreatta-Wisniewski [AnWi98, page 265]. From the snake lemma, we can see now that

$$N^{\vee}_{\mathcal{C}/X\times\Delta_{|L_i}} \simeq N^{\vee}_{C_0/X_{|L_i}}.$$

ii) Let $\bar{F}:\bar{\mathcal{X}}\to\mathcal{X}_{\mathcal{C}}$ be the blowing-up of $\mathcal{X}_{\mathcal{C}}$ at the points Q_i, for $i=1,\dots,n$, and $\bar{\Pi}:\bar{\mathcal{X}}\to\Delta$ the projection onto Δ. We denote by $\bar{X}$ the strict transform of $X_{\mathcal{C},0}$, and by $Z_i,\ i=1,\dots,n$, the exceptional divisors of $\bar{F}$. $\bar{\Pi}$ has the following fibers:

$$\bar{\Pi}^{-1}(t)=\begin{cases} X_{C_t}, & \text{if } t=0\\ \bar{X}+2Z_1+\dots 2Z_n, & \text{if } t\neq 0.\end{cases}$$

Since $\mathcal{X}_{\mathcal{C}}$ is smooth, we have $Z_i\simeq\mathbb{P}_3$, and $N_{Z_i/\bar{\mathcal{X}}}\simeq\mathcal{O}_{\mathbb{P}_3}(-1)$. The multiplicities of the Z_i's in the central fiber are caused by the ordinary double point singularities of $X_{\mathcal{C},0}$. The reduced component $\bar{X}$ is a big resolution of $X_{\mathcal{C},0}$. The induced map $\bar{X}\to X_{\mathcal{C},0}$ has n exceptional divisors, $T_i=\bar{X}\cap Z_i,\ i=1,\dots,n$, each of them isomorphic to $\mathbb{P}_1\times\mathbb{P}_1$, with $N_{T_i/\bar{X}}\simeq\mathcal{O}(-1,-1)$. Moreover, $N_{T_i/Z_i}\simeq\mathcal{O}(1,1)$, for all $i=1,\dots,n$.

To compute the conormal bundle $N^{\vee}_{E_i^*/\mathcal{X}_{\mathcal{C}}}$, we need a good understanding of the main component $\bar{X}$ of $\bar{\Pi}^{-1}(0)$. Let $\bar{p}:\bar{X}\to X$ be the natural morphism onto X. This has the following alternative description :

- Consider $p_L: X_L\to X$, the blowing up of X, along the disjoint union of curves $L_1,\dots,L_n$. Let $E_1,\dots,E_n$ denote the exceptional divisors and $\bar{C}$ denote the strict transform of C and $\{x_i\}=\bar{C}\cap E_i$. The E_i's are rational ruled surfaces over $L_i,\ E_i=\mathbb{P}_{L_i}(N^{\vee}_{L_i/X})$, with $N_{E_i/X_L}\simeq\mathcal{O}_{E_i}(-1)$. Consider $f_i\in E_i$, the fiber of E_i through x_i, for all $i=1,\dots,n$.

- Let $p_{\bar{C}} : X_{L,\bar{C}} \to X_L$ be the blowing-up of X_L along $\bar{C}$. We denote by $E_{\bar{C}}$ the exceptional divisor, and by $\bar{E}_i$ the strict transforms of E_i, for all $i = 1, \ldots, n$. Each of the $\bar{E}_i{}'$s is the blowing-up of E_i at x_i. Let ℓ_i denote the exceptional divisor of these blowing-up. $E_{\bar{C}}$ and $\bar{E}_i$ meet transversally along ℓ_i, and ℓ_i sits in $E_{\bar{C}}$ as fiber. Moreover, we have $N_{\bar{E}_i/X_{L,\bar{C}}} = p_{\bar{C}}^* N_{E_i/X_L}$ (see [Ful98]). In each $\bar{E}_i$, we denote by $\bar{f}_i$ the strict transform of f_i.

