Quantum Cohomology and symplectomorphism type of S^1-manifolds with isolated fixed points

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Abstract of the Dissertation

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In this dissertation we study the quantum cohomology ring of symplectic manifolds with semi-free circle actions and isolated fixed points by means of the Seidel elements. We also provide a gluing procedure for Hamiltonian S^1-manifolds that allow us to classify these symplectic manifolds in dimension 6.

To my parents, Sue and Séneca.

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All of you make me feel the luckiest man in Babylon.

Chapter 1

Introduction

Let (M, ω) be a $2n$ dimensional compact, connected, symplectic manifold, and let $\{\lambda_t\} = \lambda : S^1 \longrightarrow \text{Symp}(M, \omega)$ be a symplectic circle action on M, that is, if X is the vector field generating the action, then $\mathcal{L}_X\omega = d\iota_X\omega = 0$. Recall that the action is **semi-free** if it is free on $M \setminus M^{S^1}$. This is equivalent to say that the only non-zero *weights* at every fixed point are ± 1. A circle action is said to be Hamiltonian if there is a C^∞ function $H : M \longrightarrow \mathbb{R}$ such that $\iota_X\omega = -dH$. Such a function is called a Hamiltonian for the action. This function is not unique, but it is up to a constant.

In this dissertation we study symplectic manifolds with symplectic circle actions whose fixed points are isolated. We study their quantum cohomology and in dimension six we classify them up to equivariant symplectomorphism.

Tolman and Weitsman proved in [TW00] that if the action is semi-free and admits only isolated fixed points, then the action must be Hamiltonian provided that there is at least one fixed point. There is a great deal of information concerning the topology of manifolds carrying such actions. The first result in this direction is due to Hattori [Hat92]. He proves that there is an isomorphism from the cohomology ring $H^*(M;\mathbb{Z})$ to the cohomology ring of a product of n copies of S^2. Moreover, this isomorphism preserves Chern classes. In [TW00] Tolman and Weitsman generalize Hattori's result to equivariant cohomology. The the first result of this dissertation is to provide an extension to quantum cohomology. In §2.2.1 we prove that M is an almost Fano manifold, therefore we can use polynomial coefficients $\Lambda := \mathbb{Q}[q_1, \ldots, q_n]$ for the quantum cohomology ring. Then we have the following theorem.

Theorem 1.0.1 *Let (M, ω) be a $2n$-dimensional compact connected symplectic manifold. Assume M admits a semi-free circle action with a finite non-empty set of fixed points. Then there is an isomorphism of (small) quantum cohomology*

$$QH^*(M;\Lambda) \cong QH^*((S^2)^n;\Lambda).$$

Note that we can compute directly the quantum cohomology of $S^2 \times \cdots \times S^2$ to get the following result.

Corollary 1.0.2 *The (small) quantum cohomology of M is given by*

$$QH^*(M;\Lambda) \cong QH^*((S^2)^n;\Lambda) \cong \frac{\mathbb{Q}[x_1, \ldots, x_n, q_1, \ldots, q_n]}{< x_i * x_i - q_i >}$$

where $\deg(x_i) = 2$ *and* $\deg q_i = 4$.

Moreover, all other products are given by

$$x_{i_1} * \cdots * x_{i_k} = x_{i_1} \smile \cdots \smile x_{i_k}$$

for $i_1 < \cdots < i_k$. Here the product on the left is the quantum product, while *the term on the right is the usual cup product.*

Chapter 2 is dedicated to the proof of Theorem 1.0.1. To prove Theorem 1.0.1 we will construct a set of generators $\{x_i\}$ of the cohomology ring $H^*(M;\mathbb{Z})$. Then we prove in Lemma 2.3.1 that the quantum products of these generators satisfy the expected relations given in Corollary 1.0.2.

The proof of Theorem 1.0.1 strongly relies on the techniques and results developed by McDuff and Tolman on the Seidel automorphism of the quantum cohomology of symplectic manifolds with circle actions [MT04]. We apply their results in our particular case and specialize them to understand exactly how the Seidel automorphism acts on the generators $\{x_i\}$. We will see in Corollary 2.2.13 that this action do not have higher order terms, that is the automorphism acts by single homogeneous terms in quantum cohomology. Thus the Seidel automorphism is essentially a permutation of the elements in the basis. We then use this and the associativity of the quantum product to compute the quantum products of the basis $\{x_i\}$. The construction of this basis is based on the tools that Tolman and Weitsman developed to prove the following theorem.

Theorem 1.0.3 ([TW00]) *Let (M,ω) be a compact, connected symplectic manifold with a semi-free, Hamiltonian circle action with isolated fixed points. Let y be the canonical generator of $H^*(BS^1,\mathbb{Z})$. Then, there is an isomorphism of rings $H^*_{S^1}(M) \simeq H^*_{S^1}(((S^2)^n))$ which takes the equivariant Chern classes of M to those of $(\mathbb{P}^1)^n$. Therefore the equivariant cohomology ring is given by*

$$H^*_{S^1}(M) = \mathbb{Z}[x_1,\ldots,x_n,y]/(x_i y - x_i^2).$$

Here $x_i \in H^2_{S^1}(M)$ *and the equivariant Chern series is given by* $c_t(M) = \sum_i c_i(M)t^i$ *where*

$$c_t(M) = \prod_i (1 + t(2x_i - y)).$$

Although Tolman and Weitsman use equivariant cohomology for getting an invariant base for $H^*(M;\mathbb{Z})$, the results of McDuff-Tolman require a more geometric description of the basis. Therefore the crucial element in most of the results of this paper is having geometric representatives of the cycles dual to the cohomology basis. These geometric representatives are defined by the Morse complex of the Hamiltonian function.

In Chapter 3 we will study the symplectomorphism type of Hamiltonian semi-free S^1-manifolds. To minimize our notations, we will normalize the Hamiltonian H by requiring the minimum of H to be 0. This makes H unique. We will denote by $\mathbf{HSymp}_{2n}$ the class of manifolds (M, H, ω) with semi-free Hamiltonian circle actions with normalized Hamiltonians.

In Chapter 3 we will provide a mechanism that allows to classify manifolds (M, H, ω) up to **isomorphism**, i.e. equivariant symplectomorphism. The idea is to investigate how to *reconstruct* M from its *local data* $\mathcal{L}(M)$ and what type of information determines the local data. $\mathcal{L}(M)$ can be thought as an atlas for Hamiltonian S^1-actions. It will be (roughly speaking) given by neighborhoods of the critical levels of H called *critical germs* and open submanifolds where the action is free called *free slices*. We also include in $\mathcal{L}(M)$ gluing maps on the overlaps of the germs and free slices. This information will be enough to reconstruct M, as proved in Theorem 3.1.5. We show that up to certain equivalence relations, $\mathcal{L}(M)$ determines the manifold M up to isomorphism. This works for all Hamiltonian S^1-actions.

We will then describe the local data in terms of more intrinsic invariants. In order to accomplish that, one has to make certain concessions. First, to describe the germs we restrict to the case when all critical levels have (co)index at most 2. In this case, because the reduced spaces $\overline{M}_\lambda$ are all smooth, we can apply the well-known Guillemin-Sternberg cobordism theorem [GS89] (c.f. Theorem 3.2.1) to describe the germs in terms of the fixed point data, as we now define.

Definition 1.0.4 *The fixed point data of M at a critical value λ of (co)index 2 consists of the triple $(\overline{M}_\lambda, \overline{F}_\lambda, \omega_\lambda)$ as well as the bundle $P_\lambda \longrightarrow \overline{M}_\lambda$ at λ. Here $\overline{F}_\lambda$ is the fixed point set at λ as a submanifold of $\overline{M}_\lambda$ and ω_λ is the reduced symplectic structure.*

*The **fixed point data** of M is the collection of all above tuples for all λ of (co)index 2, and the maximum and minimum submanifolds $(F_{\max}, \omega_{\max})$, $(F_{\min}, \omega_{\min})$ with their respective normal bundles in M.*

Note that this fixed point data contains more information than that used by Li in [Li03] (cf. Remark 3.3.3). In general, the fixed point data do not determine the isomorphism type of the free slices. However, as we will see in Lemma 3.2.7 this isomorphism type will be unique if the reduced spaces $\overline{M}_\lambda$ and the path reduced forms ω_t on $\overline{M}_\lambda$ for regular values t have the *rigidity* property, defined as follows.

Definition 1.0.5 *Let B a symplectic manifold. A tuple $(B, \{\omega_t\}_{t\in I})$, where $\{\omega_t\}_{t\in I}$ is a family of symplectic forms on B, is said to be **rigid** if*

(i) *Every deformation between any two cohomologous forms can be homotoped with fixed endpoints into an isotopy. (See more details in §3.2)*

(ii) $\mathrm{Symp}(B, \omega_t) \cap \mathrm{Diff}_0(B)$ *is path connected for all* $t \in I$.

The gluing maps we consider are defined so that near the critical levels we can identify, *along regular levels*, the free slices and the germs. This construction is motivated by the work of [Li05] but it differs in the fact that we glue along an open submanifold instead of just one level. In this situation, the gluing does not depend on the choices. Using these methods we can prove the following result.

Theorem 1.0.6 (Weak classification Theorem) *Let* $(M, H, \omega) \in \mathbf{H}\mathrm{Symp}_{2n}$ *and let* $\mathcal{C}(M)$ *be its set of critical values. Suppose that each non-extremal* $\lambda \in \mathcal{C}(M)$ *has (co)index 2 and that all the reduced spaces* $\overline{M}_\lambda$ *are rigid. Then* (M, H, ω) *is determined by its fixed point data up to equivariant symplectomorphism.*

In order to illustrate the method in other situations, we will discuss explicit examples where even less information classifies the manifold. We restrict ourselves to the case $\dim M = 6$ since we have two important advantages, all the non-extremal critical levels are simple and the reduced spaces are four dimensional. Therefore we can use established results on the unicity of the symplectic structures and on the topology of the group of symplectomorphisms. In this case, it is sometimes possible to describe the isomorphism class of M with less information, as the following theorem shows.

Theorem 1.0.7 *Let $M \in \mathrm{HSymp}_6$. Suppose that all the reduced spaces $\overline{M}_\lambda$ are rigid and that the fixed point sets $(\overline{F}_\lambda, \omega_\lambda)$ are either surfaces or isolated fixed points. Then, the isomorphism class of M is uniquely determined by the critical values $C(M)$ and the tuples $(\overline{M}_\lambda, \overline{F}_\lambda, \omega_\lambda)$.*

The most remarkable application of Theorem 1.0.7 is the following. Let $Y^n = S^2 \times \cdots \times S^2$ be the n-fold product of spheres and let σ be the canonical area form on S^2. Provide Y^n with the product symplectic form $\lambda_1\sigma \times \cdots \times \lambda_n\sigma$ that takes the value $\lambda_i > 0$ on each of the spheres of Y^n. Let the circle act diagonally on Y^n in the standard semi-free Hamiltonian fashion. Y^n is the only known example of a $2n$-dimensional symplectic manifold that admits a semi-free circle action with isolated fixed points. Thus, it is natural to ask if this is the only manifold up to equivariant symplectomorphisms that has this property. In the case $n = 2$ the methods of Karshon [Kar99] answer this question positively. In the present paper we establish the result for $n = 3$, we have the following corollary of Theorem 1.0.7.

Corollary 1.0.8 *Let M be a 6-dimensional symplectic manifold with a semi-free circle action that has isolated fixed points. Then, M is equivariantly symplectomorphic to Y^3 with the canonical product form for some λ_i.*

Here the λ_i are in fact the critical levels of the (only) three fixed points of index 2. As we will see, our techniques will apply for $n = 2$ as well, providing an alternative proof without using Karshon's results.

Theorems 1.0.3 and 1.0.1 work in all dimensions. However the techniques used to prove 1.0.7 do not work in higher dimensions. This is because our rigidity arguments strongly rely on results in four dimensional symplectic geometry, which do not have higher dimensional versions. We finally note that the proof of Theorem 1.0.8 does not use Wall's theorem, in contrast to previous methods.

As a last remark, we point out that it seems possible to remove the semi-free assumption from Theorems 1.0.6 and Corollary 1.0.8. In this case one has to deal with generalizations of our tools to the orbifold category. Fortunately the work of L. Godinho [God01, God04] gives some hope in this direction.

Chapter 2

Quantum Cohomology and circle actions with isolated fixed points

This chapter is organized as follows. All the Morse theoretical constructions are in §2.1.1. In section 2.1.2 we use equivariant cohomology to provide an invariant basis for cohomology. Then we establish the relation with the Morse cycles. In §2.2.1 we define the quantum cohomology ring and we get results that help to reduce the quantum product formulas. In §2.2.3 we define the Seidel automorphism in quantum cohomology. In §2.2.4 we relate the Seidel automorphism with invariant chains, then we compute explicitly the Seidel element. Finally in §2.3 we provide the proof of Theorem 1.0.1.

2.1 Morse Theory and Equivariant Cohomology

In this section we establish some of the tools we need to prove Theorem 1.0.1. We start in §2.1.1 with basic definitions of Morse theory. For more details the reader can consult [AB95, Sch99].

Following the approach of [MT04], in §2.1.2 we will construct invariant Morse cycles to be able to calculate the Seidel element of M. This will be done in Section 2.2.4. We introduce equivariant cohomology to identify a basis in cohomology and describe the relation with Morse cycles. At the end, we provide several results that will be necessary in §2.2.

2.1.1 Morse Theory

Let (M, ω) be a symplectic $2n$-dimensional manifold with a S^1 action generated by a Hamiltonian function H. Thus $\iota_X \omega = -dH$ and $X = J\text{grad}(H)$, where the gradient is taken with respect to the metric $g_J(x, y) = \omega(x, Jy)$ for an ω-compatible S^1-invariant almost complex structure J. With respect to this metric, H is a (perfect) Morse function [Kar99] and the zeroes of X are exactly the critical points of H. For each fixed point $p \in M^{S^1}$, denote by $\alpha(p)$ the index of p and let $m(p)$ be the sum of weights at p. Since the action is semi-free $m(p) = n_+(p) - n_-(p)$ where $n_+(p)$ is the number of positive weights and $n_-(p)$ the number of negative ones. Then $\alpha(p) = 2n_-(p) = n - m(p)$.

In order to understand the (co)homology of M in terms of S^1-invariant cycles, we will consider the *stable* and *unstable* manifolds with respect to the gradient flow $-\text{grad}(H)$. More precisely, let p, q be critical points of H. Define the stable and unstable manifolds by

$$W^s(q) = \{\gamma : \mathbb{R} \longrightarrow M | \lim_{t \longrightarrow \infty} \gamma(t) = q\},$$

$$W^u(p) = \{\gamma : \mathbb{R} \longrightarrow M | \lim_{t \longrightarrow -\infty} \gamma(t) = p\}.$$

Here $\gamma(t)$ satisfies the negative gradient flow equation

$$\gamma'(t) = -\text{grad}H(\gamma(t)).$$

These spaces are manifolds of dimension

$$\dim W^s(q) = 2n - \alpha(q) \quad \text{and} \quad \dim W^u(p) = \alpha(p),$$

and the evaluation map $\gamma \mapsto \gamma(0)$ induces smooth embeddings into M

$$E_q : W^s(q) \longrightarrow M \ \text{ and } \ E_p : W^u(p) \longrightarrow M.$$

When these manifolds intersect transversally for all fixed points p, q, the gradient flow is said to be Morse-Smale [AB95, Sch99]. Under this circumstance we say that the pair (H, g_J) is *Morse regular*.

In [Sch99] Schwarz proved that there is a way of *partially compactifying* the stable and unstable manifolds and that there are natural extensions of the evaluation maps so that these compactifications with their evaluation maps $E_p : \overline{W^s(p)} \longrightarrow M$ and $E_q : \overline{W^u(q)} \longrightarrow M$, define *pseudocycles.* The compactification of $W^s(p)$ is made by adding *broken trajectories* through fixed points of index $\alpha(p) - 1$. When the action is semi-free and admits isolated fixed points, all the fixed points have even index, therefore $W^s(p)$ is already compact in the sense of Schwarz. Thus $W^s(p)$ is itself a pseudocycle. The same is true for $W^u(q)$. It is well known that pseudocycles define classes in homology (see [MS04]). We will denote by $[W^u(q)] \in H_{\alpha(q)}(M; \mathbb{Z})$ and $[W^s(p)] \in H_{n-\alpha(p)}(M; \mathbb{Z})$ the homology classes defined by these manifolds. To get S^1-invariant cycles representing these classes, we need to consider a special type of almost complex structures, as we explain below.

