A homology and cohomology theory for real projective varieties

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Abstract of the Dissertation

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In this thesis we develop homology and cohomology theories which play the same role for real projective varieties that Lawson homology and morphic cohomology play for projective varieties respectively. They have nice properties such as the existence of long exact sequences, the homotopy invariance, the homotopy property for trivial bundle projection, the splitting principle and the natural transformations to singular theories. The Friedlander-Lawson Moving Lemma is used to prove a duality theorem between these two theories. We consider them as extensions of singular homology and cohomology respectively, and we extend the classical Harnack-Thom Theorem to get a comparison theorem between the rank of Lawson homology groups with $\mathbb{Z}_2$-coefficients and the rank of the groups in this homology theory. We define signatures in morphic cohomology and state the Morphic Conjecture which includes the Grothendieck Standard Conjecture over $\mathbb{C}$ as a special case. We show that this conjecture implies an extension of the classical Hodge Index Theorem to morphic cohomology.

To my parents

Contents

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Chapter 1

Introduction

The study of solving polynomial equations dates back to the very beginning of mathematics. General algebraic solutions of a given equation was the original goal. This goal was achieved for equations of degree 2, 3 and 4. But it was proved by Abel and Galois that it was impossible for equations of degree 5. Galois theory was created several decades later to study some properties of the roots of equations. At the same time, people started to consider the more complicated problem of solving polynomial equations of more than one variable. The zero loci of polynomial equations, which are called algebraic varieties, are basic source of geometry and exemplify many important geometric phenomena. In this thesis, we study the properties of projective varieties which are the zero loci of homogeneous polynomials in projective spaces. Projective varieties, especially nonsingular ones, have wonderful properties such as: Hodge decomposition, Lefschetz decomposition, Weak Lefschetz property. The structure of projective varieties is so rich that even though a lot of properties have been discovered, their theory contains fundamental problems which are still unsolved. The most important of these is the Hodge Conjecture which claims that the rational (p,p)-cohomology classes of a nonsingular projective variety can be represented by algebraic cycles with rational coefficients. Algebraic cycles are some finite formal sum of irreducible subvarieties with integral coefficients. If this conjecture is true, it implies the truth of many others, including the famous Grothendieck Standard Conjecture. In examining the proof of Weil Conjecture for curves and abelian varieties over finite fields(by Weil himself), Grothendieck was led to state his two Standard Conjectures which imply the Weil Conjecture for smooth varieties defined over a finite field. There are analogous conjectures for smooth complex projective varieties. One of them, the Hodge Standard Conjecture, which asserts that there is an abstract version of the Hodge index theorem for the $\mathbb{Q}$-vector space of classes of algebraic cycles, is a consequence of hodge theory and Lefschetz decomposition over $\mathbb{C}$.

Let X be a smooth projective variety of dimension n and $C^j(X)$ be the subspace of $H^{2j}(X;\mathbb{Q})$ which is generated by algebraic cycles. By Hard Lefschetz

Theorem, we have the following commutative diagram:

$$\begin{array}{ccc} H^{2j}(X;\mathbb{Q}) & \xrightarrow{L^{n-2j}} & H^{2n-2j}(X;\mathbb{Q}) \\ \uparrow & & \uparrow \\ C^j(X) & \longrightarrow & C^{n-j}(x) \end{array}$$

where L^{n-2j} is an isomorphism for $j \leq \lfloor \frac{n}{2} \rfloor$. The Grothendieck Standard Conjecture claims that restricting L^{n-2j} gives an isomorphism between $C^j(X)$ and $C^{n-j}(X)$, or equivalently that the adjoint operator Λ maps $C^{n-j}(X)$ into $C^j(X)$. If the Hodge Conjecture is true, this will follow from the Hard Lefschetz Theorem, since then $C^j(X) = H^{j,j}(X)$ and $C^{n-j}(X) = H^{n-j,n-j}(X)$.

The group $Z_p(X)$ of p-cycles of a projective variety X encodes many properties of X. For many decades, since $Z_p(X)$ is a very large group in general, quotients of $Z_p(X)$ were studied instead. For example, the quotient of $Z_p(X)$ by rational equivalence is the Chow group $CH_p(X)$ on which the intersection theory of algebraic varieties can be built [Fu].

For a complex projective variety X, the group $Z_p(X)$ has additional natural structure. According to the Chow Theorem, there is a canonical way, by means of Chow forms, to give $\mathscr{C}_{p,d}(X)$, the set of effective p-cycles of degree d, the structure of a projective variety. Thus $\mathscr{C}_{p,d}(X)$ has a canonical analytic topology. Since $Z_p(X)$ is the group completion of the monoid $\coprod_{d\geq 0} \mathscr{C}_{p,d}(X)$, it inherits a topology from this analytic structure which makes it a topological group. From this point of view, methods from topology, especially from homotopy theory, come into play. The title of the Hirzebruch's book, "Topological methods in algebraic geometry", perfectly describes the way we study projective varieties via their cycle groups. The starting point of this approach is the Lawson Suspension Theorem, which says that there is a homotopy equivalence between $Z_p(X)$ and $Z_{p+1}(\Sigma\!\!\!/ X)$, where $\Sigma\!\!\!/ X$ is the projective cone over X (or more topologically the Thom space of the $\mathcal{O}(1)$ bundle over X). It is then natural to define the Lawson homology group $L_pH_n(X) = \pi_{n-2p}Z_p(X)$, the $(n-2p)$-th homotopy group of $Z_p(X)$. When $p = 0$, we have by the Dold-Thom Theorem that $L_0H_n(X) = H_n(X;\mathbb{Z})$ (the singular homology of X). Thus we view the Lawson homology as an extension of singular homology for projective varieties. This point of view will be strengthened after we develop a corresponding theory for real projective varieties and extend some classical theorems from singular homology.

In the past 15 years, Lawson, Friedlander, Mazur, Gabber, Michelsohn, Lam, Lima-Filho, Walker and dos Santos have discovered many properties of Lawson homology and have related it to other theories like Chow group, higher Chow group, motivic cohomology and algebraic K-theory. Notably, the morphic cohomology was established by Friedlander and Lawson, and a duality theorem between morphic cohomology and Lawson homology was proved by using their Moving Lemma [FL2], [FL3]. It has been shown that Lawson homology and morphic cohomology groups admit limit mixed Hodge structure [FM], [FL1].

Algebraic topology also reaps the bounty of this harvest. Since

$$Z_0(A^n) = \frac{Z_0(\mathbb{P}^n)}{Z_0(\mathbb{P}^{n-1})} = K(\mathbb{Z}, 2n)$$

where $K(\mathbb{Z}, 2n)$ is the Eilenberg-Mac Lane space, we are able to represent many Eilenberg-Mac Lane spaces by more concrete algebraic cycle spaces. For example, consider the natural embedding $\mathscr{G}^q(\mathbb{P}^n) \subset \mathscr{C}^q(\mathbb{P}^n)$ of the Grassmannian of codimension-q planes into the limit space $\mathscr{C}^q(\mathbb{P}^n) = \lim_{d\to\infty} \mathscr{C}^q_d(\mathbb{P}^n)$ of codimension-q cycles of $\mathbb{P}^n$. Letting n go to infinity, we get a map

$$BU_q \xrightarrow{c} K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \times \cdots \times K(\mathbb{Z}, 2q)$$

where $BU_q = lim_{n\to\infty} \mathscr{G}^q(\mathbb{P}^n)$ is the classifying space for the unitary group U_q. A beautiful theorem by Lawson and Michelsohn [LMi] says that for each $q \geq 1$, this map induces an isomorphism

$$\mathbb{Z} \cong \pi_{2q}(BU) \xrightarrow{c_*} \pi_{2q}(K(\mathbb{Z}, ev)) \cong \mathbb{Z}$$

which is multiplication by $(q-1)!$ where $K(\mathbb{Z}, ev) = \prod_{i=1}^{\infty} K(\mathbb{Z}, 2i)$. Our next example is from morphic cohomology.

- There is a join pairing in algebraic cocycle groups which induces the cup product in singular cohomology.

- The Chern classes in morphic cohomology map to the Chern classes in singular cohomology.

- The cocycle group $Z^m(X)$ of a m-dimensional smooth projective manifold is homotopy equivalent to the mapping space $Map(X, K(\mathbb{Z}, 2m))$.

The Fundamental Theorem Of Algebra says that a degree d polynomial in one variable over $\mathbb{C}$ has d zeros in complex plane counting multiplicities. A polynomial f of degree d over $\mathbb{R}$ need not have d real zeros, but if we take $Z(f)$ to be the set of d zeros of f, and $Z(f)^{av}$ to be the set of nonreal zeros of f counting multiplicities, we find that if we define $R(f) = Z(f) - Z(f)^{av}$, since nonreal zeros appear in conjugate pairs, the cardinality $|R(f)|$ of $R(f)$ and the degree d have the following relation:

$$|R(f)| \equiv d \bmod 2$$

This is the Reduced Real Fundamental Theorem of Algebra.

The principle is that everything in the complex world has a counterpart in the real world with $\mathbb{Z}_2$-coefficients. Over $\mathbb{C}$, the utility of (co)homology theory with $\mathbb{Z}$-coefficients is related to the fact that the zeros of any polynomial can be counted. But over $\mathbb{R}$, we can only count the zeros modulo 2. The bridge from the complex world to the real world passes to a quotient of the set of elements invariant under conjugation. The reduced real cycle group is defined as

$$R_p(X) = \frac{Z_p(X)_{\mathbb{R}}}{Z_p(X)^{av}}$$

where X is a real projective variety, $Z_p(X)_{\mathbb{R}}$ is the set of p-cycles which are conjugate invariant and $Z_p(X)^{av}$ are cycles of the form $c + \bar{c}$. This group first appeared in the thesis of Lam where he proved the Lawson Suspension Theorem for reduced real cycle groups. We develop this idea further to get a homology-like theory called reduced real Lawson homology and a cohomology-like theory called reduced real morphic cohomology. By using the Moving Lemma of Friedlander and Lawson, we prove a duality between them. Under the natural transformations constructed in Chapter 4, this duality is compatible with the usual Poincaré Duality with $\mathbb{Z}_2$-coefficients. As with morphic cohomology, the join pairing of cycles induces a cup product in the reduced real morphic cohomology and after passing to singular cohomology with $\mathbb{Z}_2$-coefficients it becomes the usual cup product.

The Reduced Real Fundamental Theorem of Algebra is a very special case of Harnack-Thom Theorem proved in Chapter 6. The classical Harnack-Thom Theorem compares the respective sums of $\mathbb{Z}_2$-betti of X and ReX, where X is a real projective variety and ReX is the set of real points of X. We extend this comparison to Lawson homology with $\mathbb{Z}_2$-coefficients and to the reduced real Lawson homology. When the cycle group is the zero cycle group, we recover the classical Harnack-Thom Theorem.

In the last chapter we define signatures in morphic cohomology. We discover that there is a fundamental conjecture, called the "Morphic Conjecture", which implies the Grothendieck Standard Conjecture. With the assumption of the Morphic Conjecture, we prove the Hodge Index Theorem for morphic cohomology, which gives the classical Hodge Index Theorem in the case of top dimension. We would like to establish a theorem analogous to the Rokhlin First Theorem in order to relate morphic signatures to the Euler characteristic in reduced real Lawson homology of X. We are unable to do this currently. Perhaps what is needed is a cobordism theory for morphic cohomology.

The interplays between the real world and complex world are very interesting. We hope that our study of real projective varieties would further our understanding of projective varieties in general.

Chapter 2

Reduced Real Lawson Homology Groups And Reduced Real Morphic Cohomology Groups

2.1 Topology Of Reduced Real Cycle And Cocycle Spaces

Let V be a projective variety in $\mathbb{CP}^n$. An r-cycle on V is a linear combination of irreducible subvarieties of the same dimension r in V with $\mathbb{Z}$-coefficients. Two subvarieties of V meet properly if the codimension of each component of their intersection is the sum of the codimension in V of that two subvarieties. We say that a cycle c is in reduced form if $c = \sum n_i V_i$ where n_i are nonzero integers and $V_i \neq V_j$ if $i \neq j$. When we say that a cycle c_1 meets another cycle c_2 properly we mean that in their reduced forms, each component of c_1 and c_2 meets properly.

Throughout this thesis, a real projective variety $V \subset \mathbb{CP}^n$ is a complex projective variety which is invariant under the conjugation.

Let us define some notation:

Definition Let X, Y be two projective varieties and the let the dimension of X be m. Denote the group of all r-cycles of X by $Z_r(X)$ and by $Z_r(Y)(X)$ the subgroup of $Z_{r+m}(X \times Y)$ consisting of all cycles which meet $x \times Y$ properly in $X \times Y$ for all $x \in X$. We call $Z_r(Y)(X)$ the dimension r cocycle group of X with values in Y.

Suppose now X and Y are real projective varieties. We will write the conjugate of a point $x \in X$ as $\overline{x}$. The conjugation induces a map on cycle groups. A subvariety $V \subset X \times Y$ is real if and only if $\overline{V} = V$ where $\overline{V} = \{(\overline{x}, \overline{y}) | (x, y) \in V\}$. For $f \in Z_r(Y)(X)$, we use $f(x)$ to denote the cycle-theoretical fiber of f under the projection $p : X \times Y \longrightarrow X$ over x, that is, $f(x) = f \bullet p^{-1}(x)$ where $\bullet$ is the intersection product of cycles. Define $\overline{f}(x) = \overline{f(\overline{x})}$. If f is real, then $\overline{f(\overline{x})} = \overline{f \bullet p^{-1}(\overline{x})} = \overline{f} \bullet \overline{p^{-1}(\overline{x})} = f \bullet p^{-1}(x) = f(x)$ for all $x \in X$ and if $\overline{f} = f$, for $(x, y) \in f$, $\overline{y} \in \overline{f(x)} = f(\overline{x})$. Thus f is real if and only if $\overline{f} = f$. This notation is convenient when we think of f as a map from X to the cycle space of Y.

Let $Z_r(X)_{\mathbb{R}}$ be the subgroup of $Z_r(X)$ consisting of cycles invariant under conjugation and let $Z_r(X)^{av} = \{V + \overline{V} | V \in Z_r(X)\}$ be the averaged cycle group. We define the reduced real cycle group to be

$$R_r(X) = \frac{Z_r(X)_{\mathbb{R}}}{Z_r(X)^{av}},$$

and the reduced real Y-valued cocycle group to be

$$R_r(Y)(X) = \frac{Z_r(Y)(X)_{\mathbb{R}}}{Z_r(Y)(X)^{av}}$$

where

$$Z_r(Y)(X)_{\mathbb{R}} = Z_r(Y)(X) \cap Z_{r+m}(X \times Y)_{\mathbb{R}},$$

$$Z_r(Y)(X)^{av} = Z_r(Y)(X) \cap Z_{r+m}(X \times Y)^{av}.$$

A topological group G is a group which is also a Hausdorff space and for $(g, h) \in G \times G$, the product $(g, h) \longrightarrow gh^{-1}$ is a continuous map. For a projective variety X, let $\mathscr{C}_{r,d}(X)$ be the set of degree d r-cycles of X. By Chow Theorem, $\mathscr{C}_{r,d}(X)$ has a structure as a projective variety. Thus we may consider $\mathscr{C}_{r,d}(X)$ as a complex projective variety with the analytic topology. Let $K_{r,d}(X) = \coprod_{d_1+d_2 \leq d} (\mathscr{C}_{r,d_1}(X) \times \mathscr{C}_{r,d_2}(X))/\sim$ where the equivalence relation is given by $(a_1, b_1) \sim (a_2, b_2)$ if and only if $a_1 + b_2 = a_2 + b_1$. $K_{r,d}(X)$ inherits topology from this quotient which makes $K_{r,d}(X)$ a compact Hausdorff space.

And from the filtration,

$$K_{r,1}(X) \subset K_{r,2}(X) \subset \cdots = Z_r(X),$$

we give $Z_r(X)$ the weak topology which means that a subset $A \subset Z_r(X)$ is closed if and only if $A \cap K_{r,d}(X)$ is closed for all d. This topology makes $Z_r(X)$ a topological abelian group and we will call this topology the Chow topology. In general, let

$$X_1 \subset X_2 \subset X_3 \subset \cdots = X$$

be a chain of closed inclusions of topological spaces. We define the topology on X by declaring a subset $C \subset X$ to be closed if and only if its intersection $C \cap X_i$ is closed for all $i \geq 1$. This topology is called the weak topology with respect to the subspaces.

The homotopy types of $Z_r(\mathbb{P}^n)_{\mathbb{R}}$ and $Z_r(\mathbb{P}^n)^{av}$ were computed in [LLM] which are quite complicated but the homotopy type of $R_r(\mathbb{P}^n)$ is much simpler. In the following we will see that the reduced real cycle groups are closely related to the singular homology with $\mathbb{Z}_2$-coefficients of the real points. These are some of the reasons that we work on the reduced real cycle groups.

Definition The filtration

$$K_{r,1}(X) \subset K_{r,2}(X) \subset \cdots = Z_r(X)$$

is called the canonical r-filtration of X and if X is a real projective variety,

$$K_{r,1}(X)_{\mathbb{R}} \subset K_{r,2}(X)_{\mathbb{R}} \subset \cdots = Z_r(X)_{\mathbb{R}}$$

is called the canonical real r-filtration of X where $K_{r,i}(X)_{\mathbb{R}}$ is the subset of real cycles in $K_{r,i}(X)$ and

$$K_{r,1}(X)^{av} \subset K_{r,2}(X)^{av} \subset \cdots = Z_r(X)^{av}$$

is called the canonical averaged r-filtration of X where $K_{r,i}(X)^{av}$ is the subset of averaged cycles in $K_{r,i}(X)$. If a filtration is defined by a sequence of compact sets, this filtration is called a compactly filtered filtration.

We will assume that all varieties considered in this paper are normal. For $V \in Z_r(Y)(X)$, we may view V as a cycle in $X \times Y$ with equidimensional fibers over X or a continuous morphism from X to $Z_r(Y)$. By giving compact-open topology to the set of morphisms $\mathcal{M}(X, Z_r(Y))$, Friedlander and Lawson proved that it is homeomorphic to $Z_r(Y)(X)$ by using their graphing construction. This is their viewpoint in [FL2]. In the following, we will intertwine these two viewpoints very often.

Proposition 2.1.1. *Suppose that Y is a subvariety of a projective variety X. Then $Z_r(Y)$ is a closed subgroup of $Z_r(X)$.*

Proof. This is because $Z_r(Y) \cap K_{r,d}(X) = K_{r,d}(Y)$ which is closed in $K_{r,d}(X)$. □

A filtration of a topological space T by a sequence of subspaces

$$T_0 \subset T_1 \subset \cdots \subset T_j \cdots$$

is said to be locally compact if for any compact subset $K \subset T$, there exists some $e \geq 0$ such that $K \subset T_e$.

Lemma 2.1.2. *Suppose that X is a Hausdorff space and the topology on X is given by the filtration*

$$T_0 \subset T_1 \subset \cdots \subset T_j \cdots = X$$

by the weak topology. Then this filtration is a locally compact filtration.

Proof. Let $K \subset X$ be a compact subset. Assume that K is not contained in any T_e, then we can find a sequence $\{x_i\}_{i \in I}$ where $x_i \in (T_i - T_{i-1}) \cap K$ and I is a sequence of distinct integers which goes to infinity. Let $U_k = X - \{x_i\}_{i\in I, i\neq k}$. Then $U_k \cap T_j = T_j - \{x_i | i \leq j, i \neq k\}$ which is open for all j. Thus U_k is an open subset of X and $K \subset \cup_{i\in I} U_k$ but we are unable to find a finite subcover which contradicts to the compactness of K. So K has to be contained in some T_e. □

Proposition 2.1.3. *For real projective varieties X, Y, $Z_r(Y)(X)_{\mathbb{R}}$ and $Z_r(Y)(X)^{av}$ are closed subgroups of $Z_r(Y)(X)$. In particular, $Z_r(Y)_{\mathbb{R}}$ and $Z_r(Y)^{av}$ are closed subgroups of $Z_r(Y)$.*

Proof. Define $\psi : Z_r(Y)(X) \longrightarrow Z_r(Y)(X)$ by $\psi(f) = \overline{f} - f$. Since ψ is continuous and $Z_r(Y)(X)_{\mathbb{R}}$ is the kernel of ψ, $Z_r(Y)(X)$ is closed. Suppose that $\{f_n + \overline{f_n}\}$ is a sequence in $Z_r(Y)(X)^{av}$ which converges to c. Since the canonical averaged r-filtration of X is locally compact, by Lemma 2.1.2, $A = \{f_n + \overline{f_n}\} \cup \{c\}$ is contained in $K_{r,d}(X)^{av}$ for some $d > 0$. Therefore there exist $g_n \in K_{r,d}(X)^{av}$ such that $g_n + \overline{g_n} = f_n + \overline{f_n}$ for all n. Let $\{g_{n_i}\}$ be a convergent subsequence of $\{g_n\}$ which converges to a cycle g. Then $\{\overline{g_{n_i}}\}$ converges to $\overline{g}$. Therefore $g_{n_i} + \overline{g_{n_i}}$ converges to $g + \overline{g}$ which implies $c = g + \overline{g}$. Thus $Z_r(Y)(X)^{av}$ is closed. Take X to be a point, it follows immediately that $Z_r(Y)_{\mathbb{R}}$ and $Z_r(Y)^{av}$ are closed in $Z_r(Y)$. □

Corollary 2.1.4. *For real projective varieties X, Y, $R_r(Y)(X)$ and $R_r(X)$ are topological abelian groups.*

Proof. This is from the general fact that the quotient of a topological group by any of its closed normal subgroups is a topological group. □

Proposition 2.1.5. *Suppose that G is a topological group and*

$$K_1 \subset K_2 \subset \cdots = G$$

is a filtration by compact subsets of G which generates the topology of G as above. Let H be a closed normal subgroup of G and $q : G \longrightarrow G/H$ be the quotient map. Denote the restriction of q to K_k by q_k and $M_k = q(K_k)$. We define a topology on M_k by making q_k a quotient map, for all k. Then

1. *M_k is a subspace of M_{k+1} for all k.*

2. *the weak topology of G defined by the filtration*

$$M_1 \subset M_2 \subset \cdots = G/H$$

   coincides with the quotient topology of G/H.

Proof. 1. If $C \subset M_{k+1}$ is a closed subset, since M_{k+1} is compact, C is also compact. From the commutative diagram,

$$\begin{array}{ccc} K_k & \xrightarrow{i} & K_{k+1} \\ \downarrow{\scriptstyle q_k} & & \downarrow{\scriptstyle q_{k+1}} \\ M_k & \xrightarrow{\bar{i}} & M_{k+1} \end{array}$$

we have $\bar{i}^{-1}(C) = q_k i^{-1} q_{k+1}^{-1}(C)$ which is compact thus closed. So $\bar{i}$ is continuous. Since M_k is compact and $\bar{i}$ is injective, it is an embedding.

2. Let C be a closed subset of G/H under the quotient topology. Then $q^{-1}(C)$ is closed which means $q^{-1}(C) \cap K_k$ is closed for all k. $q_k^{-1}(C \cap M_k) = q^{-1}(C) \cap q^{-1}(M_k) \cap K_k = q^{-1}(C) \cap K_k$ is closed, so $C \cap M_k$is closed for all k thus C is closed under the weak topology. On the other hand, assume that C is a closed subset of G/H under the weak topology, that is, $C \cap M_k$ is closed for all k. $q^{-1}(C) \cap K_k = q^{-1}(C \cap M_k) \cap K_k = q_k^{-1}(C \cap M_k)$ which is closed for all k, thus C is a closed subset in the quotient topology.