  We can immediately see that $N_{\bar{E}_i/X_{L,\bar{C}|\bar{f}_i}} \simeq \mathcal{O}_{\mathbb{P}_1}(-1)$, and the exact sequence:

$$0 \to N_{\bar{f}_i/\bar{E}_i} \to N_{\bar{f}_i/X_{L,\bar{C}}} \to N_{\bar{E}_i/X_{L,\bar{C}|\bar{f}_i}} \to 0$$

  yields

$$N_{\bar{f}_i/X_{L,\bar{C}}} \simeq \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1).$$

- We blow-up now $X_{L,\bar{C}}$ along $\bar{f}_i$, for all $i = 1, \ldots, n$. The resulting $3-$fold is isomorphic to $\bar{X}$, where the exceptional divisors of the last blowing-up coincide with $T_i,\ i = 1, \ldots, n$. Let $p_{\bar{F}} : \bar{X} \to X_{L,\bar{C}}$ be the blowing-up map. The map $\bar{p}$ is the composition:

$$\bar{p} = p_L \circ p_{\bar{C}} \circ p_{\bar{F}}.$$

  Denote by $R_i = p_{\bar{F}}^* \bar{E}_i - T_i$ the strict transforms of $\bar{E}_i$. Since $\bar{f}_i \subset \bar{E}_i,\ R_i$ is isomorphic to $\bar{E}_i$, the blowing-up of of E_i at x_i. Let $\bar{E}$ be the strict transform of $E_{\bar{C}}$, and $\bar{\ell}_i$ be the strict transform of ℓ_i, for all $i = 1, \ldots, n$. $\bar{E}$ is isomorphic to $E_{\bar{C}}$ blown-up at the intersection points of $E_{\bar{C}}$ with the curves $\bar{f}_i$, and intersects R_i transversally along $\bar{\ell}_i$ for all $i = 1, \ldots, n$.

First we need to determine $N_{R_i/\bar{X}}$. To do this, we have to analyze more closely the position of the exceptional divisors of the map $\bar{p}$.

- R_i and T_i meet transversally along h_i, one of the rulings of T_i which coincides with $\bar{f}_i$, under the identification of R_i with $\bar{E}_i$;
- $\bar{E}$ and T_i meet transversally along k_i, the other ruling of T_i, for all $i = 1, \ldots, n$;
- $\bar{E} \cap R_i \cap T_i = \{\text{point}\}$, for all $i = 1, \ldots, n$;
- $R_i \cap R_j = \emptyset$, for $i \neq j$.

Let $p_i : R_i \to E_i$ be the blowing up of E_i at x_i, where h_i is the strict transform of the fiber through x_i, and $\bar{\ell}_i$ denotes the exceptional divisor. Using i) we can see that E_i^* is actually the elementary transform of E_i centered at x_i. Consequently, we denote by $q_i : R_i \to E_i^*$, the blowing-down of $\bar{f}_i$, for every $i = 1, \ldots, n$.

Claim 2.37.1. $N_{R_i/\bar{X}} \simeq \mathcal{O}_{R_i}(-h_i) \otimes p_i^* \mathcal{O}_{E_i}(-1)$.

Proof of Claim. Since R_i is the blowing-up of E_i, we can write $N_{R_i/\bar{X}}$ as

$$\mathcal{O}_{R_i}(ah_i) \otimes p_i^* \mathcal{O}_{E_i}(b) \otimes p_i^* \mathcal{O}_{E_i}(cf),$$

where f is the generic fiber of the ruled surface E_i. Let $d_i = \deg N_{L_i/X}$. Let also denote by $\bar{f}$ the strict transform in $\bar{\mathcal{X}}$ of the generic fiber of E_i. From the fact that $R_i = p_{\bar{F}}^* \bar{E}_i - T_i$ and the projection formula, we compute:

$$\begin{aligned} R_i \cdot \bar{f} &= (p_{\bar{F}}^* \bar{E}_i - T_i) \cdot \bar{f} = E_i \cdot f = -1; \\ R_i \cdot \bar{\ell}_i &= (p_{\bar{F}}^* \bar{E}_i - T_i) \cdot \bar{\ell}_i = p_{\bar{C}}^* E_i \cdot \ell_i - T_i \cdot \bar{\ell}_i = -1; \\ R_i^3 &= (p_{\bar{F}}^* \bar{E}_i - T_i)^3 = \bar{E}_i^{\,3} - 3 p_{\bar{F}}^* \bar{E}_i \cdot p_{\bar{F}}^* \bar{E}_i \cdot T_i + 3 p_{\bar{F}}^* \bar{E}_i \cdot T_i^2 - T_i^3 \\ &= E_i^3 + 3(\bar{E}_i \cdot \bar{f}_i) \cdot (T_i \cdot k_i) + 2 = -d_i + 3 - 2 = -d_i + 1. \end{aligned}$$

On the other hand, computing on the surface R_i, we have:

- $R_i \cdot \bar{f} = (ah_i + p_i^* \mathcal{O}_{E_i}(b) + cp_i^* f) \cdot p_i^* f = b$, and so $b = -1$.
- $R_i \cdot \bar{\ell}_i = (ah_i + p_i^* \mathcal{O}_{E_i}(b) + cp_i^* f) \cdot \bar{\ell}_i = a(p_i^* f - \bar{\ell}_i) \cdot \bar{\ell}_i = a$, and so $a = -1$;
- $R_i^3 = (-h_i - p_i^* \mathcal{O}_{E_i}(1) + cp_i^* f)^2 = -1 - d_i + 2 - 2c = -d_i + 1 - 2c$, and so $c = 0$.

□

We compute now $N^\vee_{R_i/\bar{\mathcal{X}}}$ from the conormal sequence of the inclusions $R_i \subset \bar{X} \subset \bar{\mathcal{X}}$:

$$0 \to N^\vee_{\bar{X}/\bar{\mathcal{X}}_{|R_i}} \to N^\vee_{R_i/\bar{\mathcal{X}}} \to N^\vee_{R_i/\bar{X}} \to 0. \tag{2.10}$$

In $\bar{\mathcal{X}}$, we have $\bar{X} + 2Z_1 + \cdots 2Z_n \sim 0$, (linearly equivalence) and so

$$N^\vee_{\bar{X}/\bar{\mathcal{X}}} \simeq \mathcal{O}_{\bar{X}}(2T_1 + \cdots 2T_n).$$

Tensoring by $\mathcal{O}_{R_i}$, we get $N^\vee_{\bar{X}/\bar{\mathcal{X}}_{|R_i}} \simeq \mathcal{O}_{R_i}(2h_i)$. Hence we obtained:

$$0 \to \mathcal{O}_{R_i}(2h_i) \to N^\vee_{R_i/\bar{\mathcal{X}}} \to \mathcal{O}_{R_i}(h_i) \otimes p_i^* \mathcal{O}_{E_i}(1) \to 0. \tag{2.11}$$

On the other hand, since $\mathcal{X}_\mathcal{C}$ and E_i^* are smooth, R_i is the strict transform of E_i^* in $\bar{\mathcal{X}}$. Moreover, the restriction of blowing-up map $\bar{F}$ to R_i coincides with the blowing-up q_i with h_i as exceptional divisor. Therefore, by [Ful98, page 437],

$$N_{R_i/\bar{\mathcal{X}}} \simeq q_i^* N_{E_i^*/\mathcal{X}_\mathcal{C}} \otimes \mathcal{O}_{R_i}(-h_i).$$

From (2.11) we obtain:

$$0 \to \mathcal{O}_{R_i}(h_i) \to q_i^* N^\vee_{E_i^*/\mathcal{X}_\mathcal{C}} \to p_i^* \mathcal{O}_{E_i}(1) \to 0. \tag{2.12}$$

Lemma 2.38. *On the surface R_i, we have:*

$$p_i^* \mathcal{O}_{E_i}(1) = q_i^* \mathcal{O}_{E_i^*}(1) \otimes q_i^* \mathcal{O}_{E_i^*}(f) \otimes \mathcal{O}_{R_i}(-h_i),$$

where here f denotes the generic fiber of E_i^*.