Assume (M, ω) admits a Hamiltonian S^1-action with isolated fixed points. Each fixed point $p \in M$ has a neighborhood $U(p)$ that is diffeomorphic to a neighborhood of zero in a $2n$-dimensional Hermitian vector space $E(p) = E_1 \oplus \cdots \oplus E_n$, in such a way that the moment map H is given by

$$H(v_1, \ldots, v_n) = \sum_j \pi m_j |v_j|^2$$

and S^1 acts on E_j just by multiplication by $e^{2\pi i m_j}$. Here the numbers $m_j \in \mathbb{Z}$ are exactly the weights of the action. Under the identification above, the almost-complex structure J is the standard complex structure on the Hermitian vector space $E(p)$. Observe that $E(p)$ can be written as $E^+ \oplus E^-$ where $E^\pm$ is the sum of the E_j where $m_j > 0$ or $m_j < 0$ respectively. We can call the spaces $E^\pm$ the positive and negative normal bundles to the point p.

If we start with any compatible almost complex structure J_F near the fixed points, we can extend J to an S^1-invariant ω-compatible almost complex structure J_M on M whose restriction to the open sets $U(p)$ is J_F. Denote by $\mathcal{J}_{\text{inv}}(M)$ the set of all J that are equal to J_M near the fixed points.

The following lemma shows that it is possible to acquire regularity with generic almost-complex structures.

Lemma 2.1.1 ([**MT04**]) *Suppose that H generates a semi free S^1-action on (M, ω). Then for a generic choice of $J \in \mathcal{J}_{\text{inv}}(M)$ the pair (H, g_J) is Morse regular.*

For the rest of this paper, we will only consider Morse regular pairs (H, g_J) as in the previous lemma. We finally remark that when M is equipped with a regular pair and if there is a (broken) gradient trajectory from a fixed point p to a fixed point q, then $\alpha(p) - \alpha(q) > 0$.

2.1.2 Equivariant Cohomology

We can start with a quick review of equivariant cohomology. Let ES^1 be a contractible space where S^1 acts freely, and denote $BS^1 = ES^1/S^1$. Then $H^*(BS^1;\mathbb{Z})$ is the polynomial ring $\mathbb{Z}[y]$ where $y \in H^2(BS^1;\mathbb{Z})$.

Let S^1 act on a manifold M. The equivariant cohomology of M, denoted by $H^*_{S^1}(M)$ is defined by $H^*(M \times_{S^1} ES^1;\mathbb{Z})$. Note that $H^*(BS^1;\mathbb{Z})$ is naturally isomorphic to $H^*_{S^1}(pt)$, if $pt \in M$ is a point. Under this construction, we have two natural maps, the projection $p : M \times_{S^1} ES^1 \longrightarrow BS^1$ and the inclusion (as fiber) $i : M \longrightarrow M \times_{S^1} ES^1$. The pullback $p^* : H^*(BS^1;\mathbb{Z}) \longrightarrow H^*_{S^1}(M)$ makes $H^*_{S^1}(M)$ a $H^*(BS^1;\mathbb{Z})$ module, while the restriction $i^* : H^*_{S^1}(M) \longrightarrow H^*(M)$ is the "reduction" of invariant data to ordinary data. An immediate consequence is that $i^*(y) = 0$.

Let $j : M^{S^1} \longrightarrow M$ be the natural inclusion. In [Kar99] Kirwan proved that if the action is Hamiltonian, the induced map $j^* : H^*_{S^1}(M) \longrightarrow H^*_{S^1}(M^{S^1})$ is injective. The proof of this theorem is based on the following result, where we weaken the statement to match our needs. For a fixed point $p \in M^{S^1}$ we denote by $a|_p := (j_p)^*(a)$ where $(j_p)^* : H^*_{S^1}(M) \longrightarrow H^*_{S^1}(p)$ and j_p is the obvious inclusion.

Theorem 2.1.2 ([?]) *Let the circle act on a symplectic manifold M in a*

Hamiltonian way. Assume the action is semi-free and that there are only isolated fixed points. Let $p \in M$ be a fixed point of index $2k$. Then there exists a unique class $a_p \in H^{2k}_{S^1}(M)$ such that $a_p|_p = (-1)^k y^k$, and $a_p|_{p'} = 0$ for all other fixed points p' of index less than or equal to $2k$. Moreover, if we consider all fixed points, the classes a_p form a basis for $H^*_{S^1}(M)$ as a $H^*(BS^1;\mathbb{Z})$ module.

As a remark on the previous theorem, note that the term $(-1)^k y^k$ is the equivariant Euler class of the negative normal bundle at p.

As stated in §1, there is an isomorphism $H_*(M;\mathbb{Z}) \cong H_*(S^2 \times \cdots \times S^2;\mathbb{Z})$ if M satisfies the hypothesis of Theorem 2.1.2. Since H is perfect there are exactly $\dim(H_{2k}(M)) = \binom{n}{k}$ critical points of index $2k$. In [Hat92, TW00], the above isomorphism is proved by counting fixed points. We will not discuss the proof here.

Denote the points of index 2 by $p_1, \ldots, p_n$. In the light of Theorem 2.1.2 for each fixed point we get classes $a_1, \ldots, a_n \in H^2_{S^1}(M)$ such that

$$\begin{aligned} a_j|_{p_j} &= -y \\ a_j|_p &= 0 \qquad \text{for all other fixed points } p \text{ of index 0 or 1.} \end{aligned} \tag{2.1}$$

These classes satisfy the following Proposition.

Proposition 2.1.3 ([TW00, Prop 4.4]) *Let I be a subset of $\{1, \ldots, n\}$ with k elements. There exists a unique fixed point p_I of index $2k$ such that*

$$a_j|_{p_I} = -y \quad \textit{if and only if} \quad j \in I$$

and $a_j|_{p_I} = 0$ otherwise.

Proposition 2.1.3 identifies the fixed points in M with subsets I of $\mathcal{S} := \{1, \ldots, n\}$. Observe that the cohomology class $a_I := \prod_{i \in I} a_i \in H^{2k}_{S^1}(M)$ is the same as the class a_{p_I} mentioned in Theorem 2.1.2. Moreover this class is such that

$$a_I|_{p_J} = (-1)^k y^k \text{ if and only if } I \subseteq J \tag{2.2}$$

and it is zero otherwise.

Remark 2.1.4 *The class a_0, associated to the unique point of index zero, takes the value $1 \in H^0_{S^1}(pt)$ when restricted to any fixed point. Therefore it is the identity element in the ring $H^*_{S^1}(M)$. Denote ya_0 by y.*

If we apply the same results to the Hamiltonian function $-H$, we obtain unique classes $b_J \in H^{2n-2k}_{S^1}(M)$ associated to each p_J of index $2k$ such that $b_J|_{p_J} = (-1)^{n-k} y^{n-k}$ and is zero when restricted to all other fixed points of index greater or equal to $2k$. These classes also form a basis of $H^*_{S^1}(M)$. The next proposition establishes the relation with the former basis.

Proposition 2.1.5 *Let $I = \{i_1, \ldots, i_k\}$ and let $I^c = \{i_{k+1}, \ldots, i_n\}$ be its complement. Then the classes b_I satisfy the following relation*

$$b_I = \sigma_{n-k} + y\sigma_{n-k-1} + \cdots + y^{n-k}, \tag{2.3}$$

where σ_i is the i-th symmetric function in the variables $a_{i_{k+1}}, \ldots, a_{i_n}$.

Proof: By Proposition 2.1.3 the class $a_i + y$ is such that

$$(a_i + y)|_{p_J} = y \text{ if } i \notin J \text{ and zero otherwise.}$$

These are exactly the relations that characterize the class $b_{\{i\}^c}$, then

$$b_{\{i\}^c} = a_i + y.$$

Now, it follows that that

$$b_I = \prod_{i \notin I} b_{\{i^c\}} = \prod_{i \notin I} (a_i + y).$$

This proves the result. □

Consider a point p_I of index $2k$ and associate the class $a_I \in H^{2k}_{S^1}(M)$ as before. When we restrict a_I to M we obtain a class $a_I|_M \in H^{2k}(M;\mathbb{Z})$. By taking the Poincaré dual of $a_I|_M$, we get a homology class $p_I^+ \in H_{2n-2k}(M;\mathbb{Z})$. Similarly using the class b_I we get a homology class $p_I^- \in H_{2k}(M;\mathbb{Z})$. Here is an immediate corollary of Proposition 2.1.5.

Corollary 2.1.6 *The class p_I^- is the same as the class $p_{I^c}^+$.*

Proof: This is clear from Equation (2.3), because the variable y is mapped to zero under restriction to usual cohomology. Now use that $\sigma_{n-k} = a_{I^c}$. □

The last part of this section establishes the relation of the $p_I^\pm$ classes with the stable and unstable manifolds of §2.1.1. This is summarized in the following proposition. Remember that we are working with an almost-complex structure J in $\mathcal{J}_{\text{inv}}(M)$. This result would fail without this hypothesis.

Proposition 2.1.7 *Let p_I be a fixed point of index $2k$. Then the classes p_I^- and p_I^+ are exactly the same as the classes $[W^u(p_I)]$ and $[W^s(p_I)]$ respectively.*

Proof:

Recall that ES^1 can be taken to be the infinite dimensional sphere S^∞. Consider a finite dimensional approximation $M^N := M \times_{S^1} S^{2N+1}$ of $M \times_{S^1} ES^1 = M \times_{S^1} S^\infty$ for $N \in \mathbb{N}$ big enough . These are finite dimensional smooth compact manifolds. Since $W^s(p_I)$ is S^1-invariant, there is a natural extension $W^{N,s}(p_I) := W^s(p_I) \times_{S^1} S^{2N+1}$ of $W^s(p_I)$ to M^N. Let X^N be the Poincaré dual of $W^{N,s}(p_I)$ in M^N.

For all N, there is a natural inclusion (as fibre) $i_N : M \hookrightarrow M^N$. Since the inclusions are natural, the restriction $X^N|_M := (i_N)^*(X^N) \in H^*(M)$ is the same as the Poincaré dual of $[W^s(p_I)]$ in M.

Observe that the natural inclusions

$$M^N \hookrightarrow M^{N+1} \hookrightarrow \cdots \lim_N M^N = M \times_{S^1} ES^1$$

induce a sequence

$$\cdots \longrightarrow X^{N+2} \longrightarrow X^{N+1} \longrightarrow X^N$$

given by the restrictions. Thus, by considering the directed limit, there is an element

$$X := \lim_N X^N \in H^*(M \times_{S^1} ES^1) = H^*_{S^1}(M)$$

that restricts to X^N for all N. Naturally, if $i : M \hookrightarrow M \times_{S^1} ES^1$ is the inclusion, then $X|_M := i^*(X) = \mathrm{PD}([W^s(p_I)])$. We claim that X satisfies the same properties as the class a_I, that is, $X|_{p_I} = (-1)^k y^k$ and $X|_p = 0$ for all other fixed points p such that $\alpha(p) \leq 2k$. Therefore, by Theorem 2.1.2 we must have $X = a_I$. Then $\mathrm{PD}(X|_M) = \mathrm{PD}(a_I|_M)$ and the result will follow immediately.

Take a neighborhood $U(p_I)$ around p_I as in §2.1.1. Thus, $U(p_I)$ is isomorphic to an open neighborhood V of zero in $E^+ \oplus E^-$. It is clear that if $U(p_I)$ is small enough, $W^s(p_I) \cap U(p_I)$ is diffeomorphic to $E^+ \cap V$. Therefore, the normal bundle of $W^s(p_I)$ can locally be identified with E^-. Finally, by carrying this localization to X^N and considering the limit, we have $X|_{p_I} = e(E^-) = (-1)^k y^k$, where $e(E^-)$ is the equivariant Euler class of E^-.

To finish the proof, observe that if p is any other fixed point with index less than or equal to $2k$, there is no gradient line from p to p_I. This is because the gradient flow is Morse-Smale. Hence, by using the localization again we obtain that $X|_p = 0$. This proves the proposition.

□

Corollary 2.1.8 *By the definition of the classes a_I and b_I, we have*

$$[W^u(p_I)] = p_I^- = \mathrm{PD}(b_I|_M) \text{ and } [W^s(p_I)] = p_I^+ = \mathrm{PD}(a_I|_M),$$

therefore the product $[W^u(p_I)] \cap [W^s(p_J)]$ is given by

$$[W^u(p_I)] \cap [W^s(p_J)] = \mathrm{PD}(b_I|_M) \cap \mathrm{PD}(a_J|_M) = \mathrm{PD}(b_I a_J|_M).$$

Corollary 2.1.9 *By Corollary 2.1.6 and Proposition 2.1.7 we have the "duality" relation $[W^u(p_I)] = [W^s(p_{I^c})]$.*

Remark 2.1.10 *Let* $x_i := a_i|_M = \mathrm{PD}(p_i^+) \in H^2(M;\mathbb{Z})$. *The theory of this section proves that the elements* x_i *generate the algebra* $H^*(M;\mathbb{Z})$. *Therefore a* $\mathbb{Z}$*-basis for* $H^{2k}(M;\mathbb{Z})$ *consists of the elements* $x_I = x_{i_1} \dots x_{i_k}$ *for sets* $I = \{i_1 < i_2 \dots < i_k\}$. *Moreover, by Theorem 1.0.3 the first Chern class of* M *is given by* $c_1(M) = 2(x_1 + \dots + x_n)$, *thus the Chern numbers of the spherical classes* p_k^- *are* $c_1(p_k^-) = 2$ *for all* $k = 1, \dots, n$.

Proposition 2.1.7 also provides some information about the existence of gradient lines. More precisely we have the next proposition.

Proposition 2.1.11 *Let* $I = \{i_1, \dots, i_k\} \subset \mathcal{S}$. *Take* $i_{k+1} \notin I$ *and consider* $I' = I \cup \{i_{k+1}\}$. *Let* $A_I := \sum_{i \in I} p_i^- \in H_2(M)$. *Then,*

a) There is a gradient line from $p_{I'}$ *to* p_I. *Moreover, the homology class of the sphere generated by rotating the gradient line by the* S^1 *action is* $p_{i_{k+1}}^-$.

b) There is a broken gradient line from $p_{\mathcal{S}}$ *to* p_I. *The class* A_{I^c} *is then represented by rotating this broken line and then* $c_1(A_{I^c}) = n + m(p_I)$.

Proof:

To prove there is a gradient line from $p_{I'}$ to p_I we need to show that the intersection $W^u(p_{I'}) \cap W^s(p_I)$ is non-empty. By definition of the intersection product in terms of pseudocycles [MS04] it is enough to prove that the intersection product of the classes $[W^u(p_{I'})]$ and $[W^s(p_I)]$ is non-zero.

Consider the equivariant cohomology classes $b_{I'}$ and a_I. By Proposition 2.1.5 we get

$$b_{I'} a_I = a_{I'^c} a_I + yd$$

where $d \in H^*_{S^1}(M)$. Since $I'^c \cup I = \{i_{k+1}\}^c$,

$$a_{I'^c} a_I = a_{\{i_{k+1}\}^c}.$$

Once again by Proposition 2.1.5

$$a_{\{i_{k+1}\}^c}|_M = b_{i_{k+1}}|_M,$$

thus

$$b_{I'} a_I|_M = b_{i_{k+1}}|_M.$$

Now, using Corollary 2.1.8 we get

$$[W^u(p_{I'})] \cap [W^s(p_I)] = \mathrm{PD}(b_{I'} a_I|_M) = \mathrm{PD}(b_{i_{k+1}}|_M) = p^-_{i_{k+1}} \neq 0. \qquad (2.4)$$

Therefore, there is a gradient line, thus a whole *gradient sphere* A, just by rotating the gradient line. Note that there can be more than one gradient sphere from $p_{I'}$ to p_I. We claim that all these gradient spheres must be homologous.

It is not hard to see from the construction of A that

$$\omega(A) = \int_A \omega = H(p_{I'}) - H(p_I).$$

Therefore if A' is another gradient sphere joining $p_{I'}$ and p_I, $\omega(A) = \omega(A')$. Also observe that if ω' is any S^1-invariant form sufficiently close to ω then $\omega(A) = \omega'(A)$. Now since the symplectic condition is an open condition we can perturb ω to obtain a new symplectic form ω' close to ω. By averaging with respect to the group action, we can assume the form ω' to be S^1-invariant. This proves that the classes A' and A have the same symplectic area, that is $\omega'(A) = \omega'(A')$, for an open set of symplectic forms ω'. Since M is simply connected and there is no torsion A must be homologous to A'. Finally by Equation (2.4) this sphere must be in class $p^-_{i_{k+1}}$. Finally, recall that by Remark 2.1.10 the Chern number of the class $p^-_{i_{k+1}}$ is given by $c_1(p^-_{i_{k+1}}) = m(p_I) - m(p_{I'}) = 2$.