□

Suppose that X is a real projective variety and Y is a real subvariety of X. Denote

$$Z_r(X,Y)^{av} = \frac{Z_r(X)^{av}}{Z_r(Y)^{av}},$$

$$Z_r(X,Y)_{\mathbb{R}} = \frac{Z_r(X)_{\mathbb{R}}}{Z_r(Y)_{\mathbb{R}}}.$$

Corollary 2.1.6. *Suppose that X is a real projective variety and Y is a real subvariety of X. Consider the canonical real and averaged filtrations of X:*

$$K_{r,1}(X)_{\mathbb{R}} \subset K_{r,2}(X)_{\mathbb{R}} \subset \cdots = Z_r(X)_{\mathbb{R}}$$

$$K_{r,1}(X)^{av} \subset K_{r,2}(X)^{av} \subset \cdots = Z_r(X)^{av}$$

and the quotient maps $q_1 : Z_r(X)_R \longrightarrow R_r(X)$, $q_2 : Z_r(X)_{\mathbb{R}} \longrightarrow Z_r(X,Y)_{\mathbb{R}}$, $q_3 : Z_r(X)^{av} \longrightarrow Z_r(X,Y)^{av}$. Then

1. *for all k, $q_1(K_{r,k}(X)_{\mathbb{R}})$, $q_2(K_{r,k}(X)_{\mathbb{R}})$ and $q_3(K_{r,k}(X)^{av})$ are subspaces of $q_1(K_{r,k+1}(X)_{\mathbb{R}})$, $q_2(K_{r,k+1}(X)_{\mathbb{R}})$ and $q_3(K_{r,k+1}(X)^{av})$ respectively.*

2. *the weak topology of $R_r(X), Z_r(X,Y)_{\mathbb{R}}, Z_r(X,Y)^{av}$ induced from the filtrations above coincides with the quotient topology on them.*

Proposition 2.1.7. *For a real projective variety X, let ReX denote the set of real points of X and let $Z_0(ReX)$ denote the subgroup of $Z_0(X)$ generated by real points of X. Then*

1. *$Z_0(ReX)$ is a closed subgroup of $Z_0(X)_{\mathbb{R}}$.*

2. *$R_0(X)$ is isomorphic as a topological group to $\frac{Z_0(ReX)}{2Z_0(ReX)}$.*

Proof. 1. Since X is compact, the quotient map from $X \times \cdots \times X$(k-times) to the symmetric product $SP^k(X)$ is a closed map. $ReX \times \cdots \times ReX$

is a closed subset of $X \times \cdots \times X$ thus $SP^k(ReX)$ is a closed subset of $SP^k(X)$. Consider the following filtrations:

$$A_1 \subset A_2 \subset \cdots = Z_0(ReX)$$

where $A_n = \coprod_{k=1}^n SP^k(ReX) \times SP^k(ReX)/\sim$ and

$$B_1 \subset B_2 \subset \cdots = Z_0(X)$$

where $B_n = \coprod_{k=1}^n SP^k(X) \times SP^k(X)/\sim$ and $\sim$ is the equivalence relation from the group completion. The topology of $Z_0(ReX)$ and $Z_0(X)$ are defined by these two filtrations respectively. Since $SP^k(ReX)$ and $SP^k(ReX)$ are compact, A_n is closed in B_n. Observe that $Z_0(ReX) \cap B_n = A_n$, thus $Z_0(ReX)$ is a closed subgroup of $Z_0(X)$ and hence a closed subgroup of $Z_0(X)_{\mathbb{R}}$.

2. Let $q : Z_0(X)_{\mathbb{R}} \longrightarrow R_0(X)$ be the quotient map and $i : Z_0(ReX) \hookrightarrow Z_0(X)_{\mathbb{R}}$ be the inclusion map. Then $q \circ i : Z_0(ReX) \longrightarrow R_0(X)$ is a continuous map and $2Z_0(ReX)$ is in the kernel, thus we get a continuous map $\psi : \frac{Z_0(ReX)}{2Z_0(ReX)} \longrightarrow R_0(X)$. Since each class in $R_0(X)$ can be

represented uniquely by cycle of real points modulo 2, ψ is bijective. Let

$$K_1 \subset K_2 \subset \cdots = Z_0(X)_{\mathbb{R}}$$

be the filtration which defines the topology of $Z_0(X)_{\mathbb{R}}$ and $T_i = K_i \cap Z_0(ReX)$. $Z_0(ReX)$ is a closed subset of $Z_0(X)_{\mathbb{R}}$ and K_i are compact, thus T_i are compact for all i. Since ψ is bijective, $\psi(T_i + 2Z_0(ReX)) = K_i + Z_0(X)^{av}$ for each $i > 0$. For a closed $C \subset \frac{Z_0(ReX)}{2Z_0(ReX)}$, $\psi(C) \cap (K_i + Z_0(X)^{av}) = \psi(C \cap T_i + 2Z_0(ReX))$ which is closed, thus ψ^{-1} is continuous. Then it is easy to see that ψ is a topological group isomorphism.

□

Definition For any $f \in Z_p(X)$, let $f = \sum_{i \in I} n_i V_i$ be in the reduced form. Let

$$RP(f) = \sum_{i \in I, \overline{V_i} = V_i} n_i V_i$$

which is called the real part of f. Let

$$J = \{i \in I | V_i \text{ is not real and } \overline{V_i} \text{ is also a component of } f\}$$

and for $i \in J$, let m_i be the maximum value of the coefficients of V_i and $\overline{V_i}$.

Define the averaged part to be

$$AP(f) = \sum_{i \in J} m_i(V_i + \overline{V}_i)$$

and the imaginary part to be

$$IP(f) = f - RP(f) - AP(f).$$

Then $f \in Z_p(X)_{\mathbb{R}}$ if and only if $IP(f) = 0$.

Proposition 2.1.8. *Suppose that X, Y, Z are real projective varieties and Y is a subvariety of Z, then*

1. *the inclusion map $i : Z_r(Y)(X) \longrightarrow Z_r(Z)(X)$ induces a closed embedding $\bar{i} : R_r(Y)(X) \hookrightarrow R_r(Z)(X)$ for $r \geq 0$.*
2. *the inclusion map $i : Z_r(Y)^{av} \longrightarrow Z_r(Z)^{av}$ induces a closed embedding $\bar{i} : Z_r(Z, Y)^{av} \hookrightarrow Z_r(Z, Y)_{\mathbb{R}}$ for $r \geq 0$.*

Proof. 1. Since the inclusion $Z_r(Y)(X) \overset{i}{\hookrightarrow} Z_r(Z)(X)$ is an embedding, the restriction

$$i : Z_r(Y)(X)_{\mathbb{R}} \hookrightarrow Z_r(Z)(X)_{\mathbb{R}}$$

is also an embedding. Since $Z_r(Y)(X)^{av} \subset Z_r(Z)(X)^{av}$, i induces a map

$$\bar{i} : R_r(Y)(X) \longrightarrow R_r(Z)(X).$$

If $f \in Z_r(Z)(X)^{av} \cap Z_r(Y)(X)_{\mathbb{R}}$, then $RP(f) = 2g$ for some $g \in Z_r(Y)(X)_{\mathbb{R}}$. Thus $f = 2g + AP(f) \in Z_r(Y)(X)^{av}$ and therefore $\bar{i}$ is injective.

Let $A_1 \subset A_2 \subset \cdots = Z_r(Y)(X)$, $B_1 \subset B_2 \subset \cdots = Z_r(Z)(X)$ be the canonical r-filtrations. Let q_1, q_2 be the quotient maps from $Z_r(Y)(X)_{\mathbb{R}}$, $Z_r(Z)(X)_{\mathbb{R}}$ to $R_r(Y)(X)$ and $R_r(Z)(X)$ respectively. Let $C_k = q_1(A_k)$, $D_k = q_2(B_k)$ for all k. Since $i(A_k) \subset B_k$, $\bar{i}(C_k) \subset D_k$ and from the definition of canonical filtrations, $\bar{i}^{-1}(D_k) \subset C_k$. For any closed subset W of $R_r(Y)(X)$, $W \cap C_k$ is compact and by the injectivity of $\bar{i}$, $\bar{i}(W \cap C_k) = \bar{i}(W) \cap \bar{i}(C_k) = \bar{i}(W) \cap D_k$ which is compact and thus closed. By Corollary 2.1.6, $\bar{i}^{-1}$ is continuous. Take $W = R_r(Y)(X)$, we see that $\bar{i}(R_r(Y)(X))$ is closed in $R_r(Z)(X)$.

2. Since the inclusion $i : Z_r(Z)^{av} \hookrightarrow Z_r(Z)_{\mathbb{R}}$ is an embedding and $i(Z_r(Y)^{av}) \subset Z_r(Y)_{\mathbb{R}}$, it induces a map $\bar{i} : Z_r(Z, Y)^{av} \hookrightarrow Z_r(Z, Y)_{\mathbb{R}}$. If $\bar{i}(f + Z_r(Y)^{av}) = f + Z_r(Y)_{\mathbb{R}} = Z_r(Y)_{\mathbb{R}}$, then $f \in Z_r(Y)_{\mathbb{R}}$. This implies $RP(f) \in Z_r(Y)_{\mathbb{R}}$. But f is also an averaged cycle, so that means

$RP(f) = 2g$ for some $g \in Z_r(Y)_{\mathbb{R}}$ and therefore $f = 2g + AP(f) \in Z_r(Y)^{av}$. Hence, $\bar{i}$ is injective. For the rest, apply Corollary 2.1.6 and follow the argument above.

□

From this Proposition, when we need to take quotient, we will abuse of notation and write $\frac{R_r(Z)(X)}{R_r(Y)(X)}$ and $\frac{Z_r(Z,Y)_{\mathbb{R}}}{Z_r(Z,Y)^{av}}$ for the quotient of $R_r(Z)(X)$, $Z_r(Z,Y)_{\mathbb{R}}$ by the images of $\bar{i}$ in $R_r(Y)(X)$ and $Z_r(Z,Y)^{av}$ respectively.

2.2 Relative Theory And Long Exact Sequences

We will apply the Borel construction to define the reduced real Lawson homology and reduced real morphic cohomology for quasi projective varieties. Let us recall the definition of Borel construction and some properties that we are going to use later. For the construction of classifying spaces and some basic properties, we refer the reader to [Ben]. In the following, we assume that every topological group has the homotopy type of a CW-complex.

Definition Suppose that G is a topological group acting on a topological space X. The Borel construction is the orbit space $B(X,G) = (X \times EG)/G$ where EG is the universal bundle of G.

Proposition 2.2.1. *1. The projection $p : B(G, X) \longrightarrow BG$ is a fibration with fibre X where BG is the classifying space of G.*

$$\begin{array}{ccc} X & \longrightarrow & B(X, G) \\ & & \big\downarrow p \\ & & BG \end{array}$$

2. Suppose that $\phi : H \longrightarrow G$ is a topological group homomorphism with closed image. Then the projection $p : B(G, H) \longrightarrow G/H$ is a fibration with fibre $B(Ker\phi) = EH/Ker\phi$.

$$\begin{array}{ccc} EH/Ker\phi & \longrightarrow & B(G, H) \\ & & \big\downarrow p \\ & & G/H \end{array}$$

In particular, when ϕ is injective, $B(G, H)$ is homotopy equivalent to G/H.

3. Suppose that we have a commutative diagram of topological group homomorphisms:

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi_1} & G_1 \\ \big\downarrow \psi_1 & & \big\downarrow \psi_2 \\ H_2 & \xrightarrow{\phi_2} & G_2 \end{array}$$

Then ψ_1, ψ_2 induce a map $B(\psi_2, \psi_1) : B(G_1, H_1) \longrightarrow B(G_2, H_2)$. When

ψ_1, ψ_2 are homotopy equivalences, $B(\psi_2, \psi_1)$ is also a homotopy equivalence.

4. *Suppose that $F : I \times H_1 \longrightarrow H_2$, $F' : I \times G_1 \longrightarrow G_2$ are homotopies between $F_0 = \psi_1, F_1 = \psi_1'$ and $F_0' = \psi_2, F_1' = \psi_2'$ respectively where each F_t, F_t' are group homomorphisms. If the following diagram commutes:*

$$\begin{array}{ccc} H_1 & \xrightarrow{\phi_1} & G_1 \\ \downarrow{\scriptstyle F_t} & & \downarrow{\scriptstyle F_t'} \\ H_2 & \xrightarrow{\phi_2} & G_2 \end{array}$$

for each $t \in I$, then $B(\psi_2, \psi_1)$ is homotopic to $B(\psi_2', \psi_1')$.

Proof. (1) is a basic property of Borel construction. For (2), it is not difficult to see that the fibre is $EH/ker\phi$. If $ker\phi$ is a point, since EH is contractible, from the induced long exact sequence of that fibration and Whitehead Theorem, we get that $B(G, H)$ is homotopy equivalent to G/H. For (3), consider the long exact sequences on homotopy groups induced from the two fibrations

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_n(G_1) & \longrightarrow & \pi_n(B(G_1, H_1)) & \longrightarrow & \pi_n(BH_1) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \psi_{2*}} & & \downarrow{\scriptstyle B(\psi_2,\psi_1)_*} & & \downarrow{\scriptstyle B\psi_{1*}} & & \\ \cdots & \longrightarrow & \pi_n(G_2) & \longrightarrow & \pi_n(B(G_2, H_2)) & \longrightarrow & \pi_n(BH_2) & \longrightarrow & \cdots \end{array}$$

and then use the Whitehead Theorem which implies $B(\psi_2, \psi_1)$ is a homotopy equivalence. For (4), by (3) F_t, F_t' induce a map $B(F_t', F_t)$ and thus F and F'

induce a map $B(F', F) : I \times B(G_1, H_1) \longrightarrow B(G_2, H_2)$ which is a homotopy between $B(F'_0, F_0) = B(\psi_2, \psi_1)$ and $B(F'_1, F'_0) = B(\psi'_2, \psi'_1)$. □

Definition Suppose that X, Y are real projective varieties and the dimension of Y is n. We define the codimension t real cocycle group on X with values in Y to be

$$R^t(Y)(X) = R_{n-t}(Y)(X),$$

the codimension t reduced real cocycle group of X to be

$$R^t(X) = \frac{R_0(\mathbb{P}^t)(X)}{R_0(\mathbb{P}^{t-1})(X)}.$$

To define relative reduced real morphic cohomology, we need to show that $R^t(X)$ acts on $R^t(Y)$.

Proposition 2.2.2. *Suppose that X, Y are real projective varieties and Y is a subvariety of X. Then there is a topological group homomorphism*

$$\phi : R^t(X) \longrightarrow R^t(Y).$$

Proof. Consider the map

$$Z_0(\mathbb{P}^t)(X) \longrightarrow Z_0(\mathbb{P}^t)(Y)$$

defined by intersecting with $Y \times \mathbb{P}^t$ in $X \times \mathbb{P}^t$. Check the dimension we see that all cycles in $Z_0(\mathbb{P}^t)(X)$ intersect $Y \times \mathbb{P}^t$ properly and thus the map is continuous. We have a commutative diagram:

$$\begin{array}{ccccc} Z_0(\mathbb{P}^t)(X)^{av} & \longrightarrow & Z_0(\mathbb{P}^t)(X)_{\mathbb{R}} & \longrightarrow & R_0(\mathbb{P}^t)(X) \\ \downarrow & & \downarrow & & \\ Z_0(\mathbb{P}^t)(Y)^{av} & \longrightarrow & Z_0(\mathbb{P}^t)(Y)_{\mathbb{R}} & \longrightarrow & R_0(\mathbb{P}^t)(Y) \end{array}$$

where the maps in the middle are restriction of the map defined by intersecting with $Y \times \mathbb{P}^t$. Thus it induces a map from $R_0(\mathbb{P}^t)(X)$ to $R_0(\mathbb{P}^t)(Y)$. And from the following commutative diagram:

$$\begin{array}{ccccc} R_0(\mathbb{P}^{t-1})(X) & \longrightarrow & R_0(\mathbb{P}^t)(X) & \longrightarrow & R^t(X) \\ \downarrow & & \downarrow & & \\ R_0(\mathbb{P}^{t-1})(Y) & \longrightarrow & R_0(\mathbb{P}^t)(Y) & \longrightarrow & R^t(Y) \end{array}$$

we see that it induces a map from $R^t(X)$ to $R^t(Y)$. □

Definition Suppose now Y is a subvariety of X. Define the codimension t

relative reduced real cocycle group to be

$$R^t(X|Y) = B(R^t(Y), R^t(X))$$

by the map from Proposition 2.2.2 and define the dimension p relative reduced real cycle group to be

$$R_p(X|Y) = \frac{R_p(X)}{R_p(Y)}.$$

We say that a quasi projective variety U is real if there exist real projective varieties X and Y , $Y \subset X$ such that $U = X - Y$. We define the p-th reduced real cycle group of U to be $R_p(U) \doteq R_p(X|Y) = \frac{R_p(X)}{R_p(Y)}$. We will show later that for another pair of real projective varieties X', Y' where $Y' \subset X'$ and $U = X' - Y'$, $\frac{R_p(X)}{R_p(Y)}$ is homeomorphic to $\frac{R_p(X')}{R_p(Y')}$.

Proposition 2.2.3. *1. $R^t(X)$ is homotopy equivalent to $B(R_0(\mathbb{P}^t)(X), R_0(\mathbb{P}^{t-1})(X))$*

2. $R_p(X|Y)$ is homotopy equivalent to $B(R_p(X), R_p(Y))$.

Proof. By Proposition 2.1.8, $R_0(\mathbb{P}^{t-1})(X) \hookrightarrow R_0(\mathbb{P}^t)(X)$ and $R_p(Y) \hookrightarrow R_p(X)$ are injective. Thus by 2.2.1, $R^t(X)$ is homotopy equivalent to $B(R_0(\mathbb{P}^t)(X), R_0(\mathbb{P}^{t-1})(X))$ and $R_p(X|Y)$ is homotopy equivalent to $B(R_p(X), R_p(Y))$. □

Now we are ready to define reduced real Lawson homology groups and reduced real morphic cohomology groups.

Definition We define the p dimensional reduced real Lawson homology group of a real projective variety U to be

$$RL_pH_n(U) = \pi_{n-p}(R_p(U)),$$

the codimension t reduced real morphic cohomology group of a real projective variety X to be

$$RL^tH^k(X) = \pi_{t-k}(R^t(X)) = \pi_{t-k}(B(R_0(\mathbb{P}^t)(X), R_0(\mathbb{P}^{t-1}(X)))),$$

the reduced real bivariant Y-valued morphic cohomology to be

$$RL^tH^k(X;Y) = \pi_{t-k}R^t(Y)(X).$$

Suppose now X, Y are real projective varieties and $Y \subset X$. We define the relative reduced real morphic cohomology group to be

$$RL^tH^k(X|Y) = \pi_{t-k}(R^t(X|Y)) = \pi_{t-k}(B(R^t(Y), R^t(X)))$$

and the relative reduced real Lawson homology group to be

$$RL_pH_n(X|Y) = \pi_{n-p}(R_p(X|Y)) = \pi_{n-p}(B(R_p(X), R_p(Y))).$$

Theorem 2.2.4. *(Long exact sequences) Suppose that X, Y are real projective varieties and $Y \subset X$. Then we have the long exact sequence in reduced real Lawson homology:*

$$\cdots \longrightarrow RL_pH_n(Y) \longrightarrow RL_pH_n(X) \longrightarrow RL_pH_n(X|Y) \longrightarrow RL_pH_{n-1}(Y) \longrightarrow \cdots$$

and the long exact sequence in reduced real morphic cohomology:

$$\cdots \longrightarrow RL^tH^k(Y) \longrightarrow RL^tH^k(X|Y) \longrightarrow RL^tH^{k+1}(X) \longrightarrow RL^tH^{k+1}(Y) \longrightarrow \cdots.$$

Proof. By Borel construction, we have two fibrations

$$\begin{array}{ccc} R_p(X) & \longrightarrow & B(R_p(X), R_p(Y)) \\ & & \downarrow \\ & & B(R_p(Y)) \end{array}$$

and

$$\begin{array}{ccc} R^t(Y) & \longrightarrow & B(R^t(Y), R^t(X)) \\ & & \downarrow \\ & & B(R^t(X)) \end{array}$$

Thus they induce long exact sequences on homotopy groups and we recall that for a topological group G, $\pi_k(BG) = \pi_{k-1}(G)$. $\square$

Proposition 2.2.5. *Suppose that $X_3 \subset X_2 \subset X_1$ are real projective varieties.*

There is a long exact sequence in reduced real Lawson homology:

$$\cdots RL_pH_n(X_2|X_3) \longrightarrow RL_pH_n(X_1|X_3) \longrightarrow RL_pH_n(X_1|X_2) \longrightarrow RL_pH_{n-1}(X_2|X_3) \longrightarrow \cdots$$

Proof. It is easy to see that $R_p(X_2|X_3)$ acts on $R_p(X_1|X_3)$. By Borel construction, we have a fibration

$$\begin{array}{ccc} R_p(X_1|X_3) & \longrightarrow & B(R_p(X_1|X_3), R_p(X_2|X_3)) \\ & & \downarrow \\ & & B(R_p(X_2|X_3)) \end{array}$$

which induces a long exact sequence on homotopy groups:

$$\cdots RL_pH_n(X_2|X_3) \longrightarrow RL_pH_n(X_1|X_3) \longrightarrow RL_pH_n(X_1|X_2) \longrightarrow RL_pH_{n-1}(X_2|X_3) \longrightarrow \cdots$$

□

Corollary 2.2.6. *Suppose that U, V are real projective varieties and $V \subset U$ is closed. Then there is a (localization) long exact sequence in reduced real Lawson homology:*

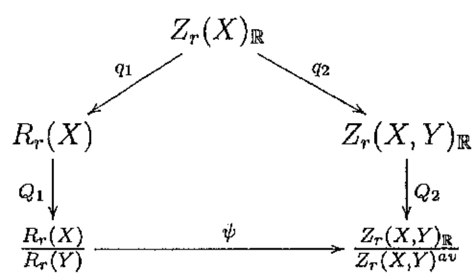
$$\cdots RL_pH_n(V) \longrightarrow RL_pH_n(U) \longrightarrow RL_pH_n(U-V) \longrightarrow RL_pH_{n-1}(V) \longrightarrow \cdots$$

Proof. Let $U = X_1 - X_3$ where X_1, X_3 are real projective varieties and take the closure $\overline{V}$ of V in X_1. Let $X_2 = \overline{V} \cup X_3$. Then $X_2 - X_3 = V$, $X_1 - X_2 = U - V$ and $X_3 \subset X_2 \subset X_1$. By the Proposition above, we get the long exact sequence. □

We need to show that the definition of reduced real cycle groups for real quasi projective varieties is independent of choices of its compactification.

Proposition 2.2.7. *Suppose that X, Y are real projective varieties and Y is a subvariety of X. Then $\frac{R_r(X)}{R_r(Y)}$ is homeomorphic to $\frac{Z_r(X,Y)_{\mathbb{R}}}{Z_r(X,Y)^{av}}$.*

Proof. Let Q_1, Q_2, q_1, q_2 be the quotient maps and $\psi(x + Z_r(X)^{av} + R_r(Y)) = x + Z_r(Y)_{\mathbb{R}} + Z_r(X,Y)^{av}$ as following:



The diagram commutes and since those quotient maps are open maps, ψ is continuous. Easy to see that ψ is bijective and the inverse of ψ is also continuous. □

Definition Suppose that X, Y, X', Y' are real projective varieties and $Y \subset X$,

$Y' \subset X'$. (X,Y) is said to be relatively isomorphic to (X',Y') if there is a real regular map $f : X \longrightarrow X'$ such that f induces an isomorphism as quasiprojective varieties between $X - Y$ and $X' - Y'$. f is called a real relative isomorphism.