Proof of Lemma. Computing the canonical line bundle of R_i in two ways, we get:

$$p_i^* \mathcal{O}_{E_i}(K_{E_i}) \otimes \mathcal{O}_{R_i}(\bar{\ell}_i) = q_i^* \mathcal{O}_{\bar{E}_i}(K_{\bar{E}_i}) \otimes \mathcal{O}_{R_i}(h_i). \tag{2.13}$$

Using the canonical bundle formula for ruled surfaces, and the fact that E_i^* is the elementary transform of E_i centered at x_i, from (2.13) we have:

$$\begin{aligned} &p_i^* \mathcal{O}_{E_i}(-2) \otimes p_i^* \mathcal{O}_{E_i}(-d_i f) \otimes \mathcal{O}_{R_i}(\bar{\ell}_i) = \\ &q_i^* \mathcal{O}_{E_i^*}(-2) \otimes q_i^* \mathcal{O}_{E_i^*}((-d_i - 1) f) \otimes \mathcal{O}_{R_i}(h_i). \end{aligned} \tag{2.14}$$

But, $\mathcal{O}_{R_i}(\bar{\ell}_i) = q_i^* \mathcal{O}_{E_i}(f) \otimes \mathcal{O}_{R_i}(-h_i)$, and $p_i^* \mathcal{O}_{E_i}(f) = q_i^* \mathcal{O}_{E_i^*}(f)$. Simplifying

(2.14) we get:

$$p_i^*\mathcal{O}_{E_i}(-2) = q_i^*\mathcal{O}_{E_i^*}(-2) \otimes q_i^*\mathcal{O}_{E_i^*}(-2f) \otimes \mathcal{O}_{R_i}(2h_i),$$

and the proof of the lemma follows. □

To finish the proof of the proposition, notice that we obtained the following exact sequence:

$$0 \to \mathcal{O}_{R_i}(h_i) \to q_i^* N^{\vee}_{E_i^*/\mathcal{X}_\mathcal{C}} \to q_i^*\mathcal{O}_{E_i^*}(1) \otimes q_i^*\mathcal{O}_{E_i^*}(f) \otimes \mathcal{O}_{R_i}(-h_i) \to 0.$$

By pushing forward on E_i^*, since $R^1 q_{i*}\mathcal{O}_{R_i}(h_i) = 0$ and $q_{i*}\mathcal{O}_{R_i}(h_i) = \mathcal{O}_{E_i^*}$, the projection formula yields:

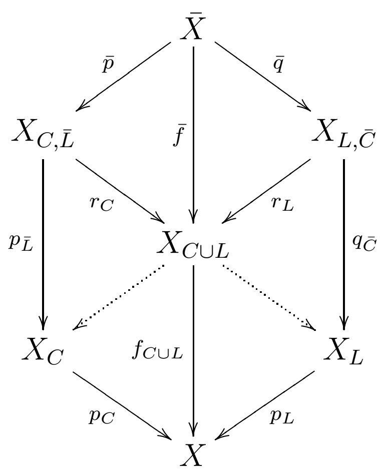
$$0 \longrightarrow \mathcal{O}_{E_i^*} \longrightarrow N^{\vee}_{E_i^*/\mathcal{X}_\mathcal{C}} \longrightarrow \mathcal{J}_{Q_i} \otimes \mathcal{O}_{E_i^*}(1) \otimes \mathcal{O}_{E_i^*}(f) \longrightarrow 0,$$

where, $\mathcal{J}_{Q_i}$ is the ideal sheaf of the point Q_i, and we are done. □

Corollary 2.39. *The Chern classes of* $N^{\vee}_{E_i^*/\mathcal{X}_\mathcal{C}}$ *are:*

- $\det(N^{\vee}_{E_i^*/\mathcal{X}_\mathcal{C}}) = \mathcal{O}_{E_i^*}(1) \otimes \mathcal{O}_{E_i^*}(f)$;
- $c_2(N^{\vee}_{E_i^*/\mathcal{X}_\mathcal{C}}) = 1$.