To prove the second part, we can do the same process for each point in $I^c = \{i_{k+1}, \ldots, i_n\}$. Then getting a sequence of gradient lines

$$p_S \xrightarrow{\gamma_1} p_{S-\{i_{n-1}\}} \cdots p_{I\cup\{i_{k+1}\}} \xrightarrow{\gamma_{n-k}} p_I.$$

It is clear now that the chain of gradient spheres obtained by rotating this broken gradient line must be in class $A_{I^c} = A_{i_{k+1}} + \cdots + A_{i_n}$. Note that we could also use a gluing argument as in [Sch93] to prove that there is an honest gradient line from p_S to p_I. Thus

$$c_1(A_{I^c}) = \sum_{l=k+1}^{n} c_1(A_{i_l}) = m(p_I) - m(p_S) = n + m(p_I).$$

□

Corollary 2.1.12 *Assume x, y are any fixed points in M such that there is a gradient line joining them. Then, the Chern class of the J-holomorphic sphere that is obtained by rotating the gradient line by the action has Chern number $|m(x) - m(y)|$.*

Proof: The proof is based on the same computations of Proposition 2.1.11. Recall that $m(x) = n - \alpha(x)$ where $\alpha(x)$ is the Morse index of x. Since there is a gradient line joining x and y we can assume that they do not have the same index, otherwise it contradicts the fact that the gradient flow is Morse-Smale. Without lost of generality assume $m(x) < m(y)$, and that $x = p_I, y = p_J$. Following the notation of the previous proposition, let A_{I^c} and A_{J^c} be the classes of the spheres obtained by rotating the gradient lines that join p_S with p_I and p_S with p_J respectively. Thus, if A is the class of the sphere obtained by rotating the gradient line that joins p_I with p_J, then $A + A_{I^c} = A_{J^c}$. Finally, the Chern numbers satisfy $c_1(A) + c_1(A_{I^c}) = c_1(A_{J^c})$, which in turn by Proposition 2.1.11 gives $c_1(A) = n + m(p_J) - n - m(p_I) = m(x) - m(y)$.

□

2.2 Quantum Cohomology and the Seidel Automorphism

2.2.1 Small Quantum Cohomology

In the literature, there are several definitions of quantum cohomology. In this section we make precise the definition of the quantum cohomology we are using, assuming the definition of genus zero Gromov-Witten invariants. We will follow entirely the approach of [MS04, Chapter 11].

Let Λ_ω be the usual *Novikov ring* of (M, ω). We recall that Λ_ω is the completion of the group ring of $H_2(M) := H_2(M;\mathbb{Z})/\text{Torsion}$. It consists of all (possibly infinite) formal sums of the form

$$\lambda = \sum_{A \in H_2(M)} \lambda_A e^A$$

where $\lambda_A \in \mathbb{R}$ and the sum satisfies the finiteness condition

$$\#\{A \in H_2(M) | \lambda_A \neq 0, \omega(A) \leq c\} < \infty$$

for every real number c. By definition, $\deg(e^A) = 2c_1(A)$, where c_1 is the first Chern class of M.

The **(small) quantum cohomology** of M with coefficients in Λ_ω is defined by

$$QH^*(M) := H^*(M) \otimes_{\mathbb{Z}} \Lambda_\omega.$$

As before $H^*(M)$ denotes the ring $H^*(M;\mathbb{Z})$ modulo torsion. We now proceed to define the **quantum product** on $QH^*(M)$. We want the quantum product to be a linear homomorphism of Λ_ω-modules

$$QH^*(M) \otimes_{\Lambda_\omega} QH^*(M) \longrightarrow QH^*(M) : (a, b) \mapsto a * b.$$

Since $QH^*(M)$ is generated by the elements of $H^*(M)$ as a Λ_ω-module, it is enough to describe the multiplication for elements in $H^*(M)$. Let $e_0, e_1, \ldots, e_n$ be a basis for $H^*(M)$ (as a $\mathbb{Z}$-module). Assume each element is homogeneous and $e_0 = 1$, the identity for the usual product. Define the integer matrix

$$g_{ij} := \int_M e_i \smile e_j.$$

Here $e_i \smile e_j$ is the usual cup product in cohomology. Let g^{ij} be the inverse matrix. The quantum product of $a, b \in H^*(M)$, is defined by

$$a * b := \sum_{B \in H_2(M)} \sum_{k,j} \mathrm{GW}^M_{B,3}(a, b, e_k) g^{kj} e_j \otimes e^B. \tag{2.5}$$

The coefficients $\mathrm{GW}^M_{B,3}$ are the usual Gromov-Witten invariants of J-holomorphic curves in class B. The terms in the sum are nonzero only if $\deg(e_k)+\deg(e_j) = \dim M$ and $\deg(a) + \deg(b) + \deg(e_k) = \dim M + 2c_1(B)$. Thus, it is enough to consider classes B such that

$$\deg(a) + \deg(b) - \dim M \leq 2c_1(B) \leq \deg(a) + \deg(b).$$

In the problem at hand, a basis for $H^*(M)$ is given by the elements x_I as in 2.1.10. Then the integrals

$$g_{IJ} = \int_M x_I \smile x_J$$

all vanish unless the sets I and J are complementary. This is because if I, $J \subset \{1, \ldots, n\}$, $x_I \smile x_J = x_S$ if and only if $I^c = J$. Here x_S is the positive generator of $H^{2n}(M; \mathbb{Z})$.

We claim that to compute the quantum product, we only need to consider in Equation (2.5) classes B such that $c_1(B) \geq 0$. More precisely, we have the proposition.

Proposition 2.2.1 *Assume (M, ω) is a symplectic manifold with a semi-free S^1-action with only isolated fixed points. Let $B \in H_2(M)$, and let $a, b, c \in H^*(M)$. If $c_1(B) < 0$, then the Gromov-Witten invariant $\mathrm{GW}^M_{B,3}(a, b, c)$ is* *zero. Moreover, if* $c_1(B) = 0$ *and some* $\mathrm{GW}^M_{B,3} \neq 0$*, then* $B = 0$*. Therefore, the expression for the quantum product (2.5) can be written as*

$$a * b = a \smile b + \sum_{B \in H_2(M), c_1(B) > 0} a_B \otimes e^B.$$

where the classes a_B *have degree* $\deg(a_B) = \deg a + \deg b - 2c_1(B)$.

Remark 2.2.2 *Note that since* $c_1(B)$ *is even, the classes* a_B *appear in the sum above by "jumps" of four in the degree.*

The rest of this section is dedicated to the proof of Proposition 2.2.1.

To compute the Gromov-Witten invariants $\mathrm{GW}^M_{B,3}(a, b, c)$ one usually constructs a regularization (virtual cycle) $\overline{\mathcal{M}}^{\nu}_{0,3}(M, J, B)$ of the moduli space $\overline{\mathcal{M}}_{0,3}(M, J, B)$. Then one computes the intersection number of the evaluation map

$$ev : \overline{\mathcal{M}}^{\nu}_{0,3}(M, J, B) \longrightarrow M^3$$

with a cycle $\alpha_1 \times \alpha_2 \times \alpha_3$ representing the class $\mathrm{PD}(a) \times \mathrm{PD}(b) \times \mathrm{PD}(c)$. This procedure can be modified in the following way. First, let $\alpha : Z \longrightarrow M^3$ be a pseudocycle that represents the product $\mathrm{PD}(a) \times \mathrm{PD}(b) \times \mathrm{PD}(c)$, then define the *cut-down* moduli space by

$$\overline{\mathcal{M}}_{0,3}(M, J, B; Z) := ev^{-1}(\overline{\alpha(Z)}).$$

Here $ev : \overline{\mathcal{M}}_{0,3}(M, J, B) \longrightarrow M^3$ is the evaluation map and $\overline{\alpha(Z)}$ is the closure in M^3 of the pseudocycle Z [MS04]. Finally, construct a regularization of the cut-down moduli space. McDuff and Tolman use this approach to calculate the Gromov-Witten invariants. The next result is proved in [MT04], it shows exactly how to compute the invariants $\mathrm{GW}^M_{B,3}$ using this procedure. Remember that an S^1 action on M can be extended to an action on J-holomorphic curves just by post-composition. Also, a pseudocycle $\alpha : Z \longrightarrow M^3$ is said to be S^1-invariant, if $\alpha(Z)$ is.

Proposition 2.2.3 *Let (M, ω) be a symplectic manifold. Then, the Gromov-Witten invariant $\mathrm{GW}^M_{B,3}(a, b, c)$ is a sum of contributions, one from each connected component of the moduli space $\overline{\mathcal{M}}_{0,3}(M, J, B; Z)$.*

Assume now that M is equipped with an S^1 action $\{\lambda_t\}$, and that $\alpha : Z \longrightarrow M^3$ and J are S^1-invariant. Then, a connected component of $\overline{\mathcal{M}}_{0,3}(M, J, B; Z)$ makes no contribution to $\mathrm{GW}^M_{B,3}(a, b, c)$ unless it contains an S^1-invariant element.

Proposition 2.2.3 shows that to compute the Gromov-Witten invariants in the presence of a circle action, one has to compute the invariant elements in the moduli spaces. The following lemma is a modification of McDuff-Tolman [MT04, Lemma 3.5]. It describes what the non-constant invariant elements in the moduli space $\mathcal{M}_{0,k}(M, J, B)$ are. We include a proof so that Corollary 2.2.5 is a natural result.

Lemma 2.2.4 *Let (M, ω) be a symplectic manifold with a Hamiltonian S^1-action. Let J be an invariant almost complex structure compatible with ω, and let g_J be the S^1-invariant metric associated to J. Suppose $[u, z_1, \ldots, z_k]$ is a class in the moduli space $\mathcal{M}_{0,k}(M, J, B)$ represented by a J-holomorphic sphere $u : \mathbb{P}^1 \longrightarrow M$, and k marked points $z_i \in \mathbb{P}^1$. Assume $[u, z_1, \ldots, z_k]$ is* *fixed by the action* $\lambda = \{\lambda_\theta\}$. *Then, if* $\mathrm{im}(u)$ *does not lie entirely in* M^{S^1}, *there are at most two marked points, i.e.* $k \leq 2$ *and there are two integers* p, q, $p \neq 0, q > 0$, *a parametrization* $u : \mathbb{R} \times S^1 \longrightarrow M$ *and a path* $\gamma : \mathbb{R} \longrightarrow M$ *joining two fixed points* $x, y \in M^{S^1}$ *so that the marked points are in* $u^{-1}\{x, y\}$ *and such that*

$$\gamma'(s) = \frac{p}{q}\,\mathrm{grad}(H) \text{ and } u(s,t) = \lambda_{\frac{pt}{q}}\gamma(s). \tag{2.6}$$

Moreover, if we fix γ, *the parametrization is unique provided*

$$x = \lim_{s \longrightarrow -\infty} \gamma(s) \text{ and } y = \lim_{s \longrightarrow \infty} \gamma(s).$$

Finally, if the action is semi-free, then q *is* 1.

Proof: If the image of u is in M^{S^1}, then the map is constant. Assume that $u : \mathbb{P}^1 \longrightarrow M$ is a non-constant and not multiply covered J-holomorphic sphere in M. Since the equivalence class $[u, z_1, \ldots, z_k]$ is fixed under the action, for each $\theta \in S^1$ the map $(u', z_1', \ldots, z_k') = (\lambda_\theta \circ u, \lambda_\theta(z_1), \ldots, \lambda_\theta(z_k))$ must be a reparametrization of $(u, z_1, \ldots, z_k)$. Thus, there is a $\phi_\theta \in \mathrm{PSL}(2, \mathbb{C})$ such that $\lambda_\theta \circ u = u \circ \phi_\theta$, and $z_i' = \phi_\theta(z_i)$. Therefore $\lambda_\theta(u(z_i)) = u(z_i)$ for all $\theta \in S^1$, then $u(z_i) \in M^{S^1}$ for all i. Since the map u is not multiply covered, ϕ_θ is unique. Then, it is easy to see that the assignment $S^1 \longrightarrow \mathrm{PSL}(2, \mathbb{C}) : \ \theta \mapsto \phi_\theta$ is a homomorphism. Using the fact that the only circle subgroups of $\mathrm{PSL}(2, \mathbb{C})$ are rotations about a fixed axis, we can see that there are exactly two points in $\mathbb{P}^1$ that are mapped by u into M^{S^1}. We can choose coordinates on $\mathbb{P}^1$ so that the rotation axis is the line joining the points $[0 : 1]$ and $[1 : 0]$. As we saw before, the image of any marked point is fixed by the action, then we have that all the marked points are contained in the set $\{[0:1],[1:0]\}$. It follows that $k \leq 2$. In the case $\mathrm{Im}(u) \cap M^{S^1} = \{x, y\}$, we may choose $u([0:1]) = x$ and $u([1:0]) = y$. Identify $\mathbb{P}^1 \backslash \{[0:1],[1:0]\}$ with the cylinder $\mathbb{R} \times S^1$ with standard coordinates (s,t) and complex structure j_0 defined by $j_0(\partial_s) = \partial_t$, $(s,t) \in \mathbb{R} \times S^1$. If $k = 2$ we identify the marked points $[0:1],[1:0]$ with the ends of the cylinder. Thus we find that for these coordinates there is a $q \neq 0$ such that

$$\phi_\theta(s,t) = (s, t + q\theta), \text{ and } (\lambda_\theta \circ u)(s,t) = u(s, t + q\theta).$$

Therefore, the isotropy group at any point in $\mathrm{im}(u)$ is given by $\mathbb{Z}_{|q|}$. The sign of q is uniquely determined by the choices we made.

Define $\gamma(s) := u(s,0)$. Then we get $u(s,t) = \lambda_{t/q}\gamma(s)$. Since u is J-homomorphic and J is invariant

$$(\lambda_{\frac{t}{q}})_*(\gamma'(s) + \frac{1}{q} JX(\gamma(s))) = \partial_s u + J\partial_t u = 0.$$

With respect to the metric g_J, the gradient flow of H is given by $\mathrm{grad} H = -JX$, thus $\gamma'(s) = \frac{1}{q}\mathrm{grad}(H)(\gamma(s))$. Now use that any sphere is a $|p|$-fold cover of a simple one. We absorb any negative sign into p rather than q.

If the action is semi-free, the isotropy groups are trivial, thus we must have $|q| = 1$. □

To exemplify the choice of signs in the previous lemma, take $M = \mathbb{P}^1$ with the standard semi-free circle action that rotates M with speed one. Clearly the Hamiltonian is the hight function and the only fixed points are $S := [0:$

1], $N := [1:0]$. Assume the holomorphic map $u : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is simple and has two marked points, which as before are just identified with S, N. If we want to parametrize this map, we have to choose a gradient line that joins N and S. Say we have chosen the gradient line that goes from the north pole N to the south pole S in this order (downwards). If we identify $\mathbb{P}^1 \backslash \{S, N\}$, with the cylinder $\mathbb{R} \times S^1$, in the standard coordinates (s, t) the action must rotate negatively. That is we have that q must be -1 and $p = 1$.

Although the unicity of the parametrization is not needed for the proof of the following result it is good to have a canonical choice of the parametrization. The next corollary will be needed in the proof of Proposition 2.2.1.

Corollary 2.2.5 *Assume the same hypothesis as in Lemma 2.2.4, and that the action is semi-free. Let u be an S^1-invariant sphere, and let $A \in H_2(M)$ be its homology class in M. Then, with the parametrization provided by Equation (2.6), the first Chern number $c_1(A)$ is given by $c_1(A) = |p(m(y) - m(x))|$ with p an integer. Therefore $c_1(A)$ is non-negative.*

Proof: If u is constant $c_1(A) = 0$. If u is not constant it must have a parametrization as the one given in Lemma 2.2.4. That is, for some fixed points x, y in M, and a gradient line γ joining them, u can be parametrized as the Equation (2.6). Assume without lost of generality that $m(x) < m(y)$, then we may choose the path γ from x to y. Then $q = -1$, and p is positive (note that we have chosen the *negative* gradient to be Morse-Smale). This proves that u is a p-cover of the simple gradient sphere B obtained by rotating the gradient line γ joining x, y. By Corollary 2.1.12 $c_1(B) = m(y) - m(x)$. And then $c_1(A) = p(m(y) - m(x))$. □

Remark 2.2.6 *Let u be an S^1-invariant holomorphic sphere, let $A \in H_2(M)$ be its homology class. Lemma 2.2.4 and Corollary 2.2.5 imply that if $c_1(A) = 0$ then A must be zero. This is because if A joins two fixed points $x, y \in M$, they must have the same index, which is not possible because the flow is assumed to be Morse-Smale.*

Note that our original goal was to understand the invariant stable maps in $\overline{\mathcal{M}}_{0,3}(M, J, B; Z)$. By Lemma 2.2.4, the non-constant components of the stable maps may carry at most two special points. Then the S^1-invariant elements in $\overline{\mathcal{M}}_{0,3}(M, J, B; Z)$ may have a *ghost* component that carries the third marked point.