Following Lima-Filho's approach in [Li1], we can show that

Proposition 2.2.8. *Suppose that $f : (X,Y) \longrightarrow (X',Y')$ is a real relative isomorphism. Then f induces a topological group isomorphism between*

1. $Z_r(X,Y)_{\mathbb{R}}$ *and* $Z_r(X',Y')_{\mathbb{R}}$
2. $Z_r(X,Y)^{av}$ *and* $Z_r(X',Y')^{av}$

Corollary 2.2.9. *Suppose that $f : (X,Y) \longrightarrow (X',Y')$ is a real relative isomorphism, then f induces a homotopy equivalence between $\frac{R_r(X)}{R_r(Y)}$ and $\frac{R_r(X')}{R_r(Y')}$.*

Proof. By Proposition 2.2.8 and Proposition 2.2.1, $B(Z_r(X,Y)_{\mathbb{R}}, Z_r(X,Y)^{av})$ is homotopy equivalent to $B(Z_r(X',Y')_{\mathbb{R}}, Z_r(X',Y')^{av})$. Thus $\frac{Z_r(X,Y)_{\mathbb{R}}}{Z_r(X,Y)^{av}}$ is homotopy equivalent to $\frac{Z_r(X',Y')_{\mathbb{R}}}{Z_r(X',Y')^{av}}$. From Proposition 2.2.7, we see that $\frac{R_r(X)}{R_r(Y)}$ and $\frac{R_r(X')}{R_r(Y')}$ are homotopy equivalent. □

Thus the reduced real cycle groups of a real quasi projective variety are well defined.

Chapter 3

Functoriality And Fundamental Properties Of Reduced Real Morphic Cohomology

3.1 Functoriality

The following proposition is an analogue of the functoriality of morphic cohomology which are established in [FL1].

Proposition 3.1.1. *Suppose that X, X', Y, Y' are real projective varieties and $f : X' \longrightarrow X$, $g : Y \longrightarrow Y'$ are morphisms of real projective varieties.*

(a) The "pullback of reduced real cocycles" determines a homomorphism

$$f^* : RL^s H^q(X;Y) \longrightarrow RL^s H^q(X';Y).$$

(b) The "pushforward of reduced real cycles" determines a homomorphism

$$g_* : RL^s H^q(X;Y) \longrightarrow RL^{s-c} H^{q-c}(X;Y')$$

where $c = dimY - dimY'$.

(c) Given morphisms $f_2 : X' \longrightarrow X$, $f_1 : X'' \longrightarrow X'$ *and* $g_1 : Y \longrightarrow Y'$, $g_2 : Y' \longrightarrow Y''$ *where* X, X', X'', Y, Y', Y'' *are all real projective varieties, then*

$$(f_2 \circ f_1)^* = (f_1)^* \circ (f_2)^* \text{ and } (g_2 \circ g_1)_* = (g_2)_* \circ (g_1)_*.$$

Proof. (a) $f^* : Z^s(Y)(X) \longrightarrow Z^s(Y)(X')$ is a continuous map which is proved in [FL1, Proposition 2.4]. So we only need to verify that f^* maps real cycles to real cycles and averaged cycles to averaged cycles. Since f is real morphism, its graph $gr(f)$ in $X' \times X$ is a real subvariety. Consider $gr(f) \times Y$ in $X' \times X \times Y$ and for any $V \in Z^s(Y)(X)$, $X' \times V \subset X' \times X \times Y$. It is easy to check that $gr(f) \times Y$ intersects $X' \times V$ properly. Let $pr : X' \times X \times Y \longrightarrow X' \times Y$ be the projection. Then $f^*V = pr_*((gr(f) \times Y) \bullet (X' \times V)) \in Z^s(Y)(X')$ where pr_* is the push-forward map of cycles induced from pr. Since intersection preserves reality of cycles, and pr is a real morphisms, f^* sends real cycles to real cycles and averaged cycles to averaged cycles. So it induces a map from $R^s(Y)(X)$ to $R^s(Y)(X')$ and therefore a group homomorphism in homotopy groups.

(b) Similar to the approach in (a).

(c) Since $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$ for Y-valued cocycle groups as proved in [FL1, Proposition 2.4], from (a), this relation passes to the quotient. Thus $(f_2 \circ f_1)^* = f_1^* \circ f_2^*$ in reduced Y-valued real cycle groups and reduced real cycle groups. A similar argument works for g_1, g_2. □

Proposition 3.1.2. *Let $f : X \longrightarrow X'$, $g : Y' \longrightarrow Y$ be morphisms of real projective varieties.*

(a) If f has equidimensional fibers, then there are Gysin homomorphisms

$$f_! : RL^sH^q(X;Y) \longrightarrow RL^{s-c}H^{q-c}(X';Y)$$

for all s, q with $s \geq q \geq c$ where $c = dim(X) - dim(X')$.

(b) If g is flat, then for all s, q with $s \geq q$, there are Gysin homomorphisms

$$g^! : RL^sH^q(X;Y) \longrightarrow RL^sH^q(X;Y').$$

(c). If $f_1 : X \longrightarrow X'$ and $f_2 : X' \longrightarrow X''$ are as in part (a), or if $g_2 : Y' \longrightarrow Y$ and $g_1 : Y'' \longrightarrow Y'$ are as in part (b), then

$$(f_2 \circ f_1)_! = (f_2)_! \circ (f_1)_! \text{ and } (g_2 \circ g_1)^! = (g_1)^! \circ (g_2)^!.$$

Proof. Since a similar result is established in [FL1, Proposition 2.5] for morphic cohomology, we only need to verify that the maps send real cycles to real cycles and averaged cycles to averaged cycles. □

3.2 Fundamental Properties

3.2.1 Homotopy Invariance

Proposition 3.2.1. *Let X, X' and Y be real projective varieties. Suppose that $F : I \times X \longrightarrow X'$ is a continuous map where I is the unit interval and for each $t \in I$, F_t is a real algebraic morphism. If $F_0(x) = f(x), F_1(x) = g(x)$ for all $x \in X$, then $f^* = g^* : RL^*H^*(X'; Y) \longrightarrow RL^*H^*(X; Y)$ and they induce the same map $f^* = g^* : RL^*H^*(X') \longrightarrow RL^*H^*(X)$.*

Proof. Consider the map

$$F^* : Z_r(Y)(X') \times \mathbb{C} \longrightarrow Z_r(Y)(X)$$

which sends (σ, t) to $f_t^*\sigma$. This map is continuous and is a homotopy between f and g. Since each F_t is a real morphism, by Proposition 3.1.1 F_t^* passes to the quotient and thus F induces a map $F^* : R_r(Y)(X') \times \mathbb{C} \longrightarrow R_r(Y)(X)$ which is a homotopy between $F_0^* = f^*$ and $F_1^* = g^*$ so they induce the same map from $RL^*H^*(X'; Y)$ to $RL^*H^*(X; Y)$. F^* induces a homotopy $R_0(\mathbb{P}^{t-1})(X') \times \mathbb{C} \longrightarrow$

$R_0(\mathbb{P}^{t-1})(X)$ and a homotopy $R_0(\mathbb{P}^t)(X') \times \mathbb{C} \longrightarrow R_0(\mathbb{P}^t)(X)$ between the two maps induced by f and g. By Proposition 2.2.1, their induced maps from $R^t(X')$ to $R^t(X)$ are homotopic. $\square$

3.2.2 The Splitting Principle

Proposition 3.2.2. *$\mathbb{RP}^d$ is homeomorphic to $SP^d(\mathbb{P}^1)_{\mathbb{R}}$ where $SP^d(\mathbb{P}^1)_{\mathbb{R}}$ is the subset of $SP^d(\mathbb{P}^1)$ consisting of conjugate invariant zero cycles.*

Proof. A proof of $\mathbb{P}^d \cong SP^d(\mathbb{P}^1)$ can be found in [L2]. The main idea is sketched in the following. For $p = \{p_1, ..., p_d\} \in SP^d(\mathbb{P}^1)$, let $p_i = [-b_i : a_i] \in \mathbb{P}^1$ and we associate p a homogeneous polynomial

$$P(x,y) = \prod_{i=1}^{d}(a_i x + b_i y) = \sum_{k=0}^{d} c_k x^k y^{d-k}$$

where

$$c_k = \sum_{|I|=k} a_I b_{I'}$$

and the sum is taken over all multi-indices $I = \{0 \leq i_1 < \cdots < i_k \leq d\}$ of length $|I| = k$ and I' is the complementary multi-index with $|I'| = d - k$. The map $\psi : SP^d(\mathbb{P}^1) \longrightarrow \mathbb{P}^d$ which maps p to the point $[c_0 : \cdots : c_d] \in \mathbb{P}^d$ is an isomorphism. To prove our statement, we observe that if p is conjugation invariant, i.e., $\bar{p} = \{\bar{p_1}, \cdots, \bar{p_d}\} = p$,

$$\sum_{k=0}^{d} \bar{c_k} x^k y^{d-k} = \prod_{i=1}^{d} (\bar{a_i} x + \bar{b_i} y) = \prod_{i=1}^{d} (a_i x + b_i y) = \sum_{k=0}^{d} c_k x^k y^{d-k},$$

thus $\bar{c_k} = c_k$ and therefore $[c_0 : \cdots : c_d] \in \mathbb{RP}^d$. For the inverse, a point $[c_0 : \cdots : c_d] \in \mathbb{RP}^d$ is sent to a point $p = \{p_1, ..., p_d\}$ by taking the roots $p_1, ..., p_d$ of the homogeneous polynomial

$$P(x, y) = \sum_{k=0}^{d} c_k x^k y^{d-k}.$$

Since c_k are real, a nonreal root and its conjugate are both roots of $P(x, y)$, thus p is conjugate invariant. $\square$

From section 2.10 of [FL1], for $1 \leq t \leq s$, there is a monoid homomorphism

$$SP(\mathbb{P}^s) \xrightarrow{p} SP(\mathbb{P}^t)$$

which is induced by the morphism

$$p : \mathbb{P}^s = SP^s(\mathbb{P}^1) \longrightarrow SP^{\binom{s}{t}} SP^t(\mathbb{P}^1)$$

defined by

$$\{x_1, ..., x_s\} \longmapsto \sum_{|I|=t} \{x_{i_1}, ..., x_{i_t}\}$$

where we identify $\mathbb{P}^s$ with $SP^s(\mathbb{P}^1)$ and then extend by linearity and compose with the natural map $SP(SP^{\binom{s}{t}}(\mathbb{P}^t)) \longrightarrow SP(\mathbb{P}^t)$. From here we see that $p(\bar{x_1}, ..., \bar{x_s}) = \overline{p(x_1, ..., x_s)}$ which means p is real.

Extend the map p by linearity to a group homomorphism

$$p : Z_0(\mathbb{P}^s) \longrightarrow Z_0(\mathbb{P}^t).$$

For a projective variety X, we get a map, by abuse of notation,

$$p : Z_0(\mathbb{P}^s)(X) \longrightarrow Z_0(\mathbb{P}^t)(X)$$

by composing with p. If X is a real projective variety, since p is real, p reduces to a map $p' : R_0(\mathbb{P}^s)(X) \longrightarrow R_0(\mathbb{P}^t)(X)$ where $1 \leq t \leq s$.

The inclusion map $i : \mathbb{P}^t \longrightarrow \mathbb{P}^s$ induces an inclusion map $i' : R_0(\mathbb{P}^t)(X) \longrightarrow R_0(\mathbb{P}^s)(X)$ by Proposition 2.1.8.

Lemma 3.2.3. *The composition* $\varphi' = p' \circ i' : R_0(\mathbb{P}^t)(X) \longrightarrow R_0(\mathbb{P}^t)(X)$ *is of the form*

$$\varphi' = Id + \psi'$$

where image $\psi' \subset R_0(\mathbb{P}^{t-1})(X)$.

Proof. $p' \circ i'(f + Z_0(\mathbb{P}^t)(X)^{av}) = p \circ i(f) + Z_0(\mathbb{P}^t)(X)^{av}$. By Lemma 2.1 in [FL1], $p \circ i(f) = (Id + \psi)(f)$ where image of ψ is contained in $Z_0(\mathbb{P}^{t-1})(X)$ and ψ is a real map, thus ψ maps $Z_0(\mathbb{P}^t)(X)_{\mathbb{R}}$ to $Z_0(\mathbb{P}^{t-1})(X)_{\mathbb{R}}$ and therefore $p' \circ i' = (Id + \psi')$ where the image of ψ' is contained in $R_0(\mathbb{P}^{t-1})(X)$. □

It is easy to see that $\psi : R^t(X) = B(R_0(\mathbb{P}^t)(X), R_0(\mathbb{P}^{t-1})(X)) \longrightarrow R^t(X)$ is homotopic to zero and by Proposition 2.13 in [FL1] with $M^t = R_0(\mathbb{P}^t)(X)$, we get the splitting principle for our reduced real theory.

Theorem 3.2.4. *(Splitting Principle) For a real projective variety X, we have a homotopy equivalence:*

$$R_0(\mathbb{P}^s)(X) \cong R^0(X) \times R^1(X) \times \cdots \times R^s(X).$$

3.2.3 The Lawson Suspension Theorem

Suppose that $X \subset \mathbb{P}^n$ are projective varieties. Let $x_\infty \in \mathbb{P}^0$. The suspension of X, $\Sigma\!\!\!/ X \subset \mathbb{P}^{n+1}$, is the complex cone over X, or equivalently, the Thom space of the hyperplane bundle $\mathcal{O}(1)$ in $\mathbb{P}^n$ restricted to X. A point in $\Sigma\!\!\!/ X$ can be written as $[t : x]$ where $t \in \mathbb{C}, x \in X$. We consider X as a subvariety of $\Sigma\!\!\!/ X$ by identifying X with the zero section of $\Sigma\!\!\!/ X$.

Proposition 3.2.5. *Suppose that $\varphi : X \longrightarrow Y$ is a morphism of projective varieties where $X \subset \mathbb{P}^n, Y \subset \mathbb{P}^m$. Then the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow & & \downarrow \\ \Sigma\!\!\!\!/\, X & \xrightarrow{\widetilde{\Sigma\!\!\!\!/}\varphi} & \Sigma\!\!\!\!/\, Y \end{array}$$

where $(\widetilde{\Sigma\!\!\!\!/}\varphi)([t : x]) = [t : \varphi(x)]$ for all $t \in C, x \in X$.

Proof. The maps in vertical arrows are inclusions and it is easy to check that the diagram commutes. □

The above commutativity induces commutativity in cycle groups.

Corollary 3.2.6. *1. Suppose that $\varphi : X \longrightarrow Y$ is a morphism of projective varieties. Then the following diagram commutes:*

$$\begin{array}{ccc} Z_r(X) & \xrightarrow{\varphi} & Z_r(Y) \\ \downarrow & & \downarrow \\ Z_r(\Sigma\!\!\!\!/\, X) & \xrightarrow{\widetilde{\Sigma\!\!\!\!/}\varphi} & Z_r(\Sigma\!\!\!\!/\, Y) \end{array} .$$

2. Suppose that $\varphi : X \longrightarrow Y$ is a morphism of real projective varieties.

Then the following diagram commutes:

$$\begin{array}{ccc} R_r(X) & \xrightarrow{\varphi} & R_r(Y) \\ \downarrow & & \downarrow \\ R_r(\Sigma\!\!\!\!/\, X) & \xrightarrow{\tilde{\Sigma\!\!\!\!/}\varphi} & R_r(\Sigma\!\!\!\!/\, Y) \end{array}$$

For $f \in Z_r(Y)(X)$, we consider f as a map from X to $Z_r(Y)$. Define the suspension map $\Sigma\!\!\!\!/_* : Z_r(Y)(X) \longrightarrow Z_{r+1}(\Sigma\!\!\!\!/\, Y)(X)$ by $(\Sigma\!\!\!\!/_* f)(x) = \Sigma\!\!\!\!/\, f(x)$, the pointwise suspension. The suspension map is continuous and in [FL1, Theorem 3.3], the Lawson Suspension Theorem for bivariant morphic cohomology is proved.

Theorem 3.2.7. $\Sigma\!\!\!\!/_* : Z_r(Y)(X) \longrightarrow Z_{r+1}(\Sigma\!\!\!\!/\, Y)(X)$ *is a homotopy equivalence.*

Let us show that the analogue Lawson Suspension Theorem is also true for bivariant reduced real cycle groups and reduced real morphic cohomology groups. The Lawson Suspension Theorem for the real cycle groups and averaged groups of projective spaces can be found in [LLM]. The equivariant version of the Lawson Suspension Theorem can be found in [LLM2].

Theorem 3.2.8. *Suppose that X, Y are real projective varieties and $Y \subset \mathbb{P}^N$. Then*

1. $\Sigma\!\!\!\!/_* : Z_r(Y)(X)_{\mathbb{R}} \longrightarrow Z_{r+1}(\Sigma\!\!\!\!/\, Y)(X)_{\mathbb{R}}$ *is a homotopy equivalence.*

2. $\Sigma\!\!\!\!/_* : Z_r(Y)(X)^{av} \longrightarrow Z_{r+1}(\Sigma\!\!\!\!/\, Y)(X)^{av}$ *is a homotopy equivalence.*

3. $\not\Sigma_* : R_r(Y)(X) \longrightarrow R_{r+1}(\not\Sigma Y)(X)$ *is a homotopy equivalence.*

Proof. Let

$$T_{r+1}(\not\Sigma Y)(X) = \{f \in Z_{r+1}(\not\Sigma Y)(X) | f(x) \text{ meets } x \times Y \text{ properly for all } x \in X\},$$

$$T_{r+1}(\not\Sigma Y)(X)_{\mathbb{R}} = T_{r+1}(\not\Sigma Y)(X) \cap Z_{r+1}(\not\Sigma Y)(X)_{\mathbb{R}},$$

$$T_{r+1}(\not\Sigma Y)(X)^{av} = T_{r+1}(\not\Sigma Y)(X) \cap Z_{r+1}(\not\Sigma Y)(X)^{av}.$$

Following Proposition 3.2 in [F1], we define $\Lambda \subset \mathbb{P}^{N+1} \times \mathbb{P}^1 \times \mathbb{P}^{N+1}$ to be the graph of the rational map $\mathbb{P}^{N+1} \times \mathbb{P}^1 \longrightarrow \mathbb{P}^{N+1}$ whose restriction to $\mathbb{P}^{N+1} \times t$ for $t \in \mathbb{A}^1 - \{0\}$ is the linear automorphism $\Theta_t : \mathbb{P}^{N+1} \longrightarrow \mathbb{P}^{N+1}$ sending $[z_0 : \cdots : z_N : z_{N+1}]$ to $[z_0 : \cdots : z_N : \frac{1}{t} z_{N+1}]$. More explicitly, Λ is the closed subvariety given by the homogeneous equations:

$$\begin{cases} X_i Y_j - X_j Y_i = 0 \\ T X_{N+1} Y_j - S X_j Y_{N+1} = 0, \quad \text{for } 0 \le i, j \le N. \end{cases}$$

and $[X_0 : \cdots : X_{N+1}] \in \mathbb{P}^{N+1}, [S : T] \in \mathbb{P}^1, [Y_0 : \cdots : Y_{N+1}] \in \mathbb{P}^{N+1}$.

For $f \in T_{r+1}(\not\Sigma Y)(X)$, $t \in \mathbb{R} - \{0\} \subset \mathbb{P}^1$, define $\phi_t(f) = Pr_{1,4*}[(X \times \Lambda) \bullet (f \times t \times \mathbb{P}^{N+1})]$ where $Pr_{1,4} : X \times \mathbb{P}^{N+1} \times \mathbb{P}^1 \times \mathbb{P}^{N+1} \longrightarrow X \times \mathbb{P}^{N+1}$ is the projection and as the proof in [F1, Proposition 3.2], $\phi_t(f)(x)$ meets $x \times Y$ properly in $x \times \not\Sigma Y$ for all $x \in X$, hence ϕ_t is a real map and $\phi_t(f) \in$

$T_{r+1}(\Sigma\!\!\!/ Y)(X)_{\mathbb{R}}$. When $t=0$, check the equations defining Λ, we see that $\phi_0(f) = \Sigma\!\!\!/ (f \bullet (X \times Y)) \in \widetilde{\Sigma\!\!\!/} Z_r(Y)(X)_{\mathbb{R}}$.

For $t \in \mathbb{R}$, since ϕ_t is real, ϕ_t maps $T_{r+1}(\Sigma\!\!\!/ Y)(X)_{\mathbb{R}}$ to $T_{r+1}(\Sigma\!\!\!/ Y)(X)_{\mathbb{R}}$ and $T_{r+1}(\Sigma\!\!\!/ Y)(X)^{av}$ to $T_{r+1}(\Sigma\!\!\!/ Y)(X)^{av}$. Therefore, $\phi_0 : T_{r+1}(\Sigma\!\!\!/ Y)(X)_{\mathbb{R}} \longrightarrow \widetilde{\Sigma\!\!\!/} Z_r(Y)(X)_{\mathbb{R}}$ and $\phi_0 : T_{r+1}(\Sigma\!\!\!/ Y)(X)^{av} \longrightarrow \widetilde{\Sigma\!\!\!/} Z_r(Y)(X)^{av}$ are strong deformation retractions.

For a cycle $c = \sum n_i V_i$ in its reduced form, we define the degree of c to be $degc = \sum |n_i| degV_i$. Designate $Z_{r+1,e}(\Sigma\!\!\!/ Y)(X), T_{r+1,e}(\Sigma\!\!\!/ Y)(X)$ to be the subset of $Z_{r+1}(\Sigma\!\!\!/ Y)(X), T_{r+1}(\Sigma\!\!\!/ Y)(X)$ respectively consisting of cycles of degree $\leq e$ and denote $Z_{r+1,e}(\Sigma\!\!\!/ Y)(X)_{\mathbb{R}} = Z_{r+1,e}(\Sigma\!\!\!/ Y)(X) \cap Z_{r+1}(\Sigma\!\!\!/ Y)(X)_{\mathbb{R}}$, $T_{r+1,e}(\Sigma\!\!\!/ Y)(X)_{\mathbb{R}} = T_{r+1}(\Sigma\!\!\!/ Y)(X)_{\mathbb{R}} \cap T_{r+1,e}(\Sigma\!\!\!/ Y)(X)$, $Z_{r+1,e}(\Sigma\!\!\!/ Y)(X)^{av} = Z_{r+1,e}(\Sigma\!\!\!/ Y)(X) \cap Z_{r+1}(\Sigma\!\!\!/ Y)(X)^{av}$ and $T_{r+1,e}(\Sigma\!\!\!/ Y)(X)^{av} = T_{r+1}(\Sigma\!\!\!/ Y)(X)^{av} \cap T_{r+1,e}(\Sigma\!\!\!/ Y)(X)$.