Remark 2.40. The description of $\bar{X}$ in the proof of the above proposition is of local nature and it comes from the well-known diagram below [EiHa00, pages 178-179]. This *commutative diagram* exhibits the relation between the two small resolutions and the natural big resolutions of a three-dimensional ordinary double points, as those appearing as singularities of $X_{\mathcal{C},0}$.



(2.15)

To simplify the notation and explain the diagram (2.15), let X be an arbitrary threefold, and $C,\ L \subset X$ be two smooth curves intersecting transversally at exactly one point $\{x\} = C \cap L$.

- $f_{C\cup L} : X_{C\cup L} \to X$ is the blowing-up of X along the ideal sheaf of $C \cup L$;

- $\bar{f} : \bar{X} \to X_{C\cup L}$ is the big resolution of $\bar{X}$ obtained by blowing-up the singular point.

- $p_C : X_C \to X$ is the blowing-up of X along C. Let f_C be the fiber of the exceptional divisor over x.

- $q_L : X_L \to X$ is the blowing-up of X along L. Let f_L be the fiber of the exceptional divisor over x.

- $p_{\bar{L}} : X_{C,\bar{L}} \to X$ is the blowing-up of X_C along $\bar{L}$, the proper transform of L in X_C. Let $\bar{f}_C$ denote the proper transform of f_C.

- $q_{\bar{C}} : X_{L,\bar{C}} \to X$ is the blowing-up of X_L along $\bar{C}$, the proper transform of C in X_L; Let $\bar{f}_L$ denote the proper transform of f_L.

- $\tilde{p} : \tilde{X} \to X_{C,\bar{L}}$ is the blowing-up of $X_{C,\bar{L}}$ along $\bar{f}_C$.

- $\tilde{q} : \tilde{X} \to X_{L,\bar{C}}$ is the blowing-up of $X_{L,\bar{C}}$ along $\bar{f}_L$.

- $r_L : X_{L,\bar{C}} \to \bar{X}$ and $r_C : X_{C,\bar{L}} \to \bar{X}$ are the two *small resolutions* of the singular point of $\bar{X}$.

We will modify the family $\Pi : \mathcal{X}_{\mathcal{C}} \to \Delta$ to produce a flat, proper map

$$\Phi : \mathcal{X} \to \Delta$$

with normal crossing central fiber, and with X_{C_t} as the general fiber. Of course, such a map can be viewed as a degeneration of X_{C_t}.

The map Φ is obtained as the composition

$$\mathcal{X} \xrightarrow{F} \mathcal{X}_{\mathcal{C}} \xrightarrow{\Pi} \Delta,$$

where $F : \mathcal{X} \to \mathcal{X}_{\mathcal{C}}$ is the blowing-up $X_{\mathcal{C}}$ along E_i^*, for $i = 1, \dots, n$.

It is easy to see that the generic fiber of Φ is X_{C_t}, the blowing-up of X along the smooth curve C_t. The central fiber of Φ is a normal crossing thereefold

$$X_0 = X_p \cup X_1 \cup \cdots \cup X_n,$$

with exactly $n + 1$ irreducible, smooth components.

To describe the main component, denoted by X_p, as before, let $X_L \to X$ denote the blowing-up of X along the curves L_i, $i = 1, \dots, n$. Then X_p is obtained by blowing-up X_L along $\bar{C}$, the strict transforms of the C. It actually coincides with the $3-$fold $X_{L,\bar{C}}$, described in the above proposition. We denote