Proof of Proposition 2.2.1: By Proposition 2.2.3 a contributing component of $\overline{\mathcal{M}}_{0,3}(M, J, B; Z)$ for the invariant $\mathrm{GW}^M_{B,3}(a, b, c)$ exists only if the moduli space has a S^1-invariant stable map $\mathbf{u}$. We can assume that for a representative $\{u_i\}$ of $\mathbf{u}$ there is at least one non-trivial component u_α. Since $\mathbf{u}$ is invariant, we can choose the representative $\{u_i\}$ so that the component u_α is invariant (up to reparametrization) under the circle action. This is clear if for instance the representative $\{u_i\}$ does not have any automorphisms that interchange its components. In the case such automorphisms exist, we can always choose a representative such that it has a component that is fixed by the action. To exemplify this, assume for simplicity that $\mathbf{u}$ has three components $u_i : \Sigma_i \simeq \mathbb{P}^1 \longrightarrow M, i = 1, 2, 3$, where u_3 is constant and carries a marked point (so that the map is stable), and u_1, u_2 map Σ_1, Σ_2 into the same image in M. Therefore, there is an automorphism of this stable map that permute the domains Σ_1, Σ_2. Let $\theta \in S^1$. Since λ_θ fixes $\mathbf{u}$, then, by the mere definition of stable map (Definition 5.1.4 in [MS04]), we have that $\lambda_\theta \circ u_1 = u_{f_\theta(1)} \circ \psi_\theta$, where ψ_θ is an element in $\mathrm{PSL}(2, \mathbb{C})$ and f_θ is an bijection on the set $\{1, 2\}$. Note that f_θ does not need to depend continuously on θ, but in any case we may change the representative of u so that for all θ, f_θ is the identity. Thus, the component u_1 is invariant (up to reparametrization) with respect to the circle action.

Therefore, following the same idea as in the proof of Corollary 2.2.5 we have that $c_1(B_\alpha) > 0$ if $B_\alpha \in H_2(M)$ is the class that u_α represents. Then $c_1(B) > 0$ and the first claim follows. Note that the second part is a direct consequence of Remark 2.2.6, because any S^1-invariant J-holomorphic map with zero Chern class must be constant.

Finally , the product $a * b$ can be written as

$$a \smile b + \sum_{c_1(B)>0} \sum_I \mathrm{GW}^M_{B,3}(a, b, x_I) x_{I^c} \otimes e^B.$$

Now take

$$a_B := \sum_I \mathrm{GW}^M_{B,3}(a, b, x_I) x_{I^c}.$$

This proves the proposition. Note that $\deg(a_B) = \deg a + \deg b - 2c_1(B)$.

□

2.2.2 Almost Fano Manifolds

Assume the hypotheses of Proposition 2.2.1. The relevant spheres (the ones that count for the GW invariants) all have positive first Chern class. Moreover, let $B \in H_2(M)$ be a class such that some invariant $\mathrm{GW}^M_{B,3} \neq 0$, then $c_1(B) \geq$

0. Then, by Proposition 2.1.11 and Lemma 2.2.4, B can be written as a combination

$$B = \sum_i d_i p_i^-$$

where the coefficients d_i are non-negative integers. Therefore, if we define $A_i := p_i^-$ and $q_i := e^{A_i}$, we may now consider the polynomial ring

$$\Lambda = \mathbb{Q}[q_1, \ldots, q_n]$$

as coefficients for the quantum cohomology. Then, if B is as before,

$$e^B = {q_1}^{d_1} \ldots {q_n}^{d_n}.$$

This will be really useful in §2.3. For the rest of this paper, we will assume Λ to be the quantum coefficient ring.

We finish this section with a discussion about the behavior of J-holomorphic curves in M. In the literature an almost complex manifold (N, J) is said to be **Fano** if the first Chern class $c_1(TN, J)$ takes positive values on the **effective cone** $K^{\text{eff}}(N, J)$, namely

$$K^{\text{eff}}(N, J) := \{A \in H_2(N) | \exists \text{ a } J\text{-holomorphic curve in class } A\}.$$

In symplectic geometry sometimes it is useful to consider the definition

$$K^{\text{eff}}(N, \omega) = \{A \in H_2(N) | \exists A_1, \ldots, A_n \in H_2(N) : A = \sum_i A_i, \mathrm{GW}^M_{A_i,3} \neq 0\}$$

for the effective cone on a symplectic manifold (N, ω, J) with a compatible almost complex structure J. Its clear that $K^{\text{eff}}(N, \omega) \subset K^{\text{eff}}(N, J)$. Then, we can say that (N, ω, J) is **almost Fano** if the first Chern class $c_1(TN, J)$ takes positive values on the effective cone $K^{\text{eff}}(N, \omega)$. We have the following corollary.

Corollary 2.2.7 *Let (M, ω) be a symplectic manifold with a semi-free S^1-action with isolated fixed points. Then (M, ω, J) is almost Fano.*

2.2.3 The Seidel Automorphism

In this paragraph we introduce the theory behind the definition of the Seidel element. The results concerning the present problem are discussed next. We will follow closely the book [MS04]. The proofs of the results exposed in this section are mostly contained in Chapters 8,9 and 11.

Let M be a symplectic manifold with a Hamiltonian circle action. We associate to M the locally trivial bundle $\widetilde{M_\lambda}$ over $\mathbb{P}^1$ with fibre M defined by the *clutching function* (action) $\lambda : S^1 \longrightarrow \text{Ham}(M, \omega)$:

$$\widetilde{M_\lambda} := S^3 \times_{S^1} M$$

We denote the fibres at $[0:1]$ and $[1:0]$ by M_0 and M_∞ respectively. Note that the isomorphism type of $\widetilde{M_\lambda}$ only depends on the homotopy class of λ.

Since λ is Hamiltonian, we can construct a symplectic form Ω on $\widetilde{M_\lambda}$. In fact the bundle $\pi : \widetilde{M_\lambda} \longrightarrow \mathbb{P}^1$ is a *Hamiltonian fibration* with fibre M, thus admitting sections ([MS04, Chapter 8]). We choose an Ω-compatible almost complex structure $\widetilde{J}$ on $\widetilde{M_\lambda}$, such that $\widetilde{J}$ is the product $J_0 \times J$ under trivializations. We can define for each fixed point $x \in M^{S^1}$ a $\widetilde{J}$ pseudoholomorphic section $\sigma_x := \{[z_0 : z_1; x] | [z_0 : z_1] \in S^2\}$.

$\widetilde{M_\lambda}$ has a canonical cohomology class, the first Chern class of the *vertical tangent bundle* or vertical class $c_{\text{vert}} = c_1(T\widetilde{M}_\lambda^{\text{vert}}) \in H^2(\widetilde{M_\lambda}, \mathbb{Z})$. If x is a fixed point for the circle action, we have that $c_{\text{vert}}(\sigma_x) = m(x)$. This follows from the fact that the normal bundle of the section σ_x is a sum of line bundles $L_i \longrightarrow \mathbb{P}^1$, one for each weight m_i of x and with Chern class $c_1(L_i) = m_i$ (see [MT04, Lemma 2.2]). Note that if B is a spherical class in M and if M is embedded in $\widetilde{M_\lambda}$ as a fibre, then $c_{\text{vert}}(i_*(B)) = c_1(B)$, where the latter is the usual Chern class of B. Then, for a given fixed point x and its associated section σ_x, the class $\sigma_x + i_*(B)$ has vertical Chern number $m(x) + c_1(B)$.

Take $\widetilde{A} \in H_2(\widetilde{M_\lambda}, \mathbb{Z})$ a section class, that is $\pi_*(\widetilde{A}) = [\mathbb{P}^1]$. Let $a_1, a_2 \in H^*(M)$. Given two fixed marked points $\mathbf{w} = (w_1, w_2), w_i \in \mathbb{P}^1$ we may think of the Poincaré dual to the class a_i as represented by a cycle Z_i in the fibre $M_i \hookrightarrow \widetilde{M_\lambda}$ over w_i. With this information it is possible to construct the Gromov-Witten invariant $\text{GW}^{\widetilde{M_\lambda}, \mathbf{w}}_{\widetilde{A},2}(a_1, a_2)$. This invariant counts the number of $\widetilde{J}$-holomorphic sections of $\widetilde{M_\lambda}$ in class $\widetilde{A}$ that pass through the cycles Z_i. This invariant is zero unless $2n + 2c_{\text{vert}}(\widetilde{A}) = \deg(a_1) + \deg(a_2)$. Now we have the following definition.

Definition 2.2.8 *Let (M, ω) be as before. Let $\sigma : \mathbb{P}^1 \longrightarrow \widetilde{M_\lambda}$ be a section, and suppose that σ has vertical class $c = c_{\text{vert}}(\sigma)$. The* **Seidel automorphism**

$$\Psi(\lambda, \sigma) : QH^*(M; \Lambda) \longrightarrow QH^{*-2c}(M; \Lambda)$$

is defined by

$$\Psi(\lambda,\sigma)(a) = \sum_{A\in H_2(M)} \sum_{k,j} \mathrm{GW}^{\widetilde{M}_\lambda,\mathrm{w}}_{[\sigma]+i_*(A),2}(a,e_k)g^{kj}e_j \otimes e^A. \qquad (2.7)$$

where $i : M \longrightarrow \widetilde{M}_\lambda$ *is an embedding (as fibre).*

In this definition we are considering a basis $\{e_i\}$ for $H^*(M)$ as in Equation (2.5). It is easy to see that the Seidel automorphism as defined above shifts the degree by $-2c$. We just need to analyze when the coefficients in Equation (2.7) are nonzero. In the particular case of $\sigma = \sigma_x$ for x a fixed point, $\Psi(\lambda, \sigma_x)$ shifts degree by $-2m(x)$. Note that this shift might be positive or negative.

From the previous definition, one can see that an important ingredient for the study of the Seidel element is the moduli space

$$\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma + i_*(A); Z, Z').$$

This moduli space, as before, is the *cut-off* moduli space of $\tilde{J}$-holomorphic sections in the class $\sigma + i_*(A)$ that pass through cycles Z, Z'. We will say more about these spaces in the next section.

If $\mathbb{1} \in QH^*(M)$ denotes the identity in the quantum cohomology ring, the homogeneous class $\Psi(\lambda,\sigma)(\mathbb{1}) \in QH^*(M)$ is called the **Seidel Element** of the action with respect to the section σ. We will use the same notation for the Seidel automorphism and the Seidel element. Thus, the Seidel automorphism is now given just by quantum multiplication by the element $\Psi(\lambda, \sigma)$ [MS04]. That is,

$$\Psi(\lambda,\sigma)(a) = \Psi(\lambda,\sigma) * a.$$

Note that the Seidel element has degree $\deg(\Psi(\lambda, \sigma)) = 2c_{\text{vert}}(\sigma)$.

2.2.4 Seidel Automorphism and Isolated Fixed Points

Consider now the present problem. That is, assume that the action is semi-free and it has isolated fixed points. Let $\sigma_{\max}$ be the section defined by the fixed point p_S. In this particular case the automorphism $\Psi(\lambda, \sigma_{\max})$ increases the degree by $-2m(p_S) = 2n$. Let $p_I \in M$ be a fixed point. Recall that we can associate to p_I classes in homology p_I^- and p_I^+, and if we consider all the fixed points, then the classes p_I^+ form a basis for $H_*(M)$.

The next theorem, due to McDuff and Tolman [MT04, Theorem 1.15, Proposition 3.4], gives the first step towards a description of the Seidel automorphism. Although they have proved this result in great generality (the fixed points are allowed to be in submanifolds rather than being isolated) and they use quantum homology rather than cohomology, it is not hard to adapt their result to our present notation.

Theorem 2.2.9 (McDuff-Tolman) *Let (M, ω) be a symplectic manifold with a semi-free circle action with isolated fixed points. Assume its associated Hamiltonian function H is such that $\int_M H\omega^n = 0$. Let $A_I \in H_2(M)$ be as considered in 2.1.11. Then, the Seidel automorphism can be expressed as*

$$\Psi(\lambda, \sigma_{\max})(\mathrm{PD}(p_I^-)) = \mathrm{PD}(p_I^+) \otimes e^{A_{I^c}} + \sum_{\omega(B)>0} a_B \otimes e^{A_{I^c}+B}.$$

where $a_B \in H^*(M)$. If $a_B \neq 0$ then $\deg \mathrm{PD}(p_I^+) - \deg a_B = 2c_1(B)$. Moreover, *if we write the sum above in terms of the basis* $\{\mathrm{PD}(p_J^+)\}$ *we get*

$$\Psi(\lambda, \sigma_{\max})(\mathrm{PD}(p_I^-)) = \mathrm{PD}(p_I^+) \otimes e^{A_{I^c}} + \sum_{\omega(B)>0} \sum_{J \subset S} C_{B,J}\, \mathrm{PD}(p_J^+) \otimes e^{A_{I^c}+B}.$$

The rational coefficients $C_{B,J}$ *can be nonzero only if* $|I| - |J| = c_1(B)$ *and the moduli space* $\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + B; \overline{W^u(p_I)}, \overline{W^u(p_J)})$ *has an* S^1*-invariant element.* σ_I *denotes the section defined by the fixed point* p_I.

We know by Corollary 2.1.6 that $p_I^- = p_{I^c}^+$. By definition $\mathrm{PD}(p_J^+) = x_J$, therefore we have the following straightforward corollary.

Corollary 2.2.10 *Let* (M, ω) *be a symplectic manifold with a semi-free circle action with isolated fixed points. Assume its associated Hamiltonian function* H *is such that* $\int_M H\omega^n = 0$. *Let* $\{x_I\}$ *be the basis for the cohomology ring as considered in Remark 2.1.10, and let* $A_I \in H_2(M)$ *as considered in 2.1.11. The Seidel automorphism can be expressed as*

$$\Psi(\lambda, \sigma_{\max})(x_{I^c}) = x_I \otimes e^{A_{I^c}} + \sum_{\omega(B)>0} \sum_{J \subset S} C_{B,J}\; x_J \otimes e^{A_{I^c}+B}. \qquad (2.8)$$

The rational coefficients $C_{B,J}$ *can be nonzero only if* $|I| - |J| = c_1(B)$ *and the moduli space* $\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + B; \overline{W^u(p_I)}, \overline{W^u(p_J)})$ *has an* S^1*-invariant element.*

Thus, the key to understand the Seidel automorphism is first to know what the S^1-invariant elements in moduli spaces

$$\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + B; Z, Z')$$

are. Here Z and Z' are closed S^1-invariant cycles in M. These elements are called *invariant chains in section class $\sigma_z + i_*(A)$ from $x \in Z$ to $y \in Z'$ with root z* [MT04]. We will explain what is the meaning of this. Recall that M is embedded in $\widetilde{M}_\lambda$ as the fibres M_0, M_∞.

Given $x, y, z \in M^{S^1}$ an invariant *principal* chain in section class $\sigma_z + i_*(A)$ from $x \in Z$ to $y \in Z'$ with root z consists of the following.

a) Two sequences of fixed points $\{x = x_1, \ldots, x_k = z\}$, $\{z = y_1, \ldots, y_s = y\}$, where we think of the sequence $\{x_i\}$ as embedded in M_0 and the sequence $\{y_i\}$ in M_∞.

b) The points x_k and y_1 are joined by the section σ_z.

c) For each $1 \leq i < k$ and each $1 \leq j < s$, the points x_i, x_{i+1} and y_j, y_{j+1} are joined by invariant $\widetilde{J}$-holomorphic spheres in classes $i_*(A'_i)$ and $i_*(A''_j)$ respectively. Here the classes A'_i, A''_j are in M.

d) If $A' := \sum_i A'_i$, $A'' := \sum_j A''_j$ then $A = A' + A''$.

An **invariant chain** in section class $\sigma_z + i_*(A)$ from $x \in Z$ to $y \in Z'$ with root z is a chain as above with additional ghost components at each of which a tree of invariant spheres is attached. In this case, A is the sum of classes represented by the principal spheres and the bubbles. As a final remark of this definition, note that since A' is invariant, then $c_1(A') \geq 0$ and $c_1(A') = 0$ if and only if $A' = 0$. The same applies for A''.