The second step is to show that the inclusion map $T_{r+1}(\Sigma\!\!\!/ Y)(X)_{\mathbb{R}} \hookrightarrow Z_{r+1}(\Sigma\!\!\!/ Y)(X)_{\mathbb{R}}$ and the inclusion map $T_{r+1}(\Sigma\!\!\!/ Y)(X)^{av} \hookrightarrow Z_{r+1}(\Sigma\!\!\!/ Y)(X)^{av}$ induce homotopy equivalences. We follow the proof in [FL1, Theorem 3.3]. Take two real points $t_0 = [0 : \cdots : 1], t_1 = [0 : \cdots : 1 : 1] \in \mathbb{P}^{N+2} - \mathbb{P}^{N+1}$. Consider $\mathbb{P}^{N+2}$ as the algebraic suspension of $\mathbb{P}^{N+1}$ to the point t_0 and let $pr : \mathbb{P}^{N+2} - \{t_1\} \longrightarrow \mathbb{P}^{N+1}$ denote the linear projection away from t_1. Then consider the partially defined function

$$\Psi_e : Z_{r+1,d}(\Sigma\!\!\!/ Y)(X) \times Div_e \longrightarrow Z_{r+1,de}(\Sigma\!\!\!/ Y)(X)$$

given by

$$\Psi_e(f, D) = (Id \times pr)((\not{\Sigma} f) \bullet (X \times D))$$

where Div_e is the set of effective real divisors of degree e on $\mathbb{P}^{N+2}$.

Let

$$\triangle(f(x)) = \{D \in Div_e : (\not{\Sigma} f(x) \bullet D) \text{ does not meet } \{x\} \times H \text{ properly}\}$$

where $H \subset \mathbb{P}^{N+2}$ is the hyperplane containing $\mathbb{P}^N \cup \{t_1\}$.

By [L1, Lemma 5.11], one has

$$codim_{\mathbb{R}}(\triangle(f(x))) \geq \binom{p+e+1}{e}$$

where $p = dim f(x)$. In particular, $codim_{\mathbb{R}}(\triangle(f(x))) \longrightarrow \infty$ as $e \longrightarrow \infty$.

We now choose $e(d)$ so that $codim_{\mathbb{R}}(\triangle(f(x))) > dim X + 1$ for all $e > e(d)$. Then setting

$$\triangle(f) = \cup_{x \in X} \triangle\,(f(x)),$$

we have $codim_{\mathbb{R}}(\triangle(f)) > 1$ for all $e > e(d)$. Consequently, for each $e > e(d)$, there must exist a line $L_e \subset Div_e$ containing $e \cdot \mathbb{P}^{N+1}$ such that

$$(L_e - e \cdot \mathbb{P}^{N+1}) \cap \triangle(f) = \emptyset.$$

It follows immediately that Ψ_e restricted to $Z_{r+1,e}(\not\Sigma Y)(X) \times (L_e - e \cdot \mathbb{P}^{N+1})$ has image in $T_{r+1,de}(\not\Sigma Y)(X)$ and this map is homotopic to the map given by multiplication by e. Since Ψ_e is a real map, by restricting Ψ_e, we get two maps:

$$\Psi_e : Z_{r+1,d}(\not\Sigma Y)(X)_{\mathbb{R}} \times (L_e - e \cdot \mathbb{P}^{N+1}) \longrightarrow T_{r+1,de}(\not\Sigma Y)(X)_{\mathbb{R}},$$

$$\Psi_e : Z_{r+1,d}(\not\Sigma Y)(X)^{av} \times (L_e - e \cdot \mathbb{P}^{N+1}) \longrightarrow T_{r+1,de}(\not\Sigma Y)(X)^{av}$$

which are both homotopic to the map given by the multiplication by e. It then follows as in [F1, Theorem 4.2] that the inclusions induce homotopy equivalences.

Consider the following commutative diagram:

$$\begin{array}{ccc} Z_r(Y)(X)^{av} & \longrightarrow & Z_r(Y)(X)_{\mathbb{R}} \\ \downarrow \not\Sigma & & \downarrow \not\Sigma \\ Z_{r+1}(\not\Sigma Y)(X)^{av} & \longrightarrow & Z_{r+1}(\not\Sigma Y)(X)_{\mathbb{R}} \end{array}$$

where the vertical arrows are homotopy equivalences induced by the suspension map. By Proposition 2.2.1, there is a homotopy equivalence induced by $\not\Sigma$ from $R_r(Y)(X)$ to $R_{r+1}(\not\Sigma Y)(X)$. □

Take X to be a real point, we have the Lawson Suspension Theorem for reduced real Lawson homology.

Corollary 3.2.9. *$R_t(X)$ is homotopy equivalent to $R_{t+1}(\not\Sigma X)$ for $t \geq 0$.*

Corollary 3.2.10. *$R_t(\mathbb{P}^n)$ is homotopy equivalent to $K(\mathbb{Z}_2, 0) \times K(\mathbb{Z}_2, 1) \times \cdots \times K(\mathbb{Z}_2, n-t)$.*

Proof. By Lawson Suspension Theorem, we have $R_t(\mathbb{P}^n) \cong R_0(\mathbb{P}^{n-t})$. Since $R_0(\mathbb{P}^{n-t}) = \frac{Z_0(\mathbb{RP}^{n-t})}{2Z_0(\mathbb{RP}^{n-t})}$ where $Z_0(\mathbb{RP}^{n-t})$ is the free abelian group generated by points of $\mathbb{RP}^{n-t}$, by Dold-Thom Theorem, $\frac{Z_0(\mathbb{RP}^{n-t})}{2Z_0(\mathbb{RP}^{n-t})}$ is homotopy equivalent to $K(\mathbb{Z}_2, 0) \times K(\mathbb{Z}_2, 1) \times \cdots \times K(\mathbb{Z}_2, n-t)$. □

The following is the Lawson Suspension for reduced real cocycles.

Corollary 3.2.11. *$R^t(X)$ is homotopy equivalent to $\frac{R_1(\mathbb{P}^{t+1})(X)}{R_1(\mathbb{P}^t)(X)}$.*

Proof. We have a commutative diagram of homotopy equivalences from the Suspension Theorem:

$$\begin{array}{ccc} R_0(\mathbb{P}^{t-1})(X) & \longrightarrow & R_0(\mathbb{P}^t)(X) \\ \downarrow & & \downarrow \\ R_1(\mathbb{P}^t)(X) & \longrightarrow & R_1(\mathbb{P}^{t+1})(X) \end{array}$$

and then by Proposition 2.2.1, they induce a homotopy equivalence between $R^t(X)$ and

$$B(R_1(\mathbb{P}^{t+1})(X), R_1(\mathbb{P}^t)(X))$$

which is homotopy equivalent to $\frac{R_1(\mathbb{P}^{t+1})(X)}{R_1(\mathbb{P}^t)(X)}$. □

3.2.4 Homotopy Property

Theorem 3.2.12. *Let X be a real quasiprojective variety and E be a real algebraic vector bundle over X of rank k, i.e., E is a real quasiprojective variety and $\pi : E \longrightarrow X$ is a complex algebraic vector bundle or rank k. Then $\pi^* : R_r(X) \longrightarrow R_{r+k}(E)$ induces a homotopy equivalence.*

Proof. If the dimension of X is 0, this is trivial. Assume that the Theorem is true for any real quasiprojective variety of dimension $\leq n$. Suppose that the dimension of X is $n+1$. Take a real quasi projective variety Y which is closed in X such that E restricted to $U = X - Y$ is trivial. From the following diagram of fibrations:

$$\begin{array}{ccccc} R_r(X) & \longrightarrow & R_r(U) & \longrightarrow & BR_r(Y) \\ \downarrow & & \downarrow & & \downarrow \\ R_{r+k}(E) & \longrightarrow & R_{r+k}(U \times \mathbb{C}^k) & \longrightarrow & BR_{r+k}(E|_Y) \end{array}$$

By the induction hypothesis, $\pi^* : BR_r(Y) \longrightarrow BR_{r+k}(E|_Y)$ is a homotopy equivalence. Thus the Theorem follows if it is true for trivial bundles. Then it is easy to see that we only need to show that the Theorem is true for the case where E is a trivial line bundle. We use induction on the dimension of X. Assume that the Theorem is true for trivial line bundle over any real quasiprojective varieties of dimension $< n$. Let the dimension of X be n and $\overline{X}$ be the closure of X. Take real projective varieties $Y, Y' \subset \overline{X}$ of dimension smaller than n such that $X = \overline{X} - Y$ and $U = \overline{X} - Y'$ where the hyperplane line bundle $\mathcal{O}_{\overline{X}}(1)|_U = U \times \mathbb{C}$. From the following diagram of fibrations:

$$\begin{array}{ccccc}
R_r(\overline{X}) & \longrightarrow & R_r(X \cap U) & \longrightarrow & BR_r(Y \cup Y') \\
\downarrow & & \downarrow & & \downarrow \\
R_{r+1}(\mathcal{O}_{\overline{X}}(1)) & \longrightarrow & R_{r+1}((X \cap U) \times \mathbb{C}) & \longrightarrow & BR_{r+1}(\mathcal{O}_{Y \cup Y'}(1))
\end{array}$$

By the Lawson Suspension Theorem, the left and right vertical arrows are homotopy equivalences, thus $\pi^* : R_r(X \cap U) \longrightarrow R_{r+1}((X \cap U) \times \mathbb{C})$ is a homotopy equivalence. Since $X = (X \cap U) \cup (X \cap Y')$, and from the following diagram:

$$\begin{array}{ccccc}
R_r(X) & \longrightarrow & R_r(X \cap U) & \longrightarrow & BR_r(X \cap Y') \\
\downarrow & & \downarrow & & \downarrow \\
R_{r+1}(X \times \mathbb{C}) & \longrightarrow & R_{r+1}((X \cap U) \times \mathbb{C}) & \longrightarrow & BR_{r+1}((X \cap Y') \times \mathbb{C})
\end{array}$$

By the induction hypothesis, the right vertical arrow is a homotopy equivalence. Therefore, $\pi^* : R_r(X) \longrightarrow R_{r+1}(X \times \mathbb{C})$. □

3.2.5 Real FL-Moving Lemma

The main ingredient of the proof of the duality theorem between morphic cohomology and Lawson homology is the Moving Lemma proved in [FL2]. Here we observe that the FL-Moving Lemma passes to reduced real morphic cohomology. Since we are not going to use the full power of FL-Moving Lemma, we state a simple form which fits our need.

Theorem 3.2.13. *Suppose that $X \subset \mathbb{P}^n$ is a real projective variety of dimension m. Fix nonnegative integers r, s, e where $r + s \geq m$. There exists a connected interval $I \subset \mathbb{R}$ containing 0 and a map*

$$\Psi : Z_r(X) \times I \longrightarrow Z_r(X)$$

such that

1. *$\Psi_0 =$ identity.*
2. *For $t \in I$, Ψ_t is a real map.*
3. *For all cycles Y, Z of dimensions r, s and degree $\leq e$ and all $t \neq 0$, any component of excess dimension(i.e., $> r + s - m$) of $Y \bullet \Psi_t(Z)$ lies in the singular locus of X for any $t \neq 0$.*

Friedlander and Lawson construct the map Ψ from the map

$$\Psi_{\underline{N}}(Z,p) = (-1)^{m+1}(M+1)\cdot R_{F^*}(Z) + \sum_{i=1}^{m}(-1)^i \pi^*_{F^i}\{\Theta_{\underline{N},p}\{p_{F^i *}(R_{F^{i-1}} \circ \cdots \circ R_{F^0}(Z))\}\} \bullet X$$

for some $\underline{N}$ large enough. We give a brief explanation of the definitions of functions involved in this map. Let $F = (f_0, \cdots, f_m)$, a (m+1)-tuples of real homogenous polynomials of degree d in $\mathbb{P}^n$ where the zero locus of F is disjoint from X. Via the Veronese embedding, we embed $\mathbb{P}^n$ into $\mathbb{P}^M$ where $M = \binom{n+d}{d} - 1$. Let $\pi_F : \mathbb{P}^M \cdots \to \mathbb{P}^m$ be the linear projection determined by F and $L(F)$ the center of π_F. F determines a finite map $p_F : X \longrightarrow \mathbb{P}^m$, see for example [Sh]. For a cycle Z, $\pi^*_F(Z) = Z \# L(F)$, $C_F(Z) = \pi^*_F(p_{F*}(Z))$, $R_F(Z) = C_F(Z) \bullet X - Z$ where $\#$ is the join. Since F is real, R_F, Θ are real maps, so $\Psi_{\underline{N}}$ is also a real map and thus Ψ is also a real map.

3.2.6 Duality Theorem

Now we are able to give a proof of a duality theorem between reduced real Lawson homology and reduced real morphic cohomology for nonsingular real projective varieties.

Theorem 3.2.14. *Suppose that X, Y are nonsingular real projective varieties and the dimension of X is m. Then*

1. $R_r(Y)(X)$ *is homotopy equivalent to* $R_{r+m}(X \times Y)$

2. $R^t(X)$ *is homotopy equivalent to* $R_m(X \times \mathbb{A}^t)$ *for any* $t \geq 0$

3. $R^t(X)$ *is homotopy equivalent to* $R_{m-t}(X)$ *for* $0 \leq t \leq m$.

Therefore, we have a group isomorphism

$$RL^tH^k(X) \cong RL_{m-t}H_{m-k}(X)$$

for $0 \leq t, k \leq m$.

Proof. For each $e \geq 0$, let

$$K_e = \coprod_{d_1+d_2 \leq e} \mathscr{C}_{r,d_1}(X \times Y) \times \mathscr{C}_{r,d_2}(X \times Y)/\sim$$

and let

$$K'_e = \coprod_{d_1+d_2 \leq e} \mathscr{C}_{r,d_1}(Y)(X) \times \mathscr{C}_{r,d_2}(Y)(X)/\sim$$

where $\sim$ is the naive group completion relation and $\mathscr{C}_{r,d}(Y)(X) = \mathscr{C}_{r+m,d}(X \times Y) \cap \mathscr{C}_r(Y)(X)$.

Let q be the quotient map from $Z_{r+m}(X \times Y)_{\mathbb{R}}$ to $R_{r+m}(X \times Y)$ and let q' be the quotient map from $Z_r(Y)(X)_{\mathbb{R}}$ to $R_r(Y)(X)$. Let $\widetilde{K_e} = q(K_e \cap Z_{r+m}(X \times Y)_{\mathbb{R}})$, $\widetilde{K'_e} = q'(K'_e \cap Z_r(Y)(X)_{\mathbb{R}})$.

By Real FL-Moving Lemma, we get two real maps

$$\Psi : Z_{r+m}(X \times Y) \times I \longrightarrow Z_{r+m}(X \times Y),$$

$$\Psi' : Z_r(Y)(X) \times I \longrightarrow Z_r(Y)(X)$$

where $I = [0,1]$ is the unit interval. Since Ψ and Ψ' preserve $Z_r(X \times Y)_{\mathbb{R}}, Z_{r+m}(X \times Y)^{av}$ and $Z_r(Y)(X)_{\mathbb{R}}, Z_r(Y)(X)^{av}$ respectively, they induce maps on $R_r(Y)(X) \times I$ and $R_{r+m}(X \times Y) \times I$. By abuse of notation, we will use the same notation Ψ, Ψ' to denote these two maps respectively. Restricting Ψ, Ψ' to $\widetilde{K_e} \times I$ and $\widetilde{K'_e} \times I$, we get two maps $R\phi_e = \widetilde{\Psi}|_{\widetilde{K}_e \times I}$, $R\phi'_e = \widetilde{\Psi}'|_{\widetilde{K'}_e \times I}$.

The inclusion map $\mathscr{D} : Z_r(Y)(X) \longrightarrow Z_{r+m}(X \times Y)$ induces an injective map

$$R\mathscr{D} : R_r(Y)(X) \longrightarrow R_{r+m}(X \times Y).$$

By Lemma 2.1.2, the filtrations

$$\widetilde{K_0} \subset \widetilde{K_1} \cdots = R_{r+m}(X \times Y)$$

and

$$\widetilde{K'_0} \subset \widetilde{K'_1} \cdots = R_r(Y)(X)$$

are locally compact and $R\mathscr{D}$ is filtration-preserving.

We have the following commutative diagrams:

$$\begin{array}{ccc} \widetilde{K'}_e \times I & \xrightarrow{R\phi'_e} & R_r(Y)(X) \\ {\scriptstyle R\mathscr{D}\times Id}\downarrow & & \downarrow{\scriptstyle R\mathscr{D}} \\ \widetilde{K}_e \times I & \xrightarrow{R\phi_e} & R_{r+m}(X \times Y) \end{array}$$

$$\begin{array}{ccc} \widetilde{K'}_e \times \{1\} & \xrightarrow{R\phi'_e} & R_r(Y)(X) \\ \downarrow & \overset{R\phi_e}{\nearrow} & \downarrow \\ \widetilde{K}_e \times \{1\} & \xrightarrow{R\phi_e} & R_{r+m}(X \times Y) \end{array}$$

and there is a map $\widetilde{\lambda}_e = R\phi_e$ from $\widetilde{K_e} \times 1$ to $R_r(Y)(X)$. Thus by Lemma [5.2] in [FL3], $R_r(Y)(X)$ is homotopy equivalent to $R_{r+m}(X \times Y)$.

Furthermore, we have a commutative diagram of fibrations:

$$\begin{array}{ccccc} R_0(\mathbb{P}^{t-1})(X) & \longrightarrow & R_0(\mathbb{P}^t)(X) & \longrightarrow & R^t(X) \\ \downarrow & & \downarrow & & \downarrow \\ R_m(X \times \mathbb{P}^{t-1}) & \longrightarrow & R_m(X \times \mathbb{P}^t) & \longrightarrow & R_m(X \times \mathbb{A}^t) \end{array}$$

The first two columns are homotopy equivalences which implies the last one is also a homotopy equivalence. If $0 \le t \le m$, by the homotopy property of trivial bundle projection, $R^t(X)$ is homotopy equivalent to $R_{m-t}(X)$. $\square$

Proposition 3.2.15. *Suppose X is a nonsingular real projective variety. Then $R_0(X \times \mathbb{A}^t)$ is homotopy equivalent to $\Omega^{-t}R_0(X)$ where $\Omega^{-t}R_0(X)$ is the t-*

fold delooping of $R_0(X)$ given the infinite loop space structure induced by the structure as a topological abelian group of $R_0(X)$.

Proof. For two Eilenberg-Mac Lane spaces $K(G,i), K(H,j)$, denote

$$K(G,i)\bigotimes K(H,j) = K(G\otimes H, i+j)$$

and

$$(\prod_{i=1}^{n}(K(G_i,i)))\bigotimes(\prod_{j=1}^{m}(K(H_j,j))) = \prod_{r=1}^{n+m}\prod_{i+j=r} K(G_i\otimes H_j, i+j).$$

From Theorem A.5. in [LLM], there is a canonical homotopy equivalence between $R_0(X)$ and $\prod_{k\geq 0} K(H_k(ReX;\mathbb{Z}_2),k)$. We will always consider the homotopic splitting of $R_0(X)$ into Eilenberg-Mac Lane spaces by this canonical homotopy equivalence. By the Künneth formula for $\mathbb{Z}_2$-coefficients and the Dold-Thom Theorem, we have $R_0(X\times\mathbb{P}^{t-1}) = R_0(X)\bigotimes R_0(\mathbb{P}^{t-1})$ and $R_0(X\times\mathbb{P}^{t}) = R_0(X)\bigotimes R_0(\mathbb{P}^{t})$. Since the inclusion map $\mathbb{P}^{t-1}\hookrightarrow\mathbb{P}^t$ induces an isomorphism $i_*: H_k(\mathbb{RP}^{t-1};\mathbb{Z}_2)\longrightarrow H_k(\mathbb{RP}^{t};\mathbb{Z}_2)$ for $0\leq k<t$ in homology, $i_*:\pi_k(R_0(\mathbb{P}^{t-1}))\longrightarrow\pi_k(R_0(\mathbb{P}^{t}))$ is an isomorphism for $0\leq k<t$ by Dold-Thom Theorem therefore $i_*:\pi_l R_0(X\times\mathbb{P}^{t-1})\longrightarrow\pi_l R_0(X\times\mathbb{P}^{t})$ is an isomorphism for $0\leq l<t$ and an injection for $l\geq t$. From the long exact

sequence

$$\cdots \longrightarrow \pi_l R_0(X \times \mathbb{P}^{t-1}) \longrightarrow \pi_l R_0(X \times \mathbb{P}^t) \longrightarrow \pi_l(R_0(X \times \mathbb{A}^t)) \longrightarrow \cdots$$

we see that $\pi_l(R_0(X \times \mathbb{A}^t)) = 0$ if $0 \leq l < t$ and $\pi_l(R_0(X \times \mathbb{A}^t)) = \frac{\pi_l R_0(X \times \mathbb{P}^t)}{\pi_l R_0(X \times \mathbb{P}^{t-1})}$ if $l \geq t$. Write $R_0(X)$ homotopy equivalent to a product of Eilenberg-Mac Lane spaces and denote the i-th component of the Eilenberg-Mac Lane space of $R_0(X)$ to be $R_{0,i}(X)$. Then for $l \geq t$, from above calculation, we have $R_{0,l}(X \times \mathbb{A}^t) = R_{0,l-t}(X) \bigotimes R_{0,t}(\mathbb{P}^t) = R_{0,l-t}(X) \bigotimes K(\mathbb{Z}_2, t) = \Omega^{-t} R_{0,l-t}(X)$. Thus $R_0(X \times \mathbb{A}^t) = \Omega^{-t} R_0(X)$. □

By taking homotopy groups, we have the following result:

Corollary 3.2.16. *For a nonsingular real projective variety X,*

$$RL_0H_k(X \times \mathbb{A}^t) = \begin{cases} RL_0H_{k-t}(X), & \text{if } k \geq t; \\ 0, & \text{if } k < t. \end{cases}$$

Corollary 3.2.17. *Suppose that X is a nonsingular real projective variety of dimension m. Then*

$$RL^tH^k(X) = RL^mH^k(X) = H_{m-k}(ReX; \mathbb{Z}_2)$$

for $t \geq m \geq k$ and $RL^tH^k(X) = 0$ for $t \geq k > m$.

Proof. $RL^tH^k(X) = \pi_{t-k}R^t(X) = \pi_{t-k}R_m(X \times \mathbb{A}^t) = \pi_{t-k}R_0(X \times \mathbb{A}^{t-m}) = \pi_{t-k}\Omega^{-(t-m)}R_0(X)$

$$= \begin{cases} 0, & \text{if } k > m \\ \pi_{m-k}R_0(X) = H_{m-k}(ReX;\mathbb{Z}_2) = RL^mH^k(X) & \text{if } k \le m \end{cases}$$

□

Corollary 3.2.18. $R^t(\mathbb{P}^n) = \begin{cases} \prod_{i=0}^{t} K(\mathbb{Z}_2, i) & \text{if } t \le n \\ \prod_{i=0}^{n} K(\mathbb{Z}_2, i+t-n) & \text{if } t > n \end{cases}$

Proof. If $t \le n$, by the Duality Theorem, $R^t(\mathbb{P}^n) = R_n(\mathbb{P}^n \times \mathbb{A}^t) = R_{n-t}(\mathbb{P}^n) = \prod_{i=0}^{t} K(\mathbb{Z}_2, i)$; if $t > n$, then $R^t(\mathbb{P}^n) = R_n(\mathbb{P}^n \times \mathbb{A}^t) = R_0(\mathbb{P}^n \times \mathbb{A}^{t-n}) = \Omega^{-(t-n)}R_0(\mathbb{P}^n) = \Omega^{-(t-n)}\prod_{i=0}^{n} K(\mathbb{Z}_2, i) = \prod_{i=0}^{n} K(\mathbb{Z}_2, i+t-n)$. □

Corollary 3.2.19. *Suppose that X is a smooth projective variety of dimension m. Then $R^t(\not\Sigma X)$ is homotopy equivalent to $R^t(X)$ for $0 \le t \le m$.*

Proof. $R^t(\not\Sigma X) \cong R_{m+1-t}(\not\Sigma X) \cong R_{m-t}(X) \cong R^t(X)$. □

Chapter 4

Natural Maps And Operators

4.1 From Reduced Real Morphic Cohomology To Singular Cohomology

Lemma 4.1.1. *Suppose that G is a topological group and X is a locally compact Hausdorff space, then with the compact-open topology, $Map(X,G)$, the set of continuous functions from X to G, is a topological group under pointwise multiplication.*

Proof. We define an operation $\ltimes$ from $Map(X,G)\times Map(X,G) \longrightarrow Map(X,G\times G)$ as following: for $f,g \in Map(X,G)$, $(f \ltimes g)(x) = (f(x), g(x))$. It is proved in [Bre] that $\ltimes : Map(X,G) \times Map(X,G) \longrightarrow Map(X, G \times G)$ is a homeomorphism. Let $+' : G \times G \longrightarrow G$ be the group operation of G which is continuous, thus $+'$ induces a continuous map $+$ from $Map(X, G \times G)$ to

$Map(X,G)$. The operation $+ : Map(X,G) \times Map(X,G) \longrightarrow Map(X,G)$ is defined by $(f+g)(x) = f(x)+g(x)$ and we see that $+ = +' \circ \ltimes$ which is continuous. Since the map which sends $g \in G$ to $-g$ is continuous, the map sending $f \in Map(X,G)$ to $-f$ is continuous. Therefore $Map(X,G)$ is a topological group. $\square$

Suppose that X, Y are real projective varieties. By using the graphing construction of Friedlander and Lawson, $Z_r(Y)(X)$ can be identified as a subspace of $Map(X, Z_r(Y))$ where $Map(X, Z_r(Y))$ denotes the space of continuous functions from X to $Z_r(Y)$ with the compact-open topology. Thus the inclusion map $i : Z_r(Y)(X)_{\mathbb{R}} \hookrightarrow Map(X, Z_r(Y))$ is an embedding. For $f \in Z_r(Y)(X)_{\mathbb{R}}$, if we restrict f to $Re(X)$, the set of real points of X, since $f(x) = \overline{f(\bar{x})} = \overline{f(x)}$, the image of f lies in $Z_r(Y)_{\mathbb{R}}$. Composing the inclusion map with the restriction map, we have a continuous map

$$\Psi' : Z_r(Y)(X)_{\mathbb{R}} \longrightarrow Map(ReX, Z_r(Y)_{\mathbb{R}}).$$

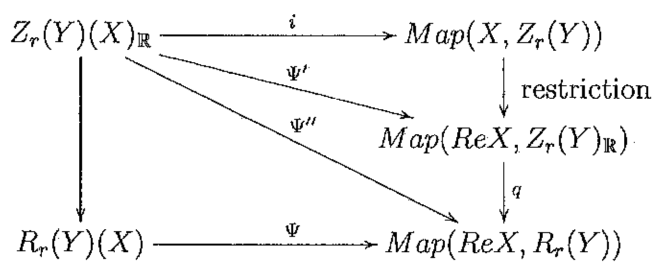
Composing again with the quotient map $q : Z_r(Y)_{\mathbb{R}} \longrightarrow R_r(Y)$, we have a continuous map

$$\Psi'' : Z_r(Y)(X)_{\mathbb{R}} \longrightarrow Map(ReX, R_r(Y)).$$

If $f = g + \overline{g}$, $\Psi''(f)(x) = g(x) + \overline{g(\bar{x})} + Z_r(Y)^{av} = g(x) + \overline{g(x)} \in Z_r(Y)^{av} = Z_r(Y)^{av}$, so $Z_r(Y)(X)^{av} \subset Ker\Psi''$ therefore Ψ'' induces a continuous map

$$\Psi : R_r(Y)(X) \longrightarrow Map(ReX, R_r(Y)).$$

We summarize the above construction in following diagram:



Now we are going to construct a map from $R^t(X)$ to $Map(ReX, R_0(\mathbb{A}^t))$.

Let $q' : R_0(\mathbb{P}^t) \longrightarrow R_0(\mathbb{A}^t)$ be the quotient map. Then q' induces a map, also denoting as q', from $Map(ReX, R_0(\mathbb{P}^t))$ to $Map(ReX, R_0(\mathbb{A}^t))$. Let $\Phi = q' \circ \Psi : R_0(\mathbb{P}^t)(X) \longrightarrow Map(ReX, R_0(\mathbb{A}^t))$. For $f \in R_0(\mathbb{P}^{t-1})(X)$, $f \in ker(\Phi)$, so Φ induces a map $\Phi^t : R^t(X) \longrightarrow Map(ReX, R_0(\mathbb{A}^t))$. We summarize the construction of Φ^t as following:

$$\Phi^t(f + Z_0(\mathbb{P}^t)(X)^{av} + R_0(\mathbb{P}^{t-1})(X)) = q' \circ q \circ (f|_{ReX}).$$

Since $R_0(\mathbb{A}^t)$ is the Eilenberg-Mac Lane space $K(\mathbb{Z}_2; t)$, taking homotopy groups, Φ^t induces a group homomorphism:

$$\Phi^{t,k} : RL^tH^k(X) \longrightarrow H^k(ReX; \mathbb{Z}_2).$$

Proposition 4.1.2. $\Phi^{t,k} : RL^tH^k(X) \longrightarrow H^k(ReX; \mathbb{Z}_2)$ *is a natural transformation for each* k *with* $0 \leq k \leq t$.

Proof. Suppose that $f : X \longrightarrow Y$ is a morphism between two real projective varieties. f induces a map $f^* : R^t(Y) \longrightarrow R^t(X)$ by mapping $\phi + R_0(\mathbb{P}^{t-1})(Y)$ to $\phi \circ f + R_0(\mathbb{P}^{t-1})(X)$. Since f is a real map, it maps $Re(X)$ to $Re(Y)$, thus f induces a map $f^* : Map(ReY, R_0(\mathbb{A}^t)) \longrightarrow Map(ReX, R_0(\mathbb{A}^t))$. It is easy to check that the following diagram commutes:

$$\begin{array}{ccc} R^t(Y) & \xrightarrow{f^*} & R^t(X) \\ \Phi^t \downarrow & & \downarrow \Phi^t \\ Map(ReY, R_0(\mathbb{A}^t)) & \xrightarrow{f^*} & Map(ReX, R_0(\mathbb{A}^t)) \end{array}$$

which then induces a commutative diagram of homotopy groups. □

4.2 S-Map In Reduced Real Morphic Cohomology

Let us give a construction of the S-map in reduced real morphic cohomology: Fix $x_\infty \in \mathbb{RP}^1$.

$$
\begin{array}{ccc}
R^t(X) \times \mathbb{RP}^1 & \longrightarrow & R^t(X) \times R_0(\mathbb{P}^1) \\
\downarrow & & \downarrow \# \\
R^t(X) \wedge \mathbb{RP}^1 & \longrightarrow & B(R_1(\mathbb{P}^{t+2})(X), R_1(\mathbb{P}^{t+1})(X)) \\
& \searrow^{s} & \downarrow \Sigma^{-1} \\
& & B(R_0(\mathbb{P}^{t+1})(X), R_0(\mathbb{P}^{t})(X)) \\
& & \downarrow \, \| \\
& & R^{t+1}(X)
\end{array}
$$

The top horizontal row sends $(f + R_0(\mathbb{P}^{t-1})(X), x) \in R^t \times \mathbb{RP}^1$ to $(f + R_0(\mathbb{P}^{t-1})(X), x - x_\infty)$ and the join on the right hand side sends $(f + R_0(\mathbb{P}^{t-1})(X), y)$ to $f\#y + R_1(\mathbb{P}^{t+1})(X)$ where $f\#y \in R_1(\mathbb{P}^{t+2})(X)$. Σ^{-1} is the homotopy inverse of the suspension map.

So we have a map from $R^t(X) \wedge S^1 \longrightarrow R^{t+1}(X)$ which induces a map in the reduced real morphic cohomology:

$$S : RL^tH^k(X) \longrightarrow RL^{t+1}H^k(X).$$

Proposition 4.2.1. *For a real projective variety X, the following diagram commutes:*

$$\begin{array}{ccc} RL^tH^k(X) & \xrightarrow{\quad S \quad} & RL^{t+1}H^k(X) \\ & {\scriptstyle \Phi^{t,k}} \searrow \quad \swarrow {\scriptstyle \Phi^{t+1,k}} & \\ & H^k(ReX;\mathbb{Z}_2) & \end{array}$$

Proof. Fix a point $x_\infty \in \mathbb{RP}^1$. Consider the following diagram:

$$\begin{array}{ccccc} R^t(X) \wedge \mathbb{RP}^1 & \longrightarrow & \frac{R_1(\mathbb{P}^{t+2})(X)}{R_1(\mathbb{P}^{t+1})(X)} & \xleftarrow{\not\Sigma \cong} & R^{t+1}(X) \\ \downarrow {\scriptstyle \Phi^t \times Id} & & \downarrow & & \downarrow {\scriptstyle \Phi^{t+1}} \\ Map(ReX, R_0(\mathbb{A}^t)) \wedge \mathbb{RP}^1 & \longrightarrow & Map(ReX, \frac{R_1(\mathbb{P}^{t+2})}{R_1(\mathbb{P}^{t+1})}) & \xleftarrow{\not\Sigma \cong} & Map(ReX, R_0(\mathbb{A}^{t+1})) \end{array}$$

The right arrow in the bottom row sends $f \wedge x$ to the map $(f \wedge x)(y) = f(y)\#(x - x_\infty)$. It is easy to see that these two squares commute. For the rest, follow the argument in Theorem 5.2, [FL1]. □

4.2.1 Filtration

For a real projective variety X, define $\mathscr{F}^t = \Phi^{t,k}(RL^tH^k(X))$. By the above proposition, we have a filtration:

$$\mathscr{F}^k \subseteq \mathscr{F}^{k+1} \subseteq \cdots \subseteq H^k(ReX;\mathbb{Z}_2)$$

which is analogous to the "topological filtration" defined by Friedlander and Mazur in [FM].

4.2.2 The H-Operator

From the construction of S-map, we have a map

$$H : R^t(X) \hookrightarrow R^t(X) \times R_0(\mathbb{P}^1) \longrightarrow R^{t+1}(X)$$

and then taking the homotopy group, we have

$$H : RL^tH^k(X) \longrightarrow RL^{t+1}H^{k+1}(X)$$

which is the H-operator in reduced real morphic cohomology.

4.3 From Reduced Real Lawson Homology To Singular Homology

In Lawson homology, there is a natural transformation

$$\Phi_{r,k} : L_rH_k(X) \longrightarrow H_k(X; \mathbb{Z})$$

defined by iterating the s-map in Lawson homology t times and then composite with the isomorphism from Dold-Thom Theorem. We show that similar construction is valid in reduced real Lawson homology. And thus we have a natural transformation from reduced real Lawson homology to singular homology of the real points with $\mathbb{Z}_2$ coefficients.

Fix a point $x_\infty \in \mathbb{RP}^1$.

$$\begin{array}{ccc} R_t(X)\times \mathbb{RP}^1 & \longrightarrow & R_t(X)\times R_0(\mathbb{P}^1) \\ \downarrow & & \downarrow \# \\ R_t(X)\wedge \mathbb{RP}^1 & \longrightarrow & R_{t+1}(X\#\mathbb{P}^1) \\ & \overset{s}{\searrow} & \downarrow \Sigma^{-2} \\ & & R_{t-1}(X) \end{array}$$

The map on the top horizontal row sends $(V,x)\in R_t(X)\times \mathbb{RP}^1$ to $V\#(x-x_\infty)\in R_t(X)\times R_0(\mathbb{P}^1)$ and since joining with a zero cycle is a zero cycle, it reduces to the smash product of $R_t(X)$ and $\mathbb{RP}^1$. Finally, we take the homotopy inverse of the suspension map twice. So we get a map $s : S^1\wedge R_t(X)\longrightarrow R_{t-1}(X)$ which induces a map in the reduced real Lawson homology

$$s : RL_tH_k(X)\longrightarrow RL_{t-1}H_k(X).$$

We iterate this map t times and then apply the Dold-Thom isomorphism

$\tau : \pi_k R_0(X) \xrightarrow{\cong} H_k(ReX; \mathbb{Z}_2)$. This gives us a map

$$\Phi_{t,k} : RL_t H_k(X) \longrightarrow H_k(ReX; \mathbb{Z}_2)$$

where $\Phi_{t,k} = \tau \circ s^t$.

Proposition 4.3.1. *For a morphism $f : X \longrightarrow Y$ between real projective varieties X, Y, the following diagram commutes:*

$$\begin{array}{ccc} RL_r H_k(X) & \xrightarrow{\Phi_{r,k}} & H_k(ReX; \mathbb{Z}_2) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ RL_r H_k(Y) & \xrightarrow{\Phi_{r,k}} & H_k(ReY; \mathbb{Z}_2) \end{array}$$

Thus Φ is a natural transformation from reduced real Lawson homology to singular homology of real points with $\mathbb{Z}_2$-coefficients.

Proof. The following diagram commutes:

$$\begin{array}{ccccc} R_r(X) \wedge \mathbb{RP}^1 & \longrightarrow & R_{r+1}(X \# \mathbb{P}^1) & \xleftarrow{\Sigma^2} & R_{r-1}(X) \\ \downarrow{\scriptstyle f_* \wedge Id} & & \downarrow{\scriptstyle (f \# Id)_*} & & \downarrow{\scriptstyle f_*} \\ R_r(Y) \wedge \mathbb{RP}^1 & \longrightarrow & R_{r+1}(Y \# \mathbb{P}^1) & \xleftarrow{\Sigma^2} & R_{r-1}(Y) \end{array}$$

Taking homotopy groups, we have the following commutative diagram:

$$\begin{array}{ccc} RL_rH_k(X) & \xrightarrow{s} & RL_{r-1}H_k(X) \\ \downarrow f_* & & \downarrow f_* \\ RL_rH_k(Y) & \xrightarrow{s} & RL_{r-1}H_k(Y) \end{array}$$

And by the functoriality of Dold-Thom isomorphism, we have the commutative diagram:

$$\begin{array}{ccc} RL_0H_k(X) & \xrightarrow{\tau} & H_k(ReX;\mathbb{Z}_2) \\ \downarrow f_* & & \downarrow f_* \\ RL_0H_k(Y) & \xrightarrow{\tau} & H_k(ReY;\mathbb{Z}_2) \end{array}$$

Applying the commutative diagram of s-map r times and applying the commutative diagram of Dold-Thom isomorphism, we get the required commutativity. □

4.3.1 Filtrations

Definition (The geometric filtration) Let X be a real projective variety and denote by $RG_jH_n(X) \subset H_n(ReX;\mathbb{Z}_2)$ the subspace of $H_n(ReX;\mathbb{Z}_2)$ generated by the images of maps $H_n(ReY;\mathbb{Z}_2) \longrightarrow H_n(ReX;\mathbb{Z}_2)$ induced from all morphisms $Y \longrightarrow X$ of real projective variety Y of dimension $\leq 2n-j$. The subspaces $RG_jH_n(X)$ form a decreasing filtration:

$$\cdots \subset RG_jH_n(X) \subset RG_{j-1}H_n(X) \subset \cdots \subset RG_0H_n(X) \subset H_n(ReX;\mathbb{Z}_2)$$

which is called the geometric filtration.

The s-map in reduced real Lawson homology enables us to define a filtration which is analogous to the topological filtration in the Lawson homology.

Definition (The topological filtration) Suppose that X is a real projective variety. Let $RT_tH_n(X)$ denote the subspace of $H_n(ReX;\mathbb{Z}_2)$ given by the image of $\Phi_{t,n}$, i.e.,

$$RT_tH_n(X) = \Phi_{t,n}(RL_tH_n(X)).$$

The subspaces $RT_tH_n(X)$ form a decreasing filtration:

$$\cdots \subset RT_tH_n(X) \subset RT_{t-1}H_n(X) \subset \cdots \subset RT_0H_n(X) = H_n(ReX;\mathbb{Z}_2),$$

and $RT_tH_n(X)$ vanishes if $t > n$. This filtration is called the topological filtration.

It was conjectured by Friedlander and Mazur in [FM] that the topological filtration and geometric filtration in Lawson homology coincide. We post the similar conjecture for reduced real Lawson homology.

Conjecture For a smooth real projective variety X, the topological filtration and the geometric filtration in reduced real Lawson homology coincide, i.e.,

$$RT_tH_n(X) = RG_tH_n(X).$$

4.3.2 The h-Operator

Let us now construct a map which is the analogue of the h-operator in Lawson homology.

$$h : R_r(X) \hookrightarrow R_r(X) \times R_0(\mathbb{P}^1) \xrightarrow{\#} R_{r+1}(X \# \mathbb{P}^1) \xrightarrow{\not\Sigma^{-2}} R_{r-1}(X)$$

which induces a map, called the h-operator, in reduced real Lawson homology:

$$h : RL_r H_k(X) \longrightarrow RL_{r-1} H_{k-1}(X).$$

Chapter 5

Operations

5.1 Cup Product

From Section 6 in [FL1], there is a continuous pairing induced by the join of varieties:

$$Z^s(Y)(X) \wedge Z^{s'}(Y')(X') \xrightarrow{\#} Z^{s+s'}(Y\#Y')(X \times X')$$

given by

$$(\varphi\#\varphi')(x,x') = (\varphi(x))\#(\varphi'(x'))$$

For a point $[x_0 : ... : x_n : y_0 : ... : y_m] \in V\#U$ in the join of two varieties V and U, $[\bar{x}_0 : ... : \bar{x}_n : \bar{y}_0 : ... : \bar{y}_m] = \overline{[x_0 : ... : x_n : y_0 : ... : y_m]} \in \overline{V\#U}$, thus $\#$ is a real pairing. Therefore, for real projective varieties X, X', Y, Y', $\#$ reduces to a continuous pairing

$$R^s(Y)(X) \wedge R^{s'}(Y')(X') \xrightarrow{\#} R^{s+s'}(Y\#Y')(X \times X').$$

Taking the homotopy groups, we have a pairing:

$$RL^sH^q(X;Y) \otimes RL^{s'}H^{q'}(X';Y') \longrightarrow RL^{s+s'}H^{q+q'}(X \times X'; Y\#Y'),$$

and when restricted to the diagonal in $X \times X$, it determines a cup product:

$$RL^sH^q(X;Y) \otimes RL^{s'}H^{q'}(X;Y') \xrightarrow{\#} RL^{s+s'}H^{q+q'}(X; Y\#Y').$$

The pairing

$$R^s(\mathbb{P}^s)(X) \wedge R^{s'}(\mathbb{P}^{s'})(X') \longrightarrow R^{s+s'}(\mathbb{P}^{s+s'+1})(X \times X')$$

by the Lawson Suspension Theorem reduces to a pairing on the reduced real cocycle groups :

$$R^s(X) \wedge R^{s'}(X') \longrightarrow R^{s+s'}(X \times X')$$

and when restricted to the diagonal in $X \times X$, we get a pairing

$$R^s(X) \wedge R^{s'}(X) \longrightarrow R^{s+s'}(X)$$

which gives us a commutative cup product in reduced real morphic cohomology:

$$RL^sH^q(X) \otimes RL^{s'}H^{q'}(X) \longrightarrow RL^{s+s'}H^{q+q'}(X).$$

Proposition 5.1.1. *Suppose that $X, X', Y, Y', W, W', Z, Z'$ are all real projective varieties and $f : X \longrightarrow X', g : W \longrightarrow W'$ are real morphisms. Then we have the following commutative diagrams:*

1.

$$\begin{array}{ccc} R^s(Y)(X') \wedge R^{s'}(Z)(W') & \xrightarrow{\#} & R^{s+s'}(Y\#Z)(X' \times W') \\ {\scriptstyle f^*\wedge g^*}\downarrow & & \downarrow{\scriptstyle (f\times g)^*} \\ R^s(Y)(X) \wedge R^{s'}(Z)(W) & \xrightarrow{\#} & R^{s+s'}(Y\#Z)(X \times W) \end{array}$$

2.

$$\begin{array}{ccc} R^s(X') \wedge R^{s'}(X') & \xrightarrow{\#} & R^{s+s'}(X') \\ {\scriptstyle f^*\wedge f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ R^s(X) \wedge R^{s'}(X) & \xrightarrow{\#} & R^{s+s'}(X) \end{array}$$

3.

$$\begin{array}{ccc} R^s(X)(Y) \wedge R^{s'}(W)(Z) & \xrightarrow{\#} & R^{s+s'}(X\#W)(Y \times Z) \\ {\scriptstyle f_*\wedge g_*}\downarrow & & \downarrow{\scriptstyle (f\times g)_*} \\ R^{s-c}(X')(Y) \wedge R^{s'-c'}(W')(Z) & \xrightarrow{\#} & R^{(s+s')-(c+c')}(X'\#W')(Y \times Z) \end{array}$$

where $c = dimX - dimX'$, $c' = dimW - dimW'$.

In short, cup product in reduced real morphic cohomology is natural with *respect to real morphisms.*

Proof. We observe that

$$(f^*\varphi)\#(g^*\varphi') = (\varphi \circ f)\#(\varphi' \circ g) = (\varphi\#\varphi') \circ (f \times g) = (f \times g)^*(\varphi\#\varphi'),$$

$$(f_*\alpha)\#(g_*\alpha') = (f \times g)_*(\alpha\#\alpha')$$

and then check that they pass to reduced real cocycles. □

From Lawson Suspension Theorem, we obtain a canonical homotopy equivalence:

$$Z^q(\mathbb{P}^n) \longrightarrow \prod^q \stackrel{\text{def}}{=} \prod_{k=0}^{q} K(\mathbb{Z}, 2k)$$

for all $n \geq q$. In [LMi], Lawson and Michelsohn showed that the complex join $\#_{\mathbb{C}} : \prod^q \times \prod^{q'} \longrightarrow \prod^{q+q'}$ has the property that

$$\#_{\mathbb{C}}^*(\iota_{2k}) = \sum_{r+s=k} \iota_{2r} \otimes \iota_{2s}$$

in integral cohomology where ι_{2k} is the generator of

$$H^{2k}(K(\mathbb{Z}, 2k); \mathbb{Z}) \cong \mathbb{Z}.$$

The cup product in cohomology is characterized by some axioms which can be found for example in [AGP]. From the above result, it is easy to check that the complex join induces the cup product in integral cohomology. Following a similar approach, Lam in [Lam] showed that the corresponding result holds for the pairing: $\# : \prod^{q}_{\mathbb{R}} \times \prod^{q'}_{\mathbb{R}} \longrightarrow \prod^{q+q'}_{\mathbb{R}}$ where $\prod^{q}_{\mathbb{R}} = K(\mathbb{Z}_2, 0) \times K(\mathbb{Z}_2, 1) \times \cdots \times K(\mathbb{Z}_2, q)$, thus $\#$ induces the cup product in $\mathbb{Z}_2$-cohomology.

We now show that the natural transformation from reduced real morphic cohomology to $\mathbb{Z}_2$-singular cohomology is a ring homomorphism.