An immediate lemma is the following.

Lemma 2.2.11 *Assume the hypotheses of Corollary 2.2.10, and suppose* $\sigma_z +$ A *is an invariant chain in the moduli space*

$$\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + B; \overline{W^u(p_I)}, \overline{W^u(p_J)}).$$

Let $A = A' + A''$ *be the decomposition of* A *as described above. Then, the first Chern classes* $c_1(A'), c_1(A'')$ *can be estimated by*

$$c_1(A') \geq |m(x) - m(z)| \textit{ and } c_1(A'') \geq |m(y) - m(z)|.$$

Therefore

$$\begin{aligned} &c_1(A) \geq |m(x) - m(z)| + |m(y) - m(z)|, \textit{ and} \\ &c_1(B) \geq c_1(A''). \end{aligned} \tag{2.9}$$

Moreover if the coefficient $C_{B,J} \neq 0$*, then* $c_1(B) > 0$.

Proof: If A_i is an invariant sphere joining x_i to x_{i+1}, Corollary 2.2.4 shows that for some integer p_i, $c_1(A_i) = |p_i(m(x_i) - m(x_{i+1}))| \geq |m(x_i) - m(x_{i+1})|$. Then $c_1(A') \geq \sum_{i=1}^k |m(x_i) - m(x_{i+1})| \geq |m(x) - m(z)|$. The inequality for $c_1(A'') > 0$ follows similarly.

Now, by assumption the invariant chain $\sigma_z + i_*(A)$ is in class $\sigma_I + i_*(B)$, thus $\sigma_z + i_*(A) = \sigma_I + i_*(B)$ and then

$$c_1(B) + m(p_I) = c_1(A) + m(z).$$

Since $x \in W^u(p_I)$, $m(x) > m(p_I)$. Using $c_1(A) = c_1(A') + c_1(A'')$ and that

$c_1(A') \geq m(x) - m(z)$, we get

$$c_1(B) > c_1(A'') + m(x) - m(z) - m(p_I) + m(z) \geq c_1(A'') \geq 0.$$

To prove the last claim, we want to see that if $C_{B,J} \neq 0$, then $c_1(B) > 0$. By contradiction assume that $C_{B,J} \neq 0$ and that $c_1(B) = 0$. By Corollary 2.2.10 there must be an invariant chain $\sigma_z + i_*(A)$ is in class $\sigma_I + i_*(B)$ and then $c_1(B) = |I| - |J| = 0$. By Equation (2.9) we have that, $c_1(A'') = 0$. Since A'' is a gradient sphere, $A'' = 0$. As explained before, A'' joins x, y, it follows that $y = z$ and then $m(y) = m(z)$.

Now, from the equality $\sigma_z + i_*(A) = \sigma_I + i_*(B)$ we get $m(z) + c_1(A') - m(p_I) = 0$. Finally, since $y \in W^u(p_J)$, $m(z) = m(y) \geq m(p_J) = m(p_I)$ and then

$$0 = c_1(A') + m(z) - m(p_I) \geq c_1(A').$$

Then $A' = 0$, and thus $A = 0$. If $A = 0$, this implies that $x = y = z$. Then, $m(z) = m(y) = m(p_I)$ which is a contradiction to the fact that that $x \in W^u(p_I)$.

□

With Lemma 2.2.11 we can simplify the expression (2.8) to get the following corollary.

Corollary 2.2.12 *Assume the same hypotheses of Corollary 2.2.10. Then the Seidel element is given by*

$$\Psi(\lambda, \sigma_{\max})(x_{I^c}) = x_I \otimes e^{A_{I^c}} + \sum_{\omega(B)>0, c_1(B)>0} \sum_{J \subset S} C_{B,J}\ x_J \otimes e^{A_{I^c}+B}. \quad (2.10)$$

Again $C_{B,J} = 0$ *unless* $|I| - |J| = c_1(B)$ *and the moduli space* $\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + B; \overline{W^u(p_I)}, \overline{W^u(p_J)})$ *has an* S^1*-invariant element.*

Note that the only difference to Equation (2.8) is that we are considering only classes B with positive Chern number.

If there are any higher order terms, that is, terms that correspond to positive first Chern classes $c_1(B) > 0$, they contribute to the sum (2.10) as an element of degree $2(|J| + c_1(A_{I^c} + B))$. Heuristically an invariant chain $A + \sigma_z$ makes a contribution only if $c_1(A)$ is big enough so that the inequalities (2.9) are satisfied. We will see in our next result that with our present hypotheses there are no such contributions. Thus there are not higher order terms. This result fails if for instance we allow the action to have fixed points along submanifolds, as we will see in the example described in §2.2.5. Observe that we can normalize our Hamiltonian function H (by adding a constant) so that $\int_M H\omega^n = 0$ without altering any of our previous results.

Theorem 2.2.13 *Let* (M, ω) *be a symplectic manifold with a semi-free circle action with isolated fixed points. Assume its associated Hamiltonian function* H *is such that* $\int_M H\omega^n = 0$. *Then, the Seidel automorphism* $\Psi(\lambda, \sigma_{\max})$ *acts on the basis* $\{x_I\}$ *by*

$$\Psi(\lambda, \sigma_{\max})(x_I) = x_{I^c} \otimes e^{A_I} \qquad (2.11)$$

Proof: Consider I^c instead of I. By Corollary 2.2.12 the Seidel automorphism can be computed

$$\Psi(\lambda, \sigma_{\max})(x_{I^c}) = x_I \otimes e^{A_{I^c}} + \sum_{c_1(B)>0, J \subset S} C_{B,J} x_J \otimes e^{A_{I^c}+B}$$

As in Proposition 2.2.1, the Chern number $c_1(B)$ is a multiple of two. Thus the terms in the sum appear with "jumps" of four in the degree. By Corollary 2.2.12, $C_{B,J}$ is nonzero only if there is a S^1-invariant element in the moduli space $\overline{\mathcal{M}}_{0,2}(\widetilde{M}_\lambda, \tilde{J}, \sigma_I + B; \overline{W^u(p_I)}, \overline{W^u(p_J)})$. We want to see that the coefficients $C_{B,J}$ are all zero.

By contradiction assume there is an invariant chain $\sigma_z + A$ in this moduli space. Therefore A goes from a fixed point $x \in \overline{W^u(p_I)}$ to a fixed point $y \in \overline{W^u(p_J)}$. This chain satisfies

$$\sigma_z + A = \sigma_I + B. \tag{2.12}$$

Since the gradient flow is Morse-Smale and there is a gradient line from p_I to x, $m(x) \geq m(p_I) = n - 2|I|$. Analogously $m(y) \geq m(p_J) = n - 2|J|$. Since $c_1(B) = |I| - |J| > 0$ and we know $c_1(A) + m(z) = m(p_I) + c_1(B)$ from Equation (2.12), we get

$$c_1(A) = 2|K| - |I| - |J|, \tag{2.13}$$

where $K \subset \mathcal{S}$ is such that $p_K = z$.

Finally, from Lemma 2.2.11 we have

$$\begin{aligned} c_1(A) &\geq |m(x) - m(z)| + |m(y) - m(z)| \\ &\geq -2m(z) + m(y) + m(x) \\ &\geq 4|K| - 2|I| - 2|J|. \end{aligned}$$

Therefore, by Equation (2.13)

$$2|K| - |I| - |J| = c_1(A) \geq 2(2|K| - |I| - |J|).$$

This is only possible if $c_1(A) = 0$, i.e $2|K| - |J| = |I|$. Then, A must be zero and $x = y = z$. Therefore $B = \sigma_z - \sigma_I$ and hence $c_1(B) = m(z) - m(p_I) = 2(|I| - |K|)$. Since $c_1(A) = 0$, Equation (2.13) implies $|I| - |K| = |K| - |J|$. Thus $0 < c_1(B) = 2(|K| - |J|)$. By hypothesis $p_K = z = y \in \overline{W^u(p_J)}$. Then we have $|K| \leq |J|$. Thus $c_1(B) \leq 0$, which is a contradiction. This proves the theorem.

□

Corollary 2.2.14 *The Seidel element $\Psi(\lambda, \sigma_{\max})$ is given by*

$$\Psi(\lambda, \sigma_{\max}) = x_S$$

and the quantum product of x_S with the element x_I is given by

$$x_S * x_I = x_{I^c} \otimes e^{A_I}. \qquad (2.14)$$

Proof: The first part is obvious since

$$\Psi(\lambda, \sigma_{\max}) = \Psi(\lambda, \sigma_{\max}) * \mathbb{1} = \Psi(\lambda, \sigma_{\max}) * x_0 = x_S \otimes e^0.$$

For the second part, observe that

$$x_{I^c} \otimes e^{A_I} = \Psi(\lambda, \sigma_{\max}) * x_I = x_S * x_I.$$

/

□

The next paragraph is dedicated to discuss an example where the symplectic manifold has a semi-free circle action but the Seidel automorphism has higher order terms when evaluated on a particular class. In this example the fixed points are along submanifolds. This illustrates that we cannot have a result similar to Theorem 2.2.13 if we weaken one of our hypothesis.

2.2.5 Example

This example is taken from [MT04, Example 5.6]. Let $M = \widetilde{\mathbb{P}^2}$ be the one point blow up of $\mathbb{P}^2$ with the symplectic form ω_μ so that on the exceptional divisor E, $0 < \omega_\mu(E) = \mu < 1$ and if $L = [\mathbb{P}^1]$ is the standard line, we have $\omega_\mu(L) = 1$. We can identify M with the space

$$\{(z_1, z_2) \in \mathbb{C}^2 |\ \mu \leq |z_1|^2 + |z_2|^2 \leq 1\}$$

where the boundaries are collapsed along the Hopf fibres. One of the collapsed boundaries is identified with the exceptional divisor. The other with L.

A basis for $H_*(M)$ is given by the class of a point pt, the exceptional divisor E, the fibre class $F = L - E$ and the fundamental class $[M]$. Note that the intersection products are given by $E \cdot E = -1$, $E \cdot F = 1$, $F \cdot F = 0$. Denote by b and f the Poincaré duals of E, F respectively. Then $b \cdot b = -1$ and $f \cdot f = 0$. It is not hard to see that the positive generator of $H^4(M)$ is $b \smile f = \mathrm{PD}(pt)$. Let us denote this class by just bf, so that a basis for the cohomology ring is $\{\mathbb{1}, b, f, bf\}$. Observe that M with the usual complex structure is Fano .

The non-vanishing Gromov-Witten invariants are given by

$$\mathrm{GW}^M_{L,3}(bf, bf, f) = \mathrm{GW}^M_{F,3}(bf, b, b) = 1;$$
$$\mathrm{GW}^M_{E,3}(c_1, c_2, c_3) = \pm 1 \text{ where } c_i = b \text{ or } f.$$

Let us consider the usual Novikov ring Λ_ω as the quantum coefficients. Then the quantum products are given by:

$$\begin{aligned} bf * bf &= (b+f)\otimes e^L & \qquad bf * f &= \mathbb{1}\otimes e^L \\ bf * b &= f\otimes e^F & b * b &= -bf + b\otimes e^E + \mathbb{1}\otimes e^F \\ b * f &= bf - b\otimes e^E & f * f &= b\otimes e^E. \end{aligned}$$

In [MT04] it is proved that the circle action on M which is given by:

$$\alpha : (z_1, z_2) \mapsto (e^{-2\pi i t} z_1, e^{-2\pi i t} z_2), \qquad \text{for } 0 \le t \le 1$$

is Hamiltonian. The maximum set of this action is exactly the points lying on the exceptional divisor E and the minimum set is the line L. After taking an appropriate reference section σ, the Seidel element $\Psi(\alpha, \sigma)$ is given by

$$\Psi(\alpha, \sigma) = b.$$

Thus, evaluating the Seidel map on the class f we have

$$\Psi(\alpha, \sigma)(f) = \Psi(\alpha, \sigma) * f = b * f = bf - b\otimes e^E.$$

Therefore the Seidel automorphism does have higher order terms when evaluated on the class f.

2.3 Proof of Theorem 1.0.1

Now we are ready for proving the main theorem. Recall that the quantum coefficient ring is $\Lambda = \mathbb{Q}[q_1, \ldots, q_n]$. We also denote the usual cup product $a \smile b$ by ab for all $a, b \in H^*(M)$.

Proof of Theorem 1.0.1:

This is an immediate consequence of the next lemma.

□

Lemma 2.3.1 *Let $I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$, and let $1 \leq i \leq n$. Then*

$$x_{i_1} * \cdots * x_{i_k} = x_I \text{ and } x_i * x_i = \mathbb{1} \otimes e^{A_i} = q_i \tag{2.15}$$

Proof: To prove the first equality we will proceed by induction. Assume we have only two elements, say x_i, x_j, with $i \neq j$. Then, by Proposition 2.2.1 and Remark 2.2.2 we have

$$x_i * x_j = x_{\{ij\}} + c\ \mathbb{1} \otimes e^B,$$

where the coefficient c is a rational number and $c_1(B) > 0$.

From Corollary 2.2.14 and the associativity of quantum multiplication we get

$$(x_S * x_i) * x_j = (x_{\{i\}^c} * x_j) \otimes e^{A_i} = \tag{2.16}$$
$$x_S * (x_i * x_j) = x_{\{ij\}^c} \otimes e^{A_{ij}} + c\, x_S \otimes e^B.$$

By Proposition 2.2.1 the term $x_{\{i\}^c} * x_j$ is of the form

$$x_{\{i\}^c} x_j + \sum_{c_1(B')>0} a_{B'} \otimes e^{B'}$$

where again $\deg(a_{B'}) = \deg(x_{\{i\}^c}) + \deg(x_j) - 2c_1(B') < 2n$. Since $j \in \{i\}^c$, the term $x_{\{i\}^c} x_j$ is zero. Thus we have

$$\sum_{c_1(B')>0} a_{B'} \otimes e^{B'} \otimes e^{A_i} = x_{\{ij\}^c} \otimes e^{A_{ij}} + c\, x_S \otimes e^B.$$

Then by comparing the degree of the coefficients in the previous equation, the constant c must vanish.

For the general case we will use the same argument. Assume the result holds for k different elements. Let $I' = \{i_{k+1}\} \cup I$. The quantum product $x_{i_1} * \cdots * x_{i_{k+1}}$ is by the inductive hypothesis, the same as $x_I * x_{i_{k+1}}$. This element can be written in terms of the basis as

$$x_I * x_{i_{k+1}} = x_{I'} + \sum_{c_1(B)>0, J \subset S} a_{B,J}\, x_J \otimes e^B$$

where $2|J| = \deg(x_J) = \deg(x_{I'}) - 2d \leq \deg(x_{I'}) - 4$ and the coefficients $a_{B,J}$ are rational.

As before, using quantum associativity and Corollary 2.2.14 we get

$$\begin{aligned}(x_S * x_I) * x_{i_{k+1}} &= (x_{I^c} * x_{i_{k+1}}) \otimes e^{A_I} \\ x_S * (x_I * x_{i_{k+1}}) &= x_{I'^c} \otimes e^{A_{I'}} + \sum_{c_1(B)>0, J\subset S} a_{B,J}\ x_{J^c} \otimes e^{A_J+B}.\end{aligned} \tag{2.17}$$

Here the degree satisfies

$$\deg(x_{J^c}) = 2n - \deg(x_{I'}) + 2d \geq 2n - \deg(x_{I'}) + 4 = 2(n - |I| + 1). \tag{2.18}$$

Now, the center term in Equation (2.17) is written as

$$(x_{I^c} x_{i_{k+1}} + \sum_{c_1(B')>0, K\subset S} c_{B',K}\ x_K \otimes e^{B'}) \otimes e^{A_I},$$

where we have

$$\deg(x_K) \leq \deg(x_{I^c}) + \deg(x_{i_{k+1}}) - 4 = 2(n - |I| - 1). \tag{2.19}$$

Since $i_{k+1} \in I^c$, $x_{I^c} x_{i_{k+1}} = 0$. Finally we have the identity

$$\sum_{c_1(B')>0, K\subset S} c_{B',K}\ x_K \otimes e^{B'+A_I} = x_{I'^c} \otimes e^{A_{I'}} + \sum_{c_1(B)>0, J\subset S} a_{B,J}\ x_{J^c} \otimes e^{A_J+B}.$$

By Equations (2.18),(2.19), the coefficients $a_{B,J}$ are zero. This proves the first part of the lemma.