Proposition 5.1.2. *Suppose that X and X' are real projective varieties. Then for all s, s' and q, q', there are commutative diagrams:*

$$\begin{array}{ccc} RL^sH^q(X) \otimes RL^{s'}H^{q'}(X') & \xrightarrow{\#} & RL^{s+s'}H^{q+q'}(X \times X') \\ \downarrow{\scriptstyle \Phi^{s,q} \otimes \Phi^{s',q'}} & & \downarrow{\scriptstyle \Phi^{s+s',q+q'}} \\ H^q(ReX; \mathbb{Z}_2) \otimes H^{q'}(ReX'; \mathbb{Z}_2) & \longrightarrow & H^{q+q'}(Re(X \times X'); \mathbb{Z}_2) \end{array}$$

where the lower horizontal arrow is the usual cup product in $\mathbb{Z}_2$-coefficients. In particular, the map

$$\Phi^{*,*} : RL^*H^*(X) \longrightarrow H^*(ReX; \mathbb{Z}_2)$$

is a graded-ring homomorphism.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} R^s(X) \wedge R^{s'}(X') & \xrightarrow{\#} & R^{s+s'}(X \times X') \\ \downarrow & & \downarrow \\ Map(ReX, R_0(\mathbb{A}^s)) \wedge Map(ReX', R_0(\mathbb{A}^{s'})) & \xrightarrow{\#} & Map(Re(X \times X'), R_0(\mathbb{A}^{s+s'})) \end{array}$$

Since the lower arrow in the diagram above is exactly the map classifying the cup product in $\mathbb{Z}_2$-cohomology, taking homotopy, proves the first assertion. Taking $X = X'$ and the naturality of Φ in the case of the diagonal map $\delta : X \longrightarrow X \times X$, we see that it is a ring homomorphism. $\square$

5.2 Slant Product

Suppose that X, Y are projective varieties. Define a product

$$Z_r(Y)(X) \times Z_p(X) \longrightarrow Z_{r+p}(Y)$$

by sending (f, V) to $Pr_*(f \bullet (V \times Y))$ where $Pr : X \times Y \longrightarrow Y$ is the projection and we consider f as a cycle in $X \times Y$ which intersects $V \times Y$ properly. It is proved in [FL1, Proposition 7.1] this is a continuous pairing.

Now suppose that X, Y are real projective varieties. Since in the definition, each operation is real, the above pairing reduces to a pairing

$$R_r(Y)(X) \times R_p(X) \longrightarrow R_{r+p}(Y)$$

and it is easy to see that it reduces again to

$$R_r(Y)(X) \wedge R_p(X) \longrightarrow R_{r+p}(Y).$$

Therefore, we have a slant product:

$$RL^r H^k(X;Y) \otimes RL_p H_n(X) \longrightarrow RL_{r+p} H_{n-k+2r}(Y).$$

Fix $r = 0, p \leq t$. For $Y = \mathbb{P}^{t-1}$, the product from the above construction sends

$$R_0(\mathbb{P}^{t-1})(X) \wedge R_p(X) \longrightarrow R_p(\mathbb{P}^{t-1}).$$

Therefore, the product

$$R_0(\mathbb{P}^t)(X) \wedge R_p(X) \longrightarrow R_p(\mathbb{P}^t)$$

reduces to

$$\frac{R_0(\mathbb{P}^t)(X)}{R_0(\mathbb{P}^{t-1})(X)} \wedge R_p(X) \longrightarrow \frac{R_p(\mathbb{P}^t)}{R_p(\mathbb{P}^{t-1})}.$$

Since $\frac{R_p(\mathbb{P}^t)}{R_p(\mathbb{P}^{t-1})} = R_p(\mathbb{A}^t) \cong R_0(\mathbb{A}^{t-p})$, we have a pairing:

$$R^t(X) \wedge R_p(X) \longrightarrow R_0(\mathbb{A}^{t-p})$$

which induces a Kronecker pairing:

$$RL^tH^k(X) \otimes RL_pH_k(X) \longrightarrow \mathbb{Z}_2$$

for $p \leq k \leq t$.

Proposition 5.2.1. *Let X be a real projective variety. Then for all $s \geq q$, the diagram*

$$\begin{array}{ccc} RL^sH^q(X) \otimes RL_0H_q(X) & & \\ & \overset{\kappa}{\searrow} & \\ \Big\downarrow {\scriptstyle \Psi^{s,q} \otimes \Psi_{0,q}} & & \mathbb{Z}_2 \\ & \underset{\kappa^{top}}{\nearrow} & \\ H^q(ReX;\mathbb{Z}_2) \otimes H_q(ReX;\mathbb{Z}_2) & & \end{array}$$

commutes; i.e., under the natural transformation to $\mathbb{Z}_2$-singular theory, the Kronecker pairing introduced above is carried to the topological one.

Proof. The canonical homeomorphism between $R_0(X)$ and $\frac{Z_0(ReX)}{2Z_0(ReX)}$ is given by sending an element $c \in R_0(X)$ to the cycle formed by real points of X that represent c which is unique modulo 2, i.e., there exist unique $x_1, ..., x_k \in ReX$ such that $c = \sum_{i=1}^k x_i + Z_0(X)^{av}$ and c is mapped to $\sum_{i=1}^k x_i + 2Z_0(ReX)$

as the proof in Proposition 2.1.7. In the following, we will assume that the representation of c is the real point representation.

Consider the following diagram:

$$\begin{array}{ccc} R^s(X) \wedge R_0(X) & & \\ \Big\downarrow{\scriptstyle \Phi^s \wedge Id} & \searrow & \\ & & R_0(\mathbb{A}^s) \\ & \nearrow & \\ Map(ReX, R_0(\mathbb{A}^s)) \wedge R_0(X) & & \end{array}$$

The slant product on the top row sends $(f + Z_0(\mathbb{P}^t)^{av} + R_0(\mathbb{P}^{t-1})(X), \sum_i x_i + Z_0(X)^{av})$ to $(\sum_i f(x_i) + Z_0(\mathbb{P}^t)^{av} + R_0(\mathbb{P}^{t-1}))$ and the pairing in the bottom row sends $(\varphi, \sum_i x_i)$ to $\sum_i \varphi(x_i)$. Under the natural maps, it is not difficult to see that the diagram commutes.

Taking homotopy groups, from the bottom row, we get a pairing:

$$\kappa : \ H^q(ReX; \mathbb{Z}_2) \otimes H_q(ReX; \mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$$

Observe that the following diagram commutes:

$$\begin{array}{ccc} Map(ReX, (\frac{Z_0(S^t)}{2Z_0(S^t)})^0) \wedge \frac{Z_0(ReX)}{2Z_0(ReX)} & \longrightarrow & (\frac{Z_0(S^t)}{2Z_0(S^t)})^0 \\ \Big\downarrow & & \Big\downarrow \\ Map(ReX, R_0(\mathbb{A}^t)) \wedge R_0(X) & \longrightarrow & R_0(\mathbb{A}^t) \end{array}$$

where the canonical quotient map $S^t \longrightarrow \mathbb{RP}^t$ induces a map from $Map(ReX, (\frac{Z_0(S^t)}{2Z_0(S^t)})^0)$ to $Map(ReX, R_0(\mathbb{A}^t))$ and a map from $(\frac{Z_0(S^t)}{2Z_0(S^t)})^0$ to $R_0(\mathbb{A}^t)$, and we use $(\frac{Z_0(S^t)}{2Z_0(S^t)})^0 = K(\mathbb{Z}_2, t)$ to denote the component of $\frac{Z_0(S^t)}{2Z_0(S^t)}$ containing the identity. The map from $\frac{Z_0(ReX)}{2Z_0(ReX)}$ to $R_0(X)$ is the canonical map which is defined in 2.1.7. Each vertical arrow of the diagram above is a homotopy equivalence. Thus to prove this Proposition it will suffice to establish the following lemma which is a generalization of Lemma 8.3 in [FL1]. □

We use the convention that $\mathbb{Z}_0 = \mathbb{Z}$ and $\frac{Z_0(Y)}{0Z_0(Y)} = Z_0(Y)$.

Lemma 5.2.2. *For any finite CW-complex Y and $p = 0$ or a prime number, the pairing*

$$Map\left(Y, (\frac{Z_0(S^t)}{pZ_0(S^t)})^0\right) \wedge \frac{Z_0(Y)}{pZ_0(Y)} \longrightarrow \left(\frac{Z_0(S^t)}{pZ_0(S^t)}\right)^0$$

sending $(f, \sum_i n_i y_i + pZ_0(Y))$ to $\sum_i n_i f(y_i) + pZ_0(S^t)$ induces a pairing

$$\kappa : H^q(Y; \mathbb{Z}_p) \otimes H_q(Y; \mathbb{Z}_p) \longrightarrow \mathbb{Z}_p$$

which is the topological Kronecker pairing κ^{top} where $K = (\frac{Z_0(S^t)}{pZ_0(S^t)})^0 = K(\mathbb{Z}_p, t)$ is the component of $\frac{Z_0(S^t)}{pZ_0(S^t)}$ containing the identity.

Proof. For a continuous map $\varphi : Y \longrightarrow Z$, from the definition, κ has following naturality property:

$$\kappa(\varphi^*\alpha, u) = \kappa(\alpha, \varphi_* u) \tag{5.1}$$

for all $\alpha \in H^q(Z;\mathbb{Z}_p)$ and $u \in H_q(Y;\mathbb{Z}_p)$.

Let $Y = S^q$, $K = (\frac{Z_0(S^t)}{pZ_0(S^t)})^0 = K(\mathbb{Z}_p, t)$. A generator of $H^q(S^q;\mathbb{Z}_p) = \pi_{t-q}Map(S^q, K)$ is given by the homotopy class of $f : S^{t-q} \longrightarrow Map(S^q, K)$ where $f(x)(y) = x \wedge y + pZ_0(S^t)$ and a generator of $H_q(S^q;\mathbb{Z}_p) = \pi_q(\frac{Z_0(S^q)}{pZ_0(S^q)})$ is given by the homotopy class of $g : S^q \longrightarrow \frac{Z_0(S^q)}{pZ_0(S^q)}$ defined by $g(x) = x + pZ_0(S^q)$. Then $f \wedge g : S^{t-q} \wedge S^q \longrightarrow K$ where

$$(f \wedge g)(x \wedge y) = f(x)(g(y)) = f(x)(y + pZ_0(S^t)) = x \wedge y + pZ_0(S^t)$$

is the generator of $H_t(K;\mathbb{Z}_p) = \mathbb{Z}_p$. Thus $\kappa = \kappa^{top}$ when Y is a sphere.

Now let $Y = (\frac{Z_0(S^q)}{pZ_0(S^q)})^0 = K(\mathbb{Z}_p; q)$. Let $i : S^q \longrightarrow Y$ denote the generator of π_q. Let $u \in H^q(Y;\mathbb{Z}_p)$ and $\sigma \in H_q(Y;\mathbb{Z}_p)$ be given. Since $i_* : H_q(S^q;\mathbb{Z}_p) \longrightarrow H_q(Y;\mathbb{Z}_p)$ is an isomorphism, there is an element $\tau \in H_q(S^q;\mathbb{Z}_p)$ with $i_*\tau = \sigma$. Then by equation 5.1 and the case of spheres, $\kappa^{top}(u, \sigma) = \kappa^{top}(i^*u, \tau) = \kappa(i^*u, \tau) = \kappa(u, \sigma)$.

For a general space Y, fix $u \in H^q(Y;\mathbb{Z}_p)$ and let $f : Y \longrightarrow K(\mathbb{Z}_p, q)$ be the map classifying u; i.e., $u = f^*\iota$ where $\iota \in H^q(K(\mathbb{Z}_p, q);\mathbb{Z}_p)$ is the canonical generator. Then for any $\tau \in H_q(Y;\mathbb{Z}_p)$, we have

$$\kappa^{top}(u,\tau) = \kappa^{top}(\iota, f_*\tau) = \kappa(\iota, f_*\tau) = \kappa(u,\tau).$$

This complete the proofs. □

5.3 The Compatibility Of The Duality Theorem With The $\mathbb{Z}_2$-Poincaré Duality

Let us recall that a projective variety X is said to have ***full real points*** if $dim_{\mathbb{R}} ReX = dim_{\mathbb{C}} X$.

Theorem 5.3.1. *Suppose that X is a real projective manifold of dimension m and X has full real points, then the duality theorem 3.2.6 is compatible with the $\mathbb{Z}_2$-Poincaré duality of its real points, i.e., the following diagram commutes:*

$$\begin{array}{ccc} RL^tH^k(X) & \xrightarrow{\mathscr{R}\mathscr{D}} & RL_{m-t}H_{m-k}(X) \\ \Phi\downarrow & & \downarrow\tau \\ H^k(ReX;\mathbb{Z}_2) & \xrightarrow{\mathscr{P}} & H_{m-k}(ReX;\mathbb{Z}_2) \end{array}$$

where $\mathscr{P}$ is the Poincaré duality map sending $\alpha \in H^k(ReX;\mathbb{Z}_2)$ to $\alpha \cap [ReX] \in H_{m-k}(ReX;\mathbb{Z}_2)$.

Proof. By Proposition 4.2.1, we reduce the problem to the case $t = m$, i.e., we need to prove the commutativity of the following diagram:

$$\begin{array}{ccc} RL^m H^k(X) & \xrightarrow{\mathscr{R}\mathscr{D}} & RL_0 H_{m-k}(X) \\ \Phi \downarrow & & \downarrow \Phi \\ H^k(ReX;\mathbb{Z}_2) & \xrightarrow{\mathscr{P}} & H_{m-k}(ReX;\mathbb{Z}_2) \end{array}$$

Since the evaluation and intersection products:

$$H^k(ReX;\mathbb{Z}_2) \otimes H_k(ReX;\mathbb{Z}_2) \xrightarrow{<,>} H_0(ReX;\mathbb{Z}_2),$$

$$H_{m-k}(ReX;\mathbb{Z}_2) \otimes H_k(ReX;\mathbb{Z}_2) \xrightarrow{\odot} H_0(ReX;\mathbb{Z}_2)$$

are perfect pairings, it suffices to prove that

$$< \Phi(\alpha), \gamma >= \tau(\mathscr{R}\mathscr{D}(\alpha)) \odot \gamma$$

for all $\alpha \in RL^m H^k(X)$ and all $\gamma \in H_k(ReX;\mathbb{Z}_2)$ where τ is the Dold-Thom isomorphism. To prove this equality, it suffices to prove the commutativity of the following diagram:

$$\begin{array}{ccc}
H^k(ReX;\mathbb{Z}_2)\otimes H_k(ReX;\mathbb{Z}_2) & \xrightarrow{<,>} & H_0(ReX;\mathbb{Z}_2) \\
\uparrow{\scriptstyle \Phi\otimes\tau} & & \uparrow{\scriptstyle \epsilon^{-1}} \\
\pi_{m-k}(R^m(X))\otimes\pi_k(R_0(X)) & \xrightarrow{\backslash} & \pi_m(R_0(\mathbb{A}^m)) \\
\downarrow{\scriptstyle \mathscr{R}\mathscr{D}\otimes pr_1^*} & & \downarrow{\scriptstyle =} \\
\pi_{m-k}(R_m(X\times\mathbb{A}^m))\otimes\pi_k(R_m(X\times\mathbb{A}^m)) & \xrightarrow{pr_2\circ\bullet} & \pi_m(R_0(\mathbb{A}^m)) \\
\uparrow{\scriptstyle pr_1^*\circ\tau^{-1}\otimes pr_1^*\circ\tau^{-1}} & & \uparrow{\scriptstyle \epsilon} \\
H_{m-k}(ReX;\mathbb{Z}_2)\otimes H_k(ReX;\mathbb{Z}_2) & \xrightarrow{\odot} & H_0(ReX;\mathbb{Z}_2)
\end{array} \qquad (5.2)$$

where

$$\epsilon = pr_{2*}\circ pr_1^* : \pi_0(R_0(X)) \longrightarrow \pi_m(R_0(X\times\mathbb{A}^m)) \longrightarrow \pi_m(R_0(\mathbb{A}^m))$$

and $\backslash$ is the slant product. The commutativity of the top square follows from the construction of slant product 5.2 and the naturality of Kronecker pairing 5.2.1.

To verify the commutativity of the middle square of the diagram 5.2, it suffices to show that the following diagram commutes:

$$\begin{array}{ccc}
R^m(X)\wedge R_0(X) & \xrightarrow{\backslash} & R_0(\mathbb{A}^m) \\
\downarrow{\scriptstyle \mathscr{R}\mathscr{D}\wedge pr_1^*} & & \downarrow{\scriptstyle =} \\
R_m(X\times\mathbb{A}^m)\wedge R_m(X\times\mathbb{A}^m) & \xrightarrow{pr_{2*}\circ\bullet} & R_0(\mathbb{A}^m)
\end{array}$$

For $f + R_0(\mathbb{P}^{m-1})(X) \in R^m(X)$, $\sum n_i x_i + Z_0(X)^{av} \in R_0(X)$, $\mathscr{R}\mathscr{D}(f +$

$R_0(\mathbb{P}^{m-1})(X)) = f + R_m(X \times \mathbb{P}^{m-1})$, $pr_1^*(\sum n_i x_i + Z_0(X)^{av}) = \sum n_i(x_i \times \mathbb{P}^m) + R_m(X \times \mathbb{P}^{m-1})$, $pr_{2*}[(f + R_m(X \times \mathbb{P}^{m-1})) \bullet (\sum n_i(x_i \times \mathbb{P}^m) + R_m(X \times \mathbb{P}^{m-1}))] = pr_{2*}(f \bullet \sum n_i(x_i \times \mathbb{P}^m)) + R_0(\mathbb{P}^{m-1})$ which is the slant product between $f + R_m(X \times \mathbb{P}^{m-1})$ and $\sum n_i x_i + Z_0(X)^{av}$.

To prove the commutativity of the bottom square, it suffices to prove the commutativity of the following diagram:

$$\begin{array}{ccc}
H_{m-k}(ReX;\mathbb{Z}_2) \otimes H_k(ReX;\mathbb{Z}_2) & \xrightarrow{(pr_1^*)^{\otimes 2}} & H^{BM}_{2m-k}(ReX \times \mathbb{R}^m;\mathbb{Z}_2) \otimes H^{BM}_{m+k}(ReX \times \mathbb{R}^m;\mathbb{Z}_2) \\
\downarrow \times & & \downarrow \times \\
H_m(ReX \times ReX;\mathbb{Z}_2) & \xrightarrow{(pr_1 \times pr_1)^*} & H^{BM}_{3m}((ReX \times \mathbb{R}^m)^2;\mathbb{Z}_2) \\
\downarrow \Delta! & & \downarrow \Delta! \\
H_0(ReX;\mathbb{Z}_2) & \xrightarrow{pr_1^*} & H^{BM}_m(ReX \times \mathbb{R}^m;\mathbb{Z}_2) \\
\downarrow = & & \downarrow pr_{2*} \\
H_0(ReX;\mathbb{Z}_2) & \xrightarrow{\epsilon} & H^{BM}_m(\mathbb{R}^m;\mathbb{Z}_2)
\end{array} \tag{5.3}$$

The composition of the maps in the right column can be identified with the map $pr_{2*} \circ \bullet$ in diagram 5.2 using the naturality of τ and the homotopy property of trivial bundle projection of reduced real Lawson homology and the composition of the maps in the left column of diagram 5.3 is the intersection pairing. Thus the commutativity of diagram 5.3 implies the commutativity of the last square of 5.2.

The evident intertwining of the external product $\times$ and the flat pull-back pr_1^* implies the commutativity of the top square. The Gysin maps and flat pullbacks commute, for a proof, for example in [FG; 3.4.d]. The commutativity of the bottom square comes from the definition of ϵ. $\square$

Chapter 6

Harnack-Thom Theorem In Reduced Real Lawson Homology

Real algebraic geometry has a very long history and under this name, it has two big branches. One direction of real algebraic geometry is to study complex projective varieties which are invariant under conjugation. Thus some familiar concepts such as two varieties with complementary dimensions must meet, projection of a variety is again a variety, are still valid in this content. But there is another possible meaning of the name "real algebraic geometry" which is the study of the real zero loci of real polynomials in real Euclidean spaces or real projective spaces. We call this type of real algebraic variety a "totally real algebraic variety" to distinguish it from the previous one. Life is more complicated in the totally real world. For example two circles in $\mathbb{R}^2$ may not intersect, the projection of a circle in $\mathbb{R}^2$ to $\mathbb{R}^1$ may not even be an algebraic variety, an irreducible smooth real variety may not be connected. It is not difficult to see that classical methods from complex algebraic geometry do not quite work here. As shown in [T], we define a real version of suspension map for totally real algebraic cycles, but the analogous Lawson Suspension Theorem becomes very complicated in this case. Even though totally real algebraic geometry does not have good properties from the classical point of view, it has its own problems. For example since totally real algebraic varieties are more flexible, we are able to ask if any smooth manifold is diffeomorphic to some nonsingular totally real algebraic varieties. This is the Nash-Tognoli Theorem and a proof can be found in [BCR].

The Harnack Theorem says that a nonsingular totally real curve of degree d in $\mathbb{RP}^2$ has at most $g(d)+1$ connected components where $g(d) = \frac{(d-1)(d-2)}{2}$. Later on Thom generalized Harnack's Theorem to all totally real projective varieties. In this section, we generalize Harnack-Thom Theorem further to a statement involving Lawson homology and reduced real Lawson homology. For 0-cycle spaces, we recover the Harnack-Thom Theorem. This result strengthens our belief that Lawson homology and reduced real Lawson homology are the right extensions of singular homology for projective varieties.

Let us recall that the real part $RP(C)$ of a cycle C, roughly speaking, is the part consisting of irreducible real subvarieties and the average part $AP(C)$

of a cycle C is the part consisting of conjugate pairs of complex cycles. The imaginary part is the part left after cancelling out the real and average parts. The precise definitions and some basic properties are given in 2.1.

In the following, we will assume that X is a real projective variety.

Proposition 6.0.2. *Suppose that the sequence $\{AP(f_i)\}$ converges to f where $f_i \in Z_p(X)$, then $RP(f) \in 2Z_p(X)_{\mathbb{R}}$.*

Proof. Since $AP(f_i) \in Z_p(X)^{av}$ and $Z_p(X)^{av}$ is closed by Proposition 2.1.3, f is in $Z_p(X)^{av}$ so $RP(f) \in 2Z_p(X)_{\mathbb{R}}$. □

Lemma 6.0.3. *The following sequence is exact:* $0 \longrightarrow \frac{Z_p(X)_{\mathbb{R}}}{2Z_p(X)_{\mathbb{R}}} \stackrel{i}{\longrightarrow} \frac{Z_p(X)}{2Z_p(X)} \stackrel{1+c_*}{\longrightarrow} \frac{Z_p(X)^{av}}{2Z_p(X)_{\mathbb{R}}} \longrightarrow 0.$

Proof. It is easy to check that $i(f + 2Z_p(X)_{\mathbb{R}}) = f + 2Z_p(X)$ is well defined and injective and $(1 + c_*)(f + 2Z_p(X)) = f + \bar{f} + 2Z_p(X)_{\mathbb{R}}$ is well defined and surjective. $1 + c_*$ sends the image of i to 0, thus the only thing we need to prove is for $f + \bar{f} \in 2Z_p(X)_{\mathbb{R}}$, $f \in Z_p(X)_{\mathbb{R}}$. Since $f + \bar{f} = 2RP(f) + 2AP(f) + IP(f) + IP(\bar{f}) \in 2Z_p(X)_{\mathbb{R}}$, this implies $IP(f) = IP(\bar{f}) = 0$ so $f \in Z_p(X)_{\mathbb{R}}$. □

Definition Let $Q_p(X)$ be the collection of all c in $Z_p(X)^{av}$ such that there exists a sequence $\{v_i\} \subset Z_p(X)_{\mathbb{R}}$ where $v_i = RP(v_i)$ for all i and v_i converges to c. It is not difficult to see that $Q_p(X)$ is a topological subgroup of $Z_p(X)^{av}$.

Let $ZQ_p(X)_{\mathbb{R}} = 2Z_p(X)_{\mathbb{R}} + Q_p(X)$ denote the internal sum of $2Z_p(X)_{\mathbb{R}}$ and $Q_p(X)$, then again $ZQ_p(X)_{\mathbb{R}}$ is a topological subgroup of $Z_p(X)^{av}$. $Q_p(X)$ is the closure of average p-cycles formed by irreducible real subvarieties, thus $ZQ_p(X)_{\mathbb{R}}$ is a closed subgroup.

The following example was given by Lawson to show that the set of 1-cycles formed by irreducible real subvarieties may not be closed which contrasts to the case of 0-cycles, i.e., $ZQ_p(X)_{\mathbb{R}}$ may not be equal to $2Z_p(X)_{\mathbb{R}}$.

Example In $\mathbb{P}^2$, consider the sequence of irreducible real subvarieties $V_\epsilon =$ the zero locus of $X^2 + Y^2 - \epsilon Z^2$. As ϵ converges to 0, V_ϵ converges to the cycle formed by two lines $X = iY$ and $X = -iY$ which have no real points.

Proposition 6.0.4. *For a real projective variety X, $ZQ_0(X)_{\mathbb{R}} = 2Z_0(X)_{\mathbb{R}}$.*

Proof. The free abelian group $Z_0(ReX)$ generated by real points of X is closed in $Z_0(X)$, so if $c \in Q_0(X)$, then $c \in 2Z_0(X)_{\mathbb{R}}$, see Proposition 2.1.7. □

Lemma 6.0.5. *Define $\widetilde{AP} : \frac{Z_p(X)_{\mathbb{R}}}{ZQ_p(X)_{\mathbb{R}}} \longrightarrow \frac{Z_p(X)^{av}}{ZQ_p(X)_{\mathbb{R}}}$ by*

$$f + ZQ_p(X)_{\mathbb{R}} \longmapsto AP(f) + ZQ_p(X)_{\mathbb{R}}.$$

Then $\widetilde{AP}$ is continuous.

Proof. There exist compact sets K_i such that

$$K_1 \subset K_2 \subset K_3 \subset \cdots = Z_p(X)_{\mathbb{R}}$$

and the topology of $Z_p(X)_{\mathbb{R}}$ is given by the weak topology induced from this filtration. Thus the filtration

$$K_1 + ZQ_p(X)_{\mathbb{R}} \subset K_2 + ZQ_p(X)_{\mathbb{R}} \subset \cdots = \frac{Z_p(X)_{\mathbb{R}}}{ZQ_p(X)_{\mathbb{R}}}$$

defines the topology of $\frac{Z_p(X)_{\mathbb{R}}}{ZQ_p(X)_{\mathbb{R}}}$.

Suppose that $f_i + ZQ_p(X)_{\mathbb{R}}$ converges to $ZQ_p(X)_{\mathbb{R}}$. Since $A = \{f_i + ZQ_p(X)_{\mathbb{R}}\} \cup \{ZQ_p(X)_{\mathbb{R}}\}$ is compact, $A \subset K_n + ZQ_p(X)_{\mathbb{R}}$ for some n. Thus there exists $g_i \in K_n$ such that under the quotient map q, $q(g_i) = f_i + ZQ_p(X)_{\mathbb{R}}$ for all i. K_n is compact, thus $\{g_i\}$ has a convergent subsequence. If for every convergent subsequence $\{g_{ij}\}$ of $\{g_i\}$, $\{\widetilde{AP}(q(g_{ij}))\}$ converges to a same point, then $\{\widetilde{AP}(q(g_i))\}$ converges.

Let $\{g_{ij}\}$ be a convergent subsequence of $\{g_i\}$ and g be the point that g_{ij} converges to, thus $g \in ZQ_p(X)_{\mathbb{R}}$. Let $\{AP(g_{ijk})\}$ be a subsequence of $\{AP(g_{ij})\}$ which converges to a real cycle h. Since $\{g_{ijk}\}$ is a subsequence of $\{g_{ij}\}$, it converges to g, hence

$$RP(g_{ijk}) = g_{ijk} - AP(g_{ijk}) \longrightarrow g - h.$$

By Proposition 6.0.2, $RP(h) \in 2Z_p(X)_{\mathbb{R}}$ and since $RP(g) \in 2Z_p(X)_{\mathbb{R}}$ we have $RP(g-h) \in 2Z_p(X)_{\mathbb{R}}$. $g-h$ is a real cycle and this implies $g-h \in Z_p(X)^{av}$. Furthermore, since $\{RP(g_{ij})\} \longrightarrow g-h$, by definition, $g-h \in Q_p(X)$. g is in $ZQ_p(X)_{\mathbb{R}}$ thus $h \in ZQ_p(X)_{\mathbb{R}}$. Passing to the quotient, we see that $AP(g_{ijk}) + ZQ_p(X)_{\mathbb{R}} \longrightarrow ZQ_p(X)_{\mathbb{R}}$ for all convergent subsequences of $\{AP(g_{ij})\}$, thus $AP(g_{ij}) + ZQ_p(X)_{\mathbb{R}} \longrightarrow ZQ_p(X)_{\mathbb{R}}$. Consequently, this implies the convergence of $\{AP(g_i) + ZQ_p(X)_{\mathbb{R}}\}$ to $ZQ_p(X)_{\mathbb{R}}$. Then $\widetilde{AP}(f_i + ZQ_p(X)_{\mathbb{R}}) = \widetilde{AP}(g_i + ZQ_p(X)_{\mathbb{R}}) = AP(g_i) + ZQ_p(X)_{\mathbb{R}} \longrightarrow ZQ_p(X)_{\mathbb{R}}$. So $\widetilde{AP}$ is continuous. □

Lemma 6.0.6. *Define* $\widetilde{RP} : \frac{Z_p(X)_{\mathbb{R}}}{Z_p(X)^{av}} \longrightarrow \frac{Z_p(X)_{\mathbb{R}}}{ZQ_p(X)_{\mathbb{R}}}$ *by*

$$f + Z_p(X)^{av} \longmapsto RP(f) + ZQ_p(X)_{\mathbb{R}}.$$

Then $\widetilde{RP}$ *is continuous.*

Proof. We proceed as in the proof above. There exist compact sets K_i such that

$$K_1 \subset K_2 \subset K_3 \subset \cdots = Z_p(X)_{\mathbb{R}}$$

and the topology of $Z_p(X)_{\mathbb{R}}$ is given by the weak topology induced from this filtration. The filtration

$$K_1 + Z_p(X)^{av} \subset K_2 + Z_p(X)^{av} \subset \cdots = \frac{Z_p(X)_{\mathbb{R}}}{Z_p(X)^{av}}$$

defines the topology of $\frac{Z_p(X)_{\mathbb{R}}}{Z_p(X)^{av}}$ and the filtration

$$K_1 + ZQ_p(X)_{\mathbb{R}} \subset K_2 + ZQ_p(X)_{\mathbb{R}} \subset \cdots = \frac{Z_p(X)_{\mathbb{R}}}{ZQ_p(X)_{\mathbb{R}}}$$

defines the topology of $\frac{Z_p(X)_{\mathbb{R}}}{ZQ_p(X)_{\mathbb{R}}}$.

Suppose that $f_i + Z_p(X)^{av}$ converges to $Z_p(X)^{av}$. Since $A = \{f_i + Z_p(X)^{av}\} \cup \{Z_p(X)^{av}\}$ is compact, $A \subset K_n + Z_p(X)^{av}$ for some n. Thus there exists $g_i \in K_n$ such that under the quotient map q, $q(g_i) = f_i + Z_p(X)^{av}$ for all i's. Let $\{g_{ij}\}$ be a convergent subsequence of $\{g_i\}$ and let g be the point that g_{ij} converges to, thus $g \in Z_p(X)^{av}$. Let $\{RP(g_{ijk})\}$ be a subsequence of $\{RP(g_{ij})\}$ which converges to a real cycle h. Since $\{g_{ijk}\}$ is a subsequence of $\{g_{ij}\}$, it converges to g, hence

$$AP(g_{ijk}) = g_{ijk} - RP(g_{ijk}) \longrightarrow g - h.$$

By Proposition 6.0.2, $RP(g-h) \in 2Z_p(X)_{\mathbb{R}}$ and since $RP(g) \in 2Z_p(X)_{\mathbb{R}}$ we have $RP(h) \in 2Z_p(X)_{\mathbb{R}}$. h is a real cycle and this implies $h \in Z_p(X)^{av}$. Furthermore, since $\{RP(g_{ij})\} \longrightarrow h$, by definition, $h \in Q_p(X)$. Passing to the quotient, we see that $RP(g_{ijk}) + ZQ_p(X)_{\mathbb{R}} \longrightarrow ZQ_p(X)_{\mathbb{R}}$. Thus $RP(g_{ij}) + ZQ_p(X)_{\mathbb{R}} \longrightarrow ZQ_p(X)_{\mathbb{R}}$ which implies that $RP(g_i) + ZQ_p(X)_{\mathbb{R}} \longrightarrow ZQ_p(X)_{\mathbb{R}}$ and therefore $\widetilde{RP}$ is continuous. □

Theorem 6.0.7. *$\frac{Z_p(X)_{\mathbb{R}}}{ZQ_p(X)_{\mathbb{R}}}$ is isomorphic as a topological group to $\frac{Z_p(X)_{\mathbb{R}}}{Z_p(X)^{av}} \times \frac{Z_p(X)^{av}}{ZQ_p(X)_{\mathbb{R}}}$*

Proof. Define $\psi : \frac{Z_p(X)_{\mathbb{R}}}{ZQ_p(X)_{\mathbb{R}}} \longrightarrow \frac{Z_p(X)_{\mathbb{R}}}{Z_p(X)^{av}} \times \frac{Z_p(X)^{av}}{ZQ_p(X)_{\mathbb{R}}}$ by $f + ZQ_p(X)_{\mathbb{R}} \longmapsto (f + Z_p(X)^{av}, AP(f) + ZQ_p(X)_{\mathbb{R}})$ and define $\phi : \frac{Z_p(X)_{\mathbb{R}}}{Z_p(X)^{av}} \times \frac{Z_p(X)^{av}}{ZQ_p(X)_{\mathbb{R}}} \longrightarrow \frac{Z_p(X)_{\mathbb{R}}}{ZQ_p(X)_{\mathbb{R}}}$ by $(f + Z_p(X)^{av}, g + ZQ_p(X)_{\mathbb{R}}) \longmapsto RP(f) + g + ZQ_p(X)_{\mathbb{R}}$. By Lemmas, ψ and ϕ are continuous and it is easy to check they are inverse of each other. □

For a group G, let $B(G)$ be the classifying space of G. From [Ben, Theorem 2.4.12], if N is a normal closed subgroup of M, then there is a fibration $B(M) \longrightarrow B(M/N)$ with fibre $B(N)$. Recall that $\pi_i(B(G)) \cong \pi_{i-1}(G)$, see [Ben, Theorem 2.4.11].

Let $A = \frac{Z_p(X)_{\mathbb{R}}}{2Z_p(X)_{\mathbb{R}}}$, $B = \frac{Z_p(X)}{2Z_p(X)}$, $C = \frac{Z_p(X)^{av}}{2Z_p(X)_{\mathbb{R}}}$, $D = \frac{Z_p(X)_{\mathbb{R}}}{ZQ_p(X)_{\mathbb{R}}}$, $E = \frac{Z_p(X)_{\mathbb{R}}}{Z_p(X)^{av}}$, $F = \frac{Z_p(X)^{av}}{ZQ_p(X)_{\mathbb{R}}}$, $G = \frac{ZQ_p(X)_{\mathbb{R}}}{2Z_p(X)_{\mathbb{R}}}$.

In the following Proposition, we use the notation T_n to denote the n-th homotopy group of $B(T)$ where T is any of the groups $A, ..., G$. We note that all these groups are $\mathbb{Z}_2$-spaces so their homotopy groups are vector spaces over $\mathbb{Z}_2$.

Proposition 6.0.8. *We have the following fibrations:*

1. $B(A) \longrightarrow B(B) \longrightarrow B(C)$

2. $B(C) \longrightarrow B(A) \longrightarrow B(E)$

3. $B(G) \longrightarrow B(A) \longrightarrow B(D)$

4. $B(G) \longrightarrow B(C) \longrightarrow B(F)$

These fibrations induce long exact sequences:

1. $\cdots \xrightarrow{c_{n+1}} A_n \xrightarrow{a_n} B_n \xrightarrow{b_n} C_n \xrightarrow{c_n} A_{n-1} \longrightarrow \cdots$

2. $\cdots \xrightarrow{e_{n+1}} C_n \xrightarrow{c'_n} A_n \xrightarrow{a'_n} E_n \xrightarrow{e_n} C_{n-1} \longrightarrow \cdots$

3. $\cdots \xrightarrow{d_{n+1}} G_n \xrightarrow{g_n} A_n \xrightarrow{a''_n} D_n \xrightarrow{d_n} G_{n-1} \longrightarrow \cdots$

4. $\cdots \xrightarrow{f_{n+1}} G_n \xrightarrow{g'_n} C_n \xrightarrow{c''_n} F_n \xrightarrow{f_n} G_{n-1} \longrightarrow \cdots$

Proof. (1) follows from 6.0.3, (2) and (3) follow from the fact that C and G are normal closed subgroups of A. (4) follows from the fact that G is a normal closed subgroup of C. □

Definition Suppose that X is a quasiprojective variety. We define the L_p total Betti number of X with $\mathbb{Z}_2$ coefficients to be $B(p)(X) = \sum_{k=0}^{\infty} rank L_p H_k(X;\mathbb{Z}_2)$ where

$$L_pH_k(X;\mathbb{Z}_2) = \pi_{2k-p}(\frac{Z_p(X)}{2Z_p(X)}).$$

If X is a real quasi projective variety, we define the real L_p total Betti number to be $\beta(p)(X) = \sum_{k=0}^{\infty} rank RL_pH_k(X)$. We call $\chi_p(X) = \sum_{k=0}^{\infty}(-1)^k rank L_pH_k(X;\mathbb{Z}_2)$ the L_p Euler characteristic of X with coefficients in $\mathbb{Z}_2$ and $\chi_p(ReX) = \sum_{k=0}^{\infty}(-1)^k rank RL_pH_k(X)$ the real L_p Euler characteristic.

Theorem 6.0.9. *Suppose that X is a real projective variety. Let $B(p)(X)$ be the L_p total Betti number of X with $\mathbb{Z}_2$ coefficients, $B(p)(X)_{\mathbb{R}} = \sum_{k=0}^{\infty} \pi_k(\frac{Z_p(X)_{\mathbb{R}}}{2Z_p(X)_{\mathbb{R}}})$, $BQ(p)(X) = \sum_{k=0}^{\infty} \pi_k(\frac{ZQ_p(X)_{\mathbb{R}}}{2Z_p(X)_{\mathbb{R}}})$. If $B(p)(X)$, $B(p)(X)_{\mathbb{R}}$ and $BQ(p)(X)$ are finite, then*

1. $\chi_p(X) \equiv \chi_p(ReX)$ *mod* 2
2. $B(p)(X) \equiv \beta(p)(X)$ *mod* 2

Proof. To simplify the notation, we use the same notation as in Proposition 6.0.8 but with different meaning. We use M_n to denote the rank of the n-th homotopy group of M, $Kerg_n$ and Img_n the rank of the kernel and the rank of the image of a homomorphism g_n respectively.

From the finiteness assumption of $B(p)(X)$, $B(p)(X)_{\mathbb{R}}$ and $BQ_p(X)$, we know that $\sum_{n=0}^{\infty} C_n$, $\sum_{n=0}^{\infty} E_n$,$\sum_{n=0}^{\infty} D_n$, $\sum_{n=0}^{\infty} F_n$ are finite from the long exact sequence 1, 2, 3, 4 respectively in Proposition 6.0.8. This implies that $\sum_{n=0}^{\infty} Imc_n$, $\sum_{n=0}^{\infty} Kerc_n$, $\sum_{n=0}^{\infty} Img_n$ and $\sum_{n=0}^{\infty} Img'_n$ are finite.

1. From the first two long exact sequences, we have $\chi_p(B) = \chi_p(A) + \chi_p(C)$ and $\chi_p(A) = \chi_p(C) + \chi_p(E)$, thus $\chi_p(B) \equiv \chi_p(E)$ mod 2.

2. From the long exact sequence 3 in Proposition 6.0.8, we get $D_n = Ima''_n + Imd_n = A_n - Kera''_n + G_{n-1} - Img_{n-1} = A_n - Img_n + G_{n-1} - Img_{n-1}$. From the long exact sequence 4, we get $F_n = Imc''_n + Imf_n = C_n -$

$Kerc''_n + G_{n-1} - Img'_{n-1} = C_n - Img'_n + G_{n-1} - Img'_{n-1}$. Since $D = E \times F$ by Theorem 6.0.7, we have $D_n = E_n + F_n$. Substitute the formulas of D_n and F_n into this equation and simplify it, we get

$$A_n = E_n + C_n + \theta_n$$

where $\theta_n = Img_n + Img_{n-1} - Img'_n - Img'_{n-1}$. From the long exact sequence 1, $C_n = Kerc_n + Imc_n$, thus $B_n = Ima_n + Imb_n = A_n - Imc_{n+1} + Kerc_n = E_n + C_n + \theta_n - Imc_{n+1} + Kerc_n = E_n + \theta_n + Imc_n - Imc_{n+1} + 2Kerc_n$. So $\sum_{n=0}^{\infty} B_n = \sum_{n=0}^{\infty} E_n + 2\sum_{n=0}^{\infty}(Img_n - Img'_n + Kerc_n)$. Therefore $\sum_{n=0}^{\infty} B_n \equiv \sum E_n$ mod 2 and by definition, $\sum_{n=0}^{\infty} B_n = B(p)(X)$ and $\sum_{n=0}^{\infty} E_n = \beta(p)(X)$.

□

Corollary 6.0.10. *In addition to the assumptions of the previous Theorem also assume that* $\frac{ZQ_p(X)_{\mathbb{R}}}{2Z_p(X)_{\mathbb{R}}}$ *is contractible, then*

1. $\beta(p)(X) \leq B(p)(X)$
2. $B(p)(X) \equiv \beta(p)(X)$ *mod* 2
3. $\chi_p(X) \equiv \chi_p(ReX)$ *mod* 2

Proof. We use the same notation $G = \frac{ZQ_p(X)_{\mathbb{R}}}{2Z_p(X)_{\mathbb{R}}}$ as above. If G is contractible,

$\pi_k(G) = 0$ for all $k \geq 0$. Thus $Img_n = Img'_n = 0$, so $\sum B_n = \sum E_n + 2\sum kerc_n$ which implies $\sum B_n \leq \sum E_n$. $\square$

Corollary 6.0.11. *(Harnack-Thom Theorem) Let X be a real projective variety and let $B(X), \chi(X)$ denote the standard total Betti number, Euler characteristic respectively of X in $\mathbb{Z}_2$-coefficients and $\beta(ReX), \chi(ReX)$ the standard total Betti number, Euler characteristic of ReX in $\mathbb{Z}_2$-coefficients. Then*

1. $\beta(ReX) \leq B(X)$
2. $B(X) \equiv \beta(ReX) \ mod\ 2$
3. $\chi(X) \equiv \chi(ReX) \ mod\ 2$

Proof. For $p = 0$, $2Q_0(X)_R = 2Z_0(X)_{\mathbb{R}}$, thus $G = \frac{ZQ_0(X)_{\mathbb{R}}}{2Z_0(X)_{\mathbb{R}}}$ is a point. By Dold-Thom Theorem, $\pi_k(\frac{Z_0(X)}{2Z_0(X)}) = H_k(X;\mathbb{Z}_2)$, $\pi_k(\frac{Z_0(X)_{\mathbb{R}}}{2Z_0(X)_{\mathbb{R}}}) = H_k(X/\mathbb{Z}_2;\mathbb{Z}_2)$ where $X/\mathbb{Z}_2$ is the orbit space of X under the action of the conjugation. Thus $B(0)(X)$ and $B(0)(X)_{\mathbb{R}}$ are finite. Then use the corollary above with $p = 0$. $\square$

Chapter 7

Signatures In Morphic Cohomology

7.1 Signatures

Suppose that $<,>$ is a symmetric bilinear form on a vector space V over $\mathbb{Q}$. Take a matrix representation A of $<,>$ and denote $\lambda^+, \lambda^-, \lambda^0$ the number of positive, negative and zero eigenvalues respectively, then the signature of $<,>$ is defined to be $\lambda^+ - \lambda^-$. It can be proved that it is independent of the basis chosen.

Before we proceed to the definition of signatures in morphic cohomology, we need the following result in which we make some modification of the original proof in [FL1, 7.8].

Proposition 7.1.1. *Suppose that X is a smooth projective variety of dimen-*

sion m. Then for any $t \geq m$, we have a commutative diagram:

$$\begin{array}{ccc} L^tH^q(X)\otimes\mathbb{Q} & \xrightarrow{\quad S \quad} & L^{t+1}H^q(X)\otimes\mathbb{Q} \\ & {\scriptstyle\Phi^{t,q}}\searrow \qquad \swarrow{\scriptstyle\Phi^{t+1,q}} & \\ & H^q(X;\mathbb{Q}) & \end{array}$$

and each map is an isomorphism.

Proof. By the Duality Theorem of morphic cohomology, we know that $Z^t(X)$ is homotopy equivalent to $Z_m(X\times\mathbb{A}^t)$. If $t \geq m$, $Z_m(X\times\mathbb{A}^t) = Z_0(X\times\mathbb{A}^{t-m})$. From the fibration:

$$\begin{array}{ccc} Z_0(X\times\mathbb{P}^{t-1}) & \longrightarrow & Z_0(X\times\mathbb{P}^t) \\ & & \downarrow \\ & & Z_0(X\times\mathbb{A}^{t-m}) \end{array}$$

we get a long exact sequence:

$$\cdots \longrightarrow H_k(X\times\mathbb{P}^{t-m-1}) \xrightarrow{i_*} H_k(X\times\mathbb{P}^{t-m}) \longrightarrow H_k^{BM}(X\times\mathbb{A}^{m-t})$$

$$\longrightarrow H_{k-1}(X\times\mathbb{P}^{t-m-1}) \longrightarrow \cdots$$

and then tensor by $\mathbb{Q}$, we get

$$\cdots \longrightarrow H_k(X\times\mathbb{P}^{t-m-1};\mathbb{Q}) \xrightarrow{i_*} H_k(X\times\mathbb{P}^{t-m};\mathbb{Q}) \longrightarrow H_k^{BM}(X\times\mathbb{A}^{m-t};\mathbb{Q})$$

$$\longrightarrow H_{k-1}(X \times \mathbb{P}^{t-m-1}; \mathbb{Q}) \longrightarrow \cdots$$

where i_* is induced from the inclusion map $i : X \times \mathbb{P}^{t-m-1} \subset X \times \mathbb{P}^{t-m}$. Since the inclusion map $ic : \mathbb{P}^{t-m-1} \subset \mathbb{P}^{t-m}$ induces an isomorphism in homology groups $ic_* : H_k(\mathbb{P}^{t-m-1}) \longrightarrow H_k(\mathbb{P}^{t-m})$ for $k \leq 2(t-m-1)$, thus i_* is injective and by Künneth formula in homology, it is not difficult to see that

$$H_k^{BM}(X \times \mathbb{A}^{t-m}; \mathbb{Q}) = \begin{cases} 0, & \text{if } k < 2(t-m) \\ H_{k-2(t-m)}(X; \mathbb{Q}), & \text{if } k \geq 2(t-m) \end{cases}$$

Therefore, if $t \geq m$, $L^tH^k(X) \otimes \mathbb{Q} = \pi_{2t-k}Z_0(X \times \mathbb{A}^{t-m}) \otimes \mathbb{Q} = \pi_{2m-k}(Z_0(X)) \otimes \mathbb{Q} = H_{2m-k}(X; \mathbb{Q}) = H^k(X; \mathbb{Q})$. By Theorem 5.2 in [FL1], we have the following commutative diagram:

$$\begin{array}{ccc} L^tH^q(X)_{\mathbb{Q}} & \xrightarrow{\quad S \quad} & L^{t+1}H^q(X)_{\mathbb{Q}} \\ & {\scriptstyle \Phi^{t,q}} \searrow \quad \swarrow {\scriptstyle \Phi^{t+1,q}} & \\ & H^q(X; \mathbb{Q}) & \end{array}$$

If $t = m$, since $\Phi^{t,q}$ is an isomorphism and by the dimension reason, S is an isomorphism, so $\Phi^{t+1,q}$ is also an isomorphism. Thus the three maps in this diagram are all isomorphisms for $t \geq m$. $\square$

Suppose now X is a nonsingular projective variety of complex dimension

$2m$. The cup product in morphic cohomology

$$L^sH^{2m}(X)\otimes L^sH^{2m}(X)\longrightarrow L^{2s}H^{4m}(X)$$

is a symmetric bilinear form for all $s\geq m$. But since it is possible that $L^sH^{2m}(X)$ has infinite rank, we do not define the signature directly.