The second part is analogous, just write

$$x_i * x_i = x_i x_i + c\ \mathbb{1} \otimes e^B = c\ \mathbb{1} \otimes e^B$$

then multiplying by $x_{\mathcal{S}}$

$$(x_{\mathcal{S}} * x_i) * x_i = (x_{\{i\}^c} * x_i) \otimes e^{A_i} = c \; x_{\mathcal{S}} \otimes e^{B}.$$

Since $x_{\{i\}^c} * x_i = x_{\mathcal{S}}$, it follows that $c = 1$ and $e^B = e^{A_i}$.

□

Chapter 3

Gluing Hamiltonian S^1-Manifolds

In this chapter we investigate the symplectomorphism type of Hamiltonian S^1-manifolds by means of the gluing.

3.1 General Setting for the Gluing

In this section we will be using the term Hamiltonian S^1-manifold referring to triples (M, H, ω), where M is a closed, smooth, connected $2n$-manifold, ω a symplectic form on M and H is a normalized Hamiltonian function on M that generates a circle action compatible with ω on M. Although we will be working with general, not necessary closed manifolds, we will always think of them as (isomorphic to) submanifolds of a closed one, say M. Therefore, we sometimes will not make explicit the presence of the symplectic form. We will use the notation $\mathbb{H}\mathrm{Symp}_{2n}$ referring to the class of Hamiltonian S^1-manifolds, closed or not, up to isomorphism. Here (M, H, ω) and (M', H', ω') are **isomorphic** if there exist an equivariant diffeomorphism $f : M \longrightarrow M'$ such that $f^*(\omega') = \omega$. Note that f is equivariant if and only if $H \circ f = H'$.

Let (M, L, ω) be a closed Hamiltonian S^1-manifold and denote by

$$\mathcal{C}(M) = \{0 = \lambda_0 < \cdots < \lambda_s\}$$

the collection of critical values of L. Thus $L(M) = [\lambda_0, \lambda_s]$. The set $\mathcal{C}(M)$ is invariant under isomorphism. We now describe how to get pieces of M by *localizing to critical levels using the hamiltonian.* This is, for each $\varepsilon > 0$ consider the neighborhood $L^{-1}(\lambda - \varepsilon, \lambda + \varepsilon)$ of the level set $L^{-1}(\lambda)$, and for $\lambda, \lambda' \in \mathcal{C}(M)$ take the open submanifold $L^{-1}(\lambda, \lambda')$. In this paper we are interested on knowing what is needed to reconstruct M if one just knows the *isomorphism type* of the pieces $L^{-1}(\lambda - \varepsilon, \lambda + \varepsilon)$ and $L^{-1}(\lambda, \lambda')$. In general one would need to specify how to glue these pieces to get back to M. To see what type of gluing maps are allowed, suppose that $(Y, H), (Z, K)$ are manifolds isomorphic to the pieces $L^{-1}(\lambda - \varepsilon, \lambda + \varepsilon)$ and $L^{-1}(\lambda, \lambda')$ respectively. Note that one is tempted to glue Y and Z along *their overlap*, that is we want to identify $H^{-1}(\lambda, \lambda + \varepsilon)$ and $K^{-1}(\lambda, \lambda + \varepsilon)$ through isomorphisms, to obtain back a manifold isomorphic to $L^{-1}(\lambda - \varepsilon, \lambda')$ such that Y and K are symplectic submanifolds. Therefore one need to consider a maximal set of gluing maps with this property. One may think of these data as a *symplectic atlas of compatible Hamiltonian charts* for M. We now provide the precise formalism that allows this to work.

Definition 3.1.1 *Let $\lambda \in \mathbb{R}$ and let $\varepsilon_0 > 0$.*

(i) A ***cobordism*** *at λ is a tuple (Y, H, ϵ) such that $0 < \epsilon < \varepsilon_0$ and Y is a Hamiltonian S^1-manifold whose Hamiltonian function H takes Y onto $I_\epsilon = (\lambda - \epsilon, \lambda + \epsilon)$ and λ is the only critical value of H. Moreover we*

require that if $\epsilon' < \epsilon$ the restriction $(H^{-1}(I_{\epsilon'}), H, \epsilon')$ is identified with (Y, H, ϵ'). Two cobordisms are equivalent, $(Y, H, \epsilon) \sim (Y', H', \epsilon')$, if and only if there is $\epsilon'' < \min(\epsilon, \epsilon')$ such that (Y, H, ϵ'') and (Y', H', ϵ'') are isomorphic. That is, there is an isomorphism

$$f : H^{-1}(I_{\epsilon''}) \longrightarrow (H')^{-1}(I_{\epsilon''}).$$

A ***critical germ*** *$G(\lambda, \varepsilon_0)$* ***at*** *$\lambda$ is an equivalence class in $\mathbb{H}\mathrm{Symp}_{2n}$ of such tuples.*

(ii) Similarly, consider tuples (Y, H, ϵ) as above where the only critical value of H is its minimum (maximum) value λ. Thus $H(Y) = [\lambda, \lambda + \epsilon)$ $(H(Y) = (\lambda - \epsilon, \lambda])$. We have a similar equivalence relation between them as before. An equivalence class $m(\lambda, \varepsilon)(M(\lambda, \varepsilon))$ is called a ***minimal (maximal) germ*** *at λ.*

We will often refer to critical, maximal or minimal germs just as germs. Note that if $\delta < \varepsilon$, there is a natural *restriction* map $G(\lambda, \varepsilon) \longrightarrow G(\lambda, \delta)$. The triples (Y, H, ϵ) as in Definition 3.1.1 (ii) are neighborhoods of the maximal and minimal sets. To see this, first note that there is a unique maximum component $F_{\max}$ of the fixed point set [AB84]. Then by using an equivariant version of the Darboux Theorem applied to points in $F_{\max}$, there is a triple of the form (Y, H, ϵ) for ϵ small enough. Moreover, its maximal germ is determined uniquely by the symplectomorphism type of $F_{\max}$ and its normal bundle. A similar remark applies to the minimum.

Definition 3.1.2 *Let I be an open interval. A* ***free slice*** *is a tuple (Z, K, I, ω)*

where $K : Z \longrightarrow I$ is a surjective moment map for a free S^1-action on the symplectic manifold (Z, ω). We say that two slices $(Z, K, I, \omega) \sim (Z', K', I, \omega')$ are equivalent if they are isomorphic. We denote by $F(I)$ an equivalence class of such slices.

Definition 3.1.3 *Let $G(\lambda, \varepsilon)$ be a germ at λ and let $F(I)$ be a class of free slices for $I = (\lambda', \lambda)$. Let $(Y, H, \epsilon) \in G(\lambda, \varepsilon)$ and $(Z, K, I) \in F(I)$. A* ***gluing map*** *$(\phi, \epsilon) : (Y, H, \epsilon) \longrightarrow (Z, K, I)$ is given by a pair (ϕ, ϵ) where $0 < \epsilon < \varepsilon$ and ϕ is an isomorphism*

$$H^{-1}(\lambda - \epsilon, \lambda) \xrightarrow{\phi} K^{-1}(\lambda - \epsilon, \lambda).$$

Two gluing maps $(\phi, \epsilon), (\phi', \epsilon')$ are equivalent if there are $\epsilon'' < min(\epsilon, \epsilon')$ and isomorphisms

$$f : Y \longrightarrow Y', g : Z \longrightarrow Z'$$

such that the following diagram is commutative.

$$\begin{array}{ccc} H^{-1}(\lambda - \epsilon'', \lambda) & \xrightarrow{\phi} & K^{-1}(\lambda - \epsilon'', \lambda) \\ \downarrow f & & \downarrow g \\ H'^{-1}(\lambda - \epsilon'', \lambda) & \xrightarrow{\phi'} & K'^{-1}(\lambda - \epsilon'', \lambda) \end{array} \tag{3.1}$$

A ***gluing class*** *$\Phi : G(\lambda, \varepsilon) \longrightarrow F(I)$ is an equivalence class of pairs (ϕ, ϵ). This maps are well defined by Diagram (3.1). Analogously one can define gluing maps for $F(I)$ and the germ $G(\lambda, \varepsilon)$ when $I = (\lambda, \lambda')$. Note that this definition also applies if we substitute the germ $G(\lambda, \varepsilon)$ by a maximal or a minimal one.*

Recall that we are interested in *building* symplectic manifolds out of germs and free slices. We now describe the simplest case where we can do that. Suppose we have $G(\lambda, \varepsilon)$ and $F(\lambda, \lambda')$ a germ and a class of free slices, we want to see how to obtain a new manifold via a gluing class $\Phi : G(\lambda, \varepsilon) \longrightarrow F(\lambda, \lambda')$. Choose representatives $(Y, H, \epsilon) \in G(\lambda, \varepsilon), (Z, K, I, \omega) \in F(I)$ and $(\phi, \epsilon) \in \Phi$. Then, consider the manifold

$$Y \cup_{(\phi,\epsilon)} Z$$

obtained by gluing Y and Z along the overlap $(\lambda, \lambda + \epsilon)$, that is

$$Y \sqcup Z / \sim \text{ where } x \sim y \iff \phi(x) = y$$

where

$$\phi : H^{-1}(\lambda, \lambda + \epsilon) \longrightarrow K^{-1}(\lambda, \lambda + \epsilon)$$

is the restricted isomorphism on the interval $(\lambda, \lambda + \epsilon)$.

If $(\phi', \epsilon'), (Y', H', \epsilon'), (Z', K', I)$ is another set of choices, there exist $0 < \epsilon'' < \min(\epsilon, \epsilon') < \varepsilon$ and isomorphisms f, g as in the commutative Diagram (3.1). Therefore, by restricting the gluing maps and f, g to the open interval $(\lambda, \lambda+\epsilon'')$ one gets that $\phi(x) = y$ if and only if $\phi'(f(x)) = g(y)$. Then (f, g) induces an isomorphism

$$Y \cup_{(\phi,\epsilon)} Z \xrightarrow{\cong} Y' \cup_{(\phi',\epsilon')} Z'. \tag{3.2}$$

Denote by

$$G(\lambda, \varepsilon) \cup_{\Phi} F(I)$$

the isomorphism class produced by this gluing. Note that the Hamiltonian function on this new manifold is the one defined by $(H,K): Y \sqcup Z \longrightarrow (\lambda - \varepsilon, \lambda')$ after passing to the quotient. Therefore this is a well defined operation in $\mathbb{H}\mathrm{Symp}_{2n}$.

Conversely, the neighborhoods $(H,K)^{-1}(\lambda-\epsilon, \lambda+\epsilon) \subset Y \cup_{(\phi,\epsilon)} Z$ are isomorphic to $H^{-1}(\lambda-\epsilon,\lambda+\epsilon)$ for all ϵ. Similarly $(H,K)^{-1}(\lambda,\lambda') \cong K^{-1}(\lambda,\lambda')$. Then we have the following lemma.

Lemma 3.1.4 *Suppose we have given a germ of cobordism $G(\lambda,\varepsilon)$, a free slice $F(\lambda,\lambda')$ and a gluing class $\Psi : G(\lambda,\varepsilon) \longrightarrow F(\lambda,\lambda')$. Then we can associate a unique isomorphism class*

$$G(\lambda,\varepsilon) \cup_\Phi F(\lambda,\lambda') \textit{ with hamiltonian } (H,K)$$

in $\mathbb{H}\mathrm{Symp}_{2n}$. Moreover, the manifolds $(H,K)^{-1}(I_\epsilon)$ and $(H,K)^{-1}(\lambda,\lambda')$ represent the classes $G(\lambda,\varepsilon)$ and $F(\lambda,\lambda')$.

It is clear that we can apply the same idea to define a gluing of the form $F(\lambda',\lambda) \cup_\Phi G(\lambda,\varepsilon)$. Similarly, when we have maximal and minimal germs $M(\lambda,\varepsilon)$, $m(\lambda',\varepsilon)$ one can construct manifolds

$$m(\lambda',\varepsilon') \cup_\Phi F(\lambda',\lambda) \text{ and } F(\lambda',\lambda) \cup_\Phi M(\lambda,\varepsilon).$$

It is important to note that order of the gluing does not matter provided ε is small enough.

To reconstruct M, we would like to define this process more generally. We want to glue more general data, as we now explain. **A set of local data $\mathcal{L}$** consists of:

- a collection $\mathcal{C} = \{0 = \lambda_0 < \cdots < \lambda_{s+1}\}$ of critical levels.

- germs $G(\lambda_i, \varepsilon_i)$ at ε_i for all $i = 1, \ldots, s$, minimal and maximal germs $m(\lambda_0, \varepsilon_0)$, $M(\lambda_{s+1}, \varepsilon_{s+1})$ respectively.

- for all $j = 0, \ldots, s$, equivalence classes of free slices $F(I_j)$ where $I_j = (\lambda_j, \lambda_{j+1})$ is a maximal open interval of regular values.

- gluing classes Φ_i^+, Φ_i^- from $G(\lambda_i, \varepsilon_i)$ to $F(I_i)$ and $F(I_{i-1})$ respectively for all $i = 1, \ldots, s$.

As an example consider $(M, \omega, H) \in \mathbb{H}\mathrm{Symp}_{2n}$. Its localizations

$$H^{-1}(\lambda - \epsilon, \lambda + \epsilon), \text{ and the free sets } H^{-1}(\lambda, \lambda')$$

for $\epsilon > 0$ and $\lambda, \lambda' \in \mathcal{C}(M)$ define the germs and free slices. Then the **local data associated to** M, denoted by $\mathcal{L}_M$, is the collection of all the isomorphism classes of these sets and the gluing classes on the overlaps. Now we have the following theorem.

Theorem 3.1.5 *Given a set of local data $\mathcal{L}$, there exists a closed Hamiltonian S^1-manifold $M_{\mathcal{L}}$ such that its associated set of local data is $\mathcal{L}$. Moreover, this manifold is unique up to isomorphism, this is we have a one-to-one association*

$$\{ \textit{ Sets of local data } \} \longrightarrow \mathbb{H}\mathrm{Symp}_{2n}$$

Proof:

Consider representatives

$$(Y_{s+1}, H_{s+1}, \epsilon_{s+1}) \in M(\lambda_{s+1}, \varepsilon_{s+1}), \quad (Y_0, H_0, \epsilon_0) \in m(\lambda_0, \varepsilon_0)$$

$$((Y_i, H_i, \epsilon_i) \in G(\lambda_i, \varepsilon_i), \varepsilon_{s+1}), \quad (Z_i, K_i, I_i) \in F(I_i)$$

and

$$(\phi_i^{\pm}, \epsilon) \in \Phi^{\pm}.$$

On the disjoint union

$$\mathcal{M}_{\mathcal{L}} = Y_0 \sqcup Z_1 \sqcup Y_1 \cdots \sqcup Z_{s+1} \sqcup Y_{s+1}$$

define the equivalence relation by

$$y \sim_{\mathcal{L}} z \iff \text{for some } j, (y \in Y_j, z \in Z_{j+1}, \phi_j^-(y) = z) \text{ or } (y \in Y_j, z \in Z_j, \phi_j^+(y) = z)$$

Define $M_{\mathcal{L}} := \mathcal{M}_{\mathcal{L}}/\sim_{\mathcal{L}}$ with hamiltonian $H_{\mathcal{L}} = [H_0, K_1, \ldots, K_{s+1}, H_{s+1}]$. Note that $H_{\mathcal{L}} : M_{\mathcal{L}} \longrightarrow [\lambda_0, \lambda_{s+1}]$ and then $M_{\mathcal{L}}$ is a closed symplectic manifold whose isomorphism class is completely determined by the local data $\mathcal{L}$, since another set of choices would give an isomorphism as in Equation (3.2). Finally, to see that the association $\mathcal{L} \mapsto M_{\mathcal{L}}$ is one-to-one, we proceed as before, by considering the localizations of $M_{\mathcal{L}}$ at the critical levels with the Hamiltonian $H_{\mathcal{L}}$. One can see that this recovers $\mathcal{L}$.

□

As a last note, we clarify that for the rest of this paper we will treat isomorphic manifolds as equal, unless we specify the contrary.

3.2 The gluing, the reduced spaces and the classification theorem

Most of the background material for this section can be found in [MS98] and [GS89]. Let $(M, H, \omega) \in \mathbf{HSymp}_{2n}$. Let $t \in \mathbb{R}$ be a regular value of H. Then S^1 acts freely on the level set $H^{-1}(t)$. The orbit space $\overline{M}_t := H^{-1}(t)/S^1$ is called the reduced space of M at the level t. This space is symplectic with the reduced symplectic form ω_t. The fibration $\pi : H^{-1}(t) \longrightarrow \overline{M}_t$ is a principal S^1 bundle over $\overline{M}_t$. Denote its total space by P_t or just by P, whenever there is no risk of confusion. Denote its Euler class by $e(P) \in H^2(\overline{M}_t)$.