Recall that there is a natural transformation $\Phi^{q,k}:L^qH^k(X)\longrightarrow H^k(X)$ for any q,k. Define

$$\widetilde{L^q}H^k(X)_{\mathbb{F}}=\frac{L^qH^k(x)\otimes\mathbb{F}}{Ker\Phi^{q,k}_{\mathbb{F}}}$$

the morphic cohomology with $\mathbb{F}$ coefficients quotient by the kernel of Φ. Now $\widetilde{L^q}H^k(X)_{\mathbb{F}}$ is finitely generated and we will very often identify $\widetilde{L^q}H^k(X)$ with the image $\Phi^{q,k}(L^qH^k(X))$.

Definition (morphic signatures) For a smooth connected projective variety X of complex dimension $2m$ and for $s\geq m$, we define the s-th morphic signature of X, denoting as σ_s, to be the signature of the symmetric bilinear form:

$$<,>:\widetilde{L^s}H^{2m}(X)_{\mathbb{Q}}\otimes\widetilde{L^s}H^{2m}(X)_{\mathbb{Q}}\longrightarrow\widetilde{L^{2s}}H^{4m}(X)_{\mathbb{Q}}=\mathbb{Q}.$$

We observe that when $s=2m$, σ_{2m} is just the usual signature of X.

7.2 The Morphic Conjecture

Recall that from Corollary 5.4 in [FL1], we know that the groups $\widetilde{L^s}H^q(X) = \Phi^{s,q}(L^sH^q(X))$ carry Hodge structure for a smooth projective variety X. Thus we are able to define the hodge numbers for morphic cohomology.

Definition Suppose that X is a smooth projective variety. Decompose

$$\widetilde{L^s}H^k(X)_{\mathbb{C}} = \bigoplus_{p+q=k} H_s^{p,q}(X)$$

and we define the morphic hodge numbers of X to be

$$h_s^{p,q}(X) = dim_{\mathbb{C}} H_s^{p,q}(X).$$

Let $\Omega \in L^1H^2(X)$ be a class coming from a very ample line bundle over X. Define an operation

$$\mathcal{L} : L^sH^q(X) \longrightarrow L^{s+1}H^{q+2}(X)$$

by $\mathcal{L}(\alpha) = \Omega \cdot \alpha$. The transformation $\Phi^{*,*}$ carries the cup product in morphic cohomology to the cup product in singular cohomology and it sends Ω to the cohomology class represented by Ω. Thus under the transformation $\Phi^{*,*}$, $\mathcal{L}$ carries over to the standard Lefschetz operator L. Thus $\widetilde{L^s}H^q(X)$ is L-

invariant.

There is a standard Hermitian inner product on $\mathscr{A}^{p,q}(X)$, the (p,q)-forms on X, called the Hodge inner product defined by

$$(\alpha, \beta) = \int_X \alpha \wedge *\bar{\beta}$$

where $*$ is the Hodge star operator. Let Λ to be the adjoint of L with respect to Hodge inner product. Since L, Λ commute with the Laplacian, they define operators on the harmonic spaces. From Hodge theory we know that a cohomology class has a unique harmonic representation. We identify the cohomology groups of X with harmonic forms on X. The Hodge inner product induces a Hermitian inner product in harmonic spaces which we also call the Hodge inner product. Restrict Hodge inner product to $\widetilde{L^s}H^q(X)_{\mathbb{C}}$, and let λ be the adjoint of $\mathcal{L}$. It seems natural to ask if the following question is true.

Conjecture (Morphic Conjecture) For a smooth projective manifold X, the adjoint operator $\lambda : H_s^{p,q}(X) \longrightarrow H_{s-1}^{p-1,q-1}(X)$ is the restriction of the standard adjoint operator, i.e.,

$$\lambda = \Lambda|_{H_s^{p,q}(X)}.$$

Let us recall the Grothendieck Standard Conjecture . For a smooth projective variety X with dimension n, let $C^j(X)$ be the subspace of $H^{2j}(X;\mathbb{Q})$

which is generated by algebraic cycles. By Hard Lefschetz Theorem, we have the following commutative diagram:

$$\begin{array}{ccc} H^{2j}(X;\mathbb{Q}) & \xrightarrow{L^{n-2j}} & H^{2n-2j}(X;\mathbb{Q}) \\ \uparrow & & \uparrow \\ C^j(X) & \longrightarrow & C^{n-j}(x) \end{array}$$

where L^{n-2j} is an isomorphism for $j \leq \lfloor \frac{n}{2} \rfloor$. Grothendieck Standard Conjecture claims that the restriction of L^{n-2j} also gives an isomorphism between $C^j(X)$ and $C^{n-j}(X)$ or equivalently, the adjoint operator Λ maps $C^{n-j}(X)$ into $C^j(X)$. This conjecture has various forms as explained in [Kle]. The case of abelian varieties was proved by Lieberman in [Lieb]. Friedlander in [F3] showed that by assuming this conjecture, the topological filtration defined by using the s-map in Lawson homology coincides with the geometric filtration. One simple observation is that the Grothendieck Standard Conjecture is a special case of the Morphic Conjecture.

Proposition 7.2.1. *The Morphic Conjecture implies the Grothendieck Standard Conjecture.*

Proof. If the Morphic Conjecture is true, then Λ^{n-2j} maps $\widetilde{L^{n-j}}H^{2n-2j}(X)_{\mathbb{C}} = C^{n-j}(X) \otimes \mathbb{C}$ into $\widetilde{L^j}H^{2j}_{\mathbb{C}}(X) = C^j(X) \otimes \mathbb{C}$, thus they are isomorphic which implies that $C^{n-j}(X)$ is isomorphic to $C^j(X)$. □

Let a, b be two nonnegative integers. Define

$$LH^{a,b}(X)_{\mathbb{F}} = \widetilde{L^a} H^0(X)_{\mathbb{F}} \oplus \widetilde{L^{a+1}} H^2(X)_{\mathbb{F}} \oplus \cdots \oplus \widetilde{L^b} H^1(X)_{\mathbb{F}} \oplus \widetilde{L^{b+1}} H^3(X)_{\mathbb{F}} \oplus \cdots$$

$$= \bigoplus_{i=0}^{n} \widetilde{L^{a+i}} H^{2i}(X)_{\mathbb{F}} \bigoplus_{j=0}^{n-1} \widetilde{L^{b+j}} H^{2j+1}(X)_{\mathbb{F}}$$

where $\mathbb{F} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

$LH^{a,b}(X)_{\mathbb{F}}$ is an L-invariant subring of the cohomology ring $\bigoplus_{i=0}^{2n} H^i(X;\mathbb{F})$.

Let us use $H^*(X;\mathbb{C})$ to denote the cohomology ring of X.

Proposition 7.2.2. *Suppose that X is a projective manifold of dimension n. Assuming the Morphic Conjecture, $LH^{a,b}(X)$ is a $sl_2(\mathbb{C})$ submodule of $H^*(X;\mathbb{C})$ thus it has a sub-Lefschetz decomposition.*

Proof. Let $h = \sum_k (n-k) Pr_k$ where Pr_k projects a form to its k-component. The $sl_2(\mathbb{C})$ structure on $H^*(X;\mathbb{C})$ is given by

$$[\Lambda, L] = h, [h, \Lambda] = 2\Lambda, [h, L] = -2L$$

and since $\mathcal{L}, \lambda$ are the restriction of L, Λ on $LH^{a,b}(X)_{\mathbb{C}}$, from the relation $h = [\Lambda, L]$, we see that h restricts to an operator on $LH^{a,b}(X)_{\mathbb{C}}$. Thus $LH^{a,b}(X)_{\mathbb{C}}$ has a $sl_2(\mathbb{C})$ structure and therefore admits a sub-Lefschetz decomposition of the Lefschetz decomposition of $H^*(X;\mathbb{C})$ □

7.3 Hodge Index Theorem In Morphic Cohomology

In the following, by assuming the Morphic Conjecture on $LH^{a,b}(X)$, we are going to generalize the classical hodge index theorem to morphic cohomology. See [Hirz] or [GH] for a proof of the classical Hodge Index Theorem.

Theorem 7.3.1. *(Hodge Index Theorem) If the Morphic Conjecture is true on $LH^{a,b}(X)_{\mathbb{C}}$ where X is a projective $2n$-manifold and a, b are nonnegative integers, then $\sigma_{a+n}(X) = \sum_{p,q}(-1)^q h^{p,q}_{s_{p,q}}$ where*

$$s_{p,q} = \begin{cases} a + \frac{p+q}{2}, & \text{if } p+q \text{ is even} \\ b + \frac{p+q-1}{2}, & \text{if } p+q \text{ is odd}. \end{cases}$$

Proof. Let $m = 2n$. Let ω be the $(1,1)$-form associated to the standard Kähler metric on X and let $\mathscr{H}^{p,q}$ be the space of harmonic (p,q)-forms on X. The Lefschetz operator $L : \mathscr{H}^{p,q} \longrightarrow \mathscr{H}^{p+1,q+1}$ is defined by $L\alpha = \omega \wedge \alpha$. Since ω is real, we have $\overline{L\alpha} = L\overline{\alpha}$, therefore L is a real operator. The conjugate of the hodge star operator $\bar{*} : \mathscr{H}^{p,q} \longrightarrow \mathscr{H}^{m-p,m-q}$ is an anti-isomorphism where $\bar{*}\alpha = \overline{*\alpha} = *\bar{\alpha}$ and the adjoint operator $\Lambda : \mathscr{H}^{p,q} \longrightarrow \mathscr{H}^{p-1,q-1}$ is given by the formula $\Lambda = (-1)^{p+q}\bar{*}L\bar{*}$. It is easy to verify that $\Lambda = (-1)^{p+q} * L *$ and $\overline{\Lambda\alpha} = \Lambda\overline{\alpha}$.

Using the Hodge structure on $\widetilde{L^s}H^k(X)$, decompose $\widetilde{L^s}H^k(X) = \bigoplus_{p+q=k} \mathscr{H}_s^{p,q}$. Now we assume that the Morphic Conjecture is true on $LH^{a,b}(X)$ and we use $\mathcal{L}, \lambda$ for the restriction of L, Λ to $LH^{a,b}(X)$ respectively. Let $h = \sum_{k=0}^{2m}(m - k)Pr_k$ where Pr_k is the projection of a form to its k-component and we use the same notation for its restriction to $LH^{a,b}(X)$. Let $B_0^{p,q} = ker\lambda : \mathscr{H}_{s_{p,q}}^{p,q} \longrightarrow \mathscr{H}_{s_{p,q}-1}^{p-1,q-1}$.

In the following, we verify each step as in the proof of the Hodge Index Theorem in [Hirz, Theorem 15.8.2].

1. $\lambda\mathcal{L}^k : B_0^{p-k,q-k} \longrightarrow B_0^{p-1,q-1}, p+q \leq m, k \geq 1$ is, up to a non-zero scalar factor, equal to $\mathcal{L}^k$.

   Proof. By using the relations $[\lambda, \mathcal{L}] = h$ and $[h, \mathcal{L}] = -2\mathcal{L}$. □

2. $\mathcal{L} : B_0^{p-k,q-k} \longrightarrow \mathscr{H}_{s_{p,q}}^{p,q}$ for $p+q \leq m$ is a monomorphism.

   Proof. If $\mathcal{L}^k\alpha = 0$, applying (1), we have $\mathcal{L}^{k-1}\alpha = c\lambda\mathcal{L}^k\alpha = 0$ and then repeating this process several times we get $\alpha = 0$. □

3. By the Lefschetz decomposition on $LH^{a,b}(X)$, we have

$$\mathscr{H}_{s_{p,q}}^{p,q} = B_0^{p,q} \oplus LB_0^{p-1,q-1} \oplus \cdots \oplus L^r B_0^{p-r,q-r}$$

   where $r = min(p,q)$.

4. For $\varphi \in \mathcal{L}^k B_0^{p-k,q-k}$ where $p+q=m$, $\bar{*}\varphi = (-1)^{q+k}\overline{\varphi}$.

5. By Lefschetz decomposition, we have

$$\widetilde{L}^{a+n} H^m(X)_{\mathbb{C}} = \bigoplus_{\substack{p+q=m \\ k \leq min(p,q)}} \mathcal{L}^k B_0^{p-k,q-k}$$

6. The summands in the above direct sum decomposition are mutually orthogonal with respect to the hodge inner product.

Proof. This is from the proof in [Hirz]. □

7.

$$\mathcal{L}^{a+n} H^m(X)_{\mathbb{R}} = \bigoplus_{\substack{p+q=m \\ k \leq p \leq q}} E_k^{p,q}$$

where $E_k^{p,q}$ is the real vector space of real harmonic forms α which can be written in the form $\alpha = \varphi + \overline{\varphi}$ with $\varphi \in \mathcal{L}^k B_0^{p-k,q-k}$.

8. $\sigma_{a+n}(X)$ is the index of the quadratic form

$$Q(\alpha, \beta) = \int_X \alpha \wedge \beta$$

where $\alpha, \beta \in \mathcal{L}^{a+n} H^m(X)_{\mathbb{R}}$.

9. By (4) and (6), the real vector space summands in the sum (7) are

mutually orthogonal with respect to Q. (4) implies that the quadratic form $(-1)^{q+k}Q(\alpha,\beta)$ is positive definite when restricted to $E^{p,q}$.

10. Therefore,

$$\sigma_{a+n}(X) = \sum_{\substack{p+q=m \\ k\leq p\leq q}} (-1)^{q+k} dim_{\mathbb{R}} E_k^{p,q}.$$

11.

$$\sigma_{a+n}(X) = \sum_{\substack{p+q=m \\ k\leq min(p,q)}} (-1)^{q+k} dim_{\mathbb{C}} \mathcal{L}^k B_0^{p,q}$$

Proof. $dim_{\mathbb{R}} E_k^{p,q} = 2 dim_{\mathbb{C}} \mathcal{L}^k B_0^{p,q}$ for $p < q$ and $dim_{\mathbb{R}} E_k^{n,n} = dim_{\mathbb{C}} \mathcal{L}^k B^{n,n}$. ☐

12. Let $h^{p,q}_{s_{p,q}} = dim_{\mathbb{C}} \mathscr{H}^{p,q}_{s_{p,q}}$. Then

$$h^{p-k,q-k}_{s_{p-k,q-k}} - h^{p-k-1,q-k-1}_{s_{p-k-1,q-k-1}} = dim_{\mathbb{C}} B_0^{p-k,q-k} = dim_{\mathbb{C}} \mathcal{L}^k B_0^{p-k,q-k}$$

for $p+q \leq m$.

Proof. From (3), we get

$$\mathscr{H}^{p-k,q-k} = B_0^{p-k,q-k} \oplus L B_0^{p-k-1,q-k-1} \oplus \cdots \oplus \mathcal{L}^r B_0^{p-k-r,q-k-r}$$

and

$$\mathscr{H}^{p-k-1,q-k-1} = B_0^{p-k-1,q-k-1} \oplus L B_0^{p-k-2,q-k-2} \oplus \cdots \oplus \mathcal{L}^{r-1} B_0^{p-k-r+1,q-k-r+1}.$$

Then by (2), we see that $dim_{\mathbb{C}} \mathcal{L}^i B_0^{p-k-i,q-k-i} = dim_{\mathbb{C}} \mathcal{L}^{i+1} B_0^{p-k-i-1,q-k-i-1}$ for $i \leq min(p-k, q-k)$. □

13. By the Morphic Conjecture, we have the Hard Lefschetz Theorem, therefore $h_{s_{p-k-1,q-k-1}}^{p-k-1,q-k-1} = h_{s_{p+k+1,q+k+1}}^{p+k+1,q+k+1}$ for $p+q=m$ and by Hodge structure, $h_{s_{r,k}}^{r,k} = h_{s_{k,r}}^{k,r} = h_{s_{m-r,m-k}}^{m-r,m-k}$.

14.

$$\sigma_{a+n}(X) = \sum_{p,q} (-1)^q h_{s_{p,q}}^{p,q}.$$

Proof. From (11) and (12), we see that

$$\sigma_{a+n}(X) = \sum_{\substack{p+q=m \\ k \leq min(p,q)}} (-1)^{q+k} (h_{s_{p-k,q-k}}^{p-k,q-k} - h_{s_{p-k-1,q-k-1}}^{p-k-1,q-k-1})$$

By (13) and some simple calculation, we get the formula. □

□

Corollary 7.3.2. *When $a = m$, the above formula gives the classical Hodge*

Index Theorem:

$$\sigma(X) = \sigma_{2m}(X) = \sum_{p,q}(-1)^q h^{p,q}_{s_{p,q}} = \sum_{p,q}(-1)^q h^{p,q}.$$

7.4 Some Discussions

7.4.1 Relations Between Signature And Euler Characteristic

The Hirzebruch Signature Formula says that the signature $\sigma(M)$ of an oriented smooth manifold M^{4k} can be represented as a linear combination of Pontrjagin numbers, i.e.,

$$\sigma(M) = L_k(p_1, ..., p_k)[M],$$

see [Hirz, chapter II.8] for the detail. We wonder if there is any formula of this kind for the morphic signatures. It will be very interesting and important to have this kind of formulas. For instance the Hodge Conjecture even for smooth hypersurfaces in $\mathbb{P}^n$ are known only for very few cases, see [Lew] for a survey. For a smooth hypersurface X of dimension $2n$ in a complex projective space $\mathbb{P}^{2n+1}$, by the Weak Lefschetz Theorem, we have $H^k(X;\mathbb{Q}) \cong H^k(\mathbb{P}^{2n+1};\mathbb{Q})$ for $k < 2n$. Thus dimension of $H^{p,p}(X;\mathbb{Q})$ is 1 for $p \neq n, 0 \leq p \leq 2n$ and $H^{p,p}(X;\mathbb{Q})$ is generated by algebraic cycles. Therefore the adjoint operator

$\Lambda : H^{2n-p,2n-p}(X;\mathbb{Q}) \longrightarrow H^{p,p}(X;\mathbb{Q})$ is an isomorphism for $0 \leq p < n$. For $p = n$, $\Lambda : H^{n,n}(X;\mathbb{Q}) \longrightarrow H^{n,n}(X;\mathbb{Q})$ which is an isomorphism. Thus the Grothendieck Standard Conjecture is trivially true for this case which is equivalent to the Morphic Conjecture for the case $LH^{0,0}(X)$, thus the signature formula above is valid for $a = 0$. By the weak Lefschetz Theorem, the cohomology groups of X are same as the cohomology groups of the projective space except the middle cohomology group, thus the signature formula has a very simple form:

$$\sigma_n(X) = 1 + (-1)^{n-1} + (-1)^n h_n^{n,n}$$

where $h_n^{n,n}$ is the dimension of the subspace of $H^{n,n}(X)$ which is generated by algebraic cycles. The Hodge Conjecture says that $h_n^{n,n} = h_{\mathbb{Q}}^{n,n}$ where $h_{\mathbb{Q}}^{n,n}$ is the dimension of $H^{n,n}(X;\mathbb{Q}) \overset{\text{def}}{=} H^{n,n}(X) \cap H^{2n}(X;\mathbb{Q})$. Thus if there is anyway to calculate $\sigma_n(X)$, it is equal to the calculation of $h_n^{n,n}$, which can be used to verify the Hodge Conjecture in the case of hypersurfaces since $h_{\mathbb{Q}}^{n,n}$ of hypersurfaces in projective spaces can all be calculated, see [Lew, Chapter 9].

The signature of oriented smooth $4k$-manifolds is a cobordism invariant. The main part of the proof of Hirzebruch Signature Formula is to apply Thom Theorem, which determines the structure of oriented cobordism with rational coefficients, to calculate the signature. Thus we hope there is a cobordism theory for morphic cohomology such that the morphic signatures are also cobordism invariants.

Let us recall a theorem by Rokhlin about a relation of the signature of a complex projective manifold and the Euler characteristic of its real points. See [Gud] for a proof and [DK] for theorems of this type.

Theorem 7.4.1. *(Rokhlin First Theorem) Suppose that X is a complex projective manifold of dimension $2n$. If X is an M-manifold, i.e., the $\mathbb{Z}_2$-betti number of X is same as the $\mathbb{Z}_2$-betti number of ReX, then*

$$\chi(ReX) \equiv \sigma(X) \ mod\ 16$$

where $\chi(ReX)$ is the $\mathbb{Z}_2$-euler characteristic of ReX and $\sigma(X)$ is the usual signature of X.

Since we believe our reduced real theory is the extension of singular theory, it is natural to ask if there is any theorem analogous to Rokhlin First Theorem in our reduced real Lawson homology and Lawson homology.

7.4.2 Relations With KR-Theory

A by-product in the joint work of Atiyah and Singer on the index theorem is the establishment of KR-theory by Atiyah in [A]. They discovered that for a smooth manifold X with smooth involution τ, they could define an involution on the cotangent sphere bundle $S(X)$ by

$$(x, \xi) \mapsto (\bar{x}, -\tau^*(\xi)).$$

But if τ is the identity involution on X, the involution on $S(X)$ is not the identity but is the anti-podal map on each fibre and this was the basic reason why the KR-theory was established. For the detail, see [A].

Karoubi and Weibel recently established an isomorphism between algebraic K-theory and KR-theory of nonsingular real algebraic varieties for some range, see [KW]. They proved a theorem, what they called the Real version of Riemann-Roch, and used Voevodsky's result of Milnor conjecture and Postnikov style tower of Friedlander and Suslin to obtain their results.

A closely related theory, semi-topological K-theory for real varieties, is established by Friedlander and Walker in [FW] which is related to the $\mathbb{Z}_2$-equivariant Lawson homology founded by Santos in [San].

We believe that there are Stiefel-Whitney classes which sends real algebraic vector bundles to reduced real morphic cohomology. The work is ongoing.

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Index