The reduced symplectic form ω_t on $\overline{M}_t$ satisfies

$$\pi^* \omega_t = i_t^* \omega \tag{3.3}$$

where $H^{-1}(t)$ is the total space of the principal S^1 bundle P_t.

Let λ be a critical value of H and let $F \subset H^{-1}(\lambda)$ be the fixed point set in this level. Although the quotient $\overline{M}_\lambda := H^{-1}(\lambda)/S^1$ may be singular, if the Morse index of λ is 2 (resp. coindex 2) it can be identified with the non-singular space $\overline{M}_{\lambda-\epsilon}$ (resp. $\overline{M}_{\lambda+\epsilon}$) [GS89] in such a way that we have a reduction principal S^1-bundle

$$P_\lambda \longrightarrow \overline{M}_\lambda.$$

Note that if $\dim M = 6$ all the non-extremal critical values have (co)index 2. If λ has (co)index 2 the fixed point set F is an embedded submanifold of $\overline{M}_\lambda$.

By considering the Hamiltonian $-H$, we can assume that all the results about index 2 critical values have similar results for index 4 as well.

The main goal of this article is to provide a method to classify symplectic manifolds up to some equivalence. We now introduce the precise definitions of common relations in symplectic geometry that we will use in this paper. Consider two symplectic forms ω_0, ω_1 on a manifold X. This forms are said to be **symplectomorphic** if there is a diffeomorphism $f : X \longrightarrow X$ such that $f^*\omega_1 = \omega_0$. A **deformation** between ω_0, ω_1 is a (smooth) family $\{\omega_s\}$ of symplectic forms that join them. A deformation is an **isotopy** if the elements in the family $\{\omega_s\}$ all lie in the same cohomology class. It is well known (Moser's lemma) that two symplectic forms are isotopic if and only if there is a family of diffeomorphisms $\{h_s\}$ on X such that $h_s^*\omega_s = \omega_0$ and $h_0 = id$. The concepts of isotopy and deformation are in general not equivalent (cf. Example 13.20 in [MS98]), but for some special cases as we will see in Theorem 3.3.1, they agree. For the objectives of the present work, manifolds where this two propierties agree will be a key ingredient as we will see in Lemma 3.2.7.

When the manifolds are equipped with circle actions, we will make the natural assumption that all the deformations and maps are S^1-equivariant.

3.2.1 Germs at critical levels of index 2 and the change in the fixed point data

We now describe the germ $G(\lambda, \varepsilon)$ when λ is a critical value of index 2 and the action is semi-free. We will follow entirely [GS89] (see also [MS98]). Let (M, ω) be a symplectic manifold and let $S \subset M$ be a closed submanifold. For $\epsilon > 0$

small enough, we denote by $Bl_S(M)$ and $\beta : Bl_S(N) \longrightarrow N$ the ϵ-symplectic blow up of M along S and the blow-down map as defined by Guillemin and Sternberg [GS89]. The construction of the manifold $Bl_S(M)$ depend on several choices[1], but its diffeomorphism type is independent of them. $Bl_S(M)$ admits a blow-up symplectic structure denoted by $\widehat{\omega}(\epsilon)$. This form is not independent of the choices[2], but its *germ* of isotopy classes is. This is, if $\widehat{\omega}'(\epsilon)$ is the form obtained by making different choices, then for some ε_0 small enough there exist a smooth family $f_\epsilon \in \mathrm{Diff}(M)$ such that $f_\epsilon^*(\widehat{\omega}(\epsilon)) = \widehat{\omega}'(\epsilon)$ for all $0 < \epsilon < \varepsilon_0$. With this in mind, assume we have the following data

i) a compact symplectic manifold $(\bar{M}, \bar{\omega})$

ii) a symplectic submanifold $F \subset \bar{M}$

iii) a principal S^1-bundle $\pi : P \longrightarrow \bar{M}$

iv) a connection 1-form α on P

v) an interval $I = (\lambda - \epsilon, \lambda + \epsilon)$ for (small) $\epsilon > 0$.

Then it is possible to create a cobordism $(Y(\lambda), H, \omega) \in \mathbf{HSymp}_{2n}$ having the following properties.

(a) H maps $Y(\lambda)$ onto I.

(b) For all $t > 0$, $H^{-1}(\lambda - t)$ is equivariantly diffeomorphic to P.

[1] To be precise, an embedding of a ball bundle and a connection on a principal $U(n)$-bundle.

[2] Not even when one blows-up a point. This is because there is a non-compact family of choices.

(c) For all $t > 0$ the symplectic reduction $Y(\lambda)$ at the level $\lambda - t$ is symplectomorphic to $\bar{M}$ with symplectic form $\bar{\omega} - td\alpha$.[3]

(d) λ is a critical level of H of index 2.

(e) For all $t > 0$ the reduction of $Y(\lambda)$ at $\lambda + t$ is the blow up $Bl_F(\bar{M})$ of $\bar{M}$ along F with symplectic structure

$$\widehat{\omega}(t) + \beta^*(td\alpha). \tag{3.4}$$

Here $\widehat{\omega}(t)$ is the t-blow up form and $\beta : Bl_F(\bar{M}) \longrightarrow \bar{M}$ is the blow down map.

(f) The fixed point set at λ is F.

Theorem 3.2.1 (Guillemin-Sternberg cobordism theorem) *Let* $(M, K, \omega) \in$ HSymp$_{2n}$ *and* λ *be an index 2 critical value. Let* $I = (\lambda - \varepsilon, \lambda + \varepsilon)$ *be a sufficiently small interval in* $\mathbb{R}$. *Then, the open submanifold* $K^{-1}(I)$ *is equivariantly symplectomorphic to the manifold* $(Y(\lambda), H, \omega)$ *as described above. Moreover, the germ* $G(\lambda, \varepsilon) \in$ HSymp$_{2n}$ *of* M *at* λ *only depends on the fixed point data* $(\overline{M}_\lambda, \overline{F}_\lambda, \omega_\lambda)$, $P_\lambda \longrightarrow \overline{M}_\lambda$ *at* λ.

Note that the reduced space of M at the level $\lambda + \epsilon$ is obtained from the reduced space at the level λ by blowing up. An immediate corollary of Theorem 3.2.1 is the following.

[3] $d\alpha$ is a 2-form on P, but it descends to B. This is the form that we consider here.

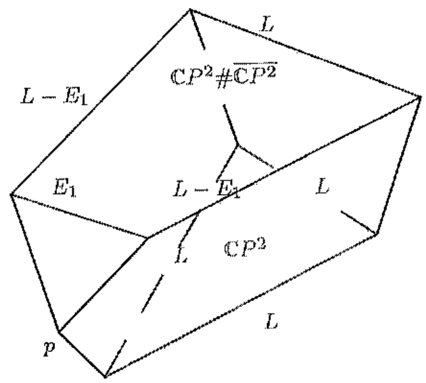

Figure 3.1: The cobordism around an isolated fixed point p of index 2. Here we assume that the minimum is also isolated. The base of the figure represents $\mathbb{C}P^2$ and the top the blow-up $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. If the fixed point is of index 4 this cobordism is up-side down.

Corollary 3.2.2 *If $(M, H, \omega) \in \mathbf{H}\mathrm{Symp}_{2n}$ is a manifold such that every non-extremal $\lambda \in \mathcal{C}(M)$ is simple, then all the germs of cobordism are completely determined by the fixed point data, as in Definition 1.0.4.*

We now describe the change in the fixed point data. Let ν be the normal bundle of the fixed point submanifold F_λ in M. The normal bundle decomposes as a sum $\nu^+ \oplus \nu^-$ of positive and negative directions. Observe that a small neighborhood of the zero section of ν is S^1-isomorphic to a neighborhood $U \cong U^+ \cup U^-$ of F_λ in M. Denote by $\overline{X}_t$ the symplectic reduction of U^- at t if $t \leq \lambda$ or the reduction of U^+ at t if $t > 0$. The diffeotype of the triple $(P_t, \overline{M}_t, \overline{X}_t)$ depends smoothly on t for $t \in (\lambda - \epsilon, \lambda]$ and it is constant. Denote it by $(P_-, \overline{M}_-, \overline{X}_-)$ where $\overline{X}_- = \overline{F}_\lambda$. Similarly, we denote the common diffeotype for $t \in (\lambda, \lambda + \epsilon)$ by $(P_+, \overline{M}_+, \overline{X}_+)$.

The relation between them is quite subtle as we now see. By Theorem 3.2.1, $\overline{M}_+$ is the blow up of $\overline{M}_-$ along $\overline{X}_-$. Let $\beta : \overline{M}_+ \longrightarrow \overline{M}_-$ be the blow down map. β restricts to a diffeomorphism $\overline{M}_+ - \overline{X}_+$ onto $\overline{M}_- - \overline{X}_-$, and when restricted to $\overline{X}_+$ it is a fibration

$$\beta : \overline{X}_+ \longrightarrow \overline{X}_-$$

whose fibers are all diffeomorphic to $\mathbb{C}P^{k-1}$. Here $2k$ is the codimension of $\overline{X}_-$ in $\overline{M}_-$. Denote by L' the line bundle on $\overline{M}_+$ whose Chern class is dual to its codimension 2 submanifold $\overline{X}_+$. Let L be the circle bundle associated to L'. Then we have [GS89, Formula 13.3]

$$P_+ = \beta^*(P_-) \otimes L \tag{3.5}$$

as circle bundles over $\overline{M}_+$. Since L is trivial on $\overline{M}_+ - \overline{X}_+$, then $P_- \cong P_+$ on $\overline{M}_+ - \overline{F}_+$.

Note that the construction of L depends only on the normal bundle of $\overline{X}_+$ in $\overline{M}_+$, and hence on the pair $(\overline{M}_-, \overline{X}_-) = (\overline{M}_\lambda, \overline{F}_\lambda)$. Then P_+ is given in terms of $P_-, (\overline{M}_-, \overline{F}_\lambda)$.

We now see examples on which the relation between P_- and P_+ is easy to depict.

Example 3.2.3 Suppose F_1 is an isolated fixed point $p \in M$. The relation between the Euler classes $e(P_1)$ and $e(P_2)$ is clear. If the index of p is 2, let E be the exceptional divisor in B_2 that is introduced after we blow up B_1 at p and let $\mathrm{PD}(E) \in H_2(M)$ be its Poincaré dual. Then we have

$$e(P_2) = \beta^* e(P_1) + \mathrm{PD}(E).$$

If p has coindex 2, then B_2 is obtained by blow down B_1 at an exceptional divisor E. In this case

$$\beta^* e(P_2) = e(P_1) + \mathrm{PD}(E).$$

Example 3.2.4 Suppose codim $F_1 = 4$ and that the index of $F_1 = 2$. By Lemma 5 in McDuff [McD88] all the reduced spaces $\overline{M}_t$ for $t \in I, t \neq \lambda_1$ are all diffeomorphic, to B_1 say. In particular B_2 is diffeomorphic to B_1. Moreover

$$e(P_2) = e(P_1) + \mathrm{PD}(F_1)$$

where F_1 is embedded in B_1. If $\dim M = 6$ this case applies when F_1 is a surface.

3.2.2 The isomorphism type of free slices

As before, let $(M, H, \omega) \in \mathbf{H}\mathrm{Symp}_{2n}$. Suppose $\lambda' < \lambda$ are two consecutive critical values of H. The interval $J = (\lambda', \lambda)$ is a maximal set of regular values of H. Thus S^1 acts freely on the open set $H^{-1}(J)$. Let $I = [t_0, t_1] \subset J$. Let $(\overline{M}_t, \omega_t)$ be the reduced spaces for $t \in I$. The classic Duistermaat-Heckman Theorem establishes that the reduced spaces $\overline{M}_t$ are all identified with the reduced space $B = \overline{M}_{t_0}$ for a fixed point $t_0 \in I$. The diffeomorphism type of $H^{-1}(I)$ is then given by $P_{t_0} \times I$. If ω_t is the reduced symplectic form at the regular level t, then they define a path of symplectic structures on B. Their cohomology classes satisfy

$$[\omega_t] = [\omega_{t_0}] + (t - t_0)e(P), t \in I. \tag{3.6}$$

Our aim is to understand the isomorphism class of $H^{-1}(I)$. To accomplish that, suppose that we have two invariant symplectic forms ω and ω' on $P \times I$ and that there is an equivariant diffeomorphism $\phi : P \times I \longrightarrow P \times I$ such that $\phi^*(\omega) = \omega'$. By considering the reduction at t, ϕ defines a family $\overline{\phi}_t : B \longrightarrow B$ of maps such that the following diagram

$$\begin{array}{ccc} P_t & \xrightarrow{\phi_t} & P_t \\ \pi \downarrow & & \downarrow \pi \\ B & \xrightarrow{\overline{\phi}_t} & B \end{array} \tag{3.7}$$

is commutative. Here we are denoting by ϕ_t the restriction of ϕ to the level t. Note that $(\overline{\phi}_t)^*(\omega_t) = \omega'_t$, where the forms ω_t, ω'_t are the symplectic reductions of ω, ω'. Moreover $\overline{\phi}_t$ preserves the Euler class $e(P)$ for all $t \in I$.

It is not hard to see that given any smooth family of maps $\overline{f}_t : B \longrightarrow B$ such that $(\overline{f}_t)^*\omega_t = \omega'_t$ and $(\overline{f}_t)^*e(P) = e(P)$ lifts to a family of isomorphisms $f_t : P \longrightarrow P$ that make Diagram (3.7) commute. Therefore, these maps bundle together to a diffeomorphism $f : P \times I \longrightarrow P \times I$ such that $f^*(\omega) = \omega'$. In fact the isomorphism type of $H^{-1}(I)$ is given by the following lemma.

Lemma 3.2.5 (Proposition 5.8 in [MS98]) *The symplectic manifold $H^{-1}(I)$ is determined by the bundle $P \longrightarrow B$ and the family ω_t up to an equivariant symplectomorphism.*

For simplicity of notation, assume that $I = [0, 1]$. The discussion above motivates the following definition.

Definition 3.2.6 *Two families $\{\omega_t\}_{t\in I}$ and $\{\omega'_t\}_{t\in I}$ of symplectic forms on B satisfying Equation (3.6) are said to be* **equivalent** *if there is a smooth family $\{\omega_{s,t}\}$ of symplectic forms such that*

$$\frac{d}{ds}[\omega_{s,t}] = 0 \quad \text{and } \omega_{0,t} = \omega_t, \omega_{1,t} = \omega'_t, \tag{3.8}$$

for $0 \leq t, s \leq 1$. We denote by $\breve{\omega}_I$ the equivalence class of the path $\{\omega_t\}_{t\in I}$ on B.

Note that if $\{\omega_t\} \sim \{\omega'_t\}$, there is an isotopy ω_s of forms on $P \times I$ that lifts $\omega_{s,t}$ and such that $\omega_0 = \omega, \omega_1 = \omega'$. By Moser's lemma, we have a family of maps $f_s : P \times I \longrightarrow P \times I$ such that $f_s^*\omega_s = \omega$.

In our case there will be only one of these equivalence classes. This is basically a property of rigid manifolds, as we now see.

Lemma 3.2.7 *Suppose B is a manifold such that every deformation between two cohomologous forms can be homotoped to an isotopy. Consider two families of symplectic forms $\{\omega'_t\}, \{\omega_t\}$ $t \in I$ on B whose cohomology classes satisfy Equation (3.6) and such that $\omega_0 = \omega'_0$. If $\mathrm{Symp}(B, \omega_t) \cap \mathrm{Diff}_0(M)$ is path connected for all $t \in I$, then $\{\omega'_t\} \sim \{\omega_t\}$.*

Proof: Since $[\omega_t] - [\omega'_t] = [\omega_0] - [\omega'_0] = 0$, we see that ω_t is cohomologous to ω'_t for all t. We want to see that there is a family of symplectic forms $\{\omega_{s,t}\}_{0\leq s,t,\leq 1}$ such that for each fixed t the path $\omega_{s,t}$ is an isotopy from ω_t to ω'_t.

First, since $\omega_0 = \omega'_0$, then for $s \in [0,1]$ the cohomology class of the form $s\omega'_t + (1-s)\omega_t$ is constant respect to s. Then by Moser's argument, for an $\epsilon > 0$ small enough there is a family $\omega_{s,t}$ satisfying Equation (3.8) but only for $0 \leq t \leq \epsilon$ (Compare with Example 3.20 in [MS98]). We want to see that under our hypotheses we can take $\epsilon = 1$. To see this, define

$$\mathcal{D} = \{T : \exists \ \{\omega_{s,t}\}_{0 \leq s \leq 1, 0 \leq t \leq T} \text{ satisfying Equation (3.8)}\} \subset [0,1].$$

We claim that $\mathcal{D}$ is actually $[0,1]$. We will do this by proving that $\mathcal{D}$ is open and closed. Let $T \in \mathcal{D}$. Since ω_T is isotopic to ω'_T, we can assume that $\omega_T = \omega'_T$. Then, by the same argument as when $T = 0$, there is an $\varepsilon > 0$ such that $T + \varepsilon \in \mathcal{D}$. Thus $\mathcal{D}$ is open.

To see that $\mathcal{D}$ is closed, take T be such that $0 < T \leq 1$ and $T - \epsilon \in \mathcal{D}$ for every $\epsilon > 0$ small. The path

$$\alpha_s = \begin{cases} \omega_{v(s)} & v(s) = (1-2s)T, 0 \leq s \leq \frac{1}{2} \\ \omega'_{u(s)} & u(s) = (2s-1)T, \frac{1}{2} \leq s \leq 1 \end{cases} \tag{3.9}$$

is a deformation between ω_T and ω'_T. Since B is assumed to be rigid and $[\omega_T] = [\omega'_T]$, α_s can be homotoped through deformations with fixed endpoints to an isotopy. Let β_s such isotopy with $\beta_0 = \omega_T$ and $\beta_1 = \omega'_T$. Again, by Moser's argument, we can suppose β_s can be extended in a small neighborhood of T. That is, for an $\epsilon > 0$ there is a family $\beta_{s,T-\epsilon}$ of symplectic forms which is homotopic to β_s.

On the other hand, because of the hypothesis on T, we have $T - \epsilon \in \mathcal{D}$.

Thus there is a family $\omega_{s,t}$ that satisfies Equation (3.8), for all $t \in [0, T-\epsilon]$. The concatenation of the two isotopies $\omega_{s,T-\epsilon}$, $-\beta_{s,T-\epsilon}$ defined by

$$\gamma_s := \begin{cases} \omega_{s,T-\epsilon} & 0 \leq s \leq 1 \\ \beta_{2-s,T-\epsilon} & 1 \leq s \leq 2 \end{cases}$$

(after smoothing) defines a loop at $\omega_{T-\epsilon}$ in the space of symplectic structures $\mathcal{S}(a)$ with fixed cohomology class $a = [\omega_{T-\epsilon}]$.

By using the fibration

$$\mathrm{Symp}(B, \omega_{T-\epsilon}) \cap \mathrm{Diff}_0(B) \longrightarrow \mathrm{Diff}_0(B) \xrightarrow{\pi} \mathcal{S}(a), \quad \pi : f \mapsto f^*(\omega_{T-\epsilon}),$$

there is a lift $\{f_s\}_{s\in[0,2]}$ in $\mathrm{Diff}_0(B)$ such that $f_0 = id$ and $f_s^*(\gamma_s) = \omega_{T-\epsilon}$ for all s. Since the fibre $\mathrm{Symp}(B, \omega_{T-\epsilon}) \cap \mathrm{Diff}_0(M)$ is path connected, we can assume that $f_2 = id$ as well.

The map

$$h_s := f_s \circ (f_{2-s})^{-1},\ 0 \leq s \leq 1$$

is such that

$$h_0 = h_1 = id, \quad h_s^*(\gamma_{2-s}) = \gamma_s$$

that is,

$$h_s^*(\beta_{2-s,T-\epsilon}) = \omega_{s,T-\epsilon}.$$

Therefore, the new family

$$\hat{\beta}_{s,t} := h_s^*(\beta_{s,t})$$

satisfies Equation (3.8) for $0 \leq s \leq 1, T-\epsilon \leq t \leq T$ and agrees with $\omega_{s,t}$ at

$t = T - \epsilon$. After smoothing, we see that $\omega_{s,t}$ can be extended to all $t \leq T$ via $\hat{\beta}_{s,t}$. Then $T \in \mathcal{D}$. This proves that $\mathcal{D}$ is closed.

We would like to emphasize where our argument fails if $\mathrm{Symp}(B, \omega_{T-\epsilon}) \cap \mathrm{Diff}_0(M)$ is not path connected. In this case, we cannot consider f_2 to be the identity and then $h_0 \neq id$. For our boundary conditions, we need the extension $\hat{\beta}_{0,t}$ to agree with ω_t for $T - \epsilon \leq t \leq T$. We cannot conclude that if h_0 is not the identity.

□

Note that by exhaustion of closed intervals I in $J = (\lambda', \lambda)$, we can assume that any two paths of forms $\omega_t, \omega'_t, t \in J$ are equivalent. In the particular case when λ has index 2 by Theorem 3.2.1 one can assume that $J = (\lambda', \lambda]$ and that $B = \overline{M}_\lambda$. In this case, by Lemma 3.2.7 any two paths of symplectic structures $\omega_t, \omega'_t, t \in (\lambda, \lambda']$ that satisfy Equation (3.6) are equivalent as well. Note that this immediately forces any two gluing maps to be equivalent. Therefore we have the following.

Proposition 3.2.8 *Suppose that λ has index 2 and that $(\overline{M}_\lambda, \{\omega_t\}_{t \in I})$ is rigid. Then, any path of symplectic forms satisfying Equation (3.6) is equivalent to ω_t. Therefore the isomorphism class of the free slice $H^{-1}(\lambda', \lambda)$ is unique up to isomorphism. Moreover, there is unique gluing class between germs at λ and free slices on (λ', λ).*

A similar result follows when λ' has index 4. Now we are ready to prove the results in the introduction.

Proof of Theorem 1.0.6: Recall that $(M, H, \omega) \in \mathbf{H}\mathrm{Symp}_{2n}$ and $\mathcal{C}(M)$ is the set of critical values. We are assuming that each non-extremal critical value $\lambda \in \mathcal{C}(M) = \{\lambda_0, \ldots, \lambda_{s+1}\}$ has (co)index 2 and that all the reduced spaces $\overline{M}_\lambda$ are rigid. Therefore, by Theorem 3.2.1 and Proposition 3.2.8 the free slices, germs, and gluing maps are completely determined by the fixed point data. Hence the local data $\mathcal{L}(M)$ will be determined as well. Then by Theorem 3.1.5 the result will follow. □

Proof of Corollary 1.0.7: Recall that we are assuming that for each $\lambda \in \mathcal{C}(M)$ the fixed point submanifolds $(F_{\lambda_i}, \omega_{\lambda_i})$ are isolated or surfaces of index 2. Its minimal germ is determined by $(F_{\lambda_0}, \omega_{\lambda_0})$. By taking the reduction at a level t close enough to λ_0 one gets the reduced bundle $P_{\lambda_1} \longrightarrow \overline{M}_{\lambda_1}$. Examples 3.2.3, 3.2.4 show that the principal bundle $P_\lambda \longrightarrow \overline{M}_{\lambda_2}$ is then determined by the fixed point data at λ_1. If we do this process at each λ we see that bundles P_λ are actually determined by the minimal information we are assuming, that is the manifolds $(F_\lambda, \omega_\lambda)$ for all $\lambda \in \mathcal{C}(M)$. This determines the fixed point data of M. □

3.3 Six dimensional examples

In this section we will apply our previous analysis to six dimensional examples. We will classify this manifolds by just knowing the minimal information required in Theorem 1.0.7. In §3.3.1 we will provide a complete analysis of the case when all the fixed points are isolated, that is we provide a proof of Theorem 1.0.8.

3.3.1 Example A

Assume that (M, ω) is a symplectic 6-manifold with a semi-free S^1-action. If all the fixed points are isolated it is not necessary to assume that the action is Hamiltonian, it would follow from [TW00]. The fixed points are given as follows. The minimum, three critical points p_1, p_2, p_3 of index two, three q_1, q_2, q_3 of index four and a maximum. We will denote by λ_i the critical values $H(p_i)$. Without lost of generality we may assume the minimum value of H is zero, and that $\lambda_1 \leq \lambda_2 \leq \lambda_3$. It is not hard to see [Gon03] that the fixed points of index 4 are in the level sets $H^{-1}(\lambda_i + \lambda_j), j \neq i$ and the maximum is the unique point in $H^{-1}(\lambda_1 + \lambda_2 + \lambda_3)$ (See Figure 3.2). Before going any further recall that we want to prove that M is isomorphic to $Y^3 = S^2 \times S^2 \times S^2$ with the product symplectic form $\varpi = \lambda_1\sigma \times \lambda_2\sigma \times \lambda_3\sigma$. Here σ is the canonical area form on S^2. We are assuming that the circle acts on Y^3 by

$$e^{2\pi it}(x, y, z) \mapsto (e^{2\pi it}x, e^{2\pi it}y, e^{2\pi it}z).$$

Proof of Theorem 1.0.8: We start by noticing that there are essentially two cases to analyze, when $\lambda_1 + \lambda_2 \leq \lambda_3$ and when $\lambda_1 + \lambda_2 > \lambda_3$. The difference of this two cases is the order in which we reach fixed points. For simplicity, one can treat the first case since the second one is completely analogous. Our aim is to prove that the fixed point data of M is equivalent to the one of Y. Then as before Theorem 3.1.5 will finish the proof. Since the fixed points are isolated, by Theorem 1.0.7 we only need to show that the reduced spaces $(\overline{Y}_t, \varpi_t)$ are rigid, this will prove that the manifold Y is unique, then the theorem will follow.

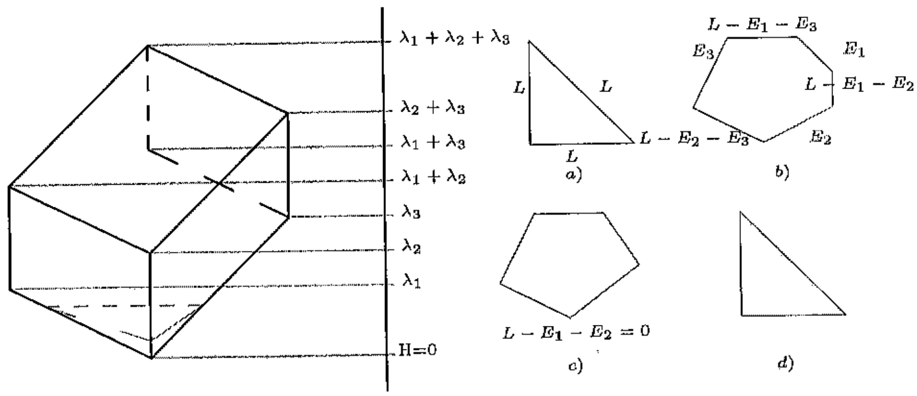

Figure 3.2: The fixed points, their critical values and some reduced spaces of a manifold with isolated fixed points (Example A). Figures $a), b), c)$ and $d)$ represent the reduced spaces for t in the intervals $(0,\lambda_1), (\lambda_3, \lambda_1+\lambda_2), (\lambda_1+\lambda_2, \lambda_1+\lambda_3), (\lambda_2+\lambda_3, \lambda_1+\lambda_2+\lambda_3)$ respectively. Note that as $t \longrightarrow \lambda_1+\lambda_2$ the exceptional sphere $L-E_1-E_2$ blows down. $d)$ is the manifold obtained after blowing down the exceptional spheres $L-E_i-E_j$. $a)$ and $d)$ are diffeomorphic (via Cremona transformation) but the Euler class of the principal bundles associated to these reduced spaces differ by a sign.

It is not hard to see that the only reduced spaces that occur are $\mathbb{C}P^2$ and the blow up $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$ for $k=1,2,3$. (See figure 3.2). Theorem 3.3.1 and Lemma 3.3.2 now finish the proof. □

Theorem 3.3.1 (McDuff, [McD98]) *Let X be a blow-up of $\mathbb{C}P^2$ or a rational surface.* [4] *Then*

i) Any deformation of two cohomologous symplectic forms on X may be homotoped through deformations with fixed endpoints to an isotopy.

[4] This theorem is actually true for more general 4-manifolds, named manifolds of non-simple SW-type.

ii) Any two cohomologous symplectic forms on X are symplectomorphic.

$\square$

Lemma 3.3.2 (Abreu-McDuff, et. al.) *Let (X, ω) be $\mathbb{C}P^2$ or the blow-up $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ with the symplectic forms ϖ_t as above. Then the group $\mathrm{Symp}(X, \varpi_t)$ is path connected for all t. Similarly if (X, ω) denotes any of the blow ups $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$ for $k \leq 3$, then the group $\mathrm{Symp}^H(X, \omega)$ of symplectomorphisms that induce the identity on $H_*(X)$ is path connected.*

We shall not discuss the proof of this lemma, since the arguments are far from the context of the present work and they are contained in several papers. The reader can consult the original articles [Gro85],[AM00], [LP04], [Pin] or the survey [McD04] for all of these cases. The technics involved are pseudo-holomorphic curves and the *inflation* procedure.

3.3.2 Example B

This is example is based on Example 1 in [Li03]. Let $Z = (\mathbb{C}P^3, \omega_{FS})$ equipped with the standard Fubini-Study form ω_{FS} and with the semi-free circle action

$$e^{2\pi it}[z_0 : z_1 : z_2 : z_3] \mapsto [e^{2\pi it}z_0 : e^{2\pi it}z_1 : z_2 : z_3],$$

whose Hamiltonian function is

$$H_Z([z_0 : z_1 : z_2 : z_3]) = \frac{1}{2}\frac{|z_0|^2 + |z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2}.$$

The fixed submanifolds are the subsets

$$F_1 := H_Z^{-1}(1) = [z_0 : z_1 : 0 : 0] \text{ and } F_0 := H_Z^{-1}(0) = [0 : 0 : z_2 : z_3].$$

being both of them copies of $\mathbb{C}P^1 \cong S^2$ with the form $\omega_{FS} = \frac{1}{4}\sigma$. The reduced spaces $\overline{Z}_t$ are all diffeomorphic to the product $S^2 \times S^2$ (see Figure 3.3.2) where the two factors $A = S^2 \times pt, B = pt \times S^2$ correspond to the spheres

$$H^{-1}(t) \cap \{z_2 = 0\}/S^1 \text{ and } H^{-1}(t) \cap \{z_3 = 0\}/S^1$$

respectively. The reduced form is given by

$$\omega_t := (1-t)\omega_{FS} \times t\ \omega_{FS} \text{ for all } t \in (1, 0).$$

Assume (M, ω) is any other Hamiltonian S^1 manifold whose minimum and maximum submanifolds $(M_0, \omega_1), (M_1, \omega_2)$ are both symplectomorphic to S^2, ω_{FS}. Suppose that $H(M_0) = 0, H(M_1) = 1$. Moreover, suppose that the normal bundle at the minimum has Chern number 2. These data is enough to prove that M is isomorphic to Z. To see this we can use Theorem 1.0.6. Note that Theorem 3.3.1 applies to $S^2 \times S^2$. The result will follow if the group of symplectomorphisms $\mathrm{Symp}(S^2 \times S^2, \omega_t)$ is path connected. This is true if A, B have different symplectic areas [McD04]. Thus is rigid, and Theorem 1.0.6 classifies these manifolds.

Remark 3.3.3 We remark that although we can sometimes ignore the reduction bundles from the fixed point data as in Examples A and B in §3.3.1, we

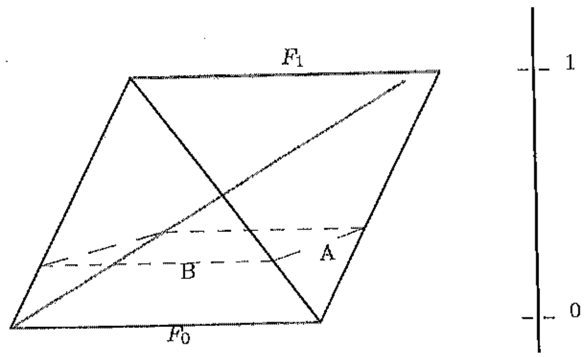

Figure 3.3: $\mathbb{C}P^3$ as in Example B. The reduced spaces are all diffeomorphic to $S^2 \times S^2$. The area of the first sphere A increases as t increases but the area of B vanishes as t reaches the maximum.

cannot do the same with all the data of Definition 1.0.4. For instance if one just knows the diffeomorphism type of the fixed submanifolds, and not its normal bundle in the reduced spaces, we cannot conclude that the isomorphism type of manifold is determined by these data. Li [Li03] has constructed two Hamiltonian S^1-manifolds with diffeomorphic fixed point data, but different isomorphism type. In Li's situation to conclude that the manifolds are (diffeomorphic), it is necessary to prescribe the *twist* type of the manifold. This is a global invariant of the manifold. Theorem 1.0.6 shows that this twist is determined if we have rigid reduced spaces and richer fixed point data. For more information see [Li03, Li05].

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